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Fuzzy Product Rule for Solving Fully Fuzzy Linear Systems

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Abstract: In this paper we construct solutions of the fuzzy matrix equation $\widehat{A}\widehat{x} = \widehat{b}$ for $\widehat{x}$ when the elements in $\widehat{A}$ and $\widehat{b}$ are MMCE-triangular fuzzy numbers. Here we apply the product rule to solve the equation without any restriction on the signs of multiplied fuzzy numbers. Then we give two examples of the fuzzy product rule.

Key Words: Neutrosophic fuzzy set, fuzzy number, fully fuzzy linear system (FFLS), fuzzy product, MMCE-representation.

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§1. Introduction

The systems of linear equations play important role in various areas of mathematics, statistic and engineering systems. Fuzzy systems represented by fuzzy numbers rather then crisp numbers have a major role for fuzzy modelling which can formulate uncertainty in real world problems. Fuzzy arithmetic operations have an essential role for treat linear systems whose parameters are all or partially fuzzy numbers. Firstly, the basic arithmetic structure for fuzzy numbers was introduced by Zadeh [11] and later this was developed many researcher such as Mizumoto and Tanaka [15], Dubois and Prade [7], Klir [9]. The fuzzy addition operation is practically easy to use. But, the other three fuzzy operations see various difficulties. Here we consider the multiplication operation for use the system of linear equations. A main disadvantage of this operation is that the shape of type fuzzy numbers (L-R, triangular or trapezoidal numbers) is not preserved. For this reason the researchers sought alternative ways for the product of fuzzy numbers.

Ma et al. introduced a new multiplicative operation of product type in [4]. They defined easily computable arithmetic operations based on split representation. But it has a drawback about the support. This problem has been solved by using middle-core-ecart representation (MCE-representation) of fuzzy numbers. Later, Zeinali and Maheri [6] introduced the modified MCE-product (MMCE, for short).

The system of linear equations $A\widehat{x} = \widehat{b}$, where A is a crisp matrix and $\widehat{b}$ is a fuzzy number vector, is called a fuzzy system of linear equation (FSLE) have been solved firstly Friedman et al. [12]. Following general model for solving such a fuzzy linear systems was proposed by many

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researchers ([13], [16], [4]). The linear system $\widehat{A}\widehat{x}=\widehat{b}$, where $\widehat{A}$ fuzzy matrix and $\widehat{b}$ is a fuzzy number vector, is called a fully fuzzy linear system (FFLS). A lot of works have been done this area with different methods ([[17], [14], [8], [10]).

In this paper we will investigate the solutions of FFLS with MMCE-representation of triangular fuzzy numbers. This paper is organized as follows. In Section 2, we briefly present the necessary preliminaries on fuzzy theory and MMCE-representation. In section 3, we summarise the definition and some properties of the FFLS. Then the solution of FFLS is constructed via MMCE-representation of fuzzy numbers. The proposed method is illustrated by solving some examples. Section 4 conclusion and some suggestions for future works are given.

§2. **Preliminaries**

In this section, we recall the basic notation of fuzzy numbers, the cross product and FFLS.

Definition 2.1([2]) *Let E be a universal set. A fuzzy subset $\widehat{A}$ of E is given by its membership function $\mu_{\widehat{A}}:E\to[0,1]$, where $\mu_{\widehat{A}}(t)$ represents the degree to which $t\in E$ belongs to $\widehat{A}$. We denote the class of the fuzzy subsets of E by the sembol $F(E)$.*

Generally, a neutrosophic fuzzy set $A(NFSA)$ is characterized by truth membership function $T_A(x)$, an indeterminacy membership functions $I_A(x)$ and a falsity membership function $F_A(x)$.

Definition 2.2([9]) *The $\alpha-$level of a fuzzy set $\widehat{A}\subseteq E$, denoted by $\left[\widehat{A}\right]^{\alpha}$, is defined as*

$$\left[\widehat{A}\right]^{\alpha}=\left\{x\in E:\widehat{A}\left(t\right)\geq\alpha\right\},\quad\forall\alpha\in\left(0,1\right].$$

Furthermore, if E is also topological space, then the $0-$level is defined as the closure of the support of $\widehat{A}$. That is,

$$\left[\widehat{A}\right]^{0}=\overline{\left\{x\in E:\widehat{A}\left(t\right)>0\right\}}.$$

The $1-$level of a fuzzy subset $\widehat{A}$ is also called as core of $\widehat{A}$ and denoted by $\left[\widehat{A}\right]^{1}=core\left(\widehat{A}\right)$.

Definition 2.3([9]) *A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy interval), whose $\alpha-$ cut set is denoted by $[\widehat{u}]^{\alpha}$, i.e., $[\widehat{u}]^{\alpha}=\{x:\widehat{u}\left(t\right)\geq 0\}$, if it satisfies two axioms:*

(i) There exists $r\in\mathbb{R}$ such that $\widehat{u}\left(r\right)=1$;

(ii) For all $0<\alpha\leq 1$, there exist real numbers $-\infty<u_{\alpha}^{-}\leq u_{\alpha}^{+}<+\infty$ such that $[\widehat{u}]^{\alpha}$ is equal to the closed interval $[u_{\alpha}^{-},u_{\alpha}^{+}]$.

Similarly, we can also define a neutrosophic fuzzy real number and fuzzy interval.

Definition 2.4([2]) *A fuzzy number $\widehat{A}$ is said to be triangular if the parametric representation of its $\alpha-$level is of the form*

$$\left[\widehat{A}\right]^{\alpha}=\left[\left(a_2-a_1\right)\alpha+a_1,a_3-\left(a_3-a_2\right)\alpha\right]$$

for all $\alpha \in [0,1]$*, where* $\left[\widehat{A}\right]^0 = [a_1, a_3]$ *and core* $\left(\widehat{A}\right) = a_2$*. A triangular fuzzy number is denoted by the triple* (a_1, a_2, a_3)*.*

The set of all fuzzy real numbers (fuzzy intervals) and triangular fuzzy numbers are denoted by $\mathbb{R}_F$ and $\mathbb{R}_T$, respectively.

Definition 2.5([1]) *An arbitrary fuzzy number* $\widehat{u}$ *in the parametric form is represented by an ordered pair of functions* $[u_\alpha^-, u_\alpha^+]$*,* $0 \le \alpha \le 1$*, which satisfy the following requirements*

(i) u_α^- *is a bounded non-decreasing left continuous function on* $(0,1]$ *and right- continuous for* $\alpha = 0$*;*

(ii) u_α^+ *is bounded non- increasing left continuous function on* $(0,1]$ *and right- continuous for* $\alpha = 0$*;*

(iii) $u_\alpha^- \le u_\alpha^+$*,* $0 < \alpha \le 1$*.*

Definition 2.6([4],[3]) *For* $\widehat{u} \in \mathbb{R}_F$*, consider the functions* $\theta_u^-, \theta_u^+ \to \mathbb{R}_+$ *defined by*

$$\theta_u^-(\alpha) = m_u - u_\alpha^-, \qquad \theta_u^+(\alpha) = u_\alpha^+ - m_u$$

where $m_u = \frac{u_1^- + u_1^+}{2}$*. Then,* $\widehat{u} = (m_u; \theta_u^-; \theta_u^+)$ *is MCE-representation of* $\widehat{u}$*. Note that the semicolon symbol makes this different from the well-known notation of a general triangular fuzzy number denoted by* (a, b, c)*. From now on, this notation is used for fuzzy numbers.*

From reference [5], clearly, $(m_u; \theta_u^-; \theta_u^+)$ represents a fuzzy number if and only if θ_u^-, θ_u^+ are bounded, positive, non-increasing, left-continuous on $(0;1]$ and right-continuous at 0.

Although the MCE-product is easy to use, it doesn't preserve the shapes of triangular and trapezoidal fuzzy numbers in reference [4].

First, we note that for a triangular fuzzy number $\widehat{u} = (a; b; c)$, MCE-representation is in the form

$$\widehat{u} = (b; (b-a)(1-\alpha); (c-b)(1-\alpha)),$$

which means that if $\widehat{u} \in \mathbb{R}_T$, then $\widehat{u}$ can be presented by $(m_u; k_u^-(1-\alpha); k_u^+(1-\alpha))$, where $k_u^-, k_u^+ \in \mathbb{R}_+$. Now, the modification of MCE-product can be done as follows:

Definition 2.7([6]) *Let* $\widehat{u} = (m_u; k_u^-(1-\alpha); k_u^+(1-\alpha))$ *and* $\widehat{v} = (m_v; k_v^-(1-\alpha); k_v^+(1-\alpha))$ *be two triangular fuzzy numbers. The modified MCE-product (denoted by MMCE-product for short) is defined by*

$$\widehat{u} \circledast \widehat{v} = (m_u m_v; k_u^- k_v^-(1-\alpha); k_u^+ k_v^+(1-\alpha)),$$

the $\alpha-$*cut of* $\widehat{u} \circledast \widehat{v}$ *is*

$$(\widehat{u} \circledast \widehat{v})_\alpha = [m_u m_v - k_u^- k_v^-(1-\alpha); m_u m_v + k_u^+ k_v^+(1-r)]$$

and its support is

$$\sup p\widehat{u} \circledast \widehat{v} = [m_u m_v - k_u^- k_v^-; m_u m_v + k_u^+ k_v^+].$$

The triple representation for $\widehat{u} \circledast \widehat{v}$ as a triangular fuzzy number is

$$\widehat{u} \circledast \widehat{v} = \left(m_u m_v - k_u^- k_v^-, m_u m_v, k_u^+ k_v^+ \right).$$

For $\widehat{u}, \widehat{v} \in \mathbb{R}_T$ and $t \in \mathbb{R}$, the sum and scalar multiplication are defined by

$$\begin{aligned} \widehat{u} \oplus \widehat{v} &= \left(m_u + m_v; k_u^- (1-\alpha) + k_v^- (1-\alpha); k_u^+ (1-\alpha) + k_v^+ (1-\alpha) \right) \\ tu &= \begin{cases} (tm_u; tk_u^- (1-\alpha); tk_u^+ (1-\alpha)), & t \geq 0, \\ (tm_u; -tk_u^+ (1-\alpha); -tk_u^- (1-\alpha)), & t < 0 \end{cases} \end{aligned}$$

and the MMCE-product is invertible and the inverse of $\widehat{u} = (m_u; k_u^- (1-\alpha); k_u^+ (1-\alpha))$ is

$$\widehat{u}^{-1} = \left(\frac{1}{m_u}; \frac{1}{k_u^-}(1-\alpha); \frac{1}{k_u^+}(1-\alpha) \right).$$

Example 2.8 Consider $\widehat{A} = (-1, 1, 3)$ and $\widehat{B} = (1, 3, 6)$, then $\widehat{A} = (1; 2(1-\alpha), 2(1-\alpha))$ and $\widehat{B} = (3; 2(1-\alpha), 3(1-\alpha))$ are MMCE triangular fuzzy numbers. The below algebraic operations can be easily done.

$$\begin{aligned} A \oplus B &= (4; 4(1-\alpha), 5(1-\alpha)) = (0, 4, 9); \\ A \ominus B &= (-2; 5(1-\alpha), 4(1-\alpha)) = (-7, -2, 2); \\ A \circledast B &= (3; 4(1-\alpha), 6(1-\alpha)) = (-1, -3, 9); \\ A \setminus B &= A \circledast B^{-1} = (-3; 2(1-\alpha), 2(1-\alpha)) = (-5, -3, -1). \end{aligned}$$

Definition 2.9([7]) *A matrix $\widehat{C} = (\widehat{c}_{ij})$ is called a fuzzy matrix,if each element of $\widehat{C}$ is a MMCE triangular fuzzy number.*

Up to rest of this paper, we represent $n \times m$ fuzzy matrix $\widehat{C} = (\widehat{c}_{ij})_{n \times m}$ with $\widehat{c}_{ij} = ((m_c)_{ij}; (k_u^-)_{ij} (1-\alpha); (k_u^+)_{ij} (1-\alpha))$ and a new notation $\widehat{C} = (C, E, F)$, where $C = (m_c)_{ij}$, $E = (k_u^-)_{ij} (1-\alpha)$ and $F = (k_u^+)_{ij} (1-\alpha)$ are three $n \times m$ crisp matrix.

Definition 2.10([7]) *Consider the $n \times n$ linear system of equations:*

$$\begin{cases} (\widehat{c}_{11} \circledast \widehat{x}_1) \oplus (\widehat{c}_{12} \circledast \widehat{x}_2) \ldots (\widehat{c}_{1n} \circledast \widehat{x}_n) = \widehat{b}_1 \\ (\widehat{c}_{21} \circledast \widehat{x}_1) \circledast (\widehat{c}_{22} \circledast \widehat{x}_2) \cdots (\widehat{c}_{2n} \circledast \widehat{x}_n) = \widehat{b}_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\widehat{c}_{n1} \otimes \widehat{x}_1) \circledast (\widehat{c}_{n2} \circledast \widehat{x}_2) \ldots (\widehat{c}_{nn} \circledast \widehat{x}_n) = \widehat{b}_n. \end{cases}$$

The matrix form of the above equations is

$$\widehat{C} \circledast \widehat{x} = \widehat{b} \tag{1}$$

where the coefficient matrix $\widehat{C} = (\widehat{c}_{ij})$ *is a* $n \times n$ *fuzzy matrix for integers* $1 \le i, j \le n$ *and* $\widehat{x}_i, \widehat{b}_i$ *are MMCE triangular fuzzy numbers,* $1 \le i \le n$. *Such a system is called a fully fuzzy linear system (FFLS).*

§3. The Solution of FFLS

In this section we solve a FFLS $\widehat{C} \circledast \widehat{x} = \widehat{b}$ using Computational methods given by [8]. However, in this fuzzy system we use MMCE triangular fuzzy numbers instead of LR fuzzy numbers. Thus, a FFLS will be solved not only for positive fuzzy numbers but also for all fuzzy numbers. In this paper, we suppose that all fuzzy numbers are MMCE triangular fuzzy numbers.

Definition 3.1([8]) *We say* $\widehat{x}$ *is a fuzzy approximate solution or more shortly, fuzzy solution of* $\widehat{C} \circledast \widehat{x} = \widehat{b}$ *with the left and right shape functions similar to that* $L(.)$ *and* $R(.)$ *which used in* $\widehat{C}$ *and* $\widehat{b}$ *if and only if* $\widehat{C} \circledast \widehat{x} = \widehat{b}$ *with approximate operators as mentioned above, i.e.* $\widehat{x} = (x, y, z)$ *is said to be fuzzy solution of* $(C, E, F) \otimes \widehat{x} = (b, q, s)$ *iff*

$$Cx = b, \qquad Cy + Ex = q, \qquad Cz + Fx = s \tag{2}$$

where the membership function of each element of $\{x \mid \mu_{\widehat{x}} > 0\}$ *can be defined with the same functions* L *and* R *which used in* $\widehat{C}$ *and* $\widehat{b}$.

Note that we use MMCE triangular fuzzy number with semicolon notation which is p-resented by $(m_u; k_u^-(1-\alpha); k_u^+(1-\alpha)) = (C; E; F)$ instead of LR fuzzy number denoted by (C, E, F) in (2).

Now, we use the Eq (1) as follow

$$\begin{aligned}&((m_c)_{ij}\,; \left(k_c^-\right)_{ij}(1-r); \left(k_c^+\right)_{ij}(1-\alpha)) \circledast ((m_x)_j\,; \left(k_x^-\right)_j(1-\alpha); \left(k_x^+\right)_j(1-\alpha))\\ &= ((m_b)_j\,; \left(k_b^-\right)_j(1-\alpha); \left(k_b^+\right)_j(1-\alpha)).\end{aligned}$$

If we rearrange the Eqs in (2), we get the following equations

$$\begin{aligned}\sum_{j=1}^{n}(m_c)_{ij}\,.\,(m_x)_j &= (m_b)_i\\ \sum_{j=1}^{n}(m_c)_{ij}\,.\left(k_x^-\right)_j(1-\alpha) + \sum_{j=1}^{n}\left(k_c^-\right)_{ij}(1-\alpha).\,(m_x)_j &= \left(k_b^-\right)_j(1-\alpha)\\ \sum_{j=1}^{n}(m_c)_{ij}\,.\left(k_x^+\right)_j(1-\alpha) + \sum_{j=1}^{n}\left(k_c^+\right)_{ij}(1-\alpha).\,(m_x)_j &= \left(k_x^+\right)_j(1-\alpha)\end{aligned}$$

where C is a nonsingular crisp matrix $(1 \le j \le n)$.

If we assume that C is a nonsingular crisp matrix, we can write similarly from reference [8] that

$$(Cx, Cy + Ex, Cz + Fx) = (b, q, s)\,.$$

So, we have

$$\begin{cases} Cx = b, \\ Cy = q - Ex, \\ Cz = s - Fx. \end{cases} \tag{3}$$

Thus, we can easily get

$$x = C^{-1}b, \tag{4}$$
$$y = C^{-1}q - C^{-1}Ex, \tag{5}$$
$$z = C^{-1}s - C^{-1}Fx, \tag{6}$$

by using the inverse matrix of C, which enables us to get the following result.

Theorem 3.2 *Let $\widehat{A} = (A; M; N)$ and $\widehat{b} = (b; g; h)$ be non-negative fuzzy matrix and non-negative fuzzy vector, respectively, and $\widehat{A}$ be the product of a permutation matrix by a diagonal matrix with positive diagonal entries. Moreover let $h \geq MA^{-1}b$, $g \geq NA^{-1}b$ and $\left(MA^{-1} + I\right) b \geq h$. Then the system $\widehat{A}\widehat{x} = \widehat{b}$ has a positive fuzzy solution.*

Proof See [8] for its proof. □

Now we consider two examples, one consisting of positive triangular fuzzy numbers and for the other example it does not matter the sign of the triangular fuzzy numbers.

Example 3.3 Consider the following FFLS for positive fuzzy numbers (Test 3.2 in [8])

$$\begin{cases} \widehat{5}\widehat{x}_1 + \widehat{6}\widehat{x}_2 = \widehat{50}, \\ \widehat{7}\widehat{x}_1 + \widehat{4}\widehat{x}_2 = \widehat{48} \end{cases}$$

where $\widehat{4} = (4, 4, 5)$, $\widehat{5} = (4, 5, 6)$, $\widehat{6} = (5, 6, 8)$, $\widehat{7} = (6, 7, 7)$, $\widehat{48} = (43, 48, 55)$ and $\widehat{50} = (40, 50, 67)$ are triangular fuzzy numbers. Using MMCE triangular fuzzy numbers instead of triangular fuzzy numbers we mean

$$\begin{cases} (5; (1-\alpha); (1-\alpha)) \circledast (x_1; y_1(1-\alpha); z_1(1-\alpha)) \oplus (6; (1-\alpha); 2\,(1-\alpha)) \\ \circledast\, (x_2; y_2(1-\alpha); z_2(1-\alpha)) = (50; 10(1-\alpha); 17\,(1-\alpha)) \\ (7; (1-\alpha); 0) \circledast (x_1; y_1(1-\alpha); z_1(1-\alpha)) \oplus (4; 0; (1-\alpha)) \circledast (x_2; y_2(1-\alpha); z_2(1-\alpha)) \\ = (48; 5(1-\alpha); 7\,(1-\alpha)) \end{cases}$$

So with Eq. (4) we get

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 48 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Similarly by Eqs. (5) and (6) we have

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} y_1\,(1-\alpha) \\ y_2\,(1-\alpha) \end{bmatrix} = \begin{bmatrix} 10\,(1-\alpha) \\ 5\,(1-\alpha) \end{bmatrix} - \begin{bmatrix} 1\,(1-\alpha) & 1\,(1-\alpha) \\ 1\,(1-\alpha) & 0\,(1-\alpha) \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/11 \\ 1/11 \end{bmatrix}$$

and

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} z_1\,(1-\alpha) \\ z_2\,(1-\alpha) \end{bmatrix} = \begin{bmatrix} 17\,(1-\alpha) \\ 7\,(1-\alpha) \end{bmatrix} - \begin{bmatrix} 1\,(1-\alpha) & 2\,(1-\alpha) \\ 0\,(1-\alpha) & 1\,(1-\alpha) \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}.$$

So, the solution is

$$\widehat{x} = \begin{bmatrix} \left(4; \frac{1}{11}\,(1-\alpha)\,; 0\right) \\ \left(5; \frac{1}{11}\,(1-\alpha)\,; \frac{1}{2}\,(1-\alpha)\right) \end{bmatrix}$$

where this solution is a fuzzy solution; also we consider that our solution is the same as the solution in [8].

Example 3.4 Consider the following FFLS for all fuzzy numbers (Example 2 in [10])

$$\begin{cases} \widehat{-1}\widehat{x}_1 + \widehat{2}\widehat{x}_2 = \widehat{-2}, \\ \widehat{-3}\widehat{x}_1 + \widehat{3}\widehat{x}_2 = \widehat{4} \end{cases}$$

where,

$$\begin{aligned} \widehat{-3} &= (-4,-3,-2), \quad \widehat{-1} = (-3,-1,2), \\ \widehat{2} &= (1,2,4), \quad \widehat{3} = (1,3,6), \\ \widehat{-2} &= (-3,-2,-1), \quad \widehat{4} = (1,4,5) \end{aligned}$$

are triangular fuzzy numbers. Using MMCE triangular fuzzy numbers instead of triangular fuzzy numbers we mean

$$\begin{cases} (-1; 2(1-\alpha); 3\,(1-\alpha)) \circledast (x_1; y_1(1-\alpha); z_1(1-\alpha)) + (2; (1-\alpha); 2\,(1-\alpha)) \\ \qquad \circledast (x_2; y_2(1-\alpha); z_2(1-\alpha)) = (-2; (1-\alpha); (1-\alpha)) \\ (-3; (1-\alpha); (1-\alpha)) \circledast (x_1; y_1(1-\alpha); z_1(1-\alpha)) + (3; 2\,(1-\alpha)\,; 3\,(1-\alpha)) \\ \qquad \circledast (x_2; y_2(1-\alpha); z_2(1-\alpha)) = (4; 3(1-\alpha); (1-\alpha)) \end{cases}$$

So with Eq. (4) we get

$$\begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -14/3 \\ -10/3 \end{bmatrix}.$$

Similarly by Eq. (5) and (6) we have

$$\begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 (1-\alpha) \\ y_2 (1-\alpha) \end{bmatrix} = \begin{bmatrix} 1 (1-\alpha) \\ 3 (1-\alpha) \end{bmatrix} - \begin{bmatrix} 2 (1-\alpha) & 1 (1-\alpha) \\ 1 (1-\alpha) & 2 (1-\alpha) \end{bmatrix} \begin{bmatrix} -14/3 \\ -10/3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/11 \\ 1/11 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} z_1 (1-\alpha) \\ z_2 (1-\alpha) \end{bmatrix} = \begin{bmatrix} 1 (1-\alpha) \\ 1 (1-\alpha) \end{bmatrix} - \begin{bmatrix} 5 (1-\alpha) & 2 (1-\alpha) \\ 1 (1-\alpha) & 3 (1-\alpha) \end{bmatrix} \begin{bmatrix} -14/3 \\ -10/3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 11.22 \\ 16.44 \end{bmatrix}.$$

So, the solution of system is

$$\widehat{x} = \begin{bmatrix} (-4.66; -8.77 (1-\alpha); 11.22 (1-\alpha)) \\ (-3.33; 8.88 (1-\alpha); 16.44 (1-\alpha)) \end{bmatrix} = \begin{bmatrix} (-8.77, -4.66, 15.33) \\ (-12.21, -3.33, 13.11) \end{bmatrix}$$

where this solution is a fuzzy solution; but the solution in [10] is not a fuzzy solution. So the method we used in this paper is more convenient.

§4. **Conclusion**

In this study, we introduced the Direct method in [8] for finding the solution of fully fuzzy linear system (FFLS) by using the MMCE triangular fuzzy numbers with the product rule instead of LR fuzzy numbers. We presented two examples to implement the given method. We verified that the sign of fuzzy numbers does not matter in the fuzzy solution of the system.

This product rule easy to use for multiplication of fuzzy numbers which are not depend on the signs . For this reason, it provides a great advantage in solving fuzzy equation systems. So, for future work, we can apply this new method to find fuzzy eigenvalues and fuzzy eigenvectors of the the system of linear equations $\widehat{A}\widehat{x} = \widehat{\lambda}\widehat{x}$.

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