

Further Results on Analytic Odd Mean Labeling of Graphs

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Abstract: Let $G = (V, E)$ be a graph with p vertices and q edges. A graph G is analytic odd mean if there exist an injective function $f : V \rightarrow \{0, 1, 3, 5 \dots, 2q - 1\}$ with an induce edge labeling $f^* : E \rightarrow Z$ such that for each edge uv with $f(u) < f(v)$,

$$f^*(uv) = \begin{cases} \left\lceil \frac{f(v)^2 - (f(u)+1)^2}{2} \right\rceil, & \text{if } f(u) \neq 0 \\ \left\lceil \frac{f(v)^2}{2} \right\rceil, & \text{if } f(u) = 0 \end{cases}$$

is injective. We say that f is an analytic odd mean labeling of G . In this paper we prove that sun graph S_n , prism D_n , helm graph H_n , the graph $C_n \circ P_2$, banana tree, bamboo tree, perfect binary tree, the graph PC_n , unicyclic graph, the caterpillar $P_k(n_0, n_1, \dots, n_{k-1})$ and spider graph are analytic odd mean graph.

Key Words: Mean labeling, analytic mean labeling, analytic odd mean labeling, Smarandachely analytic odd mean labeling, analytic odd mean graph.

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§1. Introduction

Throughout this paper we consider only finite, simple and undirected graph $G = (V, E)$ with p vertices and q edges and notations not defined here are used in the sense of Harary [2]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling. An excellent survey of graph labeling is available in [1]. The concept of analytic mean labeling was introduced in [7]. A graph G is analytic mean graph if it admits a bijection $f : V \rightarrow \{0, 1, 2, \dots, p - 1\}$ such that the induced edge labeling $f^* : E \rightarrow Z$ given by $f^*(uv) = \left\lceil \frac{f(u)^2 - f(v)^2}{2} \right\rceil$ with $f(u) > f(v)$ is injective. Motivated by the results in [7], we introduced a new mean labeling called analytic odd mean labeling in [3]. A graph G is an analytic odd mean if there exist an injective function $f : V \rightarrow \{0, 1, 3, 5 \dots, 2q - 1\}$ with an induce edge labeling $f^* : E \rightarrow Z$ such that for each edge

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uv with $f(u) < f(v)$,

$$f^*(uv) = \begin{cases} \left\lceil \frac{f(v)^2 - (f(u)+1)^2}{2} \right\rceil, & \text{if } f(u) \neq 0 \\ \left\lceil \frac{f(v)^2}{2} \right\rceil, & \text{if } f(u) = 0 \end{cases}$$

is injective. We say such an f is an analytic odd mean labeling of G . Otherwise, a Smarandachely analytic odd mean labeling of G if there exists an integer $0 \leq k \leq q$ holding with $|f^{-*}(k)| \geq 2$, where f^{-*} is the inverse of f^* . We proved that cycle C_n , path P_n , n -bistar, comb $P_n \odot K_1$, graph $L_n \odot K_1$, wheel graph W_n , flower graph Fl_n , some splitting graphs, multiple of graphs, quadrilateral snake $Q(n)$, double quadrilateral snake $DQ(n)$, coconut tree, fire cracker and some star graphs, splitting graph $spl(G)$, $P_n(1, 2, 3, \dots, n)$, the complete bipartite graph $K_{m,n}$, the graph $C_k \odot \bar{K}_n$, the square graph of P_n , C_n , $B_{n,n}$, H -graph and $H \odot mK_1$ are analytic odd mean graphs in [4], [5] and [6].

We use the following definitions in the subsequent section to prove the results.

Definition 1.1 A sun graph S_n is a cycle C_n with a pendent edge attached to each vertex of a cycle C_n .

Definition 1.2 The prism $D_n, n \geq 3$ is a cubic graph which can be represented as a Cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on n vertices.

Definition 1.3 A banana tree is a tree obtained from a family of stars by joining one end vertex of each star to a new vertex.

Definition 1.4 A tree is called a spider if it has a center vertex c of degree $R > 1$ and all the other vertex is either a leaf or with degree 2. Thus a spider is an amalgamation of k paths with various lengths. If it has x_1 's path of length a_1 , x_2 's path of length a_2, \dots . We denote the spider graph by $SP(a_1^{x_1}, a_2^{x_2}, \dots, a_m^{x_m})$ where $a_1 < a_2 < \dots < a_m$ and $x_1 + x_2 + \dots + x_m = R$.

Definition 1.5 A helm $H_n, n \geq 3$ is obtained from the wheel graph W_n by adding a pendent edge at each vertex on the wheel's rim.

Definition 1.6 A bamboo tree $(P_n \otimes K_{1,n_i})_{i=1}^k$ is an one point union of $P_n \otimes K_{1,n_i}$ where $1 \leq i \leq k$.

Definition 1.7 A caterpillar $P_k(n_1, n_2, \dots, n_k)$ is a tree in which all the vertices are within distance 1 of a central path P_k for $k \geq 1$. When $k \geq 2$, a caterpillar is obtained from a path $P_k = u_1 u_2 u_3 \dots u_k$ attaching $n_i \geq 0$ pendent vertices $v_i^j (1 \leq j \leq n_i)$ to each u_i .

Definition 1.8 A perfect binary tree is a full binary tree in which all the leaves are at the same level and in which every parent has two children.

§2. Main Results

In this section we prove that sun graph S_n , prism D_n , helm graph H_n , path union of $n - 3$

copies of C_n , the graph $C_n \circ P_2$, banaba tree, bamboo tree, the graph PC_n , unicyclic graph, perfect binary tree, the caterpillar graph $P_k(n_0, n_1, \dots, n_{k-1})$ and spider graph are analytic odd mean graphs.

Theorem 2.1 For every positive integer $n \geq 3$, the sun graph S_n is an analytic odd mean graph.

Proof Let the vertex set and edge set of the sun graph be $V(S_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(S_n) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{u_n u_1\}$. Now $|V(G)| = 2n = |E(G)|$. We define an injective map $f : V(S_n) \rightarrow \{0, 1, 3, 5, \dots, 4n-1\}$ by $f(u_i) = 2i - 1$ for $1 \leq i \leq n$ and $f(v_i) = 2n + 2i - 1$ for $1 \leq i \leq n$.

The induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(u_i u_{i+1}) &= 2i + 1 \text{ for } 1 \leq i \leq n-1, \\ f^*(u_1 u_n) &= 2n^2 - 2n - 1, \\ f^*(u_i v_i) &= 2n(n-1) + 2i(2n-1) + 1 \text{ for } 1 \leq i \leq n. \end{aligned}$$

We observe that the edge labels of $u_i u_{i+1}$ are $3, 5, \dots, 2n-1$ as i increases and the edge labels of $u_i v_i$ are increased by $4n-2$ as i increases from 1 to n . Hence all the edge labels are distinct and odd. Hence S_n admits an analytic odd mean labeling. \square

An analytic odd mean labeling of S_8 is shown in Figure 1.

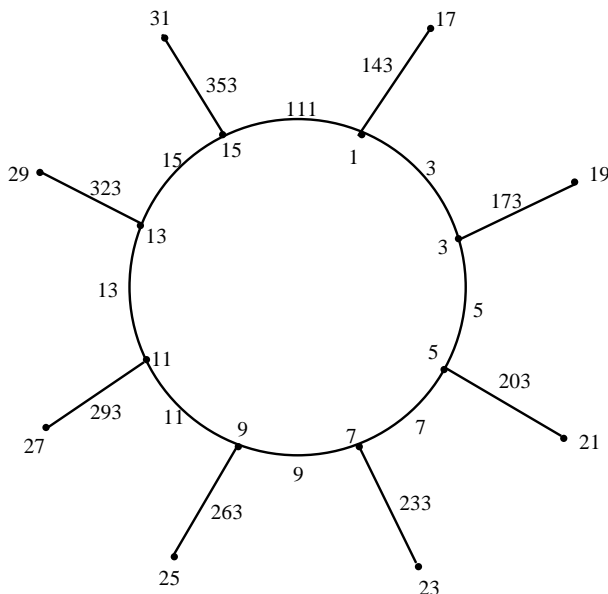


Figure 1

Theorem 2.2 For every positive $n \geq 3$, the prism D_n is an analytic odd mean graph.

Proof Let the vertex set and edge set of the prism be $V(D_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(D_n) = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\} \cup \{u_n v_1, v_n v_1\}$. Now $|V(D_n)| = 2n$ and $|E(D_n)| = 3n$. We define an injective map $f : V(D_n) \rightarrow \{0, 1, 3, 5, \dots, 6n-1\}$ by $f(u_i) = 2i - 1$ for $1 \leq i \leq n$ and $f(v_i) = 2n + 2i - 1$ for $1 \leq i \leq n$.

The induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(u_i u_{i+1}) &= 2i + 1 \text{ for } 1 \leq i \leq n - 1, \\ f^*(u_1 u_n) &= 2n^2 - 2n - 1, \\ f^*(u_i v_i) &= 2n(n - 1) + 2i(2n - 1) + 1 \text{ for } 1 \leq i \leq n, \\ f^*(v_i v_{i+1}) &= 2n + 2i + 1 \text{ for } 1 \leq i \leq n - 1, \\ f^*(v_1 v_n) &= 6n^2 - 8n - 1. \end{aligned}$$

We observe that the edge labels of $u_i u_{i+1}$ and $v_i v_{i+1}$ are $3, 5, \dots, 2n - 1$ and $2n + 3, 2n + 5, \dots, 2n + 2n - 1 = 4n - 1$ respectively as i increases and the edge labels of $u_i v_i$ are increased by $4n - 2$ as i increases from 1 to n . So all the edge labels are distinct and odd. Hence D_n admits an analytic odd mean labeling. \square

An analytic odd mean labeling of D_8 is shown in Figure 2.

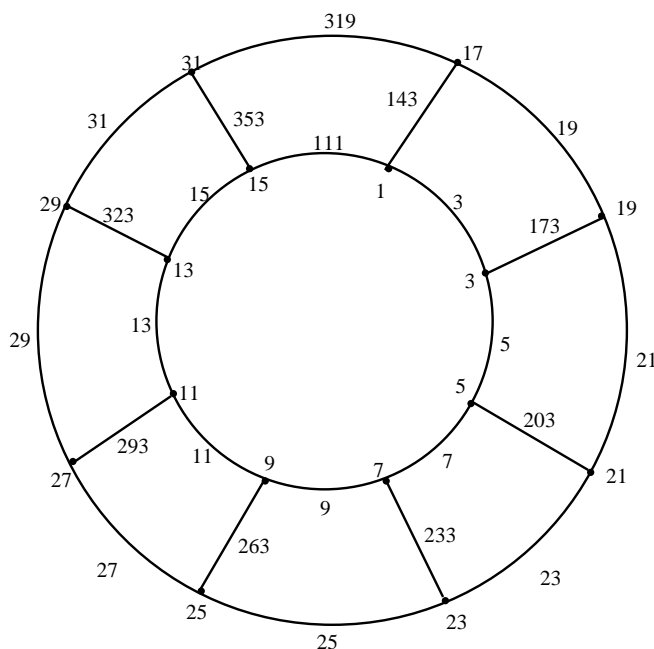


Figure 2

Theorem 2.3 *The helm graph $H_n, n \geq 3$ is an analytic odd mean graph.*

Proof Let $V(H_n) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and $E(H_n) = \{vv_i, v_i u_i : 1 \leq i \leq n\} \cup \{v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_n v_1\}$. Now $|V(G)| = 2n + 1$ and $|E(G)| = 3n$. We define an injective map $f : V(G) \rightarrow \{0, 1, 3, 5, \dots, 6n - 1\}$ by $f(v) = 0, f(v_i) = 4i - 3$ for $1 \leq i \leq n$ and $f(u_i) = 4n + 2i - 1$ for $1 \leq i \leq n$.

The induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(vv_i) &= 8i^2 - 12i + 5 \text{ for } 1 \leq i \leq n, \\ f^*(v_i v_{i+1}) &= 12i - 1 \text{ for } 1 \leq i \leq n - 1, \\ f^*(v_n v_1) &= 8n^2 - 12n + 3, \\ f^*(v_i u_i) &= 4n(2n - 1) - 6i(i - 1) + 8ni - 1 \text{ for } 1 \leq i \leq n. \end{aligned}$$

We observe that when i increases, the difference of edges $v_i u_i$ are decreased by 12. Clearly all the edge labels are odd and distinct. Hence H_n admits an analytic odd mean labeling. \square

An analytic odd mean labeling of H_{10} is shown in Figure 3.

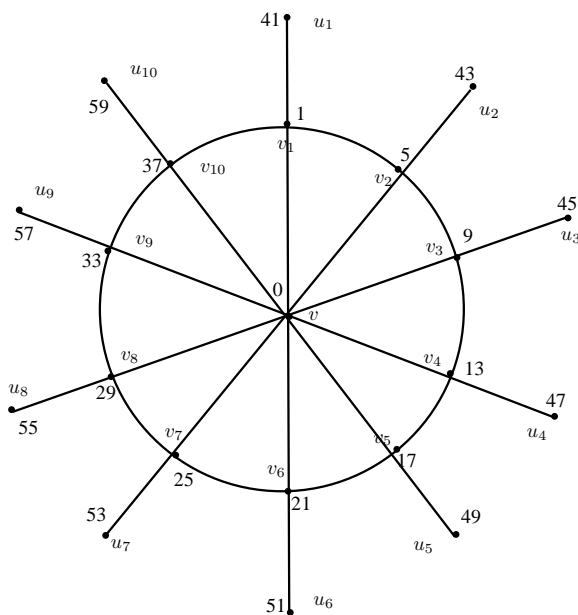


Figure 3

Theorem 2.4 *The graph PC_n ($n \geq 4$ and n is even) is obtained from $C_n = v_1 v_2 \cdots v_n v_1$ by adding the chords v_i and v_{n-i+2} for $2 \leq i \leq l$ where $l = n/2$. Then the graph PC_n is an analytic odd mean graph.*

Proof Let $G = PC_n$. Let the vertex set and edge set of G be $V(G) = \{v_i, : 1 \leq i \leq n\}$ and $E(G) = \{v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_i v_{n-i+2} : 2 \leq i \leq l\} \cup \{v_n v_1\}$. Then there are n vertices and $3n/2 - 1$ edges. We define an injective map $f : V(G) = \{0, 1, 3, \dots, 3n - 3\}$ by $f(v_i) = 2i - 1$ for $1 \leq i \leq n$.

The induced edge labeling f^* is defined as follows:

$$f^*(v_i v_{i+1}) = 2i + 1 \text{ for } 1 \leq i \leq n,$$

$$f^*(v_n v_1) = 2n^2 - 2n - 1,$$

$$f^*(v_i v_{n-i+2}) = 2n(n + 3) - 2i(2n + 3) + 5 \text{ for } 2 \leq i \leq l.$$

It can be easily verified that f is an analytic odd mean labeling and hence PC_n is an analytic odd mean graph. \square

An analytic odd mean labeling of PC_{10} is shown in Figure 4.

Theorem 2.6 *The caterpillar $P_k(n_0, n_1, \dots, n_{k-1})$ is an analytic odd mean graph for $k \geq 2, n_i \geq 0$.*

Proof Let $V(G) = \{v_i, v_i^j : 0 \leq i \leq k - 1 \text{ and } 1 \leq j \leq n_i\}$ and $E(G) = \{v_{i-1}v_i : 1 \leq i \leq k - 1\} \cup \{v_i v_i^j : 0 \leq i \leq k - 1 \text{ and } 1 \leq j \leq n_i\}$. Hence $|V| = n_0 + n_1 + \dots + n_k + k$ and $|E| = n_0 + n_1 + \dots + n_k + k - 1$. We define an injective map on the vertex set by

$$\begin{aligned} f(v_0) &= 0, f(v_i) = 2i - 1 \text{ for } 1 \leq i \leq k - 1, \\ f(v_0^j) &= 2k + 2j - 3 \text{ for } 1 \leq j \leq n_0, \\ f(v_i^j) &= 2k - 3 + 2 \sum_{r=0}^{i-1} n_r + 2j \text{ for } 1 \leq i \leq k - 1 \text{ and } 1 \leq j \leq n_i. \end{aligned}$$

The induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(v_{i-1}v_i) &= 2i - 1 \text{ for } 1 \leq i \leq k - 1, \\ f^*(v_0 v_0^j) &= 2k(k - 3) + 2j(j - 3) + 4kj + 5 \text{ for } 1 \leq j \leq n_0, \\ f^*(v_i v_i^j) &= \left[\left(2k - 3 + 2 \sum_{r=0}^{i-1} n_r + 2j - 2i \right) \left(2k - 3 + 2 \sum_{r=0}^{i-1} n_r + 2j + 2i \right) + 1 \right] \div 2 \text{ for } \\ &1 \leq i \leq k - 1 \text{ and } 1 \leq j \leq n_i. \end{aligned}$$

Clearly the edge labels are odd and distinct. Hence $P_k(n_0, n_1, \dots, n_{k-1})$ admits an analytic odd mean labeling. \square

An analytic odd mean labeling of $P_5(3, 2, 1, 5, 4)$ is shown in Figure 6.

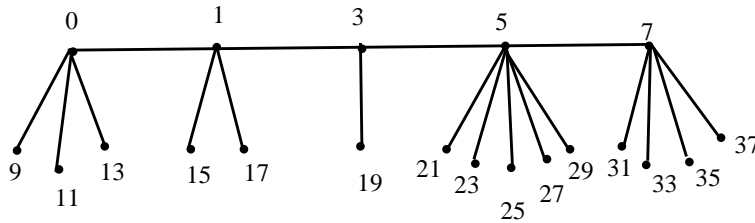


Figure 6

Theorem 2.7 *The graph $C_n \circ P_2$ is an analytic odd mean graph.*

Proof Let G be the graph $C_n \circ P_2$. Let $V(G) = \{v_i, v_{i,1}, v_{i,2} : 1 \leq i \leq n\}$ and $E(G) = \{v_i v_{i+1} : 1 \leq i \leq n - 1\} \cup \{v_n v_1\} \cup \{v_i v_{i,1}, v_i v_{i,2}, v_{i,1} v_{i,2} : 1 \leq i \leq n\}$. Now $|V(G)| = 3n$ and $|E(G)| = 4n$.

We define an injective map $f : V(G) \rightarrow \{0, 1, 3, 5, \dots, 8n - 1\}$ by

$$\begin{aligned} f(v_i) &= 2i - 1 \text{ for } 1 \leq i \leq n, \\ f(v_{i,1}) &= 4i + 2n - 3 \text{ for } 1 \leq i \leq n, \\ f(v_{i,2}) &= 4i + 2n - 1 \text{ for } 1 \leq i \leq n. \end{aligned}$$

The induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(v_i v_{i+1}) &= 2i + 1 \text{ for } 1 \leq i \leq n - 1, \\ f^*(v_n v_1) &= 2n^2 - 2n + 1, \\ f^*(v_i v_{i,1}) &= 6i(i - 2) + 2n(n - 3) + 8ni + 5 \text{ for } 1 \leq i \leq n, \\ f^*(v_i v_{i,2}) &= 2i(3i - 2) + 2n(n - 1) + 8ni + 1 \text{ for } 1 \leq i \leq n, \\ f^*(v_{i,1} v_{i,2}) &= 4i + 2n - 1 \text{ for } 1 \leq i \leq n. \end{aligned}$$

Clearly the edge labels are odd and distinct. Hence $C_n \circ P_2$ admits an analytic odd mean labeling. \square

An analytic odd mean labeling of $C_5 \circ P_2$ is shown in Figure 7.

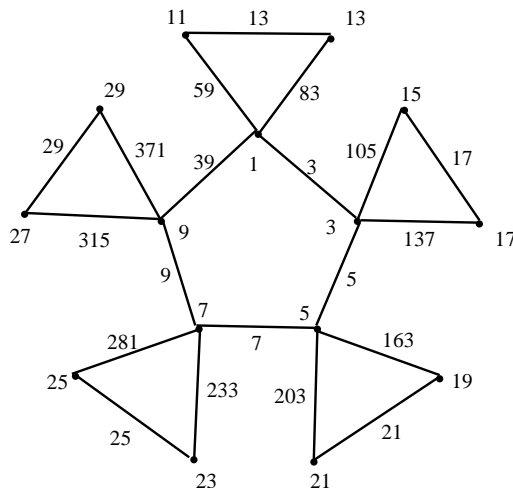


Figure 7

Theorem 2.8 Let $K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_k}$ be a family of stars with vertex sets $V(K_{1,n_j}) = \{a_j, a_j^1, a_j^2, \dots, a_j^{n_j}\}$, and $deg(a_j) = n_j, 1 \leq j \leq k$. Let $BT(n_1, n_2, \dots, n_k)$ be a banana tree obtained by adding a new vertex a and joining it to $a_1^1, a_2^1, a_3^1, \dots, a_k^1$. Then $BT(n_1, n_2, \dots, n_k)$ admits an analytic mean labeling where n_j is any positive integer.

Proof Let the vertex set and edge set be $V = \{a, a_j, a_j^r : 1 \leq j \leq k, 1 \leq r \leq n_j\}$ and $E = \{aa_j^1 : 1 \leq j \leq k\} \cup \{a_j a_j^r : 1 \leq j \leq k, 1 \leq r \leq n_j\}$ respectively. We assume $n_0 = 0$. Hence $|V| = n_1 + n_2 + \dots + n_k + k + 1$ and $|E| = n_1 + n_2 + \dots + n_k + k$.

We define an injective function f on the vertex set of banana tree as follows :

$$f(a) = 0, f(a_j) = 2j - 1 \text{ for } 1 \leq j \leq k,$$

$$f(a_j^r) = 2k - 1 + 2 \sum_{i=0}^{j-1} n_i + 2r \text{ for } 1 \leq r \leq n_j, 1 \leq j \leq k.$$

The induced edge labeling f^* is defined as follows:

$$f^*(a_j a_j^r) = 2(k^2 + r^2 - j^2) + 2\left(\sum_{i=0}^{j-1} n_i + j\right)(2k - 1 + \sum_{i=0}^{j-1} n_i) + 2j \sum_{i=0}^{j-1} n_i - 2k + 1$$

for $1 \leq r \leq n_j, 1 \leq j \leq k$ and

$$f^*(aa_j^1) = \frac{\left[\left(2k - 1 + 2 \sum_{i=0}^{j-1} n_i + 2 \right)^2 + 1 \right]}{2}$$

$$= 2k(k + 1) + 2 \sum_{i=0}^{j-1} n_i(2k + 1) + 2 \left(\sum_{i=0}^{j-1} n_i \right)^2 + 1$$

for $1 \leq j \leq k$.

Clearly the edge labels are odd and distinct. Therefore f is an analytic odd mean labeling and hence the banana tree is an analytic odd mean graph. \square

An analytic odd mean labeling of $BT(3, 5, 6, 4)$ is shown in Figure 8.

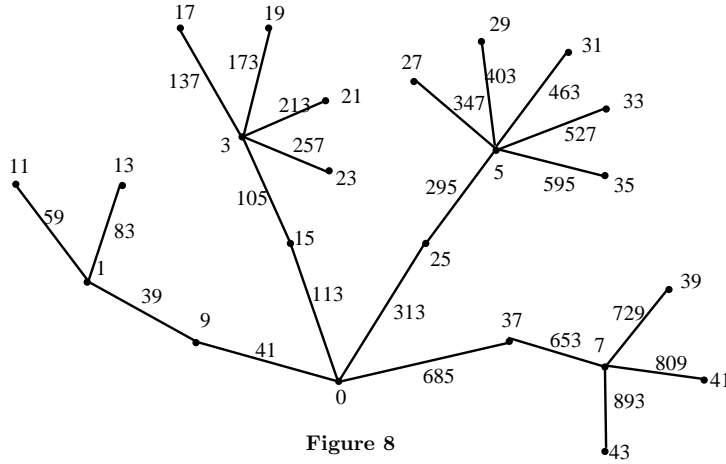


Figure 8

Theorem 2.9 Let $K_{1,n_1}, K_{1,n_2}, \dots, K_{1,n_k}, K_{1,k}$ be a family of stars with vertex sets $V(K_{1,n_j}) = \{v_j^r : 0 \leq r \leq n_j\}$ and $\deg(v_j^0) = n_j$, for $1 \leq j \leq k$, $V(K_{1,k}) = \{v, v_1, v_2, \dots, v_k\}$. The bamboo tree $(P_2 \otimes K_{1,n_j})$ for $j = 1, 2, \dots, k$ is obtained by joining the vertex v_1, v_2, \dots, v_k with $v_1^1, v_2^1, \dots, v_k^1$ respectively. Clearly the number of vertices and edges of the bamboo tree are $n_1 + n_2 + \dots + n_k + k + 1$ and $n_1 + n_2 + \dots + n_k + k$ respectively. Then the bamboo tree $(P_2 \otimes K_{1,n_j})$ for $j = 1, 2, \dots, k$ is an analytic odd mean graph.

Proof Let the vertex set and edge set be $V = \{v, v_j, v_j^r : 1 \leq j \leq k, 0 \leq r \leq n_j\}$ and $E = \{vv_j : 1 \leq j \leq k\} \cup \{v_j v_j^1 : 1 \leq j \leq k\} \cup \{v_j^0 v_j^r : 1 \leq j \leq k, 1 \leq r \leq n_j\}$ respectively. We assume $n_0 = 0$. Hence $|V| = n_1 + n_2 + \dots + n_k + 2k + 1$ and $|E| = n_1 + n_2 + \dots + n_k + 2k$.

We define an injective function f on the vertex set of bamboo tree as follows:

$$\begin{aligned} f(v) &= 0, f(v_j) = 2j - 1 \text{ for } 1 \leq j \leq k, \\ f(v_j^r) &= 4k - 1 + 2 \sum_{i=0}^{j-1} n_i + 2r \text{ for } 1 \leq r \leq n_j, 1 \leq j \leq k, \\ f(v_j^0) &= 2k + 2j - 1 \text{ for } 1 \leq j \leq k. \end{aligned}$$

The induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(vv_j) &= 2j^2 - 2j + 1 \text{ for } 1 \leq j \leq k, \\ f^*(v_j v_j^1) &= \left[(4k + 1 + 2 \sum_{i=0}^{j-1} n_i)^2 + 1 \right] \div 2 - 2j^2 \text{ for } 1 \leq j \leq k, \\ f^*(v_j^0 v_j^r) &= \left[(4k - 1 + 2 \sum_{i=0}^{j-1} n_i + 2r)^2 + 1 \right] \div 2 - 2(k^2 + j^2) - 4kj \text{ for } 1 \leq r \leq n_j, 1 \leq j \leq k. \end{aligned}$$

Clearly the edge labels are odd and distinct. Therefore f is an analytic odd mean labeling and hence the bamboo tree is an analytic odd mean graph. \square

An analytic odd mean labeling of $(P_2 \otimes K_{1,n_j})$ for $j = 1, 2, 3, 4$ with $n_1 = 5, n_2 = 4, n_3 = 3, n_4 = 7$ is shown in Figure 9.

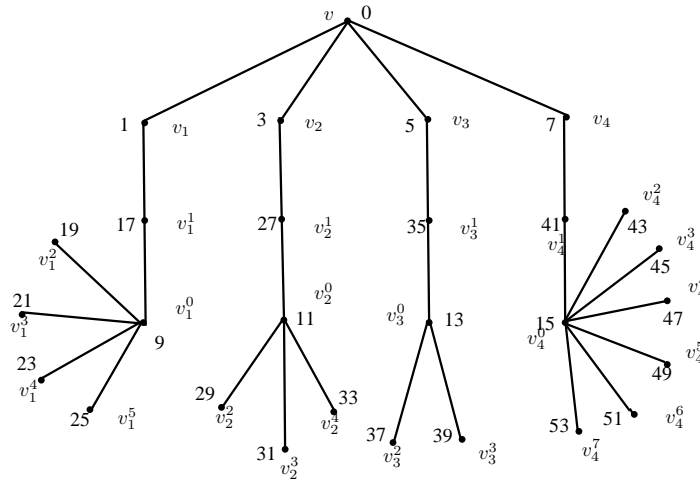


Figure 9

Theorem 2.10 *The perfect binary tree T of order p is an analytic odd mean graph.*

Proof Let $V(T) = \{v_i : 1 \leq i \leq p\}$ and $E(T) = \{v_i v_{2i}, v_i v_{2i+1} : 1 \leq i \leq q/2\}$. Hence $|V| = p$ and $|E| = p - 1$. We define an injective map $f : V(G) = \{0, 1, 3, \dots, 2p - 3\}$ by $f(v_1) = 0$ and $f(v_i) = 2i - 3$ for $2 \leq i \leq p$. The induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(v_1 v_2) &= 1, \\ f^*(v_1 v_3) &= 5, \\ f^*(v_i v_{2i}) &= 6i^2 - 8i + 3 \text{ for } 2 \leq i \leq q/2, \\ f^*(v_i v_{2i+1}) &= 6i^2 - 1 \text{ for } 2 \leq i \leq q/2. \end{aligned}$$

We observe that the difference of edge labels of $v_i v_{2i}$ and $v_i v_{2i+1}$ is $8i - 4$ as i increases from 1 to $\frac{q}{2}$. Therefore the edge labels are odd and distinct. Hence the binary tree admits an analytic odd mean labeling. \square

An analytic odd mean labeling of T of order 15 is shown in Figure 10.

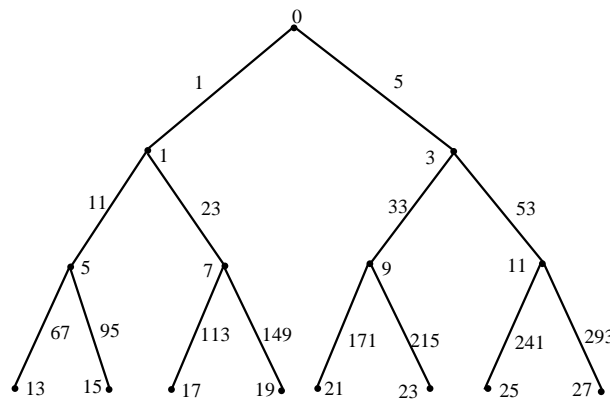


Figure 10

Theorem 2.11 *The spider graph $SP(1^n, k^m)$, $n \geq 1$, and $k, m \geq 2$ is an analytic odd mean graph.*

Proof Let $V(SP(1^n, k^m)) = \{v\} \cup \{w_a, : 1 \leq a \leq n\} \cup \{v_i^j : 1 \leq i \leq k \text{ and } 1 \leq j \leq m\}$ and $E(SP(1^n, k^m)) = \{vv_1^j, v_i^j v_{i+1}^j : 1 \leq i \leq k - 1 \text{ and } 1 \leq j \leq m\} \cup \{vw_a : 1 \leq a \leq n\}$. We define an injective map $f : V(SP(1^n, k^m)) \rightarrow \{0, 1, 3, 5, \dots, 2(km + n) - 1\}$ by $f(v) = 0$, $f(v_i^j) = (j - 1)2k + 2i - 1$ for $1 \leq i \leq k$, $1 \leq j \leq m$ and $f(w_a) = 2mk + 2a - 1$ for $1 \leq a \leq n$. The induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(vv_1^j) &= 2k(j - 1)(k(j - 1) + 1) + 1 \text{ for } 1 \leq j \leq m, \\ f^*(v_i^j v_{i+1}^j) &= 2k(j - 1) + 2i + 1 \text{ for } 1 \leq i \leq k - 1 \text{ and } 1 \leq j \leq m, \\ f^*(vw_a) &= 2mk(mk - 1) + 2a(a - 1) + 4mak + 1 \text{ for } 1 \leq a \leq n. \end{aligned}$$

It can be verified that the edge labels are odd and distinct. Hence $SP(1^n, k^m)$ is an analytic odd mean graph. \square

An analytic odd mean labeling of $SP(1^n, 6^5)$ is shown in Figure 11.

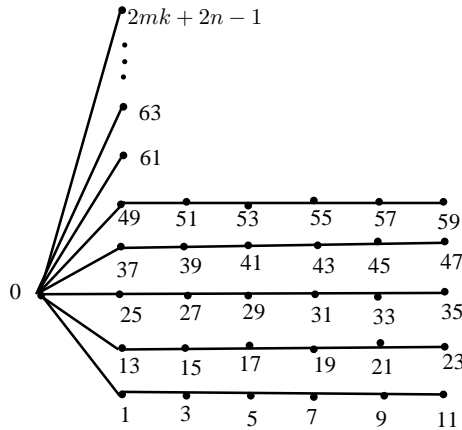


Figure 11

Theorem 2.12 *The spider graph $SP(2^n, k^m)$, $n \geq 1, k, m \geq 2$ is an analytic odd mean graph if (a) k is even and m is any integer and (b) k is odd and m is even.*

Proof Let $V(SP(2^n, k^m)) = \{v\} \cup \{w_{a,1}, w_{a,2} : 1 \leq a \leq n\} \cup \{v_i^j : 1 \leq i \leq k \text{ and } 1 \leq j \leq m\}$ and $E(SP(2^n, k^m)) = \{vv_1^j, v_i^j v_{i+1}^j : 1 \leq i \leq k \text{ and } 1 \leq j \leq m\} \cup \{vw_{a,1}, w_{a,1}w_{a,2} : 1 \leq a \leq n\}$. We define an injective map $f : V(SP(2^n, k^m)) \rightarrow \{0, 1, 3, 5, \dots, 2km + 4n - 1\}$ by $f(v) = 0$, $f(v_i^j) = (j - 1)2k + 2i - 1$ for $1 \leq i \leq k$, $1 \leq j \leq m$; $f(w_{a,1}) = 2mk + 4a - 3$ for $1 \leq a \leq n$ and $f(w_{a,2}) = 2mk + 4a - 1$ for $1 \leq a \leq n$.

The induced edge labeling f^* is defined as follows:

$$\begin{aligned} f^*(vv_1^j) &= 2k(j - 1)(k(j - 1) + 1) + 1 \text{ for } 1 \leq j \leq m, \\ f^*(v_i^j v_{i+1}^j) &= 2k(j - 1) + 2i + 1 \text{ for } 1 \leq i \leq k - 1 \text{ and } 1 \leq j \leq m, \\ f^*(vw_{a,1}) &= 2mk(mk - 3) + 4a(2a - 3) + 8mak + 5 \text{ for } 1 \leq a \leq n, \\ f^*(w_{a,1}w_{a,2}) &= 2mk + 4a - 1 \text{ for } 1 \leq a \leq n. \end{aligned}$$

It can be verified that the edge labels are odd and distinct. Hence $SP(2^n, k^m)$ is an analytic odd mean graph. \square

An analytic odd mean labeling of $SP(2^n, 5^4)$ is shown in Figure 12.

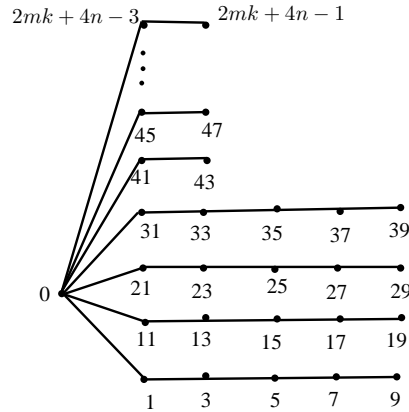


Figure 12

Theorem 2.13 *The spider graph $SP(1^s, 2^n, k^m), n, s \geq 1, k, m \geq 2$ is an analytic odd mean graph if (a) k is even and m is any integer and (b) k is odd and m is even.*

Proof Let $V(SP(1^s, 2^n, k^m)) = \{v\} \cup \{w_{a,1}, w_{a,2} : 1 \leq a \leq n\} \cup \{v_i^j : 1 \leq i \leq k \text{ and } 1 \leq j \leq m\} \cup \{u_r : 1 \leq r \leq s\}$ and $E(SP(1^s, 2^n, k^m)) = \{vv_1^j, v_i^j v_{i+1}^j : 1 \leq i \leq k \text{ and } 1 \leq j \leq m\} \cup \{vw_{a,1}, w_{a,1}w_{a,2} : 1 \leq a \leq n\}$. We define an injective map $f : V(SP(1^s, 2^n, k^m)) \rightarrow \{0, 1, 3, 5, \dots, 2km + 4n + 2s - 1\}$ by $f(u_r) = 2mk + 4n + 2r - 1$ for $1 \leq r \leq s$ and $f(v), f(v_i^j), f(w_{a,1})$ and $f(w_{a,2})$ are defined as in Theorem 2.12. Then the induced edge labeling $f^*(vv_1^j), f^*(v_i^j v_{i+1}^j), f^*(vw_{a,1})$ and $f^*(w_{a,1}w_{a,2})$ are as in Theorem 2.12 and

$$f^*(vu_r) = \frac{(2mk + 2r + 4n - 1)^2 + 1}{2}$$

for $1 \leq r \leq s$.

It can be verified that the edge labels are odd and distinct. Hence $SP(1^s, 2^n, k^m)$ is an analytic odd mean graph. \square

An analytic odd mean labeling of $SP(1^s, 2^n, 3^4)$ is shown in Figure 13.

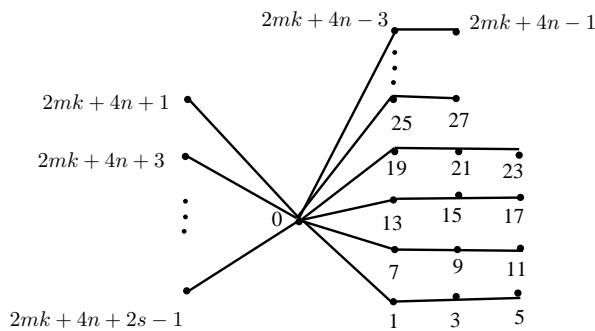


Figure 13

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