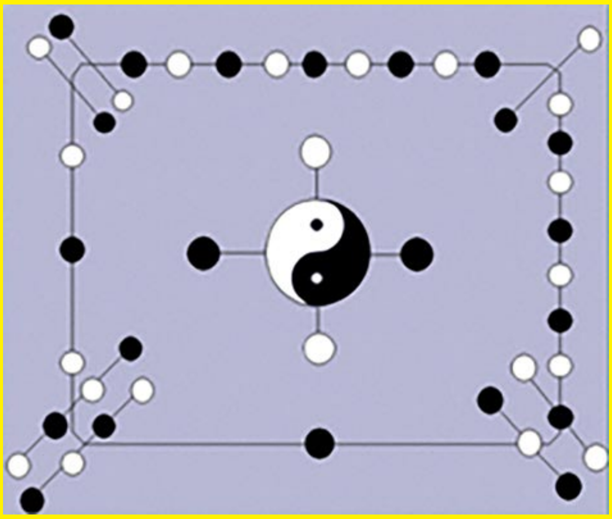
ISSN 1937 - 1055

VOLUME 1 - VOLUME 2, 2025

INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

June, 2025

Vol.1-Vol.2, 2025 ISSN 1937-1055

International Journal of

Mathematical Combinatorics

(www.mathcombin.com)

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

June, 2025

Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.L.F. MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

Smarandache multi-spaces and Smarandache geometries with applications to other sciences;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Mathematical theory on gravitational fields and parallel universes;

Applications of Combinatorics to mathematics and theoretical physics.

Generally, papers on applications of combinatorics to other mathematics and other sciences are welcome by this journal.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews (USA), Zentralblatt Math (Germany), Index EuroPub (UK), Referativnyi Zhurnal (Russia), Mathematika (Russia), EBSCO (USA), Google Scholar, Baidu Scholar, Directory of Open Access (DoAJ), International Scientific Indexing (ISI, impact factor 2.012), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA), CNKI(China).

Subscription A subscription can be ordered by an email directly to

Linfan Mao

The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science Beijing, 100190, P.R.China, and also the President of Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA

Email: maolinfan@163.com

Price: US$48.00

Famous Words:

Give me space, time and logarithms, I can create a universe.

By *Galileo Galilei*, an Italian philosopher, astronomer and mathematician.

International J.Math. Combin. Vol.1-Vol.2(2025), 01-23

Combinatorics – A Mathematical Approach for Holding on the Realty of Thing in the Universe

Linfan MAO

1. Chinese Academy of Mathematics and System Science, Beijing 100190, P.R. China
2. Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA

E-mail: maolinfan@163.com

Abstract: Usually, one holds a thing T on its appearance or characters and particularly, by mathematics. But *is the mathematical reality equal to the reality of thing T?* The answer is not certain because the recognition of human on thing T is only a local or conditional one, implied in the fable of blind men with an elephant, i.e., the sophist told the blind men that an elephant has all characteristics that they are talking about. Then, *what is the significance of this fable?* It lies essentially in the shape of an elephant and generally, the reality of a thing is a combinatorial one, i.e., combinatorics is priori to the recognition of human because all of us are similar to the blind in front of a thing. In this report, I discuss the *non-harmonious group* with *Smarandache multispace* inherited a topological graph G in first, generalize it to *G-flows* $\overrightarrow{G}^L$ or networks $\overrightarrow{N}$ with vector flows and then, *continuity flows* $\overrightarrow{G}^L$, i.e., mathematics over 1-dimensional topological graphs, which extends the classic mathematics over combinatorial structures $\overrightarrow{G}$. This report surveys how to establish such a system by viewing continuity flow $\overrightarrow{G}^L$ as a mathematical element for establishing the Banach flow space, Hilbert flow space over topological graphs $\overrightarrow{G}$ and then, how to apply it to generalize a few of important conclusions in functional analysis such as those of the inverse mapping theorem, closed graph theorem and the Hahn-Banach theorem for providing the recognition of human on the reality of things, including the subdivision of a matter M into elementary particles with a mathematical supporting, which forms a complex network on M in physics, and shows also the 12 meridians on human body in traditional Chinese medicine is an example of G-flows or generally, continuity flows with dynamic equations.

Key Words: Combinatorial notion, contradiction, non-solvable system of equations, non-harmonious group, Smarandachely denied axiom, Smarandache multispace, G-flow, continuity flow, mathematical combinatorics, 12 meridians on human body.

AMS(2010): 05C10, 05C21, 35A08, 46B25, 51D20, 51H20, 51P05.

§1. Introduction

Usually, one holding a thing T on its appearance or characters and particularly, by mathematical reality. Then, *what is the reality of a thing T?* In dictionary, the word "*reality*" is explained to

[1]Reported at *The 10th International Combinatorics and Graph Theory Conference (CGT 2025)*, May 23-25, 2025, Xian, P.R.China

[2]Received January 15, 2025. Accepted March 16, 2025

be the state of things as they actually exist, including everything that is and has been, whether or not it is observable or comprehensible. *Can one really hold on the reality of thing T?* The answer is not certain unless the mathematical reality. Generally, a thing T is multilateral or complex one but the recognition of human on thing T has certain limitations, i.e., it is only the local rather than the whole. Then, *how to solve this problem and how to cross the gap from the local to the whole?* The answer is nothing else but the combinatorics.

1.1. Combination Prior to Reductionism. As we all known, the reduction on a matter T is subdivided it into the minimum recognizable elements so that humans can understand the reality of matter T. For example, to subdivide a matter $T \to$ molecule $\to$ atom $\to$ nucleus $\to$ proton and neutron $\to$ elementary particle consisting of quarks, leptons with interaction quanta including photons and other particles of mediated interactions ([35]), and a living $L \to$ biological macromolecule $\to$ cell or gene, such as those shown in Figure 1 on subdividing of matter with elementary particles.

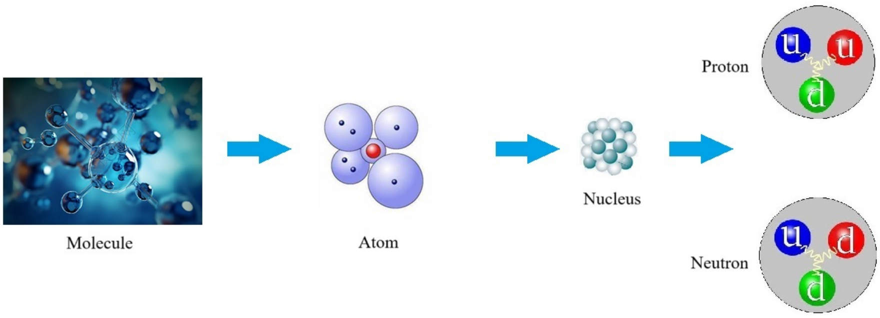


Figure 1

Actually, there is an assumption implied in reductionism without proof, i.e., the reality of matter T can be held if the behavior of elementary particles over a topological 1-dimensional structure is recognized locally by humans. For example, the models of proton, neutron are both over graphs K_3 by quarks in Figure 1. But, *is this assumption right and so, one can holds on the reality of matter T by reductionism?* The answer is not certain unless all elementary particles are in stationary or synchronization.

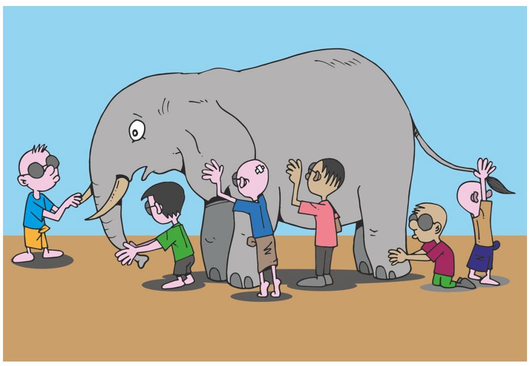


Figure 2

For holding the whole with the local, we all learned a famous fable of the *blind men with an elephant* in elementary school, which narrates that there are 6 blind men wanted to know an elephant looked like by feeling its body one by one, see Figure 2. Their recognitive process is like this, namely the 1st one touched the elephant's tooth and claimed "*an elephant is like a big, thick and smooth radish*, the 2nd one touched the elephant's trunk and claimed "*an elephant is like a tube*, the 3rd one touched the elephant's ear and claimed "*an elephant is like a big fan*, the 4th one touched the elephant's belly and claimed "*an elephant is clearly like a wall*, the 5th one touched the elephant's leg and claimed "*an elephant is clearly like a big pillar* and finally, the 6th one touched the elephant's tail and claimed "*an elephant is like* *a piece of grass rope*. Each of them believed the perception of himself on the shape of elephant is right and insisting on his own opinion, kept an endless quarrelling with others. At this time, a sophist came forward and told them "*why you are thinking about the elephant's shape different is because each of you touches the different part of the elephant's body. Essentially, an elephant has all characteristics that you are talking about!* This fable shows the limitation of recognition of blind man compared to the normal, namely the elephant shape in eyes of the blind men is very different from, even a bit ridiculous to that of the normal human. Then, *what is the elephant shape in eyes of the sophist by that of the blind men?* The answer is certainly nothing else but a union of characteristics recognized locally by the 6 blind men, namely the combination of all the local to form a whole such as those shown in Figure 3.

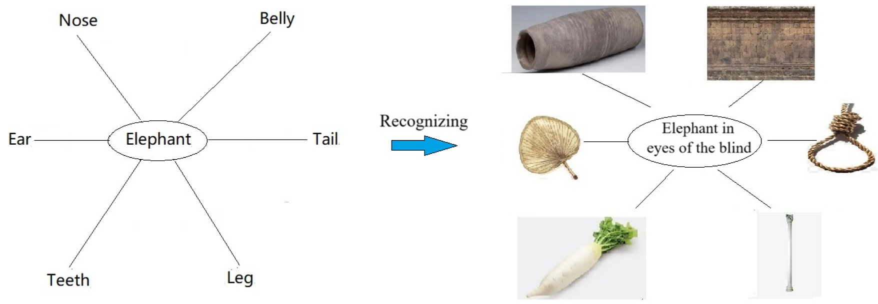


Figure 3

$$\textbf{An elephant} = \{\textit{4 big pillars}\}\bigcup\{\textit{1 gross rope}\}\bigcup\{\text{1 tubes}\}$$
$$\bigcup\{\textit{2 big fans}\}\bigcup\{\textit{1 big wall}\}\bigcup\{\textit{2 big radishes}\} \qquad (1.1)$$

with a combinatorial structure

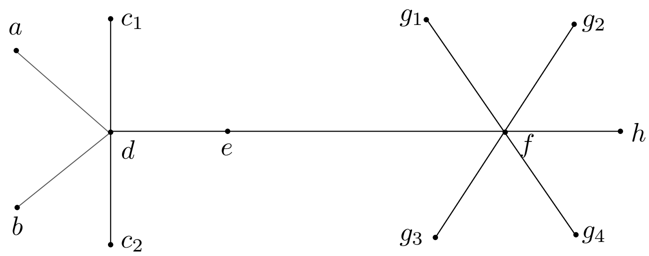


Figure 4

where a_1, a_2 = big radishes, b_1, b_2 = big fans, c= elephant's head, d = elephant's neck, e = big wall, g_1, g_2, g_3, g_4 = big pillars, f = grass rope. So, *what is the philosophical implication of the fable of blind men with an elephant?* Certainly, the elephant is existing in a 3-dimension $\mathbb{R}^3$. It is clear that the shape of an elephant can not be any combination of 2 big teeth (or big radishes), 1 trunk (or tube), 2 ears (or big fans), 1 belly (or wall), 4 legs (or big pillars) and 1 tail (or grass rope) but such a combination over the 1-dimensional topological graph G^L shown in Figure 4 in eyes of the sophist, inherited in the recognition of blind men by feeling the different parts on the elephant body.

Notice that the sophist told the blind men that "*an elephant has all characteristics that you are talking about*" is essentially a claim on the elephant shape, i.e., its shape is a union (1.1) of all characteristics of blind men hold on the elephant shape. Certainly, this is the recognitive way of human on thing T by reductionism. Generally, let the observable characteristics be $\chi_1, \chi_2, \cdots, \chi_n$ in reductionism of thing T and denote the mathematical reality of thing T by $T_{\mathcal{M}}$. So, one recognizes the mathematical reality of thing T by a union

$$T_{\mathcal{M}} = \bigcup_{1}^{n} \mathcal{R}(\chi_i) \tag{1.2}$$

of local recognitions $\mathcal{R}(\chi_i)$ of human, called a *Smarandache multispace* ([14],[36]), where $\mathcal{R}(\chi_i)$ is the mathematical reality on characteristic χ_i for integers $1 \le i \le n$. Now, *is any combination of characteristics of $\chi_1, \chi_2, \cdots, \chi_n$ necessarily the thing T?* The answer is certainly Not because one can not assert the characteristics $\chi_1, \chi_2, \cdots, \chi_n$ are complete, and the recognitive process of human on thing T is like the blind men on the elephant by feeling its partly body. Furthermore, T is existing in space, which inherits a topological 1-dimensional stricture G^L in the reductionism process and the combination is conclusively priori to the reductionism in recognition of thing T, which naturally leads to a complex network ([1]-[2]) or combinatorial fields ([10]-[12]) on thing T. For example, there are 3.6×10^{13} and 2.8×10^{13} cells respectively in a male or female body, which can be characterized by complex networks with 3.6×10^{13} or 2.8×10^{13} nodes, respectively.

1.2. Combinatorics Implied in Contradictory System. Usually, human quarrelling is because of the differences in recognition on one thing, which leads to contradiction, even by mathematics. For example, let $S_i, 1 \le i \le 6$ be the elephant shape of blind men in fable of the blind men with an elephant. They were quarrelling because their recognition are very different, i.e., $S_i \ne S_j$ if $1 \le i \ne j \le 6$. However, the contradiction arising is not due to the nature of elephant but the modeling of blind men, and the sophist told the blind men is its shape should be essentially a contradictory system holding with a *Smarandachely denied axiom* ([35],[36]), i.e., the axiom *the elephant shape S = a big radish* is simultaneously validated and invalided, or the axiom *the elephant shape S = the reality of elephant shape* only invalided but in six different shapes S_i,$1 \le i \le 6$ simultaneously, contradicting to a definite recognition on the elephant shape. Indeed, there is a fundamental question on the recognition of human, i.e., *is a contradictory system in mathematics worthless in recognition?* The answer is certainly Not because we can not asserted so, i.e., the contradiction is essentially caused by the modeling way of human, which violates the axiom adopted in the mathematical system. However, all things are harmonious in nature, namely the contradiction arising in a mathematical modeling only implies its inappropriate, not the objective of thing in nature. Particularly, the modeling of elements in a self-organized system such as those of biological population, cell system, gene, etc., i.e., all elements are self-motivated, not necessarily in stationary and synchronization are the case.

Figure 5

For example, let $A = \{C_1, C_2, C_3\}$ and $B = \{C_1', C_2', C_3'\}$ be two groups consisting of three Tom cats chasing three Jerry mice in Figure 5 along three straight lines on Euclidean plane $\mathbb{R}^2$

respectively, shown in Figure 6. Then, *how to modeling the running behavior of cats in groups A or B on plane $\mathbb{R}^2$?* For answering this question, a natural idea is to describe the running behavior of the two groups by moving orbits, i.e., lines on Euclidean plane $\mathbb{R}^2$, solve the two systems of linear equations and then, answer this question. In fact, the obits of three cats in groups A or B respectively form two systems of linear equations by cats running lines shown in Figure 6, i.e

$$(LES_3^N)\ \begin{cases} y &= 4 \\ y &= 2 \\ x+y &= 8 \end{cases} \qquad\qquad (LES_3^S)\ \begin{cases} x &= 3 \\ y &= 3 \\ x+y &= 6 \end{cases}$$

However, the system (LES_3^N) is non-solvable and the system (LES_3^S) has a solution $(3,3)$. Now, *can we conclude that the running behavior of cats in A are nothing unless an empty set $\emptyset$, and cats in B are all still at the point $(3,3)$ without moving?* Of course Not because all cats in group A and B are running on Euclidean plane $\mathbb{R}^2$.

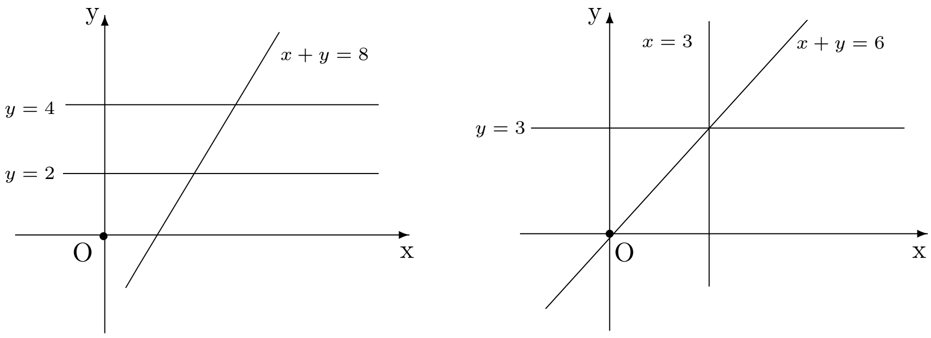


Figure 6

Then, *what is wrong with the modeling of running behavior of cats?* The answer is modeling by the solutions of systems (LES_3^N) and (LES_3^S) on running behavior of cats in groups A and B is inappropriate. Certainly, a running orbit of cat in groups A or B can be characterized by the solution of line equation, i.e., the straight line of cat in A or B on Euclidian plane $\mathbb{R}^2$ but not the solution of system of linear equations.

In this case, the orbits $\mathrm{Orb}(A)$ or $\mathrm{Orb}(B)$ should be a union of points of cats in groups A or B passing on plane $\mathbb{R}^2$, i.e.,

$$\begin{aligned} \mathrm{Orb}(A) &= \{(x,y):\ y=4\}\bigcup\{(x,y):\ y=2\}\bigcup\{(x,y):\ x+y=8\}, \\ \mathrm{Orb}(B) &= \{(x,y):\ x=3\}\bigcup\{(x,y):\ y=3\}\bigcup\{(x,y):\ x+y=6\}, \end{aligned}$$

where each of the orbits $\mathrm{Orb}(A)$ and $\mathrm{Orb}(B)$ is a Smarandache multispace, i.e., combinatorial one. For example, denote the points of a cat running on straight line $ax+by=c$ by the set $L_{a,b,c}=\{(x,y)|ax+by=c, a\neq 0 \text{ or } b\neq 0\}$ in Figure 6, the line intersections of (LES_3^N) and

(LES_3^S) for cats in groups A, B by

$$\begin{aligned} v_1 &= L_{0,1,4}\bigcap L_{1,1,8},\ v_2 = L_{0,1,2}\bigcap L_{1,1,8},\ L_{0,1,2}\bigcap L_{0,1,4}=\emptyset, \\ u_1 &= L_{1,0,3}\bigcap L_{1,1,6},\ u_2 = L_{0,1,3}\bigcap L_{1,0,3},\ u_3 = L_{0,1,3}\bigcap L_{1,1,6} \end{aligned}$$

and so, the running of cats in groups A, B can be characterized by combinatorial solutions of systems (LES_3^N) and (LES_3^S) respectively, i.e., labeled graphs P_3^L, C_3^L shown in Figure 7.

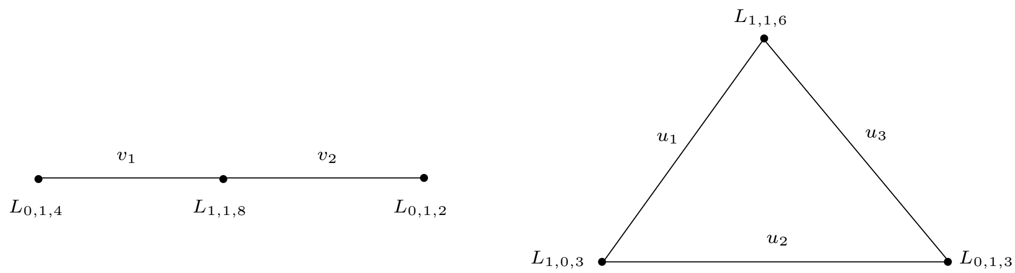


Figure 7

Generally, let $\widetilde{S}$ be a Smarandache multispace on n distinct spaces $S_1, S_2, \cdots, S_n$ for an integer $n \geq 1$. Define a labeled graph $G^L[\widetilde{S}]$ associated with $\widetilde{S}$ by

$$\begin{aligned} V\left(G^L\left[\widetilde{S}\right]\right) &= \{S_1, S_2, \cdots, S_n\} \\ E\left(G^L\left[\widetilde{S}\right]\right) &= \left\{(S_i, S_j)\,|S_i\bigcap S_j \neq \emptyset,\ 1 \leq i \neq j \leq n\right\} \end{aligned}$$

and labels on the vertex $\mathcal{S}_i$, edge $(\mathcal{S}_i, \mathcal{S}_j)$ for integers $1 \leq i \neq j \leq n$ respectively by

$$L:\ S_i \to L(S_i) \ = \ S_i \text{ and } L:\ (S_i, S_j) \to L(S_i, S_j) \ = \ S_i\bigcap S_j.$$

Certainly, a Smarandache multispace $\widetilde{S}$ is equivalent to the labeled graph $G^L[\widetilde{S}]$ by definition. However, $\widetilde{S}$ is a multiset implying the mathematical reality of thing but $G^L[\widetilde{S}]$ is sets over topological graph G which can contributes to establish a mathematics, i.e., *mathematical combinatorics* by viewing $G^L[\widetilde{S}]$ as a mathematical element. So, *could we really establish such a mathematics on elements* $G^L[\widetilde{S}]$*?* The answer is definite by combinatorics.

The main purpose of this report is to survey the establishing of mathematics on continuity flows $\overrightarrow{G}^L$ holding with vertex conservation law, show the importance of non-harmonious groups with G-solutions in recognition of the reality of thing T, generalize G-solutions to G-flows, continuity flows for constructing mathematics as those of Banach flow space, Hilbert flow space over topological graphs G and then, generalize a few of important conclusions in classic mathematics such as the inverse mapping theorem, closed graph theorem and the Hahn-Banach theorem in these flow spaces for providing the recognition of human on the reality of things, including the subdivision of a matter M into elementary particles with a mathematical supporting, and shows also the 12 meridians, Ren and Du meridians on human body in traditional Chinese medicine contributing an example of G-flows or generally, continuity flows by viewing the *Ying Qi* and

Wei Qi on meridians to be vital energy ([24]-[27], [33]).

All terminologies and notations not defined in this paper are standard such as those of algebra, complex systems, functional analysis, topology are referred to [3]-[5] and [34], topological graph is referred to [6]-[7], [15] and Smarandache multispace are referred to [14] and [36].

§2. Mathematical Reality

A mathematical reality on thing T is such an abstraction of T with a priori assumption that its evolution can be modeled consistent with the symbol behavior in a mathematical system. Usually, it is characterized by solvable equation in the classic. However, it should be not a solvable but non-solvable system, i.e., Smarandachely denied system or Smarandache multispace by Godel's incompleteness theorem on formal system, which concludes that *there exist always statements in a formal system S that can neither be proved nor disproved so long as S contains the Peano's axioms of arithmetic*, namely it can be only characterized by non-solvable system of solvable equations, i.e., *non-harmonious group* defined following.

Definition 2.1([29]) *A non-harmonious group $\mathcal{S}$ is such a system $\mathcal{S}$ consisting of elements P_i, $1 \leq i \leq m, m \geq 2$ with interactions that P_i is constrained on a system of equations*

$$(ES_m)\begin{cases} \mathscr{F}_{P_1}(\mathbf{x},\mathbf{y})=0 \\ \mathscr{F}_{P_2}(\mathbf{x},\mathbf{y})=0 \\ \cdots\cdots\cdots\cdots \\ \mathscr{F}_{P_m}(\mathbf{x},\mathbf{y})=0 \end{cases} \tag{2.1}$$

at time t, where $\mathscr{F}_i(\mathbf{x}^0,\mathbf{y}^0)=0$ and $\mathscr{F}_i$ satisfies the existence condition of implicit function theorem in a neighborhood U of point $(\mathbf{x}^0,\mathbf{y}^0)$ in Euclidean space $\mathbb{R}^n$ for integers $1 \leq i \leq m$.

Notice that each function of $\mathscr{F}_{v_1}, \mathscr{F}_{v_2}, \cdots, \mathscr{F}_{v_m}$ in equation (2.1) satisfying the condition of implicit function theorem. There must be a solution manifold $S_{\mathscr{F}_i} \subset \mathbb{R}^n$ with $\mathscr{F}_i: S_{\mathscr{F}_i} \to 0$ for integer $1 \leq i \leq m$. Then, the system (2.1) has no solution or has a solution is because of

$$\bigcap_{i=1}^{m} S_{\mathscr{F}_i} = \emptyset \quad \text{or} \quad \bigcap_{i=1}^{m} S_{\mathscr{F}_i} \neq \emptyset. \tag{2.2}$$

geometrically. So, *what is the meaning of system* (2.1) *has or has no solution?* The answer is that the solution shows the overlap state of elements $P_1, P_2, \cdots, P_m$ at time t, not the state of elements $P_1, P_2, \cdots, P_m$ because the behavior of element P_i is the solution manifold $S_{\mathscr{F}_i}$ for integer $1 \leq i \leq m$. Accordingly, the non-solvable case of system (2.1) indicates only that there is no overlap state in elements $P_1, P_2, \cdots, P_m$, not implies the state of P_i existing or not because its state is characterized by the solution manifold $S_{\mathscr{F}_i}$ for integers $1 \leq i \leq m$.

And so, *how to characterize the group behavior of $P_1, P_2, \cdots, P_m$?* the answer should be the union $\bigcup_{i=1}^{m} S_{\mathscr{F}_i}$ or Smarandache multispace on solution manifolds $S_{\mathscr{F}_i}, 1 \leq i \leq m$, not the intersection $\bigcap_{i=1}^{m} S_{\mathscr{F}_i}$ in classical mathematics for non-harmonious group $\mathcal{S}$. In other words, the solution of system (2.1) can be only applied to the recognition of thing T if all element states are the same in evolving, holds with $S_{\mathscr{F}_i} = S_{\mathscr{F}_j}, 1 \leq i, j \leq m$. It is worth noted that $\bigcup_{i=1}^{m} S_{\mathscr{F}_i} \subset \mathbb{R}^n$ is a union that characterizes the state of elements to some extent but still not completely the state of elements in group S. Then, *how should we characterize the state of group S in this case?* The answer is the *G-solution* of equations in system (2.1).

Definition 2.2([29]) *For any integer $m \geq 1$, the G-solution of system* (2.1) *on non-harmonious group S is a labeled graph G^L with vertex and edge sets defined by*

$$V\left(G^L\right) = \{S_{\mathscr{F}_i},\ 1 \leq i \leq m\},$$
$$E\left(G^L\right) = \{\ (S_{\mathscr{F}_i}, S_{\mathscr{F}_j}) \mid if\ S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j} \neq \emptyset\ for\ integers\ 1 \leq i, j \leq m\ \}$$

and labels on vertices and edges of G by

$$L:\ S_{\mathscr{F}_i} \to S_{\mathscr{F}_i}, \quad (S_{\mathscr{F}_i}, S_{\mathscr{F}_j}) \to S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j}, \quad 1 \leq i \neq j \leq m.$$

Such a G-solution $\bigcup_{i=1}^{m} S_{\mathscr{F}_i}$ is called a *combinatorial manifold* in geometry ([9]). Notice that the case of $\bigcap_{i=1}^{m} S_{\mathscr{F}_i} = \emptyset$ is meaninglessness in classical mathematics because it includes contradiction. However, it is mostly due to the overlap of element states rather than the non-existence of state of elements, namely it is more meaningful to study the G-solution of non-harmonious group S than that of classical one and get

Theorem 2.3([29]) *For any integer $m \geq 1$, a G-solution G^L of system* (2.1) *on a non-harmonious group S is always existing.*

Generally, we can apply G-solution to discuss respectively those of non-solvable systems of algebraic equations, ordinary differential equations and partial differential equations for characterizing the states of non-harmonious groups with stability of the system, see [11]-[18] for details. For example, let (LEq_m^1), $(LDES_m^1)$ be respectively a non-solvable system of linear equations and ordinary differential equations, i.e.,

$$(LEq_m^1)\ \ AX = B, \qquad (LDES_m^1)\begin{cases} \dot{X} = A_1X \\ \dot{X} = A_2X \\ \cdots\cdots\cdots \\ \dot{X} = A_mX \end{cases}$$

as examples, where matrixes $A = (a_{ij})_{m\times n}$, $A_k = \left(a_{ij}^k\right)_{m\times n}$, $X = (x_1, x_2, \cdots, x_n)^T$, $B = (b_1, b_2, \cdots, b_m)^T$ and $a_{ij}, a_{ij}^{[k]}, b_i$ are real numbers for integers $1 \leq i \leq m$, $1 \leq j \leq n$. For any integers $1 \leq i, j \leq m, i \neq j$, two linear equations

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n &= b_i, \\ a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n &= b_j \end{aligned}$$

are called *parallel* if there exists a constant c such that

$$c = \frac{a_{j1}}{a_{i1}} = \frac{a_{j2}}{a_{i2}} = \cdots = \frac{a_{jn}}{a_{in}} \neq \frac{b_j}{b_i}, \tag{2.3}$$

which is essentially the condition of parallel planes in Euclidean space $\mathbb{R}^n$. Now, let L_i be the ith linear equation in (LEq_m^1). We classify these equations $L_i,\ 1 \leq i \leq m$ to *parallel families*

$$\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s \tag{2.4}$$

with the maximal property, i.e., all linear equations in family $\mathscr{C}_i$ are parallel and there are no other equations parallel to lines in $\mathscr{C}_i$, $|\mathscr{C}_i| = n_i$ for integers $1 \leq i \leq s$. Then, the linear algebraic non-solvable system (LEq_m^1) can be easily characterized.

Theorem 2.4([16]) *Let (LEq_m^1) be a linear equation system for integers $m, n \geq 1$. Then*

$$G[LEq_m^1] \simeq K_{n_1, n_2, \cdots, n_s} \tag{2.5}$$

with $n_1 + n + 2 + \cdots + n_s = m$, where $\mathscr{C}_i$ is the parallel family with $n_i = |\mathscr{C}_i|$ for integers $1 \leq i \leq s$ in (LEq_m^1) and the system (LEq_m^1) is non-solvable if $s \geq 2$.

Particularly, if $n = 2$, let H be a planar graph with each edge of straight segment on $\mathbb{R}^2$. Define its *c-line graph* $L_C(H)$ by

$V(L_C(H)) = \{$straight lines $L = e_1e_2 \cdots e_l, s \geq 1$ in $H\}$;

$E(L_C(H)) = \{(L_1, L_2)|$ if e_i^1 and e_j^2 are adjacent in H for $L_1 = e_1^1 e_2^1 \cdots e_l^1$, $L_2 = e_1^2 e_2^2 \cdots e_s^2$, $l, s \geq 1\}$.

For example, a planar graph H with its c-line graph $L_c(H)$ is shown in Figure 8.

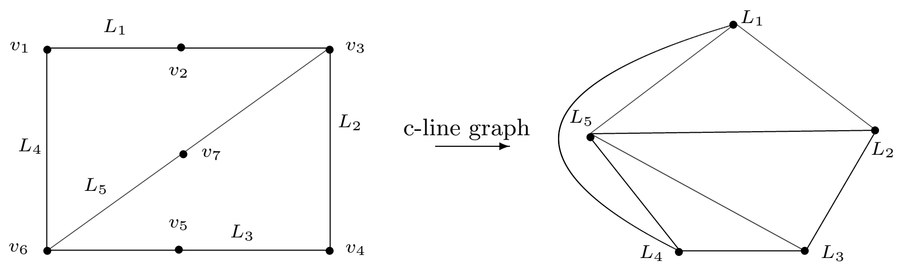


Figure 8

And so, the non-solvable system (LEq_m^1) of linear equations can also be characterized by c-line graph $L_c(H)$, i.e.,

Theorem 2.5([16]) *If $n = 2$, a linear equation system (LEq_m^1) is non-solvable if and only if*

$$G[LEq_m^1] \simeq L_C(H)), \tag{2.6}$$

where H is a planar graph of order $|H| \geq 2$ on $\mathbb{R}^2$ with each edge a straight segment

Notice that the solution of differential equation in system $(LDES_m^1)$ is a linear spaces spanned by its basic solutions. Thus, we can label each vertex of G-solution to get a *basis graph* of $(LDES_m^1)$ by its base.

Theorem 2.6([17]) *Let H be a basis graph. Then, there is a unique linear homogeneous* *differential equation system* $(LDES_m^1)$ *with G-solution* H.

For example, let $(LDES_6^1)$ be a system of homogeneous differential equations $(1)-(6)$. It is easily to know the bases of $(1)-(6)$ are respectively

$$\{e^t, e^{2t}\}, \{e^{2t}, e^{3t}\}, \{e^{3t}, e^{4t}\}, \{e^{4t}, e^{5t}\}, \{e^{5t}, e^{6t}\}, \{e^{6t}, e^t\}$$

with a basis graph $G^L[LDES_6^1]$ shown in Figure 9.

$$(LDES_6^1)\begin{cases} \ddot{x}-3\dot{x}+2x=0 & (1)\\ \ddot{x}-5\dot{x}+6x=0 & (2)\\ \ddot{x}-7\dot{x}+12x=0 & (3)\\ \ddot{x}-9\dot{x}+20x=0 & (4)\\ \ddot{x}-11\dot{x}+30x=0 & (5)\\ \ddot{x}-7\dot{x}+6x=0 & (6)\end{cases}$$

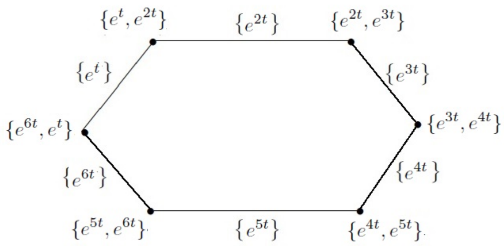


Figure 9

Furthermore, two linear spaces are isomorphic if and only if they have the same dimension. We can replace each basis by its dimension, obtain a basis graph $G^L[LDES_m^1]$ of systems $(LDES_m^1)$ labeled with integers, called an *integral graph* and then, classify systems $(LDES_m^1)$ of linear differential equations by integral graphs.

Theorem 2.7([17]) *Let* $(LDES_m^1)$, $(LDES_m^1)'$ *be two systems of linear homogeneous differential equation with integral graphs* H, H', *respectively. Then* $(LDES_m^1) \overset{\mathscr{L}}{\simeq} (LDES_m^1)'$ *if and only if the integral graph* $H = H'$.

Similarly, let $(PDES_m^1)$ be a system of partial differential equations with

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0\\ F_2(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0\\ F_m(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0\end{cases}$$

on a function $u(x_1, \cdots, x_n, t)$. Then, its *symbol* is determined by

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0\\ F_2(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_m(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0,\end{cases}$$

i.e., substitute $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_n^{\alpha_n}$ into $(PDES_m^1)$ for the term $u_{x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}}$, where $\alpha_i \geq 0$ for integers $1 \leq i \leq n$ and a non-solvable system $(PDES_m^1)$ is *algebraically contradictory* if its symbol is non-solvable. Otherwise, *differentially contradictory*. Then, we can characterize the G-solution of non-solvable systems of partial differential equations of first order, which form a non-harmonious groups and accordingly, to the reality of things.

Theorem 2.8([20]) *A Cauchy problem on systems*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} x_i|_{x_n=x_n^0} = x_i^0(s_1, s_2, \cdots, s_{n-1}) \\ u|_{x_n=x_n^0} = u_0(s_1, s_2, \cdots, s_{n-1}) \\ p_i|_{x_n=x_n^0} = p_i^0(s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases}$$

if and only if the system

$$F_k(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0, \ 1 \le k \le m$$

is algebraically contradictory, in this case there must be an integer $k_0, \ 1 \le k_0 \le m$ *such that*

$$F_{k_0}(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) \neq 0 \tag{2.7}$$

or it is differentially contradictory itself, i.e., there is an integer $j_0, \ 1 \le j_0 \le n-1$ *such that*

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0. \tag{2.8}$$

Theorem 2.9([20]) *A Cauchy problem on system* $(PDES_m^1)$ *of partial differential equations of first order with initial values* $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}, \ 1 \le i \le n$ *for the* k*th equation in* $(PDES_m^1)$*,* $1 \le k \le m$ *such that*

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^{n} p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0 \tag{2.9}$$

is uniquely G*-solvable, i.e.,* $G^L[PDES]$ *is uniquely determined.*

Now, *what is the role of* G*-solution on non-harmonious groups* S*?* Its role is on the global stability of non-harmonious group. Usually, a solution of system (ES_m) of differential equations is called stable or asymptotically stable if for all solutions $Y(t)$ of the differential equations (ES_m) with $|Y(0) - X(0)| < \delta(\varepsilon)$, exists $|Y(t) - X(t)| < \varepsilon$ for $\forall\varepsilon > 0$ and $t \ge 0$ or furthermore, $\lim\limits_{t\to 0} |Y(t) - X(t)| = 0$. However, if $\bigcap\limits_{i=1}^{m} S_{T_i} = \emptyset$ there are no solutions of (ES_m), the classical theory is failed to apply. By Theorem 2.3, any non-harmonious groups S has a G-solution $G^L[ES_m]$ of system (ES_m) whenever it is solvable or not, namely the G-solution $G^L[ES_m]$ can be used to characterize the stability of system (ES_m).

Let $G^L(t)$ be a G-solution of (ES_m) with initial values $G^L(t_0)$ and let $\omega : V\left(G^L[ES_m]\right) \to$

$\mathbb{R}$ be a functional. A system (ES_m) is said to be *ω-stable* if there exists a number $\delta(\varepsilon)$ for any number $\varepsilon > 0$ such that

$$\left\|\omega\left(G^{L_1(t)-L_2(t)}\right)\right\| < \varepsilon \tag{2.10}$$

or furthermore, *asymptotically ω-stable* if

$$\lim_{t\to\infty}\left\|\omega\left(G^{L_1(t)-L_2(t)}\right)\right\| = 0 \tag{2.11}$$

if the initial values hold with $\|L_1(t_0)(v) - L_2(t_0)(v)\| < \delta(\varepsilon)$ for $\forall v \in V\left(\overrightarrow{G}\right)$. In this case, if there is a Liapunov ω-function $L(\omega(t)) : \mathscr{O} \to \mathbb{R}, n \geq 1$ on $\overrightarrow{G}$ with open $\mathscr{O} \subset \mathbb{R}^n$ such that $L(\omega(t)) \geq 0$ with equality hold only if $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$ and $\dot{L}(\omega(t)) < 0$ if $t \geq t_0$. Denoted by $\mathbf{O}$ the zero G-solution of system (ES_m), i.e., all vertices and edges on $G^L[ES_m]$ are labeled by $\mathbf{0}$, we get a result on ω-stability of (ES_m) following.

Theorem 2.10([21]) *If there is a Liapunov ω-function $L(\omega(t)) : \mathscr{O} \to \mathbf{R}$ on $G^L[ES_m]$ of system (ES_m), then it is ω-stable, and furthermore, if $\dot{L}(\omega(t)) < 0$ for $G^L[ES_m] \neq \mathbf{O}$, then it is asymptotically ω-stable.*

For a linear system $(LDES^1_m)$ of differential equations, we can further introduce the sum-stable and prod-stable on $(LDES^1_m)$, i.e., a system $G^L[LDES^1_m]$ is *sum-stable* or *asymptotically sum-stable* if for all solutions $Y_v(t)$, $v \in V(G^L)$ of linear differential equations in $(LDES^1_m)$ with $|Y_v(0) - X_v(0)| < \delta_v$ exists

$$\left|\sum_{v\in V(H^L)} Y_v(t) - \sum_{v\in V(H^L)} X_v(t)\right| < \varepsilon \tag{2.12}$$

for all $t \geq 0$ or furthermore,

$$\lim_{t\to 0}\left|\sum_{v\in V(H^L)} Y_v(t) - \sum_{v\in V(H^L)} X_v(t)\right| = 0 \tag{2.13}$$

and *prod-stable* or *asymptotically prod-stable* if for all solutions $Y_v(t)$, $v \in V(G)$ of $(LDES^1_m)$ with $|Y_v(0) - X_v(0)| < \delta_v$ exists

$$\left|\prod_{v\in V(G)} Y_v(t) - \prod_{v\in V(G)} X_v(t)\right| < \varepsilon \tag{2.14}$$

for all $t \geq 0$, or furthermore,

$$\lim_{t\to 0}\left|\prod_{v\in V(G)} Y_v(t) - \prod_{v\in V(G)} X_v(t)\right| = 0. \tag{2.15}$$

We get criterions on sum-stable and prod-stable of the linear system $(LDES^1_m)$ following.

Theorem 2.11([17]) *A zero* **O***-solution of system* $(LDES_m^1)$ *of linear homogenous differential equation is asymptotically sum-stable if and only if* $\mathrm{Re}\alpha_v < 0$ *for each* $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *with vertex* $v \in G^L[LDES_m^1]$.

Theorem 2.12([17]) *A zero* **O***-solution of systems* $(LDES_m^1)$ *of linear homogenous differential equation is asymptotically prod-stable if and only if*

$$\sum_{v\in V(G)} \mathrm{Re}\alpha_v \ < \ 0 \tag{2.16}$$

for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *with vertex* $v \in G^L[LDES_m^1]$.

Similarly, we can also discuss non-solvable systems of differential equations by linearizing its non-linear differential parts, get criterions on the global stability of non-linear differential equations and then, apply to the stability of system (ES_m) of differential equations. For example, the stability of food web in biological systems. Notice that a food web is a complex network of interconnecting and overlapping food chains, i.e., "*eating or being eaten*" among various organisms within an ecosystem and it is more suitable characterized by labeled directed graphs $\overrightarrow{G}^L$ with by Kolmogorov model.

Theorem 2.13([22]) *A food web* $\overrightarrow{G}^L$ *with initial value* $\overrightarrow{G}^{L_0}$ *is globally stable or asymptotically stable if and only if there is an Eulerian multi-decomposition*

$$\left(\overrightarrow{G}\bigcup\overleftarrow{G}\right)^{\widehat{L}} = \bigoplus_{i=1}^{s} \overrightarrow{H}_i^L \tag{2.17}$$

with solvable stable or asymptotically stable conservative equations on labeling Eulerian subgraphs $\overrightarrow{H}_i^L$ *for integers* $1 \leq i \leq s$, *where* $\overleftarrow{G}$ *is the digraph reversing orientation on every edge in* $\overrightarrow{G}$, $\left(\overrightarrow{G}\bigcup\overleftarrow{G}\right)^{\widehat{L}}$ *is a labeled graph with labeling* $\widehat{L} : V(\overrightarrow{G}\bigcup\overleftarrow{G}) = L\left(V(\overrightarrow{G})\right)$ *and* $\widehat{L} : E\left(\overrightarrow{G}\bigcup\overleftarrow{G}\right) \to L\left(E\left(\overrightarrow{G}\bigcup\overleftarrow{G}\right)\right)$ *by* $\widehat{L}:\ (u,v) \to \{0,(x,y),yf'\},\ (v,u) \to \{xf,(x,y),0\}$ *if* $L : (u,v) \to \{xf,(x,y),yf'\}$ *for* $\forall(u,v) \in E(\overrightarrow{G})$ *such as those shown in Figure 10.*

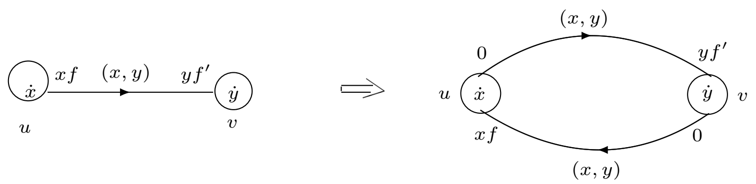


Figure 10

§3. Mathematical Combinatorics

A G-solution $G^L[ES_m]$ of system (ES_m) characterize the state of non-harmonious group $\mathcal{S}$ without direction on edges, i.e., not the action within itself. However, all things are in constantly moving and evolved by their internal action under the external with elements moving in thing, which is only in one-way. Whence, we generalize G-solution $G^L[ES_m]$ to a directed situation for modeling substance flow on the evolution of thing T, holding with conservation at each vertex, i.e., a generalization of network N to continuity flow, which is vectors in Banach space over a topological graph $\overrightarrow{G}$ with end-operator actions, i.e., *mathematical combinatorics*.

Definition 3.1([23]) *A continuity flow* $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ *is an oriented topological graph* $\overrightarrow{G}^L$ *in space* $\mathscr{S}$ *associated with a mapping* $L : v \to L(v),\ (v,u) \to L(v,u)$, *2 end-operators* $A^+_{vu} \in \mathscr{A} : L(v,u) \to L^{A^+_{vu}}(v,u)$ *and* $A^+_{uv} \in \mathscr{A} : L(u,v) \to L^{A^+_{uv}}(u,v)$ *on a Banach space* $\mathscr{B}$ *over a field* $\mathscr{F}$ *such as those shown in Figure* 11 *with* $L(v,u) = -L(u,v)$, $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ *for* $\forall (v,u) \in E(\overrightarrow{G}^L)$ *and meanwhile, holding with the continuity equation*

$$\sum_{u \in N^-_G(v)} L^{A^+_{uv}}(u,v) - \sum_{u \in N^+_G(v)} L^{A^+_{uv}}(u,v) \ = \ L(v) \tag{3.1}$$

at any vertex $v \in V(\overrightarrow{G}^L)$ *of topological graph* $\overrightarrow{G}^L$, *where* $N^-_G(v), N^+_G(v)$ *are respectively the in-neighborhood and out-neighborhood of vertex* $v \in V(\overrightarrow{G}^L)$, *namely all vertices in* $N^-_G(v) \subset N_G(v)$ *or* $N^+_G(v) \subset N_G(v)$ *flow into or out of the vertex* v *and* $N^-_G(v) \cup N^+_G(v) = N_G(v)$.

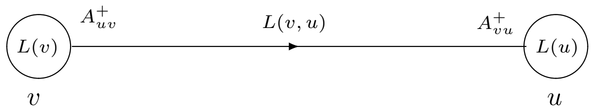


Figure 11

Notice that the continuity equations on vertices of $\overrightarrow{G}^L$ form a non-solvable system (ES_m) of equations. For example, let the $L : (v,u) \to L(v,u) \in \mathbb{R}^n \times \mathbb{R}^+$ with end-operators $A^+_{vu} = a_{vu}\dfrac{\partial}{\partial t}$ and $a_{vu} :\ \mathbb{R}^n \to \mathbb{R}$ for any edge $(v,u) \in E\left(\overrightarrow{G}\right)$ in Figure 12 following.

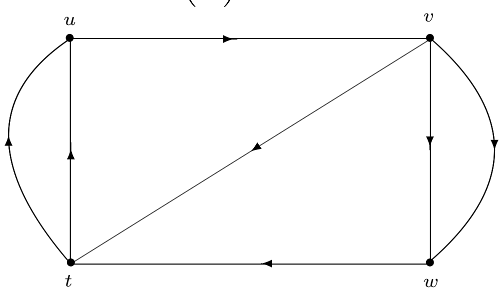


Figure 12

Then, the continuity equations on vertices of $\overrightarrow{G}^L$ are partial differential equations

$$\begin{cases} a_{tu^1}\dfrac{\partial L(t,u)^1}{\partial t} + a_{tu^2}\dfrac{\partial L(t,u)^2}{\partial t} = a_{uv}\dfrac{\partial L(u,v)}{\partial t} \\ a_{uv}\dfrac{\partial L(u,v)}{\partial t} = a_{vw^1}\dfrac{\partial L(v,w)^1}{\partial t} + a_{vw^2}\dfrac{\partial L(v,w)^2}{\partial t} + a_{vt}\dfrac{\partial L(v,t)}{\partial t} \\ a_{vw^1}\dfrac{\partial L(v,w)^1}{\partial t} + a_{vw^2}\dfrac{\partial L(v,w)^2}{\partial t} = a_{wt}\dfrac{\partial L(w,t)}{\partial t} \\ a_{wt}\dfrac{\partial L(w,t)}{\partial t} + a_{vt}\dfrac{\partial L(v,t)}{\partial t} = a_{tu^1}\dfrac{\partial L(t,u)^1}{\partial t} + a_{tu^2}\dfrac{\partial L(t,u)^2}{\partial t} \end{cases}$$

Usually, a continuity flow $(\overrightarrow{G}; L, \mathscr{A})$ is abbreviated to $\overrightarrow{G}^L$ for simplicity, which is a generalization of G-solution of non-harmonious group $\mathcal{S}$ and substance flow in physics, a more accurate model on the reality of thing T and includes most mathematical models on thing T. For example, if $L(v) = \dot{x}_v$, $v \in V(\overrightarrow{G})$, a continuity flow $\overrightarrow{G}^L$ is a *complex flow* ([23]); if x_v is a constant $\mathbf{v}_v$ dependent on v for $v \in V(\overrightarrow{G})$, a continuity flow $\overrightarrow{G}^L$ is an *action flow* ([21]); if $\mathscr{A} = \mathbb{Z}$ or $\mathbb{C}$, particularly, $\mathscr{A} = \mathbf{1}_{\mathscr{V}}$, a continuity flow $\overrightarrow{G}^L$ is $\overrightarrow{G}$*-flow* ([19]) and if $\mathscr{A} = \{\mathbf{1}_{\mathscr{B}}\}$ and $\mathscr{B}$ is the number field $\mathbb{Z}$ or $\mathbb{R}$, a continuity flow $\overrightarrow{G}^L$ is *complex network* ([4]), which was discussed extensively in complex science.

Now, *could we really establish mathematics on continuity flows $\overrightarrow{G}^L$ by view it as a mathematical element?* The answer is definite by considering $\overrightarrow{G}^L$ to be a family of vectors underlying a topological graph $\overrightarrow{G}$ with addition, multiplication and scalar multiplication for continuity flows $\overrightarrow{G}^L, \overrightarrow{G'}^{L'}$ and $\lambda \in \mathscr{F}$ defined by

$$G^L + G'^{L'} = (G \setminus G')^L \bigcup \left(G \bigcap G'\right)^{L+L'} \bigcup (G' \setminus G)^{L'}, \quad (3.2)$$

$$G^L \cdot G'^{L'} = (G \setminus G')^L \bigcup \left(G \bigcap G'\right)^{L\cdot L'} \bigcup (G' \setminus G)^{L'}, \quad (3.3)$$

$$\lambda \cdot G^L = G^{\lambda \cdot L}. \quad (3.4)$$

where, for any vertex $v \in V(G)$ and edge $(v,u) \in E(G)$, $L(v), L'(v), L(v,u), L'(v,u) \in \mathscr{B} L+L'$: $v \to L(v)+L'(v)$, $(v,u) \to L(v,u)+L'(v,u)$, $L \cdot L' : v \to L(v) \cdot L'(v)$, $(v,u) \to L(v,u) \cdot L'(v,u)$ $\lambda \cdot L : v \to \lambda \cdot L(v)$, $(v,u) \to \lambda \cdot L(v,u)$, $L(v) \cdot L'(v)$ and $L(v,u) \cdot L'(v,u)$ denotes the *Hadamard product* of vectors in Banach space $\mathscr{B}$, namely

$$(x_1, x_2, \cdots, x_n) \cdot (y_1, y_2, \cdots, y_n) = (x_1 y_1, x_2 y_2, \cdots, x_n y_n). \quad (3.5)$$

such as those shown for addition and scalar multiplication in Figure 13.

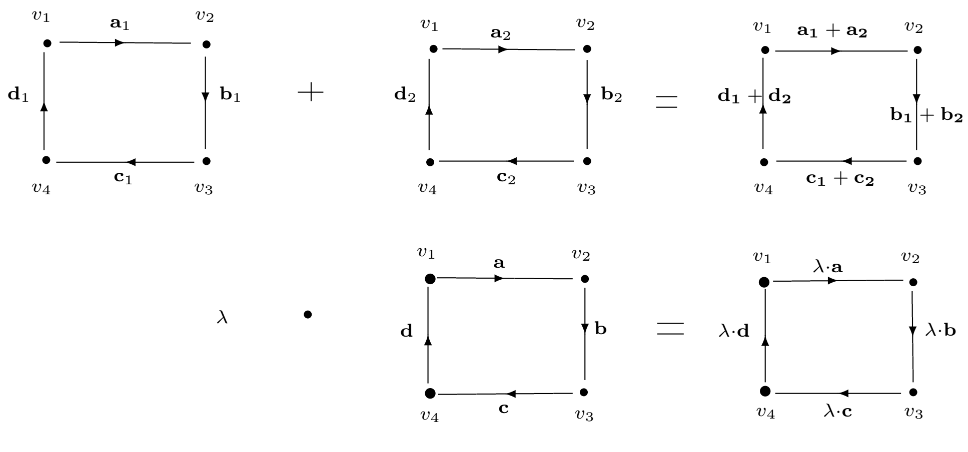


Figure 13

Notice that the addition "+", scalar multiplication "·" on vectors $\mathbf{a}, \mathbf{b}$ can be viewed as the operations on a particular continuity flow $\overrightarrow{G}^L$, i.e., path $\overrightarrow{P}^L_{n+1}$, which enables us to establish mathematics on continuity flows $\overrightarrow{G}^L$ on Banach space $\mathscr{B}$ over topological graphs $\overrightarrow{G}$.

3.1. Banach Flow Space

Let $\mathscr{G}$ be a graph family closed under the union operation of graph, $\mathscr{B}$ be a Banach space over field $\mathscr{F}$ and denoted by $\mathscr{G}_{\mathscr{B}}$ all continuity flows $\overrightarrow{G}^L$ with $\overrightarrow{G} \in \mathscr{G}$, $L : V(\overrightarrow{G}) \bigcup E(\overrightarrow{G}) \to \mathscr{B}$. Then, we have

Theorem 3.1([25],[32]) *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a linear space $(\mathscr{B}; +, \cdot)$, then, all continuity flows $(\mathscr{G}_{\mathscr{B}}; +, \cdot)$ is a linear space, and furthermore, a commutative ring if $\mathscr{B}$ is a commutative ring $(\mathscr{B}; +, \cdot)$ over a field $\mathscr{F}$.*

Assume all end-operators are continuous linear operators in $\mathscr{A}$ and define the norm of a continuity flow $\overrightarrow{G}^L$ by

$$\left\| \overrightarrow{G}^L \right\| = \sum_{(v,u) \in E\left(\overrightarrow{G}\right)} \left\| L^{A^+_{vu}}(v,u) \right\|, \tag{3.6}$$

where $\|\cdot\|$ is the norm on Banach space $\mathscr{B}$. Then, we can verify the non-negative, homogeneity and the triangle inequality hold with $\mathscr{G}_{\mathscr{B}}$ and the non-negative, conjugacy and the linearity if $\mathscr{B}$ is further a Hilbert space, i.e.,

Theorem 3.2([25],[32]) *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a Banach space $(\mathscr{B}; +, \cdot)$, then, $\mathscr{G}_{\mathscr{B}}$ with linear operators A^+_{vu}, A^+_{uv} for $\forall (v,u) \in E\left(\bigcup\limits_{G \in \mathscr{G}} \overrightarrow{G}\right)$ is a Banach space, and furthermore, $\mathscr{G}_{\mathscr{B}}$ is a Hilbert space if $\mathscr{B}$ is a Hilbert space.*

3.2. G-Isomorphic Operators on Banach Flow Space

Let $G_1^{L_1}, G_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$ be continuity flows. Usually, a mapping $f : G_1^{L_1} \to G_2^{L_2}$ is said to be a *G-isomorphic operator* between continuity flows $G_1^{L_1}$, $G_2^{L_2}$ and the continuity flow $G_1^{L_1}$ is said to be G-isomorphic to $G_2^{L_2}$ if G_1, G_2 are isomorphic in graphs, i.e., there is an isomorphism $\varphi : G_1 \to G_2$ of graph and $L_2 = f \circ \varphi \circ L_1$ for $\forall (v,u) \in E(G_1)$, i.e., $G_1^{L_1}, G_2^{L_2}$ are isomorphic of labeled graphs. Furthermore, we conventionalize that $\widehat{G}^{\widehat{L}} = G^L$ for a topological graph $\widehat{G} \supset G$ if $\widehat{L}(x) = L(x)$ for $x \in V(G) \cup E(G)$ and $\widehat{L}(x) = \mathbf{0}$ for $x \notin V(G) \cup E(G)$, which reflects the essence of continuity flow. And by this convention, a $\widehat{G}$-isomorphism between continuity flows $G_1^{L_1}, G_2^{L_2}$ can be generally defined even if $G_1^{L_1}, G_2^{L_2}$ are non-isomorphic but with a supergraph $\widehat{G}$ as $\widehat{G} \supseteq G_1 \bigcup G_2$, a G-isomorphic operator can be generally defined by

Definition 3.3([29]) *A mapping $f : G_1^{L_1} \to G_2^{L_2}$ is a G-isomorphic operator between continuity flows $G_1^{L_1}$ and $G_2^{L_2}$ if*

(1) *there is an isomorphism $\varphi : \widehat{G} \to \widehat{G}$ with $\widehat{G} \supset G_1, G_2$ in graph;*

(2) *for $\forall (v,u) \in E(G_1)$ there is $L_2 = f \circ \varphi \circ L_1$ but for $\forall (v,u) \in E(G_2 \setminus G_1)$, $f : \mathbf{0} \to L_2(v,u)$ and for $\forall (v,u) \in E(G_1 \setminus G_2)$ and $\forall (v,u) \in E(\widehat{G} \setminus (G_1 \bigcup G_2))$, $f : L(v,u) \to \mathbf{0}$.*

Notice that a G-isomorphic operator $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}'}$ is naturally commutative with end-operators in $\mathscr{A}$ on edges of continuity flows $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$ by definition. Let $G_1^{L_1}$, $G_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$ be two continuity flows with scalars $\lambda, \mu \in \mathscr{F}$. Then, a G-isomorphic operator $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}'}$ is *linear* if

$$f\left(\lambda G_1^{L_1} + \mu G_2^{L_2}\right) = \lambda f\left(G_1^{L_1}\right) + \mu f\left(G_2^{L_2}\right), \tag{3.7}$$

is *continuous* at continuity flow $G_0^{L_0}$ if for any number $\varepsilon > 0$ there always exists a real number $\delta(\varepsilon)$ such that

$$\left\| {G_1}^{L_1} - G_0^{L_0} \right\| < \delta(\varepsilon) \ \Rightarrow \ \left\| f\left(G^L\right) - f\left(G_0^{L_0}\right) \right\| < \varepsilon \tag{3.8}$$

and is *bounded* if there exists a constant $\xi \in [0, \infty)$ such that $\|f(G^L)\| \leq \xi \|G^L\|$ for any continuity flow $G^L \in \mathscr{G}_{\mathscr{B}}$. Furthermore, if

$$\left\| f\left(G_1^{L_1}\right) - f\left(G_2^{L_2}\right) \right\| \leq \xi \left\| G_1^{L_1} - G_2^{L_2} \right\|, \quad \xi \in [0, 1) \tag{3.9}$$

for two continuity flows $G_1^{L_1}, G_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$ and a constant ξ, then f is a *contraction* on continuity flow space $\mathscr{G}_{\mathscr{B}}$. And then, we can generalize a few of well-known theorems in classical functional analysis to Banach flow space following.

Theorem 3.4(Fixed Flow Theorem, [25],[29],[32]) *For a continuous G-isomorphic contractor $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}'}$ there is only one continuity flow $G^L \in \mathscr{G}_{\mathscr{B}}$ such that $f\left(G^L\right) = G^L$.*

Theorem 3.5(Banach Inverse Theorem, [25],[29],[32]) *A G-isomorphic linear operator $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ is continuous if and only if it is bounded and furthermore, if f is $1-1$ then the inverse operator f^{-1} of f is also a G-isomorphic continuous operator.*

For a G-isomorphic operator $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$, its image Grap$f$ of is defined by

$$\text{Grap}f \ = \ \left\{ \left(\overrightarrow{G}^L, f\left(\overrightarrow{G}^L\right)\right) \middle| \, \overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}} \right\} \tag{3.10}$$

and f is *closed* if the image Grap$f$ of f is closed.

Theorem 3.6(Closed Graph Theorem, [25],[29],[32]) *If $\mathbf{T} : \mathscr{G}_{\mathscr{B}_1} \to \mathscr{G}_{\mathscr{B}_2}$ is a closed linear operator with Banach spaces $\mathscr{B}_1$, $\mathscr{B}_2$, then $\mathbf{T}$ is continuous.*

Particularly, a G-isomorphic linear operator $f : \mathscr{G}_{\mathscr{B}} \to \mathbb{R}$ or $\mathbb{C}$ is called a *flow functional*, which can be applied to generalize the Hahn-Banach theorem to Banach flow space $\mathscr{G}_{\mathscr{B}}$.

Theorem 3.7(Hahn-Banach Theorem, [25],[29],[32]) *Let $\mathscr{H}_{\mathscr{B}}$ be a subspace of Banach flow space $\mathscr{G}_{\mathscr{B}}$ and let $F : \mathscr{H}_{\mathscr{B}} \to \mathbb{C}$ be a continuous linear flow functional on $\mathscr{H}_{\mathscr{B}}$. Then, there is a continuous linear flow functional $\widetilde{F} : \mathscr{G}_{\mathscr{B}} \to \mathbb{C}$ satisfies the conditions that if $\overrightarrow{G}^L \in \mathscr{H}_{\mathscr{B}}$ then $\widetilde{F}(\overrightarrow{G}^L) = F(\overrightarrow{G}^L)$ and $\|\widetilde{F}\| = \|F\|$.* Particularly, *if $\mathbf{O} \neq \overrightarrow{G}_0^{L_0} \in \mathscr{G}_{\mathscr{B}}$, there is a continuous linear flow functional F such that $\|F\| = 1$ and $\|F(\overrightarrow{G}_0^{L_0})\| = \|\overrightarrow{G}_0^{L_0}\|$.*

Then, *what is the important role of Hahn-Banach theorem in Banach flow space $\mathscr{G}_{\mathscr{B}}$?* Certainly, it can extend a flow functional from a small range to a large one. Furthermore, it convinces us that the subdividing of matter does not affect the validity of quantum hypothesises, i.e., a pure state in quantum mechanics is represented in terms of a normalized vector $|\psi\rangle$ in Hilbert space $\mathcal{H}$ with $\langle\psi|\psi\rangle = 1$, for an observable physical quantity a of quantum there exists a corresponding Hermitian operator H acting on $\mathcal{H}$ and the time evolving of state is governed by Schrödinger equation

$$i\hbar \frac{d\,|\psi\rangle}{dt} = H\,|\psi\rangle\,, \tag{3.11}$$

where $\hbar$ is the Planck's constant. Now, *can we conclude that the existence of Hermitian operator H in a quantum Q with quark structure $\overrightarrow{G}^L$ or a matter T such as the proton and neutron consist of quanta over a topological graph $\overrightarrow{G}^L$?* The answer is certainly Yes by a generalization of Fréchet-Riesz representation theorem in Banach space to Banach flow space $\mathscr{G}_{\mathscr{B}}$.

Theorem 3.8(Fréchet-Riesz Theorem, [25],[29],[32]) *Let $f : \mathscr{G}_{\mathscr{B}} \to \mathbb{C}$ be a continuous linear flow functional. For any continuous flow $G^L \in \mathscr{G}_{\mathscr{B}}$, there uniquely exists a continuous flow of $\widehat{G}^{\widehat{L}} \in \mathscr{G}_{\mathscr{B}}$ holding with $f(\overrightarrow{G}^L) = <\overrightarrow{G}^L, \widehat{G}^{\widehat{L}}>$.*

3.3. Integral and Differential on Continuity Flow

Let the isomorphism $\varphi = \mathrm{id}_G$, i.e., the identity mapping of topological graph $\overrightarrow{G}$ and let $\mathscr{B}$ be a Hilbert space and particularly, a function field on variable $\mathbf{x}$, Then, a G-isomorphic operator f is determined ([29]) by equation

$$L_2(v,u) \ = \ f \circ L_1(v,u), \quad \forall (v,u) \in E(\overrightarrow{G}) \tag{3.11}$$

which is equivalent to

$$f\left(\overrightarrow{G}^L[\mathbf{x}]\right) \ = \ \overrightarrow{G}^{f(L[\mathbf{x}])}. \tag{3.12}$$

Thereby, we can define the power

$$\overrightarrow{G}^{aL}[\mathbf{x}] = \overrightarrow{G}^{aL[\mathbf{x}]}, \quad a^{G^L[\mathbf{x}]} = \overrightarrow{G}^{a^{L[\mathbf{x}]}} \tag{3.13}$$

of continuity flow $\overrightarrow{G}^L[\mathbf{x}]$ for a number $a \in \mathbb{R}$ and respectively, the polynomial, sum and product of continuity flows by

$$\begin{aligned}
a_0 + a_1\overrightarrow{G}^L + a_2\overrightarrow{G}^{L^2} + \cdots + a_n\overrightarrow{G}^{L^n} \ &= \ \overrightarrow{G}^{a_0+a_1L+a_2L^2+\cdots+a_nL^n}, \\
a_1\overrightarrow{G}_1^{L_1} + a_2\overrightarrow{G}_2^{L_2} + \cdots + a_n\overrightarrow{G}_n^{L_n} \ &= \ \left(\bigcup_{i=1}^n \overrightarrow{G}_i\right)^{a_1L_1+a_2L_2+\cdots+a_nL_n}, \\
\left(a_1\overrightarrow{G}_1^{L_1}\right)\cdot\left(a_2\overrightarrow{G}_2^{L_2}\right)\cdots\left(a_n\overrightarrow{G}_n^{L_n}\right) \ &= \ \left(\bigcup_{i=1}^n \overrightarrow{G}_i\right)^{a_1L_1\cdot a_2L_2\cdot\cdots\cdot a_nL_n}.
\end{aligned}$$

Particularly, the 3 interesting exponential identities ([31]) for integer $n \geq 1$ following

$$\begin{aligned}
e^x \ &= \ 1 + \frac{x}{1!} + \frac{x^{2]}}{2!} + \cdots + \frac{x^n}{n!} + \cdots, &(3.14)\\
e^{tA} \ &= \ \mathbf{I} + \frac{tA}{1!} + \frac{t^2A^2}{2!} + \cdots + \frac{t^nA^n}{n!} + \cdots, &(3.15)\\
e^{G^L[\mathbf{x}]} \ &= \ \mathbf{I} + \frac{\overrightarrow{G}^L[\mathbf{x}]}{1!} + \frac{\overrightarrow{G}^{2L}[\mathbf{x}]}{2!} + \cdots + \frac{\overrightarrow{G}^{nL}[\mathbf{x}]}{n!} + \cdots, &(3.16)
\end{aligned}$$

where A is an $n \times n$ matrix, $G^L[\mathbf{x}]$ is a continuity flow that continuous in variable $\mathbf{x}$.

Furthermore, we can define the limitation of continuity flow sequence $\{\overrightarrow{G}_i^{L_i}[\mathbf{x}]\}_1^\infty$, differential and integral on continuity flow $\overrightarrow{G}^L[\mathbf{x}]$ by equality (3.12) and also, establish the calculus on $\mathscr{G}_{\mathscr{B}}$. For example, the fundamental theorem

$$\int_a^b f\frac{d}{dt}\left(\overrightarrow{G}^L[t]\right)dt = f\left(\overrightarrow{G}^L[t]\right)\Big|_{t=b} - f\left(\overrightarrow{G}^L[t]\right)\Big|_{t=a} \tag{3.17}$$

can be obtained similar to that of the calculus and introduce the variation on continuity flows $\overrightarrow{G}^L[\mathbf{x}]$, namely assume a G-isomorphic mapping $\mathscr{L} : (v,u) \in E(\overrightarrow{G}) \to \mathscr{L}\left[\mathcal{L}(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))(v,u)\right]$ is differentiable and respectively define the action $J[\overrightarrow{G}^{\mathscr{L}}[t]]$ and variation $\delta J[\overrightarrow{G}^{\mathscr{L}}[t]]$ on a continuity flow $\overrightarrow{G}^{\mathscr{L}}[t]$ by

$$J\left[\overrightarrow{G}^{\mathscr{L}}[t]\right] = \left|\int_{t_1}^{t_2}\overrightarrow{G}^{\mathscr{L}[\mathcal{L}(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))]}dt\right|, \quad \delta J\left[\overrightarrow{G}^{\mathscr{L}}[t]\right] = \left|\delta\int_{t_1}^{t_2}\overrightarrow{G}^{\mathscr{L}[\mathcal{L}(t,\mathbf{q}(t),\dot{\mathbf{q}}(t))]}dt\right|, \tag{3.18}$$

where the variation $\delta : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ is a G-isomorphic operator. And so, the Euler-Lagrange equations

$$\frac{\partial\overrightarrow{G}^{\mathscr{L}}}{\partial q_i} - \frac{d}{dt}\frac{\partial\overrightarrow{G}^{\mathscr{L}}}{\partial\dot{q}_i} = \mathbf{O}, \quad 1 \le i \le n. \tag{3.19}$$

on continuity flow $\overrightarrow{G}^{\mathscr{L}}[t]$ can be established by the least action principle $\delta J[\overrightarrow{G}^{\mathscr{L}}[t]](v,u) = 0$ for $\forall(v,u) \in E(\overrightarrow{G}^{\mathscr{L}}[t])$ with the properties of norm in Banach flow space $\mathscr{G}_{\mathscr{B}}$.

§4. An Interesting Example

Although we are all human but it is very hard to answer what a human is unless by behavioral characteristics. A more useful definition on human is by the pair $\{Y^-, Y^+\}$ of Yin and Yang with meridians running the vital energy on body in traditional Chinese medicine, including 12 meridians, i.e., the lung meridian of hand-Taiyin (LU), heart meridian of hand-Shaoyin (HT), pericardium meridian of hand-Jueyin (PC), the spleen meridian of foot-Taiyin (SP), kidney meridian of foot-Shaoyin (KI), liver meridian of foot-Jueyin (LR), large intestine meridian of hand-Yangming (LI), small intestine meridian of hand-Taiyang (SI), Sanjiao (triple burner) meridian of hand-Shaoyang (SJ), stomach meridian of foot-Yangming (ST), bladder meridian of foot Taiyang (BL), gallbladder meridian of foot-Shaoyang (GB), Ren meridian, Du meridian and 671 acupoints such as those shown in Figure 14. All of them connect the five organs and six bowels, communicating the up and down of vital energy of human running with a balanced pair $\{Y^-, Y^+\}$, called to be *Ying Qi* $\mathbf{\Psi}^-$ and *Wei Qi* $\mathbf{\Psi}^+$, i.e., an operating ruler for human body in traditional Chinese medicine, and there must be imbalance acupoints in one of the 12

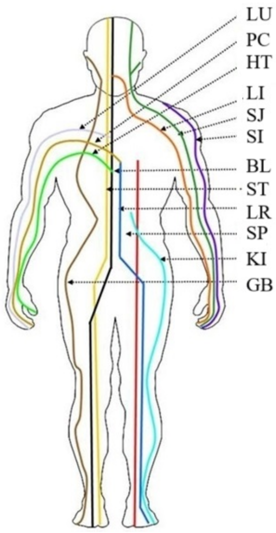


Figure 14

meridians for a patient. Then, the essence of doctor treating illness is by a natural law, i.e., *reducing the excess with supply the insufficient*, regulates on the meridians of human so that the restore balance of them by acupuncture or drugs.

Then, *how to modeling the running of a living body of human?* The answer is by the running of vital energy on 12 meridians with Ren and Du meridians in traditional Chinese medicine. Furthermore, *how to modeling the running of vital energy on 12 meridians with Ren and Du meridians?* The answer is nothing else but a G-flow $\overrightarrow{G}_{12}^L$ defined by ([24],[26])

$$
\begin{aligned}
V\left(\overrightarrow{G}_{12}\right) &= \{\text{All acupoints } v \text{ on 12 meridians with Ren and Du meridians}\},\\
E\left(\overrightarrow{G}_{12}\right) &= \{\text{All segments } (v,u) \text{ connecting adjacent points on 12 meridians with Ren}\\
&\qquad \text{and Du meridians with orientation of Ying Qi running}\}
\end{aligned}
$$

such as those shown in Figure 15

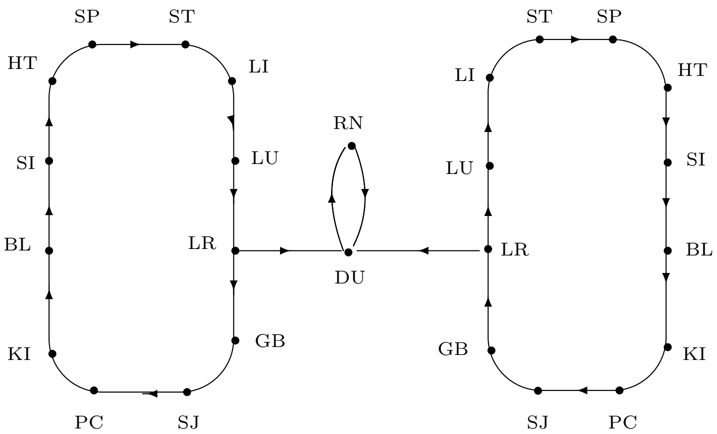


Figure 15

with a labeling $L:(v,u)\to\{\mathbf{\Psi}^-(v,u),\mathbf{\Psi}^+(v,u)\}$, $L:(v,u)\to\mathbf{\Psi}^-(v,u)$ or $L:(v,u)\to\mathbf{\Psi}^+(v,u)$ for edge $\forall(v,u)\in E(\overrightarrow{G}_{12})$. Notice that the *Ying Qi* $\mathbf{\Psi}^-$ and *Wei Qi* $\mathbf{\Psi}^+$ both are the vital energy with constant distributing c on each meridian of human body. Whence, they run in accordance with the law of conservation of energy and thereby, hold with continuity equation at each vertex of $\overrightarrow{G}_{12}^L$, i.e., $\overrightarrow{G}_{12}^L$ is a G-flow of continuity flow, which is characterized ([33]) by

$$
\begin{cases}
\dfrac{\partial \overrightarrow{G}_{12}^{\mathbf{\Psi}^-}}{\partial x_i} - \dfrac{d}{dt}\dfrac{\partial \overrightarrow{G}_{12}^{\mathbf{\Psi}^-}}{\partial \dot{x}_i} = \mathbf{O}, & 1\le i\le n,\\[2ex]
\dfrac{\partial \overrightarrow{G}_{12}^{\mathbf{\Psi}^+}}{\partial x_j} - \dfrac{d}{dt}\dfrac{\partial \overrightarrow{G}_{12}^{\mathbf{\Psi}^+}}{\partial \dot{x}_j} = \mathbf{O}, & 1\le j\le n,\\[2ex]
\overrightarrow{G}_{12}^{\mathbf{\Psi}^-} + \overrightarrow{G}_{12}^{\mathbf{\Psi}^+} = \overrightarrow{G}_{12}^{L_c}, &
\end{cases}
\tag{4.1}
$$

for integers $1\le i\le n$ if we assume that $\mathbf{\Psi}^-$ and $\mathbf{\Psi}^+$ both are Lagrangian on the vital energy field of human, where the 1st and 2nd equations are Euler-Lagrange equations (3.19), the 3rd equation is the balance equation of *Ying Qi* and *Wei Qi* on meridians of human body, and the labeling $L_c : \forall(v,u) \in E(\overrightarrow{G}_{12}) \rightarrow \mathbf{c}_{vu}$ is constant. Notice that the system (4.1) of differential equations is essentially the theoretical foundation and leads to clinical techniques of traditional Chinese medicine, i.e., the amazing acupuncture and the prescription on compatibility of traditional Chinese medicines for a disease treatment.

§5. **Conclusion**

The reality of a thing T existing in universe should be a combinatorial one in the eyes of human because of the recognitive limitation and particularly, the mathematical reality. However, there are no mathematics applicable to reality of things unless the partial or conditional. Thus, a new mathematics should be established for the recognition of human on reality of thing by combinatorics, from the local to the whole. I introduce how to do such an objective in this report from non-harmonious groups to mathematical combinatorics, i.e., mathematics over topological graph, which is a natural way for recognizing the reality of thing T because thing T is not isolated but consisted of its elements, connected also with other things in unverse and also, the recognitive results by reductionism is nothing else but a complex network, we have to establish such a mathematics over topological graphs $\overrightarrow{G}$ inherited in thing T for crossing the recognitive gap from the local to the whole, including both of the macroscopic and microscopic such as the system of celestial body, particle moving or living evolution, the digital economy devolving of international or domestic trade, namely the evolution of all system, no matter it is harmonious or self-organized can be globally characterized by *mathematical combinatorics*.

References

[1] R.Albert and A.L.Barabaśi, Topology of evolving networks: local events and universality, *Physical Review E*, Vol.85(24)(2000), 5234-5240.

[2] A.L.Barabaśi and R. Albert, Emergence of scaling in random network, *Science*, Vol.286, 5439(1999), 509-520.

[3] G.Birkhoff and S.MacLane, *A Survey of Modern Algebra* (4th edition), Macmillan Publishing Co., Inc, 1977.

[4] G.R.Chen, X.F.Wang and X.Li, *Introduction to Complex Networks – Models, Structures and Dynamics* (2nd Edition), Higher Education Press, Beijing, 2015.

[5] John B.Conway, *A Course in Functional Analysis*, Springer-Verlag New York, Inc., 1990.

[6] J.L.Gross and T.W.Tucker, *Topological Graph Theory*, John Wiley & Sons, 1987.

[7] Yanpei Liu,*Enumerative Theory of Maps*, Kluwer Academic Publisher, Dordrecht /Boston/ London (1999).

[8] Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.* Vol.1(2007), No.1, 1-19.

[9] Linfan Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[10] Linfan Mao, An introduction to Smarandache multi-spaces and mathematical combinatorics, *Scientia Magna*, Vol.3, No.1(2007), 54-80.

[11] Linfan Mao, Combinatorial fields – An introduction, *International J. Math.Combin.*, Vol.3 (2009), 1-22.

[12] Linfan Mao, Relativity in combinatorial gravitational fields, *Progress in Physics*, Vol.3,2010, 39-50.

[13] Linfan Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, First edition published by American Research Press in 2005, Second edition is a graduate textbook in mathematics, published by The Education Publisher Inc., USA, 2011.

[14] Linfan Mao, *Smarandache Multi-Space Theory*, First edition published by Hexis, Phoenix in 2006, Second edition is a graduate textbook in mathematics, published by The Education Publisher Inc., USA, 2011.

[15] Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, First edition published by InfoQuest in 2009, Second edition is a graduate textbook in mathematics, published by The Education Publisher Inc., USA, 2011.

[16] Linfan Mao, Non-solvable spaces of linear equation systems, *International J. Math.Combin.*, Vol.2, 2012, 9-23.

[17] Linfan Mao, Global stability of non-solvable ordinary differential equations with applications, *International J. Math.Combin.*, Vol.1, 2013, 1-37.

[18] Linfan Mao, Mathematics on non-mathematics – A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[19] Linfan Mao, Extended Banach G-flow spaces on differential equations with applications, *Electronic J.Mathematical Analysis and Applications*, Vol.3, No.2(2015), 59-91.

[20] Linfan Mao, Cauchy problem on non-solvable system of first order partial differential equations with applications, *Methods and Applications of Analysis*, Vol. 22, 22015, 171C200.

[21] Linfan Mao, Mathematics with natural reality – Action flows, *Bull.Cal.Math.Soc.*, Vol.107, 6(2015), 443-474.

[22] Linfan Mao, Biological n-system with global stability, *Bull.Cal.Math.Soc.*, Vol.108, 6(2016), 403-430.

[23] Linfan Mao, Complex system with flows and synchronization, *Bull.Cal.Math.Soc.*, Vol.109, 6(2017), 461C484.

[24] Linfan Mao, Mathematical 4th crisis: to reality, *International J.Math. Combin.*, Vol.3 (2018), 147-158.

[25] Linfan Mao, Harmonic flows dynamics on animals in microscopic level with balance recovery, *International J.Math. Combin.*, Vol.1(2019), 1-44.

[26] Linfan Mao, Sciences dilemma C A review on science with applications, *Progress in Physics*, Vol.15, 2(2019), 78-85.

[27] Linfan Mao, Graphs, networks and natural reality – From intuitive abstracting to theory, *International J.Math. Combin.*, Vol.4(2019), 1-18.

[28] Linfan Mao, Mathematical elements on natural reality, *Bull.Cal.Math.Soc.*, Vol.111, 6(2019), 597C618.

[29] Linfan Mao, Dynamic network with e-index applications*International J.Math.Combin.*, Vol.4 (2020), 1-35.

[30] Linfan Mao, Reality or mathematical formality – Einsteins general relativity on multi-fields, *Chinese J.Math.Sci.*, Vol.1, No.1(2021), 1-16.

[31] Linfan Mao, Reality with Smarandachely denied axiom, *International J.Math. Combin.*, Vol.3(2021), 1-19.

[32] Linfan Mao, *Combinatorial Theory on the Universe*, Global Knowledge-Publishing House, 2023.

[33] Linfan Mao, *Field Theory on the Universe – Field Action in Emperor's Inner Canon* (in Chinese), Global Knowledge-Publishing House, 2025.

[34] William S.Massey, *Algebraic Topology: An Introduction*, Seringer-Verlag, New York, 1967.

[35] Quang Ho-Kim and Pham Xuan Yem, *Elementary Particles and Their Interactions: Conceptions and Phenomena*, Springer-Verlag Berlin Heidelberg, 1998.

[36] F.Smarandache, *A Unifying Field in Logics–Neutrosopy: Neturosophic Probability, Set, and Logic*, American research Press, Rehoboth, 1999.

[37] F.Smarandache, S-denying a theory, *International J.Math. Combin.*, Vol.2(2013), 1-7.

[38] F.Smarandache, NeutroGeometry & antigeometry are alternatives and generalizations of the non-Euclidean geometries, *Neutrosophic Sets and Systems*, Vol. 46 (2021), 456-476.

[39] F.Smarandache, V.Christianto, Fu Yuhua, R.Khrapko and J.Hutchison, *Unfolding the Labyrinth: Open Problems in Physics, Mathematics, Astrophysics and Other Areas of Science*, Hexis, Phoenix, 2006.

[40] W.B.Vasantha Kandasamy and F.Smarandache, *Basic Neutrosophic Algebraic Structures and Their Applications to Fuzzy and Neutrosophic Models*, Hexis, Church Rock, 2004.

International J.Math. Combin. Vol.1-Vol.2(2025), 24-50

Enumeration the Number of Spanning Trees of the Sequence of Some Families of Graphs That Have the Same Average Degree

S. N. Daoud[1,2,*]

1. Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah 41411, Saudi Arabia
2. Department of Mathematics and Computer Sciences, Faculty of Science
Menoufia University, Shebin E1 Kom 32511, Egypt

Mohmmed Aljohani

Department of Mathematics and Computer Sciences, Faculty of Science
Menoufia University, Shebin E1 Kom 32511, Egypt

E-mail: salamadaoud@gmail.com, mhjohany@taibahu.edu.sa

Abstract: In mathematics one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can create many new graphs from a given set of graphs. In this work, we compute the explicit formulas for the number of spanning trees of sequences of families of graphs of the same average degree four by electrically equivalent transformations and rules of weighted generating function. Finally, we compare the entropy of our graphs with other studied graphs with average degree being four.

Key Words: Number of spanning tree, electrically equivalent transformations, entropy.

AMS(2010): 05C30, 05C50, 05C63.

§1. Introduction

Deriving closed formulae of the number of spanning trees for various graphs has attracted the attention of a lot of researchers. The importance of this research line is in fact due to

(1) Solving some computationally hard problems such as the Steiner tree;

(2) Problem and traveling salesman problem [1];

(3) Counting the number of Eulerian circuits in a graph [2];

(4) Deriving formulas for different type of graphs can be helpful in identifying those graphs that contain the maximum number of spanning trees.

Such an investigation has practical consequences related to network reliability [5,6]. The number of spanning trees $\tau(G)$ of a finite connected undirected graph G is an acyclic $(n-1)$ - edge spanning subgraph. There exist various methods for finding this number. Kirchhoff [7] gave the famous matrix tree theorem: if D is the diagonal matrix of the degrees of G and A denote the adjacency matrix of G, Kirchhoff matrix $L = D - A$ has all of its cofactors equal to $\tau(G)$. Another method to count the complexity of a graph is using Laplacian eigenvalues.

[1]Received July 12, 2024. Accepted April 12, 2025

[2*]Correspondence: salamadaoud@gmail.com

Let G be a connected graph with k vertices. Kelmans and Chelnoknov [8] derived the following formula

$$\tau(G) = \frac{1}{k}\prod_{i=1}^{k-1}\mu_i,$$

where $k = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_k = 0$ are the eigenvalues of the Kirchhoff matrix L.

The degeneration of the graph through successive elimination of contraction of its edges represent the core of another way to compute the complexity of a graph [9]. If $G = (V, E)$ is a multigraph with $e \in E$, then $G.e$ is the graph obtained from G by contracting the degree until its endpoints are a single vertex. The formula for computing the number of spanning trees of a multigraph G is given by:

$$\tau(G) = \tau(G - e) + \tau(G.e)$$

This formula is beautiful but not practically useful (grows exponentially with the size of the graph-may be as many as $2^{|E(G)|}$ terms. For a summary of other results for calculating the umber of the spanning trees of graphs, see [10].

§2. Electrically Equivalent Transformations

Kirchhoff's motivation was study of electrical networks: an edge-weighted graph can be regarded as an electrical network, where weights are the conductance of the respective edges. The effect conductance between two specific vertices x, y can be written as the quotient of (weighted) number of spanning trees and the (weighted) number of so-called thickets, i.e., spanning forests with exactly two components and property that each of the components contains precisely one of the vertices x, y [11-13]. In the following, we list the effect of some simple transformations on the number of spanning trees. Let H be an edge weighted graph, H' be the corresponding electrically equivalent graph, $\tau(H)$ denotes the weighted number of spanning trees H.

(i) *Parallel edges*: If two parallel edges with conductances x and y in H are merged into a single edge with conductances $x + y$ in H', then $\tau(H') = \tau(H)$.

(ii) *Serial edges*: If two serial edges with conductances x and y in H are merged into a single edge with conductance $\frac{xy}{x+y}$ in H', then

$$\tau(H') = \frac{1}{x + y}\tau(H).$$

(iii) $\triangle - Y$ *transformation*: If a triangle with conductances a, b and c in H is changed into an electrically equivalent star graph with conductances

$$x = \frac{ab + bc + ca}{a}, \ y = \frac{ab + bc + ca}{b} \text{ and } z = \frac{ab + bc + ca}{c}$$

in H', then

$$\tau(H') = \frac{(ab + bc + ca)^2}{abc}\tau(H).$$

(iv) $Y - \triangle$ transformation: If a star graph with conductances x, y and z in H is changed into an electrically equivalent triangle with conductances

$$a = \frac{yz}{x+y+z}, \quad b = \frac{az}{x+y+z} \text{ and } c = \frac{xy}{x+y+z}$$

in H', then

$$\tau(H') = \frac{1}{a+b+c}\tau(H).$$

In this work, we compute the number of spanning trees of three sequences of graphs of average degree four based on Tridiminished icosahedron graph we named it $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ and $\mathcal{D}_n$ respectively.

§3. Number of Spanning Trees in the Sequences of $\mathcal{A}_n$ Graph

Consider the sequence of graphs $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ constructed as shown in Figure 1. According to this construction, the number of total vertices $|V(\mathcal{A}_n)|$ and edges $|E(\mathcal{A}_n)|$ are $|V(\mathcal{A}_n)| = 9n-6$ and $|E(\mathcal{A}_n)| = 18n-15$, $n = 1, 2, \cdots$. The average degree of $\mathcal{A}_n$ is in the large n limit which is 4.

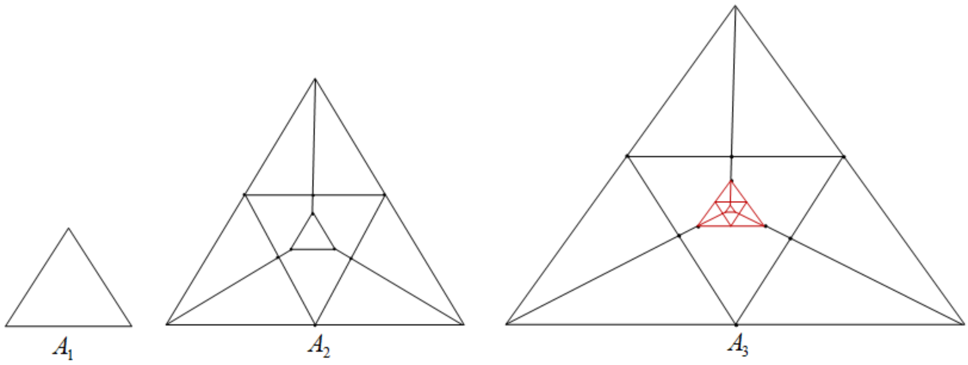


Figure 1. Some sequences of graph $\mathcal{A}_n$

Theorem 3.1 *For any integer $n \geq 1$, the number of spanning trees in the sequence of the graph $\mathcal{A}_n$ is given by*

$$\frac{2^{3n-7}\left((56-17\sqrt{14})(23+6\sqrt{14})^n+(23-6\sqrt{14})^n(56+17\sqrt{14})\right)^2\left(25(194+53\sqrt{14})+13\left(\frac{25}{1033+276\sqrt{14}}\right)^n(1798+481\sqrt{14})\right)^2}{91875\left(885+220\sqrt{14}-1325^n(1033+276\sqrt{14})^{1-n}\right)^2}$$

Proof We use the electrically equivalent transformation to transform $\mathcal{A}_i$ to $\mathcal{A}_{i-1}$. Figures $2-4$ following illustrate the transformation process from $\mathcal{A}_2$ to $\mathcal{A}_1$.

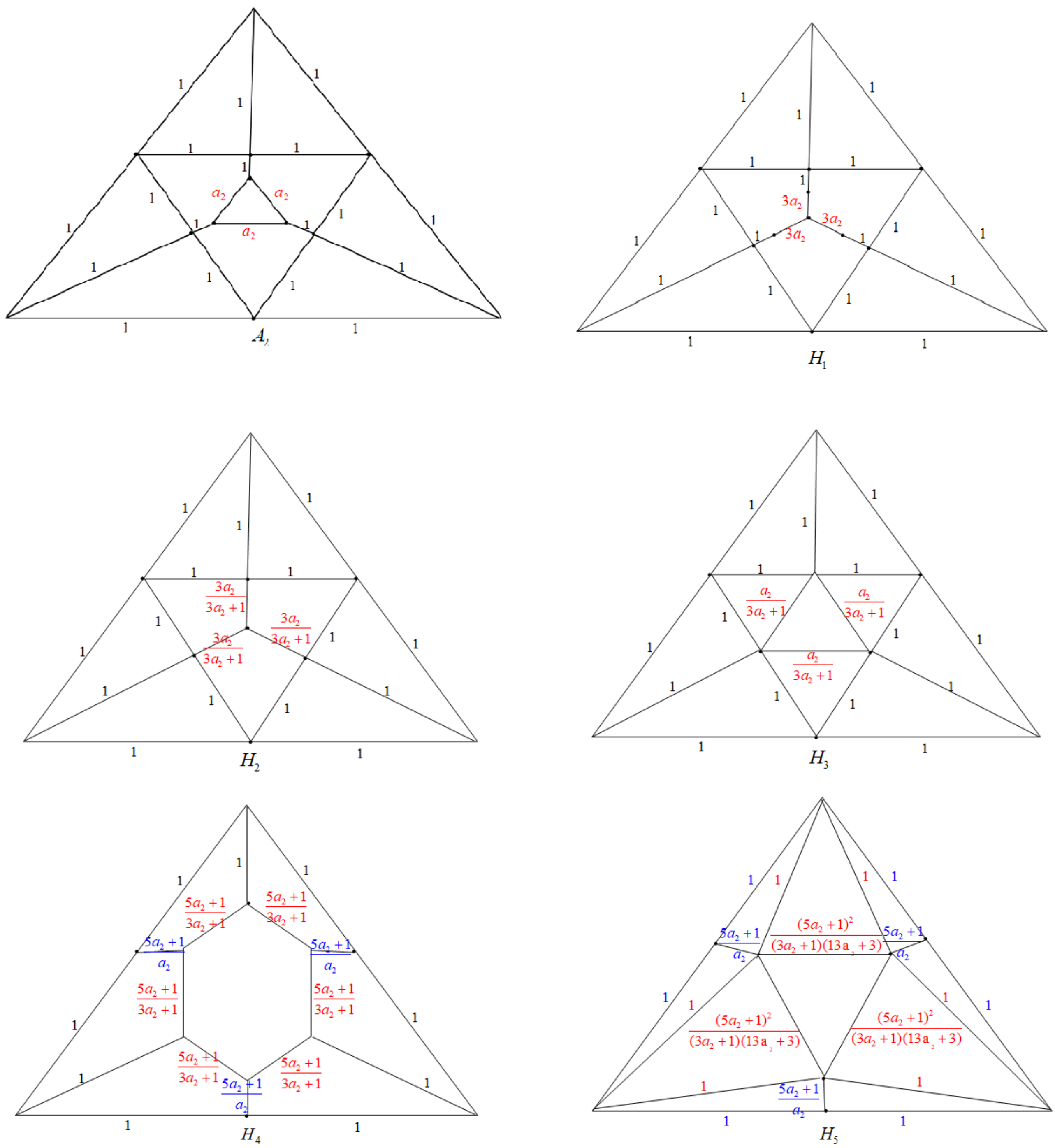
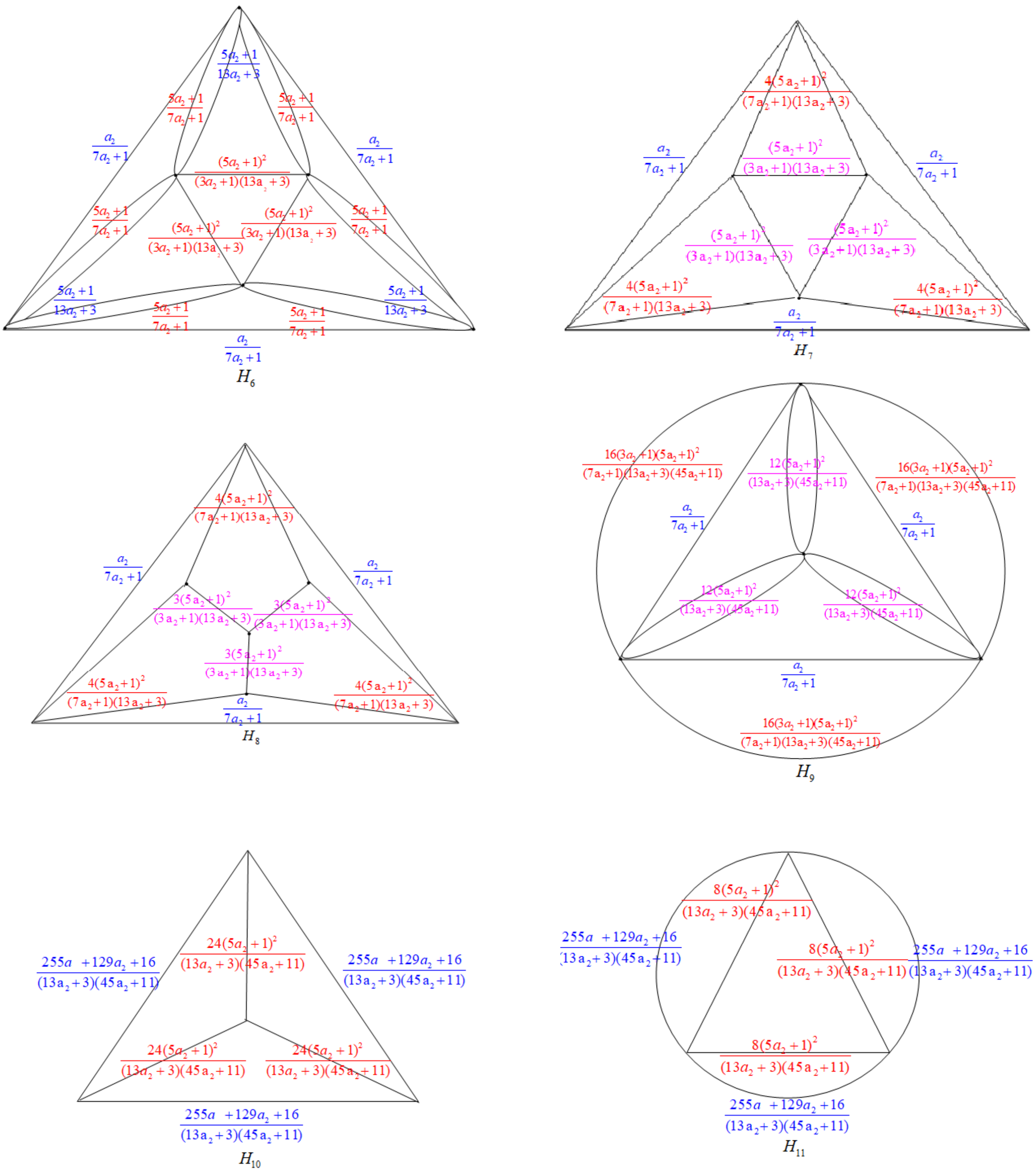


Figure 2



5a_2+1 / 13a_2+3
5a_2+1 / 7a_2+1
a_2 / 7a_2+1
(5a_2+1)^2 / (3a_2+1)(13a_2+3)
H_6
4(5a_2+1)^2 / (7a_2+1)(13a_2+3)
(5a_2+1)^2 / (3a_2+1)(13a_2+3)
a_2 / 7a_2+1
H_7
4(5a_2+1)^2 / (7a_2+1)(13a_2+3)
3(5a_2+1)^2 / (3a_2+1)(13a_2+3)
a_2 / 7a_2+1
H_8
16(3a_2+1)(5a_2+1)^2 / (7a_2+1)(13a_2+3)(45a_2+11)
12(5a_2+1)^2 / (13a_2+3)(45a_2+11)
a_2 / 7a_2+1
H_9
24(5a_2+1)^2 / (13a_2+3)(45a_2+11)
255a + 129a_2 + 16 / (13a_2+3)(45a_2+11)
H_10
8(5a_2+1)^2 / (13a_2+3)(45a_2+11)
255a + 129a_2 + 16 / (13a_2+3)(45a_2+11)
H_11

Figure 3

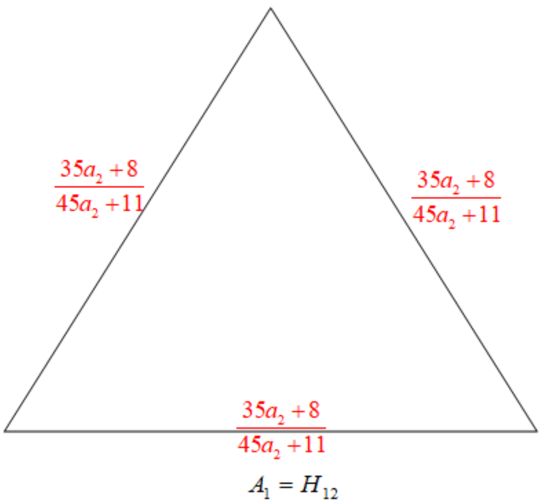


Figure 4

Using the properties given in Section 2, we have the following transformations:

$$\tau(H_1) = 9a_2\tau(\mathcal{A}_2), \qquad \tau(H_2) = \left[\frac{1}{3a_2+1}\right]^3 \tau(H_1),$$

$$\tau(H_3) = \frac{3a_2+1}{9a_2}\tau(H_2), \qquad \tau(H_4) = \left[\frac{(5a_2+1)^2}{a_2(3a_2+1)}\right]^3 \tau(H_3),$$

$$\tau(H_5) = \left[\frac{3a_2+1}{13a_2+3}\right]^3 \tau(H_4), \qquad \tau(H_6) = \left[\frac{a_2}{7a_2+1}\right]^3 \tau(H_5),$$

$$\tau(H_7) = \tau(H_6), \qquad \tau(H_8) = \frac{9(5a_2+1)^2}{(3a_2+1)(13a_2+3)}\tau(H_7),$$

$$\tau(H_9) = \left[\frac{(3a_2+1)(7a_2+1)(13a_2+3)}{(5a_2+1)(45a_2+11)}\right]^3 \tau(H_8), \quad \tau(H_{10}) = \tau(H_9),$$

$$\tau(H_{11}) = \frac{(13a_2+3)(45a_2+11)}{72(5a_2+1)^2}\tau(H_{10}) \qquad \text{and } \tau(\mathcal{A}_1) = \tau(H_{11}).$$

Combining these twelve transformations, we have

$$\tau(\mathcal{A}_2) = 8(45a_2+11)^2\tau(\mathcal{A}_1). \tag{1}$$

Further

$$\tau(\mathcal{A}_n) = \prod_{i=2}^{n} 8(45a_i+11)^2\tau(\mathcal{A}_1) = 3\times 8^{n-1}a_1^2\left[\prod_{i=2}^{n}(45a_i+11)\right]^2, \tag{2}$$

where $a_{i-1} = \frac{35a_i+8}{45a_i+11}, i = 2, 3, \ldots, n$. Its characteristic equation is $45\lambda^2 - 24\lambda - 8 = 0$, which have two roots

$$\lambda_1 = \frac{4-2\sqrt{14}}{15} \quad \text{and} \quad \lambda_2 = \frac{4+2\sqrt{14}}{15}.$$

Subtracting these two roots into both sides of $a_{i-1} = \frac{35a_i+8}{45a_i+11}$, we get

$$a_{i-1} - \frac{4-2\sqrt{14}}{15} = \frac{35a_i+8}{45a_i+11} - \frac{4-2\sqrt{14}}{15} = (23+6\sqrt{14}) \cdot \frac{a_i - \frac{4-2\sqrt{14}}{15}}{45a_i+11}, \tag{3}$$

$$a_{i-1} - \frac{4+2\sqrt{14}}{15} = \frac{35a_i+8}{45a_i+11} - \frac{4+2\sqrt{14}}{15} = (23-6\sqrt{14}) \cdot \frac{a_i - \frac{4+2\sqrt{14}}{15}}{45a_i+11}. \tag{4}$$

Let $b_i = \frac{a_i - \frac{4-2\sqrt{14}}{15}}{a_i - \frac{4+2\sqrt{14}}{15}}$. Then by Eqs. (3) and (4), we get $b_{i-1} = \left(\frac{1033+276\sqrt{14}}{25}\right) b_i$ and $b_i = \left(\frac{1033+276\sqrt{14}}{25}\right)^{n-i} b_n$. Therefore,

$$a_i = \frac{\left(\frac{(033+276\sqrt{14}}{25}\right)^{n-i} \left(\frac{4+2\sqrt{14}}{15}\right) b_n - \frac{4-2\sqrt{14}}{15}}{\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-i} b_n - 1}.$$

Thus

$$a_1 = \frac{2\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-1} (194+53\sqrt{14}) - 13(4-2\sqrt{14})}{3\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-1} (177+44\sqrt{14}) - 195}. \tag{5}$$

Using the expression $a_{n-1} = \frac{35a_n+8}{45a_n+11}$ and denoting the coefficients of $35a_n+8$ and $45a_n+11$ as α_n and β_n we have

$$45a_n + 11 = \alpha_0 (35a_n + 8) + \beta_0 (45a_n + 11),$$

$$45a_{n-1} + 11 = \frac{\alpha_1 (35a_n + 8) + \beta_1 (45a_n + 11)}{\alpha_0 (35a_n + 8) + \beta_0 (45a_n + 11)},$$

$$45a_{n-2} + 11 = \frac{\alpha_2 (35a_n + 8) + \beta_2 (45a_n + 11)}{\alpha_1 (35a_n + 8) + \beta_1 (45a_n + 11)},$$

$$\vdots$$

$$45a_{n-i} + 11 = \frac{\alpha_i (35a_n + 8) + \beta_i (45a_n + 11)}{\alpha_{i-1} (35a_n + 8) + \beta_{i-1} (45a_n + 11)}, \tag{6}$$

$$45a_{n-(i+1)} + 11 = \frac{\alpha_{i+1} (35a_n + 8) + \beta_{i+1} (45a_n + 11)}{\alpha_i (35a_n + 8) + \beta_i (45a_n + 11)}, \tag{7}$$

$$\vdots$$

$$45a_2 + 11 = \frac{\alpha_{n-2} (35a_n + 8) + \beta_{n-2} (45a_n + 11)}{\alpha_{n-3} (35a_n + 8) + \beta_{n-3} (45a_n + 11)}$$

Substituting Eq.(6) into Eq.(2), we obtain

$$\tau(\mathcal{A}_n) = 3 \times 8^{n-1} a_1^2 \left[\alpha_{n-2}(35a_n + 8) + \beta_{n-2}(45a_n + 11)\right]^2. \tag{8}$$

where $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 45, \beta_1 = 11$. By the expression $a_{n-1} = \frac{35a_n+8}{45a_n+11}$ and Eqs. (6) and (7), we have

$$\alpha_{i+1} = 46\alpha_i - 25\alpha_{i-1}; \beta_{i+1} = 46\beta_i - 25\beta_{i-1}. \tag{9}$$

The characteristic equation of Eq.(9) is $\mu^2 - 46\mu + 25 = 0$ which have two roots

$$\mu_1 = 23 + 6\sqrt{14} \quad \text{and} \quad \mu_2 = 23 - 6\sqrt{14}.$$

The general solutions of Eq. (9) are

$$\alpha_i = c_1\mu_1^i + c_2\mu_2^i; \beta_i = d_1\mu_1^i + d_2\mu_2^i.$$

Using the initial conditions $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 45, \beta_1 = 11$, yields

$$\begin{aligned} \alpha_i &= \frac{15\sqrt{14}}{56}(23+6\sqrt{14})^i - \frac{15\sqrt{14}}{56}(23-6\sqrt{14})^i; \beta_i \\ &= \left(\frac{7-\sqrt{14}}{14}\right)(23+6\sqrt{14})^i + \left(\frac{7+\sqrt{14}}{14}\right)(23-6\sqrt{14})^i. \end{aligned} \tag{10}$$

If $a_n = 1$, it means that $\mathcal{A}_n$ without any electrically equivalent transformation. Plugging Eq. (10) into Eq.(8), we have

$$\begin{aligned} \tau(\mathcal{A}_n) &= 3 \times 8^{n-1} a_1^2 \left[\left(\frac{1568 + 421\sqrt{14}}{56}\right)(23+6\sqrt{14})^{n-2}\right. \\ &\quad \left. + \left(\frac{1568 - 421\sqrt{14}}{56}\right)(23-6\sqrt{14})^{n-2}\right]^2, \ n \geq 2. \end{aligned} \tag{11}$$

When $n = 1, \tau(\mathcal{A}_1) = 3$ which satisfies Eq.(11). Therefore, the number of spanning trees in the sequence of the graph $\mathcal{A}_n$ is given by

$$\begin{aligned} \tau(\mathcal{A}_n) &= 3 \times 8^{n-1} a_1^2 \left[\left(\frac{1568 + 421\sqrt{14}}{56}\right)(23+6\sqrt{14})^{n-2}\right. \\ &\quad \left. + \left(\frac{1568 - 421\sqrt{14}}{56}\right)(23-6\sqrt{14})^{n-2}\right]^2, \ n \geq 1, \end{aligned} \tag{12}$$

where

$$a_1 = \frac{2\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-1}(194+53\sqrt{14}) - 13(4-2\sqrt{14})}{3\left(\frac{1033+276\sqrt{14}}{25}\right)^{n-1}(177+44\sqrt{14}) - 195}, \ n \geq 1. \tag{13}$$

Inserting Eq. (13) into Eq.(12) we obtain the result. □

§4. Number of Spanning Trees in the Sequences of $\mathcal{B}_n$ Graph

Consider the sequence of graphs $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ constructed shown in Figure 5. According to this construction, the number of total vertices $|V(\mathcal{B}_n)|$ and edges $|E(\mathcal{B}_n)|$ are $|\mathrm{V}(\mathcal{B}_n)| = 9n - 6$ and $|E(\mathcal{B}_n)| = 18n - 15$ for $n = 1, 2, \cdots$ The average degree of $\mathcal{B}_n$ is in the large n limit which is 4.

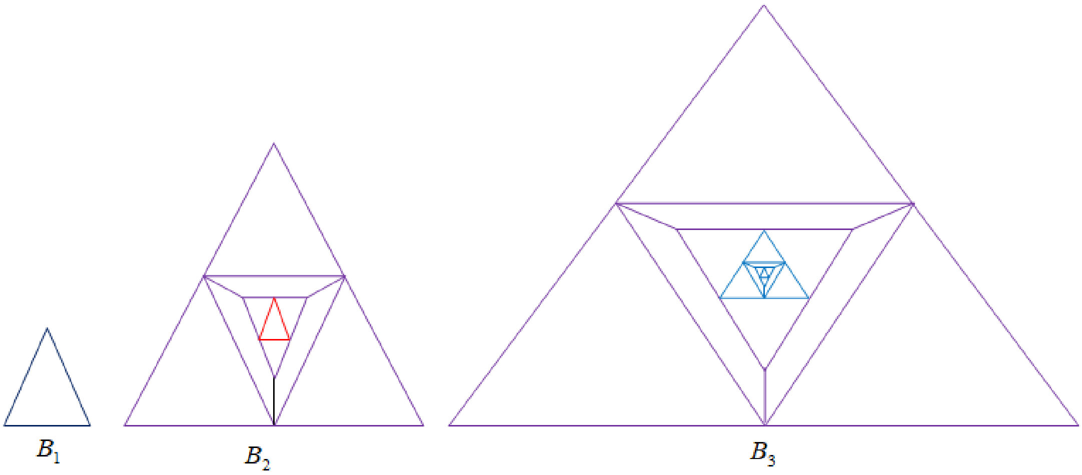


Figure 5. Some sequences of graph $\mathcal{B}_n$

Theorem 4.1 *For $n \geq 1$, the number of spanning trees in the sequence of $\mathcal{B}_n$ graph is given by*

$$\frac{(41+3\sqrt{185})^{2n}\left(72^{3n+4}(259+19\sqrt{185})+(1673+123\sqrt{185})^{n}(-2701+231\sqrt{185})-2(1673-123\sqrt{185})^{n}(1321233+97139\sqrt{185})\right)^{2}}{262848\left(2^{3n+1}(1673+123\sqrt{185})+(21+\sqrt{185})(1673+123\sqrt{185})^{n}\right)^{2}}$$

Proof We use the electrically equivalent transformation to transform $\mathcal{B}_i$ to $\mathcal{B}_{i-1}$. Figures $6-8$ illustrate the transformation process from $\mathcal{B}_2$ to $\mathcal{B}_1$.

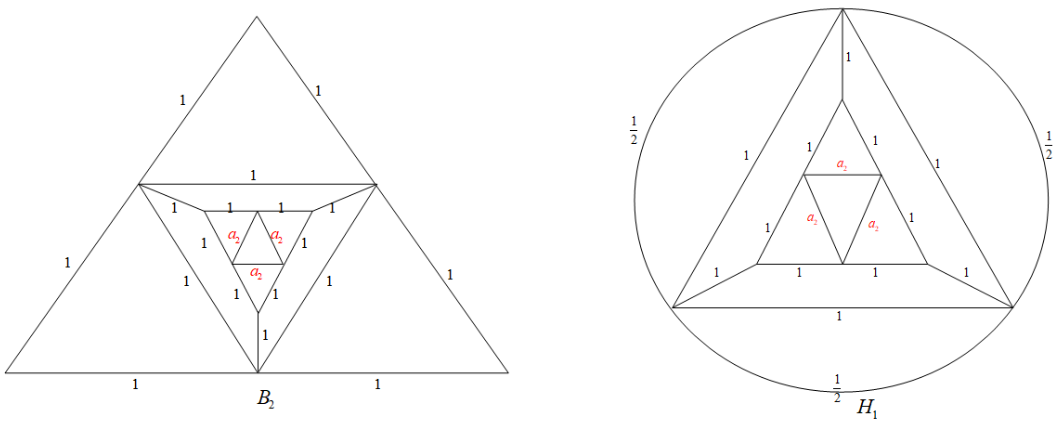
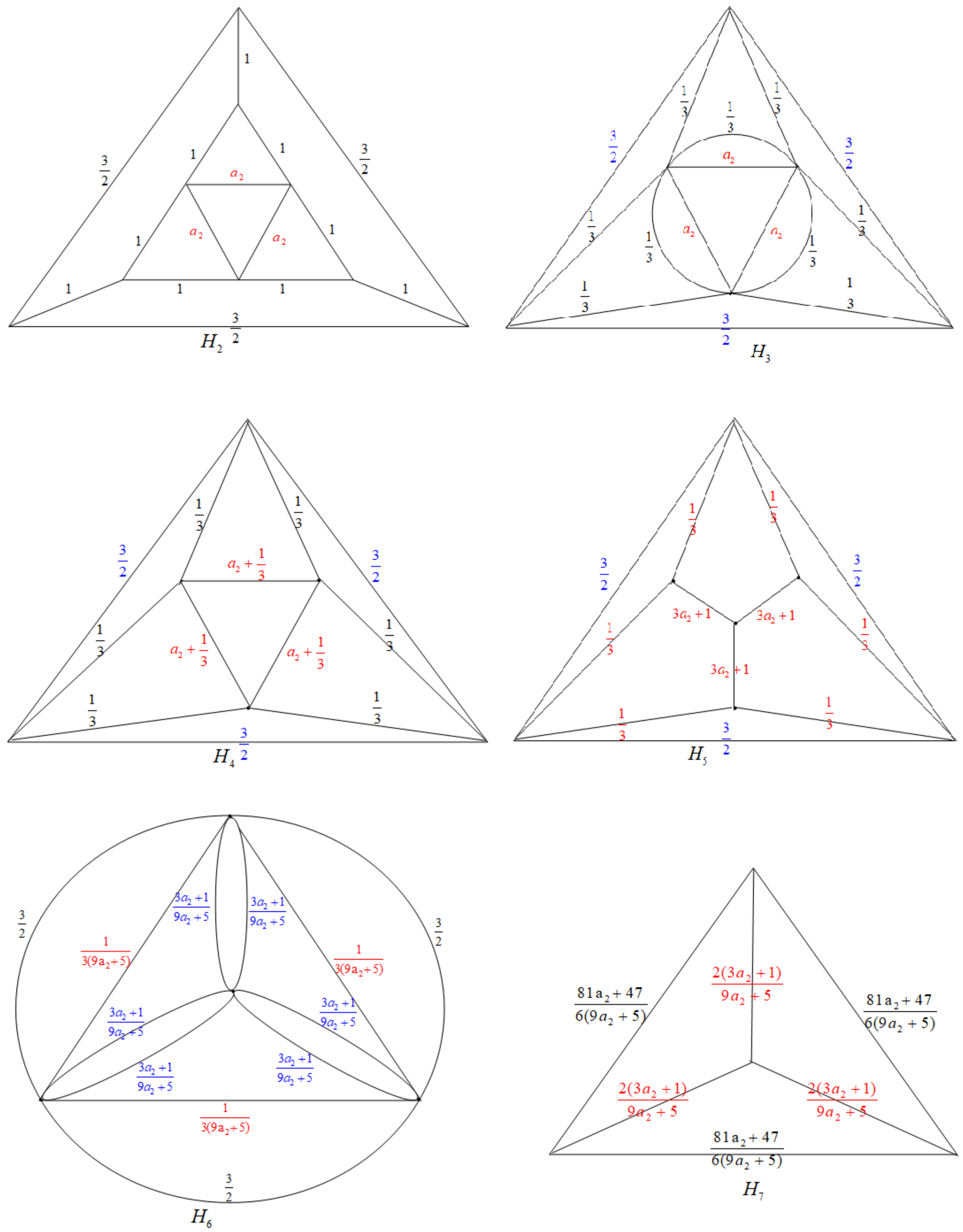


Figure 6



1
3/2
a_2
H_2
1/3
H_3
a_2 + 1/3
H_4
3a_2 + 1
H_5
(3a_2 + 1)/(9a_2 + 5)
1/(3(9a_2 + 5))
H_6
2(3a_2 + 1)/(9a_2 + 5)
(81a_2 + 47)/(6(9a_2 + 5))
H_7

Figure 7

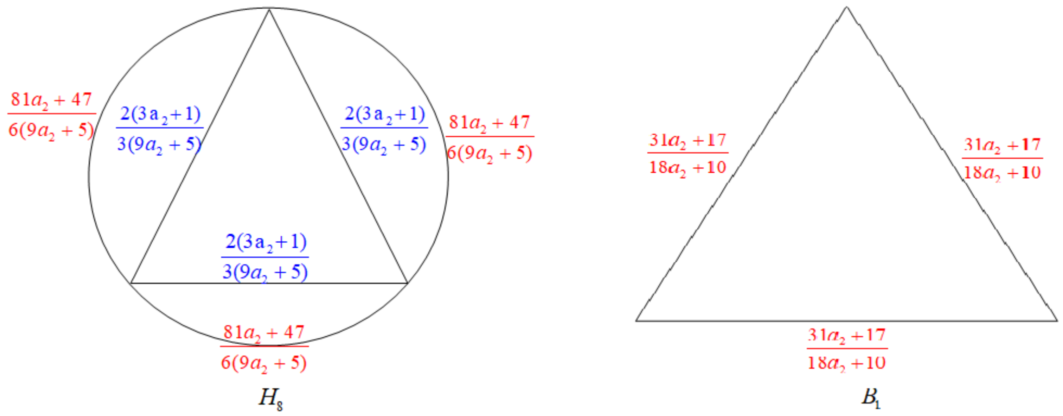


Figure 8

Using the properties given in Section 2, we have the following transformations:

$$\tau(H_1) = \left[\frac{1}{2}\right]^3 \tau(\mathcal{B}_2), \qquad \tau(H_2) = \tau(H_1),$$

$$\tau(H_3) = \left[\frac{1}{3}\right]^3 \tau(H_2), \qquad \tau(H_4) = \tau(H_3),$$

$$\tau(H_5) = (9a_2 + 3)\,\tau(H_4),$$

$$\tau(H_6) = \left[\frac{3}{9a_2 + 5}\right]^3 \tau(H_5), \qquad \tau(H_7) = \tau(H_6),$$

$$\tau(H_8) = \frac{9a_2 + 5}{6(3a_2 + 1)}\tau(H_7) \text{ and } \quad \tau(\mathcal{B}_1) = \tau(H_8).$$

Combining these nine transformations, we have

$$\tau(\mathcal{B}_2) = 4(18a_2 + 10)^2 \tau(\mathcal{B}_1). \tag{14}$$

Further

$$\tau(\mathcal{B}_n) = \prod_{i=2}^{n} 4(18a_i + 10)^2 \tau(\mathcal{B}_1) = 3 \times 4^{n-1} a_1^2 \left[\prod_{i=2}^{n} (18a_i + 10)\right]^2, \tag{15}$$

where $a_{i-1} = \frac{31a_i+17}{18a_i+10}, i = 2, 3, \cdots, n$. Its characteristic equation is $18\lambda^2 - 21\lambda - 17 = 0$ which have two roots

$$\lambda_1 = \frac{7 - \sqrt{185}}{12} \quad \text{and} \quad \lambda_2 = \frac{7 + \sqrt{185}}{12}.$$

Subtracting these two roots into both sides of $a_{i-1} = \frac{31a_i+17}{18a_i+10^2}$, we get

$$a_{i-1} - \frac{7 - \sqrt{185}}{12} = \frac{31a_i + 17}{18a_i + 10} - \frac{7 - \sqrt{185}}{12} = (41 + 3\sqrt{185}) \cdot \frac{a_i - \left(\frac{7-\sqrt{185}}{12}\right)}{2(18a_i + 10)}, \tag{16}$$

$$a_{i-1} - \frac{7+\sqrt{185}}{12} = \frac{31a_i+17}{18a_i+10} - \frac{7+\sqrt{185}}{12} = (41-3\sqrt{185}) \cdot \frac{a_i - \left(\frac{7+\sqrt{185}}{12}\right)}{2\left(18a_i+10\right)}. \tag{17}$$

Let $b_i = \frac{a_i - \frac{7-\sqrt{185}}{12}}{a_i - \frac{7+\sqrt{185}}{12}}$. Then by Eqs. (16) and (17), we get

$$b_{i-1} = \left(\frac{1673+123\sqrt{185}}{8}\right) b_i \quad \text{and} \quad b_i = \left(\frac{1673+123\sqrt{185}}{8}\right)^{n-i} b_n.$$

Therefore

$$a_i = \frac{\left(\frac{1673+122\sqrt{185}}{8}\right)^{n-i}\left(\frac{7+\sqrt{185}}{12}\right) b_n - \frac{7-\sqrt{185}}{12}}{\left(\frac{1673+123\sqrt{185}}{8}\right)^{n-i} b_n - 1}.$$

Thus,

$$a_1 = \frac{\left(\frac{1673+123\sqrt{155}}{8}\right)^{n-1}(83+7\sqrt{185})+4(7-\sqrt{185})}{3\left(\frac{1673+123\sqrt{185}}{8}\right)^{n-1}(21+\sqrt{185})+48}. \tag{18}$$

Using the expression $a_{n-1} = \frac{31a_n+17}{18a_n+10}$ and denoting the coefficients of $31a_n+17$ and $18a_n+10$ as α_n and β_n, we have

$$18a_n + 10 = \alpha_0\left(31a_n+17\right) + \beta_0\left(18a_n+10\right),$$

$$18a_{n-1} + 10 = \frac{\alpha_1\left(31a_n+17\right) + \beta_1\left(18a_n+10\right)}{\alpha_0\left(31a_n+17\right) + \beta_0\left(18a_n+10\right)},$$

$$18a_{n-2} + 10 = \frac{\alpha_2\left(31a_n+17\right) + \beta_2\left(18a_n+10\right)}{\alpha_1\left(31a_n+17\right) + \beta_1\left(18a_n+10\right)},$$

$$\vdots$$

$$18a_{n-i} + 10 = \frac{a_i\left(31a_n+17\right) + \beta_i\left(18a_n+10\right)}{\alpha_{i-1}\left(31a_n+17\right) + \beta_{i-1}\left(18a_n+10\right)}, \tag{19}$$

$$18a_{n-(i+1)} + 10 = \frac{\alpha_{i+1}\left(31a_n+17\right) + \beta_{i+1}\left(18a_n+10\right)}{\alpha_i\left(31a_n+17\right) + \beta_i\left(18a_n+10\right)}, \tag{20}$$

$$\vdots$$

$$18a_2 + 10 = \frac{\alpha_{n-2}\left(31a_n+17\right) + \beta_{n-2}\left(18a_n+10\right)}{\alpha_{n-3}\left(31a_n+17\right) + \beta_{n-3}\left(18a_n+10\right)}.$$

Substituting Eq.(19)into Eq.(15), we obtain

$$\tau\left(\mathcal{B}_n\right) = 3 \times 4^{n-1} a_1^2 \left[\alpha_{n-2}\left(31a_n+17\right) + \beta_{n-2}\left(18a_n+10\right)\right]^2. \tag{21}$$

where $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 18, \beta_1 = 10$. By the expression $a_{n-1} = \frac{31a_n+7}{18a_n+10}$ and Eqs. (19) and (20), we have

$$\alpha_{i+1} = 41\alpha_i - 4\alpha_{i-1}, \quad \beta_{i+1} = 41\beta_i - 4\beta_{i-1}.$$

Its characteristic equation is $\mu^2 - 41\mu + 4 = 0$ which have two roots

$$\mu_1 = \frac{41 + 3\sqrt{185}}{2} \quad \text{and} \quad \mu_2 = \frac{41 - 3\sqrt{185}}{2}$$

and with the general solution

$$\alpha_i = c_1\mu_1^i + c_2\mu_2^i, \quad \beta_i = d_1\mu_1^i + d_2\mu_2^i.$$

Using the initial conditions $\alpha_0 = 0, \beta_0 = 1$ and $\alpha_1 = 18, \beta_1 = 10$, yields

$$\alpha_i = \frac{6\sqrt{185}}{185}\left(\frac{41+3\sqrt{185}}{2}\right)^i - \frac{6\sqrt{185}}{185}\left(\frac{41-3\sqrt{185}}{2}\right)^i, \tag{22}$$

$$\beta_i = \left(\frac{555-21\sqrt{185}}{1110}\right)\left(\frac{41+3\sqrt{185}}{2}\right)^i + \left(\frac{555+21\sqrt{185}}{1110}\right)\left(\frac{41-3\sqrt{185}}{2}\right)^i. \tag{23}$$

If $a_n = 1$, it means that $\mathcal{B}_n$ without any electrically equivalent transformation. Plugging Eq.(23) into Eq.(21), we have

$$\begin{aligned}\tau(\mathcal{B}_n) &= 3 \times 4^{n-1}a_1^2\left[\left(\frac{518+38\sqrt{185}}{37}\right)\left(\frac{41+3\sqrt{185}}{2}\right)^{n-2}\right.\\ &\left.\quad + \left(\frac{518-38\sqrt{185}}{37}\right)\left(\frac{41-3\sqrt{185}}{2}\right)^{n-2}\right]^2, \ n \geq 2.\end{aligned} \tag{24}$$

When $n = 1, \tau(\mathcal{B}_1) = 3$ which satisfies Eq.(24).

Therefore, the number of spanning trees in the sequence of Tridiminished icosahedron graph is given by

$$\begin{aligned}\tau(\mathcal{B}_n) &= 3 \times 4^{n-1}a_1^2\left[\left(\frac{518+38\sqrt{185}}{37}\right)\left(\frac{41+3\sqrt{185}}{2}\right)^{n-2}\right.\\ &\left.\quad + \left(\frac{518-38\sqrt{185}}{37}\right)\left(\frac{41-3\sqrt{185}}{2}\right)^{n-2}\right]^2, \ n \geq 1,\end{aligned} \tag{25}$$

where

$$a_1 = \frac{\left(\frac{1673+123\sqrt{185}}{8}\right)^{n-1}(83+7\sqrt{185}) + 4(7-\sqrt{185})}{3\left(\frac{1673+123\sqrt{185}}{8}\right)^{n-1}(21+\sqrt{185}) + 48}, \ n \geq 1. \tag{26}$$

Inserting Eq.(26) into Eq.(25) we obtain the result. □

§5. Number of Spanning Trees in the Sequences of $\mathcal{C}_n$ Graph

Consider the sequence of graphs $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ constructed as shown in Figure 5. According to this construction, the number of total vertices $|V(\mathcal{C}_n)|$ and edges $|E(\mathcal{C}_n)|$ are $|\mathrm{V}(\mathcal{C}_n)| = 9n - 6$ and $|E(\mathcal{C}_n)| = 18$ for $n - 15, n = 1, 2, \cdots$. The average degree of $\mathcal{C}_n$ is in the large n limit which is 4.

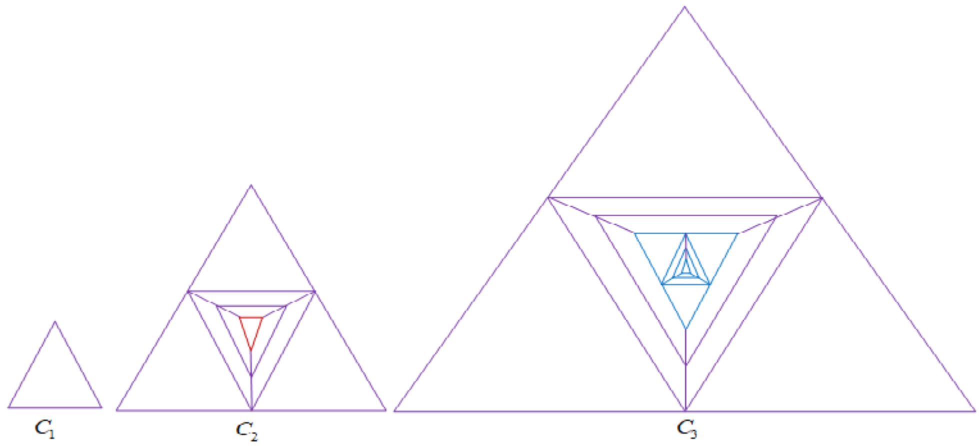


Figure 9. Some sequences of $\mathcal{C}_n$

Theorem 5.1 *For $n \geq 1$, the number of spanning trees in the sequence of $\mathcal{C}_n$ is given by*

$$\frac{32^{-7-n}(61+\sqrt{3705})^{2n}\left(\frac{1}{60}(-45+\sqrt{3705})-\frac{1}{435}\left(\frac{1}{8}(3713-61\sqrt{3705})\right)^{1-n}(600+11\sqrt{3705})\right)^{2}\left(5681-93\sqrt{3705}+\left(\frac{1}{8}(3713-61\sqrt{3705})\right)^{n}(5681+93\sqrt{3705})\right)^{2}}{61009\left(1+\frac{1}{29}2^{1-3n}(131+\sqrt{3705})(3713+61\sqrt{3705})^{n-1}\right)^{2}}$$

Proof We use the electrically equivalent transformation to transform $\mathcal{C}_i$ to $\mathcal{C}_{i-1}$. Figures $10-12$ illustrate the transformation process from $\mathcal{C}_2$ to $\mathcal{C}_1$.

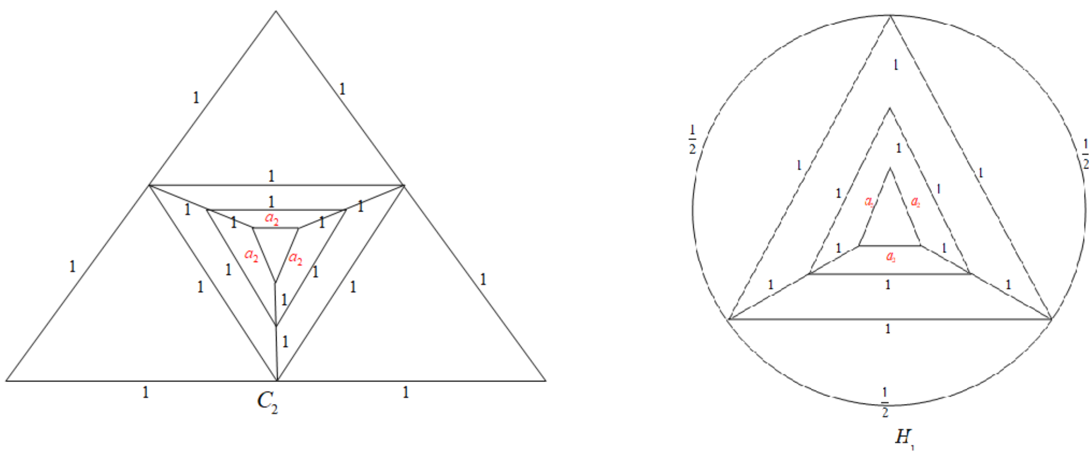


Figure 10

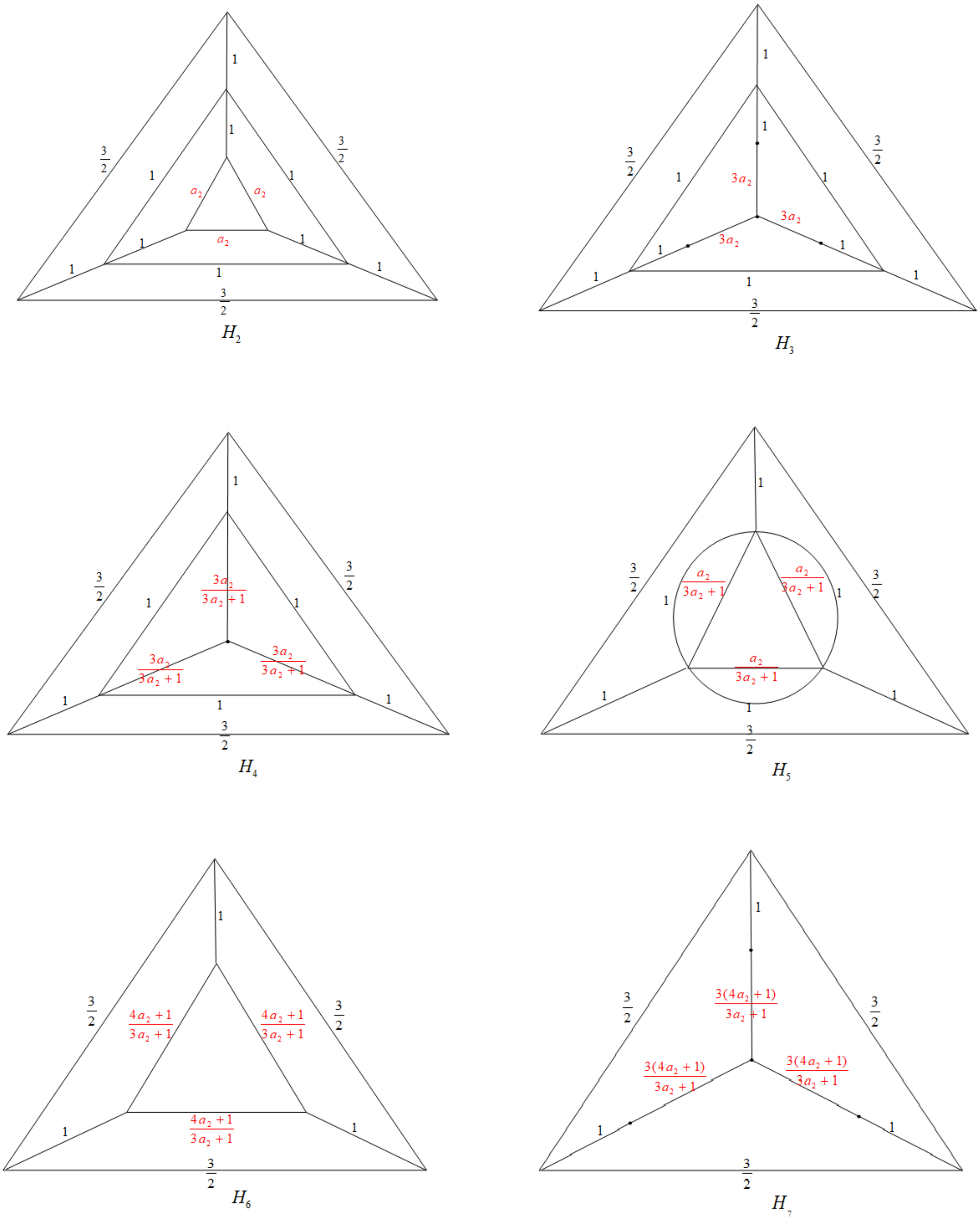

H_2
a_2
H_3
$3a_2$
H_4
$\frac{3a_2}{3a_2+1}$
H_5
$\frac{a_2}{3a_2+1}$
H_6
$\frac{4a_2+1}{3a_2+1}$
H_7
$\frac{3(4a_2+1)}{3a_2+1}$
1
$\frac{3}{2}$

Figure 11

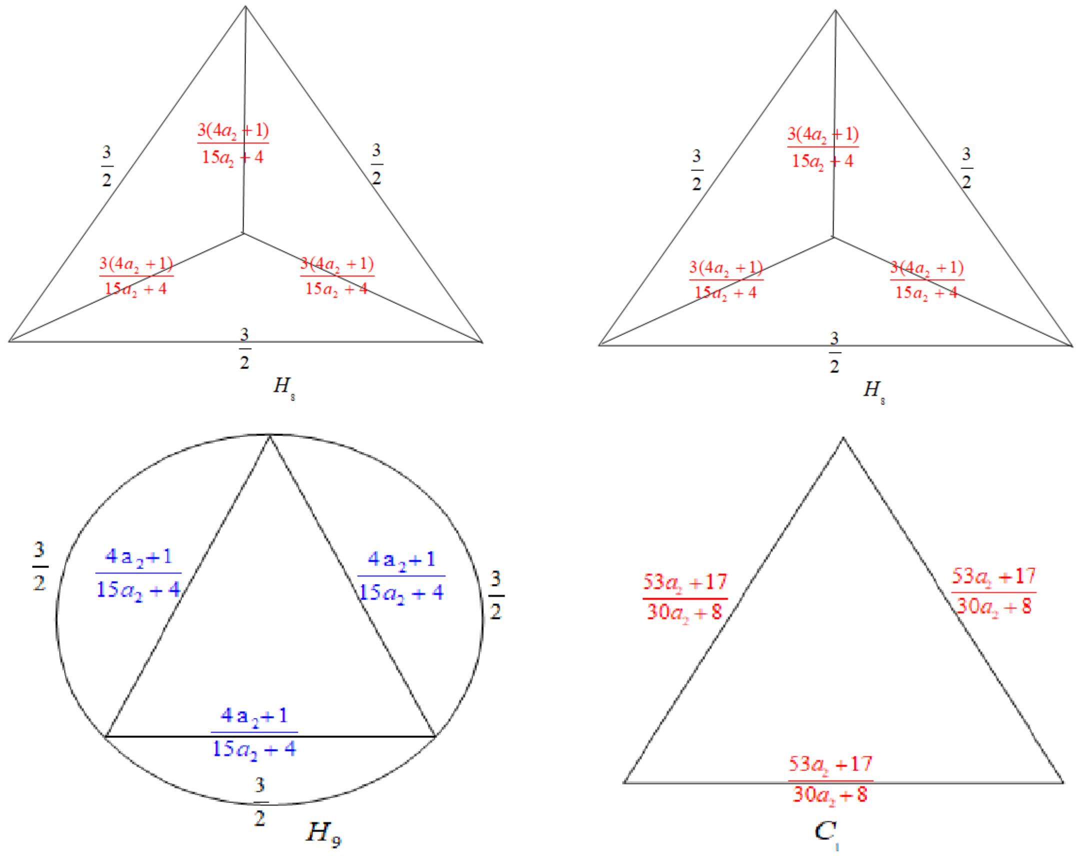


Figure 12

Using the properties given in section 2, we have the following transformations:

$$\tau(H_1) = \left[\frac{1}{2}\right]^3 \tau(\mathcal{C}_2), \qquad \tau(H_2) = \tau(H_1),$$

$$\tau(H_3) = 9a_2\tau(H_2), \qquad \tau(H_4) = \left(\frac{1}{3a_2+1}\right)^2 \tau(H_3),$$

$$\tau(H_5) = \frac{3a_2+1}{9a_2}\tau(H_4), \qquad \tau(H_6) = \tau(H_5),$$

$$\tau(H_7) = 9\left(\frac{4a_2+1}{3a_2+1}\right)\tau(H_6), \qquad \tau(H_8) = \left[\frac{(3a_2+1)}{15a_2+4}\right]^3 \tau(H_7),$$

$$\tau(H_9) = \left(\frac{15a_2+4}{9(4a_2+1)}\right)\tau(H_8), \quad \tau(\mathcal{C}_1) = \tau(H_9).$$

Combining these ten transformations, we have

$$\tau(\mathcal{C}_2) = 2(30a_2+8)^2 \tau(\mathcal{C}_1). \tag{27}$$

Further

$$\tau\left(\mathcal{C}_n\right)=\prod_{i=2}^{n} 2\left(30a_i+8\right)^2\tau\left(\mathcal{C}_1\right)=3\times 2^{n-1}a_1^2\left[\prod_{i=2}^{n}\left(30a_i+8\right)\right]^2, \tag{28}$$

where $a_{i-1}=\frac{53a_i+14}{30a_i+8}, i=2,3,\cdots,n$. Its characteristic equation is $30\lambda^2-45\lambda-14=0$ which have two roots

$$\lambda_1=\frac{45-\sqrt{3705}}{60} \quad \text{and} \quad \lambda_2=\frac{45+\sqrt{3705}}{60}.$$

Subtracting these two roots into both sides of $a_{i-1}=\frac{53a_i+14}{30a_i+8}$, we get

$$a_{i-1}-\frac{45-\sqrt{3705}}{60}=\frac{53a_i+14}{30a_i+8}-\frac{45-\sqrt{3705}}{60}=(61+\sqrt{3705})\cdot\frac{a_i-\frac{45-\sqrt{3705}}{60}}{2\left(30a_i+8\right)}, \tag{29}$$

$$a_{i-1}-\frac{45+\sqrt{3705}}{60}=\frac{53a_i+14}{30a_i+8}-\frac{45+\sqrt{3705}}{60}=(61-\sqrt{3705})\cdot\frac{a_i-\frac{45+\sqrt{3705}}{60}}{2\left(30a_i+8\right)}. \tag{30}$$

Let $b_i=\frac{a_i-\frac{45-\sqrt{3705}}{60}}{a_i-\frac{45+\sqrt{3705}}{60}}$.Then by Eqs. (29) and (30), we get

$$b_{i-1}=\left(\frac{3713+61\sqrt{3705}}{8}\right)b_i, \quad \text{and} \quad b_i=\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-i}b_n.$$

Therefore,

$$a_i=\frac{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-i}\left(\frac{45+\sqrt{3705}}{60}\right)b_n-\frac{45-\sqrt{3705}}{60}}{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-i}b_n-1}.$$

Thus

$$a_1=\frac{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-1}\left(\frac{600+11\sqrt{3705}}{435}\right)+\left(\frac{45-\sqrt{3705}}{60}\right)}{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-1}\left(\frac{131+\sqrt{3705}}{116}\right)+1}. \tag{31}$$

Using the expression $a_{n-1}=\frac{53a_n+14}{30a_n+8}$ and denoting the coefficients of $53a_n+14$ and $30a_n+8$ as α_n and β_n, we have

$$\begin{aligned}
30a_n+8&=\alpha_0\left(53a_n+14\right)+\beta_0\left(30a_n+8\right)\\
30a_{n-1}+8&=\frac{\alpha_1\left(53a_n+14\right)+\beta_1 30a_n+8)}{\alpha_0\left(53a_n+14\right)+\beta_0\left(30a_n+8\right)}\\
30a_{n-2}+8&=\frac{\alpha_2\left(53a_n+14\right)+\beta_2\left(30a_n+8\right)}{\alpha_1\left(53a_n+14\right)+\beta_1\left(30a_n+8\right)}\\
&\vdots
\end{aligned}$$

$$30a_{n-i}+8=\frac{\alpha_i\left(53a_n+14\right)+\beta_i\left(30a_n+8\right)}{\alpha_{i-1}\left(53a_n+14\right)+\beta_{i-1}\left(30a_n+8\right)}, \tag{32}$$

$$30a_{n-(i+1)}+8=\frac{\alpha_{i+1}\left(53a_n+14\right)+\beta_{i+1}\left(30a_n+8\right)}{\alpha_i\left(53a_n+14\right)+\beta_i\left(30a_n+8\right)}, \tag{33}$$

$$\vdots$$

$$30a_2+8=\frac{\alpha_{n-2}\left(53a_n+14\right)+\beta_{n-2}\left(30a_n+8\right)}{\alpha_{n-3}(53+14)+\beta_{n-3}\left(30a_n+8\right)}$$

Substituting Eq.(31) into Eq.(28), we obtain

$$\tau\left(\mathcal{C}_n\right)=3\times 2^{n-1}a_1^2\left[\alpha_{n-2}\left(53a_n+14\right)+\beta_{n-2}\left(30a_n+8\right)\right]^2, \tag{34}$$

where $\alpha_0=0,\beta_0=1$ and $\alpha_1=30,\beta_1=8$. By the expression $a_{n-1}=\frac{53a_n+14}{30a_n+8}$ and Eqs. (32) and (33), we have

$$\alpha_{i+1}=61\alpha_i-4\alpha_{i-1},\quad \beta_{i+1}=61\beta_i-4\beta_{i-1}. \tag{35}$$

The characteristic equation of Eq. (35) is $\mu^2-61\mu+4=0$ which have two roots

$$\mu_1=\frac{61+\sqrt{3705}}{2}\quad\text{and}\quad \mu_2=\frac{61+\sqrt{3705}}{2}$$

and the general solutions of Eq.(35) are

$$\alpha_i=c_1\mu_1^i+c_2\mu_2^i,\quad \beta_i=d_1\mu_1^i+d_2\mu_2^i.$$

Substituting the initial conditions $\alpha_0=0,\beta_0=1$ and $\alpha_1=30,\beta_1=8$, yields

$$\alpha_i=\frac{2\sqrt{3705}}{247}\left(\frac{61+\sqrt{3705}}{2}\right)^i-\frac{2\sqrt{3705}}{247}\left(\frac{61+\sqrt{3705}}{2}\right)^i;$$

$$\beta_i=\left(\frac{3705-45\sqrt{3705}}{7410}\right)\left(\frac{61+\sqrt{3705}}{2}\right)^i+\left(\frac{3705+45\sqrt{3705}}{7410}\right)\left(\frac{61-\sqrt{3705}}{2}\right)^i. \tag{36}$$

If $a_n=1$, it means that $\mathcal{C}_n$ without any electrically equivalent transformation. Plugging Eq. (36) into Eq.(34), we have

$$\begin{aligned}\tau\left(\mathcal{C}_n\right) &= 3\times 2^{n-1}a_1^2\left[\left(\frac{4693+771\sqrt{3705}}{247}\right)\left(\frac{61+\sqrt{3705}}{2}\right)^{n-2}\right.\\ &\quad\left.+\left(\frac{4693-771\sqrt{3705}}{247}\right)\left(\frac{61-\sqrt{3705}}{2}\right)^{n-2}\right]^2,\ n\geq 2.\end{aligned} \tag{37}$$

When $n=1,\tau\left(\mathcal{C}_1\right)=3$ which satisfies Eq. (37). Therefore, the number of spanning trees in the sequence of $\mathcal{C}_n$ graph is given by

$$\begin{aligned}\tau\left(\mathcal{C}_n\right) &= 3\times 2^{n-1}a_1^2\left[\left(\frac{4693+771\sqrt{3705}}{247}\right)\left(\frac{61+\sqrt{3705}}{2}\right)^{n-2}\right.\\ &\left.+\left(\frac{4693-771\sqrt{3705}}{247}\right)\left(\frac{61-\sqrt{3705}}{2}\right)^{n-2}\right]^2,\ n\geq 1, \end{aligned} \tag{38}$$

where

$$a_1=\frac{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-1}\left(\frac{600+11\sqrt{3705}}{435}\right)+\left(\frac{45-\sqrt{3705}}{60}\right)}{\left(\frac{3713+61\sqrt{3705}}{8}\right)^{n-1}\left(\frac{131+\sqrt{3705}}{116}\right)+1}, n\geq 1. \tag{39}$$

Inserting Eq.(39) into Eq.(38) we obtain the result. □

§6. Number of Spanning Trees in the Sequences of $\mathcal{D}_n$ Graph

Consider the sequence of graphs $\mathcal{D}_1,\mathcal{D}_2,\ldots,\mathcal{D}_n$ constructed as shown in Figure 7. According to this construction, the number of total vertices $|V(\mathcal{D}_n)|$ and edges $|E(\mathcal{D}_n)|$ are $|\mathrm{V}(\mathcal{D}_n)| = 9n-6$ and $|E(\mathcal{D}_n)| = 18n-15,\ n=1,2,\cdots$. The average degree of $\mathcal{D}_n$ is in the large n limit which is 4.

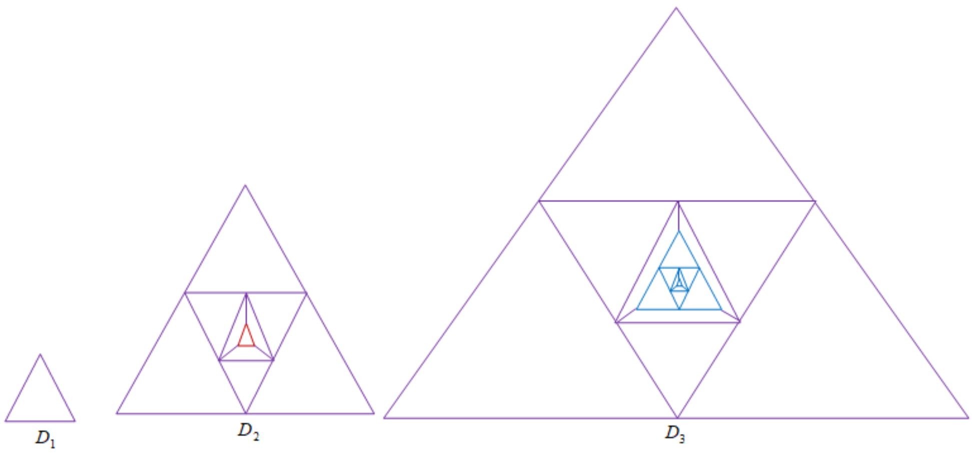
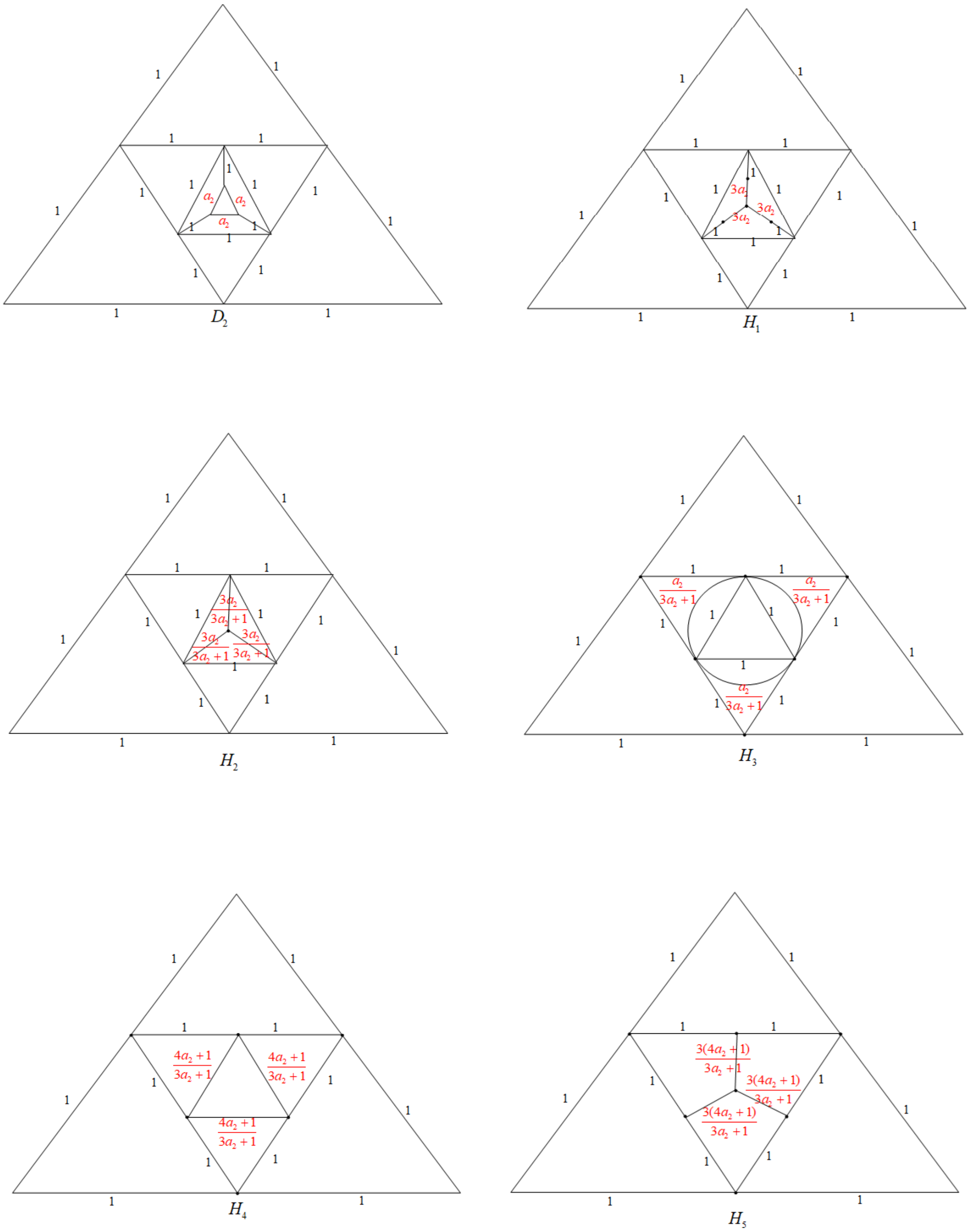


Figure 13. Some sequences of $\mathcal{D}_n$

Theorem 6.1 *For $n\geq 1$, the number of spanning trees in the sequence of $\mathcal{D}_n$ is given by*

$$\frac{\left(59+\sqrt{3477}\right)^{2n}\left(7772^{1+n}\left(50996+865\sqrt{3477}\right)+\left(3479+59\sqrt{3477}\right)^{n}\left(-7924083+129923\sqrt{3477}\right)+851\left(3479-59\sqrt{3477}\right)^{n}\left(96594537+1638137\sqrt{3477}\right)\right)^{2}}{16119372\left(-8512^{n}\left(3479+59\sqrt{3477}\right)+\left(2861+39\sqrt{3477}\right)\left(3479+59\sqrt{3477}\right)^{n}\right)^{2}}$$

Proof We use the electrically equivalent transformation to transform $\mathcal{D}_i$ to $\mathcal{D}_{i-1}$. Figure $14-16$ illustrate the transformation process from $\mathcal{D}_2$ to $\mathcal{D}_1$.



D_2
a_2
H_1
$3a_2$
H_2
$\frac{3a_2}{3a_2+1}$
H_3
$\frac{a_2}{3a_2+1}$
H_4
$\frac{4a_2+1}{3a_2+1}$
H_5
$\frac{3(4a_2+1)}{3a_2+1}$

Figure 14

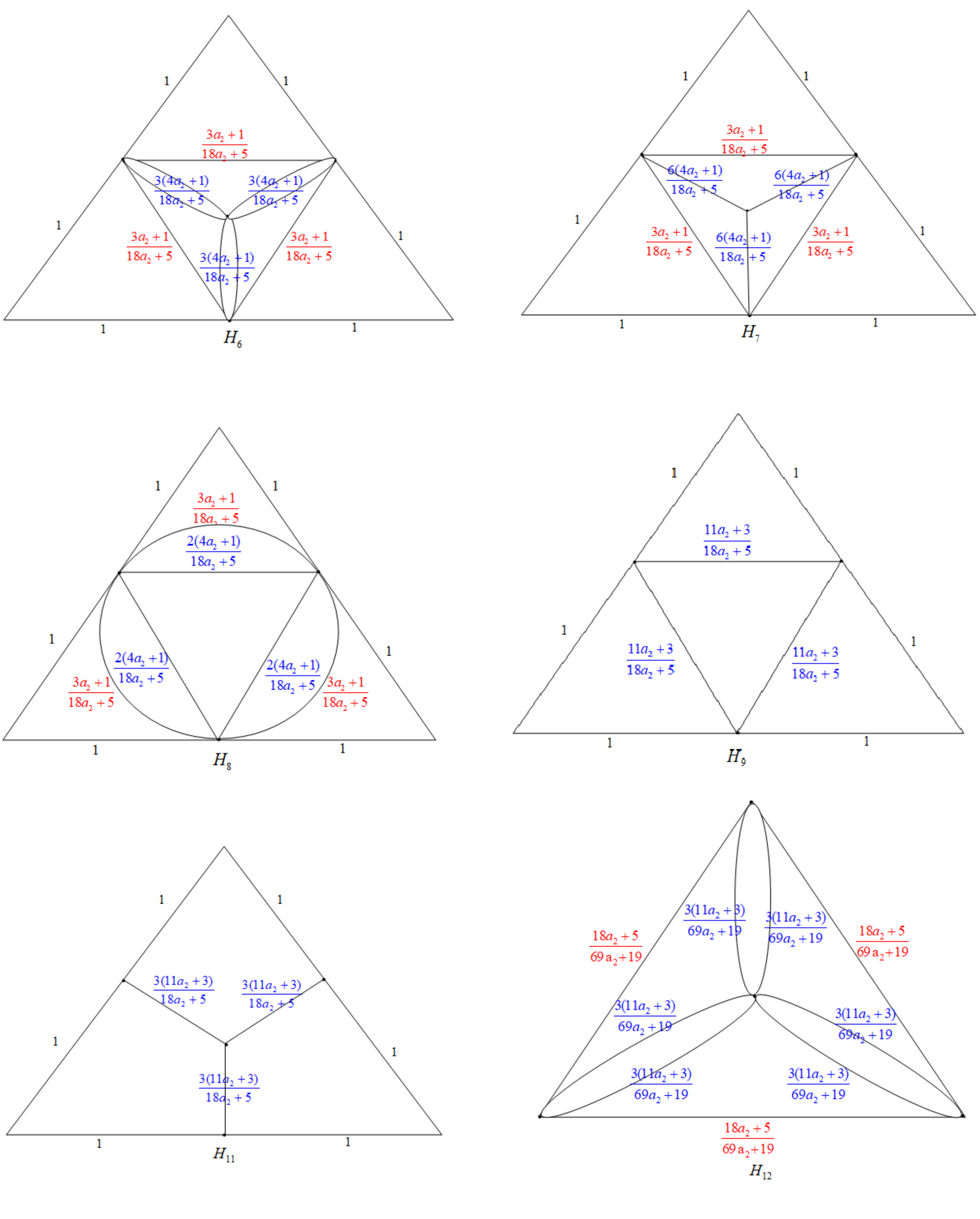

1
$\frac{3a_2+1}{18a_2+5}$
$\frac{3(4a_2+1)}{18a_2+5}$
H_6
$\frac{6(4a_2+1)}{18a_2+5}$
H_7
$\frac{2(4a_2+1)}{18a_2+5}$
H_8
$\frac{11a_2+3}{18a_2+5}$
H_9
$\frac{3(11a_2+3)}{18a_2+5}$
H_{11}
$\frac{18a_2+5}{69a_2+19}$
$\frac{3(11a_2+3)}{69a_2+19}$
H_{12}

Figure 15

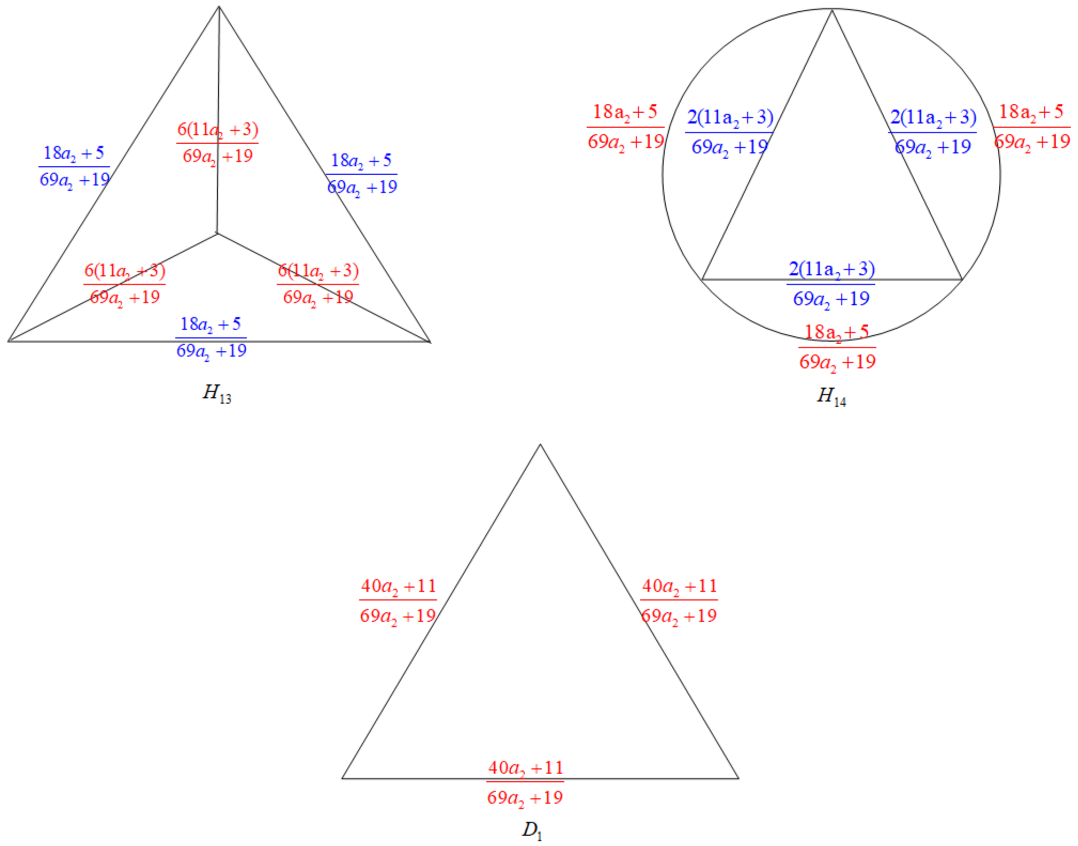


Figure 16

Using the properties given in Section 2, we have the following transformations:

$$\tau(H_1) = 9a_2\tau(\mathcal{D}_2), \qquad \tau(H_2) = \left[\frac{1}{3a_2+1}\right]^3 \tau(H_1),$$

$$\tau(H_3) = \frac{3a_2+1}{9a_2}\tau(H_2), \qquad \tau(H_4) = \tau(H_3),$$

$$\tau(H_5) = 9\left(\frac{4a_2+1}{3a_2}\right)\tau(H_4), \qquad \tau(H_6) = \left[\frac{3a_2+1}{18a_2+5}\right]^3 \tau(H_5),$$

$$\tau(H_7) = \tau(H_6), \qquad \tau(H_8) = \frac{18a_2+5}{18(4a_2+1)}\tau(H_7),$$

$$\tau(H_9) = \tau(H_8), \qquad \tau(H_{10}) = 9\left(\frac{11a_2+3}{18a_2+5)}\right)\tau(H_9),$$

$$\tau(H_{11}) = \left[\frac{18a_2+5}{69a_2+19}\right]^3 \tau(H_{10}), \qquad \tau(H_{12}) = \tau(H_{11}),$$

$$\tau(H_{13}) = \left[\frac{69a_2+19}{18(11a_2+3)}\right]\tau(H_{12}), \quad \tau(\mathcal{D}_1) = \tau(H_{13}).$$

Combining these fourteen transformations, we have

$$\tau\left(\mathcal{D}_2\right)=4\left(69a_2+19\right)^2\tau\left(\mathcal{D}_1\right). \tag{40}$$

Further

$$\tau\left(\mathcal{D}_n\right)=\prod_{i=2}^{n}4\left(69a_i+19\right)^2\tau\left(\mathcal{D}_1\right)=3\times4^{n-1}a_1^2\left[\prod_{i=2}^{n}\left(69a_i+19\right)\right]^2, \tag{41}$$

where

$$a_{i-1}=\frac{40a_i+11}{69a_i+19}, i=2,3,\ldots,n. \tag{42}$$

Its characteristic equation is $69\lambda^2-21\lambda-11=0$ which have two roots

$$\lambda_1=\frac{21-\sqrt{3477}}{138} \text{ and } \lambda_2=\frac{21+\sqrt{3477}}{138}.$$

Subtracting these two roots into both sides of $a_{i-1}=\frac{40a_i+11}{69a_i+19}$, we get

$$a_{i-1}-\frac{21-\sqrt{3477}}{138}=\frac{40a_i+11}{69a_i+19}-\frac{21-\sqrt{3477}}{138}=(59+\sqrt{3477})\cdot\frac{a_i-\frac{21-\sqrt{3477}}{138}}{2\left(69a_i+19\right)}, \tag{43}$$

$$a_{i-1}-\frac{21+\sqrt{3477}}{138}=\frac{40a_i+11}{69a_i+19}-\frac{21+\sqrt{3477}}{138}=(59-\sqrt{3477})\cdot\frac{a_i-\frac{21+\sqrt{3477}}{138}}{2\left(69a_i+19\right)}. \tag{44}$$

Let $b_i=\frac{a_i-\frac{21-\sqrt{3477}}{138}}{a_i-\frac{21+\sqrt{3477}}{138}}$. Then by Eqs. (42) and (43), we get

$$b_{i-1}=\left(\frac{3479+59\sqrt{3477}}{2}\right)b_i \text{ and } b_i=\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-i}b_n.$$

Therefore,

$$a_i=\frac{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-i}\left(\frac{21+\sqrt{3477}}{138}\right)b_n-\frac{21-\sqrt{3477}}{138}}{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-i}b_n-1}.$$

Thus,

$$a_1=\frac{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-1}\left(\frac{2127+40\sqrt{3477}}{2553}\right)-\left(\frac{21-\sqrt{3477}}{138}\right)}{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-1}\left(\frac{2861+39\sqrt{3477}}{1702}\right)-1}. \tag{45}$$

Using the expression $a_{n-1}=\frac{40a_n+11}{69a_n+19}$ and denoting the coefficients of $40a_n+11$ and $69a_n+19$ as α_n and β_n, we have

$$\begin{aligned}69a_n+19&=\alpha_0\left(40a_n+11\right)+\beta_0\left(69a_n+19\right),\\69a_{n-1}+19&=\frac{\alpha_1\left(40a_n+11\right)+\beta_1\left(69a_n+19\right)}{\alpha_0\left(40a_n+11\right)+\beta_0\left(69a_n+19\right)},\end{aligned}$$

$$69a_{n-2}+19=\frac{\alpha_2\left(40a_n+11\right)+\beta_2\left(69a_n+19\right)}{\alpha_1\left(40a_n+11\right)+\beta_1\left(69a_n+19\right)},$$

$$\vdots$$

$$69a_{n-i}+19=\frac{\alpha_i\left(40a_n+11\right)+\beta_i\left(69a_n+19\right)}{\alpha_{i-1}\left(40a_n+11\right)+\beta_{i-1}\left(69a_n+19\right)}, \tag{46}$$

$$69a_{n-(i+1)}+19=\frac{\alpha_{i+1}\left(40a_n+11\right)+\beta_{i+1}(69+19)}{\alpha_i\left(40a_n+11\right)+\beta_i\left(69a_n+19\right)}, \tag{47}$$

$$\vdots$$

$$69a_2+19=\frac{\alpha_{n-2}\left(40a_n+11\right)+\beta_{n-2}\left(69a_n+19\right)}{\alpha_{n-3}(40+11)+\beta_{n-3}\left(69a_n+19\right)}.$$

Substituting Eq.(45) into Eq.(41), we obtain

$$\tau\left(\mathcal{D}_n\right)=3\times 4^{n-1}a_1^2\left[\alpha_{n-2}\left(40a_n+11\right)+\beta_{n-2}\left(69a_n+19\right)\right]^2, \tag{48}$$

where $\alpha_0=0, \beta_0=1$ and $\alpha_1=69, \beta_1=19$. By the expression $a_{n-1}=\frac{40a_n+11}{69a_n+19}$ and Eqs. (45) and (46), we have

$$\alpha_{i+1}=59\alpha_i-\alpha_{i-1}, \quad \beta_{i+1}=59\beta_i-\beta_{i-1}$$

The characteristic equation of Eq.(48) is $\mu^2-59\mu+1=0$ which have two roots

$$\mu_1=\frac{59+\sqrt{3477}}{2} \quad \text{and} \quad \mu_2=\frac{59+\sqrt{3477}}{2}$$

and the general solutions of Eq.(48) are

$$\alpha_i=c_1\mu_1^i+c_2\mu_2^i, \quad \beta_i=d_1\mu_1^i+d_2\mu_2^i.$$

Substituting the initial conditions $\alpha_0=0, \beta_0=1$ and $\alpha_1=69, \beta_1=19$, yields

$$\alpha_i=\frac{23\sqrt{3477}}{1159}\left(\frac{59+\sqrt{3477}}{2}\right)^i-\frac{23\sqrt{3477}}{1159}\left(\frac{59-\sqrt{3477}}{2}\right)^i,$$

$$\beta_i=\left(\frac{1159-7\sqrt{3477}}{2318}\right)\left(\frac{59+\sqrt{3477}}{2}\right)^i+\left(\frac{1159+7\sqrt{3477}}{2318}\right)\left(\frac{59-\sqrt{3477}}{2}\right)^i. \tag{49}$$

If $a_n=1$, it means that $\mathcal{D}_n$ without any electrically equivalent transformation. Plugging Eq. (49) into Eq.(47), we have

$$\begin{aligned}\tau\left(\mathcal{D}_n\right) &= 3\times 4^{n-1}a_1^2\left[\left(\frac{50996+865\sqrt{3477}}{1159}\right)\left(\frac{59+\sqrt{3477}}{2}\right)^{n-2}\right.\\ &\quad\left.+\left(\frac{50996-865\sqrt{3477}}{1159}\right)\left(\frac{59-\sqrt{3477}}{2}\right)^{n-2}\right]^2, \ n\geq 2.\end{aligned} \tag{50}$$

When $n = 1, \tau(\mathcal{D}_1) = 3$ which satisfies Eq.(50). Therefore the number of spanning trees in the sequence of $\mathcal{D}_n$ graph is given by

$$\begin{aligned}\tau(\mathcal{D}_n) &= 3 \times 4^{n-1} a_1^2 \left[\left(\frac{50996 + 865\sqrt{3477}}{1159} \right) \left(\frac{59 + \sqrt{3477}}{2} \right)^{n-2} \right. \\ &\left. + \left(\frac{50996 - 865\sqrt{3477}}{1159} \right) \left(\frac{59 - \sqrt{3477}}{2} \right)^{n-2} \right]^2, \ n \geq 1, \end{aligned} \tag{51}$$

where

$$a_1 = \frac{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-1} \left(\frac{2127+40\sqrt{3477}}{2553}\right) - \left(\frac{21-\sqrt{3477}}{138}\right)}{\left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-1} \left(\frac{2861+39\sqrt{3477}}{1702}\right) - 1}, n \geq 1. \tag{52}$$

Inserting Eq.(52) into Eq.(51) we obtain the result. $\square$

§7. Numerical Results

Table 1. illustrates some values of the number of spanning trees in the graphs $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ and $\mathcal{D}_n$.

n	$\tau(\mathcal{A}_n)$	$\tau(\mathcal{B}_n)$	$\tau(\mathcal{C}_n)$	$\tau(\mathcal{D}_n)$
1	3	3	3	3
2	44376	27648	26934	31212
3	732328128	185150208	200050668	434307072
4	12101944579584	1239020203008	1485574848600	6043816558272
5	199991606950244352	8291475833499648	11031866024955312	84105744275374848
6	3304977193903255289856	55486239089142448128	81922542024547792224	1170415440635048951808

§8. Spanning Tree Entropy

After having explicit Formulas for the number of spanning trees of the sequence of the three families of graphs $\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n$ and $\mathcal{D}_n$, we can calculate its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined as in [14] as

For a graph G,

$$Z(G) = \lim_{n \to \infty} \frac{\ln \tau(G)}{|V(G)|} \tag{53}$$

and particularly,

$$Z(\mathcal{A}_n) = \frac{1}{9}(\ln[8] + 2\ln[23 + 6\sqrt{14}]) = 1.07918497,$$

$$Z(\mathcal{B}_n) = \frac{2}{9}\ln[41 + 3\sqrt{185}] = 0.9787402606,$$

$$Z(\mathcal{C}_n) = \frac{7\ln[2]}{9} - \frac{2}{9}\ln[61-\sqrt{3705}] = 0.9903046082,$$

$$Z(\mathcal{D}_n) = \frac{2}{9}\ln[59+\sqrt{3477}] = 1.060088273$$

Now we compare the value of entropy in our graphs with other graphs. It is clear that the entropy of the $\mathcal{A}_n$ graph is larger than the other three graphs and the entropy of the $\mathcal{B}_n$ graph is smaller than the other three graphs. In addition the entropy of graphs $\mathcal{A}_n$ and $\mathcal{D}_n$ is larger than the fractal scale free lattice [15] which has the entropy 1.040 and the entropy of all four graphs is smaller than two dimensional Sierpinski gasket [16] which has the entropy 1.166 of the same average degree 4.

§9. Conclusions

In this work, we enumerate the number of spanning trees in the sequences of three sequences of graphs of average degree four based on using electrically equivalent transformations. An advantage of this method lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

References

[1] Applegate D. L., Bixby, Chvtal R. E. V., Cook W. J., *The Traveling Salesman Problem: A Computational Study*, Princeton University Press, (2006).

[2] Zhang F., Yong X., Asymptotic enumeration theorems for the number of spanning trees and Eulerian trail in circulant digraphs & graphs, *Sci. China*, Ser. A43(2), (1999), 264-271.

[3] Kirby E. C., Klein D. J., Mallion R. B., Pollak, P., Sachs S. H., A theorem for counting s-panning trees in general chemical graphs and its particular application to toroidal fullerene, *Croat. Chem. Acta*, 77 (2004), 263-278.

[4] Brown T. J. N., Mallion R. B., Pollak P., Roth A., Some methods for counting the spanning trees in labelled molecular graphs, examined in relation to certain fullerenes, *Discrete Appl. Math.*, 67 (1996), 51-66.

[5] Boesch F. T., Salyanarayana A., Suffel C.L., A survey of some network reliability analysis and synthesis results, *Networks*, 54 (2009), 99-107.

[6] Boesch F. T., On unreliability polynomials and graph connectivity in reliable network synthesis, *J. Graph Theory*, 10 (1986), 339-352.

[7] Kirchhoff G. G., ber die Auflsung der Gleichungen auf welche man bei der Untersucher der linearen Verteilung galuanischer Strome gefhit wird, *Ann. Phg. Chem.*, 72 (1847), 497-508.

[8] Kelmans A. K., Chelnokov V. M., A certain polynomial of a graph and graphs with an extremal number of trees, *Journal of Combinatorial Theory* B, Vol.16 (1974), 197-214.

[9] Daoud S. N., The deletion-contraction method for counting the number of spanning trees of graphs, *European Journal of Physical Plus*, Vol.130, No. 10, Oct.(2015), 1-14.

[10] Daoud S. N., Number of spanning trees of Cartesian and composition products of graphs and Chebyshev polynomials, *IEEE Access*, Vol. 7 (2019), 71142 - 71157.

[11] Teufl E., Wagner S., Determinant identities for Laplace matrices, *Linear Algebra Appl.*, 432(2010), 441-457.

[12] Daoud S. N., Saleha W., Complexity trees of the sequence of some nonahedral graphs generated by triangle *Heliyon*, 6(9) Sep. (2020).

[13] Liu J. B. and Daoud S. N., Number of spanning trees in the sequence of some graphs, *Complexity*, Vol.2019 - Article ID 4271783—https://doi.org/10.1155/2019/4271783.

[14] Lyons R., Asymptotic enumeration of spanning trees, *Combin. Probab. Comput.*, 14 (2005), 491-522.

[15] Zhang Z., Liu H., Wu B., Zou T., Spanning trees in a fractal scale -free lattice, *Phys. Rev.* E, 83 016116 (2011).

[16] Chang S., Chen L., Yang W., Spanning trees on the Sierpinski gasket, *J. Stat. Phys.*, 126 (2007), 649-667.

International J.Math. Combin. Vol.1-Vol.2(2025), 51-66

Connected Monophonic Eccentric Domination Number of Corona Product of Some Standard Graphs

P. Titus[1], J. Ajitha Fancy[2], Santhakumaran[3] and A. Radhakrishnan[4]

1. Department of Mathematics, University College of Engineering Nagercoil, Anna University, Tirunelveli Region, Nagercoil - 629 004, India

2. Department of Mathematics, Scott Christian College (Autonomous), Nagercoil - 629 003, India

3. Department of Mathematics, Hindustan Institute of Technology and Science, Chennai - 603 103, India

4. Department of Information Technology, University College of Engineering Nagercoil, Nagercoil - 629 004, India

E-mail: titusvino@yahoo.com, ajithafancy@gmail.com, apskumar1953@yahoo.co.in, radhakrishnan.a@auttvl.ac.in

Abstract: For any two vertices u and v in a connected graph G, the monophonic distance $d_m(u,v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The monophonic eccentricity $e_m(v)$ of a vertex v in G is the maximum monophonic distance from v to a vertex of G. A set $S \subseteq V$ is a *connected monophonic eccentric dominating set* if S is a monophonic eccentric dominating set and the induced subgraph $\langle S \rangle$ is connected. The *connected monophonic eccentric domination number* $\gamma_{cme}(G)$ is the cardinality of a minimum connected monophonic eccentric dominating set of G. In this paper, we determine the connected monophonic eccentric domination number of corona product of some standard graphs.

Key Words: Monophonic eccentric vertex, monophonic eccentric dominating set, Smarandachely dominating on subgraph H^+ or H^-, monophonic eccentric domination number, connected monophonic eccentric dominating set, connected monophonic eccentric domination number.

AMS(2010): 05C12.

§1. Introduction

By a graph $G = (V, E)$ we mean a non-trivial finite undirected graph without loops and multiple edges. The *order* and *size* of G are denoted by p and q, respectively. For basic graph theoretic terminology and results we refer to [1, 4]. For any two vertices u and v in a connected graph G, the *distance* $d(u,v)$ is the length of a shortest $u-v$ path in G. For each vertex v in G, define $d^-(v) = \min\ \{d(u,v) : u \in V - \{v\}\}$. A vertex u $(\neq v)$ is called a *neighbor* of v if $d(u,v) = d^-(v)$. A vertex v is said to *dominate* a vertex u if $u = v$ or u is a neighbor of v. Since $d^-(v) = 1$ for all $v \in V$, this is equivalent to the standard definition of neighbor. A set S of vertices of G is called a *dominating set* if every vertex of G is dominated by some vertex

[1]Received February 10, 2025. Accepted May 8, 2025

in S. Equivalently, a set $S \subseteq V$ is a dominating set of G if every vertex in $V - S$ is adjacent to some vertex in S. A dominating set of G with minimum cardinality is a *minimum dominating set* and this cardinality is the *domination number* $\gamma(G)$. The topic of domination began with Berge in [1] and Ore in [5]. In 1998, a text book devoted to domination written by Teresa et. al. [9].

For any two vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of a longest $u - v$ path in G. For each vertex v in G, define $D^-(v) = \min\{D(u, v) : u \in V - \{v\}\}$. A vertex u $(\neq v)$ is called a *detour neighbor* of v if $D(u, v) = D^-(v)$. A vertex v is said to *detour dominate* a vertex u if $u = v$ or u is a detour neighbor of v. A set S of vertices of G is called a *detour dominating set* if every vertex of G is detour dominated by some vertex in S. A detour dominating set of G with minimum cardinality is a *minimum detour dominating set* and this cardinality is the *detour domination number* $\gamma_D(G)$. These concepts were introduced and studied in [3].

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(u, v) : u \in V\}$. The *monophonic radius,* $rad_m(G)$ of G is $rad_m(G) = \min\{e_m(v) : v \in V\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max\{e_m(v) : v \in V\}$. A vertex v in G is a *monophonic eccentric vertex* of u in G if $e_m(u) = d_m(u, v)$. A vertex v in G is a *monophonic central vertex* if $e_m(v) = rad_m(G)$ and the subgraph induced by the monophonic central vertices of G is the *monophonic center* of G. The monophonic distance was introduced in [6] and further studied in [7].

Let v be any vertex of a connected graph G. The set of all monophonic eccentric vertices of v is called the *monophonic eccentric neighborhood* of v and it is denoted by $N_{e_m}(v)$. The *monophonic eccentric degree* of a vertex v is defined as $deg_{e_m}(v) = |N_{e_m}(v)|$. The *minimum monophonic eccentric degree* $\delta_{e_m}(G)$ is defined as $\delta_{e_m}(G) = min\ \{deg_{e_m}(v) : v \in V\}$ and the *maximum monophonic eccentric degree* $\Delta_{e_m}(G)$ is defined as $\Delta_{e_m}(G) = max\ \{deg_{e_m}(v) : v \in V\}$. A set $S \subseteq V$ is a *monophonic eccentric dominating set* if every vertex in $V - S$ has a monophonic eccentric vertex in S. The *monophonic eccentric domination number* $\gamma_{me}(G)$ is the cardinality of a minimum monophonic eccentric dominating set of G. These concepts were introduced and studied in [10, 11]. A set $S \subseteq V$ is a *connected monophonic eccentric dominating set* if S is a monophonic eccentric dominating set and the induced subgraph $\langle S \rangle$ is connected. The *connected monophonic eccentric domination number* $\gamma_{cme}(G)$ is the cardinality of a minimum connected monophonic eccentric dominating set of G [12].

Generally, a dominating, detour dominating or monophonic eccentric dominating set S of graph G is *Smarandachely dominated on subgraph* $H \prec G$ if $G - H$ is not dominating, detour dominating, or monophonic eccentric dominating by set S, called a Smarandachely dominated on subgraph H^-, or S is not a dominating, detour dominating or monophonic eccentric dominating set of graph G but it is a dominating, detour dominating or monophonic eccentric dominating set S of graph $G + H$, called a Smarandachely dominated on subgraph H^-. Particularly, if $H = \emptyset$, a Smarandachely, detour dominating or monophonic eccentric dominating set

S on subgraph H is nothing else but dominating, detour dominating or monophonic eccentric dominating by S. For example, let $H = P_2$. If there is an edge $e \in E(G)$ such that $G - e$ is not dominating, detour dominating, or monophonic eccentric dominating by set S, then G is Smarandachely dominated on a subgraph P_2^-.

Consider the graph G given in Figure 1.1. It is easily seen that no 2-element subset of G is a connected monophonic eccentric dominating set. The sets $\{v_1, v_2, v_6\}$ and $\{v_3, v_4, v_5\}$ are the only minimum connected monophonic eccentric dominating set of G so that $\gamma_{cme}(G) = 3$.

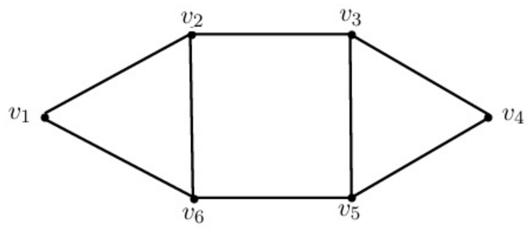


Figure 1.1

The following theorems will be used in the sequel.

Theorem 1.1([10]) *If $G = H + K_p$ or $K_p + H$, where H is any connected graph, then*

$$\gamma_{cme}(G) = \gamma_{cme}(H).$$

Theorem 1.2([10]) *Let G be a cycle of order p and let $p \equiv l(mod\ 6)$. Then,*

$$\gamma_{me}(G) = \begin{cases} \left\lceil \frac{p}{3} \right\rceil + 1 & if\ l = 2, \\ \left\lceil \frac{p}{3} \right\rceil & otherwise. \end{cases}$$

Theorem 1.3([10]) *Let G be a wheel of order p and let $p \equiv l(mod\ 6)$. Then,*

$$\gamma_{me}(G) = \begin{cases} \frac{p}{3} + 1 & if\ l = 3, \\ \left\lceil \frac{p-1}{3} \right\rceil & otherwise. \end{cases}$$

Theorem 1.4([10]) *For the complete graph K_p, $\gamma_{cme}(K_p) = 1$.*

Theorem 1.5([10]) *Let G be a wheel of order p.*

(i) If $p \leq 9$ and $p \equiv l(mod\ 6)$, then

$$\gamma_{cme}(G) = \begin{cases} \frac{p}{3} + 1 & if\ l = 3, \\ \left\lceil \frac{p-1}{3} \right\rceil & \text{otherwise.} \end{cases}$$

(ii) If $p > 9$, then $\gamma_{cme}(G) = p - 5$.

§2. Connected Monophonic Eccentric Domination Number

The corona of two graphs G_1 and G_2 is the graph $G = G_1$ o G_2 formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2. The distance related properties of corona was studied in [8] and the domination parameters of corona was studied in [2].

Theorem 2.1 *Let G be a connected graph of order m and let H be any graph of order n. Then*

$$\gamma_{cme}(G \circ H) \leq m(1 + \gamma_{me}(H)).$$

Proof Let $H_{i,n}$ be the i^{th} copy of H $(1 \leq i \leq m)$. Let S_i be a minimum monophonic eccentric dominating set of $H_{i,n}$. In G o H, it is clear that every vertex in $H_{i,n}$ is monophonic eccentric dominated by a vertex in $S = \bigcup\limits_{i=1}^{m} S_i$. Thus S is a monophonic eccentric dominating set of G o H, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S' = S \cup V(G)$. Clearly, S' is a connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G$ o $H) \leq m(1 + \gamma_{me}(H))$. □

Remark 2.2 The bounds in Theorem 2.1 is sharp. For the graph $G = C_r$ o C_s where $s = r+3$, $\gamma_{cme}(G) = m(1 + \gamma_{me}(C_s))$.

Theorem 2.3 *If $G = P_r$ o C_s $(r \geq 2)$, then*

$$\gamma_{cme}(G) = \begin{cases} 4 \quad \text{if } r = 2,3 \text{ and } s = 3, \\ r+2 \quad \text{if } (r = 2,3 \text{ and } s = 4,5)) \text{ or } (r \geq 4 \text{ and } 3 \leq s \leq \left\lceil \dfrac{r+7}{2} \right\rceil), \\ r + 2 + (2s - r - 8)\gamma_{me}(C_s) \quad \text{if } r \geq 3 \text{ and } s = \left\lceil \dfrac{r+9}{2} \right\rceil, \left\lceil \dfrac{r+11}{2} \right\rceil, \cdots, r+3, \\ r(1 + \gamma_{me}(C_s)) \quad \text{if } s > r+3. \end{cases}$$

Proof Let G be the corona product of P_r and C_s. Let $u_1, u_2, \ldots, u_r$ $(r \geq 2)$ be the vertices of P_r and let $C_{i,s}$: $v_{i,1}, v_{i,2}, \ldots, v_{i,s}, v_{i,1}$ be the i^{th} copy of C_s $(1 \leq i \leq r)$. We prove this theorem by considering three cases.

Case 1. $3 \leq s \leq \left\lceil \dfrac{r+7}{2} \right\rceil$.

Subcase 1.1 $r = 2,3$ and $s = 3$.

Let $S = \{u_1, v_{1,1}, v_{1,2}, v_{1,3}\}$. It is clear that the vertices u_i $(2 \leq i \leq r)$ and the vertices $v_{i,j}$ $(2 \leq i \leq r, 1 \leq j \leq s)$ are monophonic eccentric dominated by a vertex $v_{1,j}$. Also, the induced subgraph $\langle S \rangle$ is connected. Hence S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = 4$.

Subcase 1.2 ($r = 2, 3$ and $s = 4, 5$) or ($r \geq 4$ and $3 \leq s \leq \left\lceil \dfrac{r+7}{2} \right\rceil$).

If r is even, then the vertices u_i $(1 \leq i \leq \frac{r}{2})$ and $v_{i,j}$ $(1 \leq i \leq \frac{r}{2}, 1 \leq j \leq s)$ are monophonic eccentric dominated by a vertex $v_{r,j}$; and the vertices u_i $(\frac{r+2}{2} \leq i \leq r)$ and $v_{i,j}$ $(\frac{r+2}{2} \leq i \leq r, 1 \leq j \leq s)$ are monophonic eccentric dominated by a vertex $v_{1,j}$. Then $S = \{v_{1,j}, v_{r,j}\}$ is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \ldots, u_r\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = r + 2.$$

If r is odd, then the vertices u_i $(1 \leq i \leq \frac{r-1}{2})$ and $v_{i,j}$ $(1 \leq i \leq \frac{r-1}{2}, 1 \leq j \leq s)$ are monophonic eccentric dominated by a vertex $v_{r,j}$, the vertices u_i $(\frac{r+3}{2} \leq i \leq r)$ and $v_{i,j}$ $(\frac{r+3}{2} \leq i \leq r, 1 \leq j \leq s)$ are monophonic eccentric dominated by a vertex $v_{1,j}$, and the vertices $u_{\frac{r+1}{2}}$ and $v_{\frac{r+1}{2},j}$ $(1 \leq j \leq s)$ are monophonic eccentric dominated by both the vertices $v_{1,j}$ and $v_{r,j}$. Then $S = \{v_{1,j}, v_{r,j}\}$ is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \ldots, u_r\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = r + 2.$$

Case 2. $r \geq 3$ and $s = \left\lceil \dfrac{r+9}{2} \right\rceil, \left\lceil \dfrac{r+11}{2} \right\rceil, \cdots, r + 3.$

Subcase 2.1 r is even.

Let $m = s - \dfrac{r+8}{2}$. It can be easily seen that the vertices u_i $(1 \leq i \leq \frac{r}{2})$ and $v_{i,j}$ $(1 \leq i \leq \frac{r}{2} - m, 1 \leq j \leq s)$ are monophonic eccentric dominated by a vertex $v_{r,j}$. Similarly, the vertices u_i $(\frac{r+2}{2} \leq i \leq r)$ and $v_{i,j}$ $(\frac{r+2}{2} + m \leq i \leq r, 1 \leq j \leq s)$ are monophonic eccentric dominated by a vertex $v_{1,j}$. Let S_k $(\frac{r+2}{2} - m \leq k \leq \frac{r}{2} + m)$ be a minimum monophonic eccentric dominating set of $C_{k,s}$. It is clear that, in G, any vertex in $C_{k,s}$ $(\frac{r+2}{2} - m \leq k \leq \frac{r}{2} + m)$ is monophonic eccentric dominated by a vertex in S_k and hence

$$S = \left(\bigcup_{k=\frac{r+2}{2}-m}^{\frac{r}{2}+m} S_k \right) \bigcup \{v_{1,j}, v_{r,j}\}$$

is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \ldots, u_r\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G. Thus

$$\begin{aligned}
\gamma_{cme}(G) &= \left| S^{'} \right| \\
&= r + 2m \ \gamma_{me}(C_s) + 2 \\
&= r + 2 + \left[2(s - \frac{r+8}{2}) \right] \ \gamma_{me}(C_s) \\
&= r + 2 + (2s - r - 8) \ \gamma_{me}(C_s).
\end{aligned}$$

Subcase 2.2 r is odd.

Let $m = s - \dfrac{r+7}{2}$. It can be easily verified that the vertices u_i $(1 \le i \le \frac{r-1}{2})$ and $v_{i,j}$ $(1 \le i \le \frac{r+1}{2} - m, 1 \le j \le s)$ are monophonic eccentric dominated by a vertex $v_{r,j}$. Similarly, the vertices u_i $(\frac{r+3}{2} \le i \le r)$ and $v_{i,j}$ $(\frac{r+1}{2} + m \le i \le r, 1 \le j \le s)$ are monophonic eccentric dominated by a vertex $v_{1,j}$. Also, the vertex $u_{\frac{r+1}{2}}$ is monophonic eccentric dominated by both the vertices $v_{1,j}$ and $v_{r,j}$. Let S_k $(\frac{r+3}{2} - m \le k \le \frac{r-1}{2} + m)$ be a minimum monophonic eccentric dominating set of $C_{k,s}$. It is clear that, in G, any vertex in $C_{k,s}$ $(\frac{r+3}{2} - m \le k \le \frac{r-1}{2} + m)$ is monophonic eccentric dominated by a vertex in S_k and hence

$$S = \left(\bigcup_{k=\frac{r+3}{2}-m}^{\frac{r-1}{2}+m} S_k \right) \bigcup \{v_{1,j}, v_{r,j}\}$$

is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \ldots, u_r\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G. Thus

$$\begin{aligned}
\gamma_{cme}(G) &= \left| S^{'} \right| \\
&= r + (2m-1)\ \gamma_{me}(C_s) + 2 \\
&= r + 2 + \left[2(s - \frac{r+7}{2}) - 1 \right]\ \gamma_{me}(C_s) \\
&= r + 2 + (2s - r - 8)\ \gamma_{me}(C_s).
\end{aligned}$$

Case 3. $s > r + 3$.

If r is even, then the vertices u_i $(1 \le i \le \frac{r}{2})$ are monophonic eccentric dominated by a vertex $v_{r,j}$ $(1 \le j \le s)$ and the vertices u_i $(\frac{r+2}{2} \le i \le r)$ are monophonic eccentric dominated by a vertex $v_{1,j}$ $(1 \le j \le s)$. If r is odd, then the vertices u_i $(1 \le i \le \frac{r-1}{2})$ are monophonic eccentric dominated by a vertex $v_{r,j}$ $(1 \le j \le s)$, the vertices u_i $(\frac{r+3}{2} \le i \le r)$ are monophonic eccentric dominated by a vertex $v_{1,j}$ $(1 \le j \le s)$, and the vertex $u_{\frac{r+1}{2}}$ is monophonic eccentric dominated by both the vertices $v_{1,j}$ and $v_{r,j}$ $(1 \le j \le s)$. Let S_k $(1 \le k \le r)$ be a minimum monophonic eccentric dominating set of $C_{k,s}$. It is clear that, in G, any vertex in $C_{k,s}$ $(1 \le k \le r)$ is monophonic eccentric dominated by a vertex in S_k and hence $S = \bigcup\limits_{k=1}^{r} S_k$ is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \ldots, u_r\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = r + r\ \gamma_{me}(C_s) = r(1 + \gamma_{me}(C_s)). \qquad \square$$

The result of the above theorem contains $\gamma_{me}(C_s)$ and we can calculate $\gamma_{me}(C_s)$ using Theorem 1.2.

Note 2.4 If $r = 1$, then $G = P_1$ o C_s is a wheel. By Theorem 1.5, we have $\gamma_{cme}(G) =$

$\gamma_{cme}(W_{s+1})$.

Theorem 2.5 *If $G = P_r \ o \ W_s$ $(r \geq 2)$, then*

$$\gamma_{cme}(G) = \begin{cases} r+2 \quad \text{if } 4 \leq s \leq \left\lceil \dfrac{r+9}{2} \right\rceil, \\ r+2+(2s-r-10)\gamma_{me}(W_s) \quad \text{if } r \geq 3 \text{ and } s = \left\lceil \dfrac{r+11}{2} \right\rceil, \left\lceil \dfrac{r+13}{2} \right\rceil, \cdots, r+4, \\ r(1+\ \gamma_{me}(W_s)) \quad \text{if } s > r+4. \end{cases}$$

Proof Since $W_s = C_{s-1} + K_1$, by Theorem 1.1 we have $\gamma_{cme}(W_s) = \gamma_{cme}(C_{s-1})$. Then by Theorem 2.1, the required result can be got. □

The result of the above theorem contains $\gamma_{me}(W_s)$ and we can calculate $\gamma_{me}(W_s)$ using Theorem 1.3.

Note 2.6 If $r = 1$, then $G = P_1$ o W_s. By Theorem 1.1 we have $\gamma_{cme}(G) = \gamma_{cme}(W_s)$.

Theorem 2.7 *If $G = P_r \circ K_s$ $(r \geq 2)$, then*

$$\gamma_{cme}(G) = \begin{cases} (s+1)\left\lfloor \dfrac{r}{2} \right\rfloor \quad \text{if } \left(2 \leq r \leq 5 \quad \text{and} \quad 1 \leq s \leq 6 - \left\lfloor \dfrac{r+4}{2} \right\rfloor\right) \quad \text{or} \quad (r \geq 6 \text{ and } s = 1), \\ r+2 \quad \text{if } \left(2 \leq r \leq 5 \text{ and } s > 6 - \left\lfloor \dfrac{r+4}{2} \right\rfloor\right) \quad \text{or} \quad (r \geq 6 \quad \text{and} \quad s > 1). \end{cases}$$

Proof Let G be the corona product of P_r and K_s. Let $u_1, u_2, \cdots, u_r$ $(r \geq 2)$ and $v_{i,1}, v_{i,2}, \cdots, v_{i,s}$ be the vertices of P_r and the vertices of the i^{th} copy of K_s $(1 \leq i \leq r)$, respectively. We prove this theorem by considering two cases.

Case 1. $\left(2 \leq r \leq 5 \text{ and } 1 \leq s \leq 6 - \left\lfloor \dfrac{r+4}{2} \right\rfloor\right)$ or $(r \geq 6$ and $s = 1$.

Let $S = \{u_1, v_{1,1}, \cdots, v_{1,s}; u_2, v_{2,1}, \cdots, v_{2,s}; \cdots; u_{\lfloor \frac{r}{2} \rfloor}, v_{\lfloor \frac{r}{2} \rfloor,1}, \cdots, v_{\lfloor \frac{r}{2} \rfloor,s}\}$. It is clear that the vertices u_i $(\lceil \frac{r+1}{2} \rceil \leq i \leq r)$ and the vertices $v_{i,1}$ $(\lceil \frac{r+2}{2} \rceil \leq i \leq r)$ are monophonic eccentric dominated by the vertex $v_{1,1}$. Also, the induced subgraph $\langle S \rangle$ is connected. Hence S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = 2\left\lfloor \frac{r}{2} \right\rfloor.$$

Case 2. $\left(2 \leq r \leq 5 \text{ and } s > 6 - \left\lfloor \dfrac{r+4}{2} \right\rfloor\right)$ or $(r \geq 6$ and $s > 1)$.

It is clear that the vertices u_i $(\lceil \frac{r+1}{2} \rceil \leq i \leq r)$ and $v_{i,j}$ $(\lceil \frac{r+1}{2} \rceil \leq i \leq r, 1 \leq j \leq s)$ are monophonic eccentric dominated by a vertex $v_{1,j} (1 \leq j \leq s)$. Also, the vertices u_i $(1 \leq i \leq \lfloor \frac{r}{2} \rfloor)$ and $v_{i,j}$ $(1 \leq i \leq \lfloor \frac{r}{2} \rfloor, 1 \leq j \leq s)$ are monophonic eccentric dominated by a vertex $v_{r,j}$. Hence $S = \{v_{1,j}, v_{r,j}\}$ $(1 \leq j \leq s)$ is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \cdots, u_r\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = r + 2$. □

Note 2.8 If $r = 1$, then $G = P_1$ o K_s is a complete graph. By Theorem 1.4, $\gamma_{cme}(G) = 1$.

Theorem 2.9 *If $G = P_r$ o $K_{1,n}$ $(2 \leq r \leq 5)$, then*

$$\gamma_{cme}(G) = \begin{cases} (n+2)\left\lfloor \frac{r}{2} \right\rfloor & \text{if } 1 \leq s \leq 5 - \left\lfloor \frac{r+4}{2} \right\rfloor, \\ r+2 & \text{if } s > 5 - \left\lfloor \frac{r+4}{2} \right\rfloor. \end{cases}$$

Proof By an argument similar to Theorem 2.7, the required result can be got. □

Theorem 2.10 *If $G = P_r \circ K_{1,n}$ $(r \geq 6)$, then $\gamma_{cme}(G) = r + 2$.*

Proof Let G be the corona product of P_r and $K_{1,n}$. Let $u_1, u_2, \ldots, u_r$ $(r \geq 6)$ and $v_{i,1}, v_{i,2}, \cdots, v_{i,n+1}$ be the vertices of P_r and the i^{th} copy of $K_{1,n}$ $(1 \leq i \leq r)$, respectively. It is clear that the vertices u_i $(\lceil \frac{r+1}{2} \rceil \leq i \leq r)$ and $v_{i,j}$ $(\lceil \frac{r+1}{2} \rceil \leq i \leq r, 1 \leq j \leq n+1)$ are monophonic eccentric dominated by a vertex $v_{1,j}(1 \leq j \leq n+1)$. Also, the vertices u_i $(1 \leq i \leq \lfloor \frac{r}{2} \rfloor)$ and $v_{i,j}$ $(1 \leq i \leq \lfloor \frac{r}{2} \rfloor, 1 \leq j \leq n+1)$ are monophonic eccentric dominated by a vertex $v_{r,j}$. Hence $S = \{v_{1,j}, v_{r,j}\}$ $(1 \leq j \leq n+1)$ is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S' = \{u_1, u_2, \cdots, u_r\} \cup S$. Clearly, S' is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = r + 2$. □

Theorem 2.11 *If $G = P_r \circ K_{m,n}$ $(r, m, n \geq 2)$, then $\gamma_{cme}(G) = r + 2$.*

Proof By an argument similar to Theorem 2.10, the required result can be got. □

Note 2.12 If $G = P_1 \circ K_{m,n}$, then

$$\gamma_{cme}(G) = \begin{cases} 1 & \text{if either } m \text{ or } n = 1, \\ 2 & \text{if } m, n \geq 2. \end{cases}$$

Theorem 2.13 *Let $G = C_r \circ P_1$ $(r \geq 6)$ and let $r \equiv k(mod\ 6)$. Then,*

$$\gamma_{cme}(G) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil + r - 3 & \text{if } k = 2, 3, \\ \left\lceil \frac{r}{3} \right\rceil + r - 4 & \text{if } k = 0, 1, 4 \text{ and } 5. \end{cases}$$

Proof Let G be the corona product of C_r and P_1. Let $u_1, u_2, \cdots, u_r$ and $v_{i,1}$ be the vertices of C_r and the vertices of the i^{th} copy of P_1 $(1 \leq i \leq r)$, respectively. We prove this theorem by considering six cases.

Case 1. $r \equiv 0(mod\ 6)$.

Let $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \cdots; v_{r-5,1}, v_{r-4,1}\}$. It is easily verified that the vertices u_{r-1}, $v_{r-1,1}$, u_3 and $v_{3,1}$ are monophonic eccentric dominated by the vertex $v_{1,1}$, the vertices u_4, $v_{4,1}$, u_r and $v_{r,1}$ are monophonic eccentric dominated by the vertex $v_{2,1}$, $\cdots$, the vertices u_{r-1}, $v_{r-1,1}$, u_{r-3} and $v_{r-3,1}$ are monophonic eccentric dominated by the vertex $v_{r-5,1}$ and the vertices u_r,

$v_{r,1}$, u_{r-2} and $v_{r-2,1}$ are monophonic eccentric dominated by the vertex $v_{r-4,1}$. It is clear that S is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S\rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \cdots, u_{r-4}\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r - 4.$$

Case 2. $r \equiv 1(mod\ 6)$.

Let $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \cdots; v_{r-6,1}, v_{r-5,1}; v_{r,1}\} \cup \{u_1, u_2, \cdots, u_{r-5}; u_r\}$. By an argument similar to Case 1, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r - 4$.

Case 3. $r \equiv 2(mod\ 6)$.

Let $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \cdots; v_{r-1,1}, v_{r,1}\} \cup \{u_1, u_2, \cdots, u_{r-6}; u_{r-1}, u_r\}$. By an argument similar to Case 1, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + 1 + r - 4 = \left\lceil \frac{r}{3} \right\rceil + r - 3.$$

Case 4. $r \equiv 3(mod\ 6)$.

Let $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \cdots; v_{r-2,1}, v_{r-1,1}\} \cup \{u_1, u_2, \cdots, u_{r-7}; u_{r-2}, u_{r-1}, u_r\}$. By an argument similar to Case 1, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + 1 + r - 4 = \left\lceil \frac{r}{3} \right\rceil + r - 3.$$

Case 5. $r \equiv 4(mod\ 6)$.

Let $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \cdots; v_{r-3,1}, v_{r-2,1}\} \cup \{u_1, u_2, \cdots, u_{r-8}; u_{r-3}, u_{r-2}, u_{r-1}, u_r\}$. By an argument similar to Case 1, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r - 4.$$

Case 6. $r \equiv 5(mod\ 6)$.

Let $S = \{v_{1,1}, v_{2,1}; v_{7,1}, v_{8,1}; \cdots; v_{r-4,1}, v_{r-3,1}\} \cup \{u_1, u_2, \cdots, u_{r-9}; u_{r-4}, u_{r-3}, \cdots, u_r\}$. By an argument similar to Case 1, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r - 4$. □

Theorem 2.14 *Let $G = C_r \circ P_s$ $(r \geq 8, s \geq 2)$ and let $r \equiv k(mod\ 8)$. Then*

$$\gamma_{cme}(G) = \begin{cases} \dfrac{3r-8+k}{2} & if\ 2 \leq s \leq r+1\ and\ 0 \leq k \leq 3, \\ \dfrac{3r-k}{2} & if\ 2 \leq s \leq r+1\ and\ 3 < k < 8, \\ 2r & if\ s = r+2, \\ 3r & if\ s > r+2. \end{cases}$$

Proof Let G be the corona product of C_r and P_s. Let $u_1, u_2, \cdots, u_r$ and $v_{i,1}, v_{i,2}, \cdots, v_{i,s}$ be the vertices of C_r and the vertices of the i^{th} copy of P_s $(1 \leq i \leq r)$, respectively. We prove this theorem by considering three cases.

Case 1. $2 \leq s \leq r+1$.

Subcase 1.1 $r \equiv 0(mod\ 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \cdots; v_{r-7,j}, v_{r-6,j}, v_{r-5,j}, v_{r-4,j}\}$ $(1 \leq j \leq s)$. It is easily verified that the vertices u_{r-1}, $v_{r-1,j}$, u_3 and $v_{3,j}$ are monophonic eccentric dominated by a vertex $v_{1,j}$, the vertices u_4, $v_{4,j}$, u_r and $v_{r,j}$ are monophonic eccentric dominated by a vertex $v_{2,j}$, the vertices u_1, $v_{1,j}$, u_5 and $v_{5,j}$ are monophonic eccentric dominated by a vertex $v_{3,j}$, the vertices u_2, $v_{2,j}$, u_6 and $v_{6,j}$ are monophonic eccentric dominated by a vertex $v_{4,j}$, $\ldots$, the vertices u_{r-1}, $v_{r-1,j}$, u_{r-5} and $v_{r-5,j}$ are monophonic eccentric dominated by a vertex $v_{r-7,j}$, the vertices u_r, $v_{r,j}$, u_{r-4} and $v_{r-4,j}$ are monophonic eccentric dominated by a vertex $v_{r-6,j}$, the vertices u_1, $v_{1,j}$, u_{r-3} and $v_{r-3,j}$ are monophonic eccentric dominated by a vertex $v_{r-5,j}$ and the vertices u_2, $v_{2,j}$, u_{r-2} and $v_{r-2,j}$ are monophonic eccentric dominated by a vertex $v_{r-4,j}$. It is clear that S is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S' = \{u_1, u_2, \cdots, u_{r-4}\} \cup S$. Clearly, S' is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \frac{r}{2} + r - 4 = \frac{3r-8}{2} = \frac{3r-8+k}{2}.$$

Subcase 1.2 $r \equiv 1(mod\ 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \cdots; v_{r-8,j}, v_{r-7,j}, v_{r-6,j}, v_{r-5,j}\} \cup \{v_{r,j}\}$ $(1 \leq j \leq s)$. It is easily verified that the vertices u_{r-1}, $v_{r-1,j}$, u_3 and $v_{3,j}$ are monophonic eccentric dominated by a vertex $v_{1,j}$, the vertices u_4, $v_{4,j}$, u_r and $v_{r,j}$ are monophonic eccentric dominated by a vertex $v_{2,j}$, the vertices u_1, $v_{1,j}$, u_5 and $v_{5,j}$ are monophonic eccentric dominated by a vertex $v_{3,j}$, the vertices u_2, $v_{2,j}$, u_6 and $v_{6,j}$ are monophonic eccentric dominated by a vertex $v_{4,j}$, $\cdots$, the vertices u_{r-1}, $v_{r-1,j}$, u_{r-6} and $v_{r-6,j}$ are monophonic eccentric dominated by a vertex $v_{r-8,j}$, the vertices u_r, $v_{r,j}$, u_{r-5} and $v_{r-5,j}$ are monophonic eccentric dominated by a vertex $v_{r-7,j}$, the vertices u_{r-8}, $v_{r-8,j}$, u_{r-4} and $v_{r-4,j}$ are monophonic eccentric dominated by a vertex $v_{r-6,j}$, the vertices u_{r-7}, $v_{r-7,j}$, u_{r-3} and $v_{r-3,j}$ are monophonic eccentric dominated by a vertex $v_{r-5,j}$ and the vertices u_2, $v_{2,j}$, u_{r-2} and $v_{r-2,j}$ are monophonic eccentric dominated by a vertex $v_{r,j}$. It is clear that S is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \ldots, u_{r-5}; u_r\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \frac{r+1}{2} + r - 4 = \frac{3r-7}{2} = \frac{3r-8+k}{2}.$$

Subcase 1.3 $r \equiv 2(mod\ 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \cdots; v_{r-9,j}, v_{r-8,j}, v_{r-7,j}, v_{r-6,j}\} \cup \{v_{r-1,j}, v_{r,j}\} \cup \{u_1, u_2, \cdots, u_{r-6}; u_{r-1}, u_r\}$. By an argument similar to Subcase 1.2, it is clear that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \frac{r+2}{2} + r - 4 = \frac{3r-6}{2} = \frac{3r-8+k}{2}.$$

Subcase 1.4 $r \equiv 3(mod\ 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \cdots; v_{r-10,j}, v_{r-9,j}, v_{r-8,j}, v_{r-7,j}\} \cup \{v_{r-2,j}, v_{r-1,j}, v_{r,j}\} \cup \{u_1, u_2, \cdots, u_{r-7}; u_{r-2}, u_{r-1}, u_r\}$. By an argument similar to Subcase 1.2, it is clear that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \frac{r+3}{2} + r - 4 = \frac{3r-5}{2} = \frac{3r-8+k}{2}.$$

Subcase 1.5 $r \equiv 4(mod\ 8)$.

Let

$$\begin{aligned} S &= \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \cdots; v_{r-11,j}, v_{r-10,j}, v_{r-9,j}, v_{r-8,j}\} \\ &\quad \bigcup \{v_{r-7,j}, v_{r-6,j}, v_{r-5,j}, v_{r-4,j}\} \bigcup \{u_1, u_2, \cdots, u_{r-4}\}. \end{aligned}$$

By an argument similar to Subcase 1.2, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \frac{r+4}{2} + r - 4 = \frac{3r-4}{2} = \frac{3r-k}{2}.$$

Subcase 1.6 $r \equiv 5(mod\ 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \cdots; v_{r-12,j}, v_{r-11,j}, v_{r-10,j}, v_{r-9,j}; v_{r-4,j}, v_{r-3,j}, v_{r-2,j}, v_{r-1,j}\} \cup \{u_1, u_2, \cdots, u_{r-9}; u_{r-4}, u_{r-3}, \cdots, u_r\}$. By an argument similar to Subcase 1.2, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \frac{r+3}{2} + r - 4 = \frac{3r-5}{2} = \frac{3r-k}{2}.$$

Subcase 1.7 $r \equiv 6(mod\ 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \cdots; v_{r-13,j}, v_{r-12,j}, v_{r-11,j}, v_{r-10,j}; v_{r-5,j}, v_{r-4,j}, v_{r-3,j}, v_{r-2,j}\} \cup \{u_1, u_2, \cdots, u_{r-10}; u_{r-5}, u_{r-4}, \cdots, u_r\}$. By an argument similar to Subcase 1.2, it is clear that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \frac{r+2}{2} + r - 4 = \frac{3r-6}{2} = \frac{3r-k}{2}.$$

Subcase 1.8 $r \equiv 7(mod\ 8)$.

Let $S = \{v_{1,j}, v_{2,j}, v_{3,j}, v_{4,j}; v_{9,j}, v_{10,j}, v_{11,j}, v_{12,j}; \cdots ; v_{r-14,j}, v_{r-13,j}, v_{r-12,j}, v_{r-11,j}; v_{r-6,j}, v_{r-5,j}, v_{r-4,j}, v_{r-3,j}\} \cup \{u_1, u_2, \cdots , u_{r-11}; u_{r-6}, u_{r-5}, \cdots , u_r\}$. By an argument similar to Subcase 1.2, it is clear that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \frac{r+1}{2} + r - 4 = \frac{3r-7}{2} = \frac{3r-k}{2}.$$

Case 2. $s = r + 2$.

Let $S = \{v_{1,1}, v_{2,1}, \cdots , v_{r,1}\}$. It is easily verified that the vertices u_{r-1}, $v_{r-1,k}$, u_3, $v_{3,k}$ $(2 \le k \le s-1)$ and $v_{1,s}$ are monophonic eccentric dominated by the vertex $v_{1,1}$, the vertices u_4, $v_{4,k}$, u_r, $v_{r,k}$ $(2 \le k \le s-1)$ and $v_{2,s}$ are monophonic eccentric dominated by the vertex $v_{2,1}$, $\cdots$, the vertices u_{r-2}, $v_{r-2,k}$, u_2, $v_{2,k}$ $(2 \le k \le s-1)$ and $v_{r,s}$ are monophonic eccentric dominated by the vertex $v_{r,1}$. It is clear that S is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S\rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \ldots , u_r\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = 2r.$$

Case 3. $s > r + 2$.

Let $S = \{v_{1,1}, v_{2,1}, \cdots , v_{r,1}; v_{1,s}, v_{2,s}, \cdots , v_{r,s}\}$. In $r+2 < s < 2r$, the vertices u_{r-1}, $v_{r-1,k}$, u_3, $v_{3,k}$ $(s-r+1 \le k \le r)$ and $v_{1,l}$ $(r+1 \le l \le s)$ are monophonic eccentric dominated by the vertex $v_{1,1}$, the vertices u_4, $v_{4,k}$, u_r, $v_{r,k}$ $(s-r+1 \le k \le r)$ and $v_{2,l}$ $(r+1 \le l \le s)$ are monophonic eccentric dominated by the vertex $v_{2,1}$, $\ldots$, the vertices u_{r-2}, $v_{r-2,k}$, u_2, $v_{2,k}$ $(s-r+1 \le k \le r)$ and $v_{r,l}$ $(r+1 \le l \le s)$ are monophonic eccentric dominated by the vertex $v_{r,1}$. Also, the vertices $v_{1,l}$ $(1 \le l \le s-r)$ are monophonic eccentric dominated by the vertex $v_{1,s}$, the vertices $v_{2,l}$ $(1 \le l \le s-r)$ are monophonic eccentric dominated by the vertex $v_{2,s}$, $\ldots$, the vertices $v_{r,l}$ $(1 \le l \le s-r)$ are monophonic eccentric dominated by the vertex $v_{r,s}$.

In $s \ge 2r$, the vertices u_{r-1}, u_3 and $v_{1,l}$ $(\lfloor \frac{s+3}{2} \rfloor \le l \le s)$ are monophonic eccentric dominated by the vertex $v_{1,1}$, the vertices u_4, u_r, and $v_{2,l}$ $(\lfloor \frac{s+3}{2} \rfloor \le l \le s)$ are monophonic eccentric dominated by the vertex $v_{2,1}$, $\ldots$, the vertices u_{r-2}, u_2, and $v_{r,l}$ $(\lfloor \frac{s+3}{2} \rfloor \le l \le s)$ are monophonic eccentric dominated by the vertex $v_{r,1}$. Also, the vertices $v_{1,l}$ $(1 \le l \le \lfloor \frac{s+1}{2} \rfloor)$ are monophonic eccentric dominated by the vertex $v_{1,s}$, the vertices $v_{2,l}$ $(1 \le l \le \lfloor \frac{s+1}{2} \rfloor)$ are monophonic eccentric dominated by the vertex $v_{2,s}$, $\ldots$, the vertices $v_{r,l}$ $(1 \le l \le \lfloor \frac{s+1}{2} \rfloor)$ are monophonic eccentric dominated by the vertex $v_{r,s}$. Hence, it is clear that S is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S\rangle$ is not connected. Therefore, we consider a set $S^{'} = \{u_1, u_2, \cdots , u_r\} \cup S$. Clearly, $S^{'}$ is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = 3r$. □

Theorem 2.15 *If $G = C_r \circ P_s$ $(r \leq 7)$, then*

$$\gamma_{cme}(G) = \begin{cases} (s+1)\left\lceil \frac{r}{3} \right\rceil & if\ 1 \leq s \leq 7 - \left\lceil \frac{r+4}{2} \right\rceil, \\ \frac{3r-1}{2} & if\ r = 3, 5\ and\ 7 - \left\lceil \frac{r+4}{2} \right\rceil < s < r+2, \\ 8 & if\ (r = 4, 6\ and\ 7 - \left\lceil \frac{r+4}{2} \right\rceil < s < r+2)\ or\ (r = 7\ and\ 3 \leq s \leq 8), \\ 7 & if\ r = 7\ and\ s = 2, \\ 2r & if\ s = r+2, \\ 3r & if\ s > r+2. \end{cases}$$

Proof Let G be the corona product of C_r and P_s. Let $u_1, u_2, \cdots, u_r$ and $v_{i,1}, v_{i,2}, \cdots, v_{i,s}$ be the vertices of C_r and the vertices of the i^{th} copy of P_s $(1 \leq i \leq r)$, respectively. We prove this theorem by considering four cases.

Case 1. $1 \leq s \leq 7 - \left\lceil \frac{r+4}{2} \right\rceil$.

Let $S = \{u_1, v_{1,1}, \cdots, v_{1,s}; u_2, v_{2,1}, \cdots, v_{2,s}; \cdots; u_{\lceil \frac{r}{3} \rceil}, v_{\lceil \frac{r}{3} \rceil,1}, \cdots, v_{\lceil \frac{r}{3} \rceil,s}\}$. It is easily verified that every vertex in $V - S$ has a monophonic eccentric vertex in S and the induced subgraph $\langle S \rangle$ is connected. Hence S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = (s+1)\left\lceil \frac{r}{3} \right\rceil$.

Case 2. $7 - \left\lceil \frac{r+4}{2} \right\rceil < s < r+2$.

Subcase 2.1 $r = 3$ and 5.

If $r = 3$, let $S = \{v_{1,j}, v_{2,j}; u_1, u_2\}$. If $r = 5$, let $S = \{v_{1,j}, v_{2,j}, v_{4,j}; u_1, u_2, u_3, u_4\}$. Then by an argument similar to Case 1, it is clear that S is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = \frac{3r-1}{2}.$$

Subcase 2.2 ($r = 4$ and 6) or ($r = 7$ and $3 \leq s \leq 8$).

Let $S = \{u_1, v_{1,j}; u_2, v_{2,j}; u_3, v_{3,j}; u_4, v_{4,j}\}$. Then by an argument similar to Case 1, it is clear that S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = 8$.

Subcase 2.3 $r = 7$ and $s = 2$.

Let $S = \{u_1, v_{1,j}; u_2, v_{2,1}, v_{2,2}; u_3, v_{3,j}\}$. Then by an argument similar to Case 1, it is clear that S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = 7$.

Case 3. $s = r + 2$.

Let $S = \{v_{1,1}, v_{2,1}, \cdots, v_{r,1}; u_1, u_2, \cdots, u_r\}$. By an argument similar to Case 2 of Theorem 2.14, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = 2r$.

Case 4. $s > r + 2$.

Let $S = \{v_{1,1}, v_{2,1}, \cdots, v_{r,1}; v_{1,s}, v_{2,s}, \cdots, v_{r,s}; u_1, u_2, \cdots, u_r\}$. By an argument similar to Case 3 of Theorem 2.14, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = 3r$. □

Theorem 2.16 *If* $G = C_r \circ C_s$ $(r \geq 8)$ *and* $r \equiv k(mod\ 8)$, *then*

$$\gamma_{cme}(G) = \begin{cases} \dfrac{3r-8+k}{2} & if\ s \leq r+2\ and\ 0 \leq k \leq 3, \\ \dfrac{3r-k}{2} & if\ s \leq r+2\ and\ 3 < k < 8, \\ r(1+\gamma_{me}(C_s)) & if\ s > r+2. \end{cases}$$

Proof Let G be the corona product of C_r and C_s. Let $u_1, u_2, \cdots, u_r$ be the vertices of C_r and let $C_{i,s}$: $v_{i,1}, v_{i,2}, \cdots, v_{i,s}, v_{i,1}$ be the i^{th} copy of C_s $(1 \leq i \leq r)$, respectively. We prove this theorem by considering two cases.

Case 1. $s \leq r + 2$.

By an argument similar to Case 1 of Theorem 2.14, the required result can be got.

Case 2. $s > r + 2$.

Let S_k $(1 \leq k \leq r)$ be a minimum monophonic eccentric dominating set of $C_{k,s}$. It is clear that, in G, any vertex in $C_{k,s}$ $(1 \leq k \leq r)$ is monophonic eccentric dominated by a vertex in S_k. Also, the vertices $u_1, u_2, \cdots, u_r$ are monophonic eccentric dominated by a vertex in S_k $(1 \leq k \leq r)$ and hence $S = \bigcup_{k=1}^{r} S_k$ is a minimum monophonic eccentric dominating set of G, but the induced subgraph $\langle S \rangle$ is not connected. Therefore, we consider a set $S' = \{u_1, u_2, \cdots, u_r\} \cup S$. Clearly, S' is a minimum connected monophonic eccentric dominating set of G and so

$$\gamma_{cme}(G) = r(1 + \gamma_{me}(C_s)).$$ □

The result of the above theorem contains $\gamma_{me}(C_s)$ and we can calculate $\gamma_{me}(C_s)$ using Theorem 1.2.

Theorem 2.17 *If* $G = C_r\ o\ W_s$ $(r \geq 8)$ *and* $r \equiv k(mod\ 8)$, *then*

$$\gamma_{cme}(G) = \begin{cases} \dfrac{3r-8+k}{2} & if\ s \leq r+3\ and\ 0 \leq k \leq 3, \\ \dfrac{3r-k}{2} & if\ s \leq r+3\ and\ 3 < k < 8, \\ r(1+\gamma_{me}(W_s)) & if\ s > r+3. \end{cases}$$

Proof By Theorem 1.1 and by an argument similar to Theorem 2.16, the required result can be got. □

The result of the above theorem contains $\gamma_{me}(W_s)$ and we can calculate $\gamma_{me}(W_s)$ using Theorem 1.3.

Theorem 2.18 *If $G = C_r\ o\ C_s$ $(r \leq 7)$, then*

$$\gamma_{cme}(G) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil + r & if\ r = 3, 5\ and\ 3 \leq s \leq r+2, \\ 8 & if\ r = 4, 6, 7\ and\ 3 \leq s \leq r+2, \\ r(1 + \gamma_{me}(C_s)) & if\ s > r+2. \end{cases}$$

Proof Let G be the corona product of C_r and C_s. Let $u_1, u_2, \cdots, u_r$ be the vertices of C_r and let $C_{i,s}$: $v_{i,1}, v_{i,2}, \cdots, v_{i,s}, v_{i,1}$ be the i^{th} copy of C_s $(1 \leq i \leq r)$, respectively. We prove this theorem by considering two cases.

Case 1. $3 \leq s \leq r + 2$.

Subcase 1.1 $r = 3$ and 5.

If $r = 3$, let $S = \{v_{1,j}, v_{2,j}; u_1, u_2\}$. If $r = 5$, let $S = \{v_{1,j}, v_{2,j}, v_{4,j}; u_1, u_2, u_3, u_4\}$. By an argument similar to Case 2 of Theorem 2.15, it can be easily seen that S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = \left\lceil \frac{r}{3} \right\rceil + r$.

Subcase 1.2 $r = 4, 6$ and 7.

Let $S = \{u_1, v_{1,j}; u_2, v_{2,j}; u_3, v_{3,j}; u_4, v_{4,j}\}$. By an argument similar to Case 2 of Theorem 2.15, it clear that S is a minimum connected monophonic eccentric dominating set of G and so $\gamma_{cme}(G) = 8$.

Case 2. $s > r + 2$.

By an argument similar to Case 2 of Theorem 2.16, the required result can be got. □

The result of the above theorem contains $\gamma_{me}(C_s)$ and we can calculate $\gamma_{me}(C_s)$ using Theorem 1.2.

Theorem 2.19 *If $G = C_r \circ W_s$ $(r \leq 7)$, then*

$$\gamma_{cme}(G) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil + r & if\ r = 3, 5\ and\ 4 \leq s \leq r+3, \\ 8 & if\ r = 4, 6, 7\ and\ 4 \leq s \leq r+3, \\ r(1 + \gamma_{me}(W_s)) & if\ s > r+3. \end{cases}$$

Proof By Theorem 1.1 and by an argument similar to Theorem 2.18, the required result can be got. □

The result of the above theorem contains $\gamma_{me}(W_s)$ and we can calculate $\gamma_{me}(W_s)$ using Theorem 1.3.

Theorem 2.20 *If* $G = C_r \circ K_s$ *or* $G = C_r\ o\ K_{m,n}$ *and* $r \equiv k (mod\ 8)$*, then*

$$\gamma_{cme}(G) = \begin{cases} \left\lceil \frac{r}{3} \right\rceil + r & if\ r = 3, 5, \\ 8 & if\ r = 4, 6, 7, \\ \frac{3r - 8 + k}{2} & if\ r \geq 8\ and\ 0 \leq k \leq 3, \\ \frac{3r - k}{2} & if\ r \geq 8\ and\ 3 < k < 8. \end{cases}$$

Proof By an argument similar to Case 1 of Theorem 2.18 and Case 1 of Theorem 2.14, the required result can be got. □

References

[1] C.Berge, *Theory of Graphs and its Applications*, Methuen, London, 1962.

[2] Carmelito E. Go and Sergio R. Canoy, Domination in the corona and join of graphs, *International Mathematical Forum*, Vol. 6, No. 16 (2011), 763-771.

[3] G. Chartrand, T.W. Haynes, M.A. Henning and P. Zhang, Detour domination in graphs, *Ars Combin.*, 71 (2004), 149-160.

[4] F.Harary, *Graph Theory*, Addison-Wesley, 1969.

[5] O. Ore, *Theory of Graphs*, Amer. Math. Soc. Colloq. Publ., (38)(1962).

[6] A.P. Santhakumaran and P. Titus, Monophonic distance in graphs, *Discrete Mathematics, Algorithms and Applications*, Vol. 3, No. 2 (2011), 159-169.

[7] A.P. Santhakumaran and P. Titus, A note on "Monophonic distance in graphs", *Discrete Mathematics, Algorithms and Applications*, Vol. 4, No. 2 (2012), doi: 10.1142 / s1793830912500188.

[8] S. Sriram, Atulya K. Nagar and K.G. Subramanian, Distance-related properties of corona of certain graphs, *Int. J. Advance Soft Compu. Appl.*, Vol. 8, No. 2 (2016), 117-125.

[9] Teresa W. Haynes, Stephen T. Hedetniemi, Peter J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc, New York, 1998.

[10] P. Titus and J. Ajitha Fancy, Monophonic eccentric domination in graphs, *Commutations in Combinatorics and Optimization*, Vol.9, No.4(2024), 625-633.

[11] P.Titus, J. Fancy and A.Santhakumaran, *Monophonic eccentric domination number of corona product of complete and complete bipartite graphs with some standard graphs*, *Caribbean Journal of Science*, Vol.53, No.2,(2019), 723-737.

[12] P.Titus, J.Ajitha Fancy, Gyanendra Prasad Joshi and S. Amutha, *The connected monophonic eccentric domination number of a graph*, *Journal of Intelligent & Fuzzy Systems*, Vol.43, No.6(2022)1-10.

International J.Math. Combin. Vol.1-Vol.2(2025), 67-77

Some Fractional Product Cordial Graphs

R. Ponraj

Department of Mathematics,Sri Paramakalyani College, Alwarkurichi–627 412 , India

T. Sutharson

Research Scholar, Reg. No:241112312016
Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627 412, India
(affiliated to Manonmaniam Sundaranar University)

E-mail: ponrajmaths@gmail.com, sutharson94@gmail.com

Abstract: Let $G=(V,E)$ be a (p,q) graph,

$$M=\begin{cases}1,2,\cdots,\frac{p}{2},\frac{1}{2},\frac{1}{3},\cdots,\frac{2}{p+2}, & \text{if } p \text{ is even,}\\ 1,2,\cdots,\frac{p-1}{2},\frac{1}{2},\frac{1}{3},\cdots,\frac{2}{p+3}, & \text{if } p \text{ is odd}\end{cases}$$

and let $\chi: V(G)\to M$ be a bijection.For each edge xy assign the label $\lceil \chi(x)\chi(y)\rceil$.$\chi$ is called a fractional product cordial labeling (simply called FP-cordial labeling) if $|\Pi_\chi(0)-\Pi_\chi(1)|\leq 1$, where $\Pi_\chi(1)$ and $\Pi_\chi(0)$ respectively denotes the number of edges labelled with 1 and not labelled with 1. A graph with a fractional product cordial labeling is called a fractional product cordial graph (Simply FP-cordial graph).In this paper we investigate the fractional product cordial labeling behaviour of snake graphs, helm, sunflower graph and subdivision of the comb graphs.

Key Words: Triangular snake, quadrilateral snake, slanting ladder, triangular ladder, fan graph, flower graphs, Smarandachely FP-cordial labeling.

AMS(2010): 05C38, 05C76, 05C78.

§1. Introduction

We consider finite, simple and undirected graphs only. Several types of cordial related concept was studied in [1,4-20]. Labelled graph used in several area of science such as: coding theory, x-ray crystallography, radar, astronomy, circuit design, communication network addressing, data base management,etc [2]. The notion of FP-cordial labeling has been introduced in [12] and also FP-cordial labeling behaviour of path, cycle, complete, star, wheel, book with triangle pages, ladder, comb, double comb, bistar, subdivision of the star and subdivision of the bistar have been studied in [12]. The number of vertices of a graph G is called the order of G and number of edges is called the size of G. In this paper we investigate the FP-cordial labeling behaviour of certain graphs, like subdivision of comb, subdivision of double comb, triangular snake, quadrilateral snake, slanting ladder, triangular ladder, fan graph, flower graph, sunflower

[1]Received October 19, 2024. Accepted May 12, 2025

graph, helm and closed helm. $\lceil x \rceil$ denotes the smallest integer $\geq x$.

§2. Preliminaries

Definition 2.1([3]) *The subdivision graph $S(G)$ of a graph G is obtained from G by inserting a new vertex of degree 2 on edge of G.*

Definition 2.2([2]) *The corona graph $G_1 \odot G_2$ is the graph obtained from G_1 and G_2 by taking one copy of G_1 and n copies of G_2 and joining the i^{th} vertex of G_1 with an edge to the every vertex in the i^{th} copy of G_2 where G_1 is the graph of order n.*

Definition 2.3([3]) *Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets $E_1 E_2$ respectively.Then the join $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is given by $E_1 \cup E_2 \cup \{uv : u \in V_1 and v \in V_2\}$.*

Definition 2.4([3]) *The product graph $G_1 \times G_2$ is defined as follows: Consider any two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G_1 \times G_2$ whenever $[u_1 = v_1$ and u_2 adjacent to $v_2]$ or $[u_2 = v_2$ and u_1 adjacent to $v_1]$.*

Definition 2.5([2]) *The graph $L_n = P_n \times K_2$ is called a ladder with $2n$ vertices and $3n - 2$ edges.*

Definition 2.6([2]) *The triangular ladder, $T(L_n)$ is a graph obtained from the ladder graph L_n by adding the edges $u_j v_{j+1}, (1 \leq j \leq n-1)$ where $u_j, v_j (1 \leq j \leq n), n \geq 1$ are the vertices of L_n.*

Definition 2.7([2]) *A slanting ladder $S(L_n)(n \geq 2)$, is the graph obtained from two paths $u_1 u_2 \cdots u_n$ and $v_1 v_2 \cdots v_n$ by joining each v_j with u_{j+1}, $1 \leq j \leq n-1$.*

Definition 2.8([2]) *The graph $F_n = P_n + K_1$ is called a fan graph where P_n is a path.It has $n+1$ vertices and $2n - 1$ edges.*

Definition 2.9([2]) *The triangular snake $T_n (n \geq 2)$, is obtained from the path $P_n : u_1 u_2 \cdots u_n$ with $V(T_n) = V(P_n) \cup \{v_i : 1 \leq i \leq n-1\}$ and edge set $E(T_n) = E(P_n) \cup \{u_i v_i, u_{i+1} v_i : 1 \leq i \leq n-1\}$.*

Definition 2.10([2]) *A helm graph $H_n (n \geq 3)$, is a graph obtained from a wheel by attaching a pendent vertex at each n-cycle vertex.*

Definition 2.11([2]) *A quadrilateral snake $Q_n (n \geq 3)$, is obtained from the path $P_n : u_1 u_2 \cdots u_n$ by replacing every edge of a path by a cycle C_4, in such a way that each pair of vertices (u_i, u_{i+1}) remains adjacent. That is, it is obtained from a path P_n by joining u_i and u_{i+1} to new vertices v_i and w_i respectively, and then joining v_i and w_i by an edge for $1 \leq i \leq n-1$.*

Definition 2.12([2]) *The flower graph $Fl_n (n \geq 3)$, is the graph obtained from a helm H_n by joining each pendent vertex to the apex of the helm.*

Definition 2.13([2]) *The sunflower graph $S_n (n \geq 3)$, is obtained by taking a wheel with central vertex u and the cycle $C_n : u_1 u_2 \cdots u_n u_1$ and new vertices $v_1, v_2, \cdots, v_n$ where v_i is joined by vertices $u_i, u_{i+1(mod n)}$. Thus the sunflower graph S_n has $2n+1$ vertices and $4n$ edges.*

Definition 2.14([2]) *A closed helm $CH_n (n \geq 3)$, is the graph obtained from a helm H_n by joining each pendent vertex to form a cycle.*

Definition 2.15([2]) *The quadrilateral book graph $B(4, n)$ with n-pages is defined as n copies of cycle C_4 sharing a common edge. The common edge is called the spine or base of the book.*

Theorem 2.16([2]) *The path P_n is FP-cordial if and only if $n \notin \{3, 5\}$.*

Theorem 2.17([2]) *The cycle C_n is FP-cordial if and only if $n \geq 6$.*

§3. Fractional Product Cordial Labeling

Definition 3.1 *Let $G = (V, E)$ be a (p, q) graph,*

$$M = \begin{cases} 1, 2, \cdots, \frac{p}{2}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{p+2}, & \text{if } p \text{ is even,} \\ 1, 2, \cdots, \frac{p-1}{2}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{p+3}, & \text{if } p \text{ is odd.} \end{cases}$$

and let $\chi : V(G) \to M$ be a bijection. For each edge xy assign the label $\lceil \chi(x)\chi(y) \rceil$. Then, χ is called a fractional product cordial labeling (simply called FP-cordial labeling) if $|\Pi_\chi(0) - \Pi_\chi(1)| \leq 1$, where $\Pi_\chi(1)$ and $\Pi_\chi(0)$ respectively denotes the number of edges labelled with 1 and not labelled with 1. A graph with a fractional product cordial labeling is called a fractional product cordial graph (simply FP-cordial graph). Otherwise, if $|\Pi_\chi(0) - \Pi_\chi(1)| \geq 2$, such a labeling χ is called a Smarandachely FP-cordial labeling.

Theorem 3.2 *The subdivision of the comb $P_n \odot K_1$, $S(P_n \odot K_1)$ is FP-cordial if and only if $n \geq 2$.*

Proof Let $V(S(P_n \odot K_1)) = \{x_i, y_j, z_j : 1 \leq i \leq 2n-1, 1 \leq j \leq n\}$ and $E(S(P_n \odot K_1)) = \{x_i x_{i+1}, x_k y_j, y_j z_j : 1 \leq i \leq 2n-2, 1 \leq j \leq n, k = 1, 3, \cdots, 4n-1\}$. Then it has $4n-1$ vertices and $4n-2$ edges.

Assume $n \geq 2$. Assign labels $1, 2, \cdots, 2n-1$ to the vertices $x_1, x_2, \cdots, x_{2n-1}$ and assign the labels $\frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n+2}$ to the vertices $z_1, z_2, \cdots, z_n$. Now, assign the labels $\frac{1}{n+3}, \frac{1}{n+4}, \cdots, \frac{1}{2n+1}, \frac{1}{2}$ to the vertices $y_1, y_2, \cdots, y_n$. Therefore $\Pi_\chi(0) = 2n-1$ and $\Pi_\chi(1) = 2n-1$.

Since, $S(P_1 \odot K_1) \cong P_3$, the proof follows from Theorem 2.16. □

Theorem 3.3 *The subdivision of the double comb $P_n \odot 2K_1$, $S(P_n \odot 2K_1)$ is FP-cordial if and only if $n \geq 2$.*

Proof Let $V(S(P_n \odot 2K_1)) = \{x_i, v_j, w_j, y_j, z_j : 1 \leq i \leq 3n-1, 1 \leq j \leq n\}$ and $E(S(P_n \odot K_1)) = \{x_i x_{i+1}, x_k y_j, y_j z_j, x_k w_j, w_j v_j : 1 \leq i \leq 3n-2, 1 \leq j \leq n, k = 1, 3, \cdots, 6n-1\}$. Then it has $6n-1$ vertices and $6n-2$ edges.

Assume $n \geq 2$. Assign labels $1, 2, \cdots, n$ to the vertices $v_1, v_2, \cdots, v_n$ and assign the labels $n+1, n+2, \cdots, 2n$ to the vertices $w_1, w_2, \cdots, w_n$. Now, assign labels $2n+1, 2n+2, \cdots, 3n-1$ to the vertices $x_1, x_2, \cdots, x_{n-1}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n+1}$ to the vertices $x_n, x_{n+1}, \cdots, x_{2n-1}$. We now assign labels $\frac{1}{n+2}, \frac{1}{n+3}, \cdots, \frac{1}{2n+1}$ to the vertices $z_1, z_2, \cdots, z_n$ and assign the labels $\frac{1}{2n+2}, \frac{1}{2n+3}, \cdots, \frac{1}{3n+1}$ to the vertices $y_1, y_2, \cdots, y_n$. Therefore $\Pi_\chi(0) = 3n-1$ and $\Pi_\chi(1) = 3n-1$.

As, $S(P_1 \odot 2K_1) \cong P_5$, the proof follows from Theorem 2.16. □

Theorem 3.4 *The quadrilateral book graph $B(4,n)$ is FP-cordial if and only if $n \geq 2$.*

Proof Let $V(B(4,n)) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B(4,n)) = \{uv, uu_i, vv_i, u_iv_i : 1 \leq i \leq n\}$. Then, it has $2n+2$ vertices and $3n+1$ edges. This proof is divided into four cases.

Case 1. n is odd and $n \geq 3$.

Fix the labels $2, 3, 1$ and 4 respectively to the vertices u, v, u_1, v_1. Assign the labels $5, 7, \cdots, n$ to the vertices $u_2, u_3, \cdots, u_{\frac{n-1}{2}}$. Now, assign labels $\frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{n+2}$ to the vertices $u_{\frac{n-1}{2}+1}, u_{\frac{n-1}{2}+2}, \cdots, u_n$. We now assign labels $6, 8, \cdots, n+1$ to the vertices $v_2, v_3, \cdots, v_{\frac{n-1}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{n+1}$ to the vertices $v_{\frac{n-1}{2}+1}, v_{\frac{n-1}{2}+2}, \cdots, v_n$. Therefore, $\Pi_\chi(0) = \frac{3n+1}{2}$ and $\Pi_\chi(1) = \frac{3n+1}{2}$.

Case 2. n is even and $n > 2$.

Fix the labels $2, 3, 1$ and 4 respectively to the vertices u, v, u_1, v_1. Assign labels $5, 7, \cdots, n+1$ to the vertices $u_2, u_3, \cdots, u_{\frac{n}{2}}$. Now, assign labels $\frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{n+1}$ to vertices $u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}, \cdots, u_n$, respectively. We now assign the labels $6, 8, \cdots, n$ to the vertices $v_2, v_3, \cdots, v_{\frac{n}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{n+2}$ to the vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \cdots, v_n$. Therefore, $\Pi_\chi(0) = \frac{3n+2}{2}$ and $\Pi_\chi(1) = \frac{3n}{2}$.

Case 3. $n = 2$.

A FP-cordial labeling of $B(4,2)$ is given in Figure 1.

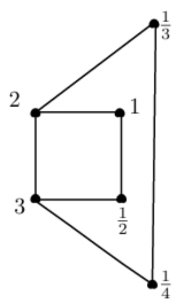


Figure 1

Case 4. $n = 1$.

Since $B(4,1) \cong C_4$, the proof follows from Theorem 2.17. □

Theorem 3.5 *The triangular snake graph T_n is FP-cordial if and only if $n \geq 4$.*

Proof Let $P_n : u_1u_2 \cdots u_n$ be the path. Let $V(T_n) = V(P_n) \cup \{y_i : 1 \leq i \leq n-1\}$ and $E(T_n) = E(P_n) \cup \{u_iy_i, y_iu_{i+1} : 1 \leq i \leq n-1\}$. Then it has $2n-1$ vertices and $3n-3$ edges. This proof is divided into three cases.

Case 1. n is even and $n \geq 4$,

Then the following subcases are arises.

Subcase 1. $n \geq 6$.

Assign labels $1, 3, \cdots, n$ to the vertices $u_1, u_2, \cdots, u_{\frac{n}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{n+1}$ to the vertices $u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}, \cdots, u_n$. We now assign the labels $2, 4, \cdots, n-2$ to the vertices $y_1, y_2, \cdots, y_{\frac{n}{2}-1}$ and assign the labels $\frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{n+2}$ to the vertices $y_{\frac{n}{2}}, y_{\frac{n-1}{2}+1}, \cdots, y_{n-1}$. Therefore $\Pi_\chi(0) = \frac{3n-2}{2}$ and $\Pi_\chi(1) = \frac{3n-4}{2}$.

Subcase 2. $n = 4$.

A FP-cordial labeling of T_4 is given in Figure 2.

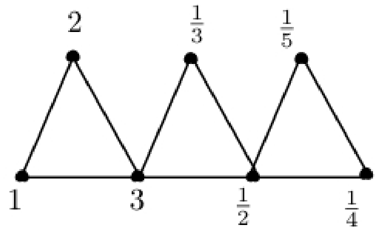


Figure 2

Case 2. n is odd and $n \geq 5$.

Assign labels $1, 3, \cdots, n-2$ to the vertices $u_1, u_2, \cdots, u_{\frac{n-1}{2}}$, respectively and assign the labels $\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{n+1}$ to the vertices $u_{\frac{n+1}{2}}, u_{\frac{n+1}{2}+1}, \cdots, u_n$. Now assign the labels $2, 4, \cdots, n-1$ to the vertices $y_1, y_2, \cdots, y_{\frac{n-1}{2}}$ and assign labels $\frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{n}$ to vertices $y_{\frac{n-1}{2}+1}, y_{\frac{n-1}{2}+2}, \cdots, y_{n-1}$, respectively. Therefore, $\Pi_\chi(0) = \frac{3n-3}{2}$ and $\Pi_\chi(1) = \frac{3n-3}{2}$.

Case 3. $n \in \{2, 3\}$.

Suppose $T_n \in \Omega_{fpc}$. If $n = 2$, since $T_2 \cong C_3$, the proof follows from Theorem 2.17. If $n = 3$, the vertex labels are $1, 2, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. Suppose 1 and 2 are the vertex labels of the non adjacent vertices then $\Pi_\chi(0) = 0$ and $\Pi_\chi(1) = 6$, a contradiction.

Suppose 1 and 2 are the vertex labels of the adjacent vertices then $\Pi_\chi(0) = 1$ and $\Pi_\chi(1) = 5$, which is also not possible. □

Theorem 3.6 *The quadrilateral snake Q_n is FP-cordial if and only if $n \geq 4$.*

Proof Let $P_n : u_1u_2 \cdots u_n$ be the path. Let $V(Q_n) = V(P_n) \cup \{v_i, w_i : 1 \leq i \leq n-1$ and $E(Q_n) = E(P_n) \cup \{u_iv_i, v_iw_i, w_iu_{i+1} : 1 \leq i \leq n-1\}$. Then, it has $3n-2$ vertices and $4n-4$ edges. This proof is divided into three cases.

Case 1. n is even and $n \geq 4$.

Assign labels $2, 3, \cdots, \frac{n+2}{2}$ to the vertices $u_1, u_2, \cdots, u_{\frac{n}{2}}$ and assign the labels $1, (\frac{n+2}{2}) + 2, (\frac{n+2}{2}) + 4, \cdots, \frac{3n-2}{2}$ to the vertices $v_1, v_2, \cdots, v_{\frac{n}{2}}$. Now assign the labels $(\frac{n+2}{2}) + 1, (\frac{n+2}{2}) + 3, \cdots, (\frac{3n-4}{2})$ to the vertices $w_1, w_2, \cdots, w_{\frac{n}{2}-1}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{n+2}$ to the vertices $u_{\frac{n}{2}+1}, u_{\frac{n}{2}+2}, \cdots, u_n$. We now assign labels $\frac{2}{n+4}, \frac{2}{n+8}, \cdots, \frac{2}{3n}$ to the vertices $w_{n-1}, w_{n-2}, \cdots, w_{\frac{n}{2}}$ and assign the labels $\frac{2}{n+6}, \frac{2}{n+10}, \cdots, \frac{2}{3n-2}$ to the vertices $v_{n-1}, v_{n-2}, \cdots, v_{\frac{n}{2}+1}$. Therefore, $\Pi_\chi(0) = 2n - 2$ and $\Pi_\chi(1) = 2n - 2$.

Case 2. n is odd and $n \geq 5$.

Assign labels $2, 3, \cdots, \frac{n+1}{2}$ to the vertices $u_1, u_2, \cdots, u_{\frac{n-1}{2}}$ and assign the labels $1, \frac{n+5}{2}, \frac{n+9}{2}, \cdots, \frac{3n-5}{2}$ to the vertices $v_1, v_2, \cdots, v_{\frac{n-1}{2}}$. Now assign the labels $\frac{n+3}{2}, \frac{n+7}{2}, \cdots, (\frac{3n-3}{2})$ to the vertices $w_1, w_2, \cdots, w_{\frac{n-1}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{n+3}$ to the vertices $u_{\frac{n-1}{2}+1}, u_{\frac{n-1}{2}+2}, \cdots, u_n$. We now assign the labels $\frac{2}{n+5}, \frac{2}{n+9}, \cdots, \frac{2}{3n-1}$ to the vertices $v_{\frac{n-1}{2}+1}, v_{\frac{n-1}{2}+2}, \cdots, v_{n-1}$ and assign the labels $\frac{2}{n+7}, \frac{2}{n+11}, \cdots, \frac{2}{3n+1}$ to the vertices $w_{\frac{n-1}{2}+1}, w_{\frac{n-1}{2}+2}, \cdots, w_{n-1}$. Therefore, $\Pi_\chi(0) = 2n - 2$ and $\Pi_\chi(1) = 2n - 2$.

Case 3. $n \in \{2, 3\}$.

Suppose $Q_n \in \Omega_{fpc}$. If $n = 2$, $Q_2 \cong C_4$. Therefore, the proof follows from Theorem 2.17.

In the case of $n = 3$, the vertex labels are $1, 2, 3, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. If $1, 2$ and 3 are the vertex labels of the non adjacent vertices, $\Pi_\chi(0) \leq 1$ a contradiction to the size of Q_3 is 12.

If $1, 2$ and 3 are the vertex labels of the adjacent vertices, $\Pi_\chi(0) \leq 3$ again a contradiction to the size of Q_3 is 12. □

Theorem 3.7 *If $n \geq 2$ then the slanting ladder $S(L_n)$ is FP-cordial.*

Proof Let $V(S(L_n)) = \{x_i, y_i : 1 \leq i \leq n\}$ and $E(S(L_n)) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1} : 1 \leq i \leq n - 1\}$. Then it has $2n$ vertices and $3n - 3$ edges. This proof is divided into two cases.

Case 1. n is odd and $n \geq 3$.

Assign labels $1, 2, \cdots, n$ to the vertices $y_1, y_2, \cdots, y_n$ and assign the labels $\frac{1}{n}, \frac{1}{n-1}, \cdots, \frac{1}{2}, \frac{1}{n+1}$ to the vertices $x_1, x_2, \cdots, x_n$. We have $\Pi_\chi(0) = \frac{3n-3}{2}$ and $\Pi_\chi(1) = \frac{3n-3}{2}$.

Case 2. n is even and $n \geq 2$.

Assign labels $1, 2, \cdots, n$ to the vertices $y_1, y_2, \cdots, y_n$ and assign the labels $\frac{1}{n+1}, \frac{1}{n}, \cdots, \frac{1}{2}$ to the vertices $x_1, x_2, \cdots, x_n$. We have $\Pi_\chi(0) = \frac{3n-4}{2}$ and $\Pi_\chi(1) = \frac{3n-2}{2}$. □

Theorem 3.8 *The triangular ladder $T(L_n)$ is FP-cordial if and only if $n \neq 2$.*

Proof Let $V(T(L_n) = \{x_i, y_i : 1 \leq i \leq n\}$ and $E(T(L_n)) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1}, x_i y_i : 1 \leq i \leq n - 1\} \cup \{x_n y_n\}$. Then it has $2n$ vertices and $4n - 3$ edges. This proof is divided into three cases.

Case 1. n is odd.

Assign labels $1, 3, \cdots, n$ to the vertices $y_1, y_2, \cdots, y_{\frac{n+1}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{n+1}$

to the vertices $y_{\frac{n+1}{2}+1}, y_{\frac{n+1}{2}+2}, \cdots, y_n$. Now, assign the labels $2, 4, \cdots, n-1$ to the vertices $x_1, x_2, \cdots, x_{\frac{n-1}{2}}$ and assign the labels $\frac{1}{n+1}, \frac{1}{n}, \cdots, \frac{2}{n+3}$ to the vertices $x_{\frac{n-1}{2}+1}, x_{\frac{n-1}{2}+2}, \cdots, x_n$. We have $\Pi_\chi(0) = 2n-2$ and $\Pi_\chi(1) = 2n-1$.

Case 2. n is even and $n \geq 4$.

Assign labels $1, 3, \cdots, n-1$ to the vertices $y_1, y_2, \cdots, y_{\frac{n}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{n+2}$ to the vertices $y_{\frac{n}{2}+1}, y_{\frac{n}{2}+2}, \cdots, y_n$. Now, assign labels $2, 4, \cdots, n$ to the vertices $x_1, x_2, \cdots, x_{\frac{n}{2}}$ and assign the labels $\frac{1}{n+1}, \frac{1}{n}, \cdots, \frac{2}{n+4}$ to the vertices $x_{\frac{n-1}{2}+1}, x_{\frac{n-1}{2}+2}, \cdots, x_n$. We have $\Pi_\chi(0) = 2n-1$ and $\Pi_\chi(1) = 2n-2$.

Case 3. $n = 2$.

Suppose $T(L_2) \in \Omega_{fpc}$. In this case the vertex labels are $1, 2, \frac{1}{2}, \frac{1}{3}$. If 1 and 2 are the vertex labels of the non adjacent vertices, $\Pi_\chi(0) = 0$ and $\Pi_\chi(1) = 5$, a contradiction. If 1 and 2 are the vertex labels of the adjacent vertices, $\Pi_\chi(0) = 1$ and $\Pi_\chi(1) = 4$, not possible. □

Theorem 3.9 *The fan graph F_n is FP-cordial if and only if $n \notin \{2, 3, 4, 6\}$.*

Proof Let $P_n : u_1u_2 \cdots u_n$ be the path. Let $V(F_n) = V(P_n) \cup \{u\}$ and $E(F_n) = E(P_n) \cup \{uu_i : 1 \leq i \leq n\}$. Then fan graph has $n+1$ vertices and $2n-1$ edges. This proof is divided into three cases.

Case 1. n is odd and $n \neq 3$.

Then, the following subcases are arises.

Subcase 1.1 $n \geq 7$.

Consider the central vertex u. Assign the label 3 to u. Now, assign labels $1, 2, 4, \cdots, \frac{n+1}{2}$ to the vertices $u_1, u_2, \cdots, u_{\frac{n-1}{2}}$ and assign the labels $u_{\frac{n-1}{2}+1}, u_{\frac{n-1}{2}+2}, \cdots, u_n$. Therefore, $\Pi_\chi(0) = n$ and $\Pi_\chi(1) = n-1$.

Subcase 1.2. $n \in \{1, 5\}$.

If $n = 1$, $F_1 \cong P_2$, the proof follows from Theorem 2.16. If $n = 5$, F_5 is FP-cordial labeling given in Figure 3.

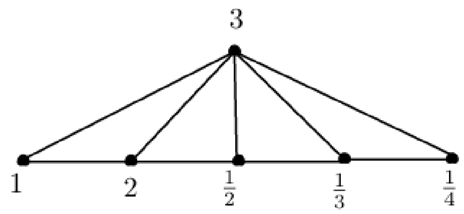


Figure 3

Case 2. n is even and $n \geq 8$.

For the central vertex u is assigned by 3. Assign labels $1, 2, 4, \cdots, \frac{n}{2}$ to the vertices $u_1, u_2, \cdots, u_{\frac{n-2}{2}}$ and assign the labels $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{n+4}$ to the vertices $u_{\frac{n-2}{2}+1}, u_{\frac{n-2}{2}+2}, \cdots, u_n$. Therefore, $\Pi_\chi(0) = n-1$ and $\Pi_\chi(1) = n$.

Case 3. $n \in \{2, 3, 4, 6\}$.

Suppose $F_n \in \Omega_{fpc}$. Then, the following subcases are arises.

Subcase 3.1 $n = 2$.

Since $F_2 \cong C_3$, the proof follows from Theorem 2.17.

Subcase 3.2 $n = 3$.

The vertex labels are $1, 2, \frac{1}{2}, \frac{1}{3}$. If 1 and 2 are the vertex labels of the non adjacent vertices, $\Pi_\chi(0) = 0$ and $\Pi_\chi(1) = 5$, a contradiction. If 1 and 2 are the vertex labels of the adjacent vertices, $\Pi_\chi(0) = 1$ and $\Pi_\chi(1) = 4$, again a contradiction.

Subcase 3.3 $n = 4$.

The vertex labels are $1, 2, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. If 1 and 2 are the vertex labels of the non adjacent vertices, $\Pi_\chi(0) = 0$ and $\Pi_\chi(1) = 7$, a contradiction. If 1 and 2 are the vertex labels of the adjacent vertices, $\Pi_\chi(0) = 1$ and $\Pi_\chi(1) = 6$, again a contradiction.

Subcase 3.4 $n = 6$.

In this case the vertex labels are $1, 2, 3, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. If the central vertex is assigned by 1 or 2 or 3, $\Pi_\chi(0) \leq 4$ a contradiction to the size of F_6 is 11. If the central vertex is assigned by $\frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$ or $\frac{1}{5}$, $\Pi_\chi(0) \leq 3$ again a contradiction to the size of F_6 is 11. □

Theorem 3.10 *The helm H_n is FP-cordial for all $n \geq 3$.*

Proof Let $V(H_n) = V(W_n) \cup \{z_i : 1 \leq i \leq n\}$ where $W_n = C_n + K_1$, C_n be the cycle $y_1 y_2 \cdots y_n y_1$, $V(K_1) = \{y\}$ and $E(H_n) = E(W_n) \cup \{y_i z_i : 1 \leq i \leq n\}$. Then it has $2n + 1$ vertices and $3n$ edges. The proof is divided into two cases.

Case 1. n is odd.

For the central vertex y is assigned by $\frac{2}{n+1}$. Assign labels $1, 2, \cdots, n$ to the vertices $y_1, y_2, \cdots, y_n$ and assign the labels

$$\frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{n-1}, \frac{2}{n+3}, \cdots, \frac{1}{n+2}$$

to the vertices $z_1, z_2, \cdots, z_n$. Therefore,

$$\Pi_\chi(0) = \frac{3n-1}{2} \text{ and } \Pi_\chi(1) = \frac{3n+1}{2}.$$

Case 2. n is even.

Consider the central vertex y. Assign labels $\frac{2}{n}$ to y. Now, assign the labels $1, 2, \cdots, n$ to the vertices $y_1, y_2, \cdots, y_n$ and assign the labels

$$\frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{n-2}, \frac{2}{n+2}, \cdots, \frac{1}{n+2}$$

to the vertices $z_1, z_2, \cdots, z_n$. Therefore,

$$\Pi_\chi(0) = \frac{3n}{2} \text{ and } \Pi_\chi(1) = \frac{3n}{2}. \qquad \square$$

Example 3.11 A FP-cordial labeling of H_{12} is shown in Figure 4.

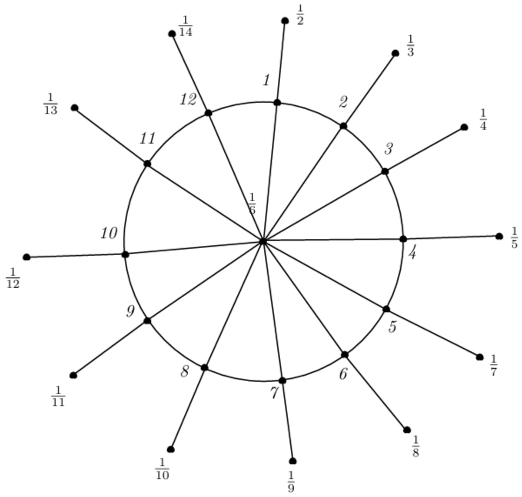


Figure 4

Theorem 3.12 *The flower graph Fl_n is FP-cordial if and only if $n \geq 4$.*

Proof Let $V(Fl_n) = \{u, v_i, w_i : 1 \leq i \leq n\}$ and $E(Fl_n) = \{uv_i, v_iv_{i+1}, v_iw_i, uw_i : 1 \leq i \leq n-1\} \cup \{uv_n, v_1v_n, v_nw_n, uw_n\}$. Then, it has $2n+1$ vertices and $4n$ edges.

Assume $n \geq 4$. Consider the central vertex u. Assign the label 1 to u. Now, assign labels $2, 3, \cdots, n-1, \frac{1}{2}, n$ to the vertices $v_1, v_2, \cdots, v_n$ and assign the labels $\frac{1}{3}, \frac{1}{4}, \cdots, \frac{1}{n+2}$ to the vertices $w_n, w_1, w_2 \cdots, w_{n-1}$. We have $\Pi_\chi(0) = 2n$ and $\Pi_\chi(1) = 2n$.

When $n = 3$, suppose $Fl_n \in \Omega_{fpc}$. Then, the vertex labels are $1, 2, 3, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. In the case $1, 2$ and 3 are the vertex labels of the non adjacent vertices then $\Pi_\chi(0) \leq 4$, a contradiction to the size of Fl_3 is 12. But in the cases $1, 2$ and 3 are the vertex labels of the adjacent vertices then $\Pi_\chi(0) \leq 1$ again a contradiction to the size of Fl_3 is 12. $\square$

Corollary 3.13 *The sunflower graph S_n is FP-cordial if and only if $n \geq 4$.*

Proof Let $V(S_n) = \{x, y_i, z_i : 1 \leq i \leq n\}$ and $E(S_n) = \{xy_i, y_iy_{i+1}, y_{i+1}z_i : 1 \leq i \leq n-1\} \cup \{xy_n, y_1z_n\}$. Then it has $2n+1$ vertices and $4n$ edges. Assume $n \geq 4$. Let

$$M = \begin{cases} 1, 2, \cdots, \frac{p}{2}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{p+2}, & \text{if } p \text{ is even,} \\ 1, 2, \cdots, \frac{p-1}{2}, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{2}{p+3}, & \text{if } p \text{ is odd} \end{cases}$$

and let $\chi : V(Fl_n) \to M$ be the FP-cordial labeling of flower graph in Theorem 3.12. Defined $\chi^* : V(S_n) \to M$ by $\chi^*(x) = \chi(u)$, $\chi^*(y_i) = \chi(v_i)$ where $1 \leq i \leq n$, $\chi^*(z_i) = \chi(w_i)$ where

$1 \leq i \leq n$. Since χ is FP-cordial labeling, χ^* is also a FP-cordial labeling of the sunflower graph S_n.

In the case of $n = 3$, there does not exist a FP-cordial labeling of the sunflower graph as in Theorem 3.12. □

Example 3.14 A FP-cordial labeling of S_{12} is shown in Figure 5.

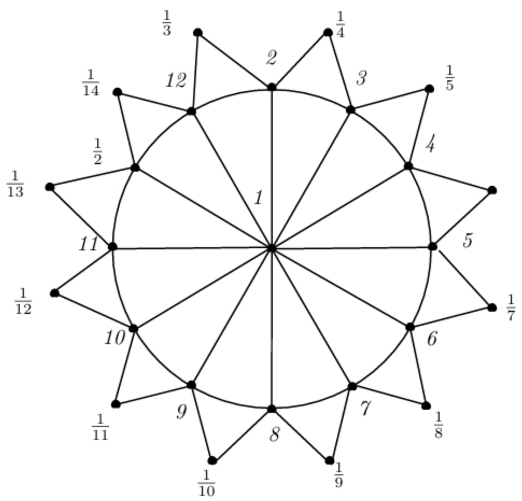


Figure 5

Corollary 3.15 *The closed helm CH_n is FP-cordial if and only if $n \geq 4$.*

Proof Let $V(CH_n) = V(H_n)$ and $E(CH_n) = E(H_n) \cup \{z_i z_{i+1} : 1 \leq i \leq n-1\} \cup \{z_1 z_n\}$. Then it has $2n+1$ vertices and $4n$ edges. Let $n \geq 4$. Take the vertex set of helm graph H_n as in Theorem 3.10. Clearly, the FP-cordial labeling χ in Theorem 3.12 is also a FP-cordial labeling of closed helm.

For the case of $n = 3$, as in Theorem 3.12, there does not exist a FP-cordial labeling of the closed helm graph. □

References

[1] I. Cahit, Cordial graphs: a weaker versionof graceful and harmonious graphs,*Ars comb.*, 23 (1987), 201-207.

[2] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 24(2021).

[3] F. Harary, *Graph Theory*, Addison Wesely, Reading Mass., 1972.

[4] W. W. Kirchherr, On the cordiality of some specific graphs, *Ars Combin.*, 31(1991), 127-138.

[5] D. Kuo, G. Chang, and Y. H. Kwong, Cordial labeling of mK_n, *Discrete Math.*, 169(1997), 121-131.

[6] R. Patrias and O. Pechenik, Path-cordial abelian groups, *Australas. J. Combin.*, 80(2021)

157-166.
[7] A. Petrano and R. Rulete, On total product cordial labeling of some graphs, *Internat. J. Math. Appl.*, 5(2B)(2017), 273-284.
[8] R. Ponraj, K. Annathurai and R. Kala , k-Remainder Cordial graphs, *Journal of Algorithms and Computation*, Vol.49, No. 2(2017), 41-52.
[9] R. Ponraj, A. Gayathri and S. Somasundaram, Pair difference cordial labeling of graphs, *J. Math. Compt. Sci*, Vol 11 (2021), 2551-2567.
[10] R. Ponraj, A. Gayathri and S. Somasundaram, Some pair difference cordial graphs, *Ikonion Journal of Mathematics*, Vol 3 (2021), 17-26.
[11] R. Ponraj and S. Prabhu, Pair mean cordial labeling of graphs, *Journal of Algorithms and Computation*, 54 issue 1(2022), 1-10.
[12] R. Ponraj and T. Sutharson, Fractional product cordial labeling of graph, *J. Appl. and Pure Math.*, Vol 7 (2025), No. 1-2, pp. 49-58.
[13] U. M. Prajapati and S. J. Gajjar, Prime cordial labeling of generalized prism graph $Y_{(m,n)}$, *Ultra Scientist*, 27(3)A,(2015), 189-204.
[14] U. M. Prajapati, and S. J. Gajjar, Cordial labeling for complement of some graphs, *Math. Today*, 30, (2015), 99-118.
[15] U. M. Prajapati, and N. B. Patel, Edge product cordial labeling of some graphs, *J. Appl. Math. Comput. Mech.*, 18(1) (2019), 69-76.
[16] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Intl. Symp. Rome 1966), Gordon and Breach, Dunod, Paris, (1967) 349-355.
[17] M. A. Seoud. and M. Aboshady, Further results on pairity combination cordial labeling, *J. Egyptian Math. Soc.*, 28(1)(2020), Paper No. 25, 10 pp.
[18] M. A. Seoud and A. E. I. Abdel Maqsoud, On cordial and balanced labeling of graphs, *J. Egyptian Math. Soe.*, 7 (1999), 127-135.
[19] M. A. Seoud and H. Jaber, Prime cordial and 3-equitable prime cordial graphs, *Util. Math.*, 111 (2019) 95-125.
[20] M. A. Seoud and M. A. Salim, Two upper bounds of prime cordial graphs, *JCMCC*, 75 (2010), 95-103.

International J.Math. Combin. Vol.1-Vol.2(2025), 78-89

Modulo Two Square Mean Labeling of Some Path and Path Related Graphs

Christopher.M

Research Scholar, PG and Research Department of Mathematics
Mannar Thirumalai Naicker College (Affiliated to Madurai Kamaraj University), Madurai, Tamilnadu, India
Department of Mathematics, SSM Institute of Engineering and Technology, Dindigul, Tamilnadu, India

Ramachandran.V

PG and Research Department of Mathematics
Mannar Thirumalai Naicker College (Affiliated to Madurai kamaraj university), Madurai, Tamilnadu, India

E-mail: christopher.mosespaul@gmail.com, me.ram111@gmail.com

Abstract: We introduce the new concept of modulo two square mean labeling. A graph is said to be modulo two square mean labeling, if there is a function ϕ from the vertex set of G to $\{1, 2, 3, \cdots n\}$, ϕ' from the edge set of G to $\{1\}$ where $\phi^{'}(uv) = \left\lceil \frac{f(u)^2+f(v)^2}{2} \right\rceil$ mod 2. In this paper we Prove that the modulo two square mean labeling of some path related graphs and $H-$graph with more than 3 vertices. Additionally, we provide a $C++$ program designed to determine the modulo two square mean labeling for the above mentioned graphs.

Key Words: Square sum labeling, mean labeling, root mean square labeling, Smarandachely mean labeling.

AMS(2010): 05C78, 05C85.

§1. Introduction

In this paper, we consider only simple, finite, undirected and non-trivial graph G = (V(G),E(G)) with the vertex set V(G) and the edge set E(G). Labeling of a graph G is an assignment integers to vertices or edges or both following certain rules. A useful survey on graph labeling by J.A.Gallian (2015) can be found in [1]. Labeled graph has its own applications in various fields such as engineering, technology, etc. A particular type of labeling becomes more interesting if there arises a number of problems that kindles the interest of the researchers. Prominent among the types of labeling is square sum labeling [2],[3], [4], [5]. In this paper we deal only finite, simple, connected and undirected graphs obtained through graph operations. Another labeling has been introduced by Somasundaram and Ponraj [6] the notion of mean labeling of graphs. A graph G with p vertices and q edges is called a mean graph if there is an injective

[1]Received November 15, 2024. Accepted May 16, 2025

function f from the vertices of G to $\{0, 1, 2, \cdots, q\}$ such that when each edge uv is labeled with

$$\begin{cases} \dfrac{f(u)+f(v)}{2}, & \text{if } f(u)+f(v) \text{ is even,} \\ \dfrac{f(u)+f(v)+1}{2}, & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$$

then the resulting edge labels are distinct. Generally, let $V' \subset V(G)$. If $G \backslash V'$ has a mean labeling ϕ, then ϕ is called a Smarandachely mean labeling respect to V' on G. The concept of root square mean labeling has been introduced by S. S. Sandhya, S. Somasundaram and S. Anusa in 2014. Meena. S and Mani. R investigated this labeling for some cycle related graphs.

§2. Basic Definitions

We use the following definitions in the subsequent section to prove the main result.

Definition 2.1([4]) *A path is a trail in which all vertices are distinct.*

Definition 2.2 *The H-Graph of a path $P_n, n \geq 3$ is obtained from two copies of $v_1, v_2, v_3, \cdots v_n$ and $u_1, u_2, u_3, \cdots u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ by an edge if n is odd and the vertices $v_{\frac{n}{2}+1}$ and $u_{\frac{n}{2}+1}$ if n is even.*

§3. Main Results

Theorem 3.1 *The graph $P_n, n \geq 2$ is a modulo two square mean labeling.*

Proof Let $v(P_n) = \{v_i / 1 \leq i \leq n\}$ be the vertex set and $E(P_n) = \{e_i = 1, 1 \leq i \leq n-1\}$is the edge set. The graph has n vertices and n-1 edges.

Let $f\colon v \to \{1, 2, \cdots n\}$ by defining the vertex labeling $f(v_i) = \{i\ for 1 \leq i \leq n\}$. Then the induced edge labels are $f(e_i) = \left\lceil \frac{f(v_i)^2 + f(v_{i+1})^2}{2} \right\rceil \bmod 2$, $1 \leq i \leq n-1$ for $e_i \in P_n$ if $n \geq 2$. Then for every $e \in E(P_n)$ is $f(e_i) = \{1\}$. Hence f is a modulo two square mean labeling. □

Illustration 3.1 A modulo two square mean labeling of P_5 is shown in Figure 1.

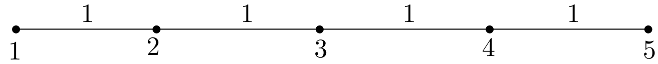


Figure 1. Modulo two square mean labeling of P_5

Program 1

```
# include <iostream>
# include <cmath>
int main()
{
int n, x[100],y[100],v[100],i;
```

```
std::cout << "Enter The Number of Vertices n = ";
std::cin>> n;
for(i = 1; i <= n; i ++)
{
std::cout<< "\n The Path of the Vertices of v[" << i << "] = " << i << "\n";
x[i]= (i*i)+((i+1)*(i+1));
y[i]=ceil(float(x[i])/2);
}
for(i = 1; i <= n; i ++)
{
std:: cout<< "\n" <<"The Square of the Edges of e(" << i << ") = " << y[i];
std:: cout<< "\t\t" <<" Edges of e(" << i << ") = " << y[i]%2;
std::cout<< "\n"; }
return 0;
}
```

Theorem 3.2 *A graph G obtained by attaching each vertex of P_n to the central vertex of $K_{1,2}$ is a modulo two square mean labeling.*

Proof Let $G = P_n \otimes K_{1,2}$ be a graph obtained from a path P_n with vertices u_i and joining the vertices v_i, w_i of $K_{1,2}$ for the vertices u_i of P_n, $1 \leq i \leq n$ respectively.

Define $f \colon V(G) \to \{1, 2, 3, \cdots 3n\}$ to be

$$f(u_i) = 3i - 1 \ 1 \leq i \leq n,$$

$$f(v_i) = 3i - 2 \ 1 \leq i \leq n,$$

$$f(w_i) = 3i \ 1 \leq i \leq n.$$

by the definition of modulo two square mean labeling. The edges get labels

$$f(u_i u_{i+1}) = \left\lceil \frac{f(u_i)^2 + f(u_{i+1})^2}{2} \right\rceil \bmod 2, \ 1 \leq i \leq n-1,$$
$$f(u_i v_i) = \left\lceil \frac{f(u_i)^2 + f(v_i)^2}{2} \right\rceil \bmod 2, \ 1 \leq i \leq n,$$
$$f(u_i w_i) = \left\lceil \frac{f(u_i)^2 + f(w_i)^2}{2} \right\rceil \bmod 2, \ 1 \leq i \leq n.$$

Clearly, f admits modulo two square mean labeling. □

Illustration 3.2 A modulo two square mean labeling of $P_3 \otimes K_{1,2}$ is shown in Figure 2.

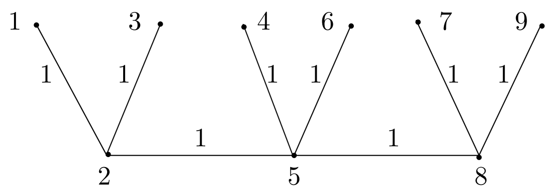


Figure 2. Modulo two square mean labeling of $P_3 \otimes K_{1,2}$

Program 2

```
# include <iostream>
# include <cmath>
int main()
{
int n, i,c=0; int v[100],u[100],w[100];
int e1[100],e2[100],e3[100],e4[100],e5[100],e6[100];
std::cout<< "Enter the Number of Vertices = ";
std::cin>>n;
for (i=1;i<=n;i++)
{
v[i]=3*i-2;
u[i]=3*i-1;
w[i]=3*i;
std::cout<< "\t The Vertices of v[" << i << "] = " << v[i];
std::cout<< "\t The Vertices of u[" << i << "] = " << u[i];
std::cout<<" \t The Vertices of w[" << i << "] = " << w[i];
std::cout<< "\n";
}
std::cout<< "\n";
for (i = 1; i <= n; i ++)
{
e3[i]=((u[i]*u[i])+(v[i]*v[i]));
e4[i]= ceil( float (e3[i])/2);
std::cout<<"\n Edges addition of e[" << i << "] = " << e3[i] << "\t\t";
std::cout<< "\t\t Edges of e[" << i << "] = " << e4[i]%2 << "\t";
}
for (i = 1; i <= n; i ++)
{
e5[i]=((u[i]*u[i])+(w[i]*w[i]));
e6[i]= ceil( float (e5[i])/2);
std::cout<< "\n Edges addition of e[" << i + n << "] = " << e5[i] << "\t\t";
std::cout<< "\t\t Edges of e[" << i + n << "] = " << e6[i]%2 << "\t";
}
for (i=1; i¡=n-1; i++)
{
e1[i]=((u[i]*u[i])+(u[i+1]*u[i+1]));
e2[i]= ceil( float (e1[i])/2);
std::cout<< "\n Edges addition of e[" << 2 * n + i << "] = " << e1[i] << "\t\t";
std::cout<< "\t\t Edges of e[" << 2 * n + i << "] = " << e2[i]%2 << "\t";
}
return 0;
```

}

Theorem 3.3 *The graph $K_{1,2} \otimes P_n$ is a modulo two square mean labeling.*

Proof Let $G = K_{1,2} \otimes P_n$ be a graph obtained by joining a pendant vertex of a path P_n with a star $K_{1,2}$ and let u be the central vertex of $K_{1,2}$. Let u_1, u_2 be the other vertices of $K_{1,2}$ and let $v_1, v_2, \cdots v_n$ be the vertices of P_n.

Define $f\colon V(G) \to \{1, 2, 3, \cdots n+2\}$ as follows:

$f(u_1) = 1$ and $f(u_2) = 2$, $f(u = v_1) = 3$,

$f(v_{i+1}) = 3 + i$. and $1 \le i \le n-1$

by the definition of modulo two square mean labeling. They are labeled as

$f\,(u_1v_1) = \left\lceil \frac{f(u_1)^2+f(v_1)^2}{2} \right\rceil$ mod 2,

$f\,(u_2v_1) = \left\lceil \frac{f(u_2)^2+f(v_1)^2}{2} \right\rceil$ mod 2,

$f\,(v_iv_{i+1}) = \left\lceil \frac{f(v_i)^2+f(v_{i+1})^2}{2} \right\rceil$ mod 2, $1 \le i \le n-1$.

Clearly, G admits modulo two square mean labeling. □

Illustration 3.3 A modulo two square mean labeling of $K_{1,2} \otimes P_3$ is shown in Figure 3.

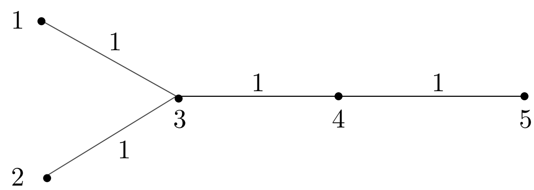


Figure 3. Modulo two square mean labeling of $K_{1,2} \otimes P_3$

Program 3

```
# include <iostream>
# include <cmath>
int main()
{
int n, i,j,c=0;
int v[100],u[100],w[100];
int e1[500],e2[500];
std::cout<< "Enter the Number of Vertices of path Graph = ";
std::cin>>n;
u[1]=1;
u[2]=2;
v[1]=3;
std::cout<< "\n The Vertices of u[" << 1 << "] = " << u[1];
std::cout<< "\n The Vertices of u[" << 2 << "] = " << u[2];
```

```
std::cout<< "\n\n The path Vertices of v[" << 1 << "] = " << v[1];
for (i = 1; i <= n − 1; i + +)
{
v[i+1]=3+i;
std::cout<< "\n The path Vertices of v[" << i + 1 << "] = " << v[i + 1];
}
std::cout<< "\n";
e1[1]=((u[1]*u[1])+(v[1]*v[1]));
e2[1]=ceil( float (e1[1])/2);
std::cout<< "\n Edges addition of e[" << 1 << "] = " << e1[1] << "\t";
std::cout<< "\t\t Edges of e[" << 1 << "] = " << e2[1]%2 << "\t";
e1[2]=((u[2]*u[2])+(v[1]*v[1]));
e2[2]= ceil( float (e1[2])/2);
std::cout<< "\t\t Edges addition of e[" << 2 << "] = " << e1[2] << "\t";
std::cout<< "\t\t Edges of e[" << 2 << "] = " << e2[2]%2 << "\t";
for(i = 1; i <= n − 1; i + +)
{
e1[i+2]=((v[i]*v[i])+(v[i+1]*v[i+1]));
e2[i+2]= ceil (float (e1[i+2])/2);
std::cout<< "\n Edges addition of e[" << i + 2 << "] = " << e1[i + 2] << "\t";
std::cout<< "\t\t Edges of e[" << i + 2 << "] = " << e2[i + 2]%2 << "\t";
}
return 0;
}
```

Theorem 3.4 *The graph H_n with odd n and $n \geq 3$, is a modulo two square mean labeling.*

Proof Let $H_n, n \geq 3$ be a H-Graph with vertex set $\{u_1, u_2, u_3, \cdots u_n, v_1, v_2, v_3, \cdots v_n\}$ and edge set $\{u_i u_{i+1}, v_i v_{i+1} \,|\, 1 \leq i \leq n-1\} \cup \{u_{\frac{n+1}{2}} \;\; v_{\frac{n+1}{2}}\, if\, n\, is\, odd\}$. Define $f : v(G) \rightarrow \{1, 2, 3, \cdots .2n\}$ by

$f(u_i) = i\ ,\quad 1 \leq i \leq n,$
$f(v_i) = n + i, \quad 1 \leq i \leq n$

by defining the edge labels

$$f(e_i) = \left\lceil \frac{f(u_i)^2 + f(u_{i+1})^2}{2} \right\rceil \bmod 2 \text{ for } 1 \leq i \leq n-1,$$

$$f(e_{i+n-1}) = \left\lceil \frac{f(v_i)^2 + f(v_{i+1})^2}{2} \right\rceil \bmod 2 \quad \text{for } 1 \leq i \leq n-1,$$

$$f(e_{2n-1}) = \left\lceil \frac{f(\frac{u_{n+1}}{2})^2 + f(\frac{v_{n+1}}{2})^2}{2} \right\rceil \bmod 2.$$

Then, for every $e \in E(P_n)$ is $f(e_i) = 1$. Hence, the graph H_n , $n \geq 3$ and n is odd has a modulo two square mean labeling. □

Illustration 3.4 A modulo two square mean labeling of H_5 is shown in Figure 4.

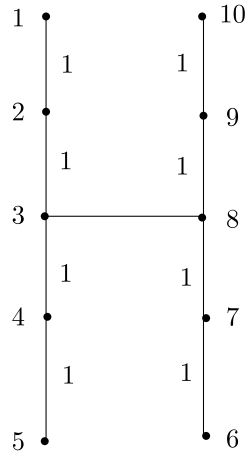


Figure 4. Modulo two square mean labeling of H_5

Program 4

```
# include <iostream>
# include <cmath>
int main()
{
int n, i,j,c=0; int v[100],u[100];
int e1[100],e2[100],e3[100],e4[100],e5[1],e6[1];
std::cout<< ”Enter the Number of Vertices of H-Graph with odd Number of vertices = ”;
std::cin>>n;
for (i = 1; i <= n; i + +)
{
u[i]=i;
v[i]=n+i;
}
for (i = 1; i <= n; i + +)
std::cout<< ”\n The Vertices of u[” << i << ”] = ” << u[i];
std::cout<< ”\n”;
for (i = 1; i <= n; i + +)
std::cout<< ”\n The Vertices of v[” << i << ”] = ” << v[i];
std::cout<< ”\n”;
for(i = 1; i <= n − 1; i + +)
{
e1[i]=((u[i]*u[i])+(u[i+1]*u[i+1]));
e2[i]= ceil( float (e1[i])/2);
std::cout<< ”\n Edges addition of e[” << i << ”] = ” << e1[i] << ”\t”;
std::cout<< ”\t\t Edges of e[” << i << ”] = ” << e2[i]%2 << ”\t”;
}
```

```
for(i = 1; i <= n − 1; i + +)
{
e3[i]=((v[i]*v[i])+(v[i+1]*v[i+1]));
e4[i]= ceil( float (e3[i])/2);
std::cout<< "\n Edges addition of e[" << n + i − 1 << "] = " << e3[i] << "\t";
std::cout<< "\t\t Edges of e[" << n + i − 1 << "] = " << e4[i]%2 << "\t";
}
e5[1]=((v[(n+1)/2]*v[(n+1)/2])+(u[(n+1)/2]*u[(n+1)/2]));
e6[1]= ceil( float (e5[1])/2);
std::cout<< "\n Middle Edges addition of e[" << n + i − 1 << "] = " << e5[1] << "\t";
std::cout<< "\t\t Edges of e[" << n + i − 1 << "] = " << e6[1]%2 << "\t";
return 0;
}
```

Theorem 3.5 *The graph $H_n \otimes K_{1,2}$ where n is odd and $n \geq 3$ is a modulo two square mean labeling.*

Proof Let $G = H_n \otimes K_{1,2}$ where H_n is a $H-$graph with vertices $u_1, u_2, u_3, \cdots u_n$ and $v_1, v_2, v_3, \cdots v_n$. for $1 \leq i \leq n$. Let t_i, s_i be the vertices of $K_{1,2}$ attached at u_i, and x_i, y_i be the vertices of $K_{1,2}$ joined at v_i. Define $f : V(G) \to \{1, 2, \cdots, 6n\}$, then the label to the vertices are as follows:

$f(t_i) = 3i - 2, 1 \leq i \leq n.$,

$f(s_i) = 3i, 1 \leq i \leq n.$,

$f(u_i) = 3i - 1, 1 \leq i \leq n.$, ,

$f(x_{i-n}) = 3i - 2, n + 1 \leq i \leq 2n.$,

$f(y_{i-n}) = 3i, n + 1 \leq i \leq 2n.$,

$f(v_{i-n}) = 3i - 1, n + 1 \leq i \leq 2n.$

by the definition of a modulo two square mean labeling. We define the edge labels are defined to be

$$f(u_it_i) = \left\lceil \frac{f(u_i)^2 + f(t_i)^2}{2} \right\rceil mod\, 2, \qquad 1 \leq i \leq n,$$

$$f(u_is_i) = \left\lceil \frac{f(u_i)^2 + f(s_i)^2}{2} \right\rceil mod\, 2, \qquad 1 \leq i \leq n,$$

$$f(v_ix_i) = \left\lceil \frac{f(v_i)^2 + f(x_i)^2}{2} \right\rceil mod\, 2, \qquad 1 \leq i \leq n,$$

$$f(v_iy_i) = \left\lceil \frac{f(v_i)^2 + f(y_i)^2}{2} \right\rceil mod\, 2, \qquad 1 \leq i \leq n,$$

$$f(u_iu_{i+1}) = \left\lceil \frac{f(u_i)^2 + f(u_{i+1})^2)}{2} \right\rceil mod\, 2, \quad 1 \leq i \leq n-1,$$

$$f(v_i v_{i+1}) = \left\lceil \frac{f(v_i)^2 + f(v_{i+1})^2)}{2} \right\rceil mod\, 2, \quad 1 \leq i \leq n-1,$$

$$f(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}) = \left\lceil \frac{f(u_{\frac{n+1}{2}})^2 + f(v_{\frac{n+1}{2}})^2}{2} \right\rceil mod\, 2, \text{ if } n \text{ is odd.}$$

Hence, the edge label satisfying the a modulo two square mean labeling and $H_n \otimes K_{1,2}$ has a modulo two square mean labeling. □

Illustration 3.5 A modulo two square mean labeling of $H_5 \otimes K_{1,2}$is shown in Figure 5.

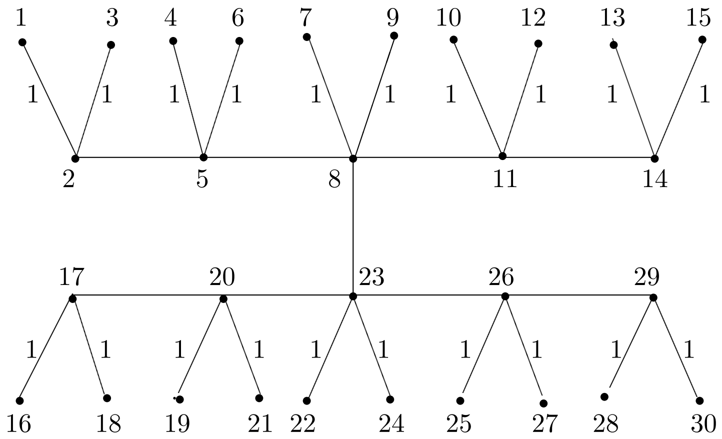


Figure 5. Modulo two square mean labeling of $H_5 \otimes K_{1,2}$

Program 5

```
# include <iostream>
# include <cmath>
int main()
{
int n, i,j,c=0; int v[100],u[100],w[100],t[100],s[100],x[100],y[100];
int e1[100],e2[100],e3[100],e4[100],e5[100],e6[100],e7[100],e8[100];
int e9[100],e10[100],e11[100],e12[100],e13[100],e14[100];
std::cout<< "Enter the Number of Vertices of H-Graph with odd Number of vertices = ";
std::cin>>n;
for (i = 1; i <= n; i + +)
{
t[i]=3*i-2;
s[i]=3*i;
u[i]=3*i-1;
}
for (i = n + 1; i <= 2 * n; i + +)
{
x[i-n]=3*i-2;
y[i-n]=3*i;
```

```
v[i-n]=3*i-1;
}
std::cout<< "\n";
for (i = 1; i <= n; i++)
{
std::cout<< "\t The Vertices of t[" << i << "] = " << t[i];
std::cout<< "\t The Vertices of u[" << i << "] = " << u[i];
std::cout<< "\t The Vertices of s[" << i << "] = " << s[i];
std::cout<< "\n";
}
std::cout<< "\n";
for (i = n + 1; i <= 2 * n; i++)
{
std::cout<< "\t The Vertices of x[" << i - n << "] = " << x[i - n];
std::cout<< "\t The Vertices of v[" << i - n << "] = " << v[i - n];
std::cout<< "\t The Vertices of y[" << i - n << "] = " << y[i - n];
std::cout<< "\n";
}
std::cout<< "\n";
for(i = 1; i <= n; i++)
{
e1[i]=((u[i]*u[i])+(t[i]*t[i]));
e2[i]= ceil(float (e1[i])/2);
std::cout<< "\n Edges addition of e[" << i << "] = " << e1[i] << "\t";
std::cout<< "\t Edges of e[" << i << "] = " << e2[i]%2 << "\t";
}
for(i = 1; i <= n; i++)
{
e3[i]=((u[i]*u[i])+(s[i]*s[i]));
e4[i]= ceil( float (e3[i])/2);
std::cout<< "\n Edges addition of e[" << n + i << "] = " << e3[i] << "\t";
std::cout<< "\t Edges of e[" << n + i << "] = " << e4[i]%2 << "\t";
}
for(i = 1; i <= n; i++)
{
e5[i]=((v[i]*v[i])+(x[i]*x[i]));
e6[i]= ceil( float (e5[i])/2);
std::cout<< "\n Edges addition of e[" << 2 * n + i << "] = " << e5[i] << "\t";
std::cout<< "\t Edges of e[" << 2 * n + i << "] = " << e6[i]%2 << "\t";
}
for (i = 1; i <= n; i++)
{
```

```
e7[i]=((v[i]*v[i])+(y[i]*y[i]));
e8[i]= ceil( float (e7[i])/2);
std::cout<< "\n Edges addition of e[" << 3*n+i << "] = " << e7[i] << "\t";
std::cout<< "\t Edges of e[" << 3*n+i << "] = " << e8[i]%2 << "\t";
}
for (i = 1; i <= n-1; i++)
{
e9[i]=((u[i]*u[i])+(u[i+1]*u[i+1]));
e10[i]= ceil( float (e9[i])/2);
std::cout<< "\n Edges addition of e[" << 4*n+i << "] = " << e9[i] << "\t";
std::cout<< "\t Edges of e[" << 4*n+i << "] = " << e10[i]%2 << "\t";
}
for(i = 1; i <= n-1; i++)
{
e11[i]=((v[i]*v[i])+(v[i+1]*v[i+1]));
e12[i]= ceil( float (e11[i])/2);
std::cout<< "\n Edges addition of e[" << 5*n-1+i << "] = " << e11[i] << "\t";
std::cout<< "\t Edges of e[" << 5*n+i-1 << "] = " << e12[i]%2 << "\t";
}
e13[1]=((v[(n+1)/2]*v[(n+1)/2])+(u[(n+1)/2]*u[(n+1)/2]));
e14[1]= ceil( float (e13[1])/2);
std::cout<< "\n Middle Edges addition of e[" << 6*n-1 << "] = " << e13[1] << "\t";
std::cout<< "\t Edges of e[" << 6*n-1 << "] = " << e14[1]%2 << "\t";
return 0;
}
```

§4. Conclusion

In this Paper, we have introduced the new concept of modulo two square mean labeling of path, path related graphs and $H-$ class graphs. This new approach will be helpful to attack standard conjectures and unsolved open problems.

References

[1] J A Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 18(2015).

[2] V Ajitha, S Arumugam, and K. A. Germina, On square sum graphs, *AKCEJ Graphs and Combinatorics*, 6(1),(2009), 1 C 10.

[3] Germina K.A and Reena Sebastian, Further results on square sum graph, *International Mathematical Forum*, Vol. 8, 2013, No. 1, 47 C 57.

[4] Germina K.A and Reena Sebastian, On square sum graphs, *Proyecciones Journal of Mathematics* 32(2013), 107-117.

[5] J. Shiama, Square sum labeling for some middle and total graphs, *International Journal of Computer Applications*, 37(4), (January 2012).

[6] S. Somasundaram and R. Ponraj, Mean labelings of graphs, *Natl. Acad, Sci. Let.*, 26 (2003) 210-213.

[7] Sandhya S.S, Somasundaram S, Anusa S, Root square mean labeling of graphs, *International Journal of Contemporary Mathematical Sciences*, Vol.9, 2014, No.14, 667- 676.

International J.Math. Combin. Vol.1-2(2025), 90-92

A Note on the Ratio of two Gamma Functions

A. Bagdasaryan[1], J. López-Bonilla[2], R. Rajendra[3] and P. Siva Kota Reddy[4]

1. College of Engineering and Technology, American University of the Middle East, Egaila-54200, Kuwait
2. ESIME-Zacatenco, Instituto Politcnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738 CDMX, Mxico
3. Department of Mathematics, Field Marshal K. M. Cariappa College (A Constituent College of Mangalore University/Kodagu University), Madikeri-571 202, India
4. Department of Mathematics, Sri Jayachamarajendra College of Engineering, JSS Science and Technology University, Mysuru-570 006, India

E-mail: armen.bagdasaryan@aum.edu.kw, jlopezb@ipn.mx, rrajendrar@gmail.com, pskreddy@jssstuniv.in

Abstract: We consider the quotient of two gamma functions and for it we obtain a simpler expression than the formula deduced by Bagdasaryan [Appl. Maths. and Comput. 256 (2015)].

Key Words: Gamma function, quotient, Pochhammer-Barnes symbol.

AMS(2010): 33B15.

§1. Introduction

Bagdasaryan [1] showed the following result involving the Pochhammer symbol [2-4] and the gamma function [5-8]:

For $k, p, q \in \mathbb{N}, p > q$, we have

$$\frac{\Gamma\left(\frac{1+qk}{p}\right)}{\Gamma\left(\frac{1-(p-q)k}{p}\right)} = \frac{1-(p-q)k}{p} \sum_{k-1=m_2+\cdots+(k-1)m_k} \frac{(-1)^{k-1+m_2+\cdots+m_k}}{p^{k-1+m_2+\cdots+m_k}} \cdot \frac{(k-1+m_2+\cdots+m_k)!}{m_2!\cdots m_k!} \cdot \left[\frac{(q)_2-(q-p)_2}{2!}\right]^{m_2} \cdots \left[\frac{(q)_{k-1}-(q-p)_{k-1}}{(k-1)!}\right]^{m_{k-1}} \left[\frac{(q)_k-(q-p)_k}{k!}\right]^{m_k}, \tag{1}$$

where the sum runs over all partitions of $(k-1)$".

In this note, we obtain a simpler expression for such ratio of gamma functions.

§2. Ratio of Gamma Functions

In fact, we know [7] the property $\Gamma(z+1) = z\Gamma(z)$. Then, it is natural the following sequence

[1]Received October 27, 2024. Accepted May 18, 2025

of products

$$
\begin{aligned}
\Gamma\left(\frac{1+qk}{p}\right) &= \Gamma\left(\frac{1-p+qk}{p}+1\right) \\
&= \frac{1-p+qk}{p}\Gamma\left(\frac{1-p+qk}{p}\right) \\
&= \frac{1-p+qk}{p}\Gamma\left(\frac{1-2p+qk}{p}+1\right), \\
&= \frac{1-p+qk}{p}\frac{1-2p+qk}{p}\Gamma\left(\frac{1-2p+qk}{p}\right) \\
&= \frac{1-p+qk}{p}\cdots\frac{1-kp+qk}{p}\Gamma\left(\frac{1-kp+qk}{p}\right),
\end{aligned}
\tag{2}
$$

and therefore

$$
\begin{aligned}
\frac{\Gamma\left(\frac{1+qk}{p}\right)}{\Gamma\left(\frac{1-(p-q)k}{p}\right)} &= \frac{1}{p^k}(1+qk-p)(1+qk-2p)(1+qk-3p)\cdots(1+qk-kp) \\
&= \frac{1}{p^k}\prod_{r=1}^{k}(1+qk-rp) \\
&= \left[\frac{1-p+qk}{p}\right]_k \\
&= \sum_{m=1}^{k} S_k^{(m)}\left(\frac{1-p+qk}{p}\right)^m,
\end{aligned}
\tag{3}
$$

as an alternative to the expression (1), with the participation of the descending factorial function and the Stirling numbers of the first kind [7], [9]. If we observe the sequence (2) it is clear that (3) is valid for p and q arbitrary real numbers with $p \neq 0$, and $k = 1, 2, 3, \cdots$.

Besides, we have the relation [7] following

$$
\frac{\Gamma(\beta)}{\Gamma(\beta-k)} = (-1)^k(1-\beta)_k,
\tag{4}
$$

whose application for $\beta = \frac{1+qk}{p}$ implies the property

$$
\frac{\Gamma\left(\frac{1+qk)}{p}\right)}{\Gamma\left(\frac{1-(p-q)k}{p}\right)} = (-1)^k\left(\frac{p-1-qk}{p}\right)_k,
\tag{5}
$$

which is compatible with (3) because we have the general relation

$$
[x]_k = (-1)^k(-x)_k.
$$

Consequently, the equalities (3) and (5) are alternatives to (1).

References

[1] A. Bagdasaryan, A combinatorial identity involving gamma function and Pochhammer symbol, *Appl. Maths. and Comput.* 256 (2015) 125-130.

[2] L. Pochhammer, Über eine klasse von integralen mit geschlossenen integrationskurven, *Math. Ann.* 37 (1890) 500-511.

[3] E. W. Barnes, On functions defined by simple hypergeometric series, *Trans. Cambridge Phil. Soc.* 20 (1908) 253-279.

[4] E. W. Barnes, A new development of the theory of hypergeometric functions, *Proc. London Math. Soc.* 6 (1908) 141-177.

[5] P. J. Davis, Leonhard Euler's integral: A historical profile of the gamma function, *Amer. Math. Monthly* 66, No. 10 (1959) 849-869.

[6] G. K. Srinivasan, The gamma function: An eclectic tour, *Amer. Math. Monthly* 114, No. 4 (2007) 297315.

[7] H. M. Srivastava, J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier, London (2012).

[8] J. Bonnar, The gamma function, *Treasure Trove of Maths.* (2017).

[9] J. Quaintance, H. W. Gould, *Combinatorial identities for Stirling numbers*, World Scientific, Singapore (2016).

International J.Math. Combin. Vol.1-Vol.2(2025), 93-102

PMC-Labeling of Certain Tree Related Graphs and Prism of Wheel Graph

R. Ponraj[1], S. Prabhu[2] and M. Sivakumar[3]

1. Department of Mathematics,Sri Paramakalyani College, Alwarkurichi–627 412, India
2. Research Scholor, Reg.No:21121232091003, Sri Paramakalyani College, Affiliated to Manonmaniam Sundaranar University, Alwarkurichi–627 412, Tenkasi, Tamilnadu, India
3. Department of Mathematics, Government Arts and Science College, Tittagudi-606106, India

E-mail: ponrajmaths@gmail.com, selvaprabhu12@gmail.com, sivamaths1975@gmail.com

Abstract: The graph $G = (V, E)$ consists of p vertices and q edges. Let

$$\rho = \begin{cases} \frac{p}{2}, & p \text{ is even} \\ \frac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $\Gamma = \{\pm 1, \pm 2, \dots, \pm \rho\}$. Consider a function $\Lambda : V \to \Gamma$ that allocates unique labels from Γ to the various vertices of V when p is even and allocates a unique labels in Γ to $p-1$ vertices of V, repeating a label for the remaining one vertex when p is odd. Then, the labeling as mentioned above is called a pair mean cordial labeling (PMC-labeling) if for every edge uv of G, there is a labeling

$$\begin{cases} \dfrac{\Lambda(u)+\Lambda(v)}{2} & \text{if } \Lambda(\mathrm{u})+\Lambda(\mathrm{v}) \text{ is even,} \\ \dfrac{\Lambda(u)+\Lambda(v)+1}{2} & \text{if } \Lambda(\mathrm{u})+\Lambda(\mathrm{v}) \text{ is odd} \end{cases}$$

such that $|\bar{\mathbb{S}}_{\Lambda_1} - \bar{\mathbb{S}}_{\Lambda_1^c}| \le 1$, and a Smarandachely PMC-labeling if $|\bar{\mathbb{S}}_{\Lambda_1} - \bar{\mathbb{S}}_{\Lambda_1^c}| \ge 2$, where $\bar{\mathbb{S}}_{\Lambda_1}$ and $\bar{\mathbb{S}}_{\Lambda_1^c}$ are denoted the number of edges labelled with 1 and the number of edges not labelled with 1, respectively. A graph G that has a pair mean cordial labeling is called a pair mean cordial graph (PMC-graph). In this research paper, we prove the existences of the PMC-labeling of some tree related graphs like the X-tree, Y-tree, prism of wheel graph, subdivision of bistar graph and coconut tree.

Key Words: X-tree, Y-tree, prism of wheel graph, subdivision of bistar graph, coconut tree.

AMS(2010): 05C38, 05C76, 05C78.

§1. Introduction

All graphs considered in this paper are simple, finite and undirected. Let $G = (V(G), E(G))$ be a graph with $p = |V(G)|$ vertices and $q = |E(G)|$ edges where $V(G)$ and $E(G)$ denote the vertex

[1]Received January 4, 2025. Accepted May 20, 2025

set and edge set of a graph G. For all terminologies and notations of graph theory, we refer the book of Harary [4]. The idea of graceful labeling was introduced by Rosa in [12]. Further results on the radio number of trees were explored in [1]. Janani and Ramachandran [5] have worked on relatively prime edge labeling of graphs. Sunitha and Sheriba [14] have been investigated the Gaussian tribonacci R-graceful labeling of the path, comb, coconut tree, regular caterpillar graph, Bistar graph and subdivision of Bistar graph. Zeen [15] proved that the existence of edge δ-graceful labeling for some cyclic related graphs like wheel graph, prism graph, double wheel graph, prism of the wheel graph, gear graph, closed helm, butterfly graph, alternate triangular cycle and friendship graph. The concept of cordial labeling was introduced by Cahit in [2]. Some new families of 3-equitable prime cordial graphs were discussed in [13]. Product cordial graph in the context of some graph operations on gear graph have been investigated in [7]. Prajapati et al. [8] have studied the SD-prime cordial labeling of K_4-snake and related graphs. For a dynamic survey on graph labeling, we follow the book of Gallian [3]. Also we have introduced a PMC-labeling in [9] and the PMC-labeling behavior of more graphs like web graph, jewel graph, sun flower graph, flower graph, tadpole graph, dumbbell graph, umbrella graph, butterfly graph, jelly fish, triangular book graph, quadrilateral book graph, triangular snake, alternate triangular snake, quadrilateral snake and alternate quadrilateral snake have been investigated in [9-11]. In this paper, we investigate the PMC-labeling behavior of some tree related graphs like the X-tree, Y-tree, prism of wheel graph, subdivision of bistar graph and coconut tree.

§2. Preliminaries

We present a few fundamental definitions that are essential for the upcoming section.

Definition 2.1([14]) *The coconut tree $CT_{m,n}$ is a graph obtained by connecting the center vertex of $K_{1,n}$ with a pendant vertex of the path P_m.*

Definition 2.2([14]) *The bistar graph $B_{m,n}$ is obtained from K_2 by attaching m pendant edges to one end of K_2 and n pendant edges to the other end of K_2.*

Definition 2.3([14]) *The subdivision of bistar graph $S(B_{m,n})$ is obtained by subdividing each edges of a bistar graph $B_{m,n}$.*

Definition 2.4([5]) *The Y-tree Y_n is a tree of three paths with exactly three vertices of degree one, one vertex of degree three and other vertices of degree two.*

Definition 2.5([5]) *The X-tree X_n is a tree of four paths with exactly four vertices of degree one, one vertex of degree four and other vertices of degree two.*

Definition 2.6([15]) *For $n \geq 3$, let $\{u_0, u_1, u_2, \ldots, u_n\}$ be the vertices of the wheel graph W_n with hub vertex u_0 and $\{v_0, v_1, v_2, \ldots, v_n\}$ be the vertices of $W_n^{'}$ a copy of the wheel graph W_n with hub vertex v_0. The prism of the wheel graph W_n, PW_n is obtained by joining u_0 of W_n to the corresponding vertex v_0 of $W_n^{'}$ and each u_i of W_n to the corresponding vertex v_i of $W_n^{'}$ for all $i = 1, 2, \ldots, n$. ie., $PW_n = K_2 \times W_n$.*

§3. Main Results

Theorem 3.1 *The X-tree X_n is a PMC-graph for all n.*

Proof The vertex set and edge set of the X-tree X_n denoted by $V(X_n) = \{u_0, u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(X_n) = \{u_0u_1, u_0v_1, u_0x_1, u_0y_1, u_iu_{i+1}, v_iv_{i+1}, x_ix_{i+1}, y_iy_{i+1} \mid 1 \leq i \leq n-1\}$ respectively. Then, X_n has $4n$ edges and $4n+1$ vertices. Let $\Lambda(u_0) = 2$, $\Lambda(x_1) = -1$, $\Lambda(x_3) = -n-1$ and $\Lambda(y_n) = 1$. We have consider the two cases.

Case 1. n is odd.

First, assign the labels $-1, -2, \cdots, \frac{-n-1}{2}$ and $3, 4, \cdots, \frac{n+3}{2}$ respectively according to the vertices $u_1, u_3, \ldots, u_n$ and $u_2, u_4, \cdots, u_{n-1}$. Then, assign the labels $\frac{n+5}{2}, \frac{n+7}{2}, \ldots, n+2$ and $\frac{-n-3}{2}, \frac{-n-5}{2}, \ldots, -n$ to the vertices $v_1, v_3, \cdots, v_n$ and $v_2, v_4, \ldots, v_{n-1}$ respectively. Also, assign the labels $-n-2, -n-3, \cdots, \frac{-3n-1}{2}$ and $n+3, n+4, \ldots, \frac{3n+1}{2}$ corresponding to the vertices $x_2, x_4, \cdots, x_{n-1}$ and $x_5, x_7, \ldots, x_n$. Assign the labels $\frac{-3n-3}{2}, \frac{-3n-5}{2}, \cdots, -2n$ and $\frac{3n+3}{2}, \frac{3n+5}{2}, \cdots, 2n$ to the vertices $y_1, y_3, \cdots, y_n$ and $y_2, y_4, \cdots, y_{n-1}$ respectively.

Case 2. n is odd.

Assign the labels $-1, -2, \cdots, \frac{-n}{2}$ and $3, 4, \cdots, \frac{n+4}{2}$ according to the vertices $u_1, u_3, \cdots, u_{n-1}$ and $u_2, u_4, \cdots, u_n$. Then, assign the labels $\frac{-n-2}{2}, \frac{-n-4}{2}, \cdots, -n$ and $\frac{n+6}{2}, \frac{n+8}{2}, \cdots, n+2$ to the vertices $v_1, v_3, \cdots, v_{n-1}$ and $v_2, v_4, \cdots, v_n$ respectively. Consequently, assign the labels $-n-2, -n-3, \cdots, \frac{-3n-2}{2}$ and $n+3, n+4, \cdots, \frac{3n}{2}$ corresponding to the vertices $x_2, x_4, \cdots, x_n$ and $x_5, x_7, \cdots, x_{n-1}$. Finally, assign the labels $\frac{3n+2}{2}, \frac{3n+4}{2}, \cdots, 2n$ and $\frac{-3n-4}{2}, \frac{-3n-6}{2}, \cdots, -2n$ to the vertices $y_1, y_3, \cdots, y_{n-1}$ and $y_2, y_4, \cdots, y_{n-2}$ respectively. In both cases, we have $\bar{\mathbb{S}}_{\Lambda_1} = 2n = \bar{\mathbb{S}}_{\Lambda_1^c}$ and the proof is complete. $\square$

Example 3.2 A PMC-labeling of the X-tree X_4 is given in Figure 1.

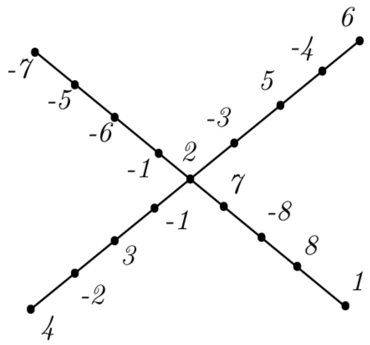


Figure 1

Theorem 3.3 *The Y-tree Y_n is a PMC-graph for all n.*

Proof The vertex set and edge set of the Y-tree Y_n are denoted by $V(Y_n) = \{u_0, u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(Y_n) = \{u_0u_1, u_0v_1, u_0x_1, u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1} \mid 1 \leq i \leq n-1\}$ respectively. Then, Y_n has $3n$ edges and $3n+1$ vertices. Let $\Lambda(u_0) = 2$ and $\Lambda(w_n) = 1$. We have consider the two cases.

Case 1. n is odd.

First, assign the labels to the vertices $u_i, 1 \leq i \leq n$ as in case (i) of Theorem 3.1. Then, assign the labels $\frac{-n-3}{2}, \frac{-n-5}{2}, \cdots, -n-1$ and $\frac{n+5}{2}, \frac{n+7}{2}, \cdots, n+1$ to the vertices $v_1, v_3, \cdots, v_n$ and $v_2, v_4, \cdots, v_{n-1}$ respectively. Subsequently, assign the labels $-n-2, -n-3, \cdots, \frac{-3n-1}{2}$ and $n+2, n+3, \cdots, \frac{3n+1}{2}$ corresponding to the vertices $w_1, w_3, \cdots, w_{n-2}$ and $w_2, w_4, \cdots, w_{n-1}$. Hence $\bar{\mathbb{S}}_{\Lambda_1} = \frac{3n-1}{2}$ and $\bar{\mathbb{S}}_{\Lambda_1^c} = \frac{3n+1}{2}$.

Case 2. n is even.

Next, assign the labels to the vertices $u_i, 1 \leq i \leq n$ as in Case 1 of Theorem 3.1. So, assign the labels $\frac{-n-2}{2}, \frac{-n-4}{2}, \cdots, -n$ and $\frac{n+4}{2}, \frac{n+6}{2}, \cdots, n+1$ to the vertices $v_1, v_3, \cdots, v_{n-1}$ and $v_2, v_4, \cdots, v_n$ respectively. Label the vertex w_1 by $-n-1$. Consequently, assign the labels $-n-2, -n-3, \cdots, \frac{-3n}{2}$ and $n+2, n+3, \cdots, \frac{3n}{2}$ corresponding to the vertices $w_2, w_4, \cdots, w_{n-2}$ and $w_3, w_5, \cdots, w_{n-1}$. Thus, $\bar{\mathbb{S}}_{\Lambda_1} = \frac{3n}{2} = \bar{\mathbb{S}}_{\Lambda_1^c}$. $\square$

Example 3.4 A PMC-labeling of the Y-tree Y_5 is given in Figure 2.

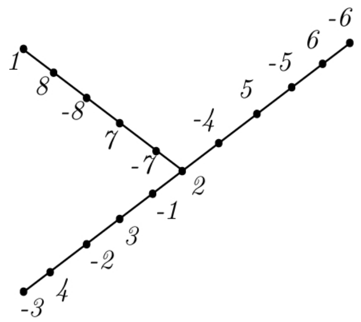


Figure 2

Theorem 3.5 *The prism of wheel graph PW_n is not PMC-graph for all $n \geq 3$.*

Proof Let us consider the prism of wheel graph PW_n, $n \geq 3$. Let $V(PW_n) = \{u_0, v_0, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(PW_n) = \{u_0v_0, u_0u_i, u_0v_i, u_iv_i \mid 1 \leq i \leq n\} \cup \{u_iu_{i+1}, u_nu_1, v_iv_{i+1}, v_nv_1 \mid 1 \leq i \leq n-1\}$ denote, respectively, the vertex set and edge set of the prism of wheel graph PW_n. Then, PW_n has $5n+1$ edges and $2n+2$ vertices. Suppose that the prism of wheel graph PW_n is a PMC-graph. We have the maximum possible number of edges designated with a label 1 is $2n-1$. Consequently, the minimum number of edges that are not designated with a label 1 is $3n+2$. Therefore, $\bar{\mathbb{S}}_{\Lambda_1^c} - \bar{\mathbb{S}}_{\Lambda_1} \geq n+3 \geq 6 > 1$, a contradiction arises. $\square$

Theorem 3.6 *The subdivision $S(B_{m,n})$ of bistar graph $B_{m,n}$ is a PMC-graph for all m and n.*

Proof Let $V(S(B_{m,n})) = \{u_0, u_i, x_i, x_0, v_0, v_j, y_j \mid 1 \leq i \leq m \,\text{and}\, 1 \leq j \leq n\}$ and $E(S(B_{m,n})) = \{u_0x_i, x_iu_i, u_0x_0, x_0v_0, v_0y_j, y_jv_j \mid 1 \leq i \leq m \,\text{and}\, 1 \leq j \leq n\}$ denote, respectively, the vertex set and edge set of the subdivision of bistar graph $S(B_{m,n})$. Then, $S(B_{m,n})$ has

$2m+2n+2$ edges and $2m+2n+3$ vertices. Let $\Lambda(u_0)=2, \Lambda(v_0)=-m-n-1$ and $\Lambda(x_0)=1$. Next we assign the labels $2,3,\ldots,m+1$ and $-1,-2,\cdots,-m$ to the vertices $u_1,u_2,\cdots,u_m$ and $x_1,x_2,\cdots,x_m$ respectively. Consequently, assign the labels $-m-1,-m-2,\cdots,-m-n$ and $m+2,m+3,\cdots,m+n+1$ corresponding to the vertices $v_1,v_2,\cdots,v_n$ and $y_1,y_2,\cdots,y_n$. Thus, $\bar{\mathbb{S}}_{\Lambda_1}=m+n+1=\bar{\mathbb{S}}_{\Lambda_1^c}$. □

Example 3.7 A PMC-labeling of the subdivision of bistar graph $S(B_{3,4})$ is given in Figure 3.

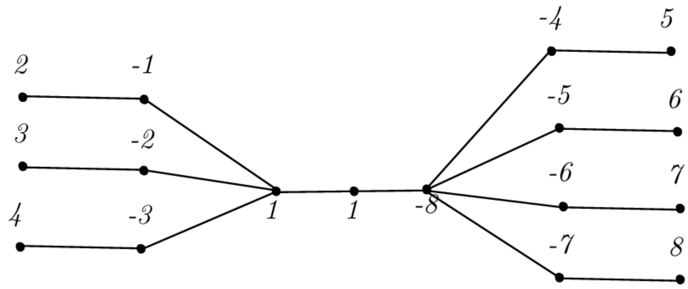


Figure 3

Theorem 3.8 *The coconut tree $CT(m,n)$ is a PMC-graph for every m,n with $|m-n|\le 3$.*

Proof Let $V(CT(m,n))=\{u_i,v_j \mid 1\le i\le m, 1\le j\le n\}$ and $E(CT(m,n))=\{u_iu_{i+1},u_nv_j : 1\le i\le m-1, 1\le j\le n\}$. Clearly, the coconut tree $CT(m,n)$ has $m+n-1$ edges and $m+n$ vertices.

Case 1. $|m-n|=0$.

Then, $m=n$. We have to show that $CT(m,n)$ is a PMC-graph.

Subcase 1.1 m is odd.

Let us assign the labels $2,3,\cdots,\frac{m+3}{2}$ and $-1,-2,\cdots,\frac{-m+1}{2}$ according to the vertices $u_1,u_3,\cdots,u_m$ and $u_2,u_4,\cdots,u_{m-1}$. Next, assign labels $\frac{-m-1}{2},\frac{-m-3}{2},\cdots,-m$ and $\frac{m+5}{2},\frac{m+7}{2},\cdots,m$ corresponding to the vertices $v_1,v_2,\cdots,v_{\frac{m+1}{2}}$ and $v_{\frac{m+3}{2}},v_{\frac{m+5}{2}},\cdots,v_{m-1}$. Label the vertex v_m by 1.

Subcase 1.2 m is even.

If $m=2$, define $\Lambda(u_1)=2$, $\Lambda(u_2)=-1$, $\Lambda(v_1)=-2$ and $\Lambda(v_2)=1$. Therefore, $\bar{\mathbb{S}}_{\Lambda_1}=1$ and $\bar{\mathbb{S}}_{\Lambda_1^c}=2$. If $m>2$, we assign the labels $2,3,\cdots,\frac{m+2}{2}$ and $-1,-2,\cdots,\frac{-m}{2}$ to the vertices $u_1,u_3,\cdots,u_{m-1}$ and $u_2,u_4,\cdots,u_m$ respectively. Further, assign the labels $\frac{-m-2}{2},\frac{-m-4}{2},\cdots,-m$ and $\frac{m+4}{2},\frac{m+6}{2},\cdots,m$ corresponding to the vertices $v_1,v_2,\cdots,v_{\frac{m}{2}}$ and $v_{\frac{m+2}{2}},v_{\frac{m+4}{2}},\cdots,v_{m-1}$. Label the vertex v_m by 1. In each cases, $\bar{\mathbb{S}}_{\Lambda_1}=m$ and $\bar{\mathbb{S}}_{\Lambda_1^c}=m-1$.

Case 2. $|m-n|=1$.

Then, $m-n=1$ or $m-n=-1$.

Subcase 2.1 $m-n=1$.

Then we have to prove that $CT(m,m-1)$ is a PMC-graph. Define $\Lambda(u_1)=1$. If m is odd, we assign the labels $2,3,\cdots,\frac{m+1}{2}$ and $-1,-2,\cdots,\frac{-m+1}{2}$ to the vertices $u_2,u_4,\cdots,u_{m-1}$ and $u_3,u_5,\cdots,u_m$ respectively. Now, assign the labels $\frac{-m-1}{2},\frac{-m-3}{2},\cdots,-m+1$ and $\frac{m+3}{2},\frac{m+5}{2},\cdots,$ $m-1$ corresponding to the vertices $v_1,v_2,\cdots,v_{\frac{m-1}{2}}$ and $v_{\frac{m+1}{2}},v_{\frac{m+3}{2}},\cdots,v_{m-2}$. Then label the vertex v_{m-1} by 2. If m is even, let us assign the labels $2,3,\cdots,\frac{m+2}{2}$ and $-1,-2,\cdots,\frac{-m+2}{2}$ to the vertices $u_2,u_4,\cdots,u_m$ and $u_3,u_5,\cdots,u_{m-1}$ respectively. Also, we assign the labels $\frac{-m}{2},\frac{-m-2}{2},\cdots,-m+1$ and $\frac{m+4}{2},\frac{m+6}{2},\cdots,m-1$ corresponding to the vertices $v_1,v_2,\cdots,v_{\frac{m}{2}}$ and $v_{\frac{m+2}{2}},v_{\frac{m+4}{2}},\cdots,v_{m-2}$. Finally label the vertex v_{m-1} by 2. Hence $\bar{\mathbb{S}}_{\Lambda_1}=m-1$ and $\bar{\mathbb{S}}_{\Lambda_1^c}=m-1$.

Subcase 2.2 $m-n=-1$.

We have to show that $CT(m,m+1)$ is a PMC-graph. If m is odd, assign the labels to the vertices u_i, $1\le i\le m$ as in Subcase 2.1 of Case 2. Next, we assign the labels $\frac{-m-1}{2},\frac{-m-3}{2},\cdots,-m$ and $\frac{m+3}{2},\frac{m+5}{2},\cdots,m$ to the vertices $v_1,v_2,\cdots,v_{\frac{m+1}{2}}$ and $v_{\frac{m+3}{2}},v_{\frac{m+5}{2}},$ $\cdots,v_m$ respectively. Then, label the vertex v_{m+1} by $\frac{m+1}{2}$. If m is even, then assign the labels to the vertices u_i, $1\le i\le m$ as in Subcase 2.1 of Case 2. Also we assign the labels $\frac{-m}{2},\frac{-m-2}{2},\cdots,-m$ and $\frac{m+4}{2},\frac{m+6}{2},\cdots,m$ to the vertices $v_1,v_2,\cdots,v_{\frac{m+2}{2}}$ and $v_{\frac{m+4}{2}},v_{\frac{m+6}{2}},\cdots,$ v_m correspondingly. Finally, label the vertex v_{m+1} by $\frac{-m}{2}$. Consequently, $\bar{\mathbb{S}}_{\Lambda_1}=m=\bar{\mathbb{S}}_{\Lambda_1^c}$.

Case 3. $|m-n|=2$.

Then, $m-n=2$ or $m-n=-2$.

Subcase 3.1 $m-n=2$.

Now we have to prove that $CT(m,m-2)$ is a PMC-graph. Assign the labels to the vertices u_i,v_j, $1\le i\le m, 1\le j\le m-2$ as in subcase (i) of case (ii). Hence $\bar{\mathbb{S}}_{\Lambda_1}=m-1$ and $\bar{\mathbb{S}}_{\Lambda_1^c}=m-2$.

Subcase 3.2 $m-n=-2$.

We have to show that $CT(m,m+2)$ is a PMC-graph. If m is odd, assign the labels $2,3,\cdots,\frac{m+3}{2}$ and $-1,-2,\cdots,\frac{-m+1}{2}$ according to the vertices $u_1,u_3,\cdots,u_m$ and $u_2,u_4,\cdots,u_{m-1}$. Next, assign the labels $\frac{-m-1}{2},\frac{-m-3}{2},\cdots,-m-1$ and $\frac{m+5}{2},\frac{m+7}{2},\cdots,m+1$ to the vertices $v_1,v_2,\cdots,v_{\frac{m+3}{2}}$ and $v_{\frac{m+5}{2}},v_{\frac{m+7}{2}},\ldots,v_{m+1}$ respectively. Then label the vertex v_{m+2} by 1. If m is even, we assign the labels $2,3,\cdots,\frac{m+2}{2}$ and $-1,-2,\cdots,\frac{-m}{2}$ corresponding to vertices $u_1,u_3,\cdots,u_{m-1}$ and $u_2,u_4,\cdots,u_m$. Moreover, we assign the labels $\frac{-m-2}{2},\frac{-m-4}{2},\cdots,-m-1$ and $\frac{m+4}{2},\frac{m+6}{2},\cdots,m+1$ according to the vertices $v_1,v_2,\cdots,v_{\frac{m+2}{2}}$ and $v_{\frac{m+4}{2}},v_{\frac{m+6}{2}},\cdots,v_{m+1}$. Finally label the vertex v_{m+2} by 1. Therefore, $\bar{\mathbb{S}}_{\Lambda_1}=m$ and $\bar{\mathbb{S}}_{\Lambda_1^c}=m+1$.

Case 4. $|m-n|=3$.

Then, $m-n=3$ or $m-n=-3$.

Subcase 4.1 $m-n=3$.

We have to show that $CT(m,m+3)$ is a PMC-graph. Define $\Lambda(u_1)=-m+2$ and $\Lambda(u_2)=1$. If m is odd, we assign the labels $2,3,\ldots,\frac{m+1}{2}$ and $-1,-2,\cdots,\frac{-m+3}{2}$ according to the vertices $u_3,u_5,\cdots,u_m$ and $u_4,u_6,\cdots,u_{m-1}$. Next, we assign the labels $\frac{-m+1}{2},\frac{-m-1}{2},\cdots,-m+3$

and $v_{\frac{m-1}{2}}, v_{\frac{m+1}{2}}, \cdots, v_{m-4}$ to the vertices $v_1, v_2, \ldots, v_{\frac{m-3}{2}}$ and $\frac{m+3}{2}, \frac{m+5}{2}, \cdots, m-2$ respectively. Then label the vertex v_{m-3} by 1. If m is even, then assign the labels $2, 3, \ldots, \frac{m}{2}$ and $-1, -2, \cdots, \frac{-m+2}{2}$ corresponding to the vertices $u_3, u_5, \cdots, u_{m-1}$ and $u_4, u_6, \cdots, u_m$. Further, we assign the labels $\frac{-m}{2}, \frac{-m-2}{2}, \cdots, -m+3$ and $\frac{m+2}{2}, \frac{m+4}{2}, \cdots, m-3$ according to the vertices $v_1, v_2, \cdots, v_{\frac{m-4}{2}}$ and $v_{\frac{m-2}{2}}, v_{\frac{m}{2}}, \cdots, v_{m-4}$. Finally, label the vertex v_{m-3} by 1. Subsequently, $\bar{\mathbb{S}}_{\Lambda_1} = \bar{\mathbb{S}}_{\Lambda_1^c} = m-2$.

Subcase 4.2 $m-n=-3$.

We have to show that $CT(m, m+3)$ is a PMC-graph. Now assign the labels to the vertices u_i, v_j, $1 \le i \le m$, $1 \le j \le m+2$ as in Subcase 3.1 of Case 3. If m is odd, then label the vertex v_{m+3} by $\frac{-m-1}{2}$. If m is even, label the vertex v_{m+3} by $\frac{m+2}{2}$. Hence $\bar{\mathbb{S}}_{\Lambda_1} = \bar{\mathbb{S}}_{\Lambda_1^c} = m+1$. □

Theorem 3.9 *The coconut tree $CT(m,n)$ is not PMC-graph for every m, n with $n-m \ge 4$.*

Proof If possible, let $CT(m,n)$ is a PMC-graph. If the edge uv receives the label 1, the possible results are either $\Lambda(u)+\Lambda(v)=1$ or $\Lambda(u)+\Lambda(v)=2$.

Case 1. $n-m$ is odd.

Then, the maximum possible number of edges designated with a label 1 is $m+1$. Subsequently, the minimum number of edges that are not designated with a label 1 is $q-(m+1) = n-2$. Therefore, $\bar{\mathbb{S}}_{\Lambda_1^c} - \bar{\mathbb{S}}_{\Lambda_1} \ge n-2-(m+1) = n-m-3 \ge 2 > 1$, we get a contradiction.

Case 2. $n-m$ is even.

Then, the maximum possible number of edges designated with a label 1 is m. Consequently, the minimum number of edges that are not designated with a label 1 is $q-m=n-1$. Thus, $\bar{\mathbb{S}}_{\Lambda_1^c} - \bar{\mathbb{S}}_{\Lambda_1} \ge n-1-m = n-m-1 \ge 3 > 1$, a contradiction arises. □

Theorem 3.10 *The coconut tree $CT(m,n)$ is a PMC-graph for every m, n with $m-n \ge 4$.*

Proof Clearly, the coconut tree $CT(m,n)$ has $m+n-1$ edges and $m+n$ vertices.

Case 1. $m \equiv 0 \pmod 4$.

Subcase 1.1 $n \equiv 0 \pmod 4$.

Let us assign the labels $2, 3, \cdots, \frac{m+n+4}{4}$ and $-1, -2, \cdots, \frac{-m-n}{4}$ according to the vertices $u_1, u_3, \cdots, u_{\frac{m+n-2}{2}}$ and $u_2, u_4, \cdots, u_{\frac{m+n}{2}}$. Label the vertex $u_{\frac{m+n+2}{2}}$ by $\frac{-m-n-4}{4}$. Also we assign the labels $\frac{-m-n-8}{4}, \frac{m+n+8}{4}$ and $\frac{-m-n-12}{4}, \frac{m+n+12}{4}$ to the vertices $u_{\frac{m+n+4}{2}}, u_{\frac{m+n+6}{2}}$ and $u_{\frac{m+n+8}{2}}, u_{\frac{m+n+10}{2}}$ respectively. This process should be repeated until the label 1 is assigned to u_m. Subsequently, assign the labels $\frac{-m-2}{2}, \frac{m+2}{2}$ and $\frac{-m-4}{2}, \frac{m+4}{2}$ corresponding to the vertices v_1, v_2 and v_3, v_4. This process should be repeated until the labels $\frac{-m-n}{2}, \frac{m+n}{2}$ are assigned to v_{n-1}, v_n.

Subcase 1.2 $n \equiv 1 \pmod 4$.

Assign the labels $2, 3, \cdots, \frac{m+n+7}{4}$ and $-1, -2, \cdots, \frac{-m-n+1}{4}$ to vertices $u_1, u_3, \cdots, u_{\frac{m+n+1}{2}}$ and $u_2, u_4, \cdots, u_{\frac{m+n-1}{2}}$ respectively. So assign the labels $\frac{-m-n-7}{4}, \frac{-m-n-3}{4}$ according to the vertices $u_{\frac{m+n+3}{2}}, u_{\frac{m+n+5}{2}}$. Assign the labels $\frac{-m-n-11}{4}, \frac{m+n+11}{4}$ and $\frac{-m-n-15}{4}, \frac{m+n+15}{4}$ to the vertices $u_{\frac{m+n+7}{2}}, u_{\frac{m+n+9}{2}}$ and $u_{\frac{m+n+11}{2}}, u_{\frac{m+n+13}{2}}$ respectively. This process should be repeated until the label 1 is assigned to u_m. Subsequently, assign the labels $\frac{-m-2}{2}, \frac{m+2}{2}$ and $\frac{-m-4}{2}, \frac{m+4}{2}$ according to the vertices v_1, v_2 and v_3, v_4 respectively. This process should be repeated until the labels $\frac{-m-n+1}{2}, \frac{m+n-1}{2}$ are assigned to v_{n-2}, v_{n-1}. Finally, label the vertex v_n by $\frac{-m-n+1}{2}$.

Subcase 1.3 $n \equiv 2 \pmod 4$.

Assign the labels $2, 3, \cdots, \frac{m+n+6}{4}$ and $-1, -2, \cdots, \frac{-m-n+2}{4}$ to the vertices $u_1, u_3, \cdots, u_{\frac{m+n}{2}}$ and $u_2, u_4, \cdots, u_{\frac{m+n-2}{2}}$ respectively. Then, assign the labels $\frac{-m-n-6}{4}, \frac{-m-n-2}{4}$ according to the vertices $u_{\frac{m+n+2}{2}}, u_{\frac{m+n+4}{2}}$. Next, assign the labels $\frac{-m-n-10}{4}, \frac{m+n+10}{4}$ and $\frac{-m-n-14}{4}, \frac{m+n+14}{4}$ to the vertices $u_{\frac{m+n+6}{2}}, u_{\frac{m+n+8}{2}}$ and $u_{\frac{m+n+10}{2}}, u_{\frac{m+n+12}{2}}$ respectively. This process should be repeated until the label 1 is assigned to u_m. Subsequently, assign the labels to the vertices v_j, $1 \leq j \leq n$ as in Subcase 1.1 of Case 1.

Subcase 1.4 $n \equiv 3 \pmod 4$.

Also assign the labels $2, 3, \cdots, \frac{m+n+5}{4}$ and $-1, -2, \cdots, \frac{-m-n-1}{4}$ according to the vertices $u_1, u_3, \cdots, u_{\frac{m+n-1}{2}}$ and $u_2, u_4, \cdots, u_{\frac{m+n+1}{2}}$. Label the vertex $u_{\frac{m+n+3}{2}}$ by $\frac{-m-n-5}{4}$. Next, assign the labels $\frac{-m-n-9}{4}, \frac{m+n+9}{4}$ and $\frac{-m-n-11}{4}, \frac{m+n+11}{4}$ according to the vertices $u_{\frac{m+n+5}{2}}, u_{\frac{m+n+7}{2}}$ and $u_{\frac{m+n+9}{2}}, u_{\frac{m+n+11}{2}}$. This process should be repeated until the label 1 is assigned to u_m. Consequently, assign the labels to the vertices v_j, $1 \leq j \leq n$ as in Subcase 1.2 of Case 1.

Case 2 $m \equiv 1 \pmod 4$.

Subcase 2.1 $n \equiv 0 \pmod 4$.

Now, assign the labels to the vertices u_i, $1 \leq i \leq m-2$ as in Subcase 1.2 of Case 1. Then, assign the labels $\frac{-m-1}{2}, 1$ according to the vertices u_{m-1}, u_m. Label the vertex v_1 by $\frac{m+1}{2}$. Assign the labels $\frac{-m-3}{2}, \frac{m+3}{2}$ and $\frac{-m-5}{2}, \frac{m+5}{2}$ to the vertices v_2, v_3 and v_4, v_5 respectively. This process should be repeated until the labels $\frac{-m-n+1}{2}, \frac{m+n-1}{2}$ are assigned to v_{n-2}, v_{n-1}. Finally, label the vertex v_n by $\frac{-m-n+1}{2}$.

Subcase 2.2 $n \equiv 1 \pmod 4$.

In this case, assign the labels to the vertices u_i, $1 \leq i \leq m-2$ as in Subcase 1.3 of Case 1. Then, assign the labels $\frac{-m-1}{2}, 1$ according to the vertices u_{m-1}, u_m. Label the vertex v_1 by $\frac{m+1}{2}$. Also, assign the labels $\frac{-m-3}{2}, \frac{m+3}{2}$ and $\frac{-m-5}{2}, \frac{m+5}{2}$ corresponding to the vertices v_2, v_3 and v_4, v_5. This process should be repeated until the labels $\frac{-m-n}{2}, \frac{m+n}{2}$ are assigned to v_{n-1}, v_n.

Subcase 2.3 $n \equiv 2 \pmod 4$.

Assign the labels to the vertices u_i, $1 \leq i \leq m-2$ as in Subcase 1.4 of Case 1. Next, assign the labels $\frac{-m-1}{2}, 1$ according to the vertices u_{m-1}, u_m. Consequently, assign the labels to the vertices v_j, $1 \leq j \leq n$ as in Subcase 2.1 of Case 2.

Subcase 2.4 $n \equiv 3 \pmod 4$.

Subsequently, assign the labels to the vertices u_i, $1 \leq i \leq m-2$ as in Subcase 1.1 of Case

1. Also, assign the labels $\frac{-m-1}{2}, 1$ corresponding to the vertices u_{m-1}, u_m. Assign the labels to the vertices v_j, $1 \leq j \leq n$ as in Subcase 2.2 of Case 2.

Case 3. $m \equiv 2 \pmod 4$.

Subcase 3.1 $n \equiv 0 \pmod 4$.

This proof is consistent with Subcase 1.3 of Case 1.

Subcase 3.2 $n \equiv 1 \pmod 4$.

This proof is consistent with Subcase 1.4 of Case 1.

Subcase 3.3 $n \equiv 2 \pmod 4$.

This proof is consistent with Subcase 1.1 of Case 1.

Subcase 3.4 $n \equiv 3 \pmod 4$.

This proof is consistent with Subcase 1.2 of Case 1.

Case 4. $m \equiv 3 \pmod 4$.

Subcase 4.1 $n \equiv 0 \pmod 4$.

This proof is consistent with Subcase 2.3 of Case 2.

Subcase 4.2 $n \equiv 1 \pmod 4$.

This proof is consistent with Subcase 2.4 of Case 2.

Subcase 4.3 : $n \equiv 2 \pmod 4$.

This proof is consistent with Subcase 2.1 of Case 2.

Subcase 4.4 $n \equiv 3 \pmod 4$.

This proof is consistent with Subcase 2.2 of Case 2. □

Example 3.11 A PMC-labeling of the coconut tree $CT(8, 4)$ is given in Figure 4.

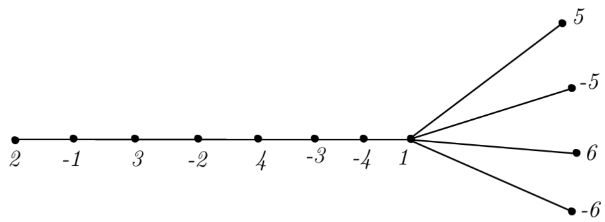


Figure 4

§4. Conclusion

The PMC-labeling behavior of some tree related graphs like the X-tree, Y-tree, prism of wheel graph, subdivision of bistar graph and coconut tree have been investigated in this paper. It is still available to future work to establish the PMC-labeling for more graph families.

References

[1] D. Bantva, Further results on the radio number of trees, *Elect. Notes Discr. Math.*, 63 (2017), 85-91.

[2] I. Cahit, Cordial graphs: a weaker versionof graceful and harmonious graphs, *Ars combin.*, 23, 201–207, (1987).

[3] J. A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 27, 1–712, (2024).

[4] F. Harary, *Graph Theory*, Addison Wesely, New Delhi, (1972).

[5] R. Janani and T. Ramachandran, On relatively prime edge labeling of graphs, *Engineering Letters*, 30(2), 2022, 1–7.

[6] V. J. Kaneria, J. R. Teraiya, and K. M. Patadiya, Some result on balanced cordial graphs, *Int. J. Math. Appl.*, 4 (2-A), (2016) 85-87.

[7] U. M. Prajapati and K. K. Raval, Product cordial graph in the context of some graph operations on gear graph, *Open J. Discrete Math.*, 6 (2016), 259-267.

[8] U. M. Prajapati and A. V. Vantiya, SD-prime cordial labeling of K_4-snake and related graphs, *Journal of Xidian University*, 14(4), 543–551, (2020).

[9] R. Ponraj and S. Prabhu, Pair mean cordial labeling of graphs, *Journal of Algorithms and Computation*, 54(1), 1–10, (2022).

[10] R. Ponraj and S. Prabhu, Pair mean cordiality of some snake graphs, *Global Journal of Pure and Applied Mathematics*, 18(1), 283–295, (2022).

[11] R. Ponraj and S. Prabhu, On pair mean cordial graphs, *J. Appl. & Pure Math.*, 5(3-4), 237–253, (2023).

[12] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs* (Internat. Symposium, Rome, July 1966) Gordon and Breach, N. Y. and Dunod Paris, 349–355, (1967).

[13] A. Sugumaran and P. Vishnu Prakash, Some new families of 3-equitable prime cordial graphs, *Internat. J. Stat. Appl. Math.*, 3(1) (2018), 45-49.

[14] K. Sunitha and M. Sheriba, Gaussian Tribonacci R-Graceful Labeling of some tree related graphs, *Ratio Mathematica*, Volume 44, 2022, 188–196.

[15] M. R. Zeen El Deen, Edge δ-graceful labeling for some cyclic related graphs, *Hindawi Advances in Mathematical Physics*, Article ID 6273245, 2020, 1–18.

International J.Math. Combin. Vol.1-Vol.2(2025), 103-106

A Short Note on an Identity of Spivey for Bell Numbers

T. Kim[1], J. López-Bonilla[2], R. Rajendra[3], P. Siva Kota Reddy[4] and M. Pavithra[5]

1. Department of Mathematics, College of Natural Science, Kwangwoon University, Seoul 139-704, Korea
2. ESIME-Zacatenco, Instituto Politcnico Nacional, Edif. 4, 1er. Piso, Col. Lindavista CP 07738 CDMX, Mxico
3. Department of Mathematics, Field Marshal K. M. Cariappa College (A constituent college of Mangalore University/Kodagu University), Madikeri-571 202, India
4. Department of Mathematics, Jayachamarajendra College of Engineering, JSS Science and Technology University, Mysuru-570 006, India
5. Department of Mathematics, Karnataka State Open University, Mysuru-570 006, India

E-mail: tkkim@kw.ac.kr, jlopezb@ipn.mx, rrajendrar@gmail.com, pskreddy@jssstuniv.in, sampavi08@gmail.com

Abstract: Spivey obtained an identity for Bell numbers, here we give an elementary proof of it and we show that it gives a recurrence relation for $\sum_{j=0}^{n} j^m S_n^{[j]}$, which shows that these quantities involving the Stirling numbers of the second kind are linear combination of the $B(k)$.

Key Words: Spivey's identity, Bell numbers, Dobinski's formula, Stirling numbers.

AMS(2010): 1B73.

§1. Introduction

Spivey [1]-[4] gave a combinatorial proof of the identity following

$$B(m+n) = \sum_{j=0}^{m} \sum_{k=0}^{n} j^{n-k} \binom{n}{k} S_m^{[j]} B(k), \tag{1}$$

for the Bell numbers [5],[7] and

$$B(n) \equiv \sum_{q=0}^{n} S_n^{[q]}, \tag{2}$$

where $S_n^{[q]}$ is a Stirling number of the second kind [6]-[12].

On the other hand, we know the Dobinski's formula [6], [7], [13]-[15] following

$$\sum_{q=0}^{n} S_n^{[q]} x^q = e^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!} x^k, \quad n \geq 0, \tag{3}$$

[1]Received November 30, 2024. Accepted May 24, 2025

which for $x = 1$ implies the expression

$$B(n) = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}. \tag{4}$$

Spivey [1] comments that the quantities

$$A_r := \sum_{j=0}^{m} j^r S_m^{[j]}, \quad r \geq 0, \tag{5}$$

for a given $m = 0, 1, \ldots$, can be expressed as a linear combination of Bell numbers. In Sec. 2 we use (4) to give an elementary proof of (1), and we deduce a recurrence relation for (5), in harmony with this affirmation of Spivey.

§2. Spivey's Identity

Here we exhibit a simple demonstration of (1). First, we perform the following calculation

$$\sum_{k=j}^{\infty}\frac{k^n}{(k-j)!} = \sum_{q=0}^{\infty}\frac{(q+j)^n}{q!} = \sum_{r=0}^{n}\binom{n}{r}j^{n-r}\sum_{q=0}^{\infty}\frac{q^r}{q!} \overset{(4)}{=} e\sum_{r=0}^{n}\binom{n}{r}j^{n-r}B(r). \tag{6}$$

Besides, let's remember the property ([6], [16])

$$k^m = \sum_{j=0}^{k}\binom{k}{j}j!S_m^{[j]} \tag{7}$$

Then

$$\begin{aligned} B(m+n) &\overset{(4)}{=} \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}k^m \\ &\overset{(7)}{=} \frac{1}{e}\sum_{j=0}^{m} j!S_m^{[j]}\sum_{k=j}^{\infty}\binom{k}{j}\frac{k^n}{k!} \\ &= \frac{1}{e}\sum_{j=0}^{m}S_m^{[j]}\sum_{k=j}^{\infty}\frac{k^n}{(k-j)!} \\ &\overset{(6)}{=} \sum_{j=0}^{m}\sum_{r=0}^{n} j^{n-r}\binom{n}{r}S_m^{[j]}B(r), \end{aligned} \tag{6}$$

in according with (1). □

From (1),

$$B(m+n) = \sum_{r=0}^{n}\binom{n}{r}B(n-r)\sum_{j=0}^{m}j^r S_m^{[j]} \overset{(5)}{=} \sum_{r=0}^{n}\binom{n}{r}B(n-r)A_r, \tag{8}$$

then in this recurrence relation for the quantities (5) we can employ $n = 0, 1, 2, \cdots$ to obtain each A_r as a linear combination of Bell numbers. In fact, for a given integer m,

$$\begin{aligned} A_0 &= B(m), \\ A_1 &= B(m+1) - B(m), \\ A_2 &= B(m+2) - 2B(m+1), \\ A_3 &= B(m+3) - 3B(m+2) + B(m), \text{and so on} \end{aligned} \tag{9}$$

Now, the Euler's operator $\left(x\frac{d}{dx}\right)^m$ ([6],[15],[16],[17]) can be applied to (3) to deduce the following explicit formula for (5),

$$A_r = \sum_{k=0}^{m} \binom{m}{k} \sum_{j=0}^{r} j! S_r^{[j]} S_{m-k}^{[j]} B(k), \tag{10}$$

which is compatible with the values (9); we can consider that (10) is the inversion of (8). The combination of (9) and (10) implies interesting identities. For example,

$$\begin{aligned} \sum_{k=0}^{n-1} 2^{n-k} \binom{n}{k} B(k) &= B(n+2) - B(n+1) - B(n), \\ \sum_{k=0}^{n-1} 3^{n-k} \binom{n}{k} B(k) &= B(n+3) - 3B(n+2) + 2B(n+1) - B(n). \end{aligned} \tag{11}$$

Spivey [18] obtained the following property

$$\sum_{k=0}^{n} (-1)^k k^m S_n^{(k)} = \sum_{j=0}^{m} (-1)^j j! S_m^{[j]} S_{n+1}^{(j+1)}, \tag{12}$$

which can be seen as companion of (10), and for $m = 1$ gives the known relation for the harmonic numbers ([6], [19], [20])

$$H_n = \frac{(-1)^n}{n!} \sum_{k=0}^{n} (-1)^k k S_n^{(k)} = \frac{(-1)^{n+1}}{n!} S_{n+1}^{(2)}, \tag{13}$$

in terms of Stirling numbers of the first kind ([3], [6], [7], [19], [20]).

References

[1] M. Z. Spivey, A generalized recurrence for Bell numbers, *J. Integer Sequences* 11, No. 2 (2008) Art. 08.2.5

[2] A. Xu, Extensions of Spivey's Bell number formula, *The Electr. J. of Combinatorics* 19, No. 2 (2012), 1-8.

[3] M. Z. Spivey, *The Art of Proving Binomial Identities*, CRC Press, Boca Raton, Fl, USA (2018).

[4] T. Kim, D. S. Kim, Generalization of Spivey's recurrence relation, *Russ. J. Math. Phys.* 31, No. 2 (2024), 218-226.

[5] H. W. Gould, T. Glatzer, Catalan and Bell numbers: Research bibliography of two special number sequences (1979), http://www.math.wvu.edu/ gould/.

[6] J. Quaintance, H. W. Gould, *Combinatorial Identities for Stirling Numbers*, World Scientific, Singapore (2016).

[7] J. Lpez-Bonilla, R. Lpez-Vzquez, J. Yalj Montiel-Prez, An identity involving Bell and Stirling numbers, *Prespacetime Journal* 8, No. 12 (2017), 1391-1393.

[8] V. Barrera-Figueroa, J. Lpez-Bonilla, R. Lpez-Vzquez, On Stirling numbers of the second kind, *Prespacetime Journal* 8, No. 5 (2017) , 572-574.

[9] B. Man Tuladhar, J. Lpez-Bonilla, O. Salas-Torres, An identity for Stirling numbers of the second kind, *Kathmandu Univ. J. of Sci.*, Eng. & Tech. 13, No. 1 (2017) 95-97.

[10] T. Kim, D. S. Kim, Some identities on degenerate r-Stirling numbers via boson operators, *Russ. J. Math. Phys.* 29, No. 4 (2022), 508-517.

[11] T. Kim, D. S. Kim, Some identities involving degenerate Stirling numbers associated with several degenerate polynomials and numbers, *Russ. J. Math. Phys.* 30, No. 1 (2023), 62-75.

[12] D. S. Kim, T. Kim, Normal ordering associated with λ-Whitney numbers of the first kind in λ-shift algebra, *Russ. J. Math. Phys.* 30, No. 3 (2023) 310-319.

[13] G. Dobinski, Archiv, *der Mathematik und Physik* 61 (1877), 333-336.

[14] J. Lpez-Bonilla, J. Yalj Montiel-Prez, S. Vidal-Beltrn, On the Melzak and Wilf identities, *J. of Middle East and North Africa Sci.* 3, No. 11 (2017), 1-3.

[15] R. Cruz-Santiago, J. Lpez-Bonilla, R. Lpez-Vzquez, On the series transformation formula of Boyadzhiev, *Prespacetime Journal* 8, No. 12 (2017), 1367-1369.

[16] T. Arakawa, T. Ibukiyama, M. Kaneko, *Bernoulli Numbers and Zeta Functions*, Springer, Japan (2014).

[17] P. M. Knopf, The operator $\left(x\frac{d}{dx}\right)^n$ and its application to series, *Math. Mag.* 76, No. 5 (2003), 364-371.

[18] M. Z. Spivey, Combinatorial sums and finite differences, *Discrete Math.* 307, No. 24 (2007), 3130-3146.

[19] A. T. Benjamin, G. O. Preston, J. J. Quinn, A Stirling encounter with harmonic numbers, *Math. Mag.* 75, No. 2 (2002), 95-103.

[20] A. Iturri-Hinojosa, J. Lpez-Bonilla, R. Lpez-Vzquez, Harmonic, Stirling and Bernoulli numbers, *Prespacetime J.* 8, No. 9 (2017), 1173-1175.

International J.Math. Combin. Vol.1-Vol.2(2025), 107-119

An Amplification of Interval-Valued Intuitionistic Fuzzy Soft Matrices for Disease Diagnosis

Mousumi Akter[1], Md Sahadat Hossain[2], Md Fazlul Hoque[1],
Rafiqul Islam[1] and Md. Nasimul Karim[2]

1. Pabna University of Science and Technology, Faculty of Science, Department of Mathematics, Pabna 6600, Bangladesh
2. University of Rajshahi, Faculty of Science, Department of Mathematics, Rajshahi 6205, Bangladesh

E-mail: mousumiakter@pust.ac.bd, fazlulmath@pust.ac.bd, rafiq.math@pust.ac.bd
sahadat@ru.ac.bd, nasimulkarim.ru@gmail.com

Abstract: The evolution of decision-making processes and medical diagnosis has been significantly influenced by advances in fuzzy soft set theory (FSS) and its extensions. This paper delves into the use of interval-valued intuitionistic fuzzy soft matrices (IVIFSMs) to improve the accuracy and reliability of medical diagnoses, focusing on viral diseases. Usually, traditional methods struggle with the uncertainties and complexities of medical data. Therefore, building upon the integration of fuzzy set theory and soft set theory, IVIFSM offers a sophisticated approach to handling uncertainties inherent in medical data. This study demonstrates how IVIFSM can provide healthcare professionals with a more robust tool to make informed decisions, thereby improving diagnostic results. Through a detailed examination of theoretical foundations, methodological innovations, and practical applications, this article underscores the potential of IVIFSM to revolutionize medical diagnosis and decision-making frameworks.

Key Words: Interval-valued intuitionistic fuzzy set (IVIFS), fuzzy soft set (FSS), interval-valued intuitionistic fuzzy soft set (IVIFSS), fuzzy soft matrix (FSM), interval-valued intuitionistic fuzzy soft matrix (IVIFSM).

AMS(2010): 06D72, 54A40.

§1. Introduction

In these days, the pattern of decision-making and medical diagnosis has undergone a significant transfiguration with the approach of fuzzy set theory and its consequent extensions. Having accurate and trustworthy decision-making tools is essential for medical diagnosis. Conventional techniques frequently find it difficult to handle the ambiguity and complexity of medical data. This has led to the research of fuzzy soft sets and their variations. However, in current days, almost all of our substantial problems in modern life, such as socio-economic development, medical discipline, and engineering fields of study, we experience uncertainties, inexact circum-

[1]Received March 1, 2025. Accepted May 26, 2025

stances, and ambiguity, which involve rough data, and a certain amount of these problems are predominantly humanistic. Over the last few years, several theories and methodologies have been exhibited to act with certain ambiguity and inexact data, like probability theory, fuzzy set theory, fuzzy sets and systems, soft set theory, etc. Lotfi Zadeh [1] introduced the notion of Fuzzy set theory, which provided a way to handle uncertainty and imprecision in various fields. Then the concept of intuitionistic fuzzy set (IFS) theory was introduced by K. T. Attanassov [2], in which each element of IFS is formed with membership and non-membership degrees between 0 and 1, offering an even preferable way to model and analyze uncertain information. After a while, Attanasov and Gargov [3] bring out the concept of interval-valued intuitionistic fuzzy set (IVIFS), which is an extension of IFS, and draw out the membership and non-membership degrees with intervals between 0 to 1. However, traditional fuzzy sets theory and fuzzy systems had limitations when addressing more complex and ambiguous scenarios. To address these challenges, Molodstov [4] introduced the soft set theory, which offers a new mathematical method to manage uncertainties without relying on traditional probabilistic approaches. Researchers like Maji [5-7] and Roy [8] furthered the idea of the soft set by combining fuzzy set theory with soft set theory, creating the concept of fuzzy soft sets (FSS) and those properties of fuzzy soft union, intersection, complement, etc. This combination allowed for more sophisticated handling of uncertainty through operations like intersection, union, and complement, making these theories more applicable to real-world problems, especially in decision-making and medical diagnosis. Afterward, Ahmad and Kharal [9] develop and enhance the analysis of Maji on FSS, which include study of the union, intersection, De Morgans Inclusions and De Morgans Laws in FSS theory. In the study of Neog and Sut [10] they demonstrated the concept of fuzzy soft sets and complement of fuzzy soft set in a new away that accommodated every obligation of complement of a classical set. In recent years, implementation of FSS in several fields of discipline and humanistic circumstances has been analyzed by many researchers [11-16]. In [13] De et.al. revised and review Sanchezs [15,16] method of medical diagnosis focuses onto intuitionistic fuzzy set. Base on intuitionistic fuzzy soft set Saikia et.al. [14] modified De et.al. [13] method. Moreover, Chetia and Das [11] analyze Sanchezs [15,16] proposed method of medical diagnosis founded on IVFSS to develop the methodology of De et.al. in [13]. The notion of interval-valued intuitionistic fuzzy soft set (IVIFSS) was introduced by mathematician Jiang et.al. [17].

Though decision-making to engineering and medical applications, matrices are very effective for addressing various everyday problems, traditional matrices, often face challenges with uncertain issues. To overcome these challenges, Chetia and Das [12] proposed intuitionistic fuzzy soft matrices (IFSM), which come with operations and properties designed to handle this uncertainty better. Hereinafter Zulqarnain et. al. created a decision-making method [18-20] known as the interval-valued fuzzy soft max-min decision-making method that uses interval-valued fuzzy soft matrices to improve decision-making and medical diagnoses and includes comparing the performance of fuzzy soft matrices with interval-valued fuzzy soft matrices. Meenakshi and Kaliraja [21] provided techniques to apply Sanchezs approach to medical diagnosis using interval-valued fuzzy matrices, in which they introduced the arithmetic mean matrix of an interval-valued fuzzy matrix and applied Sanchezs [15, 16] method directly to this mean matrix, showing how it can be effectively used for medical diagnosis. Focus on IFSMs Zulqarnain et.al. [20, 22, 23] proposed a disease diagnosis methodology, in which IFSM is utilized for diagnosis in patients who suffer from different diseases.

All these innovative ideas are used in our study to provide a unique approach to identifying diseases according to the data exhibited. The research initiative intends to increase the precision and dependability of medical diagnosis by utilizing interval-valued intuitionistic fuzzy soft matrices focus on to Zulqarnain et.al. proposed method [23] for medical diagnosis, which provides healthcare workers with improved instruments for making informed decisions. This research is built on the basic principles of IVIFSs, IVIFSMs and their combination provides a comprehensive framework for dealing with uncertainties in medical diagnosis. To do so, in section 2 we give an abridge discussion of theoretical foundations of IVIFSs, IVIFSSs and IVIFSMs. In sections 3, we explore the methodological developments of these concepts and developed an established methodology based on IVIFSMs to get an optimal result for medical diagnosis. In Section 4, we demonstrate a hypothetical example to illustrate the execution functioning of the proposed method with a particular focus on their role in improving decision-making processes and the accuracy of medical diagnosis. The discussion and consequences are narrated in Section 5.

§2. Preliminaries

In the following, we concisely review some basic concepts of IFS, IVIFS, soft set (SS), intuitionistic fuzzy soft set (IFSS), IVIFSS and IVIFSM that will be applied in the subsequent section. For convenience in this study, we assume $N_1 = \{1, 2, 3, \cdots, r\}$, $N_2 = \{1, 2, 3, \cdots, s\}$, and $N_3 = \{1, 2, 3, \cdots, t\}$, in all respects.

Definition 2.1 *Let K be the domain of objects. An IFS ν in K defined as the ordered triplet*

$$\nu = \{(\alpha, \mu_\nu(\alpha), \gamma_\nu(\alpha)) \mid \alpha \in K\}, \tag{1}$$

where $\mu_\nu : K \to [0, 1]$ and $\gamma_\nu : K \to [0, 1]$ described the membership and non-membership grade according to the objects $\alpha \in K$ and the relation $\mu_\nu(\alpha) + \gamma_\nu(\alpha) \leq 1$ holds for each $\alpha \in K$.

Definition 2.2 *Let K be the domain of objects. An IVIFS ν in K defined as the ordered triplet*

$$\nu = \{(\alpha, \mu_\nu(\alpha), \gamma_\nu(\alpha)) \mid \alpha \in K\}, \tag{2}$$

where $\mu_\nu(\alpha) = [\mu^L_\nu(\alpha), \mu^U_\nu(\alpha)] \subseteq [0, 1]$ and $\gamma_\nu(\alpha) = [\gamma^L_\nu(\alpha), \gamma^U_\nu(\alpha)] \subseteq [0, 1]$ are intervals and the relations $0 \leq \mu^L_\nu(\alpha) \leq \mu^U_\nu(\alpha) \leq 1$,$0 \leq \gamma^L_\nu(\alpha) \leq \gamma^U_\nu(\alpha) \leq 1$ and $\mu^U_\nu(\alpha) + \gamma^U_\nu(\alpha) \leq 1$ holds for each $\alpha \in K$.

Definition 2.3 *Let K be the domain of discourse and σ be the set of parameters with $\omega \subseteq \sigma$. A pair $\langle\Omega, \omega\rangle$ is studied as a soft set under the domain of discourse K, where $\Omega : \omega \to P(K)$, such as $\Omega(\epsilon) = \phi$, if $\epsilon \notin \omega$ and $P(K)$ represent the power set of K.*

By harmonizing the idea of IFS with soft set theory Maji et al. [7] present the notion of IFSS.

Definition 2.4 *Let K be the domain of discourse and σ be the set of parameters with $\omega \subseteq \sigma$. A pair $\langle \Omega, \omega \rangle$ is studied as an intuitionistic fuzzy soft set on the domain of discourse K, where*

$$\Omega : \omega \to \Im(K) \quad \text{such that } \Omega(\epsilon) = \emptyset \text{ if } \epsilon \notin \omega$$

and $\Im(K)$ represents the set of all intuitionistic fuzzy subsets of K. Conventionally, for every $\epsilon \in \omega$, $\Omega(\epsilon)$ describes an intuitionistic fuzzy subset of K and can be formed as

$$\Omega(\epsilon) = \{(\alpha, \mu_{\Omega(\epsilon)}(\alpha), \gamma_{\Omega(\epsilon)}(\alpha)) \mid \alpha \in K\},$$

where $\mu_{\Omega(\epsilon)}(\alpha)$ and $\gamma_{\Omega(\epsilon)}(\alpha)$ explain the membership and non-membership degree of α, respectively. The collection of every IFSS on K according to the parameter set $\omega \subseteq \sigma$ is acquainted as the intuitionistic fuzzy soft class and is denoted as IFSC(K, σ).

Definition 2.5 *Let K be the domain of discourse and σ be the set of parameters with $\omega \subseteq \sigma$. A pair $\langle \Omega, \omega \rangle$ is studied as an interval-valued intuitionistic fuzzy soft set on the domain of discourse K, where*

$$\Omega : \omega \to \Im\Im(K) \quad \text{such that } \Omega(\epsilon) = \emptyset \text{ if } \epsilon \notin \omega$$

and $\Im\Im(K)$ represents the set of all interval-valued intuitionistic fuzzy subsets of K. Conventionally, for every $\epsilon \in \omega$, $\Omega(\epsilon)$ describes an interval-valued intuitionistic fuzzy subset of K and can be formed as

$$\Omega(\epsilon) = \{(\alpha, \mu_{\Omega(\epsilon)}(\alpha), \gamma_{\Omega(\epsilon)}(\alpha)) \mid \alpha \in K\},$$

where $\mu_{\Omega(\epsilon)}(\alpha)$ and $\gamma_{\Omega(\epsilon)}(\alpha)$ explain the membership and non-membership degree of α, respectively. The collection of every IVIFSS on K according to the parameter set $\omega \subseteq \sigma$ is acquainted as the interval-valued intuitionistic fuzzy soft class and is denoted as IVIFSC(K, σ).

Definition 2.6 *For the set of parameters $\sigma = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, the complement of σ is elicited by $\neg\sigma$ and is illustrated as $\neg\sigma = \{\neg\varepsilon_1, \neg\varepsilon_2, \ldots, \neg\varepsilon_n\}$, where $\neg\varepsilon_i =$ not ε_i.*

Definition 2.7 *We elicit the complement of an IVIFSS $\langle \Omega, \omega \rangle$ as $\langle \Omega, \omega \rangle^{\complement}$ and is described as*

$$\langle \Omega, \omega \rangle^{\complement} = \langle \Omega^{\complement}, \neg\omega \rangle,$$

where the mapping $\Omega^c : \neg\omega \to \Im\Im(K)$ is formed as

$$\Omega^{\complement}(\varepsilon) = \{\langle \alpha, \mu_{\Omega(\neg\varepsilon)}(\alpha), \gamma_{\Omega(\neg\varepsilon)}(\alpha)\rangle \mid \alpha \in K \text{ and } \varepsilon \in \neg\omega\}.$$

Definition 2.8 *Let $K = \{\alpha_1, \alpha_2, \cdots, \alpha_m\}$ be the domain of discourse and $\sigma = \{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n\}$ be the set of parameters with $\omega \subseteq \sigma$. Presume $\langle \Omega, \omega \rangle$ to be a FSS in the fuzzy soft class (K, σ). Then, the matrix form of FSS $\langle \omega, \Omega \rangle$ is defined as $\omega_{m \times n} = \{\alpha_{ij}\}_{m \times n}$, where $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. Here*

$$\alpha_{ij} = \begin{cases} 0 & \text{if } \varepsilon_j \notin \omega \\ \mu_j(\alpha_i) & \text{if } \varepsilon_j \in \omega \end{cases}$$

and $\mu_j(\alpha_i)$ stands for the membership degree of α_i for the fuzzy set $\Omega(\varepsilon_j)$.

Definition 2.9 *Let $K = \{\alpha_i\}_{i=1}^m$ be the domain of discourse and σ be the set of parameters with $\omega \subseteq \sigma = \{\varepsilon_j\}_{j=1}^n$. Presume $\langle \Omega, \omega \rangle$ to be an IVIFSS over K. Then, the matrix form of the IVIFSS $\langle \Omega, \omega \rangle$ is defined as $\omega_{m\times n} = \{\alpha_{ij}\}_{m\times n}$, $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$. Here*

$$\alpha_{ij} = \begin{cases} \big([0,0], [1,1]\big) & \text{if } \varepsilon_j \notin \omega \\ \big(\big[\mu_j^L(\alpha_i), \mu_j^U(\alpha_i)\big], \big[\gamma_j^L(\alpha_i), \gamma_j^U(\alpha_i)\big]\big) & \text{if } \varepsilon_j \in \omega \end{cases}.$$

Here $\big[\mu_j^L(\alpha_i), \mu_j^U(\alpha_i)\big]$ and $\big[\gamma_j^L(\alpha_i), \gamma_j^U(\alpha_i)\big]$ explain the membership and non-membership degree of each α_i for the IVIFS $\Omega(\varepsilon_j)$.

Definition 2.10 *Consider two IVIFSMs $\omega_{m\times n} = \{\alpha_{ij}\}_{m\times n}$ and $\varphi_{m\times n} = \{\beta_{ij}\}_{m\times n}$. Then, the addition and subtraction of $\omega_{m\times n}$ and $\varphi_{m\times n}$ are defined as*

$$\omega + \varphi = \big([\max(\mu_\omega^L, \mu_\varphi^L), \max(\mu_\omega^U, \mu_\varphi^U)], [\min(\gamma_\omega^L, \gamma_\varphi^L), \min(\gamma_\omega^U, \gamma_\varphi^U)]\big) \tag{3}$$

and

$$\omega - \varphi = \big([\min(\mu_\omega^L, \mu_\varphi^L), \min(\mu_\omega^U, \mu_\varphi^U)], [\max(\gamma_\omega^L, \gamma_\varphi^L), \max(\gamma_\omega^U, \gamma_\varphi^U)]\big). \tag{4}$$

Definition 2.11 *Consider two IVIFSMs $\omega_{m\times n} = \{\alpha_{ij}\}_{m\times n}$ and $\varphi_{m\times n} = \{\beta_{ij}\}_{m\times n}$. Then, the max-min composition of $\omega_{m\times n}$ and $\varphi_{m\times n}$ is defined as*

$$\omega * \varphi = \big([\max(\min(\mu_\omega^L, \mu_\varphi^L)), \max(\min(\mu_\omega^U, \mu_\varphi^U))], [\min(\max(\gamma_\omega^L, \gamma_\varphi^L)), \min(\max(\gamma_\omega^U, \gamma_\varphi^U))]\big). \tag{5}$$

Definition 2.12 *Consider two IVIFSMs $\omega_{m\times n} = \{\alpha_{ij}\}_{m\times n}$ and $\varphi_{m\times n} = \{\beta_{ij}\}_{m\times n}$. Then, the geometric mean (GM) of $\omega_{m\times n}$ and $\varphi_{m\times n}$ is defined as*

$$\omega \circ \varphi = \big([\sqrt{\mu_\omega^L \cdot \mu_\varphi^L}, \sqrt{\mu_\omega^U \cdot \mu_\varphi^U}], [\sqrt{\gamma_\omega^L \cdot \gamma_\varphi^L}, \sqrt{\gamma_\omega^U \cdot \gamma_\varphi^U}]\big). \tag{6}$$

In particular, the GM of $\omega_{m\times n} = \{\alpha_{ij}\}_{m\times n}$ is given as

$$GM(\omega_{m\times n}) = \big(\sqrt{\mu_\omega^L \cdot \mu_\omega^U}, \sqrt{\gamma_\omega^L \cdot \gamma_\omega^U}\big). \tag{7}$$

Definition 2.13 *Consider two IVIFSMs $\omega_{m\times n} = \{\alpha_{ij}\}_{m\times n}$ and $\varphi_{m\times n} = \{\beta_{ij}\}_{m\times n}$. Then, the harmonic mean (HM) of $\omega_{m\times n}$ and $\varphi_{m\times n}$ is defined as*

$$\omega \bullet \varphi = \left(\left[\frac{2 \cdot \mu_\omega^L \cdot \mu_\varphi^L}{\mu_\omega^L + \mu_\varphi^L}, \frac{2 \cdot \mu_\omega^U \cdot \mu_\varphi^U}{\mu_\omega^U + \mu_\varphi^U}\right], \left[\frac{2 \cdot \gamma_\omega^L \cdot \gamma_\varphi^L}{\gamma_\omega^L + \gamma_\varphi^L}, \frac{2 \cdot \gamma_\omega^U \cdot \gamma_\varphi^U}{\gamma_\omega^U + \gamma_\varphi^U}\right]\right). \tag{8}$$

In general, the HM of $\omega_{m\times n} = \{\alpha_{ij}\}_{m\times n}$ is given as

$$HM(\omega) = \left(\frac{2 \cdot \mu_\omega^L \cdot \mu_\omega^U}{\mu_\omega^L + \mu_\omega^U}, \frac{2 \cdot \gamma_\omega^L \cdot \gamma_\omega^U}{\gamma_\omega^L + \gamma_\omega^U}\right). \tag{9}$$

§3. An Ideal Model for Medical Diagnosis Based on IVIFSS

In the following, we construct an ideal model to obtain a congenial result for medical diagnosis based on IVIFSS. Assume $S = \{\varepsilon_i\}_{i\in N_1}$ to be the set of r syndromes of s diseases $D = \{d_j\}_{j\in N_2}$. Let, $P = \{p_k\}_{k\in N_3}$ be the set of patients who view the syndromes. In accordance to the discipline the diagnosis model can be evaluated methodically like and so.

Step 1. At the outset, we describe an IVIFSS $\langle\Omega_1, \omega_1\rangle$ over $S = \{\varepsilon_i\}_{i\in N_1}$, that provide a syndrome-disease relation matrix (SDRM) $\mathcal{R}_1 = (\rho_{ij})_{r\times s}$, manifested from IVIFSS $\langle\Omega_1, \omega_1\rangle$ and is form in the following way

$$\mathcal{R}_1 = (\rho_{ij})_{r\times s} = \begin{array}{c} \\ \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_r \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \dots & d_s \end{array} \\ \begin{bmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1s} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{r1} & \rho_{r2} & \cdots & \rho_{rs} \end{bmatrix} \end{array}$$

where $\rho_{ij} = ([x_{ij}, y_{ij}], [z_{ij}, w_{ij}])$ with $0 \le x_{ij} \le y_{ij} \le 1$, $0 \le z_{ij} \le w_{ij} \le 1$ and $0 \le y_{ij}+w_{ij} \le 1$.

Correspondingly, a relation matrix $\mathcal{R}_2 = (\chi_{ij})_{r\times s}$ called non syndrome-disease relation matrix (NSDRM), is constructed from $\langle\Omega_1, \omega_1\rangle^{\complement}$, complement of $\langle\Omega_1, \omega_1\rangle$ and is formed as

$$\mathcal{R}_2 = (\chi_{ij})_{r\times s} = \begin{array}{c} \\ \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_r \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \dots & d_s \end{array} \\ \begin{bmatrix} \chi_{11} & \chi_{12} & \cdots & \chi_{1s} \\ \chi_{21} & \chi_{22} & \cdots & \chi_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{r1} & \chi_{r2} & \cdots & \chi_{rs} \end{bmatrix} \end{array}$$

where $\chi_{ij} = ([z_{ij}, w_{ij}], [x_{ij}, y_{ij}])$ in accordance to $\rho_{ij} = ([x_{ij}, y_{ij}], [z_{ij}, w_{ij}])$ and $0 \le z_{ij} \le w_{ij} \le 1$, $0 \le x_{ij} \le y_{ij} \le 1$ and $0 \le y_{ij} + w_{ij} \le 1$.

Step 2. Thereafter, we demonstrate other IVIFSS $\langle\Omega_2, \omega_2\rangle$ and its complement $\langle\Omega_2, \omega_2\rangle^{\complement}$ over $P = \{p_k\}_{k\in N_3}$. That come up with, a patient-syndrome relation matrix (PSRM) $\mathcal{N}_1 = (\xi_{ki})_{t\times r}$, and a patient-non syndrome relation matrix (PNSRM) $\mathcal{N}_2 = (\zeta_{ki})_{t\times r}$, in accordance to $\langle\Omega_2, \omega_2\rangle$ and $\langle\Omega_2, \omega_2\rangle^{\complement}$, respectively and illustrated like as

$$\mathcal{N}_1 = (\xi_{ki})_{t\times r} = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_t \end{array} \begin{array}{c} \begin{array}{cccc} \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_r \end{array} \\ \begin{bmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1r} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{t1} & \xi_{t2} & \cdots & \xi_{tr} \end{bmatrix} \end{array}$$

where $\xi_{ki} = ([\alpha_{ki}, \beta_{ki}], [\lambda_{ki}, \delta_{ki}])$, with $0 \le \alpha_{ki} \le \beta_{ki} \le 1$, $0 \le \lambda_{ki} \le \delta_{ki} \le 1$ and $0 \le \beta_{ki}+\delta_{ki} \le 1$.

Correspondingly,

$$\mathcal{N}_2 = (\zeta_{ki})_{t\times r} = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_t \end{array}\!\!\begin{array}{c} \begin{array}{cccc} \varepsilon_1 & \varepsilon_2 & \dots & \varepsilon_r \end{array} \\ \begin{bmatrix} \zeta_{11} & \zeta_{12} & \cdots & \zeta_{1r} \\ \zeta_{21} & \zeta_{22} & \cdots & \zeta_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \zeta_{t1} & \zeta_{t2} & \cdots & \zeta_{tr} \end{bmatrix} \end{array}$$

where $\zeta_{ki} = ([\lambda_{ki}, \delta_{ki}], [\alpha_{ki}, \beta_{ki}])$, according to $\xi_{ki} = ([\alpha_{ki}, \beta_{ki}], [\lambda_{ki}, \delta_{ki}])$ with $0 \leq \alpha_{ki} \leq \beta_{ki} \leq 1$, $0 \leq \lambda_{ki} \leq \delta_{ki} \leq 1$ and $0 \leq \beta_{ki} + \delta_{ki} \leq 1$.

Step 3. Hereinafter, utilizing Def.(2.11) and relation matrices $\mathcal{R}_1 = (\rho_{ij})_{r\times s}$, $\mathcal{R}_2 = (\chi_{ij})_{r\times s}$, $\mathcal{N}_1 = (\xi_{ki})_{t\times r}$ and $\mathcal{N}_2 = (\zeta_{ki})_{t\times r}$, we evaluate four relation matrices $W_1 = (\varsigma^1_{kj})_{t\times s} = \mathcal{N}_1 * \mathcal{R}_1$, $W_2 = (\varsigma^2_{kj})_{t\times s} = \mathcal{N}_1 * \mathcal{R}_2$, $W_3 = (\varsigma^3_{kj})_{t\times s} = \mathcal{N}_2 * \mathcal{R}_1$, and $W_4 = (\varsigma^4_{kj})_{t\times s} = \mathcal{N}_2 * \mathcal{R}_2$, which are manifested in the following way

$$W_1 = (\varsigma^1_{kj})_{t\times s} = \mathcal{N}_1 * \mathcal{R}_1 = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_t \end{array}\!\!\begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \dots & d_s \end{array} \\ \begin{bmatrix} \varsigma^1_{11} & \varsigma^1_{12} & \cdots & \varsigma^1_{1s} \\ \varsigma^1_{21} & \varsigma^1_{22} & \cdots & \varsigma^1_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \varsigma^1_{t1} & \varsigma^1_{t2} & \cdots & \varsigma^1_{ts} \end{bmatrix} \end{array}$$

where, each

$$\varsigma^1_{kj} = \xi_{ki} * \rho_{ij} = \big(\big[\max(\min(\alpha_{ki}, x_{ij})), \max(\min(\beta_{ki}, y_{ij}))\big], \big[(\min(\max(\lambda_{ki}, z_{ij})), \min(\max(\delta_{ki}, w_{ij}))\big]\big).$$

$$W_2 = (\varsigma^2_{kj})_{t\times s} = \mathcal{N}_1 * \mathcal{R}_2 = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_t \end{array}\!\!\begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \dots & d_s \end{array} \\ \begin{bmatrix} \varsigma^2_{11} & \varsigma^2_{12} & \cdots & \varsigma^2_{1s} \\ \varsigma^2_{21} & \varsigma^2_{22} & \cdots & \varsigma^2_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \varsigma^2_{t1} & \varsigma^2_{t2} & \cdots & \varsigma^2_{ts} \end{bmatrix} \end{array}$$

where, each

$$\varsigma^2_{kj} = \xi_{ki} * \chi_{ij} = \big(\big[\max(\min(\alpha_{ki}, z_{ij})), \max(\min(\beta_{ki}, w_{ij}))\big], \big[(\min(\max(\lambda_{ki}, x_{ij})), \min(\max(\delta_{ki}, y_{ij}))\big]\big).$$

$$W_3 = (\varsigma^3_{kj})_{t\times s} = \mathcal{N}_2 * \mathcal{R}_1 = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_t \end{array}\!\!\begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \dots & d_s \end{array} \\ \begin{bmatrix} \varsigma^3_{11} & \varsigma^3_{12} & \cdots & \varsigma^3_{1s} \\ \varsigma^3_{21} & \varsigma^3_{22} & \cdots & \varsigma^3_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \varsigma^3_{t1} & \varsigma^3_{t2} & \cdots & \varsigma^3_{ts} \end{bmatrix} \end{array}$$

where, each

$\varsigma^3_{kj} = \zeta_{ki} * \rho_{ij} = \big(\big[\max(\min(\lambda_{ki}, x_{ij})), \max(\min(\delta_{ki}, y_{ij}))\big], \big[(\min(\max(\alpha_{ki}, z_{ij})), \min(\max(\beta_{ki}, w_{ij}))\big]\big).$

$$W_4 = (\varsigma^4_{kj})_{t\times s} = \mathcal{N}_2 * \mathcal{R}_2 = \begin{array}{c} \\ p_1 \\ p_2 \\ \vdots \\ p_t \end{array} \begin{array}{c} \begin{array}{cccc} d_1 & d_2 & \dots & d_s \end{array} \\ \begin{bmatrix} \varsigma^4_{11} & \varsigma^4_{12} & \cdots & \varsigma^4_{1s} \\ \varsigma^4_{21} & \varsigma^4_{22} & \cdots & \varsigma^4_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \varsigma^4_{t1} & \varsigma^4_{t2} & \cdots & \varsigma^4_{ts} \end{bmatrix} \end{array}$$

where, each

$\varsigma^4_{kj} = \zeta_{ki} * \chi_{ij} = \big(\big[\max(\min(\lambda_{ki}, z_{ij})), \max(\min(\delta_{ki}, w_{ij}))\big], \big[(\min(\max(\alpha_{ki}, x_{ij})), \min(\max(\beta_{ki}, y_{ij}))\big]\big),$

and W_1, W_2, W_3 and W_4 are named as patient syndrome disease matrix (PSDM), patient syndrome non-disease matrix (PSNDM), patient non-syndrome disease matrix (PNSDM) and patient non-syndrome non-disease matrix (PNSNDM), gradually.

Step 4. Next, applying Eq.(7), we compute the GM of each ς^ℓ_{kj} of the exhibited relation matrices $\{W_\ell\}_{\ell=1}^4$ obtained from step 3 and that can be recount as

$$G_\ell = GM(W_\ell) = GM(\varsigma^\ell_{kj}) = \left(\sqrt{\mu^L_{\varsigma^\ell_{kj}} \cdot \mu^U_{\varsigma^\ell_{kj}}}, \sqrt{\gamma^L_{\varsigma^\ell_{kj}} \cdot \gamma^U_{\varsigma^\ell_{kj}}}\right)_{\ell=1,2,3,4}. \tag{10}$$

In this case $\mu_{\varsigma^\ell_{kj}}$ and $\gamma_{\varsigma^\ell_{kj}}$ represent the fuzzy membership degree and fuzzy reference degree of each ς^ℓ_{kj}, respectively.

Step 5. Calculate the membership value (MV) matrices of all the four G_ℓ in the following way

$$Y_\ell = MV(G_\ell) = MV(GM(\varsigma^\ell_{kj})) = \big\{(\mu_{\varsigma^\ell_{kj}} - \gamma_{\varsigma^\ell_{kj}})\big\}_{\ell=1,2,3,4}, \tag{11}$$

where $\mu_{\varsigma^\ell_{kj}} = \sqrt{\mu^L_{\varsigma^\ell_{kj}} \cdot \mu^U_{\varsigma^\ell_{kj}}}$, and $\gamma^\ell_{\varsigma_{kj}} = \sqrt{\gamma^L_{\varsigma^\ell_{kj}} \cdot \gamma^U_{\varsigma^\ell_{kj}}}$.

Step 6. Based on Step 5, the diagnosis score value matrices (DSVM) S_1 and S_2 of the obtained membership values Y_ℓ are exhibited as

$$S_1 = Y_1 - Y_3 \quad and \quad S_2 = Y_2 - Y_4. \tag{12}$$

Step 7: Using DSVMs obtained in Step 6 compute the total score value matrix (TSVM) and is derived as

$$\theta = (\tau_{kj})_{t\times s} = S_1 - S_2. \tag{13}$$

Step 8. Focus on the total scores value matrix of all t patients according to the s diseases, find

$$\psi_m = \max_j (\tau_{kj}) = \max_j (S_1(p_k, d_j) - S_2(p_k, d_j)). \tag{14}$$

which conclude that patient p_k is suffering from the disease d_m, where $1 \le m \le s$.

Step 9. If same values of ψ_m acquire in different rows, then go to Step 1 and reiterate the method by reevaluating the syndrome of the patients.

§4. Explicative Case Illustrations

In this section, we illustrate an exegetical case of study to evaluate the effectual mode of the developed model for disease diagnosis. Presume, three patients $P = \{p_1, p_2, p_3\}$ are admitted in a clinic with syndromes $S = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and the probable diseases associated with these syndromes are $D = \{d_1, d_2\}$. According to the circumstances, the developed diagnosis model can be exhibited orderly thus and so.

Step 1. Foremost, we illustrate an IVIFSS $\langle \Omega_1, D\rangle$ over $S = \{\varepsilon_i\}_{i=1}^3$, where $\Omega_1 : D \to \Im\Im(S)$. This illustration provides a SDRM $\mathcal{R}_1 = (\rho_{ij})_{3\times 2}$ obtained from IVIFSS $\langle \Omega_1, D\rangle$ and is formed as

$$\mathcal{R}_1 = \left(\rho_{ij}\right)_{3\times 2} = \begin{bmatrix} ([0.6, 0.7], [0.1, 0.2]) & ([0.6, 0.65], [0.2, 0.3]) \\ ([0.5, 0.6], [0.2, 0.3]) & ([0.4, 0.5], [0.4, 0.5]) \\ ([0.2, 0.3], [0.6, 0.7]) & ([0.7, 0.8], [0.1, 0.2]) \end{bmatrix},$$

where $\rho_{ij} = ([x_{ij}, y_{ij}], [z_{ij}, w_{ij}])$ with $0 \le x_{ij} \le y_{ij} \le 1$, $0 \le z_{ij} \le w_{ij} \le 1$, and $0 \le y_{ij} + w_{ij} \le 1$. Correspondingly, NSDRM $\mathcal{R}_2 = (\chi_{ij})_{3\times 2}$ is constructed from $\langle \Omega_1, D\rangle^{\complement}$, called the complement of $\langle \Omega_1, D\rangle$ and is formed as

$$\mathcal{R}_2 = \left(\chi_{ij}\right)_{3\times 2} = \begin{bmatrix} ([0.1, 0.2], [0.6, 0.7]) & ([0.2, 0.3], [0.6, 0.65]) \\ ([0.2, 0.3], [0.5, 0.6]) & ([0.4, 0.5], [0.4, 0.5]) \\ ([0.6, 0.7], [0.2, 0.3]) & ([0.1, 0.2], [0.7, 0.8]) \end{bmatrix},$$

where $\chi_{ij} = ([z_{ij}, w_{ij}], [x_{ij}, y_{ij}])$ in accordance to $\rho_{ij} = ([x_{ij}, y_{ij}], [z_{ij}, w_{ij}])$ and $0 \le z_{ij} \le w_{ij} \le 1, 0 \le x_{ij} \le y_{ij} \le 1, 0 \le w_{ij} + y_{ij} \le 1$.

Step 2. Next, we demonstrate the IVIFSS $\langle \Omega_2, S\rangle$ and its complement $\langle \Omega_2, S\rangle\complement$ over $P = \{p_k\}_{k=1}^3$, where $\Omega_2 : S \to \Im\Im(P)$. Those come up with a PSRM $\mathcal{N}_1 = (\xi_{kj})_{3\times 3}$ and a PNSRM $\mathcal{N}_2 = (\zeta_{kj})_{3\times 3}$ in accordance to $\langle \Omega_2, S\rangle$ and its complement $\langle \Omega_2, S\rangle\complement$, respectively, and are formed like as

$$\mathcal{N}_1 = (\xi_{kj})_{3\times 3} = \begin{bmatrix} ([0.2, 0.3], [0.6, 0.7]) & ([0.5, 0.6], [0.2, 0.3]) & ([0.3, 0.4], [0.5, 0.55]) \\ ([0.3, 0.4], [0.5, 0.55]) & ([0.4, 0.5], [0.4, 0.45]) & ([0.5, 0.6], [0.2, 0.25]) \\ ([0.6, 0.7], [0.2, 0.25]) & ([0.4, 0.6], [0.2, 0.3]) & ([0.4, 0.45], [0.45, 0.5]) \end{bmatrix},$$

where $\xi_{kj} = ([\alpha_{kj}, \beta_{kj}], [\lambda_{kj}, \delta_{kj}])$ with $0 \le \alpha_{kj} \le \beta_{kj} \le 1$, $0 \le \lambda_{kj} \le \delta_{kj} \le 1$, and $0 \le \beta_{kj} + \delta_{kj} \le 1$. Correspondingly,

$$\mathcal{N}_2 = (\zeta_{kj})_{3\times 3} = \begin{bmatrix} ([0.6, 0.7], [0.2, 0.3]) & ([0.2, 0.3], [0.5, 0.6]) & ([0.5, 0.55], [0.3, 0.4]) \\ ([0.5, 0.55], [0.3, 0.4]) & ([0.4, 0.5], [0.4, 0.5]) & ([0.2, 0.25], [0.5, 0.6]) \\ ([0.2, 0.25], [0.6, 0.7]) & ([0.2, 0.3], [0.4, 0.6]) & ([0.45, 0.5], [0.4, 0.45]) \end{bmatrix},$$

where $\zeta_{kj} = ([\lambda_{kj}, \delta_{kj}], [\alpha_{kj}, \beta_{kj}])$ according to $\xi_{kj} = ([\alpha_{kj}, \beta_{kj}], [\lambda_{kj}, \delta_{kj}])$, with $0 \leq \alpha_{kj} \leq \beta_{kj} \leq 1$, $0 \leq \lambda_{kj} \leq \delta_{kj} \leq 1$and $0 \leq \beta_{kj} + \delta_{kj} \leq 1$.

Step 3. Thereafter, employing Def.(2.11) and relation matrices $\mathcal{R}_1, \mathcal{R}_2, \mathcal{N}_1$ and $\mathcal{N}_2$ we evaluate four composition relation matrices $W_1 = (\varsigma^1_{kj})_{3\times 2} = \mathcal{N}_1 * \mathcal{R}_1$, $W_2 = (\varsigma^2_{kj})_{3\times 2} = \mathcal{N}_1 * \mathcal{R}_2$, $W_3 = (\varsigma^3_{kj})_{3\times 2} = \mathcal{N}_2 * \mathcal{R}_1$ and $W_4 = (\varsigma^4_{kj})_{3\times 2} = \mathcal{N}_2 * \mathcal{R}_2$, which are built in the following way

$$W_1 = \begin{bmatrix} ([0.5, 0.6], [0.2, 0.3]) & ([0.4, 0.5], [0.4, 0.5]) \\ ([0.5, 0.6], [0.2, 0.3]) & ([0.5, 0.6], [0.2, 0.25]) \\ ([0.6, 0.7], [0.2, 0.25]) & ([0.6, 0.65], [0.2, 0.3]) \end{bmatrix},$$

where, each

$$\varsigma^1_{kj} = \xi_{ki} * \rho_{ij} = \left[\max(\min(\alpha_{ki}, x_{ij})), \max(\min(\beta_{ki}, y_{ij}))\right], \left[\min(\max(\lambda_{ki}, z_{ij})), \min(\max(\delta_{ki}, w_{ij}))\right].$$

$$W_2 = \begin{bmatrix} ([0.3, 0.4], [0.5, 0.55]) & ([0.4, 0.5], [0.4, 0.5]) \\ ([0.5, 0.6], [0.2, 0.3]) & ([0.4, 0.5], [0.4, 0.5]) \\ ([0.4, 0.5], [0.4, 0.5]) & ([0.4, 0.45], [0.45, 0.5]) \end{bmatrix},$$

where, each

$$\varsigma^2_{kj} = \xi_{ki} * \chi_{ij} = \left[\max(\min(\alpha_{ki}, z_{ij})), \max(\min(\beta_{ki}, w_{ij}))\right], \left[\min(\max(\lambda_{ki}, x_{ij})), \min(\max(\delta_{ki}, y_{ij}))\right].$$

$$W_3 = \begin{bmatrix} ([0.6, 0.7], [0.2, 0.3]) & ([0.6, 0.65], [0.2, 0.3]) \\ ([0.5, 0.55], [0.3, 0.4]) & ([0.5, 0.55], [0.3, 0.4]) \\ ([0.2, 0.3], [0.4, 0.6]) & ([0.45, 0.5], [0.45, 0.5]) \end{bmatrix},$$

where, each

$$\varsigma^3_{kj} = \zeta_{ki} * \rho_{ij} = \left[\max(\min(\lambda_{ki}, x_{ij})), \max(\min(\delta_{ki}, y_{ij}))\right], \left[\min(\max(\alpha_{ki}, z_{ij})), \min(\max(\beta_{ki}, w_{ij}))\right].$$

$$W_4 = \begin{bmatrix} ([0.5, 0.55], [0.3, 0.4]) & ([0.2, 0.3], [0.5, 0.6]) \\ ([0.2, 0.3], [0.5, 0.6]) & ([0.4, 0.45], [0.4, 0.5]) \\ ([0.45, 0.5], [0.4, 0.45]) & ([0.2, 0.3], [0.4, 0.6]) \end{bmatrix},$$

where, each

$$\varsigma^4_{kj} = \zeta_{ki} * \chi_{ij} = \left[\max(\min(\lambda_{ki}, z_{ij})), \max(\min(\delta_{ki}, w_{ij}))\right], \left[\min(\max(\alpha_{ki}, x_{ij})), \min(\max(\beta_{ki}, y_{ij}))\right].$$

Here, W_1, W_2, W_3 and W_4 are named as PSDM, PSNDM, PNSDM and PNSNDM, respectively.

Step 4. Next, employing Eq.(10) we compute the G_ℓ of the exhibited relation matrices $\{W_\ell\}_{\ell=1}^4$ obtained from step 3 and that can be computed as

$$G_1 = \begin{bmatrix} (0.55, 0.24) & (0.45, 0.45) \\ (0.45, 0.42) & (0.55, 0.22) \\ (0.65, 0.22) & (0.62, 0.24) \end{bmatrix}, \quad G_2 = \begin{bmatrix} (0.35, 0.52) & (0.45, 0.45) \\ (0.55, 0.24) & (0.45, 0.45) \\ (0.45, 0.45) & (0.42, 0.47) \end{bmatrix},$$

$$G_3 = \begin{bmatrix} (0.65, 0.24) & (0.62, 0.24) \\ (0.52, 0.35) & (0.52, 0.35) \\ (0.24, 0.49) & (0.47, 0.47) \end{bmatrix}, \quad G_4 = \begin{bmatrix} (0.52, 0.35) & (0.24, 0.55) \\ (0.24, 0.55) & (0.42, 0.45) \\ (0.47, 0.42) & (0.47, 0.47) \end{bmatrix}.$$

Step 5. In the following, using Eq.(11) we calculate the membership value MV matrices of all the four G_ℓ thus and so

$$Y_1 = \begin{bmatrix} 0.31 & 0 \\ 0.03 & 0.33 \\ 0.43 & 0.38 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -0.17 & 0 \\ 0.31 & 0 \\ 0 & -0.05 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} 0.41 & 0.38 \\ 0.17 & 0.17 \\ -0.25 & 0 \end{bmatrix},$$

$$Y_4 = \begin{bmatrix} 0.17 & -0.31 \\ -0.31 & -0.03 \\ 0.05 & 0 \end{bmatrix}.$$

Step 6. Next, utilizing Eq.(12) the DSVMs S_1 and S_2 of the obtained membership values Y_ℓ are computed as

$$S_1 = \begin{bmatrix} -0.1 & -0.38 \\ -0.14 & 0.16 \\ 0.68 & 0.38 \end{bmatrix}, \quad S_2 = \begin{bmatrix} -0.34 & 0.31 \\ 0.62 & 0.03 \\ -0.05 & -0.05 \end{bmatrix}.$$

Step 7. Using Eq.(13) and DSVMs obtained in Step 6 determine the TSVM

$$\theta = \begin{bmatrix} 0.24 & -0.69 \\ -0.76 & 0.13 \\ 0.73 & 0.43 \end{bmatrix}.$$

Step 8. Focus on the total scores value matrix of all three patients according to the two diseases and Eq.(14), we conclude that patient p_1 and p_3 are suffering from disease d_1 and patient p_2 suffering from disease d_2.

§5. Conclusions

In our work, focus on Zulqarnain *et.al.* proposed method [23] and IVIFSM we develop a disease diagnosis model to diagnose illness in patients through a detailed examination of theoretical foundations, methodological innovations, and practical application. The established methodology provides a mathematical model in which the relation matrices are typically formed with IVIFSM, which is an extension of Zulqarnain *et.al.* proposed method [23] and increases the precision and dependability of medical diagnosis.

Acknowledgement

We admire the reviewer's insightful remarks and thoughts to enhance the paper excellence.

References

[1] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8, 1965, pp. 338-353.
[2] K. T. Atanassov, *Intuitionistic Fuzzy Sets*,P hysica-Verlag HD, 1999, pp. 1-137.
[3] K. T. Atanassov and G. Gargov, Interval-valued Intuitionistic Fuzzy Sets, *Fuzzy Sets and Systems*, 31(3), 1989, pp. 343-349.
[4] D. Molodtsov, Soft set theory first results, *Computers and Mathematics with Applications*, 37(4-5), 1999, pp.19-31.
[5] P. K. Maji, R. Biswas, and A. R. Roy, Soft set theory, *Computers and Mathematics with Applications*, 45(4-5), 2003, pp. 555-562.
[6] P. K. Maji, A. R. Roy and R. Biswas, An application of soft set in a decision making problem, *Computers and Mathematics with Applications*, 44(8-9), 2002, pp. 1077-1083.
[7] P. K. Maji, A. R. Roy and R. Biswas, On intuitionistic fuzzy soft sets, *Journal of Fuzzy Mathematics*, 12(3), 2004, pp. 669-684.
[8] A. R. Roy and P. K. Maji, A fuzzy soft set theoretic approach to decision making problems, *Journal of Computational and Applied Mathematics*, 203(2), 2007, pp. 412-418.
[9] B. Ahmad and A. Kharal, On fuzzy soft sets, *Advances in Fuzzy Systems*, 2009, pp. 1-6.
[10] T. J. Neog and D. K. Sut, Theory of fuzzy soft sets from a new perspective, *International Journal of Latest Trends in Computing*, 2(3), 2011, pp. 439-450.
[11] B. Chetia and P. K. Das, An application of interval valued fuzzy soft set in medical diagnosis, *International Journal of Contemporary Mathematical Sciences*, 5(38), 2010, pp. 1887-1894.
[12] B. Chetia and P. K. Das, Some results of intuitionistic fuzzy soft matrix theory, *Advances in Applied Science Research*, 3(1), 2012, pp. 412-423.
[13] S. K. De, R. Biswas, and A. R. Roy, An application of intuitionistic fuzzy sets in medical diagnosis, *Fuzzy Sets and Systems*, 117(2), 2001, pp. 209-213.
[14] B. K. Saikia, P. K. Das, and A. K. Borkakati, An application of intuitionistic fuzzy soft sets in medical diagnosis, *Bio Science Research Bulletin*, 19(2), 2003, pp. 121-127.
[15] E. Sanchez, Resolution of composite fuzzy relation equations, *Information and Control*, 30(1), 1976, pp. 38-48.
[16] E. Sanchez, Inverse of fuzzy relations, application to possibility distributions and medical diagnosis, *Fuzzy Sets and Systems*, 2(1), 1979, pp. 75-86.
[17] Y. Jiang, Y. Tang, Q. Chen, H. Liu and J. Tang, Interval-valued intuitionistic fuzzy soft sets and their properties, *Computers and Mathematics with Applications*, 60(3), 2010, pp. 906-918.
[18] M. Zulqarnain and M. Saeed, A new decision making method on interval valued fuzzy soft matrix (IVFSM), *British Journal of Mathematics & Computer Science*, 20(5), 2017, pp. 1-17.

[19] M. Zulqarnain and M. Saeed, An application of interval valued fuzzy soft matrix in decision making, *Science International (Lahore)*, 28(3), 2016, pp. 2261-2264.

[20] R. M. Zulqarnain, X. L. Xin, M. Saeed, N. Ahmad, Application of interval-valued fuzzy soft max-min decision making method in medical diagnosis, *International Journal of Pharmaceutical Sciences and Review and Research*, 62(1), 2020, pp. 56-60.

[21] A. R. Meenakshi and M. Kaliraja, An application of interval valued fuzzy matrices in medical diagnosis, *International Journal of Mathematical Analysis*, 5(36), 2011, pp. 1791-1802.

[22] R. M. Zulqarnain, X. L. Xin, M. Saeed, N. Ahmad, F. Dayan, and B. Ahmad, Recruitment of medical staff in health department by using TOPSIS method, *International Journal of Pharmaceutical Sciences and Review Research*, 62(1), 2020, pp. 1-7.

[23] R.M. Zulqarnain, M. Saeed, M.I. Ahamad, S. Abdal, Z. Zafar and M. Aslam,, Application of intuitionistic fuzzy soft matrices for disease diagnosis, *International Journal of Discrete Mathematics*, 5(1), 2020, pp. 4-9.

Famous Words

Mathematics is a kind of spirit, a kind of rational spirit. It is this spirit that inspires, promotes, encourages and drives human thinking to the fullest extent, and it is this spirit that attempts to decisively influence the material, moral and social life of man; trying to answer questions about human beings' own existence trying to understand and control nature; try to explore and establish the knowledge has been the most profound and most perfect connotation.

By *Felix Christian Klein*, a Germany mathematician.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats latex, eps, pdf may be submitted electronically to one member of the Editorial Board for consideration in the **International Journal of Mathematical Combinatorics** (*ISSN 1937-1055*). An effort is made to publish a paper duly recommended by a referee within a period of $2-4$ months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Theory on the Universe*, Global Knowledge-Publishing House, USA, 2023.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as JPG, EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

June, 2025

Contents

Combinatorics - A Mathematical Approach for Holding on the Realty of Thing in the Universe By Linfan Mao 01

Enumeration the Number of Spanning Trees of the Sequence of Some Families of Graphs That Have the Same Average Degree
By S. N. Daoud and Mohmmed Aljohani 24

Connected Monophonic Eccentric Domination Number of Corona Product of Some Standard Graphs
By P. Titus, J. Ajitha Fancy, Santhakumaran and A. Radhakrishnan 51

Some Fractional Product Cordial Graphs
By R. Ponraj and T. Sutharson 67

Modulo Two Square Mean Labeling of Some Path and Path Related Graphs
By Christopher M. and Ramachandran V. 78

A Note on the Ratio of two Gamma Functions
By A. Bagdasaryan, J. López-Bonilla, R. Rajendra and P. Siva Kota Reddy 90

PMC-Labeling of Certain Tree Related Graphs and Prism of Wheel Graph
By R. Ponraj, S. Prabhu and M. Sivakumar 93

A Short Note on an Identity of Spivey for Bell Numbers
By T. Kim, J. López-Bonilla, R. Rajendra, P. Siva Kota Reddy and M. Pavithra 103

An Amplification of Interval-Valued Intuitionistic Fuzzy Soft Matrices for Disease Diagnosis By Mousumi Akter, Md Sahadat Hossain, Md Fazlul Hoque, Rafiqul Islam and Md. Nasimul Karim 107

An International Journal on Mathematical Combinatorics

ISSN 1937-1055