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Further Results on Level Matrix

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Abstract: A level index is a numerator of distance based Gini index which was introduced in 2017. Then, the level index is used to evaluate the balance of rooted trees. Level matrix and level characteristic polynomial concepts were introduced recently for rooted trees. The level characteristic polynomial of rooted binary caterpillars was computed in terms of distance characteristic polynomial of paths. In this paper, we compute the level index of rooted binary caterpillars as Octahedral numbers and show that level characteristic polynomials of the mentioned graphs can be computed by a recurrence relation in terms of Chebyshev polynomials of first- and second-kind. So, we can give an affirmative answer to a recent manuscript including a question about the computation of the level characteristic polynomial of a graph with recursive formula.

Key Words: Level index, level characteristic polynomial, rooted trees, binary caterpillar, Smarandachely rooted tree.

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§1. Introduction

Corrado Gini [5] introduced in 1912 several economic summary statistics, among them what is now known as the Gini index, which it is a parameter that measures how equitably a resource is distributed throughout a population (for more details, see [4] and [7]. More recent, Balaji and Mahmoud [1] presented two distance-based molecular descriptors level index and Gini index. Clearly, the level index is a numerator of the distance-based Gini index which is used to evaluate the measure for a tree. Balaji and Mahmoud obtained a general phrase of the level index of trees. Level matrix and level characteristic polynomial concepts introduced recently [3]. Level characteristic polynomials of rooted stars, rooted double stars, and rooted binary caterpillars were obtained [3]. The level characteristic polynomials of rooted binary caterpillars were computed in terms of distance characteristic polynomials of paths. Moreover, distance characteristic polynomials of paths were obtained by Hosoya et al. in 1973 [6]. The spectrum of the level matrix was studied very recently in [2].

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In this paper, we compute the level characteristic polynomial of rooted binary caterpillar by a recurrence relation in terms of Chebyshev polynomials of the first-kind and second-kind. Therefore, we can present an affirmative answer to the question (Question 4) which appeares in [2] looking for a rooted tree that its level characteristic polynomial can be computed by a recursive formula.

§2. **Preliminaries**

We only consider simple, connected, and undirected graphs. A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$. The notation $d(u,v)$ is used to show the distance between two vertices u and v in a graph. Generally, a *Smarandachely rooted tree* T^S is a rooted tree T with s rooted vertices and the level l_u^S of $u \in V(T)$ is the minimum distance between u to rooted vertices. Particularly, if there are only one rooted vertex, such a T^S is nothing else but a rooted tree T. For a rooted tree T, the level of a vertex u is abbreviated to l_u, i.e., the distance between u and the rooted vertex, which is the objective in this paper. Certainly, the same question can be also considered on Smarandachely rooted tree T^S for a few of typical rooted vertices.

Definition 2.1([1]) *The level index of a rooted tree T, denoted by $LI(T)$, is given by*

$$LI(T) = \sum_{1 \le i < j \le n} |l_i(T) - l_j(T)|,$$

where $l_i(T)$ shows the level of the vertex v_i in T.

Definition 2.2([1]) *The level index of a rooted tree T of maximum level h is computed by the following equation such that N_i and N_{i+j} showing the number of vertices at level i and $i+j$ with the difference in depth of j*

$$LI(T) = \sum_{i=0}^{h} \sum_{j=0}^{h-i} j N_i N_{i+j}$$

Definition 2.3([3]) *Let T be a rooted tree and let its vertices be labeled as $v_1, v_2, \cdots, v_n$. The level of $v \in V(T)$ is the distance from the root of T to v. The level matrix of T is defined as the square matrix $L = L(T) = [l_{ij}]$ where l_{ij} is the absolute value of the levels' difference of vertices v_i and v_j in T.*

A rooted binary caterpillar T_k is obtained from a k vertex path by attaching a vertex to each vertex of the path which is illustrated in Figure 1.

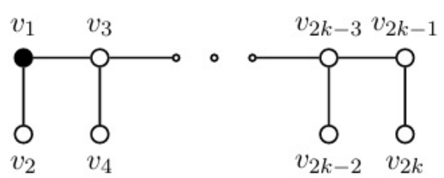

Figure 1. The rooted binary caterpillar T_k

Definition 2.4 *The level characteristic matrix of rooted binary caterpillar T_k is defined by Matrix M_k of order $2k$:*

$$M_k = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_k \\ A_{-2} & A_1 & A_2 & \ddots & \vdots \\ A_{-3} & A_{-2} & A_1 & \ddots & A_3 \\ \vdots & \ddots & \ddots & \ddots & A_2 \\ A_{-k} & \dots & A_{-3} & A_{-2} & A_1 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} x & -1 \\ -1 & x \end{pmatrix}, \quad \text{and} \quad A_k = \begin{pmatrix} -k+1 & -k \\ -k+2 & -k+1 \end{pmatrix}$$

for all integers $k \geq 2$, and $A_{-k} = A_k^t$ (transpose), for all $k \geq 1$.

For instance, the level characteristic matrix of T_5 is presented as follows

$$M_5 = \left(\begin{array}{cc|cc|cc|cc|cc} x & -1 & -1 & -2 & -2 & -3 & -3 & -4 & -4 & -5 \\ -1 & x & 0 & -1 & -1 & -2 & -2 & -3 & -3 & -4 \\ \hline -1 & 0 & x & -1 & -1 & -2 & -2 & -3 & -3 & -4 \\ -2 & -1 & -1 & x & 0 & -1 & -1 & -2 & -2 & -3 \\ \hline -2 & -1 & -1 & 0 & x & -1 & -1 & -2 & -2 & -3 \\ -3 & -2 & -2 & -1 & -1 & x & 0 & -1 & -1 & -2 \\ \hline -3 & -2 & -2 & -1 & -1 & 0 & x & -1 & -1 & -2 \\ -4 & -3 & -3 & -2 & -2 & -1 & -1 & x & 0 & -1 \\ \hline -4 & -3 & -3 & -2 & -2 & -1 & -1 & 0 & x & -1 \\ -5 & -4 & -4 & -3 & -3 & -2 & -2 & -1 & -1 & x \end{array}\right).$$

Theorem 2.5([6]) *The distance characteristic polynomial of a path P_n is given by*

$$\mathcal{C}_n(x) = x^n - \sum_{k=2}^{n} 2^{k-2}(k-1)\frac{n^2(n^2-1)(n^2-2^2)\dots(n^2-(k-1)^2)}{k^2(k^2-1)(k^2-2^2)\dots(k^2-(k-1)^2)}x^{n-k}.$$

Theorem 2.6([3]) *For $k > 2$, the characteristic polynomial of the rooted tree T_k is given by*

$$\varphi(\lambda) = (2\lambda)^{k-1}\big(\lambda \cdot \mathcal{C}_k(\lambda/2) + \mathcal{C}_{k+1}(\lambda/2)\big),$$

where $\mathcal{C}_n(x)$ is given in Theorem 2.5.

In order to obtain a general result, we can define a family Ŧ of trees such that $N_0 = 1$,$N_1 = N_2 = ... = N_{k-1} = 2$, and $N_k = 1$. Clearly $T_k \in$ Ŧ.

§3. Main Result

In this section, we first compute the level index of the rooted trees of the family Ŧ that the level indices of the members of Ŧ equal to Octahedral numbers. Moreover, we show that the determinant of the matrix M_k equals to level characteristic polynomial of only the members of Ŧ with level k. Finally, we obtain the determinant of M_k in terms of Chebyshev polynomials of the first-kind and second-kind.

Theorem 3.1 *If a tree $T \in$ Ŧ with level k, then the level index of T is given by*

$$LI(T) = \frac{k(2k^2+1)}{3}.$$

Proof By Definition 2.2, the level index of T can be computed by the following equation such that the numbers of the vertices at level i are ordered as follows $N_0 = 1$, $N_1 = N_2 = \cdots = N_{k-1} = 2$, and $N_k = 1$. Then

$$\begin{aligned} LI(T) &= \sum_{i=0}^{k}\sum_{j=0}^{k-i} jN_iN_{i+j} = k + 4\sum_{i=1}^{k-1} i(k-i) \\ &= k + 4k\sum_{i=1}^{k-1} i - 4\sum_{i=1}^{k-1} i^2 \\ &= k + 4k \times \frac{k(k-1)}{2} - 4 \times \frac{(k-1)k(2k-1)}{6} \\ &= \frac{4k^3+2k}{6} = \frac{k(2k^2+1)}{3}, \end{aligned}$$

as claimed. □

For instance, by Theorem 3.1, the initial terms of the level index of T_k are $LI(T_1) = 1$, $LI(T_2) = 6$, $LI(T_3) = 19$, $LI(T_4) = 44$, $LI(T_5) = 85$, and $LI(T_6) = 146$ (see Sequence A005900 in [8]).

Lemma 3.2 *Let T be a rooted tree. Then the level characteristic polynomial of T equals to determinant of M_k if and only if $T \epsilon$ Ŧ with level k.*

Proof The sufficient condition is clear. Then we will prove only the necessary condition. If $T \in$ Ŧ, then there are two vertices at each level except the last level k and there is one vertex at the last level k. It means that every two rows of the level matrix of T are equal because the vertices appeared on the same level. Since T is a rooted tree and there is one vertex at level k, the first row and the last row of the level matrix of T are equal in the reverse order.

Now assume that $T \notin$ Ŧ. It implies that there are at least three vertices at a level Then, at least three rows of the level matrix of T are the same is a contradiction. It is obtained that the level characteristic polynomial of T equals to determinant of M_k if and only if $T \in$ Ŧ. □

In order to find the determinant of the matrix M_k, we define the following matrices

- A_k (respectively, B_k, C_k) is the obtained matrix from M_k by removing the first row and first column (respectively, second column, $2k$th column).
- D_k is the obtained matrix from C_k by removing the first and last rows, and first and second columns.
- E_k is the obtained matrix from C_k by removing the first, second, and last rows, and the first three columns.

Then, by evaluating the determinant of the matrices $M_k, A_k, B_k, C_k, D_k, E_k$ according to the first row, we obtain the following recurrences

$$\begin{aligned}
\det(M_k) &= 2(x+1)\det(A_k) + x\det(B_k) + \det(C_k),\\
\det(A_k) &= 2x\det(M_{k-1}) - x^2\det(A_{k-1}),\\
\det(B_k) &= -x\det(M_{k-1}) + x\det(C_{k-1}),\\
\det(C_k) &= x^2\det(C_{k-1}) - 2x\det(D_{k-1}) + x^2\det(E_{k-1}),\\
\det(D_k) &= 2(x+1)\det(E_k) - x^2\det(D_{k-1}),\\
\det(E_k) &= 2x\det(D_{k-1}) - x^2\det(E_{k-1}).
\end{aligned}$$

By the recurrence of $\det(D_k)$ and $\det(E_k)$, we obtain

$$\det(D_k) = 2x(x+2)\det(D_{k-1}) - x^4\det(D_{k-2})$$

where $\det(D_1) = 2(1+x)$ and $\det(D_2) = 4x(1+x)(2+x)$. Hence, by induction on k, we have

$$\det(D_k) = 2(1+x)x^{2k-2}U_{k-1}(y),$$

where $y = 1+2/x$ and U_m is the mth Chebyshev polynomials of the second kind. Thus, by the recurrence of $\det(D_k)$, we have

$$\det(E_k) = x^{2k-2}(U_{k-1}(y) + U_{k-2}(y)).$$

By the recurrence of $\det(C_k)$, we have

$$\begin{aligned}
\det(C_k) &= -(1+x)x^{2k-2} + \sum_{j=0}^{k-2} x^{2j}(-2x\det(D_{k-1-j}) + x^2\det(E_{k-1-j}))\\
&= -(1+x)x^{2k-2} + x^{2k-1}\sum_{j=0}^{k-2} x^{2j}(xU_{k-3-j}(y) - (3x+4)U_{k-2-j}(y)).
\end{aligned}$$

Hence, by the recurrences of $\det(M_k)$, $\det(A_k)$, $\det(B_k)$ and $\det(C_k)$, we obtain that the sequence $\det(M_k)$ satisfies

$$\det(M_k) = 2x(x+2)\det(M_{k-1}) - x^4\det(M_{k-2}) + \det(C_k) + 2x^2\det(C_{k-1}) + x^4\det(C_{k-2}),$$

where $\det(M_1) = x^2 - 1$ and $\det(M_2) = x(x+2)(x^2 - 2x - 4)$. Moreover, the sequence $\det(C_k)$ satisfies

$$\det(C_k) = x(3x+4)\det(C_{k-1}) - x^3(3x+4)\det(C_{k-2}) + x^6\det(C_{k-3}),$$

where $\det(C_1) = -1 - x$, $\det(C_2) = -x(x+2)^2$, $\det(C_3) = -x^2(x+1)(x+4)^2$, and $\det(C_4) = -x^3(x^2+8x+8)^2$. Hence,

$$\sum_{k\geq} \det(C_k)t^k = \frac{t(t^2x^5 - 2tx^3 - 3tx^2 + x + 1)}{(tx^2-1)(t^2x^4 - 2tx^2 - 4tx + 1)},$$

which implies the following result. □

Theorem 3.3 *The generating function* $\sum_{k\geq} \det(M_k)t^k$ *is given by*

$$\frac{t(t^4x^{10} - 4t^3x^8 - 8t^3x^7 - t^3x^6 + 6t^2x^6 + 16t^2x^5 + 13t^2x^4 - 4tx^4 - 8tx^3 - 3tx^2 + x^2 - 1)}{(1-tx^2)(t^2x^4 - 2tx^2 - 4tx + 1)^2}.$$

On the other hand, we define the sequence m_k as follows

$$m_{2k} = x^{2k-1}\frac{\left(1+v\right)^{2k} + \left(1-v\right)^{2k}}{2} \times \frac{\left(2kv+x\right)\left(1-v\right)^{2k} - \left(2kv-x\right)\left(1+v\right)^{2k}}{2},$$

$$m_{2k+1} = -x^{2k}\frac{\left(1+v\right)^{2k+1} - \left(1-v\right)^{2k+1}}{2} \times \frac{\left((2k+1)v+x\right)\left(1-v\right)^{2k+1} + \left((2k+1)v-x\right)\left(1+v\right)^{2k+1}}{2},$$

where $v = \sqrt{x+1}$.

By finding the generating function for the sequence m_k, namely $\sum_{k\geq 1} m_k t^k$, and comparing the result with Theorem 3.3, we obtain that $\det(M_k) = m_k$. Thus, we can state the following result.

Proposition 3.4 *For all* $k \geq 1$, $\det(M_k) = m_k$.

Note that the m_k can be written in terms of Chebyshev polynomials of the second-kind, $U_k(y)$, and of the first-kind, $T_k(y)$, which, by Proposition 3.4, leads to the following result.

Theorem 3.5 *For all* $k \geq 1$,

$$\det(M_{2k}) = x^{4k-2}T_k(y)(x^2T_k(y) - 4k(1+x)U_{k-1}(y)),$$

$$\det(M_{2k+1}) = x^{4k-2}(1+x)(2U_k(y) - xT_{k+1}(y)) \times (2(2k+1)(1+x)U_k(y) - x(2k+1)T_{k+1}(y) + 2xU_k(y) - x^2T_{k+1}(y)).$$

By this way we can give an affirmative answer to an open problem (Question 4) of the paper [2].

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