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Fixed Point Theorems for Nonlinear Rational Contractions in Extended A_b-Metric Spaces

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Abstract: Fixed point theory plays a fundamental role in nonlinear analysis and has wide applications in integral equations, optimization, computer science, and dynamical systems. The classical Banach contraction principle has inspired numerous generalizations to accommodate mappings that are not strict contractions in the usual metric sense. Among such developments, A-metric and A_b-metric spaces provide a broad framework that unifies and extends classical metric spaces, S-metric spaces, and b-metric spaces. In this article, we introduce the notion of an extended A_b-metric space and investigate fixed point results for several well-known contraction principles within the framework of extended A_b-metric spaces. In particular, we establish Banach-type, Kannan-type, Reich-type, and nonlinear rational contraction results and prove the existence and uniqueness of fixed points under suitable orbit-control conditions. Our approach relies on generalized triangle inequalities associated with A_b-metrics together with the construction of Picard sequences to establish convergence. The obtained results not only extend several classical fixed point theorems to a broader class of generalized metric spaces but also unify various contraction conditions within a single framework. Several illustrative examples are presented to support the main results, and an application to a nonlinear integral equation is provided to demonstrate the effectiveness of the proposed approach. These findings contribute to the growing theory of generalized metric spaces and their applications in nonlinear analysis.

Key Words: Extended A_b-metric space, Banach contraction, Kannan contraction, rational contraction, fixed point, common fixed point.

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§1. Introduction

Fixed point theory has long been regarded as one of the fundamental branches of nonlinear analysis, with important applications in functional analysis, topology, optimization, differential equations, and applied mathematics. A pioneering contribution to this theory was made by Banach [1] through the celebrated contraction principle, now known as the Banach contraction

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principle, which guarantees the existence and uniqueness of fixed points for contraction mappings on complete metric spaces. Owing to its simplicity and effectiveness, this theorem has become a powerful tool in both pure and applied mathematics and has motivated numerous generalizations in more abstract settings.

A major advancement in this direction was the introduction of partial metric spaces by Matthews [2], where the self-distance of a point need not be zero. This concept provided a suitable framework for applications in domain theory and theoretical computer science. Later, Bukatin et al. [3] developed several foundational properties of partial metric spaces, while Oltra and Valero [11] established various fixed point results in this setting. Subsequently, Shukla [12] introduced partial b-metric spaces, extending the classical metric structure by incorporating a relaxation parameter.

Parallel to these developments, several modifications of the Banach contraction condition were proposed. Kannan [6, 7], Reich [24, 23], and Ćirić [4] introduced alternative contractive conditions that guarantee fixed points without requiring the standard Banach contraction assumption. Later, Khan et al. [5] introduced altering distance functions, while Dass and Gupta [22] investigated rational-type contractions. These generalized contractions significantly broadened the applicability of fixed point theory and unified many classical results.

Further extensions of fixed point theory have been investigated in several generalized spaces. Contributions by Jungck [13], Edelstein [14], Pant [9], and Jaggi [10] enriched the theory of compatible mappings and common ffxed points. More recently, fixed point results have been studied in S-metric spaces [16, 17, 18], A-metric spaces [19], and Ab-metric spaces [20, 21], thereby extending the scope of generalized metric structures.

The concept of extended S_b-metric spaces introduced by Mlaiki [15] further demonstrated how generalized distance functions can provide a unified framework for studying nonlinear contractions. Such generalized spaces preserve essential topological and convergence properties despite weakened metric axioms. Consequently, these developments have attracted considerable attention in nonlinear functional analysis, integral equations, optimization theory, and computer science.

Recently, several authors have investigated generalized contractions in ordered and generalized metric spaces. Bhaskar and Lakshmikantham [27] established important fixed point results in partially ordered metric spaces, while Aghajani et al. [28], Jachymski [29], and Huang et al. [30] obtained various common fixed point and coincidence point results in generalized b-metric settings. Moreover, Taş and Özgür [31] studied generalized fixed point results in S_b-metric spaces.

Motivated by these ongoing developments, in this paper we introduce the notion of an extended A_b-metric space and investigate several fixed point results involving Banach-type [1], Kannan-type [6, 7], Reich-type [24], Ćirić-type [4], and rational-type contractions [22] in this setting. The proposed framework generalizes both A-metric and A_b-metric spaces and provides greater flexibility through the incorporation of a control function. Our results unify and extend several existing theorems in the literature. Moreover, an application to a nonlinear integral equation is presented to illustrate the effectiveness and applicability of the obtained results.

§2. **Preliminaries and Notations**

In this section, we recall some basic definitions and fix the notation that will be used throughout the paper. We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{R}$ the set of real numbers.

The notion of an A-metric space was introduced as a natural generalization of metric and S-metric spaces [19].

Definition 2.1 *Let X be a nonempty set and let $n \geqslant 2$. A function*

$$A : X^n \longrightarrow [0, \infty)$$

is called an A-metric on X if the following conditions hold for all $\xi_1, \xi_2, \cdots, \xi_n, \zeta \in X$:

$(A1)$

$$A(\xi_1, \xi_2, \cdots, \xi_n) = 0 \iff \xi_1 = \xi_2 = \cdots = \xi_n.$$

$(A2)$

$$A(\xi_1, \xi_2, \cdots, \xi_n) = A(\xi_{\pi(1)}, \xi_{\pi(2)}, \cdots, \xi_{\pi(n)})$$

for every permutation π of $[n]$.

$(A3)$ (Generalized triangle inequality)

$$A(\xi_1, \xi_2, \cdots, \xi_n) \leqslant A(\xi_1, \xi_2, \cdots, \xi_{n-1}, \zeta) + A(\underbrace{\zeta, \zeta, \cdots, \zeta}_{n-1}, \xi_n).$$

Then, (X, A) is called an A-metric space.

When $n = 2$, this definition reduces to the classical metric. For $n = 3$, it recovers the S-metric structure. Thus, the A-metric provides a unified framework that generalizes several existing notions of distance.

To allow greater flexibility, the concept of an A_b-metric space was introduced as an extension of A-metric spaces [20, 21].

Definition 2.2 *Let X be a nonempty set, $b \geqslant 1$, and $n \geqslant 2$. A function $A_b : X^n \longrightarrow [0, \infty)$ is called an A_b-metric on X if the following conditions hold for all $\xi_1, \xi_2, \cdots, \xi_n, \zeta \in X$:*

$(Ab1)$

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = 0 \iff \xi_1 = \xi_2 = \cdots = \xi_n.$$

$(Ab2)$

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = A_b(\xi_{\pi(1)}, \xi_{\pi(2)}, \cdots, \xi_{\pi(n)})$$

for every permutation π of $[n]$.

$(Ab3)$ (Generalized triangle inequality)

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) \leq b\Big(A_b\big(\xi_1, \xi_2, \cdots, \xi_{n-1}, \zeta\big) + A_b\big(\underbrace{\zeta, \zeta, \cdots, \zeta}_{n-1}, \xi_n\big)\Big).$$

Then, (X, A_b) is called an A_b-metric space.

Clearly, every A-metric is an A_b-metric with $b = 1$, but the converse does not necessarily hold. The presence of the control parameter b provides additional flexibility in verifying contractive conditions.

Definition 2.3 *A sequence $\{\xi_m\} \subseteq X$ is said to converge to $\xi \in X$ if*

$$\lim_{m\to\infty} A_b(\underbrace{\xi_m, \xi_m, \cdots, \xi_m}_{n-1}, \xi) = 0.$$

In this case, we write

$$\xi_m \to \xi \qquad as \qquad m \to \infty.$$

Definition 2.4 *A sequence $\{\xi_m\} \subseteq X$ is called a Cauchy sequence in (X, A_b) if*

$$\lim_{m,p\to\infty} A_b(\underbrace{\xi_m, \xi_m, \ldots, \xi_m}_{n-1}, \xi_p) = 0.$$

Definition 2.5 *An A_b-metric space (X, A_b) is said to be complete if every Cauchy sequence $\{\xi_m\} \subseteq X$ converges to some $\xi \in X$.*

§3. New Examples of A_b-Metric Spaces

Example 3.1 Let $X = \mathbb{R}$ and fix an integer $n \geqslant 2$. Define

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) := \max_{1\leqslant i<j\leqslant n} |\xi_i - \xi_j|^2, \qquad \xi_1, \xi_2, \cdots, \xi_n \in \mathbb{R}.$$

We verify that A_b defines an A_b-metric on X.

$(Ab1)$ Clearly,

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) \geqslant 0.$$

Moreover,

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = 0$$

if and only if

$$|\xi_i - \xi_j| = 0$$

for all i, j, which implies

$$\xi_1 = \xi_2 = \cdots = \xi_n.$$

$(Ab2)$ Since the maximum of all pairwise squared differences is independent of the ordering of the coordinates, we have

$$A_b(\xi_1, \xi_2, \ldots, \xi_n) = A_b(\xi_{\pi(1)}, \xi_{\pi(2)}, \cdots, \xi_{\pi(n)})$$

for every permutation π of $\{1, 2, \cdots, n\}$.

$(Ab3)$ Let $\zeta \in X$. Using the inequality

$$|a+b|^2 \leqslant 2(|a|^2 + |b|^2),$$

we obtain

$$|\xi_i - \xi_j|^2 = |(\xi_i - \zeta) + (\zeta - \xi_j)|^2 \leqslant 2\big(|\xi_i - \zeta|^2 + |\zeta - \xi_j|^2\big).$$

Taking the maximum over all $1 \leqslant i < j \leqslant n$, it follows that

$$\begin{aligned} A_b(\xi_1, \xi_2, \cdots, \xi_n) \leqslant 2\Big(&A_b(\xi_1, \xi_2, \cdots, \xi_{n-1}, \zeta) \\ &+ A_b(\underbrace{\zeta, \zeta, \cdots, \zeta}_{n-1}, \xi_n)\Big). \end{aligned}$$

Hence, the generalized triangle inequality holds with $b = 2$. Therefore, (X, A_b) is an A_b-metric space.

§4. Extended A_b-Metric Space

In this section, we introduce the notion of an extended A_b-metric space, which generalizes both A-metric and A_b-metric spaces through the incorporation of a variable control function.

Definition 4.1(Extended A_b-Metric Space) *Let X be a nonempty set, $n \geqslant 2$, and let*

$$A_b : X^n \to [0, \infty)$$

be a mapping together with a control function

$$\Theta : X^n \to [1, \infty).$$

Then, A_b is called an extended A_b-metric on X if for all $\xi_1, \xi_2, \cdots, \xi_n, \alpha \in X$, the following conditions hold:

$(EAb1)$

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = 0 \iff \xi_1 = \xi_2 = \cdots = \xi_n.$$

$(EAb2)$

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = A_b(\xi_{\pi(1)}, \xi_{\pi(2)}, \cdots, \xi_{\pi(n)})$$

for every permutation π of $\{1, 2, \cdots, n\}$.

$(EAb3)$ (Generalized triangle inequality)

$$\begin{aligned} A_b(\xi_1, \xi_2, \cdots, \xi_n) \leqslant \Theta(\xi_1, \xi_2, \cdots, \xi_n)\Big(&A_b(\xi_1, \xi_2, \cdots, \xi_{n-1}, \alpha) \\ &+ A_b(\underbrace{\alpha, \alpha, \cdots, \alpha}_{n-1}, \xi_n)\Big). \end{aligned}$$

Then (X, A_b) is called an extended A_b-metric space.

Clearly, every A_b-metric space is an extended A_b-metric space whenever

$$\Theta(\xi_1, \xi_2, \ldots, \xi_n) = b$$

for all $\xi_1, \xi_2, \ldots, \xi_n \in X$. Moreover, when $\Theta \equiv 1$, the structure reduces to an A-metric space.

Definition 4.2 *Let (X, A_b) be an extended A_b-metric space with fixed arity $n \geqslant 2$ and control function Θ. Let $\{\xi_k\} \subset X$ be a sequence and let $\xi \in X$. Then the following notions are defined.*

1. Convergence. *The sequence $\{\xi_k\}$ is said to converge to ξ (written $\xi_k \to \xi$) if for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that*

$$A_b(\underbrace{\xi_k, \xi_k, \cdots, \xi_k}_{n-1}, \xi) < \varepsilon \qquad \text{for all } k \geqslant N(\varepsilon).$$

2. Cauchy sequence. *The sequence $\{\xi_k\}$ is called an A_b-Cauchy sequence if for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that*

$$A_b(\underbrace{\xi_k, \xi_k, \cdots, \xi_k}_{n-1}, \xi_m) < \varepsilon \qquad \text{for all } k, m \geqslant N(\varepsilon).$$

3. Completeness. *The space (X, A_b) is said to be complete if every A_b-Cauchy sequence in X converges to some point of X.*

4. Diameter. *For a nonempty subset $Y \subset X$, define its diameter by*

$$\operatorname{diam}(Y) := \sup \{A_b(\xi_1, \xi_2, \cdots, \xi_n) : \xi_1, \xi_2, \cdots, \xi_n \in Y\}.$$

Definition 4.3 *Let (X, A_b) be an extended A_b-metric space. For $\xi \in X$ and $r > 0$, define the open and closed A_b-balls centered at ξ, respectively, by*

$$\mathcal{B}_r^{A_b}(\xi) := \left\{ \eta \in X : A_b(\underbrace{\eta, \eta, \cdots, \eta}_{n-1}, \xi) < r \right\},$$

and

$$\overline{\mathcal{B}}_r^{A_b}(\xi) := \left\{ \eta \in X : A_b(\underbrace{\eta, \eta, \cdots, \eta}_{n-1}, \xi) \leqslant r \right\}.$$

Furthermore, the following notions hold.

1. Open set. *A subset $U \subset X$ is called open if for every $u \in U$, there exists $\varepsilon > 0$ such that*

$$\mathcal{B}_\varepsilon^{A_b}(u) \subseteq U.$$

2. Closed set. *A subset $V \subset X$ is called closed if whenever a sequence $\{\nu_k\} \subset V$*

converges to some $\nu \in X$, then $\nu \in V$.

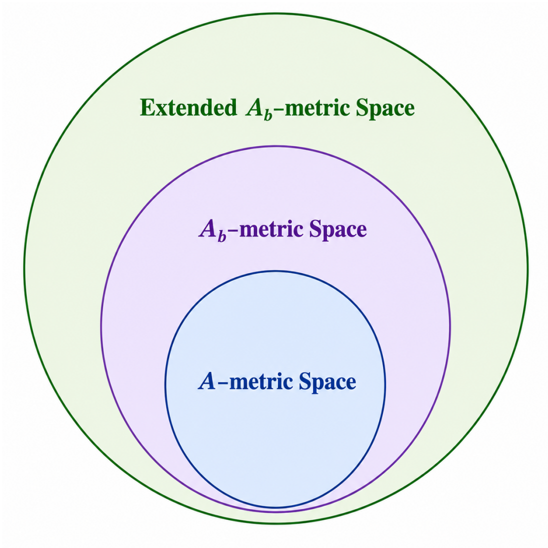

Figure 1. Hierarchical relationship among A-metric, A_b-metric, and extended A_b-metric spaces

§5. Comparative Study of A-Metric, A_b-Metric, and Extended A_b-Metric Spaces

Feature	**A-metric Space**	**A_b-metric Space**	**Extended A_b-metric Space**
Introduced by	Mujahid Abbas et al. (2015)	Manoj Ughade et al. and subsequent extensions	Generalized framework proposed in the present work
Underlying mapping	$A : X^n \to [0, \infty)$	$A_b : X^n \to [0, \infty)$	$A_b : X^n \to [0, \infty)$ with control function $\Theta : X^n \to [1, \infty)$
Parameter involved	None	Constant $b \geqslant 1$	Variable control function Θ
Non-negativity	$A(\xi_1, \ldots, \xi_n) \geqslant 0$	$A_b(\xi_1, \cdots, \xi_n) \geqslant 0$	$A_b(\xi_1, \cdots, \xi_n) \geqslant 0$
Identity property	$A(\xi_1, \cdots, \xi_n) = 0 \iff \xi_1 = \cdots = \xi_n$	Same as the A-metric case	Same condition retained
Symmetry	Invariant under every permutation of coordinates	Invariant under every permutation of coordinates	Invariant under every permutation of coordinates

Feature	**A-metric Space**	**A_b-metric Space**	**Extended A_b-metric Space**
Triangle-type inequality	$A(\xi_1, \cdots, \xi_n) \leqslant A(\xi_1, \cdots, \xi_{n-1}, \zeta) + A(\zeta, \cdots, \zeta, \xi_n)$	$A_b(\xi_1, \cdots, \xi_n) \leqslant b\Big(A_b(\xi_1, \cdots, \xi_{n-1}, \zeta) + A_b(\zeta, \cdots, \zeta, \xi_n)\Big)$	$A_b(\xi_1, \cdots, \xi_n) \leqslant \Theta(\xi_1, \cdots, \xi_n)\Big(A_b(\xi_1, \cdots, \xi_{n-1}, \alpha) + A_b(\alpha, \cdots, \alpha, \xi_n)\Big)$
Flexibility level	Least flexible	More flexible through the constant b	Most flexible through the variable control function Θ
Special cases recovered	Metric space ($n = 2$) and S-metric space ($n = 3$)	Every A-metric space when $b = 1$	Both A-metric and A_b-metric spaces
Nature of control	Fixed structure	Globally controlled by a constant b	Locally controlled by Θ, depending on the points involved
Suitability for fixed point theory	Classical generalized contractions	Wider classes of contractions	Highly adaptable framework for nonlinear contractive mappings
Generalization hierarchy	Base structure	Generalization of A-metric spaces	Generalization of both A-metric and A_b-metric spaces

§6. Examples of Extended A_b-Metric Spaces

In this section, we present several examples of functions

$$A_b : X^n \to [0, \infty)$$

that satisfy the axioms of an extended A_b-metric space introduced in the previous section. In each case, the associated control function Θ is specified and the required properties are verified.

Example 6.1 Let S be a nonempty set and let $\Gamma := S^* \cup S^\omega$ denote the collection of all finite and infinite sequences over S. Fix $n \geqslant 2$. For $\alpha_1, \alpha_2, \cdots, \alpha_n \in \Gamma$, define

$$A_b(\alpha_1, \alpha_2, \cdots, \alpha_n) := 2^{-\ell(\alpha_1, \alpha_2, \cdots, \alpha_n)},$$

where $\ell(\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}_0 \cup \{\infty\}$ denotes the length of the longest common prefix shared by the sequences $\alpha_1, \alpha_2, \cdots, \alpha_n$. By convention,

$$2^{-\infty} = 0$$

and choose the control function $\Theta \equiv 1$.

Proof The nonnegativity is immediate since powers of 2^{-k} are nonnegative.

Moreover,

$$A_b(\alpha_1, \alpha_2, \cdots, \alpha_n) = 0$$

if and only if

$$\ell(\alpha_1, \alpha_2, \cdots, \alpha_n) = \infty,$$

which occurs precisely when all sequences coincide.

Symmetry follows because the common-prefix length is invariant under permutations of the arguments.

For the generalized triangle inequality, let $\zeta \in \Gamma$ and define

$$k_i := \ell(\alpha_i, \zeta), \qquad i = 1, 2, \cdots, n.$$

Then,

$$\ell(\alpha_1, \alpha_2, \cdots, \alpha_n) \geqslant \min_{1 \leqslant i \leqslant n} k_i.$$

Hence,

$$\begin{aligned} A_b(\alpha_1, \alpha_2, \cdots, \alpha_n) &= 2^{-\ell(\alpha_1, \alpha_2, \cdots, \alpha_n)} \\ &\leqslant 2^{-\min_{1 \leqslant i \leqslant n} k_i} \\ &\leqslant \sum_{i=1}^{n} 2^{-k_i} = \sum_{i=1}^{n} A_b(\alpha_i, \cdots, \alpha_i, \zeta). \end{aligned}$$

Therefore, the generalized triangle inequality holds with $\Theta \equiv 1$. Consequently, (Γ, A_b) is an extended A_b-metric space. □

Remark 6.2 The above example provides a natural realization of an extended A_b-metric on a sequence space. The value

$$A_b(\alpha_1, \alpha_2, \cdots, \alpha_n) = 2^{-\ell(\alpha_1, \alpha_2, \cdots, \alpha_n)}$$

depends on the length of the common initial segment shared by the sequences. Consequently, sequences with larger common prefixes are regarded as closer to one another.

An important feature of this construction is that the distance is determined by structural similarity rather than by numerical magnitude. This makes the example suitable for symbolic systems, coding structures, formal languages, and tree-like models where proximity is governed by shared patterns.

Moreover, the choice $\Theta \equiv 1$ shows that the generalized triangle inequality is satisfied without any additional relaxation factor. Hence, the space behaves as a particular case of an extended A_b-metric space with the strongest control condition.

It is also worth mentioning that when $n = 2$, the mapping reduces to the classical prefix-type metric commonly used in sequence analysis and ultrametric constructions. Therefore, this example demonstrates that extended A_b-metric spaces naturally extend several well-known generalized distance frameworks while preserving meaningful geometric behavior.

Example 6.3 Let $\Gamma := [0, \infty)$ and fix constants $c > 0$ and $n \geqslant 2$. Define

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) := c \max\{\xi_1, \xi_2, \cdots, \xi_n\},$$

for all $\xi_i \in \Gamma$ and take $\Theta \equiv 1$.

Proof The nonnegativity and symmetry are immediate.

Furthermore,

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = 0$$

implies

$$\max\{\xi_1, \xi_2, \cdots, \xi_n\} = 0,$$

and hence

$$\xi_1 = \xi_2 = \cdots = \xi_n = 0.$$

For any $\alpha \in \Gamma$,

$$\begin{aligned}\sum_{i=1}^{n} A_b(\xi_i, \cdots, \xi_i, \alpha) &= c \sum_{i=1}^{n} \max\{\xi_i, \alpha\} \\ &\geqslant c \max_{1 \leqslant i \leqslant n} \xi_i = A_b(\xi_1, \xi_2, \cdots, \xi_n).\end{aligned}$$

Hence, the generalized triangle inequality is satisfied with $\Theta \equiv 1$. Therefore, (Γ, A_b) is an extended A_b-metric space. □

Example 6.4 Let (M, d) be a metric space and set $\Gamma := M$.

For $x_1, x_2, \cdots, x_n \in M$, define

$$A_b(x_1, x_2, \cdots, x_n) := \max_{1 \leqslant i < j \leqslant n} d(x_i, x_j).$$

Choose the constant control function $\Theta \equiv n$.

Proof The nonnegativity and symmetry follow directly from the properties of the metric d.

Moreover,

$$A_b(x_1, x_2, \cdots, x_n) = 0$$

if and only if $d(x_i, x_j) = 0$ for all i, j, which implies $x_1 = x_2 = \cdots = x_n$.

Let $\alpha \in M$. By the triangle inequality,

$$d(x_i, x_j) \leqslant d(x_i, \alpha) + d(\alpha, x_j).$$

Taking the maximum over all $i < j$, we obtain

$$A_b(x_1, \cdots, x_n) \leqslant 2 \max_{1 \leqslant k \leqslant n} d(x_k, \alpha).$$

Since

$$\max_{1\leqslant k\leqslant n} d(x_k,\alpha)\leqslant \sum_{k=1}^{n} d(x_k,\alpha),$$

it follows that

$$A_b(x_1,\cdots,x_n)\leqslant 2\sum_{k=1}^{n} d(x_k,\alpha).$$

Now observe that

$$A_b(x_k,\cdots,x_k,\alpha)=d(x_k,\alpha),$$

for every $k=1,2,\cdots,n$. Hence,

$$A_b(x_1,\cdots,x_n)\leqslant n\sum_{k=1}^{n} A_b(x_k,\cdots,x_k,\alpha).$$

Thus, the generalized triangle inequality holds with $\Theta\equiv n$. Consequently, (M,A_b) is an extended A_b-metric space. □

Example 6.5 Let (M,d) be a metric space and define

$$A_b(x_1,x_2,\cdots,x_n):=\sum_{1\leqslant i<j\leqslant n} d(x_i,x_j),$$

for all $x_1,x_2,\cdots,x_n\in M$.

Take the control function $\Theta\equiv n-1$.

Proof The nonnegativity and symmetry are immediate.

Moreover,

$$A_b(x_1,\cdots,x_n)=0$$

if and only if all pairwise distances vanish, which implies $x_1=x_2=\cdots=x_n$.

For any $\alpha\in M$, using the metric triangle inequality,

$$d(x_i,x_j)\leqslant d(x_i,\alpha)+d(\alpha,x_j).$$

Summing over all pairs $1\leqslant i<j\leqslant n$, we obtain

$$A_b(x_1,\cdots,x_n)\leqslant (n-1)\sum_{k=1}^{n} d(x_k,\alpha).$$

Since

$$A_b(x_k,\cdots,x_k,\alpha)=d(x_k,\alpha),$$

we conclude that

$$A_b(x_1,\cdots,x_n)\leqslant (n-1)\sum_{k=1}^{n} A_b(x_k,\cdots,x_k,\alpha).$$

Hence, the generalized triangle inequality holds with $\Theta\equiv n-1$. Therefore, (M,A_b) is an extended A_b-metric space. □

Example 6.6 Let K be a compact set and let $\Gamma := C(K)$, the space of all continuous real-valued functions on K. For $f_1, f_2, \cdots, f_n \in C(K)$, define

$$A_b(f_1, f_2, \cdots, f_n) := \sup_{x\in K} \max_{1\leqslant i<j\leqslant n} |f_i(x) - f_j(x)|$$

and choose $\Theta \equiv n$.

Proof The nonnegativity and symmetry are immediate.

Moreover,

$$A_b(f_1, f_2, \cdots, f_n) = 0$$

if and only if $f_1 = f_2 = \cdots = f_n$.

Let $g \in C(K)$. By the pointwise triangle inequality,

$$|f_i(x) - f_j(x)| \leqslant |f_i(x) - g(x)| + |g(x) - f_j(x)|.$$

Taking maxima and then supremum over K, we obtain

$$A_b(f_1, \cdots, f_n) \leqslant 2 \sup_{x\in K} \max_{1\leqslant k\leqslant n} |f_k(x) - g(x)|.$$

Furthermore,

$$A_b(f_k, \cdots, f_k, g) = \sup_{x\in K} |f_k(x) - g(x)|.$$

Hence,

$$A_b(f_1, \cdots, f_n) \leqslant n \sum_{k=1}^{n} A_b(f_k, \cdots, f_k, g).$$

Therefore, the generalized triangle inequality holds with $\Theta \equiv n$. Consequently, $(C(K), A_b)$ is an extended A_b-metric space. □

Let Γ be a nonempty set and let $A_b : \Gamma^n \to [0, \infty)$ be an extended A_b-metric with control function $\Theta : \Gamma^n \to [1, \infty)$. For $\xi, \eta \in \Gamma$, define

$$A_\eta(\xi) := A_b(\underbrace{\eta, \cdots, \eta}_{n-1}, \xi),$$

whenever no confusion arises.

§7. Main Results

Theorem 7.1 *Let (X, A_b) be a complete extended A_b-metric space of arity $n \geqslant 2$ with control* *function* $\Theta : X^n \to [1, \infty)$. *Assume that* A_b *is continuous. Let* $T : X \to X$ *satisfy*

$$A_b\big(T\xi, T\eta, \underbrace{T\xi, \ldots, T\xi}_{n-2}\big) \leqslant k\, A_b\big(\underbrace{\xi, \ldots, \xi}_{n-1}, \eta\big), \qquad \text{for all } \xi, \eta \in X, \tag{1}$$

for some constant $k \in [0, 1)$.

Assume the orbit-control condition: for each $x_0 \in X$*, the Picard orbit*

$$x_m := T^m x_0$$

satisfies

$$\lim_{m,p\to\infty} \prod_{i=m}^{p} \Theta(x_i, \ldots, x_i, x_p) =: L(x_0) < \frac{1}{k}.$$

Then, T *has a unique fixed point* $z \in X$*, and*

$$T^m x_0 \to z$$

for every $x_0 \in X$.

Proof Fix $x_0 \in X$ and define the Picard sequence

$$x_{m+1} := T x_m, \qquad m \geqslant 0.$$

For every $m \geqslant 0$, put

$$d_m := A_b\big(x_{m+1}, x_{m+2}, \underbrace{x_{m+1}, \ldots, x_{m+1}}_{n-2}\big).$$

Our goal is to show that $d_m \to 0$ geometrically (up to bounded multiplicative factors), and then deduce that (x_m) is Cauchy.

Applying (1) with

$$\xi = x_m, \qquad \eta = x_{m+1},$$

gives

$$\begin{aligned} d_m &= A_b\big(Tx_m, Tx_{m+1}, \underbrace{Tx_m, \cdots, Tx_m}_{n-2}\big) \\ &\leqslant k\, A_b\big(\underbrace{x_m, \cdots, x_m}_{n-1}, x_{m+1}\big). \end{aligned} \tag{2}$$

Thus, it remains to estimate

$$A_b\big(\underbrace{x_m, \cdots, x_m}_{n-1}, x_{m+1}\big)$$

in terms of d_{m-1}.

Using the generalized triangle inequality repeatedly, we obtain

$$A_b\big(\underbrace{x_m, \cdots, x_m}_{n-1}, x_{m+1}\big) \leqslant C_m d_{m-1},$$

where

$$C_m = \prod_{\ell=1}^{r_m} \Theta(\mathbf{t}_{m,\ell}),$$

for suitable tuples $\mathbf{t}_{m,\ell} \in \{x_0, \cdots, x_{m+1}\}^n$.

Combining this estimate with (2), we obtain

$$d_m \leqslant k C_m d_{m-1}. \tag{3}$$

Iterating (3), we get

$$d_m \leqslant d_0\, k^m \prod_{j=1}^{m} C_j.$$

By the orbit-control hypothesis,

$$k^m \prod_{j=1}^{m} C_j \to 0 \qquad (m \to \infty),$$

and hence

$$d_m \to 0. \tag{4}$$

Next, repeated application of the generalized triangle inequality yields

$$A_b\big(\underbrace{x_q, \cdots, x_q}_{n-1}, x_p\big) \leqslant M_{q,p} \sum_{j=q}^{p-1} d_j, \tag{5}$$

for suitable constants $M_{q,p} \geqslant 1$.

Since $d_j \to 0$, the series $\sum_{j=q}^{\infty} d_j$ converges, and therefore (x_m) is a Cauchy sequence in the A_b-sense.

Notice that the completeness of (X, A_b) implies the existence of $z \in X$ such that $x_m \to z$. Using continuity of A_b and passing to the limit in (1), we obtain

$$A_b\big(Tz, Tz, \underbrace{Tz, \cdots, Tz}_{n-2}\big) \leqslant k A_b\big(\underbrace{z, \cdots, z}_{n}\big).$$

Since $0 \leqslant k < 1$, it follows that

$$A_b\big(\underbrace{Tz, \cdots, Tz}_{n-1}, z\big) = 0,$$

and hence

$$Tz = z.$$

To prove the uniqueness, let $u, v \in X$ be fixed points of T. Then

$$\begin{aligned} A_b(\underbrace{u, \cdots, u}_{n-1}, v) &= A_b(Tu, Tv, \underbrace{Tu, \cdots, Tu}_{n-2}) \\ &\leqslant kA_b(\underbrace{u, \cdots, u}_{n-1}, v). \end{aligned}$$

Since $k < 1$, we conclude that

$$A_b(\underbrace{u, \cdots, u}_{n-1}, v) = 0,$$

which implies $u = v$. Therefore, T has a unique fixed point, and every Picard sequence converges to it. □

Theorem 7.2 *Let (X, A_b) be a complete extended A_b-metric space with continuous A_b. Suppose there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that for all $\xi, \eta \in X$,*

$$A_b(\underbrace{T\xi, \cdots, T\xi}_{n-1}, T\eta) \leqslant \alpha\Big(A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, T\xi) + A_b(\underbrace{\eta, \cdots, \eta}_{n-1}, T\eta)\Big).$$

Assume moreover that, for each $x_0 \in X$, the Picard orbit

$$x_m := T^m x_0$$

satisfies the orbit-control condition

$$\lim_{m,p\to\infty} \prod_{i=m}^{p} \Theta(x_i, \cdots, x_i, x_p) =: L(x_0) < \frac{1}{\alpha}.$$

Then, T has a unique fixed point $z \in X$, and

$$T^m x_0 \to z \qquad \text{for every } x_0 \in X.$$

Proof Let $x_0 \in X$ and define the Picard sequence

$$x_{m+1} := Tx_m, \qquad m \geqslant 0.$$

Set

$$u_m := A_b(x_m, x_{m+1}, \underbrace{x_m, \cdots, x_m}_{n-2}).$$

Applying the Kannan-type inequality with

$$\xi = x_m, \qquad \eta = x_{m+1},$$

we obtain

$$u_{m+1} \leqslant \alpha\Big(A_b\big(\underbrace{x_m, \cdots, x_m}_{n-1}, x_{m+1}\big) + A_b\big(\underbrace{x_{m+1}, \cdots, x_{m+1}}_{n-1}, x_{m+2}\big)\Big).$$

Using the generalized triangle inequality repeatedly, each diagonal term can be estimated by a finite product of Θ-values multiplied by u_m or u_{m+1}. Hence there exist finite constants $C_m, C_{m+1} \geqslant 1$ such that

$$(1 - \alpha C_{m+1})u_{m+1} \leqslant \alpha C_m u_m.$$

By the orbit-control hypothesis, the products of Θ-terms remain controlled along the orbit. Consequently, there exists $q \in [0, 1)$ such that $u_{m+1} \leqslant q\, u_m$ for all sufficiently large m.

Therefore,

$$u_m \to 0 \qquad (m \to \infty).$$

Proceeding as in Theorem 7.1, one shows that the Picard sequence (x_m) is Cauchy in the A_b-sense. Since (X, A_b) is complete, there exists $z \in X$ such that $x_m \to z$.

Using the continuity of A_b and passing to the limit in the contractive inequality, we obtain

$$Tz = z.$$

To prove uniqueness, let $u, v \in X$ be fixed points of T. Then

$$\begin{aligned} A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big) &= A_b\big(\underbrace{Tu, \cdots, Tu}_{n-1}, Tv\big) \\ &\leqslant \alpha\Big(A_b\big(\underbrace{u, \cdots, u}_{n-1}, u\big) + A_b\big(\underbrace{v, \cdots, v}_{n-1}, v\big)\Big) = 0. \end{aligned}$$

Hence $u = v$. Therefore, T has a unique fixed point, and every Picard sequence converges to it. □

Theorem 7.3 *Let (X, A_b) be a complete extended A_b-metric space with continuous A_b. Suppose there exist constants $\alpha, \beta, \gamma \geqslant 0$ satisfying*

$$\alpha + \beta + \gamma < 1,$$

such that for all $\xi, \eta \in X$,

$$A_b\big(T\xi, T\eta, \underbrace{T\xi, \cdots, T\xi}_{n-2}\big) \leqslant \alpha A_b\big(\underbrace{\xi, \cdots, \xi}_{n-1}, \eta\big) + \beta A_b\big(\underbrace{\xi, \cdots, \xi}_{n-1}, T\xi\big) + \gamma A_b\big(\underbrace{\eta, \cdots, \eta}_{n-1}, T\eta\big).$$

Assume further that, for each $x_0 \in X$, the Picard orbit $x_m := T^m x_0$ satisfies

$$\lim_{m,p\to\infty} \prod_{i=m}^{p} \Theta(x_i, \cdots, x_i, x_p) =: L(x_0) < \frac{1}{\alpha + \beta + \gamma}.$$

Then, T has a unique fixed point in X.

Proof Let

$$x_{m+1} := Tx_m, \qquad m \geqslant 0,$$

and define

$$v_m := A_b\big(x_m, x_{m+1}, \underbrace{x_m, \cdots, x_m}_{n-2}\big).$$

Applying the Reich-type inequality with

$$\xi = x_m, \qquad \eta = x_{m+1},$$

gives

$$v_{m+1} \leqslant \alpha v_m + \beta A_b\big(\underbrace{x_m, \cdots, x_m}_{n-1}, x_{m+1}\big) + \gamma v_{m+1}.$$

Using the generalized triangle inequality, the diagonal term is bounded by a finite Θ-multiple of v_m. Hence there exists a finite constant $C_m \geqslant 1$ such that

$$(1-\gamma)v_{m+1} \leqslant (\alpha + \beta C_m)v_m.$$

By the orbit-control condition, the products of the corresponding Θ-terms remain bounded along the orbit. Consequently, there exists $q \in [0,1)$ such that $v_{m+1} \leqslant qv_m$ for all sufficiently large m.

Therefore,

$$v_m \to 0 \qquad (m \to \infty).$$

Proceeding as in Theorem 7.1, one shows that the Picard sequence (x_m) is Cauchy in the A_b-sense. Since (X, A_b) is complete, there exists $z \in X$ such that $x_m \to z$.

Passing to the limit in the contractive condition and using continuity of A_b, we obtain

$$Tz \;=\; z.$$

To prove uniqueness, let $u, v \in X$ be fixed points of T. Then

$$\begin{aligned}A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big) &= A_b\big(Tu, Tv, \underbrace{Tu, \cdots, Tu}_{n-2}\big)\\ &\leqslant \alpha A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big) + \beta A_b\big(\underbrace{u, \cdots, u}_{n-1}, u\big) + \gamma A_b\big(\underbrace{v, \cdots, v}_{n-1}, v\big).\end{aligned}$$

Since

$$A_b\big(\underbrace{u, \cdots, u}_{n-1}, u\big) = A_b\big(\underbrace{v, \cdots, v}_{n-1}, v\big) = 0,$$

we obtain

$$A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big) \leqslant \alpha A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big).$$

Because $\alpha < 1$, it follows that

$$A_b(\underbrace{u, \cdots, u}_{n-1}, v) = 0,$$

and hence $u = v$. Therefore, T has a unique fixed point in X. □

Theorem 7.4 *Let (X, A_b) be a complete extended A_b-metric space with continuous A_b. Suppose there exists $\lambda \in (0, 1)$ such that for all $\xi, \eta \in X$,*

$$A_b\big(T\xi, T\eta, \underbrace{T\xi, \cdots, T\xi}_{n-2}\big) \leqslant \frac{\lambda \max\Big\{A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, \eta),\ A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, T\xi),\ A_b(\underbrace{\eta, \cdots, \eta}_{n-1}, T\eta)\Big\}}{1 + A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, \eta)}.$$

Assume further that the Picard orbit satisfies the same orbit-control condition as in Theorem 7.1*. Then, T has a unique fixed point in X.*

Proof Let

$$x_{m+1} :=\ Tx_m, \qquad m \geqslant 0,$$

and define

$$w_m := A_b\big(x_{m+1}, x_{m+2}, \underbrace{x_{m+1}, \cdots, x_{m+1}}_{n-2}\big).$$

Applying the rational-type inequality with

$$\xi = x_m, \qquad \eta = x_{m+1},$$

yields

$$w_m \leqslant \frac{\lambda}{1 + A_b(\underbrace{x_m, \cdots, x_m}_{n-1}, x_{m+1})} \max\Big\{A_b(\underbrace{x_m, \cdots, x_m}_{n-1}, x_{m+1}), A_b(\underbrace{x_m, \cdots, x_m}_{n-1}, Tx_m), A_b(\underbrace{x_{m+1}, \cdots, x_{m+1}}_{n-1}, x_{m+2})\Big\}.$$

If $w_m = 0$ for some m, then the Picard sequence becomes stationary and reaches a fixed point. Otherwise, repeated use of the generalized triangle inequality together with the orbit-control condition allows the Θ-factors to be absorbed into a contraction coefficient. Hence, there exists $\lambda' \in (0, 1)$ such that

$$w_m \leqslant \lambda' w_{m-1}$$

for all sufficiently large m. Therefore,

$$w_m \to 0 \qquad (m \to \infty).$$

Proceeding as in Theorem 7.1, one shows that the Picard sequence (x_m) is Cauchy in the

A_b-sense. Since (X, A_b) is complete, there exists $z \in X$ such that $x_m \to z$.

Passing to the limit in the contractive inequality and using continuity of A_b, we obtain

$$Tz \;=\; z.$$

To prove uniqueness, let $u, v \in X$ be fixed points of T. Then

$$A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big) = A_b\big(Tu, Tv, \underbrace{Tu, \cdots, Tu}_{n-2}\big) \leqslant \frac{\lambda \max\Big\{A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big), 0, 0\Big\}}{1 + A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big)}.$$

Hence

$$A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big) = 0,$$

which implies $u = v$. Therefore, T has a unique fixed point in X. □

We now present an example illustrating the applicability of Theorem 7.1.

Example 7.5 Let $X = \mathbb{R}$ and define

$$A_b(x_1, x_2, \cdots, x_n) := \max_{1 \leqslant i < j \leqslant n} |x_i - x_j|, \qquad x_1, x_2, \cdots, x_n \in \mathbb{R}.$$

Choose the control function $\Theta \equiv n$. Define the mapping

$$T : X \to X, \qquad T(x) = sx,$$

where $0 \leqslant s < 1$. Then, for every $\xi, \eta \in X$, we have

$$A_b\big(T\xi, T\eta, \underbrace{T\xi, \cdots, T\xi}_{n-2}\big) = |s\xi - s\eta| = s|\xi - \eta| = s\, A_b\big(\xi, \eta, \underbrace{\xi, \cdots, \xi}_{n-2}\big).$$

Hence,

$$A_b\big(T\xi, T\eta, \underbrace{T\xi, \cdots, T\xi}_{n-2}\big) \leqslant k\, A_b\big(\xi, \eta, \underbrace{\xi, \cdots, \xi}_{n-2}\big),$$

where $k = s \in [0, 1)$. Thus, the contractive condition (1) of Theorem 7.1 is satisfied.

Moreover, for the Picard sequence $x_m = s^m x_0$, the control values $\Theta(x_i, \cdots, x_i, x_p) = n$ remain constant along the orbit. Hence the orbit-control condition is also satisfied.

Therefore, by Theorem 7.1, the mapping T has a unique fixed point, namely 0, and

$$T^m x_0 \to 0 \qquad \text{as } m \to \infty$$

for every $x_0 \in \mathbb{R}$.

Example 7.6 Let $X = \mathbb{R}$ and define

$$A_b(x_1, x_2, \cdots, x_n) := \max_{1 \leqslant i < j \leqslant n} |x_i - x_j|.$$

Define the nonconstant control function

$$\Theta(x_1, x_2, \cdots, x_n) := 1 + \frac{1}{2 + \sum_{i=1}^{n} |x_i|}.$$

Clearly,

$$1 < \Theta(x_1, x_2, \cdots, x_n) \leqslant \frac{3}{2}$$

for all $x_1, x_2, \cdots, x_n \in X$.

Now define the mapping

$$T : X \to X, \qquad T(x) = sx,$$

where $0 \leqslant s < 1$.

For every $\xi, \eta \in X$, we have

$$A_b\big(T\xi, T\eta, \underbrace{T\xi, \cdots, T\xi}_{n-2}\big) = |s\xi - s\eta| = s|\xi - \eta| = s\, A_b\big(\xi, \eta, \underbrace{\xi, \cdots, \xi}_{n-2}\big).$$

Hence, the Banach-type contractive condition holds with $k = s \in [0, 1)$.

Next, consider the Picard sequence $x_m = T^m x_0 = s^m x_0$. Then

$$\begin{aligned} \Theta(x_m, \cdots, x_m, x_p) &= 1 + \frac{1}{2 + (n-1)|x_m| + |x_p|} \\ &= 1 + \frac{1}{2 + (n-1)|s^m x_0| + |s^p x_0|}. \end{aligned}$$

Since

$$s^m x_0 \to 0 \qquad (m \to \infty),$$

we obtain

$$\Theta(x_m, \cdots, x_m, x_p) \to \frac{3}{2}.$$

Also,

$$1 < \Theta(x_m, \cdots, x_m, x_p) \leqslant \frac{3}{2},$$

for all $m, p \in \mathbb{N}$. Therefore, the corresponding finite products of Θ-values remain bounded along the Picard orbit. Consequently, the orbit-control condition required in Theorem 7.1 is satisfied. Hence all assumptions of Theorem 7.1 hold. Therefore, the mapping T has a unique fixed point 0, and

$$T^m x_0 \to 0 \qquad \text{for every } x_0 \in \mathbb{R}.$$

Example 7.7 Let (X, A_b) be an extended A_b-metric space with control function $\Theta : X^n \to [1, \infty)$.

Fix a point $c \in X$ and define the constant mapping

$$T : X \to X, \qquad T(x) = c, \qquad x \in X.$$

Then, for every $\xi, \eta \in X$, we have

$$A_b(\underbrace{T\xi, \cdots, T\xi}_{n-1}, T\eta) = A_b(\underbrace{c, \cdots, c}_{n-1}, c) = 0.$$

Hence,

$$A_b(\underbrace{T\xi, \cdots, T\xi}_{n-1}, T\eta) \leqslant \alpha\Big(A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, T\xi) + A_b(\underbrace{\eta, \cdots, \eta}_{n-1}, T\eta)\Big),$$

for every $\alpha \in \left[0, \frac{1}{2}\right)$. Therefore, the Kannan-type contractive condition of Theorem 7.2 is satisfied.

Moreover, the Picard sequence generated by T is

$$x_m = c, \qquad m \geqslant 0,$$

which is constant. Consequently,

$$\Theta(x_m, \cdots, x_m, x_p) = \Theta(c, \cdots, c),$$

for all $m, p \in \mathbb{N}$. Hence, the orbit-control condition is trivially satisfied.

Therefore, by Theorem 7.2, the mapping T possesses the unique fixed point c, and every Picard sequence converges to c.

Next, we present an example illustrating Theorem 7.3.

Example 7.8 Let $X = \mathbb{R}$ and define

$$A_b(x_1, x_2, \cdots, x_n) := \max_{1 \leqslant i < j \leqslant n} |x_i - x_j|.$$

Define the nonconstant control function

$$\Theta(x_1, x_2, \cdots, x_n) := 1 + \frac{1}{2 + \sum_{i=1}^{n} |x_i|}.$$

Clearly,

$$1 < \Theta(x_1, x_2, \cdots, x_n) \leqslant \frac{3}{2},$$

for all $x_1, x_2, \cdots, x_n \in \mathbb{R}$.

Now define the mapping

$$T : X \to X, \qquad T(x) = sx,$$

where $0 \leqslant s < 1$. Then, for every $\xi, \eta \in X$, we have

$$A_b(T\xi, T\eta, \underbrace{T\xi, \cdots, T\xi}_{n-2}) = |s\xi - s\eta| = s|\xi - \eta| = s\, A_b(\xi, \eta, \underbrace{\xi, \cdots, \xi}_{n-2}).$$

Choose

$$\alpha = s, \qquad \beta = \gamma = 0.$$

Then,

$$\alpha + \beta + \gamma = s < 1,$$

and hence the Reich-type contractive condition of Theorem 7.3 is satisfied.

Now consider the Picard sequence $x_m = T^m x_0 = s^m x_0$.

Then,

$$\begin{aligned}\Theta(x_m, \cdots, x_m, x_p) &= 1 + \frac{1}{2 + (n-1)|x_m| + |x_p|} \\ &= 1 + \frac{1}{2 + (n-1)|s^m x_0| + |s^p x_0|}.\end{aligned}$$

Since

$$s^m x_0 \to 0, \qquad (m \to \infty),$$

we obtain

$$\Theta(x_m, \cdots, x_m, x_p) \to \frac{3}{2}.$$

Also,

$$1 < \Theta(x_m, \cdots, x_m, x_p) \leqslant \frac{3}{2},$$

for all $m, p \in \mathbb{N}$.

Hence, the corresponding products of Θ-values remain bounded along the Picard orbit, and the orbit-control condition is satisfied.

Therefore, all assumptions of Theorem 7.3 hold. Consequently, the mapping T admits the unique fixed point 0, and

$$T^m x_0 \to 0 \qquad \text{for every } x_0 \in \mathbb{R}.$$

Finally, we present an example illustrating Theorem 7.4.

Example 7.9 Let (X, A_b) be an extended A_b-metric space with control function $\Theta : X^n \to [1, \infty)$. Fix a point $p \in X$ and define the constant mapping

$$T : X \to X, \qquad T(x) = p, \qquad x \in X.$$

Then, for every $\xi, \eta \in X$, we have

$$A_b(\underbrace{T\xi, \cdots, T\xi}_{n-1}, T\eta) = A_b(\underbrace{p, \cdots, p}_{n-1}, p) = 0.$$

Therefore,

$$A_b(\underbrace{T\xi, \cdots, T\xi}_{n-1}, T\eta) \leqslant \frac{\lambda \max\left\{A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, \eta),\ A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, T\xi),\ A_b(\underbrace{\eta, \cdots, \eta}_{n-1}, T\eta)\right\}}{1 + A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, \eta)},$$

for every $\lambda \in (0,1)$. Hence, the rational-type contractive condition of Theorem 7.4 is satisfied.

Now consider the Picard sequence

$$x_m := T^m x_0, \qquad x_0 \in X.$$

Since T is constant,

$$x_m = p, \qquad m \geqslant 1.$$

Thus,

$$\Theta(x_m, \cdots, x_m, x_p) = \Theta(p, \cdots, p),$$

for all $m, p \in \mathbb{N}$.

Consequently, the orbit-control condition is trivially satisfied.

Therefore, all assumptions of Theorem 7.4 hold. Hence, the mapping T possesses the unique fixed point p.

§8 **Application of Theorem 7.4**

Let $X := C([0,1])$, the Banach space of all real-valued continuous functions on $[0,1]$, equipped with the supremum norm

$$\|f\|_\infty := \sup_{x \in [0,1]} |f(x)|.$$

Define $A_b : X^n \to [0, \infty)$ by

$$A_b(f_1, f_2, \cdots, f_n) := \max_{1 \leqslant i < j \leqslant n} \|f_i - f_j\|_\infty, \qquad f_1, f_2, \cdots, f_n \in X.$$

Choose the control function $\Theta \equiv n$. Then, (X, A_b) forms a complete extended A_b-metric space. Indeed, the convergence and Cauchy properties induced by A_b coincide with the usual uniform convergence on $C([0,1])$, and $C([0,1])$ is complete with respect to the supremum norm.

Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous kernel, $h \in C([0,1])$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous nonlinear function.

Assume that there exist constants

$$K \geqslant 0 \qquad \text{and} \qquad L \geqslant 0$$

such that

$$\sup_{x \in [0,1]} \int_0^1 |k(x,y)|\, dy \leqslant K,$$

and

$$|\varphi(u) - \varphi(v)| \leqslant L|u - v|, \qquad u, v \in \mathbb{R}.$$

Assume further that

$$\lambda := KL \in (0,1)$$

and define the operator $T : X \to X$ by

$$(Tf)(x) := \int_0^1 k(x,y)\,\varphi(f(y))\,dy + h(x), \qquad x \in [0,1].$$

We show that T satisfies the rational-type contractive condition of Theorem 7.4.

Let $f, g \in X$. Then, for every $x \in [0,1]$,

$$\begin{aligned}
|(Tf)(x) - (Tg)(x)| &= \left| \int_0^1 k(x,y)\big(\varphi(f(y)) - \varphi(g(y))\big)\,dy \right| \\
&\leqslant \int_0^1 |k(x,y)|\,|\varphi(f(y)) - \varphi(g(y))|\,dy \\
&\leqslant L \int_0^1 |k(x,y)|\,|f(y) - g(y)|\,dy \\
&\leqslant L\|f - g\|_\infty \int_0^1 |k(x,y)|\,dy.
\end{aligned}$$

Taking the supremum over $x \in [0,1]$, we obtain

$$\|Tf - Tg\|_\infty \leqslant KL\|f - g\|_\infty = \lambda\|f - g\|_\infty.$$

Hence,

$$A_b\big(Tf, Tg, \underbrace{Tf, \cdots, Tf}_{n-2}\big) = \|Tf - Tg\|_\infty \leqslant \lambda A_b\big(f, g, \underbrace{f, \cdots, f}_{n-2}\big).$$

Since

$$1 + A_b\big(f, g, \underbrace{f, \cdots, f}_{n-2}\big) \geqslant 1,$$

it follows that

$$A_b\big(Tf, Tg, \underbrace{Tf, \cdots, Tf}_{n-2}\big) \leqslant \frac{\lambda \max\Big\{A_b\big(f, g, \underbrace{f, \cdots, f}_{n-2}\big),\ A_b\big(f, Tf, \underbrace{f, \cdots, f}_{n-2}\big),\ A_b\big(g, Tg, \underbrace{g, \cdots, g}_{n-2}\big)\Big\}}{1 + A_b\big(f, g, \underbrace{f, \cdots, f}_{n-2}\big)}.$$

Thus, T satisfies the rational-type contractive condition of Theorem 7.4.

Moreover, the orbit-control condition required in Theorem 7.4 is satisfied since $\Theta \equiv n$ is constant along every Picard orbit. Hence, the corresponding finite products of Θ-values remain bounded. Therefore, all assumptions of Theorem 7.4 are satisfied. Consequently, T possesses a unique fixed point $u \in X$ such that

$$Tu \;=\; u.$$

Equivalently, the nonlinear integral equation

$$u(x) = \int_0^1 k(x,y)\,\varphi(u(y))\,dy + h(x), \qquad x \in [0,1],$$

admits a unique continuous solution $u \in C([0,1])$. Furthermore, for every initial function $u_0 \in C([0,1])$, the Picard iterative sequence $u_{m+1} := Tu_m$ converges uniformly to the unique solution u and therefore, the above nonlinear integral equation provides an application of Theorem 7.4 in the framework of extended A_b-metric spaces.

§9. **Conclusion**

In this paper, we investigated fixed point theory in the setting of extended A_b-metric spaces, which generalize several well-known structures including metric spaces, b-metric spaces, extended b-metric spaces, S-metric spaces, S_b-metric spaces, extended S_b-metric spaces, A-metric spaces, and A_b-metric spaces through the introduction of a variable control function. This generalized framework provides greater flexibility for the study of nonlinear contractive mappings and convergence analysis.

Within this setting, we established fixed point results for various classes of contractions, including Banach-type, Kannan-type, Reich-type, and nonlinear rational-type contractions. The obtained theorems were proved under suitable orbit-control conditions that ensure the convergence of Picard iterative sequences. Consequently, our results unify and extend many classical fixed point theorems available in the existing literature on generalized metric spaces.

To demonstrate the applicability of the developed theory, several illustrative examples of extended A_b-metric spaces were presented. In addition, an application to nonlinear integral equations was discussed, showing that the proposed framework is not only theoretically significant but also useful in practical problems arising in nonlinear analysis.

The results obtained in this work open several directions for future research. One may investigate fixed point theory in fuzzy, probabilistic, or stochastic extended A_b-metric spaces, study multivalued and hybrid contractions, or explore applications in optimization theory, differential equations, and computational mathematics. We believe that extended A_b-metric spaces provide a fruitful and flexible framework for further developments in modern fixed point theory and its applications.

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