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Gröbner-Shirshov Bases Theory for the Right Ideals of Left-Commutative Algebras

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Abstract: In this paper, we establish a Composition-Diamond lemma for the right ideals of free left-commutative algebras. As an application, we prove that the membership problems for the right ideals of free left-commutative algebras are decidable.

Key Words: Basis, Gröbner-Shirshov basis, left-commutative algebras.

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§1. Introduction

Gröbner bases and Gröbner-Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [33, 35] (see also [9, 10]) , free Lie algebras [34, 35] and implicitly free associative algebras [34, 35] (see also [3, 4]), by H. Hironaka [30] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [20] for ideals of the polynomial algebras.

Gröbner bases and Gröbner-Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra, see, for example, the books [1, 19, 21, 22, 26, 28], the papers [2, 3, 4], and the surveys [5, 6, 14, 16, 17, 18].

Up to now, different versions of Composition-Diamond lemma are known for the following classes of algebras apart those mentioned above: Lie p-algebras [32], associative conformal algebras [15], modules [25, 31] (see also [24]), right-symmetric algebras [8], dialgebras [11], associative algebras with multiple operators [13], matabelian Lie algebras [23], Rota-Baxter algebras [7], semirings [12], integro-differential algebras [29], and so on.

Let k be a field, A a non-associative algebra over k. We call A a left-commutative algebra over k, if A satisfies the following identity: $x(yz) = y(xz)$, $x, y, z \in A$. The variety of Novikov algebras and the variety of dual Leibniz algebras are subvarieties of the variety of left-commutative algebras. Free left-commutative algebras were firstly studied by A. Dzhu-

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madil'daev and C. Löfwall [27]. They constructed a monomial basis for free left-commutative algebras. In this paper, we establish Gröbner-Shirshov bases theory for the right ideals of left-commutative algebras. Using this theory, we prove the decidability of the membership problems for the right ideals of free left-commutative algebras.

§2. Free Left-Commutative Algebras

Let X be a well ordered set. Each letter $x_i \in X$ is called a non-associative word of degree 1. Suppose that u is a non-associative word of degree m and v is a non-associative word of degree n. Then (uv) is called a non-associative word of degree $m+n$. Denote by $d(u)$ the degree of the non-associative word u.

Let $u, v \in X^{**}$ be non-associative words. Then we say that $u > v$ if $d(u) > d(v)$. If $d(u) = d(v) \geq 2$ and $u = (u_1u_2), v = (v_1v_2)$, then we say that $u > v$ if either $u_2 > v_2$ or $u_2 = v_2$ and $u_1 > v_1$. This ordering is called non-associative degree inverse lexicographic ordering. Unless otherwise stated, the non-associative degree inverse lexicographic ordering is used throughout this paper.

Definition 2.1 *Each letter $x_i \in X$ is called a regular word of degree 1. Suppose that $u = (vw)$ is a non-associative word of degree $m, m > 1$. Then $u = (vw)$ is called a regular word of degree m if it satisfies the following conditions:*

$(S1)$ *both v and w are regular words, and*

$(S2)$ *if $w = (w_1w_2)$, then $v \geq w_1$.*

Let k be a field, $N(X)$ the set of all regular words on X, $kN(X)$ the k-linear space spanned by $N(X)$. Let $u, v \in N(X)$. Then we define a product $u \cdot v$ on $kN(X)$ by the following way: if $v = x_i \in X$, then $u \cdot v := (ux_i)$; if $v = (v_1v_2)$ and $u \geq v_1$, then $u \cdot v := (u(v_1v_2))$; if $v = (v_1v_2)$ and $u < v_1$, then $u \cdot v := (v_1(u \cdot v_2))$.

Theorem 2.2([27]) *Let $LC(X)$ be the free left-commutative algebra generated by X. Then the algebra $kN(X)$ is isomorphic to $LC(X)$.*

According to Theorem 2.2, each non-zero element f in $LC(X)$ can be uniquely presented as

$$f = \alpha_1 u_1 + \alpha_2 u_2 + \ldots + \alpha_m u_m,$$

where $\alpha_i \in k$, $u_i \in N(X)$ for all i, $\alpha_1 \neq 0$, $u_1 > u_2 > \ldots > u_m$. Here, the regular word u_1 is called the leading term of f, denoted by $\bar{f}$ and α_1 the leading coefficient of f, denoted by $\alpha_{\bar{f}}$. If $\alpha_{\bar{f}} = 1$, then f is called a monic polynomial.

For every $f \in LC(X)$ denote by L_f the operator of left multiplication by f acting on $LC(X)$, i.e., $L_f(g) = fg$ for all $g \in LC(X)$. In particular, if $f_1, f_2, \cdots, f_m, g \in LC(X)$, then $L_{f_m} \ldots L_{f_2} L_{f_1}(g) = (f_m(\cdots(f_2(f_1g))\cdots))$.

Lemma 2.3([27]) *Let $u \in N(X)$ be a regular word. Then u can be uniquely presented as*

$$u = L_{u_n} \dots L_{u_1}(x_i),$$

where $x_i \in X$, $u_n \geq \dots \geq u_1, u_j \in N(X), 1 \leq j \leq n, n \geq 0$.

Lemma 2.4 *Let $u, v \in N(X)$ be regular words and $v = L_{v_n} \dots L_{v_1}(x_i)$, where $n \geq 1, x_i \in X$. Then*

$$u \cdot v = L_{v_n} \cdots L_{v_t} L_u L_{v_{t-1}} \cdots L_{v_1}(x_i),$$

where $v_n \geq \cdots \geq v_t > u \geq v_{t-1} \geq \dots \geq v_1$.

Proof Let us use induction on n. If $n = 1$ and $u \geq v_1$, then $u \cdot v = L_u L_{v_1}(x_i)$. If $n = 1$ and $u < v_1$, then $u \cdot v = L_{v_1} L_u(x_i)$. Suppose that $n > 1$. If $u \geq v_n$, then $u \cdot v = L_u L_{v_n} \cdots L_{v_1}(x_i)$. If $u < v_n$, then $u \cdot v = v_n(u \cdot L_{v_{n-1}} \cdots L_{v_1}(x_i))$. By the inductive hypothesis, $u \cdot L_{v_{n-1}} \cdots L_{v_1}(x_i) = L_{v_{n-1}} \cdots L_{v_t} L_u L_{v_{t-1}} \cdots L_{v_1}(x_i)$, where $v_{n-1} \geq \cdots \geq v_t > u \geq v_{t-1} \geq \cdots \geq v_1$. Therefore,

$$u \cdot v = L_{v_n} \cdots L_{v_t} L_u L_{v_{t-1}} \cdots L_{v_1}(x_i),$$

where $v_n \geq \cdots \geq v_t > u \geq v_{t-1} \geq \cdots \geq v_1$. □

Lemma 2.5([27]) *If $u, v, w \in N(X)$ and $u > v$, then $u \cdot w > v \cdot w, w \cdot u > w \cdot v$.*

From Lemma 2.5, it follows that

Corollary 2.6 *If $f, g \in LC(X)$, then $\overline{(f \cdot g)} = \overline{(\overline{f} \cdot \overline{g})}$.*

§3. Composition-Diamond Lemma for Right Ideals of Free Left-Commutative Algebras

Definition 3.1 *Let $S \subset LC(X)$ be a set of monic polynomials. Each polynomial $s \in S$ is called an S-word of s-length one. Suppose that $(u)_s$ is an S-word of s-length m and v is a regular word of degree n. Then $(u)_s \cdot v$ is an S-word of s-length $m + n$.*

Definition 3.2 *Let $S \subset LC(X)$ be a set of monic polynomials. Each polynomial $s \in S$ is called a normal S-word of s-length one. Suppose that $(u)_s$ is a normal S-word of s-length m and $x_i \in X, v_j \in N(X), 1 \leq j \leq n, 0 \leq n$. Then $L_{v_n} \cdots L_{v_t} L_{(u)_s} L_{v_{t-1}} \cdots L_{v_1}(x_i)$ is called a normal S-word of s-length $m + 1 + \sum_j d(v_j)$ if $v_n \geq \cdots \geq v_t > \overline{(u)_s} \geq v_{t-1} \geq \cdots \geq v_1$. We denote $(u)_s$ by $[u]_s$ if $(u)_s$ is a normal S-word.*

Lemma 3.3 *For each S-word $(u)_s$, there exists a normal S-word $[v]_s$ such that $(u)_s = [v]_s$.*

Proof Suppose that the s-length of $(u)_s$ is m. Let us use induction on m. If $m = 1$, then $(u)_s = s$ and the lemma holds clearly. Suppose that $(u)_s = (v)_s \cdot w$, where $w \in N(X)$ and $(v)_s$ is an S-word with s-length less than m. By the induction hypothesis, these exists a normal S-word $[v']_s$ such that $(v)_s = [v']_s$. If $w = x_i \in X$, then the lemma holds clearly. Let us assume that $w = L_{w_l} \cdots L_{w_1}(x_i)$, where $x_i \in X$, $w_l \geq \cdots \geq w_1, w_j \in N(X), 1 \leq j \leq l, 1 \leq l$. Then by Lemma 2.4 we have

$$(u)_s = (v)_s \cdot w = [v']_s \cdot w = L_{w_l} \cdots L_{w_t} L_{[v']_s} L_{w_{t-1}} \cdots L_{w_1}(x_i),$$

where $w_l \geq \cdots \geq w_t > \overline{[v']_s} \geq w_{t-1} \geq \cdots \geq w_1$. This completes our proof. □

From Corollary 2.6, it follows that $\overline{[u]_s} = [u]_{\overline{s}}$.

Definition 3.4 *Let f, g be monic polynomials in $LC(X)$. If there exists a normal g-word $[u]_g$ such that $\bar{f} = \overline{[u]_g}$, then the polynomial $f - [u]_g$ is called a composition of inclusion of f and g, and denoted by $(f, g)_{\bar{f}}$.*

Let S be a given nonempty subset of $LC(X)$. The composition of inclusion $(f, g)_{\bar{f}}$ is said to be trivial modulo $(S, \bar{f})$ if

$$(f, g)_{\bar{f}} = \sum_i \alpha_i [u_i]_{s_i},$$

where $\alpha_i \in k,\ s_i \in S,\ [u_i]_{s_i}$ are normal S-words and $\overline{[u_i]_{s_i}} < \bar{f}$. If this is the case, then we write

$$(f, g)_{\bar{f}} \equiv 0 \quad mod(S, \bar{f}).$$

In general, for any regular word w and $f, g \in LC(X)$, we write

$$f \equiv g \quad mod(S, w)$$

which means that $f - g = \sum \alpha_i [u_i]_{s_i}$, where $\alpha_i \in k,\ s_i \in S$ and $\overline{[u_i]_{s_i}} < w$.

Definition 3.5 *Let $S \subset LC(X)$ be a nonempty set of monic polynomials and $Id_r(S)$ the right ideal of $LC(X)$, generated by S. Then the set S is called a Gröbner-Shirshov basis for $Id_r(S)$ if any composition of inclusion in S is trivial modulo S.*

Lemma 3.6 *Let $[u_1]_{s_1},\ [u_2]_{s_2}$ be normal S-words. If S is a Gröbner-Shirshov basis for $Id_r(S)$ and $w = \overline{[u_1]_{s_1}} = \overline{[u_2]_{s_2}}$, then*

$$[u_1]_{s_1} \equiv [u_2]_{s_2} \quad mod(S, w).$$

Proof If $[u_1]_{s_1} = s_1$ or $[u_2]_{s_2} = s_2$, then the lemma holds since S is a Gröbner-Shirshov basis for $Id_r(S)$.

Suppose that

$$[u_1]_{s_1} = L_{v_l} \cdots L_{v_p} L_{[v]_{s_1}} L_{v_{p-1}} \cdots L_{v_1}(x_i),$$

$$[u_2]_{s_2} = L_{w_m} \cdots L_{w_q} L_{[w]_{s_2}} L_{w_{q-1}} \cdots L_{w_1}(x_j),$$

where $v_l \geq \cdots \geq v_p > \overline{[v]_{s_1}} \geq v_{p-1} \geq \cdots \geq v_1$ and $w_m \geq \cdots \geq w_q > \overline{[w]_{s_2}} \geq w_{q-1} \geq \cdots \geq w_1$. From $\overline{[u_1]_{s_1}} = \overline{[u_2]_{s_2}}$ and Lemma 2.3, it follows that $x_i = x_j, l = m$ and either $p = q, v_1 = w_1, v_2 = w_2, \cdots, v_l = w_l, \overline{[v]_{s_1}} = \overline{[w]_{s_2}}$ or $p \neq q$,(Here without loss of generality we may assume $p > q$), $v_1 = w_1, v_2 = w_2, \cdots, v_{q-1} = w_{q-1}, v_q = \overline{[w]_{s_2}}, v_{q+1} = w_q, \cdots, v_{p-1} = w_{p-2}, \overline{[v]_{s_1}} =$

$w_{p-1}, v_p = w_p, \cdots, v_l = w_l$.

If $p = q, v_1 = w_1, v_2 = w_2, \cdots, v_l = w_l, \overline{[v]_{s_1}} = \overline{[w]_{s_2}}$, then

$$[u_1]_{s_1} - [u_2]_{s_2} = L_{v_l} \cdots L_{v_p} L_{([v]_{s_1} - [w]_{s_2})} L_{v_{p-1}} \cdots L_{v_1}(x_i).$$

By induction on w, $[v]_{s_1} \equiv [w]_{s_2} \ \ mod(S, \overline{[v]_{s_1})})$. From Lemma 3.3, it follows that $[u_1]_{s_1} \equiv [u_2]_{s_2} \ \ mod(S, w)$.

Suppose that $p > q, v_1 = w_1, v_2 = w_2, \cdots, v_{q-1} = w_{q-1}, v_q = \overline{[w]_{s_2}}, v_{q+1} = w_q, \cdots, v_{p-1} = w_{p-2}, \overline{[v]_{s_1}} = w_{p-1}, v_p = w_p, \cdots, v_l = w_l$. Then

$$\begin{aligned}
[u_1]_{s_1} - [u_2]_{s_2} &= L_{v_l} \cdots L_{v_p} L_{[v]_{s_1}} L_{v_{p-1}} \cdots L_{v_{q+1}} L_{v_q} L_{v_{q-1}} \cdots L_{v_1}(x_i) \\
&\quad - L_{v_l} \cdots L_{v_p} L_{[v]_{s_1}} L_{v_{p-1}} \cdots L_{v_{q+1}} L_{[w]_{s_2}} L_{v_{q-1}} \cdots L_{v_1}(x_i) \\
&\quad + L_{v_l} \cdots L_{v_p} L_{[v]_{s_1}} L_{v_{p-1}} \cdots L_{v_{q+1}} L_{[w]_{s_2}} L_{v_{q-1}} \cdots L_{v_1}(x_i) \\
&\quad - L_{v_l} \cdots L_{v_p} L_{w_{p-1}} L_{v_{p-1}} \cdots L_{v_{q+1}} L_{[w]_{s_2}} L_{v_{q-1}} \cdots L_{v_1}(x_i) \\
&= L_{v_l} \cdots L_{v_p} L_{([v]_{s_1} - w_{p-1})} L_{v_{p-1}} \cdots L_{v_{q+1}} L_{[w]_{s_2}} L_{v_{q-1}} \cdots L_{v_1}(x_i) \\
&\quad - L_{v_l} \cdots L_{v_p} L_{[v]_{s_1}} L_{v_{p-1}} \cdots L_{v_{q+1}} L_{([w]_{s_2} - v_q)} L_{v_{q-1}} \cdots L_{v_1}(x_i).
\end{aligned}$$

Since $\overline{[v]_{s_1} - w_{p-1}}, \overline{[w]_{s_2} - v_q} < w$, by Lemmas 2.5 and 3.3, we conclude that

$$[u_1]_{s_1} \equiv [u_2]_{s_2} \ \ mod(S, w).$$

This completes our proof. □

Theorem 3.7 *Let $S \subset LC(X)$ be a nonempty set of monic polynomials, $N(X)$ the set of all regular words on X and $<$ the non-associative degree inverse lexicographic ordering on $N(X)$. Let $Id_r(S)$ be the right ideal of $LC(X)$ generated by S. Then the following statements are equivalent:*

(i) S is a Gröbner-Shirshov basis for $Id_r(S)$;

(ii) $f \in Id_r(S) \Rightarrow \overline{f} = [u]_{\bar{s}}$ for some $s \in S$, where $[u]_s$ is a normal S-word;

(iii) $f \in Id_r(S) \Rightarrow f = \alpha_1[u_1]_{s_1} + \alpha_2[u_2]_{s_2} + \cdots$, where $\alpha_i \in k$, $\overline{[u_1]_{s_1}} > \overline{[u_2]_{s_2}} > \cdots$, and $[u_i]_{s_i}$ are normal S-words.

Proof (i) $\Rightarrow$ (ii). Let S be a Gröbner-Shirshov basis and $0 \neq f \in Id_r(S)$. We may assume, by Lemma 3.3, that

$$f = \sum_{i=1}^{n} \alpha_i [u_i]_{s_i},$$

where $\alpha_i \in k$, and $[u_i]_{s_i}$ are normal S-words. Let

$$w_i = \overline{[u_i]_{s_i}}, \ w_1 = w_2 = \cdots = w_l > w_{l+1} \geq \cdots .$$

We will use the induction on l and w_1 to prove that $\overline{f} = \overline{[u]_s}$ for some normal S-word $[u]_s$.

If $l=1$, then $\overline{f}=\overline{[u_1]_{s_1}}$ and hence the statement holds. Assume that $l\geq 2$. Then

$$\alpha_1[u_1]_{s_1}+\alpha_2[u_2]_{s_2}=(\alpha_1+\alpha_2)[u_1]_{s_1}-\alpha_2([u_1]_{s_1}-[u_2]_{s_2})$$

and by Lemma 3.6, we have

$$[u_1]_{s_1}\equiv[u_2]_{s_2}\ \ mod(S,[w_1]).$$

Thus, if $\alpha_1+\alpha_2\neq 0$ or $l>2$, the result follows from the induction on l. For the case that $\alpha_1+\alpha_2=0$ and $l=2$, we shall use induction on w_1 and then the result follows.

$(ii)\Rightarrow(iii)$. Assume that (ii) and $0\neq f\in Id_r(S)$. Let $f=\alpha_1\overline{f}+\cdots$. Then, by (ii), $\overline{f}=\overline{[u_1]_{s_1}}$. Therefore,

$$f_1=f-\alpha_1[u_1]_{s_1},\ \overline{f_1}<\overline{f},\ f_1\in Id_r(S).$$

Now, by using induction on $\overline{f}$, we have (iii).

$(iii)\Rightarrow(i)$. Suppose that $(f,g)_{\bar{f}}=f-[u]_g$ is a composition of inclusion of f and g, $f,g\in S$. It is clear that $(f,g)_{\bar{f}}\in Id_r(S)$. Then, by (iii), we have $(f,g)_{\bar{f}}=\alpha_1[u_1]_{s_1}+\alpha_2[u_2]_{s_2}+\cdots$, where $\alpha_i\in k$, $\bar{f}>\overline{(f,g)_{\bar{f}}}=\overline{[u_1]_{s_1}}>\overline{[u_2]_{s_2}}>\cdots$. This completes the proof. □

Theorem 3.8 *The membership problems for the right ideals of free left-commutative algebras are decidable.*

Proof Let X be a finite set and $N(X)$ all regular words on X. Let

$$T=\{(u_1,u_2,\cdots,u_l)|u_i\in N(X),u_1\geq u_2\geq\cdots\geq u_l,1\leq l\}.$$

For $(u_1,u_2,\cdots,u_p),(v_1,v_2,\cdots,v_q)\in T$, we define $(u_1,u_2,\cdots,u_p)>(v_1,v_2,\cdots,v_q)$ if either $p>q$ or $p=q$ and $(u_1,u_2,\cdots,u_p)>(v_1,v_2,\cdots,v_q)$ lexicographically. Clearly, this ordering is a well ordering on T.

Let $S=\{f_1,\ldots,f_m\}\in LC(X),1\leq m$. Let us assume that $\overline{f_1}\geq\overline{f_2}\geq\cdots\geq\overline{f_m}$. Then we set $\psi(S)=(\overline{f_1},\overline{f_2},\cdots,\overline{f_m})$. If there exists a composition of inclusion $(f_i,f_j)_{\overline{f_i}}$ of f_i and f_j, $i<j$, then we replace f_i by $(f_i,f_j)_{\overline{f_i}}$ and then we obtain a new set S_1. Clearly, $Id_r(S)=Id_r(S_1)$ and $\psi(S)>\psi(S_1)$. Since the ordering on T is a well ordering, we may obtain a finite Gröbner-Shirshov basis S_c for the right ideal $Id_r(S)$ of $LC(X)$.

Now, we show that the membership problem for the right ideal $Id_r(S)$ is decidable. We may assume, without loss of generality, that S is a finite Gröbner-Shirshov basis for the right ideal $Id_r(S)$. For an element $g\in LC(X)$, if there is no normal S-word $[u]_{f_i}$ such that $\bar{g}=\overline{[u]_{f_i}}$, then by Theorem 3.7 we may conclude that $g\notin Id_r(S)$. Otherwise, we let $g_1=g-[u]_{f_i}$. Clearly, $g\in Id_r(S)$ if and only if $g_1\in Id_r(S)$. Since $\overline{g_1}<\bar{g}$, we may complete the proof of this theorem by the induction on $\bar{g}$. □

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