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Geometry of Chain of Spheres Inside an Ellipsoidal Fragment

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Abstract: The objective of this article is to establish a condition by which we are able to state that an ellipsoidal fragment formed by a plane cutting the ellipsoid can always contain a sphere in any position inside in it. A method to construct a chain of mutually tangent spheres inscribed in the ellipsoidal segment has been proposed. The locus of the centroid as well as the radii of the mutually tangent spheres have been computed. The prime concern of our work is to explore some geometrical properties of such a chain of spheres which includes the condition of inscribability of a sphere in any position inside the ellipsoid along with the computation of points of tangency between consecutive spheres.

Key Words: Spherical chain, ellipsoidal segment, ellipsoid.

AMS(2010): 51M04, 51M05.

§1. Introduction

The proposed problem can be considered as a novel problem as there is not so much information present about this in the literature. The proposed problem is the enhancement of the same type of problem in 2-dimensions in which a chain of circles was considered in an elliptical segment. The purpose of this article is to extend the same problem to 3-dimensions in which a chain of spheres are considered to be inscribed in an ellipsoidal segment formed by a cutting plane to the ellipsoid. Lucca (2009) described the properties of the chain of mutually tangent circles inside a circular segment. In this paper, the authors discussed the locus of the centers of mutually tangent circles inside a circle. They also explored the points of tangency of the these circles and later they derived the recursive and non recursive formula for centers and radii of the circles in the chain. In their paper, Poelaert et al. (2011) discussed about the surface area and curvature of a general ellipsoid. They also derived the expressions for mean and Gaussian curvature of the ellipsoid. Pal et al. (2016) explored the properties related to the chain of mutually tangent spheres inside a spherical segment. In this article all the properties that has been found for a circular chain is recalled for a chain of spheres. Finally, Lucca (2021) enhanced his previous work to explore the properties of mutually tangent circles inside an elliptical segment.

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The organization of the proposed article is as follows. The introduction is given in section 1. Section 3 contains the basic concepts used to formulate the results. In section 4, radii and centers of the chain spheres are derived. In section 5, the condition for inscribing the chain of spheres inside the ellipsoidal segment has been obtained. Some geometrical properties are derived in section 6. Section 7 contains the conclusion.

§2. **Motivation of Work**

The research work done in the above articles motivates us to extend this idea for a 3-dimensional objects like ellipsoid. The novelty of our work is the extension of the geometrical properties of the objects inscribed in a conic to the properties in a conicoid. In this paper, we have considered an ellipsoid cutting by a plane vertically to form an ellipsoidal segment. A vertical chain of mutually tangent spheres are considered inside the ellipsoidal fragment to describe various properties like point of tangency, locus of centroid of the spheres.

§3. **Basic Concepts**

Let us consider a chain of spheres inscribed in an ellipsoid. It is assumed that a plane cutting the ellipsoid to form an ellipsoidal fragment MQN to which the spheres are inscribed. Now our aim is to explore some geometrical properties of the chain of mutually tangent spheres in an ellipsoidal segment. For this, it is better to deal with the problem in spherical coordinates. Therefore we consider the coordinate system as

$$\begin{aligned} x &= \rho(\theta,\phi)\cos\theta\sin\phi \\ y &= \rho(\theta,\phi)\sin\theta\sin\phi \\ z &= \rho(\theta,\phi)\cos\theta. \end{aligned}$$

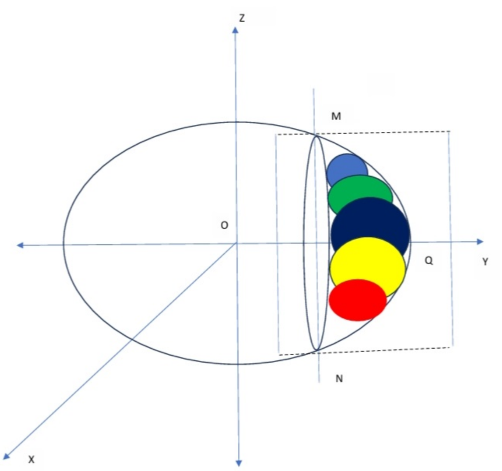

Figure 1. A chain of spheres inscribed in an ellipsoidal fragment

The equation of ellipsoid with principle semi axes a, b and $c(a \geqslant b \geqslant c)$ and with eccentric anomalies $(0 \leqslant \theta \leqslant 180)$ and $(0 \leqslant \phi < 360)$ is given by

$$\rho_e(\theta, \phi) = \frac{abc}{\sqrt{b^2c^2 \sin^2\theta \cos^2\phi + a^2c^2 \sin^2\theta \sin^2\phi + a^2b^2 \cos^2\theta}}. \tag{1}$$

The equation of a plane cutting the ellipsoid in spherical coordinates is given by

$$\rho_r(\theta, \phi) = \frac{p}{l \sin\theta \cos\phi + m \sin\theta \sin\phi + n \cos\theta}, \tag{2}$$

where l, m, n be the direction cosines of the line perpendicular to the plane and p be the distance of the plane from the origin.

Equating equations (1) and (2) and simplifying, we get the expression for p as

$$p = \frac{abc(l \cos\phi + m \sin\phi + n \cot\theta)}{\sqrt{b^2c^2 \cos^2\phi + c^2a^2 \sin^2\phi + a^2b^2 \cot^2\theta}}. \tag{3}$$

§4. Radii and Centers of the Spheres Under Two Tangent Planes

In order to inscribe a generic sphere inside an ellipsoidal segment, it is obvious to determine its radius and center. For this, it is mandatory that the centers of the spheres must lie on the bisector of the angle formed by the plane intersecting the ellipsoid and the tangent plane to the ellipsoid in the point of tangency between the spheres and the ellipsoid.

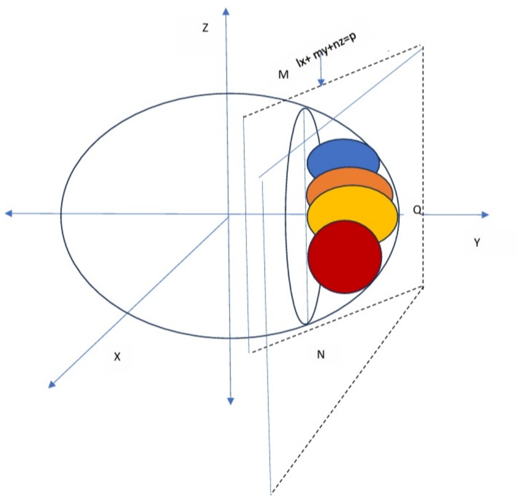

Figure 2. Spherical chain inside the ellipsoidal fragment under two planes

Theorem 1 *The radii $r_i(\theta, \phi)$ and centers $[X_c(\theta, \phi), Y_c(\theta, \phi), Z_c(\theta, \phi)]$ of spheres inscribed in an ellipsoidal segment under two tangent planes are*

$$\frac{kG}{W} \quad \textit{and} \quad \left[\frac{(kb^2c^2+1)abc \sin\theta \cos\phi}{W}, \frac{(kc^2a^2+1)abc \sin\theta \sin\phi}{W}, \frac{(ka^2b^2+1)abc \cos\theta)}{W}\right]$$

respectively, where

$$\begin{aligned}
k &= \frac{GM+N}{abc(GS+T)},\\
G &= abc\sqrt{b^4c^4\sin^2\theta\cos^2\phi + a^4c^4\sin^2\theta\sin^2\phi + a^4b^4\cos^2\theta},\\
M &= pW - abc(l\sin\theta\cos\phi + m\sin\theta\sin\phi + n\cos\theta),\\
N &= a^2b^2c^2(W^2 - b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta),\\
S &= lb^2c^2\sin\theta\cos\phi + mc^2a^2\sin\theta\sin\phi + na^2b^2\cos\theta,\\
T &= abc(b^4c^4 - a^4c^4 - a^4b^4),\\
W &= \sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}.
\end{aligned}$$

Proof Let us consider a point Q be the generic tangancy point of the sphere with the ellipsoid. The coordinates of Q are

$$\begin{aligned}
x_e(\theta,\phi) &= \frac{abc\sin\theta\cos\phi}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}},\\
y_e(\theta,\phi) &= \frac{abc\sin\theta\sin\phi}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}},\\
z_e(\theta,\phi) &= \frac{abc\cos\theta}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}}.
\end{aligned}$$

The equation of tangent plane to the ellipsoid at Q is given by

$$\frac{xx_e(\theta,\phi)}{a^2} + \frac{yy_e(\theta,\phi)}{b^2} + \frac{zz_e(\theta,\phi)}{c^2} = 1. \tag{4}$$

The equation of a plane cutting the ellipsoid is

$$lx + my + nz = p. \tag{5}$$

The equation of angle bisector between the planes (4) and (5) is given by

$$lx + my + nz - p - \frac{a^2b^2c^2 - b^2c^2xx_e(\theta,\phi) - c^2a^2yy_e(\theta,\phi) - a^2b^2zz_e(\theta,\phi)}{\sqrt{b^4c^4x_e^2(\theta,\phi) + c^4a^4y_e^2(\theta,\phi) + a^4b^4z_e^2(\theta,\phi)}} = 0. \tag{6}$$

The equation of normal to the ellipsoid at Q given by equation

$$\frac{x - x_e(\theta,\phi)}{b^2c^2x_e(\theta,\phi)} = \frac{y - y_e(\theta,\phi)}{a^2c^2y_e(\theta,\phi)} = \frac{z - z_e(\theta,\phi)}{a^2b^2z_e(\theta,\phi)} = k(say). \tag{7}$$

Now, substituting the values of $x_e(\theta,\phi)$, $y_e(\theta,\phi)$ and $z_e(\theta,\phi)$ in equation (7), we get the coordinates of the centers $[X_c(\theta,\phi), Y_c(\theta,\phi), Z_c(\theta,\phi)]$ of the spheres inside the the ellipsoidal segment. Next using the distance formula between the points $(x_e(\theta,\phi), y_e(\theta,\phi), z_e(\theta,\phi))$ and $(X_c(\theta,\phi), Y_c(\theta,\phi), Z_c(\theta,\phi))$, we get the radii $r_i(\theta,\phi)$ of the spheres inside the ellipsoidal segment. □

§5. Inscribability Condition

In this section, we derived the condition for inscribability of a sphere inside an ellipsoidal segment.

Theorem 2 *A generic sphere can always be inscribed in an ellipsoidal segment formed by a vertical plane cutting the ellipsoid if*

$$A^2 \sin^2\theta + B^2 \cos^2\theta = (C\sin\theta + D\cos\theta - U)^2$$

where,

$$\begin{aligned} A^2 &= k^2[b^4c^4\cos^2\phi + c^4a^4\sin^2\phi], \\ B^2 &= k^2a^4b^4, \\ C &= l(kb^2c^2+1)\cos\phi + m(kc^2a^2+1))\sin\phi, \\ D &= n(ka^2c^2+1), \\ U &= \frac{pW}{abc}. \end{aligned}$$

Proof Notice that the equation of sphere having center $(X_c(\theta,\phi), Y_c(\theta,\phi), Z_c(\theta,\phi))$ and radius $r(\theta,\phi)$ is

$$(x - X_c(\theta,\phi))^2 + (y - Y_c(\theta,\phi))^2 + (z - Z_c(\theta,\phi))^2 = r^2(\theta,\phi). \tag{8}$$

and the equation of the ellipsoid circumscribing the sphere is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{9}$$

Considering a generic sphere touches the ellipsoid at the point $(x_e(\theta,\phi), y_e(\theta,\phi), z_e(\theta,\phi))$ of Q in Figure 2 and $lx + my + nz = p$ be the plane cutting the ellipsoid and also touching the sphere. It is obvious that a sphere will be completely inscribed inside the ellipsoid if the distance between the center of the sphere from the point Q is equal to the length of the perpendicular from the center to the plane $lx + my + nz = p$.

Now, the distance between the center of the sphere and point Q is

$$\sqrt{(X_c(\theta,\phi) - x_e(\theta,\phi))^2 + (Y_c(\theta,\phi) - y_e(\theta,\phi))^2 + (Z_c(\theta,\phi) - z_e(\theta,\phi))^2}$$

and the length of perpendicular on the given plane from the center of the sphere is

$$\frac{lX_c(\theta,\phi) + mY_c(\theta,\phi) + nZ_c(\theta,\phi) - p}{\sqrt{l^2 + m^2 + n^2}}.$$

Equating the above two expressions and squaring both the sides, we have

$$\begin{aligned}(X_c(\theta,\phi) - x_e(\theta,\phi))^2 + (Y_c(\theta,\phi) &- y_e(\theta,\phi))^2 + (Z_c(\theta,\phi) - z_e(\theta,\phi))^2 \\ &= (lX_c(\theta,\phi) + mY_c(\theta,\phi) + nZ_c(\theta,\phi) - p)^2, \end{aligned} \quad (10)$$

where $l^2 + m^2 + n^2 = 1$.

Substituting the values of $X_c(\theta,\phi), Y_c(\theta,\phi), Z_c(\theta,\phi)$ and $x_e(\theta,\phi), y_e(\theta,\phi), z_e(\theta,\phi)$ in the above expression, we have

$$A^2 \sin^2\theta + B^2 \cos^2\theta = (C\sin\theta + D\cos\theta - U)^2, \quad (11)$$

which is the desired result. □

§6. Geometrical Properties of a Spherical Chain Inside an Ellipsoidal Segment

In this section we have explored some of the properties of a spherical chain inside an ellipsoidal segment.

Theorem 3 *The locus of the centers of mutually tangent spheres inscribed in an ellipsoidal fragment formed by a plane cutting the ellipsoid is*

$$(I + t^2 + z^2)^2 = 4(I + J)$$

where, $I = a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta$ *and* $J = t^2 + y^2 + z^2$.

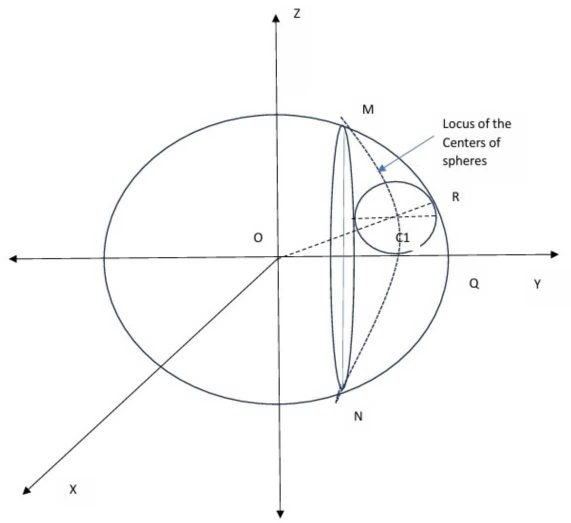

Figure 3. Locus of centers of Chain of spheres inscribed in an ellipsoidal fragment

Proof Let us consider a chain of mutually tangent spheres inscribed inside an ellipsoidal fragment formed by plane cutting the ellipsoid and tangent to the spheres. Let the origin O be the center of the ellipsoid. Now a generic point R on ellipsoid will be $(a\sin\theta\cos\phi, b\sin\theta\sin\phi, c\cos\theta)$. Let (t, y, z) be the center C_1 of spheres inside the ellipsoidal fragment MNQ. Now the line OR can be defined as

$$OR = \sqrt{a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta}.$$

Similarly, the line $OC_1 = \sqrt{t^2+y^2+z^2}$.

Using the geometry, we have

$$\sqrt{a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta} - y \;=\; \sqrt{t^2+y^2+z^2},$$

i.e.,

$$y \;=\; \sqrt{a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta} - \sqrt{t^2+y^2+z^2}.$$

Squaring both the sides, we have

$$\begin{aligned} y^2 \;=\;& a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta + t^2 + y^2 + z^2 \\ & -2\sqrt{(a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta)(t^2+y^2+z^2)}. \end{aligned}$$

Again, squaring both the sides and simplifying the above expression, we have the required result. □

Theorem 4 *The locus of points of tangency between consecutive spheres of the chain lie on*

$$PT_i^2 - t^2 - b^2 - 2r_ib - z_i^2 = 0,$$

where r_i be the radii of the spheres inscribed in the ellipsoidal segment and P is a point on ellipsoid in y-axis and T be the point of tangency.

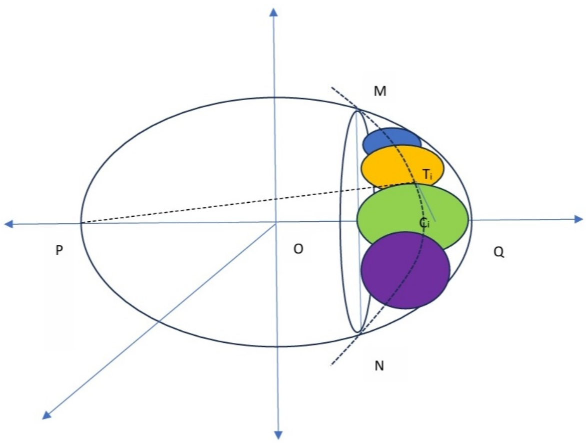

Figure 4. Point of tangancy of the spheres inside ellipsoidal fragment

Proof Let us assume that the two neighbouring spheres having centers $C_i(t, y_i, z_i)$ and

$T_i(t, y_i+1, z_i+1)$ with respective radii r_i and r_i+1 and tangent to each other at T_i and also touching the ellipsoidal fragment and the plane Y=0. From the above figure it is observed that the coordinate of point P is $(0, -b, 0)$. Then we have,

$$PC_i^2 \;=\; t^2 + (y_i + b)^2 + z_i^2 = t^2 + b^2 + y_i^2 + 2y_i b + z_i^2.$$

But it is obvious that $r_i^2 = y_i^2$ and hence using it we can write

$$PC_i^2 \;=\; t^2 + b^2 + r_i^2 + 2r_i b + z_i^2.$$

Now, using the Pythagoras theorem in the right angled triangle PC_iT_i, we have

$$PT_i^2 \;=\; PC_i^2 - r_i^2 = t^2 + b^2 + 2r_i b + z_i^2.$$

This proves the theorem. □

§7. Conclusion

In this paper, we have analyzed various properties of a chain of spheres inscribed in an ellipsoidal segment formed by a vertical plane cutting the ellipsoid. We have derived the radii and coordinates of centers of mutually tangent spheres inside the ellipsoidal segment. An inscribability condition for the vertical chain of spheres along with the locus of the centers of such a chain has been also derived. Finally some geometrical properties are also developed for such an arrangement. From a very short literature review, it has been observed that not so much work has been done so far in this field. A symmetrical extension has been done by Pal et al. (2016) of the work done by Lucca (2009) which pulls the properties of chain of circles inside a circular segment to the chain of spheres inside spherical segment. In this article, we have accomplished the task of unsymmetrical extension which extends the properties of chain of circle inside an ellipse to the chain of spheres inside an ellipsoidal fragment.

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Appendix A

Substituting the values of $x_e(\theta,\phi)$, $y_e(\theta,\phi)$, $z_e(\theta,\phi)$ in *equation*(7), we have

$$
\begin{aligned}
x &= \frac{(kb^2c^2+1)abc\sin\theta\cos\phi}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}} = \frac{(kb^2c^2+1)abc\sin\theta\cos\phi}{W},\\
y &= \frac{(kc^2a^2+1)abc\sin\theta\sin\phi}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}} = \frac{(kc^2a^2+1)abc\sin\theta\sin\phi}{W},\\
z &= \frac{(ka^2b^2+1)abc\cos\theta}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}} = \frac{(ka^2b^2+1)abc\cos\theta}{W}.
\end{aligned}
$$

Substituting the above values of x, y and z along with $x_e(\theta,\phi)$, $y_e(\theta,\phi)$ and $z_e(\theta,\phi)$ in equation (6), we have

$$
\begin{aligned}
&\frac{l(kb^2c^2+1)abc\sin\theta\cos\phi}{W} + \frac{m(kc^2a^2+1)abc\sin\theta\sin\phi}{W}\\
&+\frac{n(ka^2b^2+1)abc\cos\theta}{W} - \frac{Wabc}{\sqrt{b^4c^4\sin^2\theta\cos^2\phi + c^4a^4\sin^2\theta\sin^2\phi + a^4b^4\cos^2\theta}}\\
&+\frac{b^2c^2k(b^2c^2+1)abc\sin^2\theta\cos^2\phi}{W\sqrt{b^4c^4\sin^2\theta\cos^2\phi + c^4a^4\sin^2\theta\sin^2\phi + a^4b^4\cos^2\theta}}\\
&+\frac{a^2c^2k(a^2c^2+1)abc\sin^2\theta\sin^2\phi}{W\sqrt{b^4c^4\sin^2\theta\cos^2\phi + c^4a^4\sin^2\theta\sin^2\phi + a^4b^4\cos^2\theta}}\\
&+\frac{a^2b^2k(a^2b^2+1)abc\cos^2\theta}{W\sqrt{b^4c^4\sin^2\theta\cos^2\phi + c^4a^4\sin^2\theta\sin^2\phi + a^4b^4\cos^2\theta}} = p.
\end{aligned}
$$

Simplifying the above expression for k, we have the desired value of k.

Appendix B

Substituting the values of $(X_c(\theta,\phi), Y_c(\theta,\phi), Z_c(\theta,\phi))$ and $((x_e(\theta,\phi), y_e(\theta,\phi), z_e(\theta,\phi))$ in equation (10), we have

$$
\begin{aligned}
&\frac{k^2b^4c^4a^2b^2c^2\sin^2\theta\cos^2\phi}{W^2} + \frac{k^2a^4c^4a^2b^2c^2\sin^2\theta\sin^2\phi}{W^2} + \frac{k^2a^4b^4a^2b^2c^2\cos^2\theta}{W^2}\\
&= \left[\frac{l(kb^2c^2+1)abc\sin\theta\cos\phi}{W} + \frac{m(ka^2c^2+1)abc\sin\theta\sin\phi)}{W} + \frac{n(ka^2b^2+1)abc\cos\theta}{W} - p\right]^2\\
&\frac{k^2a^2b^2c^2}{W^2}((b^4c^4\cos^2\phi + c^4a^4\sin^2\phi)\sin^2\theta + a^4b^4\cos^2\theta)\\
&= \frac{a^2b^2c^2}{W^2}\left[(l(kb^2c^2+1)\cos\phi + m(kc^2a^2+1)\sin\phi)\sin\theta + n(ka^2b^2+1)\cos\theta - \frac{pW}{abc}\right]^2\\
&k^2((b^4c^4\cos^2\phi + c^4a^4\sin^2\phi)\sin^2\theta + a^4b^4\cos^2\theta)\\
&= \left[(l(kb^2c^2+1)\cos\phi + m(kc^2a^2+1)\sin\phi)\sin\theta + n(ka^2b^2+1)\cos\theta - \frac{pW}{abc}\right]^2.
\end{aligned}
$$