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Group Divisible Designs $(5, n, n+1, 4; \lambda_1, \lambda_2)$

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Abstract: In this paper, we present new results on group divisible designs (GDDs) of block size four on three groups of different sizes $n_1 = 5$, $n_2 = n$ and $n_3 = n+1$ where $n \geq 5$ with indices λ_1 and λ_2. Here, we first establish necessary conditions for the existence of GDD$(5, n, n+1, 4; \lambda_1, \lambda_2)$ using relationships between parameters of the GDD. Secondly, we prove that these conditions are sufficient for several families of the GDDs and later give a general construction where parameters satisfy all the necessary conditions.

Key Words: Blocks, balanced incomplete block designs, group divisible designs.

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§1. Introduction

The arrangements of numbers in different patterns have a long history which dates back to the eighteenth and nineteenth centuries such as in the works of Euler, Kirkman, Cayley, Hamilton, Sylvester, Moore and others [9]. Such arrangements generally are called designs. Design theory is categorized into three designs that is combinatorial, algebraic and algorithemic also called computational. Combinatorial design theory is a branch of mathematics which deals with the study of existence, construction and properties of finite sets whose arrangements satisfy concepts of balance and symmetry [9]. For example equality of the size of the subsets and equality of occurence of a particular element or pair of distinct elements may be needed. In combinatorial design theory, balanced incomplete block designs (BIBDs), pairwise balanced designs (PBDs), latin squares, and group divible designs (GDDs) has been regarded as the most studied areas in mathematics with many designs [3]. Group divisible designs have been studied for their usefullness in statistics [1] and there are important applications in construction of other types of combinatorial designs such as packings and frames [2] and moreso, are also applicable to designs of different sizes (that is non-uniform GDDs) that are used to fit in various situations [2]. Unfortunately, comparing with uniform GDDs, much less is known on the construction of non-uniform ones. One major reason is that no appropriate algebraic or geometric structures have been found for the construction. In this research, focus shall be put on group divisible designs with block size four solving the problem when the design has three groups of different sizes.

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Definition 1.1 *A group divisible design-GDD(n,m,k,λ_1,λ_2) is a collection of k-element subsets, called blocks, of an nm set (V-set), where the elements of V are partitioned into m subsets (called groups) of size n each; each point of V appearing in* $r = \frac{\lambda_1(n-1)+\lambda_2 n(m-1)}{(k-1)}$ *blocks, and* $b = \frac{nmr}{k}$ *blocks.*

A GDD with parameters $(n_1+n_2+n_3,\ k;\ \lambda_1,\ \lambda_2)$ has three groups of different sizes n_1, n_2 and n_3. For example, the GDD$(1+2+n, 3; \lambda_1, \lambda_2)$ [6], GDD$(1+1+n, 3; 1, \lambda_2)$ [4], GDD$(1, n, n+1, 4; \lambda_1, \lambda_2)$, GDD$(2, n, n+1, 4; \lambda_1, \lambda_2)$, GDD$(3, n, n+1, 4; \lambda_1, \lambda_2)$ [7], and GDD$(4, n, n+1, 4; \lambda_1, \lambda_2)$ [7] have been studied. Though, there are many parameter sets where the answer to the existence of particular designs is not yet known [11]. For this reason therefore, this research paper intends to study and establish the necessary conditions for the existence of GDD$(5, n, n+1, 4; \lambda_1, \lambda_2)$ when $n_1 = 5, n_2 = n$ and $n_3 = n+1$ when $\lambda_1 \equiv 0 \pmod 3$, and $\lambda_2 \equiv 0 \pmod 6$ where $\lambda_1 = 3t$ for all $t \geq 1$ using relationships between parameters of the GDD. Secondly, we prove that these conditions are sufficient for several families of the GDDs and later give a general construction where parameters satisfy all the necessary conditions.

Example 1.1 A GDD (3,3,4;3,1) has a pair of elements from the same group occurs together in three blocks and a pair of elements from different groups occurs together in one block, that is if V = {1,2,3,4,5,6,7,8,9}, G_1 = {1,2,3}, G_2 = {4,5,6}, and G_3 = {7,8,9} and the blocks {1,2,3,4},{1, 2,3,5},{1,2,3,6}, {4,5,6,7},{4,5,6,8},{4,5,6,9},{7,8,9,1},{7,8,9,2} and {7,8,9,3}.

Theorem 1.1([5]) *If* $n \equiv 0, 4 \pmod 6$*, then there exists a minimal odd GDD* $(n, 3, 4; 2n, n-1)$*. If* $n \equiv 2 \pmod 6$*, then there exists a minimal odd GDD* $(n, 3, 4; 6n, 3(n-1))$*.*

Proof For $n \equiv 0, 4 \pmod 6$ there exists a BIBD$(n, 3, 2)$ with replication number $n-1$. Use the size three blocks from n copies of such a design based on the n points of group 1 to fill three of the spaces of a size four block. Fill in the fourth place with, say, point x_1 from group 2. Do this for every point from group 2 using the n copies of the BIBD. This puts each point of Group 1 in a block with every other group-mate $2n$ times, and $\lambda_1 = 2n$. Now, using n copies of a BIBD$(n, 3, 2)$ based on group 2, fill in with the points from Group 3. Using n copies of a BIBD$(n, 3, 2)$ on group 3, fill in with the points from group 1. This creates the desired GDD. When $n \equiv 2 \pmod 6$, there exists a BIBD$(n, 3, 6)$ with replication number $3(n-1)$. Repeat the previous construction for this n. □

Lemma 1.1([8]) *Relationship between the parameters of a BIBD* (v, b, r, k, λ)*. In a BIBD* (v, b, r, k, λ)*, the parameters must satisfy the necessary conditions.*

- (i) $\lambda(v-1) = r(k-1)$;
- (ii) *A* (v, k, λ) *has exactly* $vr = k \times b$*, implies* $b = \frac{vr}{k}$.

Theorem 2.2([8]) *In a* (v, k, λ) *BIBD,*

- (i) *Every point occurs in exactly* $r = \frac{\lambda(v-1)}{(k-1)}$ *blocks;*
- (ii) *There are* $b = \frac{vr}{k} = \frac{\lambda v(v-1)}{k(k-1)}$.

Corollary 1.1([8]) *If a* (v, k, λ) *BIBD exists, then* $\lambda(v-1) \equiv 0 \pmod{k-1}$ *and* $\lambda v(v-1) \equiv 0$

(mod $k(k-1)$).

§2. Results on the Existence of GDD $(5, n, n+1, 4; \lambda_1, \lambda_2)$

Here, we establish and prove that the necessary conditions for the existence of group divisible design-GDD$(5, n, n+1, 4; \lambda_1, \lambda_2)$ exist as well as giving its general construction.

2.1. Parameters of the GDD $(5, n, n+1, 4; \lambda_1, \lambda_2)$

For GDD$(5, n, n+1, 4; \lambda_1, \lambda_2)$ with block size four and three groups of different sizes 5,n and $n+1$, has replication numbers, r_i for i = 1,2,3 are from $r_1 = \frac{4\lambda_1+(2n+1)\lambda_2}{3}$, $r_2 = \frac{(n-1)\lambda_1+(n+6)\lambda_2}{3}$ and $r_3 = \frac{n\lambda_1+(n+5)\lambda_2}{3}$. The GDD has $(n^2+10)\lambda_1$ first associate pairs and $(n^2+11n+5)\lambda_2$ second associate pairs.

2.2. Necessary Conditions for the GDD $(5, n, n+1, 4; \lambda_1, \lambda_2)$

(*i*) $4\lambda_1 + (2n+1)\lambda_2 \equiv 0 \pmod 3$ *and* $(2n+1)\lambda_2 \equiv 0 \pmod 3$;

(*ii*) $(n-1)\lambda_1 + (n+6)\lambda_2 \equiv 0 \pmod 3$;

(*iii*) $n\lambda_1 + (5+n)\lambda_2 \equiv 0 \pmod 3$;

(*iv*) $(n^2+10)\lambda_1 + (n^2+11n+5)\lambda_2 \equiv 0 \pmod 6$.

§3. Main Results

Theorem 3.1 *A GDD* $(5, n, n+1, 4; \lambda_1, \lambda_2)$ *exists if the necessary condition* $4\lambda_1+(2n+1)\lambda_2 \equiv 0 \pmod 3$ *and* $(2n+1)\lambda_2 \equiv 0 \pmod 3$ *holds.*

Proof By counting the replication numbers r_i for elements of the i^{th} group, the replication number for elements in G_1 is obtained from $r_1 = \frac{4\lambda_1+(2n+1)\lambda_2}{3}$. Since r_1 is a positive integer, then $4\lambda_1$+(2n+1)$\lambda_2 \equiv 0$ (mod 3). Again $3|4\lambda_1$ remains and for the case of $3|(2n+1)\lambda_2$, gives (2n+1)$\lambda_2 \equiv 0$ (mod 3). Now, consider the parameters of the GDD$(v, g, m, k; \lambda_1, \lambda_2)$ where block size $k = 4$, group size $g = n$ and number of groups $m = n+1$, let us proceed with the proof. Assuming that a GDD$(5, n, n+1, 4; \lambda_1, \lambda_2)$ exists, then the design has the following parameters: block size $k = 4$, group size $g = n$, number of groups $m = n+1$, total number of elements $V = gm = n(n+1)$ together with indices λ_1, and λ_2. Consider an arbitrary element x in the GDD. let r be the number of blocks containing x. Now, let us consider the possible values of $n \pmod 3$.

Case 1. $n \equiv 0 \pmod 3$.

The first congruence becomes $(0-1)\lambda_1 + 0^2\lambda_2 \equiv -\lambda_1 \equiv 0$ and this implies $\lambda_1 \equiv 0 \pmod 3$. The second congruence becomes: $\lambda_1 + (2(0)+1)\lambda_2 \equiv 0 + \lambda_2 \equiv \lambda_2 \pmod 3$. Since $\lambda_1 \equiv 0$, from $3r = (n-1)\lambda_1 + n^2\lambda_2$, we have $3r = (-1)(0) + (0)\lambda_2 = 0$, which gives no direction constraint on λ_2 . However, for the GDD to exist, the parameters must be consistent.

Case 2. $n \equiv 1 \pmod 3$.

The first congruence becomes $(1-1)\lambda_1 + 1^2\lambda_2 \equiv 0\lambda_1 + \lambda_2 \equiv \lambda_2 \equiv 0 \pmod 3$. The second congruence becomes: $\lambda_1 + (2(1)+1)\lambda_2 \equiv \lambda_1 + 3\lambda_2 \equiv \lambda_1 + 0\lambda_2 \equiv \lambda_1 \pmod 3$. From $3r = (n-1)\lambda_1 + n^2\lambda_2$, we have $3r = (0)\lambda_1 + 1\lambda_2$. Since $3r \equiv 0 \pmod 3$, then $\lambda_2 \equiv 0 \pmod 3$ which is consistent and thus $\lambda_1 \equiv 0 \pmod 3$.

Case 3. $n \equiv 2 \pmod 3$.

The first congruence becomes $(2-1)\lambda_1 + 2^2\lambda_2 \equiv \lambda_1 + 4\lambda_2 \equiv \lambda_1 + \lambda_2 \equiv 0$. This implies $\lambda_1 \equiv -\lambda_2 \equiv 2\lambda_2 \pmod 3$. The second congruence becomes: $\lambda_1 + (2(2)+1)\lambda_2 \equiv \lambda_1 + 5\lambda_2 \equiv \lambda_1 + 2\lambda_2 \pmod 3$. Now, substituting $\lambda_1 \equiv 2\lambda_2$, we get $2\lambda_2 + 2\lambda_2 \equiv 4\lambda_2 \equiv \lambda_2 \pmod 3$. From $3r = (n-1)\lambda_1 + n^2\lambda_2$, we have $3r = (1)\lambda_1 + (1)\lambda_2 = \lambda_1 + \lambda_2$. Since $3r \equiv 0$, $\lambda_1 + \lambda_2 \equiv 0$, which is consistent with $\lambda_1 \equiv -\lambda_2 \pmod 3$. Thus, $\lambda_2 \equiv 0 \pmod 3$, which implies $\lambda_1 \equiv 0 \pmod 3$ and this satisfies the necessary condition. □

Remark 3.1 *In all cases, the condition* $(n-1)\lambda_1 + 2^2\lambda_2 \equiv 0 \pmod 3$ *and this leads to* $\lambda_1 + (2n+1)\lambda_2 \equiv 0 \pmod 3$ *which gives the necessary condition* $4\lambda_1 + (2n+1)\lambda_2 \equiv 0 \pmod 3$ *and* $(2n+1)\lambda_2 \equiv 0 \pmod 3$.

Theorem 3.2 *A GDD* $(5, n, n+1, 4; \lambda_1, \lambda_2)$ *exists if the necessary condition* $(n-1)\lambda_1 + (n+6)\lambda_2 \equiv 0 \pmod 3$ *holds.*

Proof The replication number for elements in G_2 is obtained from $r_2 = \frac{(n-1)\lambda_1 + (6+n)\lambda_2}{3}$. Since r_2 is a positive integer, then $(n-1)\lambda_1 + (6+n)\lambda_2 \equiv 0 \pmod 3$. First, we prove that the necessary conditions for the GDD$(5, n, n+1, 4; \lambda_2, \lambda_2)$ exists. Let $V = n(n+1)$ be the total number of elements. Let b be the number of blocks. We derive the necessary conditions by considering the counting of pairs. Counting pairs involving a specific element: Consider an arbitrary element x which belongs to one group of size $n+1$. Within its group, there are n other elements . Each pair involving x and another element in the same group appears in λ_1 blocks. There are $n-1$ other groups, each of size $n+1$. So, there are $(n-1)(n+1)$ elements in the other groups. Each pair involving x and an element from a different group appears in λ_2 blocks. Now, let us count how many times x appears in all the blocks. Let r be the number of blocks containing x. Each block has size 4, so counting the pairs involve x in two ways, i.e.,

$$\begin{aligned} (4-1)r &= (n)\lambda_1 + (n-1)(n+1)\lambda_2, \\ 3r &= (n)\lambda_1 + (n^2-1)\lambda_2. \end{aligned} \tag{1}$$

Since the total number of pairs is $\binom{n(n+1)}{2}$ and each block contains $\binom{4}{2} = 6$ pairs, we also relate the total number of pairs to λ_1 and λ_2, i.e.,

$$\begin{aligned} b\binom{4}{2} &= n\binom{n+1}{2}\lambda_1 + \binom{n}{2}(n+1)^2\lambda_2, \\ 6b &= n(\frac{(n+1)n}{2}\lambda_1) + \frac{n(n-1)(n+1)^2}{2}\lambda_2, \\ 12b &= n^2(n+1)\lambda_1 + n(n-1)(n+1)^2\lambda_2 \end{aligned} \tag{2}$$

Also, by counting the total number of elements in all blocks, we have $4b = vr = n(n+1)r$, so $b = \frac{n(n+1)r}{4}$, substituting this into

$$12\frac{n(n+1)r}{4} = n^2(n+1)\lambda_1 + n(n-1)(n+1)^2\lambda_2, \tag{3}$$

i.e.,

$$3nr(n+1) = n^2(n+1)\lambda_1 + n(n-1)(n+1)^2\lambda_2.$$

Dividing it by $n(n+1)$ (assuming $n \geq 1$), we get

$$3r = n\lambda_1 + (n-1)(n+1)\lambda_2 = n\lambda_1 + (n^2-1)\lambda_2.$$

However, this is consistent with equation (1). Now, let us look at the modular conditions. The proof involves counting arguments modulo 3, considering the number of blocks and the nature of the design. The condition $(n-1)\lambda_1 + (n+6)\lambda_2 \equiv 0 \pmod 3$ can be re-written as $(n-1)\lambda_1 + n\lambda_2 \equiv 0 \pmod 3$ (since $6 \equiv 0 \pmod 3$). Given $\lambda_1 \equiv 0 \pmod 3$, this simplifies to n$\lambda_1 \equiv 0 \pmod 3$. From th second derived condition $(2n+1)\lambda_2 \equiv (-n+1)\lambda_2 \equiv 0 \pmod 3$. If the $gcd(-n+1, n) = gcd(-n+1+n, n) = gcd(1, n) = 1$, then we must have $\lambda_2 \equiv 0 \pmod 3$, which satisfies $n\lambda_2 \equiv 0 \pmod 3$. If the $gcd(-n+1, n) \neq 1$, then $n \equiv 1 \pmod 3$. In this case, $(-n+1)\lambda_2 \equiv 0 \pmod 3$, and the second condition is satisfied regardless of λ_2. The first condition becomes: $(1)\lambda_2 \equiv \lambda_2 \equiv 0$ (mod), so $n\lambda_2 \equiv 1.0 \equiv 0 \pmod 3$. Thus, the necessary condition $(n-1)\lambda_1 + (n+6)\lambda_2 \equiv 0 \pmod 3$ hold. □

Theorem 3.3 *A GDD* $(5, n, n+1, 4; \lambda_1, \lambda_2)$ *exists if the necessary condition* $n\lambda_1 + (5+n)\lambda_2 \equiv 0 \pmod 3$ *holds.*

Proof The replication number for elements in G_3 is obtained from $r_3 = \frac{n\lambda_1+(5+n)\lambda_2}{3}$. Since r_3 is a positive integer, then nλ_1+(5+n)$\lambda_2 \equiv 0 \pmod 3$. For a GDD$(5, n, n+1, 4 : \lambda_1, \lambda_1)$, we have the fundamental equations: $3r = (n)\lambda_1 + (n^2-1)\lambda_2$ and $12b = n^2(n+1)\lambda_1 + n(n-1)(n+1)^2\lambda_2$ and the derived necessary modular condition: $4\lambda_1 + (2n+1)\lambda_2 \equiv 0 \pmod 3$ and this implies $\lambda_1 + (2n+1)\lambda_2 \equiv 0 \pmod 3$ which gives $(2n+1)\lambda_2 \equiv 0 \pmod 3$. From these, we deduce that $\lambda_1 \equiv 0 \pmod 3$. Now, we want to show that $n\lambda_1 + (5+n)\lambda_2 \equiv 0 \pmod 3$ must hold. This condition simplifies to

$$n\lambda_1 + (2+n)\lambda_2 \equiv 0 \pmod 3. \tag{$*$}$$

Substituting $\lambda_1 \equiv 0 \pmod 3$ into (*), gives $n(0) + (2+n)\lambda_2 \equiv 0 \pmod 3$ and this yields

$$(2+n)\lambda_2 \equiv 0 \pmod 3. \tag{$**$}$$

Now, let us consider the second derived modular in necessary condition one $(2n+1)\lambda_2 \equiv 0 \pmod 3$, which is equivalent to $(-n+1)\lambda_2 \equiv 0 \pmod 3$. We have two congruences involving λ_2: $(n+2)\lambda_2 \equiv 0 \pmod 3$ and $(-n+1)\lambda_2 \equiv 0 \pmod 3$. Adding these two congruences gives $(n+2)\lambda_2 + (-n+1)\lambda_2 \equiv 0 + 0 \pmod 3$ and this gives $3\lambda_2 \equiv 0 \pmod 3$ and this implies $0 \equiv 0 \pmod 3$. This does not directly give information about n or λ_2. Now, let us consider the possible values of $n \pmod 3$.

Case 1. $n \equiv 0 \pmod 3$.

From (**), $(2+0)\lambda_2 \equiv 2\lambda_2 \equiv 0 \pmod 3$ and this implies $\lambda_2 \equiv 0 \pmod 3$. The target condition (*) becomes: $0(0) + (2+0)(0) \equiv 0 \pmod 3$, which holds.

Case 2. $n \equiv 1 \pmod 3$.

From (**), $(2+1)\lambda_2 \equiv 3\lambda_2 \equiv 0 \pmod 3$. This gives no information about λ_2. Now, from second derived condition: $(-1+1)\lambda_2 \equiv 0\lambda_2 \equiv 0 \pmod 3$. This also gives no information about λ_2. The target condition (*) becomes: $(1(0) + (2+1))\lambda_2 \equiv 3\lambda_2 \equiv 0 \pmod 3$, which holds.

Case 3. $n \equiv 2 \pmod 3$.

From (**), $(2+2)\lambda_2 \equiv 4\lambda_2 \equiv 0 \pmod 3$. The target condition (*) becomes: $(2)(0) + (2+2)(0) \equiv 0 \pmod 3$, which holds. □

Remark 3.2 *In all cases, the necessary condition* $n\lambda_1 + (5+n)\lambda_2 \equiv 0 \pmod 3$ *is satisfied and thus it is a necessary condition for the existence of the the GDD.*

Theorem 3.4 *A GDD* $(5, n, n+1, 4; \lambda_1, \lambda_2)$ *exists if the necessary condition* $(n^2+10)\lambda_1 + (n^2+11n+5)\lambda_2 \equiv 0 \pmod 6$ *holds.*

Proof The number of blocks of the GDD is obtained from b $= \frac{(n^2+10)\lambda_1 + (n^2+11n+5)\lambda_2}{6}$ and that $(n^2+10)\lambda_1 + (n^2+11n+5)\lambda_2 \equiv 0(mod 6)$. For a GDD$(5, n, n+1, 4 : \lambda_1, \lambda_1)$, we have the fundamental equations: $3r = (n)\lambda_1 + (n^2-1)\lambda_2$ and $12b = n^2(n+1)\lambda_1 + n(n-1)(n+1)^2\lambda_2$ and the necessary modular conditions derived earlier: For the number of blocks b to be an integer, $12b \equiv 0 \pmod{12}$, which implies $n^2(n+1)\lambda_1 + n(n-1)(n+1)^2\lambda_2 \equiv 0 \pmod{12}$. This congruence modulo 12 gives us information modulo its divisors, including 2. Let us consider the equation modulo 2: $n^2(n+1)\lambda_1 + n(n-1)(n+1)^2\lambda_2 \equiv 0 \pmod 2$.

Case 1. $n \equiv 0 \pmod 2$.

We have, $0^2(0+1)\lambda_1 + (0)(0-1)(0+1)^2\lambda_2 \equiv 0 \pmod 2$, which implies $0 \equiv 0 \pmod 2$. This gives no constraint on λ_1 or λ_2 when n is even.

Case 2. $n \equiv 1 \pmod 2$.

We have, $1^2(1+1)\lambda_1 + (1)(1-1)(1+1)^2\lambda_2 \equiv 1 \pmod 2$, which implies $1(0)\lambda_1 + (1)(0)(1+1)^2\lambda_2 \equiv 0 \pmod 2$ and this gives $0 \equiv 0 \pmod 2$. This also gives no constraint on λ_1 or λ_2 when n is odd. Now, let us consider the target conditions modulo 2 and 3 separately.

Modulo 2. In this case, we have $(n^2+0)\lambda_1 + (n^2+n+1)^2\lambda_2 \equiv 0 \pmod 2$, which implies $(n)^2\lambda_1 + (n^2+n+1)^2\lambda_2 \equiv 0 \pmod 2$.

If n is even, $0\lambda_1 + (0^2+0+1)\lambda_2 \equiv 0 \pmod 2$, which gives $\lambda_2 \equiv 0 \pmod 2$; If n is odd, $1\lambda_1 + (1^2+1+1)\lambda_2 \equiv 0 \pmod 2$ and this gives $\lambda_1 + (1^2+1+1)\lambda_2 \equiv 0 \pmod 2 \equiv \lambda_1 + 2\lambda_2 \equiv \lambda_1 + \lambda_2 \equiv 0 \pmod 2$. So, $\lambda_1 \equiv \lambda_2 \pmod 2$.

Modulo 3. In this case, we have $(n^2+1)\lambda_1 + (n^2+2n+2)\lambda_2 \equiv 0 \pmod 3$. Using $\lambda_1 \equiv 0$

(mod 3); , $(n^2+1)(0)+(n^2+2n+2)\lambda_2 \equiv 0 \pmod 3$ and this implies $(n^2+2n+2)\lambda_2 \equiv 0 \pmod 3$. Now, we know that $n\lambda_2 \equiv 0 \pmod 3$, then,

If $n \equiv 0 \pmod 3$, $(0+0+2)\lambda_2 \equiv 2\lambda_2 \equiv \lambda_2 \equiv 0 \pmod 3$; If $n \equiv 1 \pmod 3$: then, $(1+2+2)\lambda_2 \equiv 5\lambda_2 \equiv 2\lambda_2 \equiv \lambda_2 \equiv 0 \pmod 3$. If $n \equiv 2 \pmod 3$: then, $(4+4+2)\lambda_2 \equiv 10\lambda_2 \equiv 2\lambda_2 \equiv \lambda_2 \equiv 0 \pmod 3$. So, we have $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$. Now, let us check the target condition modulo 6 with these congruences: $(n^2+10)\lambda_1+(n^2+11n+5)\lambda_2 \equiv 0 \pmod 6$. If $\lambda_1 \equiv 0 \pmod 3$ and $\lambda_2 \equiv 0 \pmod 3$, then the target condition becomes $(n^2+10)(3k_1)+(n^2+11n+5)(3k_2) \equiv 3\big[(n^2+10)k_1+(n^2+11n+5)k_2\big] \equiv 0 \pmod 3$. We need to show that it is also 0 (mod 2). If λ_1 and λ_2 are even, then the target condition is 0 (mod 2). If λ_1 and λ_2 are odd, then the modulo 2 becomes: $(n^2+0)(1)+(n^2+n+1)(1) \equiv (n^2+n^2+n+1) \equiv n+1 \equiv 0 \pmod 2$, so n is odd and thus, the necessary condition $(n^2+10)\lambda_1+(n^2+11n+5)\lambda_2 \equiv 0 \pmod 6$ hold. □

Remark 3.3. *Combining these conditions modulo 2 and 3 gives the required necessary conditions* $(n^2+10)\lambda_1+(n^2+11n+5)\lambda_2 \equiv 0 \pmod 6$ *for* λ_1 *and* λ_2 *either being even or odd.*

These necessary conditions on b and r_i determine possibilities for the parameter n and the indices λ_1 and λ_2 which are summarized in the Table 3.1 where "Does not" means that the design does not exist for any value of n.

Table 1. The restrictions on n for GDD$(5,n,n+1,4;\lambda_1,\lambda_2)$

$\lambda_1\backslash\lambda_2$	3	6	9	12	15	18
3	Does not	Exist	Does not	Exist	Does not	Exist
6	Does not	Exist	Does not	Exist	Does not	Exist
9	Does not	Exist	Does not	Exist	Does not	Exist
12	Does not	Exist	Does not	Exist	Does not	Exist
15	Does not	Exist	Does not	Exist	Does not	Exist
18	Does not	Exist	Does not	Exist	Does not	Exist

Theorem 3.5 *Necessary conditions are sufficient for a GDD* $(n,3,4;\lambda,2\lambda)$ *for* $\lambda_1=\lambda_2 \equiv 0 \pmod 6$*, then a BIBD* $(5+n,4,\lambda)$*, a BIBD* $(6+n,4,\lambda)$ *and a BIBD* $(2n+1,4,\lambda)$ *exists.*

Proof Let $\lambda_1=t$, and $\lambda_2=2t$. The blocks of t copies of a 4-$(n,4,6)$ on G_1 as well as on G_2 and G_3 give the required 4-GDDs where groups $G_1=a_1,\cdots,a_n$, $G_2=b_1,\cdots,b_n$ and $G_3=c_1,\cdots,c_n$. It is well known that a BIBD $(n,4,6)$ exists for $n \geq 5$ [10] and thus a GDD$(5,n,n+1,4;3,6)$ will always exist. Hence a GDD$(5,n,n+1,4;3t,6t)$ always exists for all positive integers, t. □

Here, we typically denote these groups as $G_1,G_2,G_3,\cdots,G_m$ where m is the number of groups, G_i is a subset of treatment set, the union of all the groups is the entire treatment set $\bigcup_{i=1}^{m} G_i = V$ written as $V=G_1\cup G_2\cup\cdots\cup G_i$, and $G_i\cap G_j=\emptyset$ for$i\neq j$.

Example 3.1 Construction of GDD(5,6,7,4;3,6) exists with $r_1=30$, $r_2=29$, and $r_3=28$. We now show that if $n \equiv 0 \pmod 6$. Using our necessary conditions from theorem 3.4 then a GDD (5,6,7,4;3,6) exists through the following construction. When $\lambda_1 = 3$, $\lambda_2 \equiv 0 \pmod 6$ implies $\lambda_2 = 6$t for t ≥ 1. The GDD has groups, G_1 = {0,1,2,3,4}, G_2 = {5,6,7,8,9,10} and G_3 = {11, 12,13,14,15,16,17}. Then using direct construction of blocks of a BIBD(40,4,1), we get 130 blocks.

[0, 1, 2, 12]	[0, 3, 6, 9]	[0, 4, 8, 10]	[0, 5, 7, 11]	[0, 13, 26, 39]	[0, 14, 25, 28]
[0, 15, 27, 38]	[0, 16, 22, 32]	[0, 17, 23, 34]	[0, 18, 24, 33]	[0, 19, 29, 35]	[0, 20, 31, 37]
[0, 21, 30, 36]	[1, 3, 8, 11]	[1, 4, 7, 9]	[1, 5, 6, 10]	[1, 13, 28, 38]	[1, 14, 27, 39]
[1, 15, 25, 26]	[1, 16, 24, 34]	[1, 17, 22, 33]	[1, 18, 23, 32]	[1, 19, 31, 36]	[1, 20, 30, 35]
[1, 21, 29, 37]	[2, 3, 7, 10]	[2, 4, 6, 11]	[2, 5, 8, 9]	[2, 13, 25, 27]	[2, 14, 26, 38]
[2, 15, 28, 39]	[2, 16, 23, 33]	[2, 17, 24, 32]	[2, 18, 22, 34]	[2, 19, 30, 37]	[2, 20, 29, 36]
[2, 21, 31, 35]	[3, 4, 5, 12]	[3, 13, 32, 35]	[3, 14, 34, 37]	[3, 15, 33, 36]	[3, 16, 29, 39]
[3, 17, 25, 31]	[3, 18, 30, 38]	[3, 19, 22, 26]	[3, 20, 23, 28]	[3, 21, 24, 27]	[4, 13, 34, 36]
[4, 14, 33, 35]	[4, 15, 32, 37]	[4, 16, 31, 38]	[4, 17, 30, 39]	[4, 18, 25, 29]	[4, 19, 24, 28]
[4, 20, 22, 27]	[4, 21, 23, 26]	[5, 13, 33, 37]	[5, 14, 32, 36]	[5, 15, 34, 35]	[5, 16, 25, 30]
[5, 17, 29, 38]	[5, 18, 31, 39]	[5, 19, 23, 27]	[5, 20, 24, 26]	[5, 21, 22, 28]	[6, 7, 8, 12]
[6, 13, 22, 29]	[6, 14, 23, 31]	[6, 15, 24, 30]	[6, 16, 26, 35]	[6, 17, 28, 37]	[6, 18, 27, 36]
[6, 19, 32, 39]	[6, 20, 25, 34]	[6, 21, 33, 38]	[7, 13, 24, 31]	[7, 14, 22, 30]	[7, 15, 23, 29]
[7, 16, 28, 36]	[7, 17, 27, 35]	[7, 18, 26, 37]	[7, 19, 34, 38]	[7, 20, 33, 39]	[7, 21, 25, 32]
[8, 13, 23, 30]	[8, 14, 24, 29]	[8, 15, 22, 31]	[8, 16, 27, 37]	[8, 17, 26, 36]	[8, 18, 28, 35]
[8, 19, 25, 33]	[8, 20, 32, 38]	[8, 21, 34, 39]	[9, 10, 11, 12]	[9, 13, 16, 19]	[9, 14, 17, 20]
[9, 15, 18, 21]	[9, 22, 35, 39]	[9, 23, 25, 37]	[9, 24, 36, 38]	[9, 26, 29, 32]	[9, 27, 30, 33]
[9, 28, 31, 34]	[10, 13, 17, 21]	[10, 14, 18, 19]	[10, 15, 16, 20]	[10, 22, 37, 38]	[10, 23, 36, 39]
[10, 24, 25, 35]	[10, 26, 30, 34]	[10, 27, 31, 32]	[10, 28, 29, 33]	[11, 13, 18, 20]	[11, 14, 16, 21]
[11, 15, 17, 19]	[11, 22, 25, 36]	[11, 23, 35, 38]	[11, 24, 37, 39]	[11, 26, 31, 33]	[11, 27, 29, 34]
[11, 28, 30, 32]	[12, 13, 14, 15]	[12, 16, 17, 18]	[12, 19, 20, 21]	[12, 22, 23, 24]	[12, 25, 38, 39]
[12, 26, 27, 28]	[12, 29, 30, 31]	[12, 32, 33, 34]	[12, 35, 36, 37]	*130 Blocks.*	

(1) **Necessary conditions are sufficient for** a GDD $(n, 3, 4; \lambda_1, \lambda_2)$ when $\lambda_1 = \lambda_2$.

We can view that the existence of a GDD with unequal group sizes as a consequence of the existence of GDDs with equal group sizes. Applying Wilson' s Existence Theorem for GDDs, a GDD $(k, n, m; \lambda_1, \lambda_1)$ exists for sufficiently large $v = nm$ if necessary conditio
ns are satisfied. For small order v, known existence results cover specific cases. Now, our total $v = 5+6+7 = 18$ is the modest with $\lambda_1 = 6$ and $\lambda_2 = 6$. Since both pair frequencies are equal, this GDD behaves similarly to a balanced incomplete block design (BIBD) within and between groups, but with group structure restrictions. Thus, we construct 13 blocks of BIBD (13,4,1), add 63 blocks of BIBD (28,4,1). The remaining blocks are formed by constructing 77 blocks of BIBD (22,4,2). This is because from from known theorem 2.2 $b = \frac{vr}{k} = \frac{\lambda v(v-1)}{k(k-1)}$, which gives in total 130 blocks.

Remark 3.4 *A GDD(*$5, n, n+1, 4; 0, \lambda_2$*) does not exist. This is because there are only three groups and the block size is four. So, each block must contain at least a pair from the same group (*$\lambda_1 \geq 6$*) to complete the block size. A GDD*$(5, n, n+1, 4; \lambda_1, 0)$ *exists as a* $(2n+6, 4, \lambda_1)$ *BIBD for particular values of* n *and* λ_1*. So, a GDD*$(5, n, n+1, 4; 0, 0)$ *does not exist.*

Example 3.2 Construction of GDD (5,6,7,4;6,6) exists with $r_1 = 34$, $r_2 = 34$, $r_3 = 34$ and $b = 153$ blocks. The GDD has groups, $G_1 = \{0, 1, 2, 3, 4\}$, $G_2 = \{5, 6, 7, 8, 9, 10\}$ and $G_3 = \{11, 12, 13, 14, 15, 16, 17\}$. The total number of blocks can be obtained by adding 13 blocks of a BIBD(13,4,1) together with 63 blocks of a (28,4,1) BIBD and then plus 77 blocks of a BIBD(22,4,2).

In general, a GDD$(5, 6t, 6t+1, 4; 6t, 6t)$ exists with $r_1 = 24t^2 + 10t$, $r_2 = 24t^2 + 10t$, $r_3 = 24t^2 + 10t$ and $b = 72t^3 + 66t^2 + 15t$ where t is a positive integer.

(2) Necessary conditions are sufficient for a GDD $(5, n, n+1, 4; 2\lambda_1, \lambda_2)$ **when** $\lambda_1 \geq \lambda_2$

Theorem 3.6 *A GDD*$(5, n, n+1, 4; \lambda_1, \lambda_2)$ *exist for* $\lambda_1 \geq \frac{(n^2+10)\lambda_1 + (n^2+11n+5)\lambda_2}{6}$.

Proof The design has three groups of size $n_1 \geq 5$ and block size 4, then each block must have at least one first associate pair. This means that the total number of first associate pairs is at least equal to the number of blocks. Since there are $(n^2+10)\lambda_1$ first associate pairs and $\frac{(n^2+10)\lambda_1 + (n^2+11n+5)\lambda_2}{6}$ blocks, and so

$$\begin{aligned} (n^2+10)\lambda_1 &\geq \frac{(n^2+10)\lambda_1 + (n^2+11n+5)\lambda_2}{6}, \\ 5(n^2+10)\lambda_1 &\geq (n^2+11n+5)\lambda_2, \\ \lambda_1 &\geq \frac{(n^2+11n+5)\lambda_2}{(5n^2+50)}. \end{aligned}$$

This completes the proof. □

Corollary 3.1 *From* $b = \frac{vr}{k}$*, if GDD*$(5, n, n+1, 4; \lambda_1, \lambda_2)$ *exists, then it has* $(n^2+10)\lambda_1$ *first associate pairs and* $(n^2+11n+5)\lambda_2$ *second associate pairs with* $b \leq (n^2+10)\lambda_2$.

Proof The design has b blocks and $(n^2+10)\lambda_2$ first associate pairs. The total number of blocks cannot exceed the total number of first associate pairs. Thus $b \leq (n^2+10)\lambda_2$. □

Remark 3.5 *A GDD* $(5, n, n+1, 4; 2\lambda, \lambda)$ *exists if and only if BIBD*$(5+n, 4, \lambda)$ *and BIBD*$(2n+1, 4, \lambda)$ *exists. Here, a BIBD(*$n, 4, 6$*) exists for* $n \geq 5$ *and thus a GDD*$(5, n, n+1, 4; 12, 6)$ *will always exist. Hence a GDD*$(5, n, n+1, 4; 12t, 6t)$ *always exists for all positive integers,* t.

Example 3.3 Construction of GDD(5,6,7,4;9,12) exists with $r_1 = 64$, $r_2 = 63$, $r_3 = 62$ and $b = 283$ blocks. The groups of the GDD are G_1 = {0,1,2,3,4}, G_2 = {5,6,7,8,9,10} and G_3 = {11,12,13,14,15,16,17}. We construct the total number of blocks by taking the 18 blocks of a BIBD(9,4,3) on $G_1 \cup G_2$ add 55 blocks of a BIBD(11,4,6) on $G_1 \cup G_3$ together with two copies of the 105 blocks of a BIBD(15,4,6) on $G_2 \cup G_3$.

Example 3.4 Construction of GDD (5,6,7,4;18,18) exists with $r_1 = 102$, $r_2 = 102$, $r_3 =$

102 and $b = 459$ blocks. The total number of blocks can be obtained by constructing 210 blocks as 203 blocks of BIBD(29,4,3) and then add 7 blocks of BIBD(7,4,2) by normalizing a regular hadamard matrix of order 8 plus 242 blocks which has groups, $G_1 = \{0,1,2,3,4\}$, $G_2 = \{5,6,7,8,9,10\}$ and $G_3 = \{11,12,13,14,$
$15,16,17\}$ that will form 153 blocks and these blocks can be obtained by adding 13 blocks of BIBD(13,4,1) plus 63 blocks of (28,4,1)BIBD together with 77 blocks of BIBD(22,4,2) and then plus 50 blocks of BIBD(25,4,1) together with 39 blocks of BIBD (13,4,3). We generalise that, a GDD$(5, 6t, 6t + 1, 4; 18t, 18t)$ exists with $r_1 = 72t^2{+}30t$, $r_2 = 72t^2{+}30t$, $r_3 = 72t^2{+}30t$ and $b = 216t^3 + 198t^2 + 45t$ where t is a positive integer.

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