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# Generalized Perfect Neighborhood Number of a Graph

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**Abstract**: In this article, we generalize the perfect neighborhood number of a graph $G = (V, E)$ as a $k$ - perfect neighborhood number $\eta_{kp}(G)$. Here many bounds and exact values of some specific family of graphs are obtained. Also, its relationship with other graph theoretic parameters are investigated.

**Key Words**: Graph, domination number, neighborhood number, perfect neighborhood number, $k$ - perfect neighborhood number.

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## §1. Introduction

Let $G = (V, E)$ be simple, undirected, and nontrivial graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Also $|V| = n$ and $|E| = m$ denote number of vertices and number of edges in $G$. The open neighborhood $N(v)$ of vertex $v$ denotes number of vertices adjacent to $v$ and its closed neighborhood $N[v] = N(v) \cup \{v\}$. The $\beta_1(G)$ is the minimum number of edges in a maximal independent set of edge of $G$. For notation and graph theoretic terminology, we generally follow [11].

A set $D \subseteq V$ is a dominating set if every vertex not in $S$ is adjacent to one or more vertices in $D$. The cardinality of a smallest dominating set of $G$, denoted by $\gamma(G)$, is the domination number of $G$. For more details on domination theory, we refer to [12], [13], [14] and [19]. In 1985, E. Sampathkumar and P. S. Neeralagi [17], introduced an innovative concept of domination between the vertices and the edges, and vice-versa. They introduced a new parameter called the *neighborhood number* of a graph, as follows. A set $S \subseteq V$ is a neighborhood set of $G$, if $G = \bigcup_{v\in s}\langle N[v]\rangle$, where $\langle N[v]\rangle$ is the sub graph of $G$ induced by $v$ and all vertices adjacent to $v$. The neighborhood number $\eta(G)$ is the minimum cardinality of a neighborhood set of $G$. For more information on neighborhood number, we refer to [3], [15] and [16].

In 1993, Cockayne et al [7] introduced the concept of perfect domination following in this sense. A subset $D \subseteq V$ is a perfect dominating set of $G$ if any vertex of $G$ not in $D$ is adjacent to exactly one vertex of $D$. In 2010, Chaluvaraju et al [4] and [5] was generalized perfect domination as follows. A vertex subset $D$ of $G$ is called a $k$ - perfect dominating set of $G$, if any vertex $v$ of $V$ not in $D$ is adjacent to exactly $k$ - vertices of $D$. The minimum cardinality of a $k$ - perfect dominating set of $G$ is the $k$ - perfect domination number $\gamma_{kp}(G)$.

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The following known results from [4] and [5] are used in the sequel.

**Theorem** 1.1 *Let $T$ be a tree and $G$ be a connected graph. Then,*

(*i*) $\frac{kn}{\Delta(G)+k} \leq \gamma_{kP}(G)$;
(*ii*) $\alpha(T) \leq \gamma_{kP}(T)$;
(*iii*) $\beta_1(T) \leq \gamma_{kP}(T)$;
(*iv*) $\lceil \frac{diam(G)}{2} \rceil \leq \gamma_{2P}(G)$.

**Theorem** 1.2 *Let $k = \Delta(G) - 1$. Then the graph $G$ is a $kPD$ - graph if and only if $G$ satisfy one of the following conditions:*

(*i*) *there exists at least two adjacent vertices $u$ and $v$ in a graph $G$ such that $deg(u) = deg(v) = \Delta(G)$;*
(*ii*) *there exists a vertex $u$ such that $deg(u) = \Delta(G) - 1$.*

Several papers have been written on the subject of $k$ - domination in graphs and when they exists, cf. [2], [6], [9] and [10]. Further, let $k$ be a positive integer and $G$ be a graph. A subset $S$ of vertices in a graph $G$ is a $k$ - neighborhood set, if every vertex of $V - S$ is adjacent to at least $k$ - vertices in $S$. The $k$ - neighborhood number $\eta_k(G)$ is the minimum cardinality of a $k$ - neighborhood set of a graph $G$. Hence for $k = 1$, 1 - neighborhood sets are the classical neighborhood set of a graph $G$, see [17].

Analogously, here we generalize the perfect neighborhood number as follows: A subset $S$ of vertices in a graph $G$ is a $k$ - neighborhood set, if every vertex of $V - S$ is adjacent to exactly $k$ - vertices in $S$. The $k$ - perfect neighborhood number $\eta_{kp}(G)$ is the minimum cardinality of a $k$ - perfect neighborhood set of a graph $G$. Hence for $k = 1$, 1 - perfect neighborhood sets are the usual perfect neighborhood sets. The concept of perfect neighborhood number was initiated by Sampatkumar and Neerlagi [18].

Note that every nontrivial graph $G$ has a $k$ - perfect neighborhood set, since the entire vertex set is such a set and there are graphs whose only $k$ - perfect neighborhood set is $V(G)$. A graph $G$ for which $\eta_{kp}(G) < n$ is called a $k$ - perfect neighborhood graph, abbreviated $kPN$ - graph and a tree $T$ for which $\eta_{kp}(T) < n$ is called a $kPN$ - tree.

## §2. Specific Families of Graphs

**Proposition** 2.1 *For any complete graph $K_n$; $n \geq 2$ vertices with $1 \leq k \leq \Delta(G)$,*

$$\eta_{kP}(K_n) = k.$$

**Proposition** 2.2 *For any path $P_n$ with $n \geq 3$ vertices,*

(*i*) $\eta_{1P}(P_n) = n - 2$;
(*ii*) $\eta_{2P}(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even.} \end{cases}$

**Proposition** 2.3 *For any Cycle $C_n$ with $n \geq 4$ vertices,*

(*i*) $\eta_{1P}(C_n)$ *does not exist;*

(*ii*) $\eta_{2P}(C_n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$

**Proposition** 2.4 *For any Fan graph $F_n = P_n + K_1$ with $n \geq 1$ vertices,*

(*i*) $\eta_{1P}(F_n) = 1$;

(*ii*) $\eta_{2P}(F_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n = 3t + 1 \text{ for } t \geq 1 \\ \lceil \frac{n}{3} \rceil + 1, & \text{otherwise;} \end{cases}$

(*iii*) $\eta_{3P}(F_n) = \lceil \frac{n}{2} \rceil + 1$ *if* $n \geq 4$;

(*iv*) $\eta_{(n-1)P}(F_n) = n - 1$ *if* $n \geq 3$.

**Proposition** 2.5 *For any wheel graph $W_n = C_n + K_1$ with $n \geq 4$ vertices,*

(*i*) $\eta_{1P}(W_n) = 1$;

(*ii*) $\eta_{2P}(W_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1, & if\ n = 3t,\ t \geq 2, \\ \lfloor \frac{n}{2} \rfloor, & \text{otherwise;} \end{cases}$

(*iii*) $\eta_{3P}(W_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd;} \end{cases}$

(*iv*) $\eta_{(n-1)P}(W_n) = n - 1$.

**Proposition** 2.6 *For any complete bipartite graph $K_{r,s}$ with $n = r + s$ vertices,*

(*i*) $\eta_{1P}(K_{1,s}) = 1$ *if r = 1 and s = n - 1;*

(*ii*) $\eta_{sP}(K_{1,s}) = s$ *if r = 1 and s = n - 1;*

(*iii*) $\eta_{rP}(K_{r,s}) = r$ *if* $2 \leq r \leq s$;

(*iv*) $\eta_{sP}(K_{r,s}) = s$ *if* $2 \leq r \leq s$.

## §3. Properties and Bounds

**Property** 3.1 *For every graph $G$ and positive integer $k$, every vertex with degree at most $k-1$ belongs to every $k$ - perfect neighborhood set.*

**Property** 3.2 *Since $v \in V - S$ should be adjacent to $k$ - vertices in $S$, the graph $G$ is not a $kPN$ - graph for $k \geq \Delta(G)$.*

**Property** 3.3 *Let $v$ be a vertex wit $deg(v) = \Delta(G)$ and let $k = \Delta(G)$. Then $V - \{v\}$ is a $\Delta$ - perfect neighborhood set of $G$. Thus $G$ is a $kPN$ - graph for $k = \Delta(G)$.*

**Property** 3.4 *A graph will have two disjoint $k$ - perfect neighborhood sets only if $k \leq \delta(G)$, since all the vertices with degree less than $k$ belongs to every $k$ - perfect neighborhood set.*

**Property** 3.5 *If $S$ is a $k$ - neighborhood set of a graph $G$, then $S$ is a $t$ - neighborhood set for* *every $t \leq k$. But this is not true in case of $k$ - perfect neighborhood set.*

**Property** 3.6 *Every $k$ - perfect neighborhood set is a $k$ neighborhood set of a graph $G$ and hence $\eta_k(G) \leq \eta_{kP}(G)$.*

**Observation** 3.1 An $k$- perfect neighborhood set is a $k$ - perfect dominating set, and hence $\gamma_{kP}(G) \leq \eta_{kP}(G)$ for every graph $G$ and positive integer $k$.

By above observation and Theorem 1.1, we have the following lower bounds.

**Theorem** 3.1 *Let $T$ be a tree and $G$ be a connected graph. Then,*

(*i*) $\frac{kn}{\Delta(G)+k} \leq \eta_{kP}(G)$;

(*ii*) $\alpha(T) \leq \eta_{kP}(T)$;

(*iii*) $\beta_1(T) \leq \eta_{kP}(T)$;

(*iv*) $\left\lceil \frac{diam(G)}{2} \right\rceil \leq \eta_{2P}(G)$.

**Theorem** 3.2 *Let $G$ be a $kPN$ - graph with $n \geq 2$ vertices. Then*

$$n - (m/k) \leq \eta_{kP}(G) \leq n - 1.$$

*Proof* Let $S$ be a $\eta_{kP}$ - set of a nontrivial graph $G$ and $|V - S| = t$. Then there are $t$ - times of $k$ - edges from $V - S$ to $S$ with $\eta_{kP}(G)= |S|$. Since $m > tk$ , the lower bound follows. By the definition of $kPN$ - graph, the upper bound follows. □

**Theorem** 3.3 *Let $\{x_1, x_2, \cdots, x_n\}$ be the degree sequence of a graph $G$ with $degv_i = x_i$ for $i = 1, 2, \cdots, n$. If $k$ is an integer such that $k \in \{x_1, x_2, \cdots, x_n\}$, Then $G$ is a $kPN$ - graph.*

*Proof* Let $k \in \{x_1, x_2, \ldots, x_n\}$. If $S = V - v$, where $v$ is a vertex of degree $k$ in a graph $G$, then $S$ is a $k$ - $PNS$ of a graph $G$. Therefore $G$ is a $kPN$ - graph. □

**Observation** 3.2 The converse of above theorem is not true.

For example, we consider the following graph $G_1$.

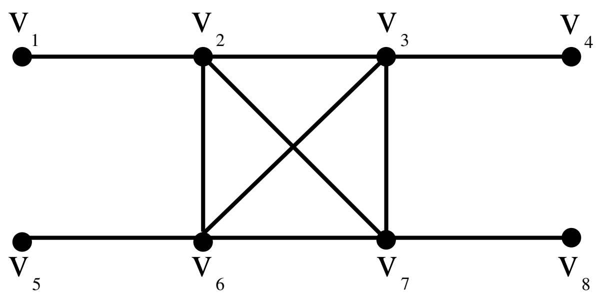


**Figure 1.** The graph $G_1$.

Here, the degree sequence of $G_1$ is $\{1, 4, 4, 1, 1, 4, 4, 1\}$, we have

(*i*) If $k = 2$, then $\eta_{2P}$ - set $S$ is $\{v_1, v_2, v_4, v_5, v_8\}$ and $V - S$ is $\{v_3, v_6, v_7\}$.

($ii$) If $k = 3$, then $\eta_{3P}$ - set $S$ is $\{v_1, v_2, v_3, v_4, v_5, v_8\}$ and $V - S$ is $\{v_6, v_7\}$.

Clearly, these graphs are $2PN$ - graph and $3PN$ - graph. But $k = 2$ and 3 does not belong to the degree sequence of a graph $G_1$.

**Theorem** 3.4 *For any connected graph $G$,*

$$\frac{kn}{\Delta(G) + k} \leq \eta_{kP}(G).$$

**Theorem** 3.5 *Let $G$ be a connected graph with $\eta_{kP}(G) = k$. Then*

$$\Delta(G) \geq Max.\{k, n - k\}.$$

*Proof* Let $S$ be a $\eta_{kP}$ - set of a graph $G$ with $\eta_{kP}(G) = k$. Then we have the following cases.

**Case 1.** Suppose if $v \in V - S$, then the degree of $v$ is greater than $|S| = k$. There fore $\Delta(G) \geq k$.

**Case 2.** Suppose if $v \notin V - S$, then the degree of $v \in S$ is greater than $|V - S| = n - k$. There fore $\Delta(G) \geq n - k$.

Thus, the result follows. □

## §4. Concluding Remarks and Further Scope

Different graph theorists have defined wide varieties of neighborhood related graph parameters by imposing extra conditions on the neighborhood set $S$ of a graph $G$, because the neighborhood number is closely related to the domination number of $G$. To stimulate further understanding or advancement in this generalized perfect neigbor based graph parameters, we pose the following open problems:

($i$) *Obtain the complexity issues of $\eta_{kP}(G)$;*

($ii$) *Characterize the class of graphs when $\gamma_{kP}(G) = \eta_{kP}(G)$?*

($iii$) *Obtain some bound and characterization on $\eta_{kP}(G)$ in terms of other domination related parameters such as total domination, connected domination, independence domination and so on.,*

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