

Fixed Point Results for $\mathcal{F}_{(S,\mathcal{T})}$ -Contraction in S -Metric Spaces Using Implicit Relation with Applications

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Abstract: The main purpose of this paper is to study and establish some fixed point theorems for $\mathcal{F}_{(S,\mathcal{T})}$ -contraction in the setting of S -metric space via an implicit relation. The results presented in this paper extend, unify and generalize several known results from the existing literature. Also, we give one of the possible applications of our result to well-posed and limit shadowing property of fixed point problems.

Key Words: Fixed point, implicit relation, $\mathcal{F}_{(S,\mathcal{T})}$ -contraction, S -metric space.

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§1. Introduction

Fixed point theory is one of the most important topic in the development of nonlinear analysis. As it is well known, one of the most useful theorems in nonlinear analysis is the Banach contraction principle [9]. A mapping $\mathcal{T}: X \rightarrow X$ where (X, d) is a metric space, is said to be a contraction if there exists $c \in [0, 1)$ such that for all $x, y \in X$,

$$d(\mathcal{T}(x), \mathcal{T}(y)) \leq c d(x, y). \quad (1.1)$$

If the metric space (X, d) is complete then the mapping satisfying (1.1) has a unique fixed point. Inequality (1.1) implies continuity of \mathcal{T} . Many authors generalized this famous result in different ways. In recent time the study of fixed point theory in metric space is very interesting field and attract many researchers to investigated different results on it.

In 2006, Mustafa and Sims [2] introduced a new notion of generalized metric space, called G -metric space and gave a modification to the contraction principle of Banach. After then, several authors studied various fixed and common fixed point problems for adequate classes of contractive mappings in generalized metric spaces (see, [1, 2, 3, 4, 8, 10, 12, 16, 22, 23, 24, 26, 33, 40, 41, 42]).

In 2012, Sedghi et al. [38] introduced the notion of S -metric space which is a generalization of a G -metric space and D^* -metric space. In [38] the authors proved some basic properties of S -metric spaces. Also, they obtained some fixed point theorems in S -metric space for a self-

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map. Afterwards, a multitude of results was obtained in these spaces (see, e.g., [13, 39, 32]) and many others.

Sedghi et al. [38] introduced the notion of S -metric spaces as follows:

Definition 1.1([38]) *Let X be a nonempty set and $S: X^3 \rightarrow \mathbb{R}^+$ be a function satisfying the following conditions*

(S1) $S(x, y, z) = 0$ if and only if $x = y = z$;

(S2) $S(x, y, z) \leq S(x, x, t) + S(y, y, t) + S(z, z, t)$ for all $x, y, z, t \in X$, where $\mathbb{R}^+ = [0, \infty)$,

Then, the function S is called an S -metric on X and the pair (X, S) is called an S -metric space or simply SMS.

Example 1.2([38]) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X , then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S -metric on X .

Example 1.3([38]) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ a norm on X , then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S -metric on X .

Example 1.4([39]) Let $X = \mathbb{R}$ be the real line. Then $S(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an S -metric on X . This S -metric on X is called the usual S -metric on X .

Lemma 1.5 ([38], Lemma 2.5) *If (X, S) be an S -metric space, then we have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

Lemma 1.6 ([38], Lemma 2.12) *Let (X, S) be an S -metric space. If $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$ then $S(x_n, x_n, y_n) \rightarrow S(x, x, y)$ as $n \rightarrow \infty$.*

Definition 1.7([38]) *Let (X, S) be an S -metric space.*

(a1) *A sequence $\{x_n\}$ in X converges to $x \in X$ if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \varepsilon$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow \infty$;*

(a2) *A sequence $\{x_n\}$ in X is called a Cauchy sequence if $S(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \varepsilon$;*

(a3) *The S -metric space (X, S) is called complete if every Cauchy sequence in (X, S) is convergent in (X, S) .*

Definition 1.8 *Let T be a self mapping on an S -metric space (X, S) . Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X with $x_n \rightarrow x$ implies that $Tx_n \rightarrow Tx$ as $n \rightarrow \infty$.*

Definition 1.9([38]) *Let (X, S) be an S -metric space. A mapping $T: X \rightarrow X$ is said to be a contraction if there exists a constant $0 \leq L < 1$ such that*

$$S(Tx, Ty, Tz) \leq LS(x, y, z) \tag{1.2}$$

for all $x, y, z \in X$.

Notice that if the S -metric space (X, S) is complete, then the mapping defined in the Definition 1.9 has a unique fixed point ([38]).

Moradi and Beiranvand [19] introduced the following notion.

Definition 1.10([19]) *Let (X, d) be a metric space and $f, \mathcal{T}: X \rightarrow X$ be two mappings. The mapping f is said to be a $\mathcal{F}_{\mathcal{T}}$ -contraction if there exists $a \in [0, 1)$ such that for all $x, y \in X$*

$$\mathcal{F}\left(d(\mathcal{T}fx, \mathcal{T}fy)\right) \leq a\mathcal{F}\left(d(\mathcal{T}x, \mathcal{T}y)\right), \quad (1.3)$$

where,

- (1) $\mathcal{F}: [0, \infty) \rightarrow [0, \infty)$, \mathcal{F} is nondecreasing continuous from the right and $\mathcal{F}^{-1}(0) = \{0\}$;
- (2) \mathcal{T} is one to one and graph closed.

We introduce the definition of $\mathcal{F}_{(S,\mathcal{T})}$ -contraction following.

Definition 1.11 *Let (X, S) be an S -metric space and $\mathcal{T}: X \rightarrow X$ be a mapping. The mapping \mathcal{T} is said to be a $\mathcal{F}_{(S,\mathcal{T})}$ -contraction if there exists $a \in [0, 1)$ such that for all $x, y, z \in X$ and*

$$\mathcal{F}\left(S(\mathcal{T}x, \mathcal{T}y, \mathcal{T}z)\right) \leq a\mathcal{F}\left(S(x, y, z)\right), \quad (1.4)$$

where $\mathcal{F}: [0, \infty) \rightarrow [0, \infty)$ is a function satisfying the following conditions

- (\mathcal{F}_1) \mathcal{F} is nondecreasing;
- (\mathcal{F}_2) \mathcal{F} is continuous from the right and;
- (\mathcal{F}_3) $\mathcal{F}^{-1}(0) = \{0\}$.

Remark 1.12 *If we take $\mathcal{F}(t) = t$ in equation (1.4), then we obtain Banach contraction type condition (1.2) in S -metric space (X, S) with $a = L$ and if X is complete then \mathcal{T} has a unique fixed point.*

Now, we introduce an implicit relation to investigate some fixed point theorems in S -metric spaces.

Definition 1.13 (Implicit Relation) *Let Φ be the family of all real valued continuous functions $\phi: \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$, for three variables. For some $h \in [0, 1)$, we consider the following conditions*

- (R1) For $x, y \in \mathbb{R}_+$, if $x \leq \phi(y, y, x)$, then $x \leq hy$;
- (R2) For $x \in \mathbb{R}_+$, if $x \leq \phi(0, 0, x)$, then $x = 0$;
- (R3) For $x \in \mathbb{R}_+$, if $x \leq \phi(x, 0, 0)$, then $x = 0$ since $h \in [0, 1)$.

The main purpose of this paper is to study $\mathcal{F}_{(S,\mathcal{T})}$ -contraction in S -metric space and establish some fixed point theorems under an implicit relation. The results presented in this paper extend, generalize and unify several known results from the existing literature. Also, we give one of the possible applications of our result to well-posed and limit shadowing property of fixed point problems.

§2. Main Results

In this section, we shall prove some fixed point theorems for $\mathcal{F}_{(S,\mathcal{T})}$ -contraction under an implicit relation in the setting of S -metric spaces.

Theorem 2.1 *Let (X, S) be a complete S -metric space and $\mathcal{T}: X \rightarrow X$ be a mapping. If for all $x, y \in X$ and*

$$\mathcal{F}\left(S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)\right) \leq \phi\left\{\mathcal{F}\left(S(x, x, y)\right), \mathcal{F}\left(S(x, x, \mathcal{T}x)\right), \mathcal{F}\left(S(y, y, \mathcal{T}y)\right)\right\} \quad (2.1)$$

where $\mathcal{F}: [0, \infty) \rightarrow [0, \infty)$ is nondecreasing continuous function and $\mathcal{F}(t) = 0$ if and only if $t = 0$ and some $\phi \in \Phi$. Then, we have

- (1) If ϕ satisfies the conditions (R1) and (R2), then \mathcal{T} has a fixed point;
- (2) If ϕ satisfies the condition (R3) and \mathcal{T} has a fixed point, then the fixed point is unique.

Proof (1) Let $x_0 \in X$ be an arbitrary point and $x_n = \mathcal{T}x_{n-1} = \mathcal{T}^n x_0$, $n = 0, 1, 2, \dots$. Now, from (2.1) we have

$$\begin{aligned} \mathcal{F}\left(S(x_{n+1}, x_{n+1}, x_{n+2})\right) &= \mathcal{F}\left(S(\mathcal{T}x_n, \mathcal{T}x_n, \mathcal{T}x_{n+1})\right) \\ &\leq \phi\left\{\mathcal{F}\left(S(x_n, x_n, x_{n+1})\right), \mathcal{F}\left(S(x_n, x_n, \mathcal{T}x_n)\right), \mathcal{F}\left(S(x_{n+1}, x_{n+1}, \mathcal{T}x_{n+1})\right)\right\} \\ &= \phi\left\{\mathcal{F}\left(S(x_n, x_n, x_{n+1})\right), \mathcal{F}\left(S(x_n, x_n, x_{n+1})\right), \mathcal{F}\left(S(x_{n+1}, x_{n+1}, x_{n+2})\right)\right\}. \end{aligned} \quad (2.2)$$

Since ϕ satisfies the condition (R1), there exists $h \in [0, 1)$ such that

$$\begin{aligned} \mathcal{F}\left(S(x_{n+1}, x_{n+1}, x_{n+2})\right) &\leq h\mathcal{F}\left(S(x_n, x_n, x_{n+1})\right) \leq \dots \\ &\leq h^{n+1}\mathcal{F}\left(S(x_0, x_0, x_1)\right). \end{aligned} \quad (2.3)$$

Thus, for all $n < m$, by using (S2) Lemma 1.5 and equation (2.3) we have

$$\begin{aligned} \mathcal{F}\left(S(x_n, x_n, x_m)\right) &\leq \mathcal{F}\left(2S(x_n, x_n, x_{n+1}) + S(x_m, x_m, x_{n+1})\right) \\ &= \mathcal{F}\left(2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_m)\right) \\ &\dots \\ &\leq \mathcal{F}\left(2[h^n + \dots + h^{m-1}]S(x_0, x_0, x_1)\right) \\ &\leq \mathcal{F}\left(\left(\frac{2h^n}{1-h}\right)S(x_0, x_0, x_1)\right). \end{aligned}$$

Taking the limit as $n, m \rightarrow \infty$ and using the property of \mathcal{F} , we get $\mathcal{F}\left(S(x_n, x_n, x_m)\right) \rightarrow 0^+$, since $0 < h < 1$. As \mathcal{F} is continuous, we obtain $S(x_n, x_n, x_m) = 0$. This proves that the sequence $\{x_n\}$ is a Cauchy sequence in the complete S -metric space (X, S) . By the completeness of the space, there exists $v \in X$ such that $\{x_n\}$ converges to $v \in X$. Now,

we prove that v is a fixed point of \mathcal{T} . Again by using inequality (2.1), we obtain

$$\begin{aligned} \mathcal{F}\left(S(x_{n+1}, x_{n+1}, \mathcal{T}v)\right) &= \mathcal{F}\left(S(\mathcal{T}x_n, \mathcal{T}x_n, \mathcal{T}v)\right) \\ &\leq \phi\left\{\mathcal{F}\left(S(x_n, x_n, v)\right), \mathcal{F}\left(S(x_n, x_n, \mathcal{T}x_n)\right), \mathcal{F}\left(S(v, v, \mathcal{T}v)\right)\right\} \\ &= \phi\left\{\mathcal{F}\left(S(x_n, x_n, v)\right), \mathcal{F}\left(S(x_n, x_n, x_{n+1})\right), \mathcal{F}\left(S(v, v, \mathcal{T}v)\right)\right\}. \end{aligned} \quad (2.4)$$

Indeed, as \mathcal{F} is continuous and note that $\phi \in \Phi$, then using the property of \mathcal{F} and taking the limit as $n \rightarrow \infty$, we get

$$\begin{aligned} \mathcal{F}\left(S(v, v, \mathcal{T}v)\right) &\leq \phi\left\{\mathcal{F}\left(S(v, v, v)\right), \mathcal{F}\left(S(v, v, v)\right), \mathcal{F}\left(S(v, v, \mathcal{T}v)\right)\right\} \\ &= \phi\left\{0, 0, \mathcal{F}\left(S(v, v, \mathcal{T}v)\right)\right\}. \end{aligned}$$

Since ϕ satisfies the condition (R2), then $\mathcal{F}\left(S(v, v, \mathcal{T}v)\right) \leq h \cdot 0 = 0$. This implies that $S(v, v, \mathcal{T}v) = 0$. Thus, $v = \mathcal{T}v$. Hence v is a fixed point of \mathcal{T} .

(2) Let u_1, u_2 be fixed points of f with $u_1 \neq u_2$. We shall prove that $u_1 = u_2$. It follows from equation (2.1) and property of \mathcal{F} that

$$\begin{aligned} \mathcal{F}\left(S(u_1, u_1, u_2)\right) &= \mathcal{F}\left(S(\mathcal{T}u_1, \mathcal{T}u_1, \mathcal{T}u_2)\right) \\ &\leq \phi\left\{\mathcal{F}\left(S(u_1, u_1, u_2)\right), \mathcal{F}\left(S(u_1, u_1, \mathcal{T}u_1)\right), \mathcal{F}\left(S(u_2, u_2, \mathcal{T}u_2)\right)\right\} \\ &= \phi\left\{\mathcal{F}\left(S(u_1, u_1, u_2)\right), \mathcal{F}\left(S(u_1, u_1, u_1)\right), \mathcal{F}\left(S(u_2, u_2, u_2)\right)\right\} \\ &= \phi\left\{\mathcal{F}\left(S(u_1, u_1, u_2)\right), 0, 0\right\}. \end{aligned}$$

Since ϕ satisfies the condition (R3), we get

$$\begin{aligned} \mathcal{F}\left(S(u_1, u_1, u_2)\right) &\leq h \mathcal{F}\left(S(u_1, u_1, u_2)\right) \\ &\Rightarrow \mathcal{F}\left(S(u_1, u_1, u_2)\right) = 0, \text{ because of } 0 < h < 1. \end{aligned}$$

This implies that $S(u_1, u_1, u_2) = 0$. Thus, $u_1 = u_2$. This shows that the fixed point of \mathcal{T} is unique. This completes the proof. \square

Theorem 2.2 *Let (X, S) be a complete S -metric space such that for positive integer n , \mathcal{T}^n satisfies the contraction condition (2.1) for all $x, y \in X$, where \mathcal{F} and ϕ are as in Theorem 2.1. Then \mathcal{T} has a unique fixed point in X .*

Proof From Theorem 2.1, let u_0 be the unique fixed point of \mathcal{T}^n . Then

$$\mathcal{T}(\mathcal{T}^n u_0) = \mathcal{T}u_0 \quad \text{or} \quad \mathcal{T}^n(\mathcal{T}u_0) = \mathcal{T}u_0,$$

which gives $\mathcal{T}u_0 = u_0$. This shows that u_0 is a unique fixed point of \mathcal{T} . This completes the proof. \square

In Theorem 2.1, if we consider \mathcal{F} is an identity map, then we obtain the following result as corollary.

Corollary 2.3 *Let (X, S) be a complete S -metric space and $\mathcal{T}: X \rightarrow X$ be a mapping satisfying the inequality*

$$S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) \leq \phi \left\{ S(x, x, y), S(x, x, \mathcal{T}x), S(y, y, \mathcal{T}y) \right\}$$

for all $x, y \in X$ and some $\phi \in \Phi$. If ϕ satisfies the conditions (R1), (R2) and (R3), then \mathcal{T} has a unique fixed point in X .

Next, we give an analogues of fixed point theorems in metric spaces for S -metric spaces by combining Theorem 2.1 with $\phi \in \Phi$ and ϕ satisfies conditions (R1), (R2) and (R3). The following corollary is an analogue of Banach's type contraction principle.

Corollary 2.4 *Let (X, S) be a complete S -metric space and $\mathcal{T}: X \rightarrow X$ be a mapping. If for all $K_1 \in [0, 1)$ and $x, y \in X$ and satisfying the inequality*

$$\mathcal{F}\left(S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)\right) \leq K_1 \mathcal{F}\left(S(x, y, z)\right)$$

where $\mathcal{F}: [0, \infty) \rightarrow [0, \infty)$ is nondecreasing continuous function and $\mathcal{F}(t) = 0$ if and only if $t = 0$. Then \mathcal{T} has a unique fixed point in X .

Proof The assertion follows using Theorem 2.1 with $\phi(p, q, r) = K_1 p$ for some $K_1 \in [0, 1)$ and all $p, q, r \in \mathbb{R}_+$. \square

The following corollary is an analogue of R. Kannan's type result [15].

Corollary 2.5 *Let (X, S) be a complete S -metric space and $\mathcal{T}: X \rightarrow X$ be a mapping. If for all $K_2 \in [0, \frac{1}{2})$ and $x, y \in X$ and satisfying the inequality*

$$\mathcal{F}\left(S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)\right) \leq K_2 \left[\mathcal{F}\left(S(x, x, \mathcal{T}x)\right) + \mathcal{F}\left(S(y, y, \mathcal{T}y)\right) \right]$$

where $\mathcal{F}: [0, \infty) \rightarrow [0, \infty)$ is nondecreasing continuous function and $\mathcal{F}(t) = 0$ if and only if $t = 0$. Then \mathcal{T} has a unique fixed point in X .

Proof The assertion follows using Theorem 2.1 with $\phi(p, q, r) = K_2(q + r)$ for some $K_2 \in [0, \frac{1}{2})$ and all $p, q, r \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(y, y, x) = K_2(y + x)$. So, if $x \leq \phi(y, y, x)$, then $x \leq \left(\frac{K_2}{1-K_2}\right)y$ with $\left(\frac{K_2}{1-K_2}\right) < 1$. Thus, \mathcal{T} satisfies the condition (R1).

Next, if $x \leq \phi(0, 0, x) = K_2(0 + x) = K_2x$, then $x = 0$, since $K_2 < \frac{1}{2} < 1$. Thus, \mathcal{T} satisfies the condition (R2).

Finally, if $x \leq \phi(x, 0, 0) = K_2 \cdot 0 = 0$, then $x = 0$. Thus, \mathcal{T} satisfies the condition (R3). \square

The following corollary is an analogue of S. Reich's type result [34].

Corollary 2.6 *Let (X, S) be a complete S -metric space and $\mathcal{T}: X \rightarrow X$ be a mapping. If for*

all $A_1, A_2, A_3 \geq 0$ with $A_1 + A_2 + A_3 < 1$ and $x, y \in X$ and satisfying the inequality

$$\begin{aligned} \mathcal{F}\left(S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y)\right) &\leq A_1 \mathcal{F}\left(S(x, x, y)\right) + A_2 \mathcal{F}\left(S(x, x, \mathcal{T}x)\right) \\ &\quad + A_3 \mathcal{F}\left(S(y, y, \mathcal{T}y)\right) \end{aligned}$$

where $\mathcal{F}: [0, \infty) \rightarrow [0, \infty)$ is nondecreasing continuous function and $\mathcal{F}(t) = 0$ if and only if $t = 0$. Then \mathcal{T} has a unique fixed point in X .

Proof The assertion follows using Theorem 2.1 with $\phi(p, q, r) = A_1p + A_2q + A_3r$ for some $A_1, A_2, A_3 \geq 0$ are constants with $A_1 + A_2 + A_3 < 1$ and all $p, q, r \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(y, y, x) = A_1y + A_2y + A_3x$. So, if $x \leq \phi(y, y, x)$, then $x \leq \left(\frac{A_1+A_2}{1-A_3}\right)y$ with $\left(\frac{A_1+A_2}{1-A_3}\right) < 1$. Thus, \mathcal{T} satisfies the condition (R1).

Next, if $x \leq \phi(0, 0, x) = A_1 \cdot 0 + A_2 \cdot 0 + A_3x = A_3x$, then $x = 0$ since $A_3 < 1$. Thus, \mathcal{T} satisfies the condition (R2).

Finally, if $x \leq \phi(x, 0, 0) = A_1 \cdot x + A_2 \cdot 0 + A_3 \cdot 0 = A_1x$, then $x = 0$ since $A_1 < 1$. Thus, \mathcal{T} satisfies the condition (R3). \square

Again, we give an analogues of fixed point theorems in metric spaces for S -metric spaces by combining Corollary 2.3 with $\phi \in \Phi$ and ϕ satisfies conditions (R1), (R2) and (R3). The following corollary is an analogue of Banach's type contraction principle.

Corollary 2.7 *Let (X, S) be a complete S -metric space. Suppose that the mapping $\mathcal{T}: X \rightarrow X$ satisfies the following condition:*

$$S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) \leq L S(x, x, y)$$

for all $x, y \in X$, where $L \in [0, 1)$ is a constant. Then \mathcal{T} has a unique fixed point in X . Moreover, \mathcal{T} is continuous at the fixed point.

Proof The assertion follows using Corollary 2.3 with $\phi(p, q, r) = Lp$ for some $L \in [0, 1)$ and all $p, q, r \in \mathbb{R}_+$. \square

The following corollary is an analogue of R. Kannan's type result [15].

Corollary 2.8 *Let (X, S) be a complete S -metric space. Suppose that the mapping $\mathcal{T}: X \rightarrow X$ satisfies the following condition:*

$$S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) \leq M [S(x, x, \mathcal{T}x) + S(y, y, \mathcal{T}y)]$$

for all $x, y \in X$, where $M \in [0, \frac{1}{2})$ is a constant. Then \mathcal{T} has a unique fixed point in X . Moreover, \mathcal{T} is continuous at the fixed point.

Proof The assertion follows using Corollary 2.3 with $\phi(p, q, r) = M(q + r)$ for some $M \in [0, \frac{1}{2})$ and all $p, q, r \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(y, y, x) = M(y + x)$. So, if $x \leq \phi(y, y, x)$, then $x \leq \left(\frac{M}{1-M}\right)y$ with $\left(\frac{M}{1-M}\right) < 1$. Thus, f satisfies the condition (R1).

Next, if $x \leq \phi(0, 0, x) = M(0 + x) = Mx$, then $x = 0$, since $M < \frac{1}{2} < 1$. Thus, \mathcal{T} satisfies the condition (R2).

Finally, if $x \leq \phi(x, 0, 0) = M.0 = 0$, then $x = 0$. Thus, \mathcal{T} satisfies the condition (R3). \square

The following corollary is an analogue of S. Reich's type result [34].

Corollary 2.9 *Let (X, S) be a complete S -metric space. Suppose that the mapping $\mathcal{T}: X \rightarrow X$ satisfies the following condition:*

$$S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) \leq k_1 S(x, x, y) + k_2 S(x, x, \mathcal{T}x) + k_3 S(y, y, \mathcal{T}y)$$

for all $x, y \in X$, where $k_1, k_2, k_3 \geq 0$ are constants with $k_1 + k_2 + k_3 < 1$. Then \mathcal{T} has a unique fixed point in X . Moreover, if $k_3 < \frac{1}{2}$, then \mathcal{T} is continuous at the fixed point.

Proof The assertion follows using Corollary 2.3 with $\phi(p, q, r) = k_1 p + k_2 q + k_3 r$ for some $k_1, k_2, k_3 \geq 0$ are constants with $k_1 + k_2 + k_3 < 1$ and all $p, q, r \in \mathbb{R}_+$. Indeed, ϕ is continuous. First, we have $\phi(y, y, x) = k_1 y + k_2 y + k_3 x$. So, if $x \leq \phi(y, y, x)$, then $x \leq \left(\frac{k_1 + k_2}{1 - k_3}\right)y$ with $\left(\frac{k_1 + k_2}{1 - k_3}\right) < 1$. Thus, \mathcal{T} satisfies the condition (R1).

Next, if $x \leq \phi(0, 0, x) = k_1.0 + k_2.0 + k_3.x = k_3.x$, then $x = 0$ since $k_3 < 1$. Thus, \mathcal{T} satisfies the condition (R2).

Finally, if $x \leq \phi(x, 0, 0) = k_1.x + k_2.0 + k_3.0 = k_1.x$, then $x = 0$ since $k_1 < 1$. Thus, \mathcal{T} satisfies the condition (R3). \square

Example 2.10 Let $X = \mathbb{R}$ be the usual S -metric space as in Example 1.4. Now, we consider the mapping $\mathcal{T}: X \rightarrow X$ by $\mathcal{T}(x) = \frac{x}{10}$ for all $x \in [0, 1]$. Then

$$\begin{aligned} S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) &= |\mathcal{T}x - \mathcal{T}y| + |\mathcal{T}x - \mathcal{T}y| \\ &= 2|\mathcal{T}x - \mathcal{T}y| = 2\left|\left(\frac{x}{10}\right) - \left(\frac{y}{10}\right)\right| \\ &= \frac{1}{5}|x - y| \leq \frac{2}{5}|x - y| \\ &= \frac{1}{5}(2|x - y|) = \alpha S(x, x, y) \end{aligned}$$

where $\alpha = \frac{1}{5} < 1$. Thus \mathcal{T} satisfies all the conditions of Corollary 2.7 and clearly $0 \in X$ is the unique fixed point of \mathcal{T} .

Example 2.11 Let $X = \mathbb{R}$ be the usual S -metric space as in Example 1.4. Now, we consider the mapping $T: X \rightarrow X$ by $\mathcal{T}(x) = \frac{x}{5}$ for all $x \in [0, 1]$. Then

$$\begin{aligned} S(\mathcal{T}x, \mathcal{T}x, \mathcal{T}y) &= |\mathcal{T}x - \mathcal{T}y| + |\mathcal{T}x - \mathcal{T}y| \\ &= 2|\mathcal{T}x - \mathcal{T}y| = 2\left|\left(\frac{x}{5}\right) - \left(\frac{y}{5}\right)\right| \\ &= \frac{2}{5}|x - y| \leq \frac{8}{15}|x - y| \\ &\leq \frac{1}{3}\left[\frac{8}{5}|x| + \frac{8}{5}|y|\right]. \end{aligned}$$

$$\begin{aligned} S(x, x, Tx) &= 2|x - Tx| = \frac{8}{5}|x|, \\ S(y, y, Ty) &= 2|y - Ty| = \frac{8}{5}|y|. \end{aligned}$$

Now, we have

$$\begin{aligned} S(Tx, Tx, Ty) &\leq \frac{1}{3}[S(x, x, Tx) + S(y, y, Ty)] \\ &= \beta[S(x, x, Tx) + S(y, y, Ty)] \end{aligned}$$

where $\beta = \frac{1}{3} < \frac{1}{2}$. Thus \mathcal{T} satisfies all the conditions of Corollary 2.8 and clearly $0 \in X$ is the unique fixed point of \mathcal{T} .

Example 2.12 Let $X = [0, 1]$. We define $S: X^3 \rightarrow \mathbb{R}_+$ by

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{if otherwise.} \end{cases}$$

for all $x, y, z \in X$. Then (X, S) is a complete S -metric space. Let $\mathcal{T}: X \rightarrow X$ be a mapping defined as $\mathcal{T}(x) = \frac{x}{2}$ for all $x \in X$.

Without loss of generality we may assume that $x > y > z$, then we have

$$\begin{aligned} S(Tx, Tx, Ty) &= \max\left\{\frac{x}{2}, \frac{x}{2}, \frac{y}{2}\right\} = \frac{x}{2}, \\ S(x, x, y) &= \max\{x, x, y\} = x, \\ S(x, x, Tx) &= \max\left\{x, x, \frac{x}{2}\right\} = x, \\ S(y, y, Ty) &= \max\left\{y, y, \frac{y}{2}\right\} = y, \end{aligned}$$

Now, we consider the inequality of Corollary 2.9, we have

$$S(Tx, Ty, Tz) = \frac{x}{2} \leq k_1 \cdot x + k_2 \cdot x + k_3 \cdot y,$$

taking $x = 1$ and $y = 0$ in the above inequality, we obtain

$$\frac{1}{2} \leq k_1 + k_2,$$

the above inequality is satisfied for $k_1 = \frac{1}{4}$, $k_2 = \frac{2}{5}$ and $k_3 = 0$ with $k_1 + k_2 + k_3 = \frac{13}{20} < 1$. Thus \mathcal{T} satisfies all the conditions of Corollary 2.9 and clearly $0 \in X$ is the unique fixed point of \mathcal{T} .

§3. Application to Well Posedness and Limit Shadowing of Fixed Point Problem

The concept of well posedness of a fixed point problem has generated much interest to several mathematicians, for example [6, 7, 11, 18, 30, 31, 35]. Here, we study well posedness and limit

shadowing of a fixed point problem of mappings in Theorem 2.1.

Definition 3.1([11]) *Let (X, d) be a metric space and $\mathcal{T}: X \rightarrow X$ be a mapping. The fixed point problem of \mathcal{T} is said to be well-posed if*

- (i) \mathcal{T} has a unique fixed point z in X ;
- (ii) for any sequence $\{x_n\}$ of points in X such that $\lim_{n \rightarrow \infty} d(\mathcal{T}x_n, x_n) = 0$, $\lim_{n \rightarrow \infty} d(x_n, z) = 0$.

The limit shadowing property of fixed point problems has been discussed in the papers [27, 28, 36] and others.

Definition 3.2([29]) *Let (X, d) be a metric space and $\mathcal{T}: X \rightarrow X$ be a mapping. The fixed point problem of \mathcal{T} is said to have limit shadowing property in X if assuming that sequence $\{x_n\}$ in X satisfies $d(\mathcal{T}x_n, x_n) = 0$ as $n \rightarrow \infty$ it follows that there exists $x \in X$ such that $d(\mathcal{T}^n x, x_n) = 0$ as $n \rightarrow \infty$.*

Now, we define the above notion in S -metric space.

Definition 3.3 *Let (X, S) be a S -metric space and $\mathcal{T}: X \rightarrow X$ be a mapping. The fixed point problem of \mathcal{T} is said to be well-posed if*

- (i) \mathcal{T} has a unique fixed point z in X ;
- (ii) for any sequence $\{x_n\}$ of points in X such that

$$\lim_{n \rightarrow \infty} S(\mathcal{T}x_n, \mathcal{T}x_n, x_n) = 0 = \lim_{n \rightarrow \infty} S(x_n, x_n, \mathcal{T}x_n),$$

we have $\lim_{n \rightarrow \infty} S(x_n, x_n, z) = 0 = \lim_{n \rightarrow \infty} S(z, z, x_n)$.

Definition 3.4 *Let (X, S) be an S -metric space and $\mathcal{T}: X \rightarrow X$ be a mapping. The fixed point problem of \mathcal{T} is said to have limit shadowing property in X if assuming that sequence $\{x_n\}$ in X satisfies $\lim_{n \rightarrow \infty} S(\mathcal{T}x_n, \mathcal{T}x_n, x_n) = 0 = \lim_{n \rightarrow \infty} S(x_n, x_n, \mathcal{T}x_n)$ it follows that there exists $z \in X$ such that $\lim_{n \rightarrow \infty} S(\mathcal{T}^n z, \mathcal{T}^n z, x_n) = 0 = \lim_{n \rightarrow \infty} S(x_n, x_n, \mathcal{T}^n z)$.*

Concerning the well-posedness and limit shadowing of the fixed point problem for a mapping in a S -metric space satisfying the conditions of Theorem 2.1, we have the following result.

Theorem 3.5 *Let $\mathcal{T}: X \rightarrow X$ be a self mapping as in Theorem 2.1. Then, the fixed point problem for \mathcal{T} is well posed.*

Proof According to Theorem 2.1, we know that \mathcal{T} has a unique fixed point $z = \mathcal{T}z \in X$. Let $\{x_n\} \subset X$ be such that $\lim_{n \rightarrow \infty} S(\mathcal{T}x_n, \mathcal{T}x_n, x_n) = 0 = \lim_{n \rightarrow \infty} S(x_n, x_n, \mathcal{T}x_n)$. Then, we have

$$\begin{aligned} S(x_n, x_n, z) &\leq 2S(x_n, x_n, \mathcal{T}x_n) + S(z, z, \mathcal{T}x_n) \\ &= 2S(x_n, x_n, \mathcal{T}x_n) + S(\mathcal{T}x_n, \mathcal{T}x_n, \mathcal{T}z). \end{aligned}$$

Now, we have

$$\begin{aligned}\mathcal{F}\left(S(x_n, x_n, z)\right) &\leq \mathcal{F}\left(2S(x_n, x_n, \mathcal{T}x_n) + S(\mathcal{T}x_n, \mathcal{T}x_n, z)\right) \\ &= \mathcal{F}\left(2S(x_n, x_n, \mathcal{T}x_n) + S(\mathcal{T}x_n, \mathcal{T}x_n, \mathcal{T}z)\right).\end{aligned}$$

Indeed, as \mathcal{F} is continuous, then using the property of \mathcal{F} and taking the limit as $n \rightarrow \infty$, we get

$$\mathcal{F}\left(S(x_n, x_n, z)\right) \leq \mathcal{F}\left(S(\mathcal{T}x_n, \mathcal{T}x_n, \mathcal{T}z)\right).$$

Now, using inequality (2.1) we obtain

$$\begin{aligned}\mathcal{F}\left(S(x_n, x_n, z)\right) &\leq \phi\left\{\mathcal{F}\left(S(x_n, x_n, z)\right), \mathcal{F}\left(S(x_n, x_n, \mathcal{T}x_n)\right), \mathcal{F}\left(S(z, z, \mathcal{T}z)\right)\right\} \\ &= \phi\left\{\mathcal{F}\left(S(x_n, x_n, z)\right), \mathcal{F}\left(S(x_n, x_n, \mathcal{T}x_n)\right), \mathcal{F}\left(S(z, z, z)\right)\right\}.\end{aligned}$$

Since \mathcal{F} is continuous, then using the property of \mathcal{F} and taking the limit as $n \rightarrow \infty$ in the above inequality, we get

$$\mathcal{F}\left(S(x_n, x_n, z)\right) \leq \phi\left\{\mathcal{F}\left(S(x_n, x_n, z)\right), 0, 0\right\}.$$

Because ϕ satisfies the condition (R3) by assumption, we obtain

$$\begin{aligned}\mathcal{F}\left(S(x_n, x_n, z)\right) &\leq h\mathcal{F}\left(S(x_n, x_n, z)\right) \\ \Rightarrow \mathcal{F}\left(S(x_n, x_n, z)\right) &= 0 \text{ because of } 0 < h < 1.\end{aligned}$$

Using the property of \mathcal{F} , this implies that $S(x_n, x_n, z) \rightarrow 0$ as $n \rightarrow \infty$ which is equivalent to saying that $x_n \rightarrow z$ as $n \rightarrow \infty$. This completes the proof. \square

Theorem 3.6 *Let $\mathcal{T}: X \rightarrow X$ be a self mapping as in Theorem ???. Then \mathcal{T} has the limit shadowing property.*

Proof According to Theorem 2.1, we know that \mathcal{T} has a unique fixed point $z = \mathcal{T}z \in X$. Let $\{x_n\} \subset X$ be such that $\lim_{n \rightarrow \infty} S(\mathcal{T}x_n, \mathcal{T}x_n, x_n) = 0 = \lim_{n \rightarrow \infty} S(x_n, x_n, \mathcal{T}x_n)$. Then, as in the previous proof,

$$\mathcal{F}\left(S(x_n, x_n, z)\right) \leq \phi\left\{\mathcal{F}\left(S(x_n, x_n, z)\right), 0, 0\right\}.$$

Since ϕ satisfies the condition (R3), then we obtain

$$\begin{aligned}\mathcal{F}\left(S(x_n, x_n, z)\right) &\leq h\mathcal{F}\left(S(x_n, x_n, z)\right) \\ \Rightarrow \mathcal{F}\left(S(x_n, x_n, z)\right) &= 0 \text{ because of } 0 < h < 1.\end{aligned}$$

Using the property of \mathcal{F} , it follows that $S(x_n, x_n, \mathcal{T}^n z) = S(x_n, x_n, z) \rightarrow 0$ as $n \rightarrow \infty$.

This completes the proof. □

§4. Conclusion

In this paper, we establish some fixed point theorems for $\mathcal{F}_{(S,\mathcal{T})}$ -contraction under an implicit relation in the framework of complete S -metric spaces and obtained some well-known results as corollaries. Also, we give some examples in support of our results and one of the possible applications of our result to well-posed and limit shadowing property of fixed point problems.

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