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Generalized h-Kropina Change of Finsler Metric

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Abstract: The purpose of the present paper is to find the necessary and sufficient conditions under which a generalized $h-$Kropina change of Finsler metric becomes a projective change. The condition under which a generalized $h-$Kropina change of Finsler metric of Douglas space leads to a Douglas space have also been found.

Key Words: Finsler metric, Kropina change, generalized Kropina change, $h-$vector, projective change, Douglas space.

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§1. Introduction

Let $F^n = (M^n, L)$ be an $n-$dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function $L(x, y)$.

In the paper [3], Shukla, Pandey and Mishra have investigated the necessary and sufficient conditions under which a generalized Kropina change of Finsler metric becomes a projective change. They have also obtained the condition under which such change of metric of a Douglas space give rise to a Douglas space.

The generalized Kropina change of Finsler metric is given by

$$\overline{L}(x,y) = \frac{L^{m+1}}{\beta^m}, \qquad \text{where} \quad \beta = b_i(x)\, y^i \tag{1.1}$$

and m is a constant not equal to -1, 0.

In the present paper we have considered the transformation (1.1), in which $b_i(x)$ in β has been replaced by $h-$vector $b_i(x, y)$ so that $\frac{\partial b_i}{\partial y^j}$ is proportional to the angular metric tensor h_{ij}.

Let

$$\frac{\partial b_i}{\partial y^j} = \rho h_{ij}, \tag{1.2}$$

where ρ is any scalar function of x, y and $h_{ij} = g_{ij} - l_i\, l_j$. It has been shown by Shukla, Pandey and Joshi in [2] that

$$\dot{\partial}_k \rho = -\frac{\rho}{L}\, l_k, \quad \text{for} \quad n > 2, \tag{1.3}$$

where $\dot{\partial}_k \equiv \frac{\partial}{\partial y^k}$.

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We shall use the equation (1.3) without quoting it in the present paper.

Let $\beta = b_i(x,y)\, y^i$ be defined on the manifold M^n. Then $L \to \frac{L^{m+1}}{\beta^m}$ is called generalized $h-$Kropina change of Finsler metric. If we write $\overline{L} = \frac{L^{m+1}}{\beta^m}$ and $\overline{F}^n = (M^n, \overline{L})$ then the Finsler space $\overline{F}^n$ is said to be obtained from F^n by a generalized $h-$Kropina change.

If $m = 1$, then generalized $h-$Kropina change reduces to $h-$Kropina change of Finsler metric. The quantities corresponding to $\overline{F}^n$ will be denoted by putting bar over those quantities.

The fundamental quantities of F^n are given by

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}, \qquad l_i = \frac{\partial L}{\partial y^i} \qquad \text{and} \qquad h_{ij} = L\frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - l_i l_j.$$

We shall denote the partial derivative with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$$L_i = \dot{\partial}_i L, \qquad L_{ij} = \dot{\partial}_i \dot{\partial}_j L, \qquad L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L.$$

Then

$$L_i = l_i, \qquad L^{-1}\, h_{ij} = L_{ij}$$

The geodesic of F^n are given by the system of differential equations

$$\frac{d^2 x^i}{ds^2} + 2G^i\left(x, \frac{dx}{ds}\right) = 0,$$

where $G^i(x,y)$ are positively homogeneous of degree two in y^i and are given by

$$2G^i = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \qquad F = \frac{L^2}{2}$$

where g^{ij} are the inverse of g_{ij}.

Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of Finsler space $F^n = (M^n, L)$ is given by [5]

$$G^i_j = \frac{\partial G^i}{\partial y^j}, \qquad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.$$

The Cartan's connection $C\Gamma = (F^i_{jk}, G^i_j, G^i_{jk})$ is constructed from L with the help of following axioms [5]:

1. Cartan connection $C\Gamma$ is $v-$metrical;
2. Cartan connection $C\Gamma$ is $h-$metrical;
3. The $(v)v$ torsion tensor field S of Cartan connection vanishes;
4. The $h(h)$ torsion tensor field T of Cartan connection vanishes;
5. The deflection tensor field D of Cartan connection vanishes.

The $h-$ and $v-$covariant derivatives with respect to Cartan connection are denoted by $_{|k}$ and $|_k$ respectively. It is clear that the $h-$covariant derivative of L with respect to $B\Gamma$ and $C\Gamma$ is the same and vanishes identically. Furthermore, the $h-$covariant derivatives of L_i, L_{ij} with respect to $C\Gamma$ are also zero.

We shall write

$$2r_{ij} = b_{i|j} + b_{j|i}, \qquad 2s_{ij} = b_{i|j} - b_{j|i}.$$

§2. Difference Tensor $\mathbf{D}^i$

The generalized $h-$Kropina change of Finsler metric L is given by

$$\overline{L} = \frac{L^{m+1}}{\beta^m}, \quad \text{where} \quad \beta(x,y) = b_i(x,y)\, y^i \quad \text{and} \quad m \neq -1, 0. \tag{2.1}$$

We may put

$$\overline{G}^i = G^i + D^i. \tag{2.2}$$

Then $\overline{G}^i_j = G^i_j + D^i_j$ and $\overline{G}^i_{jk} = G^i_{jk} + D^i_{jk}$, where $D^i_j = \dot{\partial}_j D^i$ and $D^i_{jk} = \dot{\partial}_k D^i_j$. The tensors D^i, D^i_j and D^i_{jk} are positively homogeneous in y^i of degree two, one and zero respectively.

To find D^i we deal with equation $L_{ij|k} = 0$, [4] i.e.

$$\partial_k L_{ij} - L_{ijr} G^r_k - L_{rj} F^r_{ik} - L_{ir} F^r_{jk} = 0. \tag{2.3}$$

Since $\dot{\partial}_i \beta = b_i$, from (2.1), we have

$$\text{(a)} \qquad \overline{L}_i = (m+1)\frac{L^m}{\beta^m}\, L_i - m\frac{L^{m+1}}{\beta^{m+1}}\, b_i; \tag{2.4}$$

$$\begin{aligned}\text{(b)} \qquad \overline{L}_{ij} = {} & \frac{L^m}{\beta^{m+1}}[(m+1)\beta - \rho m L^2] L_{ij} + m(m+1)\frac{L^{m-1}}{\beta^m} L_i L_j \\ & -m(m+1)\frac{L^m}{\beta^{m+1}}(L_i b_j + L_j b_i) + m(m+1)\frac{L^{m+1}}{\beta^{m+2}} b_i b_j;\end{aligned}$$

$$\begin{aligned}\text{(c)} \qquad \partial_j \overline{L}_i = {} & m(m+1)\frac{L^{m-1}}{\beta^{m+1}}(\beta L_i - L b_i)\partial_j L + m(m+1)\frac{L^m}{\beta^{m+2}}(L b_i - \beta L_i)\partial_j \beta \\ & +(m+1)\frac{L^m}{\beta^m}\partial_j L_i - m\frac{L^{m+1}}{\beta^{m+1}}\partial_j b_i;\end{aligned}$$

$$\begin{aligned}\text{(d)} \qquad \partial_k \overline{L}_{ij} = {} & \frac{L^m}{\beta^{m+1}}[(m+1)\beta - \rho m L^2]\partial_k L_{ij} + \big[\frac{m L^{m-1}}{\beta^{m+1}}((m+1)\beta \\ & -\rho(m+2)L^2)L_{ij} + m(m^2-1)\frac{L^{m-2}}{\beta^m} L_i L_j - m^2(m+1)\frac{L^{m-1}}{\beta^{m+1}}(L_i b_j \\ & + L_j b_i) + m(m+1)^2\frac{L^m}{\beta^{m+2}} b_i b_j\big]\, \partial_k L + \big\{(\rho L^2 - \beta) m(m+1)\frac{L^m}{\beta^{m+2}} L_{ij} \\ & -m^2(m+1)\frac{L^{m-1}}{\beta^{m+1}} L_i L_j + m(m+1)^2\frac{L^m}{\beta^{m+2}}(L_i b_j + L_j b_i) \\ & -m(m+1)(m+2)\frac{L^{m-1}}{\beta^{m+3}} b_i b_j\big\}\partial_k \beta + m(m+1)\frac{L^{m-1}}{\beta^{m+1}}(\beta L_j - L b_j)\, \partial_k L_i \\ & +m(m+1)\frac{L^{m-1}}{\beta^{m+1}}(\beta L_i - L b_i)\partial_k L_j + m(m+1)\frac{L^m}{\beta^{m+2}}(L b_j - \beta L_j)\partial_k b_i \\ & +m(m+1)\frac{L^m}{\beta^{m+2}}(L b_i - \beta L_i)\partial_k b_j - m\frac{L^{m+2}}{\beta^{m+1}} L_{ij}\partial_k \rho\end{aligned}$$

and

$$\text{(e)} \qquad \overline{L}_{ijk} = \frac{L^m}{\beta^{m+1}}[(m+1)\beta - \rho m L^2] L_{ijk} + m(m+1)\frac{L^{m-1}}{\beta^{m+1}}(\beta - \rho L^2)(L_i L_{jk} +$$

$$+L_jL_{ik}+L_kL_{ij})+m(m+1)\frac{L^m}{\beta^{m+2}}(\rho L^2-\beta)(b_iL_{jk}+b_jL_{ik}+b_kL_{ij})$$
$$-m^2(m+1)\frac{L^{m-1}}{\beta^{m+1}}(L_iL_jb_k+L_iL_kb_j+L_jL_kb_i)$$
$$+m(m+1)^2\frac{L^m}{\beta^{m+2}}(L_ib_jb_k+L_jb_kb_i+L_kb_ib_j)$$
$$+m(m^2-1)\frac{L^{m-2}}{\beta^m}L_iL_jL_k-m(m+1)(m+2)\frac{L^{m+1}}{\beta^{m+3}}b_ib_jb_k.$$

Since $\overline{L}_{ij|k}=0$ in $\overline{F}^n$, after using (2.2), we have

$$\partial_k\overline{L}_{ij}-\overline{L}_{ijr}\overline{G}^r_k-\overline{L}_{jr}F^r_{ik}-\overline{L}_{ir}F^r_{jk}=0,$$

where $\overline{F}^i_{jk}=F^i_{jk}+{}^cD^i_{jk}$ [1].

Substituting in the above equation the values of $\partial_k\overline{L}_{ij}$, $\overline{L}_{ir}$ and $\overline{L}_{ijr}$ from (2.4) and using (2.3) and then contracting the resulting equation with y^k, we get

$$\begin{aligned}2\overline{L}_{ijr}D^r+\overline{L}_{jr}D^r_i+\overline{L}_{ir}D^r_j-Lw(Lb_j-\beta L_j)(r_{i0}+s_{i0})-Lw(Lb_i-\\ -\beta L_i)(r_{j0}+s_{j0})-\big\{m(m+1)\frac{L^m}{\beta^{m+2}}(\rho L^2-\beta)L_{ij}-m^2(m+1)\frac{L^{m-1}}{\beta^{m+1}}\\ L_iL_j+m(m+1)^2\frac{L^m}{\beta^{m+2}}(L_ib_j+L_jb_i)-m(m+1)(m+2)\frac{L^{m+1}}{\beta^{m+3}}b_ib_j\big\}r_{00}\\ +m\frac{L^{m+2}}{\beta^{m+1}}\rho_0L_{ij}+2\rho m\frac{L^{m+1}}{\beta^{m+1}}L_rL_{ij}G^r=0.\end{aligned}\tag{2.5}$$

where '0' stands for the contraction with y^k viz. $r_{j0}=r_{jk}y^k$, $r_{00}=r_{ij}y^iy^j$ and we have use the fact that $D^i_{jk}y^k={}^cD^i_{jk}y^k=D^i_j$ [4].

Next, we deal with $\overline{L}_{i|j}=0$, that is $\partial_j\overline{L}_i-\overline{L}_{ir}\overline{G}^r_j-\overline{L}_r\overline{F}^r_{ij}=0$, then we have

$$\partial_j\overline{L}_i-\overline{L}_{ir}(G^r_j+D^r_j)-\overline{L}_r(F^r_{ij}+{}^cD^r_{ij})=0.\tag{2.6}$$

Putting the values of $\partial_j\overline{L}_i$, $\overline{L}_{ir}$ and $\overline{L}_r$ from (2.4) in (2.6) and using equation $L_{i|j}=\partial_jL_i-L_{ir}G^r_j-L_rF^r_{ij}=0$, and rearranging the terms, we get

$$-m\frac{L^{m+1}}{\beta^{m+1}}b_{i|j}=\overline{L}_{ir}D^r_j+\overline{L}_r{}^cD^r_{ij}+m(m+1)\frac{L^m}{\beta^{m+2}}(\beta L_i-Lb_i)(r_{0j}+s_{0j}),$$

which after using $2r_{ij}=b_{i|j}+b_{j|i}$ and $2s_{ij}=b_{i|j}-b_{j|i}$, we get

$$\begin{aligned}-2m\frac{L^{m+1}}{\beta^{m+1}}r_{ij}=&\overline{L}_{ir}D^r_j+\overline{L}_{jr}D^r_i+2\overline{L}_r{}^cD^r_{ij}+m(m+1)\frac{L^m}{\beta^{m+2}}\times\\ &\times(\beta L_i-Lb_i)(r_{0j}+s_{0j})+m(m+1)\frac{L^m}{\beta^{m+2}}(\beta L_j-Lb_j)(r_{i0}+s_{i0})\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}-2m\frac{L^{m+1}}{\beta^{m+1}}s_{ij} =& \overline{L}_{ir}D^r_j - \overline{L}_{jr}D^r_i + m(m+1)\frac{L^m}{\beta^{m+2}}(\beta L_i - Lb_i)\times \\ & \times (r_{0j}+s_{0j}) - m(m+1)\frac{L^m}{\beta^{m+2}}(\beta L_j - Lb_j)(r_{i0}+s_{i0}).\end{aligned} \tag{2.8}$$

Subtracting (2.7) from (2.5) and contracting the resulting equation with y^i, we get

$$-2\overline{L}_{jr}D^r + m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_j - \beta L_j)\, r_{00} - 2m\frac{L^{m+1}}{\beta^{m+1}}r_{0j} = 2\overline{L}_r D^r_j. \tag{2.9}$$

Contracting (2.9) with y^j, we get

$$\left[(m+1)\frac{L^m}{\beta^m}\, L_r - m\frac{L^{m+1}}{\beta^{m+1}}\, b_r\right]\, D^r = -\frac{1}{2}m\frac{L^{m+1}}{\beta^{m+1}}r_{00}. \tag{2.10}$$

Subtracting (2.8) from (2.5) and contracting the resulting equation with y^j, we have

$$\begin{aligned}&\left[\frac{L^m}{\beta^{m+1}}\{(m+1)\beta - \rho m L^2\}L_{ir} + m(m+1)\frac{L^{m-1}}{\beta^m}L_iL_r\right.\\ &\left. - m(m+1)\frac{L^m}{\beta^{m+1}}(L_ib_r + L_rb_i) + m(m+1)\frac{L^{m+1}}{\beta^{m+2}}b_ib_r\right] D^r\\ &= -m\frac{L^{m+1}}{\beta^{m+1}}s_{i0} + \frac{1}{2}m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_i - \beta L_i)r_{00}.\end{aligned} \tag{2.11}$$

In view of $LL_{ir} = g_{ir} - L_iL_r$, the equation (2.11) may be written as

$$\begin{aligned}&\left[(m+1)\frac{L^{m-1}}{\beta^m} - \rho m\frac{L^{m+2}}{\beta^{m+1}}\right]g_{ir}D^r + \left[\frac{L^{m-1}}{\beta^{m+1}}\{(m^2-1)\beta + \rho m L^2\}L_i\right.\\ &\left. - m(m+1)\frac{L^m}{\beta^{m+1}}b_i\right]L_rD^r + m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_i - \beta L_i)\, b_rD^r\\ &\qquad = -m\frac{L^{m+1}}{\beta^{m+1}}s_{i0} + \frac{1}{2}m(m+1)\frac{L^m}{\beta^{m+2}}(Lb_i - \beta L_i)r_{00}.\end{aligned} \tag{2.12}$$

Contracting (2.12) with $b^i(= g^{ij}b_j)$, we get

$$\begin{aligned}&-2\frac{L^m}{\beta^{m+1}}[m(m+1)L^2\triangle + (m+1)\beta^2 - \rho m L^2\beta]L_rD^r\\ &+ 2\frac{L^{m+1}}{\beta^{m+2}}[m(m+1)L^2\triangle + (m+1)\beta^2 - \rho m L^2\beta]b_rD^r\\ &\qquad = \frac{L^{m+3}}{\beta^{m+2}}[-2m\beta s_0 + m(m+1)\triangle r_{00}],\end{aligned} \tag{2.13}$$

where $\triangle = b^2 - \frac{\beta^2}{L^2}$ and $s_0 = s_{r0}b^r$.

The equations (2.10) and (2.13) constitute the system of algebraic equations in L_rD^r and b_rD^r whose solution is given by

$$L_rD^r = \frac{-mL[\{(m+1)\beta - \rho m L^2\}r_{00} + 2mL^2s_0]}{2[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}} \tag{2.14}$$

and

$$b_r D^r = -\frac{-2m(m+1)L^2\beta s_0 + [m(m+1)(\triangle L^2 - \beta^2) + \rho m^2 L^2 \beta] r_{00}}{2[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]}. \tag{2.15}$$

Contracting (2.12) by g^{ij} and putting the values of $b_r D^r$ and $L_r D^r$ from (2.14) and (2.15) respectively, we get

$$D^i = \frac{m[\{(m+1)\beta - \rho m L^2\} r_{00} + 2mL^2 s_0]\times}{2\{m+1)\beta - \rho m L^2\}[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]} \tag{2.16}$$

$$\frac{[(m+1)L^2 b^i - \{2(m+1)\beta - \rho m L^2\}\, y^i]}{} - \frac{mL^2}{(m+1)\beta - \rho m L^2} s^i_0,$$

where $l^i = \dfrac{y^i}{L}$.

Proposition 2.1 *The difference tensor* $D^i = \overline{G}^i - G^i$ *of generalized* $h-$*Kropina change of Finsler metric is given by (2.16).*

Remark *The difference tensor for* $h-$*Kropina change of Finsler metric is obtained by putting* $m = 1$ *in equation (2.16).*

§3. Conditions for Projective Change

The Finsler space $\overline{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\overline{F}^n$. It is well known that the change $L \to \overline{L}$ is projective if $\overline{G}^i = G^i + P(x,y)y^i$, where $P(x,y)$ is a homogeneous scalar function of degree one in y^i, called projective factor [6].

Thus from (2.2) it follows that $L \to \overline{L}$ is projective iff $D^i = Py^i$. Now we consider that the generalized $h-$Kropina change $L \to \overline{L} = \frac{L^{m+1}}{\beta^m}$ is projective. Then from equation (2.16), we have

$$Py^i = \frac{m[\{(m+1)\beta - \rho m L^2\} r_{00} + 2mL^2 s_0]\times}{2\{m+1)\beta - \rho m L^2\}[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]} \tag{3.1}$$

$$\frac{[(m+1)L^2 b^i - \{2(m+1)\beta - \rho m L^2\}\, y^i]}{} - \frac{mL^2}{(m+1)\beta - \rho m L^2} s^i_0,$$

Contracting (3.1) with $y_i (= g_{ij}y^j)$ and using the fact that $s^i_0 y_i = 0$ and $y_i y^i = L^2$, we get

$$P = \frac{-m[\{(m+1)\beta - \rho m L^2\} r_{00} + 2mL^2 s_0]}{2[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]}. \tag{3.2}$$

Putting the value of P from (3.2) in (3.1), we get

$$\frac{m(m+1)[\{(m+1)\beta - \rho m L^2\} r_{00} + 2mL^2 s_0](\beta y^i - L^2 b^i)}{2\{m+1)\beta - \rho m L^2\}[m(m+1)L^2\triangle + \beta\{(m+1)\beta - \rho m L^2\}]} \tag{3.3}$$

$$= -\frac{mL^2}{(m+1)\beta - \rho m L^2} s^i_0,$$

Transecting (3.3) by b_i, we get

$$r_{00} = \frac{2\beta\, s_0}{(m+1)\,\triangle}, \quad \text{where} \quad \triangle = b^2 - \frac{\beta^2}{L^2} \neq 0. \tag{3.4}$$

Putting the value of r_{00} from (3.4) in (3.2), we get

$$P = -\frac{m s_0}{(m+1)\,\triangle}. \tag{3.5}$$

Eliminating P and r_{00} from (3.5), (3.4) and (2.16), we get

$$s^i_0 = \left[b^i - \frac{\beta}{L^2} y^i\right] \frac{s_0}{\triangle}. \tag{3.6}$$

The equations (3.4) and (3.6) give the necessary conditions under which a generalized $h-$Kropina change becomes a projective change.

Conversely, if conditions (3.4) and (3.6) are satisfied, then putting these conditions in (2.16), we get

$$D^i = -\frac{m s_0}{(m+1)\triangle}\, y^i \ \text{ i.e. } \ D^i = P y^i, \quad \text{where} \quad P = -\frac{m s_0}{(m+1)\triangle}.$$

Thus $\overline{F}^n$ is projective to F^n.

Theorem 3.1 *The generalized $h-$Kropina change of a Finsler space is a projective change iff equations (3.4) and (3.6) hold the projective factor P is given by equation (3.5).*

If $m = 1$, then the equations (3.4) and (3.6) are reduced to the equations

$$r_{00} = \frac{\beta s_0}{\triangle}, \tag{3.7}$$

and

$$s^i_0 = \left[b^i - \frac{\beta}{L^2} y^i\right] \frac{s_0}{\triangle} \tag{3.8}$$

respectively and the projective factor is given by $P = -\frac{s_0}{2\triangle}$. Thus, we have

Corollary 3.1 *The $h-$Kropina change of Finsler metric is projective iff the conditions (3.7) and (3.8) hold.*

§4. Douglas Space

The Finsler space F^n is called a Douglas space if and only if $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i [7]. We shall write $hp(r)$ to denote a homogeneous polynomial in y^i of degree r. If we write $B^{ij} = D^i y^j - D^j y^i$, then from (2.16), we get

$$B^{ij}=\frac{m(m+1)L^2[\{(m+1)\beta-\rho mL^2\}r_{00}+2mL^2s_0](b^iy^j-b^jy^i)}{2\{m+1)\beta-\rho mL^2\}[m(m+1)L^2\triangle+\beta\{(m+1)\beta-\rho mL^2\}]}$$

$$-\frac{mL^2}{(m+1)\beta-\rho mL^2}(s^i_0y^j-s^j_0y^i). \tag{4.1}$$

From (4.1), we find if a Douglas space is transformed to a Douglas space by generalized $h-$Kropina change of Finsler metric, then B^{ij} must be $hp(3)$ and if B^{ij} is $hp(3)$ then generalized $h-$Kropina change transforms a Douglas space into a space of the same kind.

Theorem 4.1 *The generalized $h-$Kropina change of Douglas space leads to a Douglas space iff B^{ij} given by (4.1) is* $hp(3)$.

If $m=1$, then the equation (4.1) becomes

$$B^{ij}=\frac{L^2[(2\beta-\rho L^2)r_{00}+2L^2s_0](b^iy^j-b^jy^i)}{(2\beta-\rho L^2)[2L^2\triangle+\beta(2\beta-\rho L^2)]} \tag{4.2}$$

$$-\frac{L^2}{(2\beta-\rho L^2)}(s^i_0y^j-s^j_0y^i).$$

Thus, we have

Corollary 4.1 *The $h-$Kropina change of Douglas space leads to a Douglas space iff B^{ij} given by (4.2) is* $hp(3)$.

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