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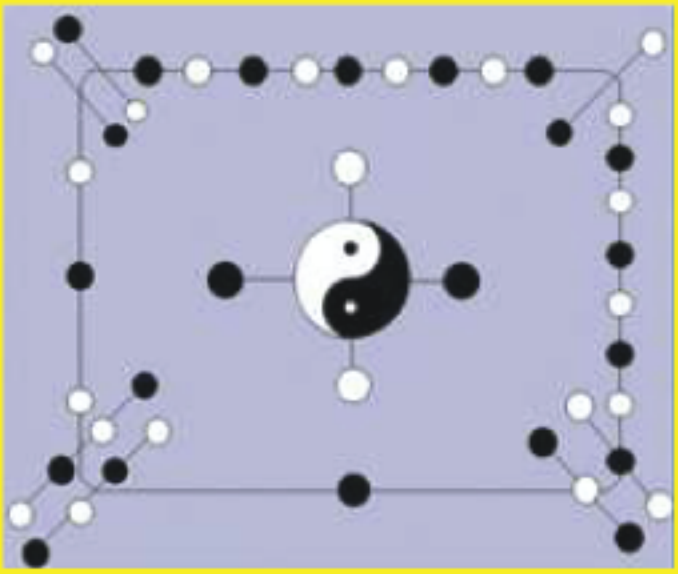

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Aims and Scope: The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

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If the facts don't fit the theory, change the facts.

By Albert Einstein, an American theoretical physicist.

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Singed Total Domatic Number of a Graph

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Abstract: Let G be a finite and simple graph with vertex set $V(G)$, $k \geq 1$ an integer and let $f : V(G) \to \{-k, k-1, \cdots, -1, 1, \cdots, k-1, k\}$ be $2k$ valued function. If $\sum\limits_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the open neighborhood of v, then f is a Smarandachely k-Signed total dominating function on G. A set $\{f_1, f_2, \ldots, f_d\}$ of Smarandachely k-Signed total dominating function on G with the property that $\sum\limits_{i=1}^{d} f_i(x) \leq k$ for each $x \in V(G)$ is called a Smarandachely k-Signed total dominating family (function) on G. Particularly, a Smarandachely 1-Signed total dominating function or family is called signed total dominating function or family on G. The maximum number of functions in a signed total dominating family on G is the signed total domatic number of G. In this paper, some properties related signed total domatic number and signed total domination number of a graph are studied and found the sign total domatic number of certain class of graphs such as fans, wheels and generalized Petersen graph.

Key Words: Smarandachely k-signed total dominating function, signed total domination number, signed total domatic number.

AMS(2000): 05C69

§1. Terminology and Introduction

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants [1] and [4]. We considered finite, undirected, simple graphs $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The order of G is given by $n = |V(G)|$. If $v \in V(G)$, then the open neighborhood of v is $N(v) = \{u \in V(G) | uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. The number $d_G(v) = d(v) = |N(v)|$ is the *degree* of the vertex $v \in V(G)$, and $\delta(G)$ is the *minimum degree* of G. The *complete graph* and the *cycle* of order n are denoted by K_n and C_n respectively. A fan and a wheel is a graph obtained from a path and a cycle by adding a new vertex and edges joining it to all the vertices of the path and cycle respectively. The generalized Petersen graph $P(n, k)$ is defined to be a graph on $2n$ vertices with $V(P(n, k)) = \{v_i u_i : 1 \leq i \leq n\}$ and $E(P(n, k)) =$

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$\{v_iv_{i+1}, v_iu_i, u_iu_{i+k} : 1 \leq i \leq n$, subscripts modulo $n\}$. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ in to some set of numbers, then $f(A) = \sum_{x \in A} f(x)$.

Let $k \geq 1$ be an integer and let $f : V(G) \to \{-k, k-1, \cdots, -1, 1, \cdots, k-1, k\}$ be $2k$ valued function. If $\sum\limits_{x \in N(v)} f(x) \geq k$ for each $v \in V(G)$, where $N(v)$ is the open neighborhood of v, then f is a Smarandachely k-Signed total dominating function on G. A set $\{f_1, f_2, \ldots, f_d\}$ of Smarandachely k-Signed total dominating function on G with the property that $\sum\limits_{i=1}^{d} f_i(x) \leq k$ for each $x \in V(G)$ is called a Smarandachely k-Signed total dominating family (function) on G. Particularly, a Smarandachely 1-Signed total dominating function or family is called signed total dominating function or family on G. The singed total dominating function is defined in [6] as a two valued function $f : V(G) \to \{-1, 1\}$ such that $\sum_{x \in N(v)} f(x) \geq 1$ for each $v \in V(G)$. The minimum of weights $w(f)$, taken over all signed total dominating functions f on G, is called the signed total domination number $\gamma_t^s(G)$. Signed total domination has been studied in [3].

A set $\{f_1, f_2, \ldots, f_d\}$ of signed total dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leq 1$ for each $x \in V(G)$, is called a signed total dominating family on G. The maximum number of functions in a signed total dominating family is the signed total domatic number of G, denoted by $d_t^s(G)$. Signed total domatic number was introduced by Guan Mei and Shan Er-fang [2]. Guan Mei and Shan Er-fang [2] have determined the basic properties of $d_t^s(G)$. Some of them are analogous to those of the signed domatic number in [5] and studied sharp bounds of the signed total domatic number of regular graphs, complete bipartite graphs and complete graphs. Guan Mei and Shan Er-fang [2] presented the following results which are useful in our investigations.

Proposition 1.1([6]) *For Circuit C_n of length n we have $\gamma_t^s(C_n) = n$.*

Proof Here no other signed total dominating exists than the constants equal to 1. □

Theorem 1.2([3]) *Let T be a tree of order $n \geq 2$. then, $\gamma_t^s(T) = n$ if and only if every vertex of T is a support vertex or is adjacent to a vertex of degree 2.*

Proposition 1.3([2]) *The signed total domatic number $d_t^s(G)$ is well defined for each graph G.*

Proposition 1.4([2]) *For any graph G of order n ,$\gamma_t^s(G) \cdot d_t^s(G) \leq n$.*

Proposition 1.5([2]) *If G is a graph with the minimum degree $\delta(G)$, then $1 \leq d_t^s(G) \leq \delta(G)$.*

Proposition 1.6([2]) *The signed total domatic number is an odd integer.*

Corollary 1.7([2]) *If G is a graph with the minimum degree $\delta(G) = 1$ or 2, then $d_t^s(G) = 1$. In particular, $d_t^s(C_n) = d_t^s(P_n) = d_t^s(K_{1,n-1}) = d_t^s(T) = 1$, where T is a tree.*

§2. Properties of the Signed Total Domatic Number

Proposition 2.1 *If G is a graph of order n and $\gamma_t^s(G) \geq 0$ then, $\gamma_t^s(G) + d_t^s(G) \leq n+1$ equality* *holds if and only if G is isomorphic to C_n or tree T of order $n \geq 2$.*

Proof Let G be a graph of order n. The inequality follows from the fact that for any two non-negative integers a and b, $a + b \leq ab + 1$. By Proposition 1.4 we have,

$$\gamma_t^s(G) + d_t^s(G) \leq \gamma_t^s(G) \cdot d_t^s(G) + 1 \leq n + 1$$

Suppose that $\gamma_t^s(G) + d_t^s(G) = n + 1$ then, $n + 1 = \gamma_t^s(G) + d_t^s(G) \leq \gamma_t^s(G) \cdot d_t^s(G) + 1 \leq n + 1$.

This implies that $\gamma_t^s(G) + d_t^s(G) = \gamma_t^s(G) \cdot d_t^s(G) + 1$. This shows that $\gamma_t^s(G) \cdot d_t^s(G) = n$ Solving equations 1 and 2 simultaneously, we have either $\gamma_t^s(G) = 1$ and $d_t^s(G) = n$ or $\gamma_t^s(G) = n$ and $d_t^s(G) = 1$. If $\gamma_t^s(G) = 1$ and $d_t^s(G) = n$ then $n = d_t^s(G) \leq \delta(G)$ There fore, $\delta(G) \geq n$ a contradiction.

If $\gamma_t^s(G) = n$ and $d_t^s(G) = 1$ then by Proposition 1.1 and Proposition 1.2, we have $\gamma_t^s(C_n) = n$ and $d_t^s(C_n) = 1$ and By Theorem 1.2, If T is a tree of order $n \geq 2$ then, $\gamma_t^s(T) = n$ if and only if every vertex of T is a support vertex or is adjacent to a vertex of degree 2 and $d_t^s(T) = 1$. □

Theorem 2.2 *Let G be a graph of order n then $d_t^s(G) + d_t^s(\bar{G}) \leq n - 1$.*

Proof Let G be a regular graph order n, By Proposition 1.5 we have $d_t^s(G) \leq \delta(G)$ and $d_t^s(\bar{G}) \leq \delta(\bar{G})$. Thus we have,

$$d_t^s(G) + d_t^s(\bar{G}) \leq \delta(G) + \delta(\bar{G}) = \delta(G) + (n - 1 - \Delta(G)) \leq n - 1.$$

Thus the inequality holds. □

§3. Signed Total Domatic Number of Fans, Wheels and Generalized Petersen Graph

Proposition 3.1 *Let G be a fan of order n then $d_t^s(G) = 1$.*

Proof Let $n \geq 2$ and let $x_1, x_2, \ldots, x_n$ be the vertex set of the fan G such that $x_1, x_2, \ldots, x_n, x_1$ is a cycle of length n and x_n is adjacent to x_i for each $i = 2, 3, \ldots, n-2$. By Proposition 1.5 and Proposition 1.6, $1 \leq d_t^s(G) \leq \delta(G) = 2$, which implies $d_t^s(G) = 1$ which proves the result.□

Proposition 3.2 *If G is a wheel of order n then $d_t^s(G) = 1$.*

Proof Let $x_1, x_2, \ldots, x_n$ be the vertex set of the wheel G such that $x_1, x_2, \ldots, x_{n-1}, x_1$ is a cycle of length $n - 1$ and x_n is adjacent to x_i for each $i = 1, 2, 3, \ldots, n - 1$. According to the Proposition 1.5 and Proposition 1.6, we observe that either $d_t^s(G) = 1$ or $d_t^s(G) = 3$. Suppose to the contrary that $d_t^s(G) = 3$. Let $\{f_1, f_2, f_3\}$ be a corresponding signed total dominating family. Because of $f_1(x_n) + f_2(x_n) + f_3(x_n) \leq 1$, there exists at least one function say f_1 with $f_1(x_n) = -1$ The condition $\sum_{x \in N(v)} f_1(x) \geq 1$ for each $v \in (V(G) - \{x_n\})$ yields $f_1(x) = 1$ for each some $i \in \{1, 2, \ldots, n-1\}$ and $t = 2, 3$ then it follows that $f_t(x_{i+1}) = f_t(x_{i+2}) = 1$, where the indices are taken taken modulo $n - 1$ and $f_t(x_n) = 1$. Consequently, the function f_t has at most $\lfloor \frac{n}{2} \rfloor - 1$ for n is odd and $\frac{n}{2} - 1$ for n is even number of vertices $x \in V(G)$ such that

$f_t(x) = -1$. Thus there exist at most $\lfloor \frac{n}{2} \rfloor - 1$ for n is odd and $\frac{n}{2} - 1$ for n is even number of vertices $x \in V(G)$ such that $f_t(x) = -1$ for at least one $i = 1, 2, 3$. Since $n \geq 4$, we observe that $2(\lfloor \frac{n}{2} \rfloor + 1) = 2\left(\frac{n}{2} - 1\right) + 1 < n$ for n is odd and $2\left(\frac{n}{2} - 1\right) + 1 < n$, a contradiction to $f_1(x_n) + f_2(x_n) + f_3(x_n) \leq 1$ for each $x \in V(G)$. □

Proposition 3.3 *Let $G = P(n, k)$ be a generalized Petersen graph then for $k = 1, 2$, $d_t^s(G) = 1$.*

Proof The generalized Petersen graph $P(n, 1)$ is a graph on $2n$ vertices with

$$V(P(n, k)) = \{v_i u_i : 1 \leq i \leq n\}$$

and $E(P(n, k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+1} : 1 \leq i \leq n$, subscripts modulo $n\}$. According to the Proposition 1.5, Proposition 1.6, we observe that $d_t^s(G) = 1$ or $d_t^s(G) = 3$.

Case 1: $k = 1$

Let $\{f_1, f_2, f_3\}$ be a corresponding signed total dominating functions. Because of $f_1(v_n) + f_2(v_n) + f_3(v_n) \leq 1$ for each $i \in \{1, 2, \ldots, 2n\}$, there exist at least one number $j \in \{1, 2, 3\}$ such that $f_j(v_i) = -1$. Let, for example, $f_1(v_k) = -1$ for for any $t \in \{1, 2, \ldots, 2n\}$ then $\sum_{x \in N(v_t)} f_1(v) \geq 1$ implies that $f_1(v_k) = f_1(v_{k+1}) = -1$ for $k \cong 0, 1 mod 4$ and $f_1(v_k) = -1$ for $k \cong 0 mod 3$. This implies, there exist at most $8r$, $8r + 2$, $8r + 4$, $8r + 6$, $r \geq 1$ vertices such that $f_t(v) = -1$ for each $t = 2, 3$ when $P(n, 1)$ is of order $2(6r + l)$ for $0 \leq l \leq 2$, $2(6r + 3)$, $2(6r + 4)$, $2(6r + 5)$ respectively. Thus there exist $3(8r) = 3(8(\frac{n}{12} - \frac{l}{6}) < n$ (similarly $< n$ for all values of vertex set) a contradiction to $f_1(v_n) + f_2(v_n) + f_3(v_n) \leq 1$ for each $v \in V(G)$.

Case 2: $k = 2$

Similar to the proof of Case 1, we can prove the claim in this case. □

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Radio Number of Cube of a Path

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Abstract: Let G be a connected graph. For any two vertices u and v, let $d(u,v)$ denotes the distance between u and v in G. The maximum distance between any pair of vertices is called the diameter of G and is denoted by $diam(G)$. A Smarandachely k-radio labeling of a connected graph G is an assignment of distinct positive integers to the vertices of G, with $x \in V(G)$ labeled $f(x)$, such that $d(u,v) + |f(u) - f(v)| \geq k + diam(G)$. Particularly, if $k = 1$, such a Smarandachely radio k-labeling is called radio labeling for abbreviation. The radio number $rn(f)$ of a radio labeling f of G is the maximum label assignment to a vertex of G. The radio number $rn(G)$ of G is minimum $\{rn(f)\}$ over all radio labelings of G. In this paper, we completely determine the radio number of the graph P_n^3 for all $n \geq 4$.

Keywords: Smarandachely radio k-labeling, radio labeling, radio number of a graph.

AMS(2010): 05C78, 05C12, 05C15

§1. Introduction

All the graphs considered here are undirected, finite, connected and simple. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d_G(u,v)$ or simply $d(u,v)$. We use the standard terminology, the terms not defined here may be found in [1].

The *eccentricity* of a vertex v of a graph G is the distance from the vertex v to a farthest vertex in G. The minimum eccentricity of a vertex in G is the *radius* of G, denoted by $r(G)$, and the of maximum eccentricity of a vertex of G is called the diameter of G, denoted by $diam(G)$. A vertex v of G whose eccentricity is equal to the radius of G is a *central vertex*.

For any real number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x and $\lfloor x \rfloor$

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denotes the greatest integer less than or equal to x. We recall that k^{th} power of a graph G, denoted by G^k is the graph on the vertices of G with two vertices u and v are adjacent in G^k whenever $d(u,v) \leq k$. The graph G^3 is called a cube of G.

A labeling of a connected graph is an injection $f : V(G) \to Z^+$, while a Smarandache k-radio labeling of a connected graph G is an assignment of distinct positive integers to the vertices of G, with $x \in V(G)$ labeled $f(x)$, such that $d(u,v)+|f(u)-f(v)| \geq k+diam(G)$. Particularly, if $k = 1$, such a Smarandache radio k-labeling is called radio labeling for abbreviation. The radio number $rn(f)$ of a radio labeling f of G is the maximum label assigned to a vertex of G. The *radio number* $rn(G)$ of G is $min\{rn(f)\}$, over all radio labelings f of G. A radio labeling f of G is a *minimal radio labeling* of G if $rn(f) = rn(G)$.

Radio labeling is motivated by the channel assignment problem introduced by Hale et al [10] in 1980. The radio labeling of a graph is most useful in FM radio channel restrictions to overcome from the effect of noise. This problem turns out to find the minimum of maximum frequencies of all the radio stations considered under the network.

The notion of radio labeling was introduced in 2001, by G. Chartrand, David Erwin, Ping Zhang and F. Harary in [2]. In [2] authors showed that if G is a connected graph of order n and diameter two, then $n \leq rn(G) \leq 2n-2$ and that for every pair k, n of integers with $n \leq k \leq 2n-2$, there exists a connected graph of order n and diameter two with $rn(G) = k$. Also, in the same paper a characterization of connected graphs of order n and diameter two with prescribed radio number is presented.

In 2002, Ping Zhang [15] discussed upper and lower bounds for a radio number of cycles. The bounds are showed to be tight for certain cycles. In 2004, Liu and Xie [5] investigated the radio number of square of cycles. In 2007, B. Sooryanarayana and Raghunath P [12] have determined radio labeling of cube of a cycle, for all $n \leq 20$, all even $n \geq 20$ and gave bounds for other cycles. In [13], they also determined radio number of the graph C_n^4, for all even n and odd $n \leq 25$.

A radio labeling is called *radio graceful* if $rn(G) = n$. In [12] and [13] it is shown that the graph C_n^3 is radio graceful if and only if $n \in \{3,4,5,6,7,9,10,11,12,13,18,19\}$ and C_n^4 is radio graceful if and only if $n \in \{3,4,5,6,7,8,9,11,12,13,14,15,16,17,23,24,25\}$.

In 2005, D. D. F. Liu and X. Zhu [7] completely determined radio numbers of paths and cycles. In 2006, D. D. F. Liu [8] obtained lower bounds for the radio number of trees, and characterized the trees achieving this bound. Moreover in the same paper, he gave another lower bound for the radio number of the trees with at most one vertex of degree more than two (called spiders) in terms of the lengths of their legs and also characterized the spiders achieving this bound.

The results of D. D. F. Liu [8] generalizes the radio number for paths obtained by D. D. F. Liu and X. Zhu in [7]. Further, D.D.F. Liu and M. Xie obtained radio labeling of square of paths in [6]. In this paper, we completely determine the radio labeling of cube of a path. The main result we prove in this paper is the following Theorem 1.1. The lower bound is established in section 2 and a labeling procedure is given in section 3 to show that the lower bounds achieved in section 3 are really the tight upper bounds.

Theorem 1.1 *Let P_n^3 be a cube of a path on n ($n \geq 6$ and $n \neq 7$) vertices. Then*

$$rn(P_n^3) = \begin{cases} \frac{n^2+12}{6}, & if\ n \equiv 0\ (mod\ 6) \\ \frac{n^2-2n+19}{6}, & if\ n \equiv 1\ (mod\ 6) \\ \frac{n^2+2n+10}{6}, & if\ n \equiv 2\ (mod\ 6) \\ \frac{n^2+15}{6}, & if\ n \equiv 3\ (mod\ 6) \\ \frac{n^2-2n+16}{6}, & if\ n \equiv 4\ (mod\ 6) \\ \frac{n^2+2n+13}{6}, & if\ n \equiv 5\ (mod\ 6) \end{cases}.$$

We recall the following results for immediate reference.

Theorem 1.2(Daphne Der-Fen Liu, Xuding Zhu [6]) *For any integer $n \geq 4$,*

$$rn(P_n) = \begin{cases} 2k^2+3, & if\ \ n = 2k+1; \\ 2k^2-2k+2 & if\ n = 2k \end{cases}$$

Lemma 1.3(Daphne Der-Fen Liu, Melanie Xie [7]) *Let P_n^2 be a square path on n vertices with $k = \lfloor \frac{n}{2} \rfloor$. Let $\{x_1, x_2, \ldots, x_n\}$ be a permutation of $V(P_n^2)$ such that for any $1 \leq i \leq n-2$,*

$$min\{d_{P_n}(x_i, x_{i+1}),\ d_{P_n}(x_{i+1}, x_{i+2}) \leq k+1,$$

and if k is even and the equality in the above holds, then $d_{P_n}(x_i, x_{i+1})$ and $d_{P_n}(x_{i+1}, x_{i+2})$ have different parities. Let f be a function, $f : V(P_n^2) \to \{0, 1, 2, \ldots\}$ with $f(x_1) = 0$, and $f(x_{i+1}) - f(x_i) = k+1-d(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$. Then f is a radio-labeling for P_n^2.

§2. Lower Bound

In this section we establish the lower bound for Theorem 1.1. Throughout, we denote a path on n vertices by P_n, where $V(P_n) = \{v_1, v_2, v_3\ \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} \mid i = 1, 2, ..., n-1\}$. A path on odd length is called an *odd path* and that of even length is called an *even path.*

Observation 2.1 By the definition of P_n^3, for any two vertices u, $v \in V(P_n^3)$, we get

$$d_{P_n^3}(u, v) = \left\lceil \frac{d_{P_n}(u,v)}{3} \right\rceil \text{ and } diam(P_n^3) = \left\lceil \frac{n-1}{3} \right\rceil$$

Observation 2.2 An odd path P_{2k+1} on $2k+1$ vertices has exactly one center namely v_{k+1}, while an even path P_{2k} on $2k$ vertices has two centers v_k and v_{k+1}.

For each vertex $u \in V(P_n^3)$, the level of u, denoted by $l(u)$, is the smallest distance in P_n from u to a center of P_n. Denote the level of the vertices in a set A by $L(A)$.

Observation 2.3 For an even n, the distance between two vertices v_i and v_j in P_n^3 is given by their corresponding levels as;

$$d(v_i, v_j) = \begin{cases} \left\lceil \frac{|l(v_i)-l(v_j)|}{3} \right\rceil, & \text{whenever } 1 \leq i, j \leq \frac{n}{2} \text{ or } \frac{n+2}{2} \leq i, j \leq n \\ \\ \left\lceil \frac{l(v_i)+l(v_j)+1}{3} \right\rceil, & \text{otherwise} \end{cases} \tag{1}$$

Observation 2.4 For an odd n, the distance between two vertices v_i and v_j in P_n^3 is given by their corresponding levels as;

$$d(v_i, v_j) = \begin{cases} \left\lceil \frac{|l(v_i)-l(v_j|)}{3} \right\rceil, & \text{whenever } 1 \leq i, j \leq \frac{n+1}{2} \text{ or } \frac{n+1}{2} \leq i, j \leq n \\ \\ \left\lceil \frac{l(v_i)+l(v_j)}{3} \right\rceil, & \text{otherwise} \end{cases} \tag{2}$$

Observation 2.5 If n is even, then

$$L(V(P_n^3)) = \left\{\tfrac{n}{2}-1, \tfrac{n}{2}-2, \ldots, 2, 1, 0, 0, 1, 2, \ldots, \tfrac{n}{2}-2, \tfrac{n}{2}-1\right\}.$$

Therefore

$$\sum_{v_i \in V(P_n^3)} l(v_i) = 2\left[1+2+\cdots+\frac{n}{2}-1\right] = \frac{n}{2}\left\{\frac{n}{2}-1\right\} = \frac{n^2-2n}{4} \tag{3}$$

Observation 2.6 If n is odd, then

$$L(V(P_n^3) = \left\{\tfrac{n-1}{2}, \tfrac{n-1}{2}-1, \ldots 2, 1, 0, 1, 2, \ldots \tfrac{n-1}{2}-1, \tfrac{n-1}{2}\right\}.$$

Therefore

$$\sum_{v_i \in V(P_n^3)} l(v_i) = 2\left[1+2+\cdots+\frac{n-1}{2}\right] = \frac{n-1}{2}\left\{\frac{n+1}{2}\right\} = \frac{n^2-1}{4} \tag{4}$$

Let f be a radio labeling of the graph P_n^3. Let $x_1, x_2, \ldots, x_n$ be the sequence of the vertices of P_n^3 such that $f(x_{i+1}) > f(x_i)$ for every $i, 1 \leq i \leq n-1$. Then we have

$$f(x_{i+1}) - f(x_i) \geq diam(P_n^3) + 1 - d(x_{i+1}, x_i) \tag{5}$$

for every $i, 1 \leq i \leq n-1$.

Summing up $n-1$ inequalities in (5), we get

$$\sum_{i=1}^{n-1}[f(x_{i+1}) - f(x_i)] \geq \sum_{i=1}^{n-1}[diam(P_n^3)+1] - \sum_{i=1}^{n-1} d(x_{i+1}, x_i) \tag{6}$$

The terms in the left hand side of the inequality (6) cancels each other except the first and the last term, therefore, inequality (6) simplifies to

$$f(x_n) - f(x_1) \geq (n-1)[diam(P_n^3)+1] - \sum_{i=1}^{n-1} d(x_{i+1}, x_i) \tag{7}$$

If f is a minimal radio labeling of P_n^3, then $f(x_1) = 1$ (else we can reduce the span of f by $f(x_n) - f(x_1) + 1$ by reducing each label by $f(x_1) - 1$). Therefore, inequality (7) can be written as

$$f(x_n) \geq (n-1)[diam(P_n^3) + 1] - \sum_{i=1}^{n-1} d(x_{i+1}, x_i) + 1 \tag{8}$$

By the observations 2.3 and 2.4, for every i, $1 \leq i \leq n-1$ it follows that

$$d(x_{i+1}, x_i) \leq \left\lceil \frac{l(x_{i+1}) + l(x_i) + 1}{3} \right\rceil \leq \frac{l(x_{i+1}) + l(x_i)}{3} + 1 \tag{9}$$

whenever n is even. And

$$d(x_{i+1}, x_i) \leq \left\lceil \frac{l(x_{i+1}) + l(x_i)}{3} \right\rceil \leq \frac{l(x_{i+1}) + l(x_i)}{3} + \frac{2}{3} \tag{10}$$

whenever n is odd.

Inequalities (9) and (10), together gives,

$$\sum_{i=1}^{n-1} d(x_{i+1}, x_i) \leq \sum_{i=1}^{n-1} \left[\frac{l(x_{i+1}) + l(x_i)}{3} + k \right]$$

where $k = 1$, if n is even and $k = \frac{2}{3}$ if n is odd.

$$\Rightarrow \sum_{i=1}^{n-1} d(x_{i+1}, x_i) \quad \leq \quad \frac{1}{3} \times 2 \sum_{i=1}^{n} l(x_i) - \frac{1}{3} [l(x_n) + l(x_1)] + k(n-1)$$

$$\Rightarrow \sum_{i=1}^{n-1} d(x_{i+1}, x_i) \quad \leq \quad \frac{2}{3} \sum_{i=1}^{n} l(x_i) + k(n-1) - \frac{1}{3} [l(x_1) + l(x_n)] \tag{11}$$

From the inequalities 8 and 11, we get

$$f(x_n) \geq (n-1)[diam(P_n^3) + 1] - \frac{2}{3} \sum_{i=1}^{n} l(x_i) + 1 - k(n-1) + \frac{1}{3} [l(x_1) + l(x_n)]$$

$$\Rightarrow f(x_n) \geq (n-1) diam(P_n^3) - \frac{2}{3} \sum_{i=1}^{n} l(x_i) + 1 + (1-k)(n-1) + \frac{1}{3} [l(x_1) + l(x_n)] \tag{12}$$

We now observe that the equality between the second and third terms in (9) holds only if $l(x_{i+1}) + l(x_i) \equiv 0 \pmod 3$ and the equality between the second and third terms in holds only if(10) $l(x_{i+1}) + l(x_i) \equiv 1 \pmod 3$. Therefore, there are certain number of pairs (x_{i+1}, x_i) for which the strict inequality holds. That is, the right hand side of (9) as well as (10) will exceed by certain amount say ξ. Thus, the right hand side of (12) can be refined by adding an amount ξ as;

$$f(x_n) \geq \left\lceil (n-1) diam(P_n^3) - \frac{2}{3} \sum_{i=1}^{n} l(x_i) + 1 + (1-k)(n-1) + \eta + \xi \right\rceil \tag{13}$$

where $\eta = \frac{1}{3}\left[l(x_i) + l(x_{i+1})\right]$.

We also see that the value of ξ increases heavily if we take a pair of vertices on same side of the central vertex. So, here onwards we consider only those pairs of vertices on different sides of a central vertex.

Observation 2.7 All the terms in the right side of the inequality (13), except ξ and η, are the constants for a given path P_n. Therefore, for a tight lower bound these quantities must be minimized. If n is even, we have two cental vertices and hence a minimal radio labeling will start the label from one of the central vertices and end at the other vertex, so that $l(x_1) = l(x_n) = 0$. However, if n is odd, as the graph P_n has only one central vertex, either $l(x_1) > 0$ or $l(x_n) > 0$. Thus, $\eta \geq 0$ for all even n, and $\eta \geq \frac{1}{3}$ for all odd n.

The terms η and ξ included in the inequality (13) are not independent. The choice of initial and final vertices for a radio labeling decides the value of η, but at the same time it (this choice) also effect ξ (since ξ depends on the levels in the chosen sequence of vertices). Thus, for a minimum span of a radio labeling, the sum $\eta + \xi$ to be minimized rather than η or ξ.

Observation 2.8 For each j, $0 \leq j \leq 2$, define $L_j = \{v \in V(P_n^3) | l(v) \equiv j(mod\ 3)\}$ and for each pair $(x_{i+1}, x_i), 1 \leq i \leq n-1$ of vertices of $V(P_n^3)$, let

$\xi_i = \left\{\frac{l(x_{i+1})+l(x_i)}{3} + 1\right\} - \left\lceil \frac{l(x_{i+1})+l(x_i)+1}{3} \right\rceil$, if n is even, or
$\xi_i = \left\{\frac{l(x_{i+1})+l(x_i)}{3} + \frac{2}{3}\right\} - \left\lceil \frac{l(x_{i+1})+l(x_i)}{3} \right\rceil$, if n is odd.

Then there are following three possible cases:

Possibility 1: Either (i) both $x_{i+1}, x_i \in L_0$ or (ii) one of them is in L_1 and the other is in L_2. In this case

$$\xi_i = \begin{cases} 0, & if\ n\ is\ even \\ \frac{2}{3}, & if\ n\ is\ odd \end{cases}.$$

Possibility 2: Either (i) both $x_{i+1}, x_i \in L_2$ or (ii) one of them is in L_0 and the other is in L_1. In this case

$$\xi_i = \begin{cases} \frac{1}{3}, & if\ n\ is\ even \\ 0, & if\ n\ is\ odd \end{cases}.$$

Possibility 3: Either (i) both $x_{i+1}, x_i \in L_1$ or (ii) one of them is in L_0 and the other is in L_2. In this case

$$\xi_i = \begin{cases} \frac{2}{3}, & if\ n\ is\ even \\ \frac{1}{3}, & if\ n\ is\ odd \end{cases}.$$

Observation 2.9 For the case n is even, the Possibility 1 given in the Observation 2.8 holds for every pair of consecutive vertices in the sequence of the form either $l_{\alpha_1}, r_{\alpha_2}, l_{\alpha_3}, r_{\alpha_4}, l_{\alpha_5}, r_{\alpha_6}, \ldots,$ or $l_{\beta_1}, r_{\gamma_1}, l_{\beta_2}, r_{\gamma_2}, l_{\beta_3}, r_{\gamma_3}, \ldots,$, where l_{α_i}, l_{β_i}, l_{γ_i} denote the vertices in the left of a central vertex and at a level congruent to 0, 1, 2 under modulo 3 respectively, and, r_{α_i},r_{β_i}, r_{γ_i} denote the corresponding vertices in the right side of a central vertex of the path P_n. The first sequence covers only those vertices of P_n^3 which are at a level congruent to 0 under modulo 3, and, the second sequence covers only those vertices of L_1 (or L_2) which lie entirely on one side of a central vertex. Now, as the sequence $x_1, x_2, \ldots, x_n$ covers the entire vertex set of P_n^3, the sequence should have at least one pair as in Possibility 2 (taken this case for minimum ξ_i) to link a vertex in level congruent to 0 under modulo 3 with a vertex not at a level congruent to 0 under modulo 3. For this pair $\xi_i \geq \frac{1}{3}$. Further, to cover all the left as well as right vertices in the same level congruent to $i, 1 \leq i \leq 2$, we again require at least one pairs as in Possibility 2 or 3. Thus, for this pair again we have $\xi_i \geq \frac{1}{3}$. Therefore,

$$\xi = \sum_{i=1}^{n} \xi_i \geq \frac{2}{3}$$

for all even n.

The above Observation 2.9 can be visualize in the graph called *level diagram* shown in Figures 1 and 2. A Hamilton path shown in the diagram indicates a sequence $x_1, x_2, \ldots, x_n$ where thin edges join the pair of vertices as in Possibility 1 indicted in Observation 2.8 and the bold edges are that of Possibility 2 or 3. Each of the subgraphs $G_{0,0}$, $G_{1,2}$ and $G_{2,1}$ is a complete bipartite graph having only thin edges and $s = \lceil \frac{n-4}{6} \rceil$.

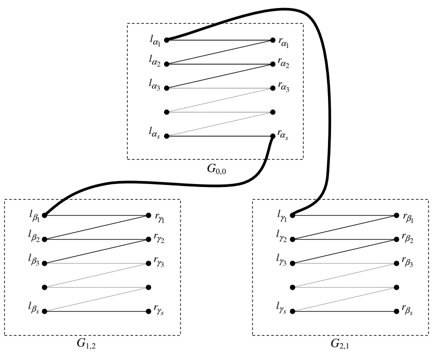

Figure 1: For P_n^3 when $n \equiv 0$ or 2 (mod 6).

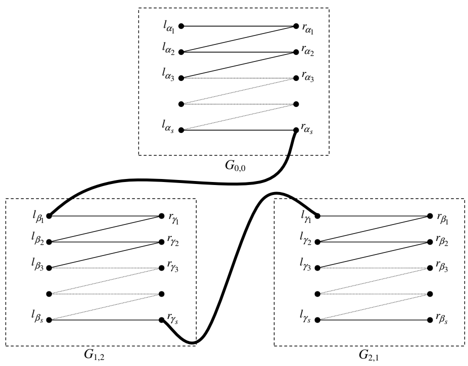

Figure 2: For P_n^3 when $n \equiv 0$ or 2 (mod 6).

If $\xi = \frac{2}{3}$ and $n \equiv 0 \pmod 6$, then only two types of Hamilton paths are possible as shown in Figures **??** and **??**. In each of the case either $l(x_1) > 0$ or $l(x_n) > 0$, therefore $\eta \geq \frac{1}{3}$. Hence $\eta + \xi \geq 1$ in this case.

If $\eta = 0$, then both the starting and the ending vertices should be in the subgraph $G_{0,0}$. Thus, one of the thin edges in $G_{0,0}$ to be broken and one of its ends to be joined to a vertex in $G_{1,2}$ and the other to a vertex in $G_{2,1}$ with bold edges. These two edges alone will not connect the subgraphs, so to connect $G_{1,2}$ and $G_{2,1}$ we need at least one more bold edge. Therefore, $\xi \geq 1$ and hence $\eta + \xi \geq 1$ in this case also.

In all the other possibilities for the case $n \equiv 0$ or 2 under modulo 6, we have $\eta \geq \frac{1}{3}$ and $\xi \geq \frac{2}{3}$, so clearly $\eta + \xi \geq 1$.

The situation is slightly different for the case when $n \equiv 4 \pmod 6$. In this case;

If $\xi = \frac{2}{3}$, then there is one and only one possible type of Hamilton path as shown in Figure **??**, so $l(x_1) > 0$ and $l(x_n) > 0$ implies that $\eta \geq \frac{2}{3}$ and hence $\eta + \xi \geq \frac{4}{3}$.

Else if, $\eta = 0$, then two bold edges are required. One edge is between a vertex of $G_{1,2}$ and a vertex of $G_{0,0}$, and, the other edge between a vertex of $G_{2,1}$ and a vertex of $G_{0,0}$ (for each such edges $\xi_i \geq \frac{1}{3}$). These two edges will not connect all the subgraphs. For this, we require an edge between a vertex of $G_{1,2}$ and a vertex of $G_{2,1}$, which can be done minimally only by an edge between a pair of vertices as in Possibility 3 indicated in observation 2.8 (for such an edge $\xi_i = \frac{2}{3}$). Thus, $\xi \geq 2 \times \frac{1}{3} + \frac{2}{3} = \frac{4}{3}$.

If $\xi = 1$, then the possible Hamilton path should contain at least either (i) one edge between $G_{1,2}$ and $G_{2,1}$, and, another edge from $G_{0,0}$, or, (ii) three edges from $G_{0,0}$. The first case is impossible because we can not join the vertices that lie on the same side of a central vertex with $\xi = 1$ and the second case is possible only if $\eta \geq \frac{1}{3}$.

Hence, for all even n, we get

$$\eta + \xi \geq 1, \quad if \quad n \equiv 0 \; or \; 2 \; (mod \; 6) \tag{14}$$

$$\eta + \xi \geq \frac{4}{3}, \quad if \quad n \equiv 4 \; (mod \; 6) \tag{15}$$

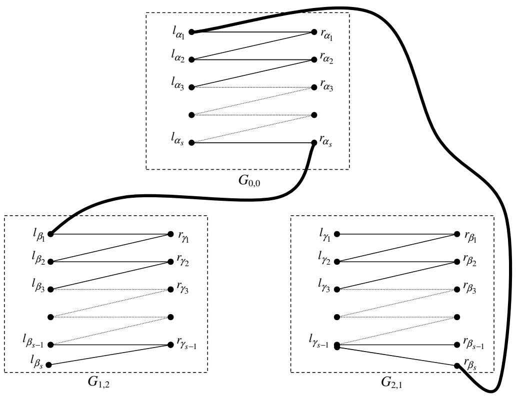

Figure 3: A Hamilton path in a level graph for the case $n \equiv 4 \pmod 6$

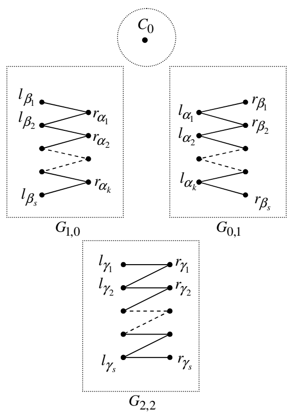

Figure 4: Level graph for the case $n \equiv 3$ or 5 (mod 6).

Observation 2.10 For the case n is odd, the Possibility 2 given in observation 2.8 holds for every pair of consecutive vertices in the sequence of the form l_{β_1}, r_{α_1}, l_{β_2}, r_{α_2}, l_{β_3}, r_{α_3}, $\ldots$, or r_{β_1}, l_{α_1}, r_{β_2}, l_{α_2}, r_{β_3}, l_{α_3}, $\ldots$, or l_{γ_1}, r_{γ_1} and l_{γ_2}, r_{γ_2}, l_{γ_3}, r_{γ_3}, $\ldots$, where l_{α_i}, l_{β_i}, l_{γ_i} denote the vertices in the left of a central vertex and at a level congruent to 0, 1, 2 under modulo 3 respectively, and, r_{α_i},r_{β_i} and r_{γ_i} denote the corresponding vertices in the right side of a central vertex of the path P_n. Let C_0 be the central vertex. Then C_0 can be joined to one of the first two sequences or the first sequence can be combined with second sequence through C_0. The third sequence covers only those vertices of P_n^3 which are at a level congruent to 2 under modulo

3, and, the first two sequences are disjoint. Hence to get a Hamilton path $x_1, x_2, \ldots, x_n$ to cover the entire vertex set of P_n^3, it should have at least a pair as in Possibility 3, (if the vertex C_0 combines first and second sequences) or at least two pairs that are not as in Possibility 1. Therefore, as the graph contains only one center vertex,

$$\eta \geq \frac{1}{3} \quad \text{and} \quad \xi \geq \frac{1}{3}$$

The above observation 2.10 will be visualized in the Figure 4.

In either of the cases, we claim that $\eta + \xi \geq \frac{5}{3}$

We note here that, if we take more than three edges amongst $G_{1,0}$, $G_{0,1}$ and $G_{2,2}$ in the level graphs shown in Figure 4, then $\xi \geq 4 \times \frac{1}{3}$, so the claim follows immediately as $\eta \geq \frac{1}{3}$.

Case 1: If $\eta = \frac{1}{3}$, then $l(x_1) = 0$, so the vertex C_0 is in either first sequence or in the second sequence (as mentioned in the Observation 2.10), but not in both. Hence at least two edges are required to get a Hamilton path. The minimum possible edges amongst $G_{1,0}$, $G_{0,1}$ and $G_{2,2}$) are discussed in the following cases.

Subcase 1.1: With two edges

The only possible two edges (in the sense of minimum ξ) are shown in Figure 5. Thus, $\xi \geq \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$. Hence the claim.

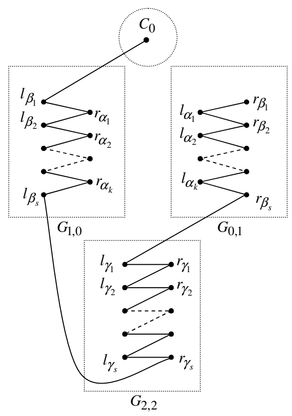

Figure 5: Level graph($n \equiv 3$ or $5 \pmod 6$).

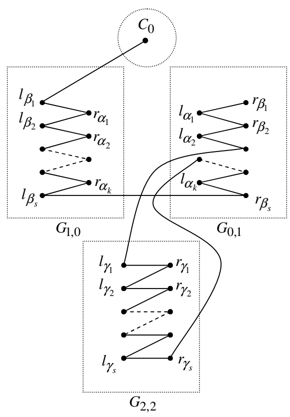

Figure 6: hamilton cycle($\eta = \frac{1}{3}$, $n \equiv 3,\ 5 (\mathrm{mod}\ 6)$).

Subcase 1.2: With three edges

The only possible three edges are shown in Figures 5 and 6. In each case, $\xi \geq \frac{4}{3}$. Hence the claim.

Case 2: If $\eta = \frac{2}{3}$, then either $l(x_1) = 0$ and $l(x_n) = 2$, or, $l(x_1) = 1$ and $l(x_n) = 1$. In the first case at least two edges are necessary, both these edges can not be as in Possibility 3 (because two such edges disconnect $G_{0,1}$ or disconnect $G_{2,2}$ or form a tree with at least three end vertices as shown in Figure 7. Similar fact holds true for the second case also (Follows easily from Figure 8.

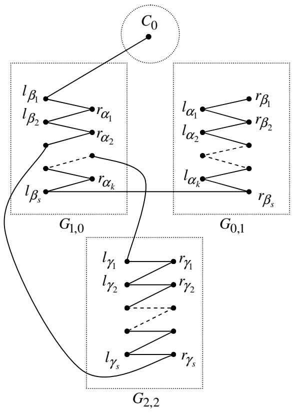

Figure 7: hamilton cycle for the case $\eta = \frac{1}{3}$ and $n \equiv 3$ or $5 (\text{mod } 6)$.

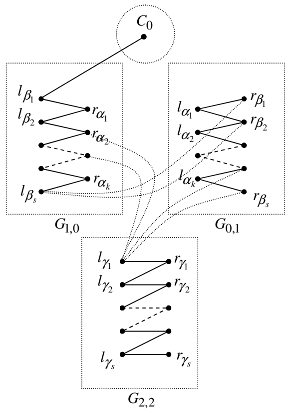

Figure 8: hamilton cycle for the case $\eta = \frac{2}{3}$ and $n \equiv 3$ or $5 (\text{mod } 6)$.

Hence $\xi \geq \frac{1}{3} + \frac{2}{3}$. Therefore,

$$\xi + \eta \geq \frac{5}{3} \quad for \quad n \equiv 3 \; or \; 5 \; (mod \; 3) \tag{16}$$

The case $n \equiv 1 (mod \; 3)$ follows similarly.

We now prove the necessary part of the Theorem 1.1.

Case 1: $n \equiv 0$ (mod 6) and $n \geq 6$

Substituting the minimum possible bound for $\eta + \xi = 1$ (as in equation (14)) $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-2n}{4}$ (follows by Observation 2.5) and $k = 1$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n}{3} - \frac{2}{3}\left(\frac{n^2-2n}{4}\right) + 1 + 1 \right\rceil$$

$$\Rightarrow f(x_n) \geq \left\lceil \frac{n^2}{6} + 2 \right\rceil = \frac{n^2}{6} + 2 \tag{17}$$

Hence $rn(P_n^3) \geq \frac{n^2+12}{6}$, whenever $n \equiv 0$ (mod 6) and $n \geq 6$.

Case 2: $n \equiv 1$ (mod 6) and $n \geq 13$

Substituting the minimum possible bound for $\eta + \xi = \frac{5}{3}$ (as in equation (14)), $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n-1}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-1}{4}$ and $k = \frac{2}{3}$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\left(\frac{n-1}{3}\right) - \frac{2}{3}\left(\frac{n^2-1}{4}\right) + 1 + \frac{1}{3}(n-1) + \frac{5}{3} \right\rceil$$

$$\Rightarrow f(x_n) \geq (n-1)\left(\frac{n-1}{3}\right) - \frac{2}{3}\left(\frac{n^2-1}{4}\right) + \frac{1}{3}(n-1) + 3 = \frac{n^2-2n+19}{6} \tag{18}$$

Hence $rn(P_n^3) \geq \frac{n^2-2n+19}{6}$, whenever $n \equiv 1 \pmod 6$ and $n \geq 13$.

Case 3: $n \equiv 2 \pmod 6$ and $n \geq 8$

Substituting the minimum possible bound for $\eta+\xi = 1$ (as in equation (14)), $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n+1}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-2n}{4}$ and $k = 1$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n+1}{3} - \frac{2}{3}\left(\frac{n^2-2n}{4}\right) + 1 + 1 \right\rceil$$

$$\Rightarrow f(x_n) \geq \left\lceil \frac{(n-2)^2}{6} + n + 1 \right\rceil = \frac{(n-2)^2}{6} + n + 1 = \frac{n^2+2n+10}{6} \tag{19}$$

Hence $rn(P_n^3) \geq \frac{n^2+2n+10}{6}$, whenever $n \equiv 2 \pmod 6$ and $n \geq 8$.

Case 4: $n \equiv 3 \pmod 6$ and $n \geq 9$

Substituting $\eta + \xi = \frac{5}{3}$, $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-1}{4}$ and $k = \frac{2}{3}$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n}{3} - \frac{2}{3}\left(\frac{n^2-1}{4}\right) + 1 + \frac{1}{3}(n-1) + \frac{5}{3} \right\rceil$$

$$\Rightarrow f(x_n) \geq \left\lceil \frac{(n-3)^2}{6} + n + 1 \right\rceil = \frac{(n-3)^2}{6} + n + 1 = \frac{n^2+15}{6} \tag{20}$$

Hence $rn(P_n^3) \geq \frac{n^2+15}{6}$, whenever $n \equiv 3 \pmod 6$ and $n \geq 9$.

Case 5: $n \equiv 4 \pmod 6$ and $n \geq 10$

Substituting the minimum possible bound for $\eta+\xi = \frac{4}{3}$ (as in equation (15)), $diam(P_n^3) = \lceil \frac{n-1}{3} \rceil = \frac{n-1}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-2n}{4}$ and $k = 1$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n-1}{3} - \frac{2}{3}\left(\frac{n^2-2n}{4}\right) + 1 + \frac{4}{3} \right\rceil$$

$$f(x_n) \geq \left\lceil \frac{(n-4)^2}{6} + n \right\rceil = \frac{(n-4)^2}{6} + n = \frac{n^2-2n+16}{6} \tag{21}$$

Hence $rn(P_n^3) \geq \frac{n^2-2n+16}{6}$, whenever $n \equiv 4 \pmod 6$ and $n \geq 10$.

Case 6: $n \equiv 5 \pmod 6$ and $n \geq 11$

Substituting $\eta + \xi = \frac{5}{3}$, $diam(G) = \lceil \frac{n-1}{3} \rceil = \frac{n+1}{3}$, $\sum_{i=1}^{n} l(x_i) = \frac{n^2-1}{4}$ and $k = \frac{2}{3}$ in the inequality (13), we get

$$f(x_n) \geq \left\lceil (n-1)\frac{n+1}{3} - \frac{2}{3}\left(\frac{n^2-1}{4}\right) + 1 + \frac{1}{3}(n-1) + \frac{5}{3} \right\rceil = \left\lceil \frac{n^2+2n+13}{6} \right\rceil$$

$$\Rightarrow f(x_n) \geq \left\lceil \frac{(n-5)^2}{6} + 2(n-1) \right\rceil = \frac{(n-5)^2}{6} + 2(n-1) = \frac{n^2+2n+13}{6} \tag{22}$$

Hence $rn(P_n^3) \geq \frac{n^2+n+13}{6}$, whenever $n \equiv 5 \pmod 6$ and $n \geq 11$.

§3. Upper Bound and Optimal Radio-Labelings

We now establish Theorem 1.1, it suffices to give radio-labelings that achieves the desired spans. Further, we will prove the following lemma similar to the Lemma 1.3 of Daphne Der-Fen Liu and Melanie Xie obtained in [7].

Lemma 3.1 *Let P_n^3 be a cube path on n ($n \geq 6$) vertices with $k = \lceil \frac{n-1}{3} \rceil$. Let $\{x_1, x_2, \ldots, x_n\}$ be a permutation of $V(P_n^3)$ such that for any $1 \leq i \leq n-2$,*

$$min\{d_{P_n}(x_i, x_{i+1}),\ d_{P_n}(x_{i+1}, x_{i+2})\} \leq 3\tfrac{k}{2} + 1$$

if k is even and the equality in the above holds, then the sum of the parity congruent to 0 under modulo 3. Let f be a function, $f : V(P_n^3) \to \{1, 2, 3, \ldots\}$ with $f(x_1) = 1$, and $f(x_{i+1}) - f(x_i) = k + 1 - d(x_i, x_{i+1})$ for all $1 \leq i \leq n-1$, where $d(x_i, x_{i+1}) = d_{P_n^3}(x_i, x_{i+1})$. Then f is a radio-labeling for P_n^3.

Proof Recall, $diam(P_n^3) = k$. Let f be a function satisfying the assumption. It suffices to prove that $f(x_j) - f(x_i) \geq k + 1 - d(x_i, x_j)$ for any $j \geq i + 2$. For $i = 1, 2, \ldots, n-1$, set

$$f_i = f(x_{i+1}) - f(x_i).$$

Since the difference in two consecutive labeling is at least one it follows that $f_i \geq 1$. Further, for any $j \geq i + 2$, it follows that

$$f(x_j) - f(x_i) = f_i + f_{i+1} + \cdots + f_{j-1}.$$

Suppose $j = i + 2$. Assume $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$. (The proof for $d(x_{i+1}, x_{i+2}) \geq d(x_i, x_{i+1})$ is similar.) Then, $d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$. Let $x_i = v_a$, $x_{i+1} = v_b$, and $x_{i+2} = v_c$. It suffices to consider the following cases.

Case 1: $b < a < c$ or $c < a < b$

Since $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$, we obtain $d(x_i, x_{i+1}) = d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2}$ and $d_{P_n}(x_i, x_{i+2}) \leq 2$ so, $d(x_i, x_{i+2}) = 1$. Hence,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= k + 1 - d(x_i, x_{i+1}) + k + 1 - d(x_{i+1}, x_{i+2}) \\ &\geq 2k + 2 - 2\left(\frac{k+2}{2}\right) \\ &= k + 1 - 1 \\ &= k + 1 - d(x_i, x_{i+2}) \end{aligned}$$

Case 2: $a < b < c$ or $c < b < a$

In this case, $d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) - 1$ and hence

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= k + 1 - d(x_i, x_{i+1}) + k + 1 - d(x_{i+1}, x_{i+2}) \\ &= 2k + 2 - \{d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2})\} \\ &= 2k + 2 - \{d(x_i, x_{i+2}) + 1\} \\ &= 2k + 1 - d(x_i, x_{i+2}) \\ &\geq k + 1 - d(x_i, x_{i+2}) \end{aligned}$$

Case 3: $a < c < b$ or $b < c < a$

Assume $min\{d_{P_n}(x_i, x_{i+1}),\ d_{P_n}(x_{i+1}, x_{i+2})\} < 3\frac{k}{2} + 1$, then we have $d(x_{i+1}, x_{i+2}) < \frac{k+2}{2}$ and by triangular inequality,

$$d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2})$$

Hence,

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= k + 1 - d(x_i, x_{i+1}) + k + 1 - d(x_{i+1}, x_{i+2}) \\ &= 2k + 2 - [d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2})] - 2d(x_{i+1}, x_{i+2}) \\ &\geq 2k + 2 - [d(x_i, x_{i+2})] - 2d(x_{i+1}, x_{i+2}) \\ &> 2k + 2 - d(x_i, x_{i+2}) - 2\left(\frac{k+2}{2}\right) \\ &= k - d(x_i, x_{i+2}) \end{aligned}$$

Therefore,

$$f(x_{i+2}) - f(x_i) \geq k + 1 - d(x_i, x_{i+2})$$

If $min\{d_{P_n}(x_i, x_{i+1}), d_{P_n}(x_{i+1}, x_{i+2})\} = 3\frac{k}{2} + 1$, then by our assumption, it must be that $d_{P_n}(x_{i+1}, x_{i+2}) = 3\frac{k}{2} + 1$ (so k is even) ,and, sum of $d_{P_n}(x_i, x_{i+1})$ and $d_{P_n}(x_{i+1}, x_{i+2})$ is congruent to 0 under modulo 3 implies that $d_{P_n}(x_i, x_{i+1}) \not\equiv 0 \pmod 3$. Hence, we have

$$d(x_i, x_{i+2}) = d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) + 1.$$

This implies

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= f_i + f_{i+1} \\ &= 2(k+1) - [d(x_i, x_{i+2})] - d(x_{i+1}, x_{i+2}) - d(x_{i+1}, x_{i+2}) + 1 \\ &\geq 2k + 2 - 2[d(x_{i+1}, x_{i+2})] - d(x_i, x_{i+2}) + 1 \\ &\geq 2k + 2 - 2\left(\frac{k+2}{2}\right) - d(x_i, x_{i+2}) + 1 \\ &= k + 1 - d(x_i, x_{i+2}) \end{aligned}$$

Let $j = i + 3$. First, we assume that the sum of some pairs of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, $d(x_{i+2}, x_{i+3})$ is at most $k + 2$. Then

$$d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + d(x_{i+2}, x_{i+3}) \leq (k+2) + k = 2k + 2$$

and hence,

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= 3k + 3 - d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) - d(x_{i+2}, x_{i+3}) \\ &\geq 3k + 3 - (2k+2) \\ &= k + 1 > k + 1 - d(x_i, x_{i+3}). \end{aligned}$$

Next, we assume that the sum of every pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$ and $d(x_{i+2}, x_{i+3})$ is greater than $k+2$. Then, by our hypotheses, it follows that

$$d(x_i, x_{i+1}), d(x_{i+2}, x_{i+3}) \geq \frac{k+2}{2} \quad \textit{and} \quad d(x_{i+1}, x_{i+2}) \leq \frac{k+2}{2} \tag{23}$$

Let $x_i = v_a$, $x_{i+1} = v_b$, $x_{i+2} = v_c$, $x_{i+3} = v_d$. Since $diam(P_n^3) = k$, by equation (23) and our assumption that the sum of any pair of the distances, $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, $d(x_{i+2}, x_{i+3})$, is greater than $k+2$, it must be that $a < c < b < d$ (or $d < b < c < a$). Then

$$d(x_i, x_{i+3}) \geq d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+3}) - d(x_{i+1}, x_{i+2}) - 1.$$

So,

$$\begin{aligned} d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + d(x_{i+2}, x_{i+3}) &\leq d(x_i, x_{i+3}) + d(x_{i+1}, x_{i+2}) + 1 \\ &\leq d(x_i, x_{i+3}) + \frac{k+2}{2} + 1 \\ &= d(x_i, x_{i+3}) + \frac{k}{2} + 2 \end{aligned}$$

By equation **??**, we have

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= 3k + 3 - d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}) - d(x_{i+2}, x_{i+3}) \\ &\geq 3k + 3 - 2 - \frac{k}{2} - d(x_i, x_{i+3}) \\ &= k + 1 - d(x_i, x_{i+3}). \end{aligned}$$

Let $j \geq i+4$. Since $min\{d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2})\} \leq \frac{k+2}{2}$, and $f_i \geq k + 1 - d(x_i, x_{i+1})$ for any i, we have $max\{f_i, f_{i+1}\} \geq \frac{k}{2}$ for any $1 \leq i \leq n-2$. Hence,

$$\begin{aligned} f(x_j) - f(x_i) &\geq f_i + f_{i+1} + f_{i+2} + f_{i+3} \\ &\geq \left\{1 + \frac{k}{2}\right\} + \left\{1 + \frac{k}{2}\right\} \\ &> k + 1 > k + 1 - d(x_i, x_j) \end{aligned}$$

□

To show the existence of a radio-labeling achieving the desired bound, we consider cases separately. For each radio-labeling f given in the following, we shall first define a permutation (line-up) of the vertices $V(P_n^3) = \{x_1, x_2, \ldots, x_n\}$, then define f by $f(x_1) = 1$ and for $i = 1, 2, \ldots, n-1$:

$$f(x_{i+1}) = f(x_i) + diam(P_n^3) + 1 - d_{P_n^3}(x_i, x_{i+1}). \tag{24}$$

For the case $n \equiv 0 \pmod 6$

Let $n = 6p$. Then $k = \left\lceil \frac{n-1}{3} \right\rceil = 2p \Rightarrow \frac{k}{2} = p$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+1}, x_2 = v_2, x_3 = v_{3p+3}, \ldots, x_n = v_{3p}$ as shown in the Table 1.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \leq i \leq n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 1 - d_{P_n^3}(x_i, x_{i+1}) \tag{25}$$

The function f defined in equation (25) chooses the vertices one from the right of the central vertex and other from the left for the consecutive labeling. The difference between two adjacent vertices in P_n is shown above the arrow. Since the minimum of any two consecutive distances is lesser than $3p+1$ and equal to $3p+1$ only if their sum is divisible by 3, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p-1]} v_2 \xrightarrow{[3p+1} v_{3p+3} \xrightarrow{3p-2} v_5 \xrightarrow{3p+1} v_{3p+6} \xrightarrow{3p-2} v_8$$

$$\xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1} v_{6p-3} \xrightarrow{3p-2]} v_{3p-1} \xrightarrow{[3p+1]} v_{6p} \xrightarrow{[6p-1]} v_1$$

$$\xrightarrow{[3p+1} v_{3p+2} \xrightarrow{3p-2} v_4 \xrightarrow{3p+1} v_{3p+5} \xrightarrow{3p-2} v_7 \xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1}$$

$$v_{6p-4} \xrightarrow{3p-2]} v_{3p-2} \xrightarrow{[3p+1]} v_{6p-1} \xrightarrow{[6p-4]} v_3 \xrightarrow{[3p+1} v_{3p+4} \xrightarrow{3p-2}$$

$$v_6 \xrightarrow{3p+1} v_{3p+7} \xrightarrow{3p-2} v_9 \xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p-2]} v_{3p} = x_n$$

Table 1: A radio-labeling procedure for the graph P_n^3 when $n \equiv 0 \pmod 6$

For the labeling f defined above we get

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left\lceil \frac{3p-1}{3} \right\rceil + \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-1) + \left\lceil \frac{3p+1}{3} \right\rceil + \\
&\quad \left\lceil \frac{6p-1}{3} \right\rceil + \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-1) + \left\lceil \frac{3p+1}{3} \right\rceil + \\
&\quad \left\lceil \frac{6p-4}{3} \right\rceil + \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-1) \\
&= p + (2p+1)(p-1) + (p+1) + 2p + (2p+1) + (p-1) + \\
&\quad (p+1) + (2p-1) + (2p+1)(p-1) \\
&= 6p^2 + 4p - 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x_n) &= (n-1)(diam P_2^3 + 1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\
&= (6p-1)(2p+1) - (6p^2 + 4p - 2) + 1 \\
&= 6p^2 + 2 = \frac{n^2}{2} + 2.
\end{aligned}$$

Hence,

$$rn(P_n^3) \leq \frac{n^2+12}{6}, \text{ if } n \equiv 0 \pmod 6$$

An example for this case is shown in Figure 9.

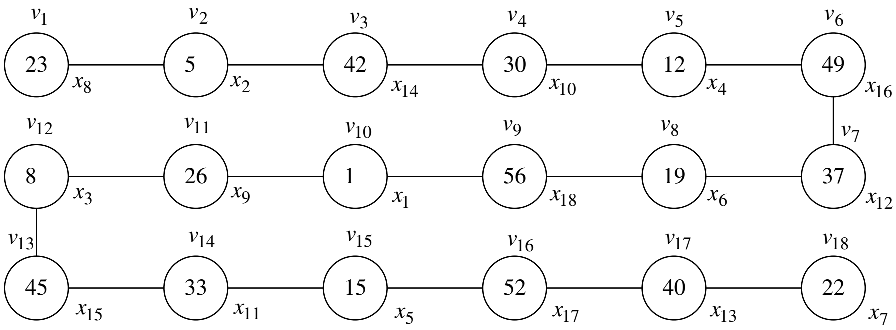

Figure 9: A minimal radio labeling of the graph P_{18}^3.

For the case $n \equiv 1$ (mod 6)

Let $n = 6p+1$. Then $k = \lceil \frac{n-1}{3} \rceil = 2p \Rightarrow \frac{k}{2} = p$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+1}, x_2 = v_3, x_3 = v_{3p+4}, \ldots, x_n = v_{6p-1}$ as shown in the Table 2.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \leq i \leq n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 1 - d_{P_n^3}(x_i, x_{i+1}). \tag{26}$$

For the function f defined in equation (26), The minimum difference between any two adjacent vertices in P_n is shown in Table 2 is less than $3p+1$ and equal to $3p+1$ only if their sum is divisible by 3, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p-2} v_3 \xrightarrow{3p+1} v_{3p+4} \xrightarrow{3p-2} v_6 \xrightarrow{3p+1} v_{3p+7} \xrightarrow{3p-2} v_9$$

$$\xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p-2} v_{3p} \xrightarrow{3p+1]} v_{6p+1} \xrightarrow{6p-1} v_2 \xrightarrow{[3p+1} v_{3p+3} \xrightarrow{3p-2}$$

$$v_5 \xrightarrow{3p+1} v_{3p+6} \xrightarrow{3p-2} \bullet\bullet\bullet \xrightarrow{3p-2} v_{3p-1} \xrightarrow{3p+1]} v_{6p} \xrightarrow{[6p-1]} v_1 \xrightarrow{[3p+1}$$

$$v_{3p+2} \xrightarrow{3p-2} v_4 \xrightarrow{3p+1} v_{3p+5} \xrightarrow{3p-2} \bullet\bullet\bullet \xrightarrow{3p-2]} v_{3p-2} \xrightarrow{3p+1} v_{6p-1} = x_n$$

Table 2: A radio-labeling procedure for the graph P_n^3 when $n \equiv 1 \pmod 6$

For the labeling f defined above we get

$$\begin{aligned}\sum_{i=1}^{n} d(x_i,x_{i+1}) &= \left\lceil\frac{3p-3}{3}\right\rceil+\left\lceil\frac{3p+1}{3}\right\rceil+\left(\left\lceil\frac{3p-2}{3}\right\rceil+\left\lceil\frac{3p+1}{3}\right\rceil\right)(p-1)+\left\lceil\frac{6p-4}{3}\right\rceil+\\ &\quad\left(\left\lceil\frac{3p+1}{3}\right\rceil+\left\lceil\frac{3p-2}{3}\right\rceil\right)(p-1)+\left\lceil\frac{3p+1}{3}\right\rceil+\left\lceil\frac{6p-1}{3}\right\rceil+\\ &\quad\left(\left\lceil\frac{3p+1}{3}\right\rceil+\left\lceil\frac{3p-2}{3}\right\rceil\right)(p-1)+\left\lceil\frac{3p+1}{3}\right\rceil+\left\lceil\frac{6p-1}{3}\right\rceil+\left\lceil\frac{3p+1}{3}\right\rceil\\ &= 2p+(2p+1)(p-2)+2p-1+(2p+1)(p-1)+3p+1+(2p+1)(p-1)+4p+2\\ &= 6p^2+6p-2.\end{aligned}$$

Therefore,

$$\begin{aligned}f(x_n) &= (n-1)(diamP_2^3+1)-\sum_{i=1}^{n} d(x_i,x_{i+1})+f(x_1)\\ &= (6p)(2p+1)-(6p^2+6p-2)+1\\ &= 6p^2+3=\frac{n^2-2n+19}{6}.\end{aligned}$$

Hence,

$$rn(P_n^3)\le \tfrac{n^2-2n+19}{6}, \text{ if } n\equiv 1 \pmod 6 \text{ and } n\ge 13$$

An example for this case is shown in Figure 10.

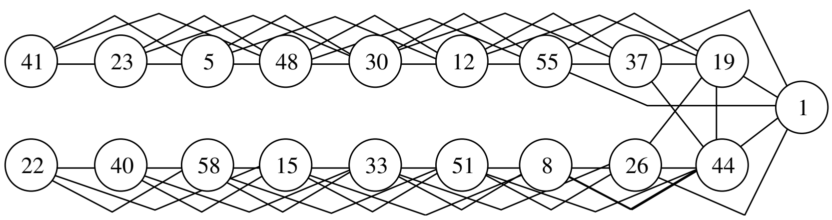

Figure 10: A minimal radio labeling of the graph P_{19}^3.

For the case $n\equiv 2 \pmod 6$

Let $n=6p+2$. Then $k=\left\lceil\frac{n-1}{3}\right\rceil=2p+1\Rightarrow 3p+2<3\frac{k}{2}+1$. Arrange the vertices of the graph P_n^3 as $x_1=v_{3p+1},x_2=v_{6p+2},x_3=v_{3p-2},\ldots,x_n=v_{3p+3}$ as shown in the Table 3.

Define a function f by $f(x_1)=1$ and for all $i, 1\le i\le n-1$,

$$f(x_{i+1})=f(x_i)+2p+2-d_{P_n^3}(x_i,x_{i+1}) \tag{27}$$

For the function f defined in equation (27), the minimum difference between any two adjacent vertices in P_n is shown in Table 3 is not greater than $3p+1$ and k is odd, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p+1} v_{6p+2} \xrightarrow{3p+4} v_{3p-2} \xrightarrow{3p+1} v_{6p-1} \xrightarrow{3p+4} v_{3p-5} \xrightarrow{3p+1}$$

$$v_{6p-4} \xrightarrow{3p+4} \bullet\bullet\bullet \xrightarrow{3p+1} v_{3p+5} \xrightarrow{3p+4]} v_1 \xrightarrow{[3p+1]} v_{3p+2} \xrightarrow{[2]} v_{3p}$$

$$\xrightarrow{[3p+1} v_{6p+1} \xrightarrow{3p+4} v_{3p-3} \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p+4} v_{3p-6} \xrightarrow{3p+1} v_{6p-5}$$

$$\xrightarrow{3p+4} v_{3p-9} \bullet\bullet\bullet \xrightarrow{3p+4]} v_3 \xrightarrow{[3p+1]} v_{3p+4} \xrightarrow{[5]} v_{3p-1} \xrightarrow{[3p+1} v_{6p}$$

$$\xrightarrow{3p+4} v_{3p-4} \xrightarrow{3p+1} v_{6p-3} \xrightarrow{3p+4} v_{3p-7} \xrightarrow{3p+1} v_{6p-6} \bullet\bullet\bullet \xrightarrow{3p+4} v_5$$

$$\xrightarrow{3p+1} v_{3p+6} \xrightarrow{3p+4]} v_2 \xrightarrow{3p+1} v_{3p+3} = x_n$$

Table 3: A radio-labeling procedure for the graph P_n^3 when $n \equiv 2 \pmod 6$

For the labeling f defined above we get

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left(\left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{3p+4}{3}\right\rceil\right)(p) + \left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{2}{3}\right\rceil + \\
&\quad \left(\left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{3p+4}{3}\right\rceil\right)(p-1) + \left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{5}{3}\right\rceil + \\
&\quad \left(\left\lceil\frac{3p+1}{3}\right\rceil + \left\lceil\frac{3p+4}{3}\right\rceil\right)(p-1) + \left\lceil\frac{3p+1}{3}\right\rceil \\
&= (2p+3)(p) + (p+1) + 1 + (2p+3)(p-1) + (p+1) + \\
&\quad 2 + (2p+3)(p-1) + (p+1) \\
&= 6p^2 + 8p.
\end{aligned}$$

Therefore,

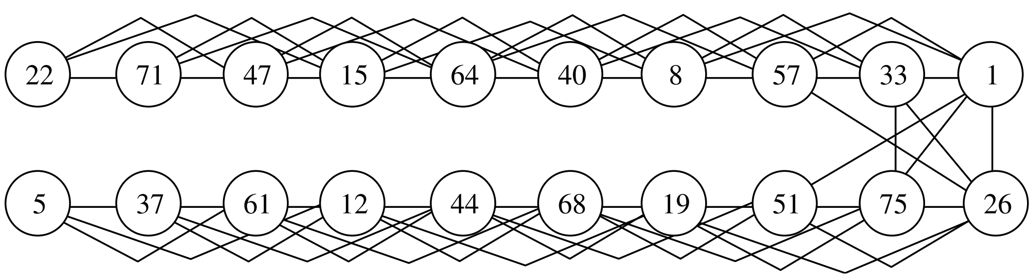

Figure 11: A minimal radio labeling of the graph P_{20}^3.

$$
\begin{aligned}
f(x_n) &= (n-1)(diam P_2^3 + 1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\
&= (6p+1)(2p+2) - (6p^2+8p) + 1 \\
&= 6p^2 + p + 3 = \frac{n^2+2n+10}{6}.
\end{aligned}
$$

Hence,

$$rn(P_n^3) \leq \tfrac{n^2+2n+10}{6}., \text{ if } n \equiv 2 \pmod 6$$

An example for this case is shown in Figure 11.

For the case $n \equiv 3 \pmod 6$

Let $n = 6p+3$. Then $k = \lceil \frac{n-1}{3} \rceil = 2p+1 \Rightarrow 3p+2 < 3\frac{k}{2}+1$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+1}, x_2 = v_{6p+2}, x_3 = v_1, \ldots, x_n = v_{3p+4}$ as shown in the Table 4.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \leq i \leq n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 2 - d_{P_n^3}(x_i, x_{i+1}) \tag{28}$$

For the function f defined in equation (28), the minimum difference between any two adjacent vertices in P_n is shown in Table 4 is not greater than $3p+1$ and k is odd, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p+1]} v_{6p+2} \xrightarrow{[6p+1]} v_1 \xrightarrow{[3p+1]} v_{3p+2} \xrightarrow{[3p+1]} v_{6p+3} \xrightarrow{[6p+1]}$$

$$v_2 \xrightarrow{[3p+1} v_{3p+3} \xrightarrow{3p-2} v_5 \xrightarrow{3p+1} v_{3p+6} \xrightarrow{3p-2} v_8 \xrightarrow{3p+1} v_{3p+9} \bullet\bullet\bullet$$

$$v_{6p-3} \xrightarrow{3p-2]} v_{3p-1} \xrightarrow{[3p+1]} v_{6p} \xrightarrow{[6p-4]} v_4 \xrightarrow{[3p+1} v_{3p+5} \xrightarrow{3p-2} v_7$$

$$\xrightarrow{3p+1} v_{3p+8} \xrightarrow{3p-2} \bullet\bullet\bullet\, v_{6p-4} \xrightarrow{3p-2]} v_{3p-2} \xrightarrow{[3p+1]} v_{6p-1} \xrightarrow{[3p-1]}$$

$$v_{3p} \xrightarrow{[3p+1} v_{6p+1} \xrightarrow{3p+4} v_{3p-3} \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p+4} v_{3p-6} \xrightarrow{3p+1}$$

$$v_{6p-5} \bullet\bullet\bullet\, v_{3p+7} \xrightarrow{3p+4]} v_3 \xrightarrow{[3p+1]} v_{3p+4} = x_n$$

Table 4: A radio-labeling procedure for the graph P_n^3 when $n \equiv 3 \pmod 6$.

For the labeling f defined above we get

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{6p+1}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{6p+1}{3} \right\rceil + \\
&\quad \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-1) + \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{6p-4}{3} \right\rceil + \\
&\quad \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-2}{3} \right\rceil \right)(p-2) + \left\lceil \frac{3p+1}{3} \right\rceil + \\
&\quad \left\lceil \frac{3p-1}{3} \right\rceil + \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p-4}{3} \right\rceil \right)(p-1) + \left\lceil \frac{3p+1}{3} \right\rceil \\
&= (p+1) + (2p+1) + (p+1) + (p+1)(2p+1) + \\
&\quad (2p+1)(p-1) + (p+1) + (2p-1) + \\
&\quad (2p+1)(p-2) + (p+1) + p + (2p+3)(p-1) + (p+1) \\
&= 6p^2 + 10p + 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x_n) &= (n-1)(diam P_2^3 + 1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\
&= (6p+2)(2p+2) - (6p^2 + 10p + 1) + 1 \\
&= 6p^2 + 6p + 4 = \frac{n^2 + 15}{6}.
\end{aligned}$$

Hence,

$$rn(P_n^3) \leq \tfrac{n^2+15}{6}, \text{ if } n \equiv 3 \pmod 6$$

An example for this case is shown in Figure 12.

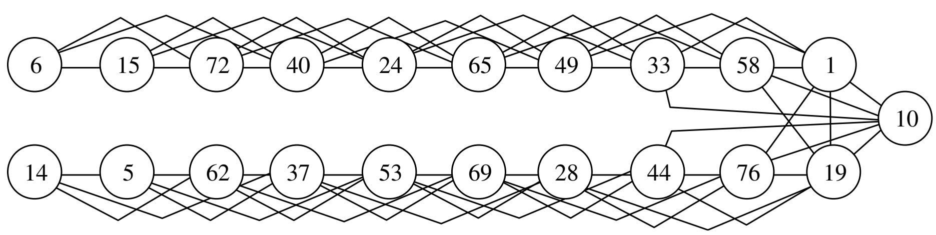

Figure 12: A minimal radio labeling of the graph P_{21}^3

For the case $n \equiv 4$ (mod 6)

Let $n = 6p + 4$. Then $k = \lceil \frac{n-1}{3} \rceil = 2p + 1 \Rightarrow 3p + 2 \leq 3\frac{k}{2} + 1$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+1}, x_2 = v_{6p+2}, x_3 = v_{3p-2}, \ldots, x_n = v_{6p+4}$ as shown in the Table 5.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \leq i \leq n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 2 - d_{P_n^3}(x_i, x_{i+1}). \tag{29}$$

For the function f defined in equation (29), the minimum difference between any two adjacent vertices in P_n is shown in Table 5 is not greater than $3p+4$ and k is odd, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+1} \xrightarrow{[3p+1]} v_{6p+2} \xrightarrow{[3p+4]} v_{3p-2} \xrightarrow{[3p+1} v_{6p-1} \xrightarrow{3p+4} v_{3p-5}$$

$$\xrightarrow{3p+1} v_{6p-4} \xrightarrow{3p+4} v_{3p-8} \xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1} v_{3p+5} \xrightarrow{3p+4]} v_1$$

$$\xrightarrow{[3p+2]} v_{3p+3} \xrightarrow{[3p+1} v_2 \xrightarrow{3p+4} v_{3p+6} \xrightarrow{3p+1} v_5 \xrightarrow{3p+4} \bullet\bullet\bullet$$

$$\xrightarrow{3p+4} v_{6p} \xrightarrow{3p+1} v_{3p-1} \xrightarrow{3p+1]} v_{6p+3} \xrightarrow{[3p+1]} v_{3p+2} \xrightarrow{[3p+2]} v_{6p+4}$$

$$\xrightarrow{[3p+4} v_{3p} \xrightarrow{3p+1} v_{6p+1} \xrightarrow{3p+4} v_{3p-3} \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p+4} v_{3p-6}$$

$$\xrightarrow{3p+1} \bullet\bullet\bullet\ v_{3p+7} \xrightarrow{3p+4} v_3 \xrightarrow{3p+1]} v_{3p+4} = x_n$$

Table 5: A radio-labeling procedure for the graph P_n^3 when $n \equiv 4 \pmod 6$.

For the labeling f defined above we get

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+4}{3} \right\rceil + \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+4}{3} \right\rceil \right)(p-1) + \\
&\quad \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+4}{3} \right\rceil \right)(p) + \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+2}{3} \right\rceil + \\
&\quad \left(\left\lceil \frac{3p+4}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil \right)(p) \\
&= (p+1)+(p+2)+(2p+3)(p-1)+(p+1)+(2p+3)p+ \\
&\quad (p+1)+(p+1)+(2p+3)p = 6p^2+12p+3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x_n) &= (n-1)(diam P_2^3 + 1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\
&= (6p+3)(2p+2) - (6p^2+12p+3) + 1 \\
&= 6p^2+6p+4 = \frac{n^2-2n+16}{6}.
\end{aligned}$$

Hence,

$$rn(P_n^3) \leq \tfrac{n^2-2n+16}{6}, \text{ if } n \equiv 4 \pmod 6$$

An example for this case is shown in Figure 13.

For the case $n \equiv 5 \pmod 6$

Let $n = 6p+5$. Then $k = \lceil \frac{n-1}{3} \rceil = 2p+2 \Rightarrow 3p+4 \leq 3\frac{k}{2}+1$. Arrange the vertices of the graph P_n^3 as $x_1 = v_{3p+3}, x_2 = v_2, x_3 = v_{3p+6}, \ldots, x_n = v_{3p+4}$ as shown in the Table 6.

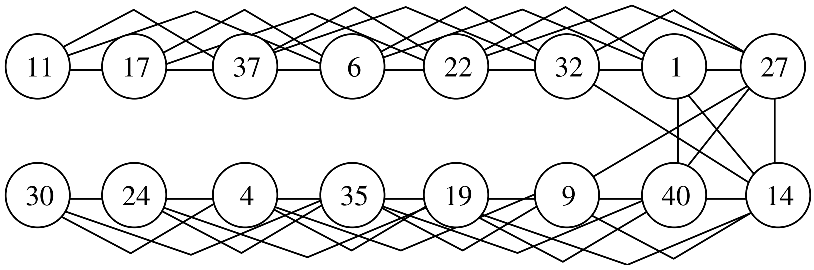

Figure 13: A minimal radio labeling of the graph P_{16}^3.

Define a function f by $f(x_1) = 1$ and for all $i, 1 \leq i \leq n-1$,

$$f(x_{i+1}) = f(x_i) + 2p + 3 - d_{P_n^3}(x_i, x_{i+1}) \tag{30}$$

For the function f defined in equation (30), the maximum difference between any two adjacent vertices in P_n is shown in Table 6 is less than or equal to $3p+4$ and the equality holds only if their sum is divisible by 3, by Lemma 3.1, it follows that f is a radio labeling.

$$x_1 = v_{3p+3} \xrightarrow{[3p+1} v_2 \xrightarrow{3p+4} v_{3p+6} \xrightarrow{3p+1} v_5 \xrightarrow{3p+4} v_{3p+9} \xrightarrow{3p+1} \bullet\bullet\bullet$$

$$\xrightarrow{3p+1} v_{3p-1} \xrightarrow{3p+4]} v_{6p+3} \xrightarrow{[3p+1]} v_{3p+2} \xrightarrow{[3p+3]} v_{6p+5} \xrightarrow{[3p+4} v_{3p+1}$$

$$\xrightarrow{3p+1} v_{6p+2} \xrightarrow{3p+4} v_{3p-2} \xrightarrow{3p+1} v_{6p-1} \xrightarrow{3p+4} \bullet\bullet\bullet \xrightarrow{3p+4} v_4$$

$$\xrightarrow{3p+1]} v_{3p+5} \xrightarrow{[3p+4]} v_1 \xrightarrow{[6p+3]} v_{6p+4} \xrightarrow{[3p+4} v_{3p} \xrightarrow{3p+1} v_{6p+1}$$

$$\xrightarrow{3p+4} v_{3p-3} \xrightarrow{3p+1} v_{6p-2} \xrightarrow{3p+4} v_{3p-6} \xrightarrow{3p+1} \bullet\bullet\bullet \xrightarrow{3p+1} v_{3p+7}$$

$$\xrightarrow{3p+4} v_3 \xrightarrow{3p+1]} v_{3p+4} = x_n$$

Table 6: A radio-labeling procedure for the graph P_n^3 when $n \equiv 5 \pmod 6$

For the labeling f defined above we get

$$\begin{aligned}
\sum_{i=1}^{n} d(x_i, x_{i+1}) &= \left(\left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+4}{3} \right\rceil\right)(p) + \left\lceil \frac{3p+1}{3} \right\rceil + \left\lceil \frac{3p+3}{3} \right\rceil + \\
&\quad \left(\left\lceil \frac{3p+4}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil\right)(p) + \left\lceil \frac{3p+4}{3} \right\rceil + \left\lceil \frac{6p+3}{3} \right\rceil + \\
&\quad \left(\left\lceil \frac{3p+4}{3} \right\rceil + \left\lceil \frac{3p+1}{3} \right\rceil\right)(p) \\
&= (2p+3)p + (p+1) + (p+1) + (2p+1)p + (p+2) + \\
&\quad (2p+1) + (2p+3)(p) = 6p^2 + 14p + 5.
\end{aligned}$$

Therefore,

$$\begin{aligned} f(x_n) &= (n-1)(diamP_2^3+1) - \sum_{i=1}^{n} d(x_i, x_{i+1}) + f(x_1) \\ &= (6p+4)(2p+3) - (6p^2+14p+5) + 1 \\ &= 6p^2+12p+8 = \frac{n^2+2n+13}{6}. \end{aligned}$$

Hence,

$$rn(P_n^3) \leq \tfrac{n^2+2n+13}{6}, \text{ if } n \equiv 5 \pmod 6$$

An example for this case is shown in Figure 14.

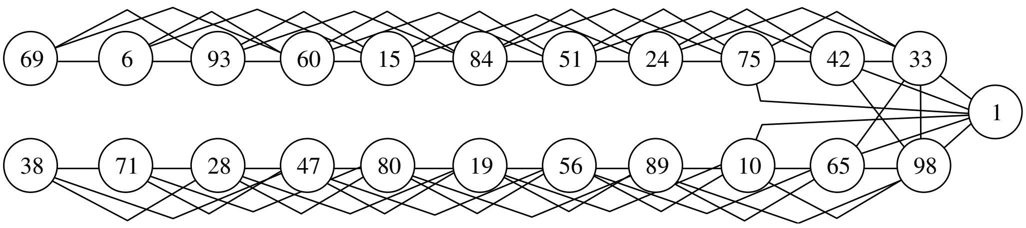

Figure 14: A minimal radio labeling of the graph P_{23}^3.

§4. Radio labeling of P_n^3 for $n \leq 5$ or $n = 7$

In this section we determine radio numbers of cube path of small order as a special case.

Theorem 4.1 *For any integer n, $1 \leq n \leq 5$, the radio number of the graph P_n^3 is given by*

$$rn(P_n^2) = \begin{cases} n, & if\ n = 1,2,3,4 \\ 8, & if\ n = 5,7 \end{cases}$$

Proof If $n \leq 4$, the graph is isomorphic to K_n and hence the result follows immediately. Now consider the case $n = 5$, we see that there is exactly one pair of vertices at a distance 2 and all other pairs are adjacent, so maximum value of $\sum_1^4 d(x_i, x_{i+1}) = 2+1+1+1 = 5$.

Now, consider a radio labeling f of P_5^3 and label the vertices as x_1, x_2, x_3, x_4, x_5 such that $f(x_i) < f(x_{i+1})$, then

$$\begin{aligned} f(x_n) - f(x_1) &\geq (n-1)(diamP_5^3+1) - \sum_1^4 d(x_i, x_{i+1}) \\ &\geq 4(3) - 5 = 7 \\ \Rightarrow f(x_n) &\geq 7 + f(x_1) = 8 \end{aligned}$$

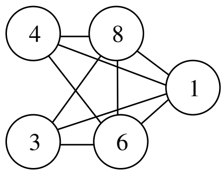

Figure 15: A minimal radio labeling of P_5^3.

includegraphics[width=8cm]figlast2.eps

Figure 16: A minimal radio labeling of P_7^3

On the other hand, In the Figure 15, we verify that the labels assigned for the vertices serve as a radio labeling with span 8, so $rn(P_n^3) = 8$.

, similarly if $n = 7$, then, as the central vertex of P_n^3 is adjacent to every other vertex, maximum value of $\sum^6 i = 1d(x_i, x_{i+1}) = 2 \times 5 + 1 = 11$. So, as above, $f(x_n) \geq (6)(2+1) - 11 + 1 = 8$. The reverse inequality follows by the Figure 16. Hence the theorem. $\square$

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Biprimitive Semisymmetric Graphs on $PSL(2,p)$

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Abstract: A simple undirected graph is said to be *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. It is easy to see that every semisymmetric graph is necessarily bipartite, with the two parts having equal size and the automorphism group acting transitively on each of these two parts. A semisymmetric graph is called *biprimitive* if its automorphism group acts primitively on each part. This paper gives a classification of biprimitive semisymmetric graphs arising from the action of the group $PSL(2,p)$ on cosets of A_5, where $p \equiv 1 \,(\mathrm{mod}10)$ is a prime. By the way, the structure of the suborbits of $PGL(2,p)$ on the cosets of A_5 is determined.

Keywords: Smarandache multi-group, group, semisymmetric graph, Biprimitive semisymmetric graph, suborbit.

AMS(2010): 05C25, 20B25

§1. Introduction

For the group- and graph-theoretic terminology we refer the reader to [1,7]. All graphs considered in this paper are finite, undirected and simple. For a graph X, we use $V(X)$, $E(X)$, $A(X)$ and $Aut(X)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. If X be a bipartite with bipartition $V(X) = U(X) \cup W(X)$. Set

$$A^+ = \langle g \in A \mid U(X)^g = U(X), W(X)^g = W(X) \rangle .$$

Clearly, if X is connected then either $|A : A^+| = 2$ or $A = A^+$, depending on whether or not there exists an automorphism which interchanges the two parts $U(X)$ and $W(X)$. Suppose G is a subgroup of A^+. Then X is said to be *G-semitransitive* if G acts transitively on both $U(X)$ and $W(X)$, and *semitransitive* if X is A^+-semitransitive. Also X is said to be *biprimitive* if A^+ acts primitively on each part. We call a graph *semisymmetric* if it is regular and edge-transitive but not vertex-transitive. It is easy to see that every semisymmetric graph is a bipartite graph with two parts of equal size and is semitransitive.

The first person who studied semisymmetric graphs was Folkman. In 1967 he constructed

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several infinite families of such graphs and proposed eight open problems (see [6]). Afterwards, Bouwer, Titov, Klin, A.V. Ivanov, A.A. Ivanov and others did much work on semisymmetric graphs (see [2-3,8-10,13]). They gave new constructions of such graphs and nearly solved all of Folkman's open problems. In particular, Iofinova and Ivanov [9] in 1985 classified biprimitive semisymmetric cubic graphs using group-theoretical methods; this was the first classification theorem for such graphs. More recently, following some deep results in group theory which depend on the classification of finite simple groups, some new methods and results in vertex-transitive graphs and semisymmetric graphs have appeared. In [5], for example, the authors give a classification of semisymmetric graphs of order $2pq$ where p and q are distinct primes. It is shown that there are 131 examples of such graphs, which are biprimitive. In [4] a classification is given, of biprimitive semisymmetric graphs arising from the action of the group $PSL(2,p)$, $p \equiv 1\,(\mathrm{mod} 8)$ a prime, on cosets of S_4. In this paper, we will classify all biprimitve graphs arising from the action of the group $PSL(2,p)$, $p \equiv 1\,(\mathrm{mod} 10)$ a prime, on cosets of A_5. To prove the classification theorem, we have to determine the suborbits of $PGL(2,p)$ acts on the cosets of A_5 and such a determination will certainly be useful for other problems.

Throughout the paper, Z_n and D_n denote the cyclic group of order n and the dihedral group of order n, respectively. A semidirect product of the group N by the group H will be denoted by $N : H$. Given a group G and a subgroup H of G, we use $[G : H]$ to denote the set of right cosets of H in G. The action of G on $[G : H]$ is always assumed to be the right multiplication action. More precisely, for $g \in G$, we use $R(g)$ to denote the effect of right multiplication of g on $[G : H]$ and let $R(G) = \{R(g)|g \in G\}$. However, for convenience, in most cases we will identify $R(g)$ with g, except for the special cases to be stated.

A Smarandache multi-group $\mathscr{G}$ is an union of groups $(G_1; \circ_1), (G_2; \circ_2), \cdots, (G_n; \circ_n)$, different two by two for an integer $n \geq 1$. Particularly, if $n = 1$, then $\mathscr{G}$ is just a group. A Smarandache multi-group $\mathscr{G}$ is naturally acting on its underlying graph $G[\mathscr{G}]$. In [5], the authors gave a group-theoretic construction of semitransitive graphs by introducing the definition of so called *bi-coset graph* as following: Let G be a group, let L and R be subgroups of G and let D be a union of double cosets of R and L in G, namely, $D = \cup_i Rd_iL$. Define a bipartite graph $X = \mathbf{B}(G, L, R; D)$ with bipartition $V(X) = [G : L] \cup [G : R]$ and edge set $E(X) = \{\{Lg, Rdg\}|g \in G, d \in D\}$. This graph is called the bi-coset graph of G with respect to L, R and D.

Note that in the above construction of semitransitive graphs, if L and R are the same subgroup, then we still use Lg and Rg to denote different vertices in the two parts of $V(X)$. It is proved in [5] that (1) the graph $X = \mathbf{B}(G, L, R; D)$ is a well-defined bipartite graph, and under the right multiplication action on $V(X)$ of G, the graph X is G-semitransitive; (2) every G-semitranstive graph is isomorphic to one of such bi-coset graphs.

Now we state the main theorem of this paper.

Theorem 1.1 *Let $p \equiv 1(\mathrm{mod}10)$, $G = PSL(2,p)$ and $Q = PGL(2,p)$. Let Y be a biprimitive semisymmetric graph with a subgroup G of $\mathrm{Aut}(Y)$ acting edge-transitively on Y and having A_5 as a vertex stabilizer. Then Y is isomorphic to one of the following graphs:*

(i) *$B(G, L, L; D)$, where $L \cong A_5$ and D is a double coset corresponds to a non-self-paired* *suborbit of G relative to L.*

(ii) $B(G, L, L^\sigma; D)$, *where σ is an involution in $Q \setminus G$ and $L\sigma dL$ corresponds to a non-self-paired suborbit of Q relative to L.*

Moreover, each such graph Y is of order $\frac{p^3-p}{60}$ and valency 60, and with the automorphism group $PSL(2,p)$. Table 1 lists the total numbers n_1 and n_2 of nonisomorphic semisymmetric graphs $B(G, L, L; D)$ and $B(G, L, L^\sigma, D)$ for each of the congruence classes of p.

TABLE 1.

$p(\text{mod}120)$	n_1	n_2
1	$\frac{p^3-60p^2+1077p-15418}{14400}$	$\frac{p^3-60p^2+1197p-1138}{14400}$
-1	$\frac{p^3-60p^2+1197p-13142}{14400}$	$\frac{p^3-60p^2+1077p+1138}{14400}$
11	$\frac{p^3-60p^2+1197p-7238}{14400}$	$\frac{p^3-60p^2+1077p-5918}{14400}$
-11	$\frac{p^3-60p^2+1077p-8362}{14400}$	$\frac{p^3-60p^2+1197p-7024}{14400}$
31	$\frac{p^3-60p^2+1197p-9238}{14400}$	$\frac{p^3-60p^2+1077p-5518}{14400}$
-31	$\frac{p^3-60p^2+1077p-8762}{14400}$	$\frac{p^3-60p^2+1197p-5042}{14400}$
41	$\frac{p^3-60p^2+1077p-12218}{14400}$	$\frac{p^3-60p^2+1197p-2738}{14400}$
-41	$\frac{p^3-60p^2+1197p-11542}{14400}$	$\frac{p^3-60p^2+1077p-2062}{14400}$
61	$\frac{p^3-60p^2+1077p-11818}{14400}$	$\frac{p^3-60p^2+1197p-4738}{14400}$
-61	$\frac{p^3-60p^2+1197p-9542}{14400}$	$\frac{p^3-60p^2+1077p-2462}{14400}$
71	$\frac{p^3-60p^2+1197p-10838}{14400}$	$\frac{p^3-60p^2+1077p-2318}{14400}$
-71	$\frac{p^3-60p^2+1077p-11962}{14400}$	$\frac{p^3-60p^2+1197p-3442}{14400}$
91	$\frac{p^3-60p^2+1197p-5638}{14400}$	$\frac{p^3-60p^2+1077p-9118}{14400}$
-91	$\frac{p^3-60p^2+1077p-5162}{14400}$	$\frac{p^3-60p^2+1197p-8642}{14400}$
101	$\frac{p^3-60p^2+1077p-8618}{14400}$	$\frac{p^3-60p^2+1197p-6338}{14400}$
-101	$\frac{p^3-60p^2+1197p-7942}{14400}$	$\frac{p^3-60p^2+1077p-5662}{14400}$

§2. Preliminaries

In this section, some preliminary results are given. The first two propositions give some properties of the groups $PSL(2,p)$ and $PGL(2,p)$.

Proposition 2.1 ([11], Lemma 2.1) *Let p be an odd prime. Then*

(1) *the maximal subgroups of $PSL(2,p)$ are:*

One class of subgroups isomorphic to $Z_p : Z_{\frac{p-1}{2}}$; one class isomorphic to D_{p-1}, when $p \geqslant 13$; *one class isomorphic to D_{p+1}, when $p \neq 7$; two classes isomorphic to A_5, when* $p \equiv 1(\text{mod}10)$; *two classes isomorphic to S_4, when* $p \equiv 1(\text{mod}8)$; *and one class isomorphic to A_4, when* $p = 5$ *or* $p \not\equiv 1(\text{mod}8)$.

(2) *The maximal subgroups of $PGL(2,p)$ are:*

One class of subgroups isomorphic to $Z_p : Z_{p-1}$; one class isomorphic to $D_{2(p-1)}$, when

$p \geqslant 7$; one class isomorphic to $D_{2(p+1)}$; one class isomorphic to S_4, when $p = 5$ or $p \not\equiv 1(\mathrm{mod}40)$ and $p \geqslant 5$; and one subgroup $PSL(2,p)$.

Proposition 2.2 ([5], Lemma 3.9) *Any extension of $PSL(2,p)$ by Z_2 is isomorphic to either $PGL(2,p)$ or $PSL(2,p) \times Z_2$. In both cases the extension is split.*

Proposition 2.3 ([5], Lemma 2.3) *The graph $X = \mathbf{B}(G,L,R;D)$ is a well-defined bipartite graph. Under the right multiplication action on $V(X)$ of G, the graph X is G-semitransitive. The kernel of the action of G on $V(X)$ is $\mathrm{Core}_G(L) \cap \mathrm{Core}_G(R)$, the intersection of the cores of the subgroups L and R in G. Furthermore, we have*

(i) X is G-edge-transitive if and only if $D = RdL$ for some $d \in G$;

(ii) the degree of any vertex in $[G:L]$ (resp. $[G:R]$) is equal to the number of right cosets of R (resp. L) in D (resp. D^{-1}), so X is regular if and only if $|L| = |R|$;

(iii) X is connected if and only if G is generated by elements of $D^{-1}D$;

(iv) $X \cong \mathbf{B}(G, L^a, R^b; D')$ where $D' = \bigcup_i R^b\ (b^{-1}d_ia)L^a$, for any $a, b \in G$.

The next proposition provides one general and three particular conditions, each of which is sufficient for a G-semitransitive graph to be vertex-transitive.

Proposition 2.4 ([5], Lemma 2.6) *Let $X = \mathbf{B}(G,L,R;D)$. If there exists an involutory automorphism σ of G such that $L^\sigma = R$ and $D^\sigma = D^{-1}$, then X is vertex-transitive. In particular,*

(i) *If G is abelian and acts regularly on both parts of X, then X is vertex-transitive. In other words, bi-Cayley graphs of abelian groups are vertex-transitive.*

(ii) *If there exists an involutory automorphism σ of G such that $L^\sigma = R$, and the lengths of the orbits of L on $[G:R]$ (or the orbits of R on $[G:L]$) are all distinct, then X is vertex-transitive.*

(iii) *If the representations of G on the two parts of X are equivalent and all suborbits of G relative to L are self-paired, then X is vertex-transitive.*

The link between groups and graphs that we use is the concept of the orbital graph of a permutation group. For the terminology of orbital graph we refer the reader to [12].

The following group theoretical results will be used later.

Proposition 2.5 ([11], Lemma 2.1) *Let G be a transitive group on Ω and let $H = G_\alpha$ for some $\alpha \in \Omega$. Suppose that $K \leqslant G$ and at least one G-conjugate of K is contained in H. Suppose further that the set of G-conjugates of K which are contained in H form t conjugacy classes of H with representatives $K_1, K_2, \cdots, K_t$. Then K fixes $\sum_{i=1}^t |N_G(K_i) : N_H(K_i)|$ points of Ω.*

Proposition 2.6 ([11], Lemma 2.2) *Let G be a primitive permutation group on Ω, and let $H = G_\alpha$ for some $\alpha \in \Omega$. Suppose that $H = A_5$ and let $K_1, \ldots, K_7$ be seven subgroups of H satisfying $K_1 \cong A_4$, $K_2 \cong D_{10}$, $K_3 \cong D_6$, $K_4 \cong Z_5$, $K_5 \cong Z_3$, $K_6 \cong D_4$ and $K_7 \cong Z_2$. Let k_i be the number of points in Ω fixed by K_i, for $i = 1, 2, \ldots, 7$. Then G has 1 suborbit of length 1, $k_1 - 1$ suborbits of length 5, $k_2 - 1$ suborbits of length 6, $k_3 - 1$ suborbits of length 10, $\frac{1}{2}(k_4 - k_2)$ suborbits of length 12, $\frac{1}{2}(k_5 - 2k_1 - k_3 + 2)$ suborbits of length 20, $\frac{1}{3}(k_6 - k_1)$ suborbits of length 15, $\frac{1}{2}(k_7 - 2k_2 - 2k_3 - k_6 + 4)$ suborbits of length 30, and all the other suborbits have length 60.*

Proposition 2.7 ([11], Lemma 2.3) *Let $D = D_{2n}$ be the dihedral group of order $2n$, considered* *as a permutation group of degree n generated by $a = (1, 2, \cdots, n)$ and $b = (1)(2, n)(3, n-1)\cdots(i, n+2-i)\cdots$, for any $n \geqslant 2$. Then the nontrival orbitals of D are $\Gamma_i = (1, i)^D = (1, n+2-i)^D$, for $2 \leqslant i \leqslant (n+2)/2$. Each of these orbitals is self-paired. Moreover, for all points i, j, with $i \neq j$, there is an involution in D which interchanges i and j.*

Proposition 2.8 ([11], Lemma 2.4) *Let G be a transitive group on Ω and let $H = G_\alpha$ for some $\alpha \in \Omega$. Suppose that G has t conjugacy classes of involutions, say $\mathcal{C}_1, \cdots, \mathcal{C}_t$. Suppose further that a representative u_j in $\mathcal{C}_j$ has N_j cycles of length 2, and that the centralizer of u_j in G has order c_j. Also for a nontrivial self-paired suborbit Δ relative to α and a point $\mathbf{B} \in \Delta$, let $inv(\Delta)$ be the number of involutions in G with a 2-cycle $(\sigma, \mathbf{B})$. Then $\sum_{j=1}^{t} \frac{N_j}{c_j} = \frac{1}{2|H|} \sum_{\Delta = \Delta^*} |\Delta(\alpha)| inv(\Delta)$, where c_j is the order of the centralizer of u_j.*

§3. Proof of Theorem 1.1

Now we begin the proof of Theorem 1.1. From now on we shall assume that $G = PSL(2, p)$ and $Q = PGL(2, p)$, where $p \equiv 1\,(\mathrm{mod} 10)$. Clearly, $Q = G : \langle\sigma\rangle$ for some involution $\sigma \in Q \setminus G$. Let Y be a semisymmetric biprimitive graph with a subgroup G of $\mathrm{Aut}(Y)$ acting edge-transitively on Y and having A_5 as a vertex stabilizer. Let $U(Y)$ and $W(Y)$ be the bipartition of $V(Y)$. Then $|U(Y)| = |W(Y)| = \frac{p^3 - p}{120}$ and $G_v \cong A_5$ for any $v \in U(Y)$ and $v \in W(Y)$. Now Y is isomorphic to the bi-coset graph $X = B(G, L, R; D)$, where $L \cong R \cong A_5$. With our notation, $V(X) = U(X) \cup W(X) = [G : L] \cup [G : R]$. We will treat the following two cases separately:

(1) Suppose the representations of G on $U(X)$ and $W(X)$ are equivalent. In this case, by Proposition 2.3 (iv), no loss of any generality, we may assume $L = R \cong A_5$. With the completely similar arguments as in [5, Lemma 4.1], we may show that X is semisymmetric if and only if $D^{-1} \neq D$, that is, D corresponds to a non-self-paired suborbit of G relative to L, and two such bi-coset graphs defined (for the same group G) by distinct double cosets D_1 and D_2 are isomorphic if and only if D_1 and D_2 are paired with each other in G, or more precisely, $D_1 = D_2^{-1}$.

(2) Suppose the representations of G on $U(X)$ and $W(X)$ are inequivalent. Let $Q = PGL(2, p) = \langle G, \sigma\rangle$, where $\sigma \in Q \setminus G$ and $\sigma^2 = 1$. By the Proposition 2.1, G has two conjugacy classes of subgroups isomorphic to A_5, which are fused by σ. Therefore, we may let $R = L^\sigma$ so that $X = \mathbf{B}(G, L, L^\sigma; D)$ where $D = L^\sigma dL$ for some $d \in G$. With the similar arguments as in [5, Lemma 4.2], X is semisymmetric if and only if the suborbit $L\sigma dL$ of Q relative to L is not self-paired, and two such graphs $X_1 = \mathbf{B}(G, L, R; D_1)$ and $X_2 = \mathbf{B}(G, L, R; D_2)$ defined by distinct double cosets $D_1 := Rd_1L$ and $D_2 := Rd_2L$ respectively are isomorphic if and only if $D_1' := L\sigma d_1 L$ and $D_2' := L\sigma d_2 L$ are paired with each other in $Q = PGL(2, p)$.

Following the above two cases, we need to determine non-self-paired suborbits of G relative to L and non-self-paired suborbits of Q relative to L which are contained in $[Q : L] \setminus [G : L]$. Noting that the number of non-self-paired suborbits of G relative to L is the same as the number of non-self-paired suborbits of Q relative to L which are contained in $[G : L]$. From now on let $\Omega = [Q : L]$, $\Omega_1 = [G : L]$ and $\Omega_2 = [Q : L] \setminus [G : L]$. We will consider the action of Q on Ω and find all non-self-paired suborbits of Q contained in Ω_1 and in Ω_2 as well. We shall do this only for the case where $G = PSL(2, p)$ and $L = A_5$, $p \equiv 1(\mathrm{mod}\ \ 120)$, and for the other cases, similar arguments and computations lead to the data listed in Appendix: TABLE $4-1$-TABLE $4-4$.

Let K_i (for $1 \leqslant i \leqslant 7$) be the representatives of the seven conjugacy classes of nontrivial subgroups of L isomorphic to A_4, D_{10}, D_6, Z_5, Z_3, D_4 and Z_2, respectively, and let $K_8 = 1$. Clearly any nontrivial subgroup K of L with a fixed point on Ω must be conjugate to one of these K_i. For each $i \in \{1, \ldots, 8\}$, let k_i, k_{i1} and k_{i2} denote the respective numbers of fixed points of K_i in Ω, Ω_1 and Ω_2. Among of all the suborbits with the L-stabilizer K_i, let x_{i1} and x_{i2} denote the respective numbers of the suborbits contained in Ω_1 and Ω_2; let y_i, y_{i1}, $y_{i2} = y_i - y_{i1}$ denote the respective numbers of self-paired suborbits contained in Ω, Ω_1 and Ω_2; and let $h_{i1} = x_{i1} - y_{i1}$ and $h_{i2} = x_{i2} - y_{i2}$ denote the respective numbers of non-self-paired suborbits contained in Ω_1 and Ω_2.

First we determine the values of x_{i1} and x_{i2}. For $i \in \{1, \ldots, 7\}$, these values are given in TABLE 2 and are obtained in the following way. After having determined the respective normalizers of each K_i in L and in G (resp. Q), we apply Proposition 2.5 to calculate k_{i1} (resp. k_i). Then $k_{i2} = k_i - k_{i1}$ can be found also. By Proposition 2.6, we can determine the values of x_{i1} and x_{i2}, $1 \leq i \leq 7$.

TABLE 2.

i	1	2	3	4	5	6	7
K_i	A_4	D_{10}	D_6	Z_5	Z_3	D_4	Z_2
$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	A_4	D_4
$N_G(K_i)$	S_4	D_{20}	D_{12}	D_{p-1}	D_{p-1}	S_4	D_{p-1}
k_{i1}	2	2	2	$\frac{p-1}{10}$	$\frac{p-1}{6}$	2	$\frac{p-1}{4}$
x_{i1}	1	1	1	$\frac{p-1}{20}-1$	$\frac{p-1}{12}-2$	0	$\frac{p-1}{8}-3$
$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p-1)}$	S_4	$D_{2(p-1)}$
k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p-1}{3}$	2	$\frac{p-1}{2}$
k_{i2}	0	0	0	$\frac{p-1}{10}-1$	$\frac{p-1}{6}$	0	$\frac{p-1}{4}$
x_{i2}	0	0	0	$\frac{p-1}{20}$	$\frac{p-1}{12}$	0	$\frac{p-1}{8}$

Finally,

$$
\begin{aligned}
x_{81} &= \frac{1}{60}\left(\frac{p^3-p}{120} - 1 - \sum_{i=1}^{7} x_{i1}\frac{60}{|K_i|}\right) \\
&= \frac{1}{60}\left(\frac{p^3-p}{120} - 1 - 1\cdot 5 - 1\cdot 6 - 1\cdot 10 - \left(\frac{p-1}{20} - 1\right)\cdot 12\right. \\
&\left. - \left(\frac{p-1}{12} - 2\right)\cdot 20 - \left(\frac{p-1}{8} - 3\right)\cdot 30\right) \\
&= \frac{p^3 - 723p + 15122}{7200}
\end{aligned}
$$

and a similar computation gives

$$
x_{82} = \frac{1}{60}\left(\frac{p^3-p}{120} - 1 - \sum_{i=2}^{7} x_{i2}\frac{60}{|K_i|}\right)
$$

$$= \frac{1}{60}\left(\frac{p^3-p}{120} - 1 - \frac{p-1}{20}\cdot 12 - \frac{p-1}{12}\cdot 20 - \frac{p-1}{8}\cdot 30\right)$$
$$= \frac{p^3 - 723p + 722}{7200}.$$

Next we determine the values of h_{i1} and h_{i2}. We claim all the non-regular suborbits of Q are self-paired, so that $h_{i1} = h_{i2} = 0$ for $1 \leqslant i \leqslant 7$. For example, let $i = 7$ and let Δ be a suborbit with L-stabilizer $K_7 = Z_2$, and take $v \in \Delta$. We consider the action of $N_Q(K_7) \cong D_{2(p-1)}$ on $Fix(K_7)$, the set of fixed points of K_7 on Ω. This action is transitive and the kernel is Z_2. Since $|Fix(K_7)| = \frac{p-1}{2}$, by Proposition 2.7, there exists an element in $N_Q(K_7)$ interchanging $u = L$ and v. So Δ is self-paired, or equivalently, $h_{71} = h_{72} = 0$.

It remains to determine h_{81} and h_{82}, the numbers of non-self-paired suborbits of Q in Ω_1 and in Ω_2 respectively. For these it suffices to calculate y_{81} and y_8, the numbers of self-paired regular suborbits of Q in Ω_1 and in Ω, since $h_{81} = x_{81} - y_{81}$, $h_{82} = x_{82} - y_{82}$ and $y_8 = y_{81} + y_{82}$. By Proposition 2.8, in order to calculate y_{81} (resp. y_8), we need the value of $inv(\Delta)$, which is defined in Proposition 2.8 for all self-paired suborbits Δ of G (resp. Q). Furthermore, to calculate $inv(\Delta)$ we need to know G_{uv} and $G_{\{u,v\}}$ (resp. Q_{uv} and $Q_{\{u,v\}}$), where $u = L$ and $v \in \Delta$.

The lengths l_i ($1 \leqslant i \leqslant 8$) of self-paired suborbits with point stabilizer K_i, the numbers y_{i1} and y_i, the groups G_{uv}, $G_{\{u,v\}}$ and Q_{uv} and $Q_{\{u,v\}}$, and the value of $inv(\Delta)$ for each Δ are listed in the following table.

TABLE 3.

i	l_i	y_{i1}	y_i	$G_{uv} = Q_{uv}$	$G_{\{u,v\}} = Q_{\{u,v\}}$	$inv(\Delta)$
1	5	1	1	A_4	S_4	6
2	6	1	1	D_{10}	D_{20}	6
3	10	1	1	D_6	D_{12}	4
4	12	$\frac{p-1}{20} - 1$	$\frac{p-1}{10} - 1$	Z_5	D_{10}	5
5	20	$\frac{p-1}{12} - 2$	$\frac{p-1}{6} - 2$	Z_3	D_6	3
7	30	$\frac{p-1}{8} - 3$	$\frac{p-1}{4} - 3$	Z_2	D_4	2
8	60	y_{81}	y_8	1	Z_2	1

Next we shall calculate y_{81} and y_8 using Proposition 2.8. We know that Q has two conjugacy classes of involutions. A representative of the first class, say $u_1 \in G$, fixes $\frac{p-1}{2}$ points, and so u_1 contains $N_1 = \frac{\frac{p^3-p}{60} - \frac{p-1}{2}}{2} = \frac{p^3-31p+30}{120}$ cycles of length 2. Further, $C_Q(u_1) \cong D_{2(p-1)}$ has order $c_1 = 2(p-1)$. A representative of the second class, say $u_2 \in Q \setminus G$, has no fixed point and so u_2 contains $N_2 = \frac{p^3-p}{120}$ cycles of length 2. Also $C_Q(u_2) \cong D_{2(p+1)}$ has order $c_2 = 2(p+1)$. By Proposition 2.8 and TABLE 3, we have

$$\frac{p^3 - 31p + 30}{240(p-1)} + \frac{p^3-p}{240(p+1)} = \frac{1}{2\cdot 60}(1\cdot 5\cdot 6 + 1\cdot 6\cdot 6 + 1\cdot 10\cdot 4$$
$$+ \left(\frac{p-1}{10} - 1\right)\cdot 12\cdot 5 + \left(\frac{p-1}{6} - 2\right)\cdot 20\cdot 3 + \left(\frac{p-1}{4} - 3\right)\cdot 30\cdot 2 + 60y_8\right).$$

It follows that $y_8 = \frac{p^2-31p+270}{60}$.

To determine y_{81} and y_{82}, we turn to the group G. Note that G has only one conjugacy class of involutions, and each involution u has precisely $\frac{p-1}{4}$ fixed points in Ω_1 and so has $N = \frac{\frac{p^3-p}{120} - \frac{p-1}{4}}{2} = \frac{p^3-31p+30}{240}$ cycles of length 2. Also $C_G(u) \cong D_{p-1}$ has order $c = p - 1$. By Proposition 2.8 and TABLE 3, we may calculcate $y_{81} = \frac{p^2-30p+509}{120}$. Hence $y_{82} = y_8 - y_{81} = \frac{p^2-32p+31}{120}$ and so $h_{81} = x_{81} - y_{81} = \frac{p^3-60p^2+1077p-15418}{7200}$ and $h_{82} = x_{82} - y_{82} = \frac{p^3-60p^2+1197p-1138}{7200}$.

Hence we find that Q has $\frac{p^3-60p^2+1077p-15418}{7200}$ non-self-paired regular suborbits, which have length 60 and are contained in Ω_1 and Q has $\frac{p^3-60p^2+1197p-1138}{7200}$ non-self-paired regular suborbits, which have length 60 and are contained in Ω_2. So we have $\frac{p^3-60p^2+1077p-15418}{14400}$ semisymmetric graphs X with valency 60 in case (i) and $\frac{p^3-60p^2+1197p-1138}{14400}$ semisymmetric graphs X with valency 60 in case (ii) , as listed in TABLE 1.

Thus we finish the proof of Theorem 1.1. □

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Appendix:

TABLE 4-1

	i	1	2	3	4	5	7	8
	i	1	2	3	4	5	7	8
	K_i	A_4	D_{10}	D_6	Z_5	Z_3	Z_2	1
	$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	D_4	
	$N_G(K_i)$	S_4	D_{20}	D_{12}	D_{p+1}	D_{p+1}	D_{p+1}	
	k_{i1}	2	2	2	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	1	1	$\frac{p+1}{20}-1$	$\frac{p+1}{12}-2$	$\frac{p+1}{8}-3$	$\frac{p^3-723p+13678}{7200}$
	y_{i1}	1	1	1	$\frac{p+1}{20}-1$	$\frac{p+1}{12}-2$	$\frac{p+1}{8}-3$	$\frac{p^2-32p+447}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-13142}{7200}$
$p\equiv -1(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p+1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p+1}{3}$	$\frac{p+1}{2}$	
	k_{i1}	0	0	0	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	0	0	0	$\frac{p+1}{20}$	$\frac{p+1}{12}$	$\frac{p+1}{8}$	$\frac{p^3-723p-722}{7200}$
	y_{i2}	0	0	0	$\frac{p+1}{20}$	$\frac{p+1}{12}$	$\frac{p+1}{8}$	$\frac{p^2-30p-31}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p+1138}{7200}$
	$N_G(K_i)$	A_4	D_{10}	D_{12}	D_{p-1}	D_{p+1}	D_{p+1}	
	k_{i1}	1	1	2	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	0	0	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^3-723p+6622}{7200}$
	y_{i1}	0	0	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^2-32p+231}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-7238}{7200}$
$p\equiv 11(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p+1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p+1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	1	0	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	1	1	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^3-723p+6622}{7200}$
	y_{i2}	1	1	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^2-30p+209}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-5918}{7200}$
	$N_G(K_i)$	A_4	D_{10}	D_{12}	D_{p+1}	D_{p-1}	D_{p-1}	
	k_{i1}	1	1	2	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	0	0	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^3-723p+7778}{7200}$
	y_{i1}	0	0	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^2-32p+269}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-8362}{7200}$
$p\equiv -11(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p-1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	1	1	0	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	1	1	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^3-723p+7778}{7200}$
	y_{i2}	1	1	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^2-32p+247}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-7042}{7200}$

TABLE 4-2

	i	1	2	3	4	5	7	8
	K_i	A_4	D_{10}	D_6	Z_5	Z_3	Z_2	1
	$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	D_4	
	$N_G(K_i)$	S_4	D_{10}	D_6	D_{p-1}	D_{p-1}	D_{p+1}	
	k_{i1}	2	1	1	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	0	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-1$	$\frac{p^3-723p+7022}{7200}$
	y_{i1}	1	0	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-1$	$\frac{p^2-32p+271}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-9238}{7200}$
$p\equiv 31(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p-1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	0	1	1	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	0	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-2$	$\frac{p^3-723p+7022}{7200}$
	y_{i2}	0	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-2$	$\frac{p^2-30p+209}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-5518}{7200}$
	$N_G(K_i)$	S_4	D_{10}	D_6	D_{p+1}	D_{p+1}	D_{p-1}	
	k_{i1}	2	1	1	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	1	0	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-1$	$\frac{p^3-723p+7378}{7200}$
	y_{i1}	1	0	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-1$	$\frac{p^2-30p+269}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-8762}{7200}$
$p\equiv -31(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p+1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p+1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	0	1	1	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	0	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-2$	$\frac{p^3-723p+7378}{7200}$
	y_{i2}	0	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-2$	$\frac{p^2-32p+207}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-5042}{7200}$
	$N_G(K_i)$	S_4	D_{20}	D_6	D_{p-1}	D_{p+1}	D_{p-1}	
	k_{i1}	2	2	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	1	1	0	$\frac{p-1}{20}-1$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-2$	$\frac{p^3-723p+11122}{7200}$
	y_{i1}	1	1	0	$\frac{p-1}{20}-1$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-2$	$\frac{p^2-30p+389}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-12218}{7200}$
$p\equiv 41(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p+1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p+1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	0	0	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	0	0	1	$\frac{p-1}{20}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-1$	$\frac{p^3-723p+3922}{7200}$
	y_{i2}	0	0	1	$\frac{p-1}{20}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-1$	$\frac{p^2-32p+111}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-2738}{7200}$
	$N_G(K_i)$	S_4	D_{20}	D_6	D_{p+1}	D_{p-1}	D_{p+1}	
	k_{i1}	2	2	1	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	1	0	$\frac{p+1}{20}-1$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-2$	$\frac{p^3-723p+10478}{7200}$
	y_{i1}	1	1	0	$\frac{p+1}{20}-1$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-2$	$\frac{p^2-32p+367}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-11542}{7200}$
$p\equiv -41(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p-1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	0	0	1	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	0	0	1	$\frac{p+1}{20}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-1$	$\frac{p^3-723p+3278}{7200}$
	y_{i2}	0	0	1	$\frac{p+1}{20}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-1$	$\frac{p^2-30p+89}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-2062}{7200}$

TABLE 4-3

	i	1	2	3	4	5	7	8
	i	1	2	3	4	5	7	8
	K_i	A_4	D_{10}	D_6	Z_5	Z_3	Z_2	1
	$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	D_4	
	$N_G(K_i)$	A_4	D_{20}	D_{12}	D_{p-1}	D_{p-1}	D_{p-1}	
	k_{i1}	1	2	2	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	0	1	1	$\frac{p-1}{20}-1$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{5}{2}$	$\frac{p^3-723p+11522}{7200}$
	y_{i1}	0	1	1	$\frac{p-1}{20}-1$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{5}{2}$	$\frac{p^2-30p+389}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-11818}{7200}$
$p\equiv 61(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p-1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p-1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	1	0	0	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	1	0	0	$\frac{p-1}{20}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{1}{2}$	$\frac{p^3-723p+4322}{7200}$
	y_{i2}	1	0	0	$\frac{p-1}{20}$	$\frac{p-1}{12}-1$	$\frac{p-1}{8}-\frac{1}{2}$	$\frac{p^2-32p+151}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-4738}{7200}$
	$N_G(K_i)$	A_4	D_{20}	D_{12}	D_{p+1}	D_{p+1}	D_{p+1}	
	k_{i1}	1	2	2	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	0	1	1	$\frac{p+1}{20}-1$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{5}{2}$	$\frac{p^3-723p+10078}{7200}$
	y_{i1}	0	1	1	$\frac{p+1}{20}-1$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{5}{2}$	$\frac{p^2-32p+327}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-9542}{7200}$
$p\equiv -61(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p+1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p+1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	0	0	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	1	0	0	$\frac{p+1}{20}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{1}{2}$	$\frac{p^3-723p+2878}{7200}$
	y_{i2}	1	0	0	$\frac{p+1}{20}$	$\frac{p+1}{12}-1$	$\frac{p+1}{8}-\frac{1}{2}$	$\frac{p^2-30p+89}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-2462}{7200}$
	$N_G(K_i)$	S_4	D_{10}	D_{12}	D_{p-1}	D_{p+1}	D_{p+1}	
	k_{i1}	2	1	2	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	0	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-2$	$\frac{p+1}{8}-2$	$\frac{p^3-723p+10222}{7200}$
	y_{i1}	1	0	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-2$	$\frac{p+1}{8}-2$	$\frac{p^2-32p+351}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-10838}{7200}$
$p\equiv 71(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p+1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p+1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	0	1	0	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	0	1	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}$	$\frac{p+1}{8}-1$	$\frac{p^3-723p+3022}{7200}$
	y_{i2}	0	1	0	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p+1}{12}$	$\frac{p+1}{8}-1$	$\frac{p^2-30p+89}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-2318}{7200}$
	$N_G(K_i)$	S_4	D_{10}	D_{12}	D_{p+1}	D_{p-1}	D_{p-1}	
	k_{i1}	2	1	2	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	1	0	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-2$	$\frac{p-1}{8}-2$	$\frac{p^3-723p+11378}{7200}$
	y_{i1}	1	0	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-2$	$\frac{p-1}{8}-2$	$\frac{p^2-30p+389}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-11962}{7200}$
$p\equiv -71(\mathrm{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p-1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	0	1	0	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	0	1	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}$	$\frac{p-1}{8}-1$	$\frac{p^3-723p+4178}{7200}$
	y_{i2}	0	1	0	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p-1}{12}$	$\frac{p-1}{8}-1$	$\frac{p^2-32p+127}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-3442}{7200}$

TABLE 4-4

	i	1	2	3	4	5	7	8
	K_i	A_4	D_{10}	D_6	Z_5	Z_3	Z_2	1
	$N_L(K_i)$	A_4	D_{10}	D_6	D_{10}	D_6	D_4	
	$N_G(K_i)$	A_4	D_{10}	D_6	D_{p-1}	D_{p-1}	D_{p+1}	
	k_{i1}	1	1	1	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	1	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-\frac{1}{2}$	$\frac{p^3-723p+3422}{7200}$
	y_{i1}	1	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-\frac{1}{2}$	$\frac{p^2-32p+151}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-5638}{7200}$
$p\equiv 91(\text{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p-1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p-1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	1	1	$\frac{p-1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	1	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-\frac{5}{2}$	$\frac{p^3-723p+10622}{7200}$
	y_{i2}	1	1	1	$\frac{p-1}{20}-\frac{1}{2}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-\frac{5}{2}$	$\frac{p^2-30p+329}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-9118}{7200}$
	$N_G(K_i)$	A_4	D_{10}	D_6	D_{p+1}	D_{p+1}	D_{p-1}	
	k_{i1}	1	1	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	1	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-\frac{1}{2}$	$\frac{p^3-723p+3778}{7200}$
	y_{i1}	1	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-\frac{1}{2}$	$\frac{p^2-30p+149}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-5162}{7200}$
$p\equiv -91(\text{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p+1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p+1}{3}$	$\frac{p-1}{2}$	
	k_{i2}	1	1	1	$\frac{p+1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	1	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-\frac{5}{2}$	$\frac{p^3-723p+10978}{7200}$
	y_{i2}	1	1	1	$\frac{p+1}{20}-\frac{1}{2}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-\frac{5}{2}$	$\frac{p^2-32p+327}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-8642}{7200}$
	$N_G(K_i)$	A_4	D_{20}	D_6	D_{p-1}	D_{p+1}	D_{p-1}	
	k_{i1}	1	2	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i1}	0	1	0	$\frac{p-1}{20}-1$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^3-723p+7522}{7200}$
	y_{i1}	0	1	0	$\frac{p-1}{20}-1$	$\frac{p+1}{12}-\frac{1}{2}$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^2-30p+269}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-8618}{7200}$
$p\equiv 101(\text{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p-1)}$	$D_{2(p+1)}$	$D_{2(p-1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	0	1	$\frac{p-1}{10}$	$\frac{p+1}{6}$	$\frac{p-1}{4}$	
	x_{i2}	1	0	1	$\frac{p-1}{20}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^3-723p+7522}{7200}$
	y_{i2}	1	0	1	$\frac{p-1}{20}$	$\frac{p+1}{12}-\frac{3}{2}$	$\frac{p-1}{8}-\frac{3}{2}$	$\frac{p^2-32p+231}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-6338}{7200}$
	$N_G(K_i)$	A_4	D_{20}	D_6	D_{p+1}	D_{p-1}	D_{p+1}	
	k_{i1}	1	2	1	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i1}	0	1	0	$\frac{p+1}{20}-1$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^3-723p+6878}{7200}$
	y_{i1}	0	1	0	$\frac{p+1}{20}-1$	$\frac{p-1}{12}-\frac{1}{2}$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^2-32p+247}{120}$
	h_{i1}	0	0	0	0	0	0	$\frac{p^3-60p^2+1197p-7942}{7200}$
$p\equiv -101(\text{mod}120)$	$N_Q(K_i)$	S_4	D_{20}	D_{12}	$D_{2(p+1)}$	$D_{2(p-1)}$	$D_{2(p+1)}$	
	k_i	2	2	2	$\frac{p+1}{5}$	$\frac{p-1}{3}$	$\frac{p+1}{2}$	
	k_{i2}	1	0	1	$\frac{p+1}{10}$	$\frac{p-1}{6}$	$\frac{p+1}{4}$	
	x_{i2}	1	0	1	$\frac{p+1}{20}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^3-723p+6878}{7200}$
	y_{i2}	1	0	1	$\frac{p+1}{20}$	$\frac{p-1}{12}-\frac{3}{2}$	$\frac{p+1}{8}-\frac{3}{2}$	$\frac{p^2-30p+209}{120}$
	h_{i2}	0	0	0	0	0	0	$\frac{p^3-60p^2+1077p-5662}{7200}$

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A Note on Path Signed Digraphs

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Abstract: A *Smarandachely k-signed digraph* (*Smarandachely k-marked digraph*) is an ordered pair $S = (D, \sigma)$ $(S = (D, \mu))$ where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of S* and $\sigma : \mathcal{A} \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed digraph or Smarandachely 2-marked digraph is called abbreviated a *signed digraph* or a *marked digraph*. In this paper, we define the path signed digraph $\overrightarrow{P_k}(S) = (\overrightarrow{P_k}(D), \sigma')$ of a given signed digraph $S = (D, \sigma)$ and offer a structural characterization of signed digraphs that are switching equivalent to their 3-path signed digraphs $\overrightarrow{P_3}(S)$. The concept of a line signed digraph is generalized to that of a path signed digraphs. Further, in this paper we discuss the structural characterization of path signed digraphs $\overrightarrow{P_k}(S)$.

Key Words: Smarandachely k-Signed digraphs, Smarandachely k-marked digraphs, signed digraphs, marked digraphs, balance, switching, path signed digraphs, line signed digraphs, negation.

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§1. Introduction

For standard terminology and notion in digraph theory, we refer the reader to the classic text-books of Bondy and Murty [2]and Harary et al. [4]; the non-standard will be given in this paper as and when required.

A *Smarandachely k-signed digraph* (*Smarandachely k-marked digraph*) is an ordered pair $S = (D, \sigma)$ $(S = (D, \mu))$ where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of S* and σ : $\mathcal{A} \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed digraph or Smarandachely 2-marked digraph is called abbreviated a *signed digraph* or a *marked digraph*. A *signed digraph* is an ordered pair $S = (D, \sigma)$, where

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$D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \to \{+, -\}$ is a function. A *marking* of S is a function $\mu : V(D) \to \{+, -\}$. A signed digraph S together with a marking μ is denoted by S_μ. A signed digraph $S = (D, \sigma)$ is *balanced* if every semicycle of S is positive (See [4]). Equivalently, a signed digraph is balanced if every semicycle has an even number of negative arcs. The following characterization of balanced signed digraphs is obtained in [9].

Proposition 1.1(E. Sampathkumar et al. [9]) *A signed digraph $S = (D, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each arc $\overrightarrow{uv}$ in S satisfies $\sigma(\overrightarrow{uv}) = \mu(u)\mu(v)$.*

In [9], the authors define switching and cycle isomorphism of a signed digraph as follows:

Let $S = (D, \sigma)$ and $S' = (D', \sigma')$, be two signed digraphs. Then S and S' are said to be isomorphic, *if there exists an isomorphism $\phi : D \to D'$ (that is a bijection $\phi : V(D) \to V(D')$ such that if $\overrightarrow{uv}$ is an arc in D then $\overrightarrow{\phi(u)\phi(v)}$ is an arc in D') such that for any arc $\overrightarrow{e} \in D$, $\sigma(\overrightarrow{e}) = \sigma'(\phi(\overrightarrow{e}))$.*

Given a marking μ of a signed digraph $S = (D, \sigma)$, *switching* S with respect to μ is the operation changing the sign of every arc $\overrightarrow{uv}$ of S' by $\mu(u)\sigma(\overrightarrow{uv})\mu(v)$. The signed digraph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called μ *switched signed digraph* or just *switched signed digraph*.

Further, a signed digraph S switches to signed digraph S' (or that they are switching equivalent to each other), written as $S \sim S'$, whenever there exists a marking of S such that $\mathcal{S}_\mu(S) \cong S'$.

Two signed digraphs $S = (D, \sigma)$ and $S' = (D', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : D \to D'$ such that the sign $\sigma(Z)$ of every semicycle Z in S equals to the sign $\sigma(\phi(Z))$ in S'.

Proposition 1.2(E. Sampathkumar et al. [9]) *Two signed digraphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Path Signed Digraphs

In [3], Harary and Norman introduced the notion of line digraphs for digraphs. The *line digraph* $L(D)$ of a given digraph $D = (V, \mathcal{A})$ has the arc set $\mathcal{A} := \mathcal{A}(D)$ of D for its vertex set and (e, f) is an arc in $L(D)$ whenever the arcs e and f in D have a vertex in common in such a way that it is the head of e and the tail of f; hence, a given digraph H is called a *line digraph* if there exists a digraph D such that $L(D) \cong H$. By a natural way, Broersma and Li [1] generalized the concept of line digraphs to that of directed path graphs.

Let k be a positive integer, and denote $\overrightarrow{P_k}$ or $\overrightarrow{C_k}$ a directed path or a directed cycle on k vertices, respectively. Let D be a digraph containing at least one directed path $\overrightarrow{P_k}$. Denote $\Pi_k(D)$, the set of all $\overrightarrow{P_k}$'s of D. Then the *directed $\overrightarrow{P_k}$-graph* of D, denoted by $\overrightarrow{P_k}(D)$, is the digraph with vertex set $\Pi_k(D)$; pq is an arc of $\overrightarrow{P_k}(D)$ if, and only if, there is a $\overrightarrow{P_{k+1}}$ or $\overrightarrow{C_k} = (v_1v_2...v_{k+1})$ in D (with $v_1 = v_{k+1}$ in the case of a $\overrightarrow{C_k}$) such that $p = v_1v_2...v_k$ and

$q = v_2...v_kv_{k+1}$. Note that $\overrightarrow{P_1}(D) = D$ and $\overrightarrow{L}(D)$. In [7], the authors proposed an open problem for further study, i.e., how to give a characterization for directed $\overrightarrow{P_3}$-graphs.

We extend the notion of $\overrightarrow{P_k}(D)$ to the realm of signed digraphs. In a signed digraph $S = (D, \sigma)$, where $D = (V, \mathcal{A})$ is a digraph called *underlying digraph of* S and $\sigma : \mathcal{A} \rightarrow \{+, -\}$ is a function. The *path signed digraph* $\overrightarrow{P_k}(S) = (\overrightarrow{P_k}(D), \sigma')$ of a signed digraph $S = (D, \sigma)$ is a signed digraph whose underlying digraph is $\overrightarrow{P_k}(D)$ called *path digraph* and sign of any arc $e = \overrightarrow{P_k}\overrightarrow{P_k'}$ in $\overrightarrow{P_k}(S)$ is $\sigma'(\overrightarrow{P_k}\overrightarrow{P_k'}) = \sigma(\overrightarrow{P_k})\sigma(\overrightarrow{P_k'})$. Further, a signed digraph $S = (G, \sigma)$ is called *path signed digraph*, if $S \cong \overrightarrow{P_k}(S')$, for some signed digraph S'. At the end of this section, we discuss the structural characterization of path signed digraphs $\overrightarrow{P_k}(S)$.We now gives a straightforward, yet interesting, property of path signed digraphs.

Proposition 2.1 *For any signed digraph $S = (D, \sigma)$, its path signed digraph $\overrightarrow{P_k}(S)$ is balanced.*

Proof Since sign of any arc $\sigma'(e = \overrightarrow{P_k}\overrightarrow{P_k'})$ in $\overrightarrow{P_k}(S)$ is $\sigma(\overrightarrow{P_k})\sigma(\overrightarrow{P_k'})$, where σ is the marking of $\overrightarrow{P_k}(S)$, by Proposition 1.1, $\overrightarrow{P_k}(S)$ is balanced. $\square$

Remark: For any two signed digraphs S and S' with same underlying digraph, their path signed digraphs are switching equivalent.

In [9], the authors defined line signed digraph of a signed digraph $S = (D, \sigma)$ as follows:

A line signed digraph $L(S)$ of a signed digraph $S = (D, \sigma)$ is a signed digraph $L(S) = (L(D), \sigma')$ where for any arc $\overrightarrow{ee'}$ in $L(D)$, $\sigma'(\overrightarrow{ee'}) = \sigma(\overrightarrow{e})\sigma(\overrightarrow{e'})$ (see also, E. Sampathkumar et al. [8]).

Hence, we shall call a given signed digraph S a *line signed digraph* if it is isomorphic to the line signed digraph $L(S')$ of some signed digraph S'. By the definition of path signed digraphs, we observe that $\overrightarrow{P_2}(S) = L(S)$.

Corollary 2.2 *For any signed digraph $S = (G, \sigma)$, its $\overrightarrow{P_2}(S)$ (=$L(S)$) is balanced.*

In [9], the authors obtain structural characterization of line signed digraphs as follows:

Proposition 2.3(E. Sampathkumar et al. [9]) *A signed digraph $S = (D, \sigma)$ is a line signed digraph (or $\overrightarrow{P_2}$-signed digraph) if, and only if, S is balanced signed digraph and its underlying digraph D is a line digraph (or $\overrightarrow{P_2}$-digraph).*

Proof Suppose that S is balanced and D is a line digraph. Then there exists a digraph D' such that $L(D') \cong D$. Since S is balanced, by Proposition 1.1, there exists a marking μ of D such that each arc $\overrightarrow{uv}$ in S satisfies $\sigma(\overrightarrow{uv}) = \mu(u)\mu(v)$. Now consider the signed digraph $S' = (D', \sigma')$, where for any arc $\overrightarrow{e}$ in D', $\sigma'(\overrightarrow{e})$ is the marking of the corresponding vertex in D. Then clearly, $L(S') \cong S$. Hence S is a line signed digraph.

Conversely, suppose that $S = (D, \sigma)$ is a line signed digraph. Then there exists a signed digraph $S' = (D', \sigma')$ such that $L(S') \cong S$. Hence D is the line digraph of D' and by Corollary 2.2, S is balanced. $\square$

We strongly believe that the above Proposition can be generalized to path signed digraphs

$\overrightarrow{P_k}(S)$ for $k \geq 3$. Hence, we pose it as a problem:

Problem 2.4 *If $S = (D, \sigma)$ is a balanced signed digraph and its underlying digraph D is a path digraph, then S is a path signed digraph.*

§3. Switching Equivalence of Signed Digraphs and Path Signed Digraphs

Broersma and Li [1] concluded that the only connected digraphs D with $\overrightarrow{P_3}(D) \cong D$ consists of a directed cycle with in-trees or out-trees attached to its vertices, with at most non-trivial trees, where a directed tree T of D is an *out-tree* of D if $V(T) = V(D)$ and precisely one vertex of T has in-degree zero (the root of T), while all other vertices of T have in-degree one, and an *in-tree* of D is defined analogously with respect to out-degrees.

Proposition 3.1(Broersma and Hoede [1]) *Let D be connected digraph without sources or sinks. If D has an in-tree or out-tree, then $\overrightarrow{P_3}(D) \cong D$ if, and only if, $D \cong \overrightarrow{C_n}$ for some $n \geq 3$. Hence, if D is strongly connected, then $\overrightarrow{P_3}(D) \cong D$ if, and only if, $D \cong \overrightarrow{C_n}$ for some $n \geq 3$.*

In the view of the above result, we now characterize signed digraphs that are switching equivalent to their $\overrightarrow{P_3}$-signed digraphs.

Proposition 3.2 *For any strongly connected signed digraph $S = (D, \sigma)$, $S \sim \overrightarrow{P_3}(S)$ if, and only if, S is balanced and $D \cong \overrightarrow{C_n}$ for some $n \geq 3$.vskip 3mm*

Proof Suppose $S \sim L(S)$. This implies, $D \cong L(D)$ and hence by Proposition 3.1, $D \cong \overrightarrow{C_n}$. Now, if S is signed digraph, then by Corollary 2.2, implies that $L(S)$ is balanced and hence if S is unbalanced its line signed digraph $L(S)$ being balanced cannot be switching equivalent to S in accordance with Proposition 1.2. Therefore, S must be balanced.

Suppose that S is balanced and $D \cong \overrightarrow{C_n}$ for some $n \geq 3$. Then, by Proposition 2.1, $\overrightarrow{P_3}(S)$ is balanced, the result follows from Proposition 1.2. $\square$

In [9], the authors defined a signed digraph S is *periodic*, if $L^{n+k}(S) \sim L^n(S)$ for some positive integers n and k.

Analogous to the line signed digraphs, we defined periodic for $\overrightarrow{P_3}(S)$ as follows:

For some positive integers n and k, define that a path signed digraph $\overrightarrow{P_3}(S)$ is periodic, if $\overrightarrow{P_3^{n+k}}(S) \sim \overrightarrow{P_3^n}(S)$.

Proposition 3.3(Broersma and Hoede [1]) *If D is strongly connected digraph and $\overrightarrow{P_3^n}(D) \cong D$ for some $n \geq 1$, then $\overrightarrow{P_3}(D) \cong D$ and D is a directed cycle.*

The following result is follows from Propositions 2.1,3.2 and 3.3.

Proposition 3.4 *If S is strongly connected signed digraph, and $\overrightarrow{P_3^n}(S) \sim S$ for some $n \geq 1$, then $\overrightarrow{P_3}(S) \sim S$ and D is a directed cycle.*

The *negation* $\eta(S)$ of a given signed digraph S defined as follows: $\eta(S)$ has the same underlying digraph as that of S with the sign of each arc opposite to that given to it in S.

However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

For a signed digraph $S = (D, \sigma)$, the $\overrightarrow{P_k}(S)$ is balanced (Proposition 2.1). We now examine, the condition under which negation of $\overrightarrow{P_k}(S)$ (i.e., $\eta(\overrightarrow{P_k}(S))$) is balanced.

Proposition 3.5 *Let $S = (D, \sigma)$ be a signed digraph. If $\overrightarrow{P_k}(D)$ is bipartite then $\eta(\overrightarrow{P_k}(S))$ is balanced.*

proof Since, by Proposition 2.1, $\overrightarrow{P_k}(S)$ is balanced, then every semicycle in $\overrightarrow{P_k}(S)$ contains even number of negative arcs. Also, since $\overrightarrow{P_k}(G)$ is bipartite, all semicycles have even length; thus, the number of positive arcs on any semicycle C in $\overrightarrow{P_k}(S)$ are also even. This implies that the same thing is true in negation of $\overrightarrow{P_k}(S)$. Hence $\eta(\overrightarrow{P_k}(S))$ is balanced. □

Proposition 3.2 provides easy solutions to three other signed digraph switching equivalence relations, which are given in the following results.

Corollary 3.6 *For any signed digraph $S = (D, \sigma)$, $\eta(S) \sim \overrightarrow{P_3}(S)$ if, and only if, S is an unbalanced signed digraph on any odd semicycle.*

Corollary 3.7 *For any signed digraph $S = (D, \sigma)$ and for any integer $k \geq 1$, $\overrightarrow{P_k}(\eta(S)) \sim \overrightarrow{P_k}(S)$.*

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A Combinatorial Decomposition of Euclidean Spaces $\mathbf{R}^n$ with Contribution to Visibility

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Abstract: The visibility of human beings only allows them to find objects in $\mathbf{R}^3$ at a time t. That is why physicists prefer to adopt the Euclidean space $\mathbf{R}^3$ being physical space of particles until last century. Recent progress shows the geometrical space of physics maybe $\mathbf{R}^n$ for $n \geq 4$, for example, $n = 10$, or 11 in *string theory*. Then *how to we visualize an object in* $\mathbf{R}^n$ *for* $n \geq 4$? This paper presents a combinatorial model, i.e., *combinatorial Euclidean spaces* established on Euclidean spaces $\mathbf{R}^3$ and prove any such Euclidean space $\mathbf{R}^n$ with $n \geq 4$ can be decomposed into such combinatorial structure. We also discuss conditions for realization $\mathbf{R}^n$ in mathematics or physical space by combinatorics and show the space $\mathbf{R}^{10}$ in string theory is a special case in such model.

Key Words: Smarandache multi-space, combinatorial Euclidean space, combinatorial fan-space, spacetime, p-brane, parallel probe, ultimate theory for the Universe.

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§1. Introduction

A Euclidean space $\mathbf{R}^n$ is the point set $\{(x_1, x_2, \cdots, x_n) |\ x_i \in \mathbf{R}, 1 \leq i \leq n\}$ for an integer $n \geq 1$. The structure of our eyes determines that one can only detect particles in an Euclidean space $\mathbf{R}^3$, which gave rise to physicists prefer $\mathbf{R}^3$ as a physical space. In fact, as showed in the references [2], [18] and [21], our visible geometry is the *spherical geometry*. This means that we can only observe parts of a phenomenon in the Universe if its topological dimensional≥ 4 ([1], [14]). It should be noted that if parallel worlds [6], [20] exist the dimensional of Universe must≥ 4. Then,

Can we establish a model for detecting behaviors of particles in $\mathbf{R}^n$ *with* $n \geq 4$*?*

This paper suggests a combinatorial model and a system for visualizing phenomenons in the space $\mathbf{R}^n$ with $n \geq 4$. For this object, we establish the decomposition of $\mathbf{R}^n$ underlying a connected graph G in Sections 2 and 3, then show how to establish visualizing system in such combinatorial model and acquire its global properties, for example, the Einstein's gravitational equations in Section 4. The final sections discusses conditions of its physical realization. Terminologies and notations not defined here are followed in [1], [3] and [4] for topology, gravitational fields and graphs.

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§2. Combinatorial Euclidean Spaces

Definition 2.1([13]) *A combinatorial system $\mathscr{C}_G$ is a union of mathematical systems $(\Sigma_1;\mathcal{R}_1)$,$(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ for an integer m, i.e.,*

$$\mathscr{C}_G = (\bigcup_{i=1}^{m}\Sigma_i;\ \bigcup_{i=1}^{m}\mathcal{R}_i)$$

with an underlying connected graph structure G, i.e., a particular Smarandache multi-space([8]), where

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$
$$E(G) = \{\ (\Sigma_i, \Sigma_j)\ |\ \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

Definition 2.2 *A combinatorial Euclidean space is a combinatorial system $\mathscr{C}_G$ of Euclidean spaces $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ with an underlying structure G, denoted by $\mathscr{E}_G(n_1, \cdots, n_m)$ and abbreviated to $\mathscr{E}_G(r)$ if $n_1 = \cdots = n_m = r$.*

It should be noted that a combinatorial Euclidean space is itself a Euclidean space. This fact enables us to decomposition a Euclidean space $\mathbf{R}^n$ into Euclidean spaces $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ underlying a graph G but with less dimensions, which gives rise to a packing problem on Euclidean spaces following.

Problem 2.1 *Let $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ be Euclidean spaces. In what conditions do they consist of a combinatorial Euclidean space $\mathscr{E}_G(n_1, \cdots, n_m)$?*

Notice that a Euclidean space $\mathbf{R}^n$ is an n-dimensional vector space with a normal basis $\overline{\epsilon}_1 = (1, 0, \cdots, 0)$, $\overline{\epsilon}_2 = (0, 1, 0 \cdots, 0)$, $\cdots$, $\overline{\epsilon}_n = (0, \cdots, 0, 1)$, namely, it has n orthogonal orientations. So if we think any Euclidean space $\mathbf{R}^n$ is a subspace of a Euclidean space $\mathbf{R}^{n_\infty}$ with a finite but sufficiently large dimension n_∞, then two Euclidean spaces $\mathbf{R}^{n_u}$ and $\mathbf{R}^{n_v}$ have a non-empty intersection if and only if they have common orientations. Whence, we only need to determine the number of different orthogonal orientations in $\mathscr{E}_G(n_1, \cdots, n_m)$.

Denoted by $X_{v_1}, X_{v_2}, \cdots, X_{v_m}$ consist of these orthogonal orientations in $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, respectively. An intersection graph $G[X_{v_1}, X_{v_2}, \cdots, X_{v_m}]$ of $X_{v_1}, X_{v_2}, \cdots, X_{v_m}$ is defined by ([5])

$$V(G[X_{v_1}, X_{v_2}, \cdots, X_{v_m}]) = \{v_1, v_2, \cdots, v_m\},$$
$$E[X_{v_1}, X_{v_2}, \cdots, X_{v_m}] = \{(v_i, v_j)|X_{v_i} \bigcap X_{v_j} \neq \emptyset, 1 \leq i \neq j \leq m\}.$$

By definition, we know that

$$G\ \cong\ G[X_{v_1}, X_{v_2}, \cdots, X_{v_m}],$$

which transfers the Problem 2.1 of Euclidean spaces to a combinatorial one following.

Problem 2.2 *For given integers κ, $m \geq 2$ and $n_1, n_2, \cdots, n_m$, find finite sets $Y_1, Y_2, \cdots, Y_m$ with their intersection graph being G such that $|Y_i| = n_i, 1 \leq i \leq m$, and $|Y_1 \cup Y_2 \cup \cdots \cup Y_m| = \kappa$.*

2.1 The maximum dimension of $\mathscr{E}_G(n_1, \cdots, n_m)$

First, applying the *inclusion-exclusion principle*, we get the next counting result.

Theorem 2.1 *Let $\mathscr{E}_G(n_1, \cdots, n_m)$ be a combinatorial Euclidean space of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ with an underlying structure G. Then*

$$\dim\mathscr{E}_G(n_1, \cdots, n_m) = \sum_{\langle v_i \in V(G)|1\leq i\leq s\rangle \in CL_s(G)} (-1)^{s+1}\dim(\mathbf{R}^{n_{v_1}} \bigcap \mathbf{R}^{n_{v_2}} \bigcap \cdots \bigcap \mathbf{R}^{n_{v_s}}),$$

where n_{v_i} denotes the dimensional number of the Euclidean space in $v_i \in V(G)$ and $CL_s(G)$ consists of all complete graphs of order s in G.

Proof By definition, $\mathbf{R}^{n_u} \cap \mathbf{R}^{n_v} \neq \emptyset$ only if there is an edge $(\mathbf{R}^{n_u}, \mathbf{R}^{n_v})$ in G. This condition can be generalized to a more general situation, i.e., $\mathbf{R}^{n_{v_1}} \cap \mathbf{R}^{n_{v_2}} \cap \cdots \cap \mathbf{R}^{n_{v_l}} \neq \emptyset$ only if $\langle v_1, v_2, \cdots, v_l\rangle_G \cong K_l$.

In fact, if $\mathbf{R}^{n_{v_1}} \cap \mathbf{R}^{n_{v_2}} \cap \cdots \cap \mathbf{R}^{n_{v_l}} \neq \emptyset$, then $\mathbf{R}^{n_{v_i}} \cap \mathbf{R}^{n_{v_j}} \neq \emptyset$, which implies that $(\mathbf{R}^{n_{v_i}}, \mathbf{R}^{n_{v_j}}) \in E(G)$ for any integers $i, j,\ 1 \leq i, j \leq l$. Therefore, $\langle v_1, v_2, \cdots, v_l\rangle_G$ is a complete graph of order l in the intersection graph G.

Now we are needed to count these orthogonal orientations in $\mathscr{E}_G(n_1, \cdots, n_m)$. In fact, the number of different orthogonal orientations is

$$\dim\mathscr{E}_G(n_1, \cdots, n_m) = \dim(\bigcup_{v\in V(G)} \mathbf{R}^{n_v})$$

by previous discussion. Applying the inclusion-exclusion principle, we find that

$$\begin{aligned}
\dim\mathscr{E}_G(n_1, \cdots, n_m) &= \dim(\bigcup_{v\in V(G)} \mathbf{R}^{n_v}) \\
&= \sum_{\{v_1,\cdots,v_s\}\subset V(G)} (-1)^{s+1}\dim(\mathbf{R}^{n_{v_1}} \bigcap \mathbf{R}^{n_{v_2}} \bigcap \cdots \bigcap \mathbf{R}^{n_{v_s}}) \\
&= \sum_{\langle v_i \in V(G)|1\leq i\leq s\rangle \in CL_s(G)} (-1)^{s+1}\dim(\mathbf{R}^{n_{v_1}} \bigcap \mathbf{R}^{n_{v_2}} \bigcap \cdots \bigcap \mathbf{R}^{n_{v_s}}).
\end{aligned}$$

□

Notice that $\dim(\mathbf{R}^{n_{v_1}} \cap \mathbf{R}^{n_{v_2}} \cap \cdots \cap \mathbf{R}^{n_{v_s}}) = n_{v_1}$ if $s = 1$ and $\dim(\mathbf{R}^{n_{v_1}} \cap \mathbf{R}^{n_{v_2}}) \neq 0$ only if $(\mathbf{R}^{n_{v_1}}, \mathbf{R}^{n_{v_2}}) \in E(G)$. We get a more applicable formula for calculating $\dim\mathscr{E}_G(n_1, \cdots, n_m)$ on K_3-free graphs G by Theorem 2.1.

Corollary 2.1 *If G is K_3-free, then*

$$\dim\mathscr{E}_G(n_1, \cdots, n_m) = \sum_{v\in V(G)} n_v - \sum_{(u,v)\in E(G)} \dim(\mathbf{R}^{n_u} \bigcap \mathbf{R}^{n_v}).$$

Particularly, if $G = v_1v_2\cdots v_m$ a circuit for an integer $m \geq 4$, then

$$\dim\mathscr{E}_G(n_1, \cdots, n_m) = \sum_{i=1}^{m} n_{v_i} - \sum_{i=1}^{m} \dim(\mathbf{R}^{n_{v_i}} \bigcap \mathbf{R}^{n_{v_{i+1}}}),$$

where each index is modulo m.

Now we determine the maximum dimension of combinatorial Euclidean spaces of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ with an underlying structure G.

Theorem 2.2 *Let $\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})$ be a combinatorial Euclidean space of $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$ with an underlying graph G, $V(G)=\{v_1,v_2,\cdots,v_m\}$. Then the maximum dimension $\dim_{max}\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})$ of $\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})$ is*

$$\dim_{max}\mathscr{E}_G(n_{v_1},\cdots,n_{v_m})=1-m+\sum_{v\in V(G)}n_v$$

with conditions $\dim(\mathbf{R}^{n_u}\cap\mathbf{R}^{n_v})=1$ for $\forall(u,v)\in E(G)$.

Proof Let $X_{v_1},X_{v_2},\cdots,X_{v_m}$ consist of these orthogonal orientations in $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, respectively. Notice that

$$|X_{v_i}\bigcup X_{v_j}|=|X_{v_i}|+|X_{v_j}|-|X_{v_i}\bigcap X_{v_j}|$$

for $1\leq i\neq j\leq m$ by Theorem 1.5.1 in the case of $n=2$. We immediately know that $|X_{v_i}\cup X_{v_j}|$ attains its maximum value only if $|X_{v_i}\cap X_{v_j}|$ is minimum. Since X_{v_i} and X_{v_j} are nonempty sets, we find that the minimum value of $|X_{v_i}\cap X_{v_j}|=1$ if $(v_i,v_j)\in E(G)$.

The proof is finished by the inductive principle. Not loss of generality, assume $(v_1,v_2)\in E(G)$. Then we have known that $|X_{v_1}\bigcup X_{v_2}|$ attains its maximum

$$|X_{v_1}|+|X_{v_2}|-1$$

only if $|X_{v_1}\cap X_{v_2}|=1$. Since G is connected, not loss of generality, let v_3 be adjacent with $\{v_1,v_2\}$ in G. Then by

$$|X_{v_1}\bigcup X_{v_2}\bigcup X_{v_3}|=|X_{v_1}\bigcup X_{v_2}|+|X_{v_3}|-|(X_{v_1}\bigcup X_{v_2})\bigcap X_{v_3}|,$$

we know that $|X_{v_1}\cup X_{v_2}\cup X_{v_3}|$ attains its maximum value only if $|X_{v_1}\cup X_{v_2}|$ attains its maximum and $|(X_{v_1}\cup X_{v_2})\cap X_{v_3}|=1$ for $(X_{v_1}\cup X_{v_2})\cap X_{v_3}\neq\emptyset$. Whence, $|X_{v_1}\cap X_{v_3}|=1$ or $|X_{v_2}\cap X_{v_3}|=1$, or both. In the later case, there must be $|X_{v_1}\cap X_{v_2}\cap X_{v_3}|=1$. Therefore, the maximum value of $|X_{v_1}\cup X_{v_2}\cup X_{v_3}|$ is

$$|X_{v_1}|+|X_{v_2}|+|X_{v_3}|-2.$$

Generally, we assume the maximum value of $|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}|$ to be

$$|X_{v_1}|+|X_{v_2}|+\cdots+|X_{v_k}|-k+1$$

for an integer $k\leq m$ with conditions $|X_{v_i}\cap X_{v_j}|=1$ hold if $(v_i,v_j)\in E(G)$ for $1\leq i\neq j\leq k$. By the connectedness of G, without loss of generality, we choose a vertex v_{k+1} adjacent with $\{v_1,v_2,\cdots,v_k\}$ in G and find out the maximum value of $|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}\cup X_{v_{k+1}}|$. In fact, since

$$\begin{aligned}|X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}\cup X_{v_{k+1}}| &= |X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k}|+|X_{v_{k+1}}|\\ &- |(X_{v_1}\cup X_{v_2}\cup\cdots\cup X_{v_k})\bigcap X_{v_{k+1}}|,\end{aligned}$$

we know that $|X_{v_1} \cup X_{v_2} \cup \cdots \cup X_{v_k} \cup X_{v_{k+1}}|$ attains its maximum value only if $|X_{v_1} \cup X_{v_2} \cup \cdots \cup X_{v_k}|$ attains its maximum and $|(X_{v_1} \cup X_{v_2} \cup \cdots \cup X_{v_k}) \bigcap X_{v_{k+1}}| = 1$ for $(X_{v_1} \cup X_{v_2} \cup \cdots \cup X_{v_k}) \cap X_{v_{k+1}} \neq \emptyset$. Whence, $|X_{v_i} \cap X_{v_{k+1}}| = 1$ if $(v_i, v_{k+1}) \in E(G)$. Consequently, we find that the maximum value of $|X_{v_1} \cup X_{v_2} \cup \cdots \cup X_{v_k} \cup X_{v_{k+1}}|$ is

$$|X_{v_1}| + |X_{v_2}| + \cdots + |X_{v_k}| + |X_{v_{k+1}}| - k.$$

Notice that our process searching for the maximum value of $|X_{v_1} \cup X_{v_2} \cup \cdots \cup X_{v_k}|$ does not alter the intersection graph G of $X_{v_1}, X_{v_2}, \cdots, X_{v_m}$. Whence, by the inductive principle we finally get the maximum dimension $\dim_{max}\mathscr{E}_G$ of $\mathscr{E}_G$, that is,

$$\dim_{max}\mathscr{E}_G(n_{v_1}, \cdots, n_{v_m}) = 1 - m + n_1 + n_2 + \cdots + n_m$$

with conditions $\dim(\mathbf{R}^{n_u} \cap \mathbf{R}^{n_v}) = 1$ for $\forall (u, v) \in E(G)$. $\square$

2.2 The minimum dimension of $\mathscr{E}_G(n_1, \cdots, n_m)$

Determining the minimum value $\dim_{min}\mathscr{E}_G(n_1, \cdots, n_m)$ of $\mathscr{E}_G(n_1, \cdots, n_m)$ is a difficult problem in general case. But for some graph families we can determine its minimum value.

Theorem 2.3 *Let $\mathscr{E}_G(n_{v_1}, n_{v_2}, \cdots, n_{v_m})$ be a combinatorial Euclidean space of $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$ with an underlying graph G, $V(G) = \{v_1, v_2, \cdots, v_m\}$ and $\{v_1, v_2, \cdots, v_l\}$ an independent vertex set in G. Then*

$$\dim_{min}\mathscr{E}_G(n_{v_1}, \cdots, n_{v_m}) \geq \sum_{i=1}^{l} n_{v_i}$$

and with the equality hold if G is a complete bipartite graph $K(V_1, V_2)$ with partite sets $V_1 = \{v_1, v_2, \cdots, v_l\}$, $V_2 = \{v_{l+1}, v_{l+2}, \cdots, v_m\}$ and

$$\sum_{i=1}^{l} n_{v_i} \geq \sum_{i=l+1}^{m} n_{v_i}.$$

Proof Similarly, we use $X_{v_1}, X_{v_2}, \cdots, X_{v_m}$ to denote these orthogonal orientations in $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, respectively. By definition, we know that

$$X_{v_i} \bigcap X_{v_j} = \emptyset, \quad 1 \leq i \neq j \leq l$$

for $(v_i, v_j) \notin E(G)$. Whence, we get that

$$|\bigcup_{i=1}^{m} X_{v_i}| \geq |\bigcup_{i=1}^{l} X_{v_i}| = \sum_{i=1}^{l} n_{v_i}.$$

By the assumption,

$$\sum_{i=1}^{l} n_{v_i} \geq \sum_{i=l+1}^{m} n_{v_i},$$

we can partition $X_{v_1}, X_{v_2}, \cdots, X_{v_m}$ to

$$X_{v_1} = (\bigcup_{i=l+1}^{m} Y_i(v_1)) \bigcup Z(v_1),$$

$$X_{v_2} = (\bigcup_{i=l+1}^{m} Y_i(v_2)) \bigcup Z(v_2),$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$X_{v_l} = (\bigcup_{i=l+1}^{m} Y_i(v_l)) \bigcup Z(v_l)$$

such that $\sum\limits_{k=1}^{l} |Y_i(v_k)| = |X_{v_i}|$ for any integer $i,\ l+1 \leq i \leq m$, where $Z(v_i)$ maybe an empty set for integers $i,\ 1 \leq i \leq l$. Whence, we can choose

$$X'_{v_i} = \bigcup_{k=1}^{l} Y_i(v_k)$$

to replace each X_{v_i} for any integer $i,\ 1 \leq i \leq m$. Notice that the intersection graph of $X_{v_1}, X_{v_2}, \cdots, X_{v_l}, X'_{v_{l+1}}, \cdots, X'_{v_m}$ is still the complete bipartite graph $K(V_1, V_2)$, but

$$|\bigcup_{i=1}^{m} X_{v_i}| = |\bigcup_{i=1}^{l} X_{v_i}| = \sum_{i=1}^{l} n_i.$$

Therefore, we get that

$$\dim_{min}\mathscr{E}_G(n_{v_1}, \cdots, n_{v_m}) = \sum_{i=1}^{l} n_{v_i}$$

in the case of complete bipartite graph $K(V_1, V_2)$ with partite sets $V_1 = \{v_1, v_2, \cdots, v_l\}$, $V_2 = \{v_{l+1}, v_{l+2}, \cdots, v_m\}$ and

$$\sum_{i=1}^{l} n_{v_i} \geq \sum_{i=l+1}^{m} n_{v_i}. \qquad \square$$

Although the lower bound of $\dim\mathscr{E}_G(n_{v_1}, \cdots, n_{v_m})$ in Theorem 2.3 is sharp, but it is not better if G is given in some cases. Consider a complete system of r-subsets of a set with less than $2r$ elements. We know the next conclusion if $G = K_m$.

Theorem 2.4 *For any integer $r \geq 2$, let $\mathscr{E}_{K_m}(r)$ be a combinatorial Euclidean space of $\underbrace{\mathbf{R}^r, \cdots, \mathbf{R}^r}_{m}$, and there exists an integer $s,\ 0 \leq s \leq r-1$ such that*

$$\binom{r+s-1}{r} < m \leq \binom{r+s}{r}.$$

Then

$$\dim_{min}\mathscr{E}_{K_m}(r) = r + s.$$

Proof We denote by $X_1, X_2, \cdots, X_m$ these sets consist of orthogonal orientations in m Euclidean spaces $\mathbf{R}^r$. Then each $X_i,\ 1 \leq i \leq m$, is an r-set. By assumption,

$$\binom{r+s-1}{r} < m \leq \binom{r+s}{r}$$

and $0 \leq s \leq r-1$, we know that two r-subsets of an $(r+s)$-set must have a nonempty intersection. So we can determine these m r-subsets $X_1, X_2, \cdots, X_m$ by using the complete system of r-subsets in an $(r+s)$-set, and these m r-subsets $X_1, X_2, \cdots, X_m$ can not be chosen in an $(r+s-1)$-set. Therefore, we find that $|\bigcup\limits_{i=1}^{m} X_i| = r+s$, i.e., if $0 \leq s \leq r-1$, then $\dim_{min}\mathscr{E}_{K_m}(r) = r+s$. $\square$

For general combinatorial spaces $\mathscr{E}_{K_m}(n_{v_1}, \cdots, n_{v_m})$ of $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, we get their minimum dimension if n_{v_m} is large enough.

Theorem 2.5 *Let $\mathscr{E}_{K_m}$ be a combinatorial Euclidean space of $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, $n_{v_1} \geq n_{v_2} \geq \cdots \geq n_{v_m} \geq \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}}-1}) \rceil + 1$ and $V(K_m) = \{v_1, v_2, \cdots, v_m\}$. Then*

$$\dim_{min}\mathscr{E}_{K_m}(n_{v_1}, \cdots, n_{v_m}) = n_{v_1} + \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}-1}}) \rceil.$$

Proof Let $X_{v_1}, X_{v_2}, \cdots, X_{v_m}$ be sets consist of these orthogonal orientations in $\mathbf{R}^{n_{v_1}}$, $\mathbf{R}^{n_{v_2}}$, $\cdots$, $\mathbf{R}^{n_{v_m}}$, respectively and

$$2^{s-1} < \frac{m}{2^{k+1}-1} + 1 \leq 2^s$$

for an integer s, where $k = n_{v_1} - n_{v_2}$. Then we find that

$$\lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}-1}}) \rceil = s.$$

We construct a family $\{Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}\}$ with none being a subset of another, $|Y_{v_i}| = |X_{v_i}|$ for $1 \leq i \leq m$ and its intersection graph is still K_m, but with

$$|Y_{v_1} \bigcup Y_{v_2} \bigcup \cdots \bigcup Y_{v_m}| = n_{v_1} + s.$$

In fact, let $X_{v_1} = \{x_1, x_2, \cdots, x_{n_{v_2}}, x_{n_{v_2}+1}, \cdots, x_{n_{v_1}}\}$ and $U = \{u_1, u_2, \cdots, u_s\}$, such as those shown in Fig.2.1 for $s = 1$ and $n_{v_1} = 9$.

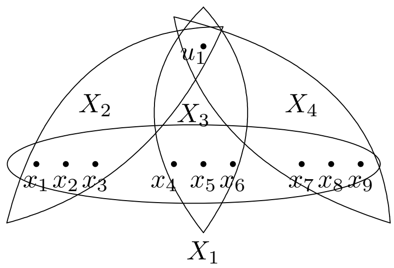

Fig.2.1

Choose g elements $x_{i_1}, x_{i_2}, \cdots, x_{i_g} \in X_{v_1}$ and $h \geq 1$ elements $u_{j_1}, u_{j_2}, \cdots, u_{j_h} \in U$. We construct a finite set

$$X_{g.h} = \{x_{i_1}, x_{i_2}, \cdots, x_{i_g}, u_{j_1}, u_{j_2}, \cdots, u_{j_h}\}$$

with a cardinal $g+h$. Let $g+h = |X_{v_1}|, |X_{v_2}|, \cdots, |X_{v_m}|$, respectively. We consequently find such sets $Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}$. Notice that there are no one set being a subset of another in the family $\{Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}\}$. So there must have two elements in each Y_{v_i}, $1 \leq i \leq m$ at least such that one is in U and another in $\{x_{n_{v_2}}, x_{n_{v_2}+1}, \cdots, x_{n_{v_1}}\}$. Now since $n_{v_m} \geq \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}}-1})\rceil + 1$, there are

$$\sum_{i=1}^{k+1}\sum_{j=1}^{s}\binom{k+1}{i}\binom{s}{j} = (2^{k+1}-1)(2^s-1) \geq m$$

different sets $Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}$ altogether with $|X_{v_1}| = |Y_{v_1}|, \cdots, |X_{v_m}| = |Y_{v_m}|$. None of them is a subset of another and their intersection graph is still K_m. For example,

$$X_{v_1}, \quad \{u_1, x_1, \cdots, x_{n_{v_2}-1}\},$$
$$\{u_1,\ x_{n_{v_2}-n_{v_3}+2},\ \cdots,\ x_{n_{v_2}}\},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$\{u_1, x_{n_{v_{k-1}}-n_{v_k}+2},\ \cdots,\ x_{n_{v_k}}\}$$

are such sets with only one element u_1 in U. See also in Fig.4.1.1 for details. It is easily to know that

$$|Y_{v_1}\bigcup Y_{v_2}\bigcup\cdots\bigcup Y_{v_m}| = n_{v_1} + s = n_{v_1} + \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}}-1})\rceil$$

in our construction.

Conversely, if there exists a family $\{Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}\}$ such that $|X_{v_1}| = |Y_{v_1}|, \cdots, |X_{v_m}| = |Y_{v_m}|$ and

$$|Y_{v_1}\bigcup Y_{v_2}\bigcup\cdots\bigcup Y_{v_m}| < n_{v_1} + s,$$

then there at most

$$\sum_{i=1}^{k+1}\sum_{j=1}^{s}\binom{k+1}{i}\binom{s-1}{j} = (2^{k+1}-1)(2^{s-1}-1) < m$$

different sets in $\{Y_{v_1}, Y_{v_2}, \cdots, Y_{v_m}\}$ with none being a subset of another. This implies that there must exists integers $i, j, 1 \leq i \neq j \leq m$ with $Y_{v_i} \subset Y_{v_j}$, a contradiction. Therefore, we get the minimum dimension $\dim_{min}\mathscr{E}_{K_m}$ of $\mathscr{E}_{K_m}$ to be

$$\dim_{min}\mathscr{E}_{K_m}(n_{v_1}, \cdots, n_{v_m}) = n_{v_1} + \lceil \log_2(\frac{m+1}{2^{n_{v_1}-n_{v_2}}-1})\rceil.$$

$\square$

As we introduce in Section 1, the combinatorial space of $\mathbf{R}^3$ is particularly interested in physics. In the case of K_m, we can determine its minimum dimension.

Theorem 2.5 *Let* $\mathscr{E}_{K_m}(3)$ *be a combinatorial Euclidean space of* $\underbrace{\mathbf{R}^3, \cdots, \mathbf{R}^3}_{m}$*. Then*

$$\dim_{min}\mathscr{E}_{K_m}(3) = \begin{cases} 3, & \text{if} \quad m = 1, \\ 4, & \text{if} \quad 2 \le m \le 4, \\ 5, & \text{if} \quad 5 \le m \le 10, \\ 2 + \lceil \sqrt{m} \rceil, & \text{if} \quad m \ge 11. \end{cases}$$

Proof Let $X_1, X_2, \cdots, X_m$ be these sets consist of orthogonal orientations in m Euclidean spaces $\mathbf{R}^3$, respectively and $|X_1 \cup X_2 \cup \cdots \cup X_m| = l$. Then each $X_i,\ 1 \le i \le m$, is a 3-set.

In the case of $m \le 10 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, any s-sets have a nonempty intersection. So it is easily to check that

$$\dim_{min}\mathscr{E}_{K_m}(3) = \begin{cases} 3, & \text{if} \quad m = 1, \\ 4, & \text{if} \quad 2 \le m \le 4, \\ 5, & \text{if} \quad 5 \le m \le 10. \end{cases}$$

We only consider the case of $m \ge 11$. Let $X = \{u, v, w\}$ be a chosen 3-set. Notice that any 3-set will intersect X with 1 or 2 elements. Our discussion is divided into three cases.

Case 1 *There exist* 3*-sets* X_1', X_2', X_3' *such that* $X_1' \cap X = \{u, v\}$*,* $X_2' \cap X = \{u, w\}$ *and* $X_3' \cap X = \{v, w\}$ *such as those shown in Fig.*2.2*, where each triangle denotes a* 3*-set.*

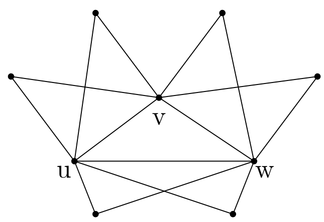

Fig.2.2

Notice that there are no 3-sets X' such that $|X' \cap X| = 1$ in this case. Otherwise, we can easily find two 3-sets with an empty intersection, a contradiction. Counting such 3-sets, we know that there are at most $3(v - 3) + 1$ 3-sets with their intersection graph being K_m. Thereafter, we know that

$$m \le 3(l - 3) + 1, \quad i.e., \quad l \ge \lceil \frac{m - 1}{3} \rceil + 3.$$

Case 2 *There are* 3*-sets* X_1', X_2' *but no* 3*-set* X_3' *such that* $X_1' \cap X = \{u, v\}$*,* $X_2' \cap X = \{u, w\}$ *and* $X_3' \cap X = \{v, w\}$ *such as those shown in Fig.*2.3*, where each triangle denotes a* 3*-set.*

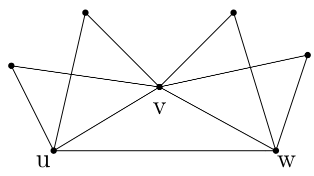

Fig.2.3

In this case, there are no 3-sets X' such that $X' \cap X = \{u\}$ or $\{w\}$. Otherwise, we can easily find two 3-sets with an empty intersection, a contradiction. Enumerating such 3-sets, we know that there are at most

$$2(l-1) + \binom{l-3}{2} + 1$$

3-sets with their intersection graph still being K_m. Whence, we get that

$$m \leq 2(l-1) + \binom{l-3}{2} + 1, \quad i.e., \quad l \geq \lceil \frac{3+\sqrt{8m+17}}{2} \rceil.$$

Case 3 *There are a* 3*-set* X_1' *but no* 3*-sets* X_2', X_3' *such that* $X_1' \cap X = \{u, v\}$, $X_2' \cap X = \{u, w\}$ *and* $X_3' \cap X = \{v, w\}$ *such as those shown in Fig.*2.4*, where each triangle denotes a* 3*-set.*

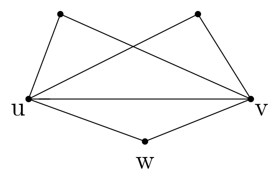

Fig.2.4

Enumerating 3-sets in this case, we know that there are at most

$$l - 2 + 2\binom{l-2}{2}$$

such 3-sets with their intersection graph still being K_m. Therefore, we find that

$$m \leq l - 2 + 2\binom{l-2}{2}, \quad i.e., \quad l \geq 2 + \lceil \sqrt{m} \rceil.$$

Combining these Cases $1-3$, we know that

$$l \geq \min\{\lceil \frac{m-1}{3} \rceil + 3, \lceil \frac{3+\sqrt{8m+17}}{2} \rceil, 2 + \lceil \sqrt{m} \rceil\} = 2 + \lceil \sqrt{m} \rceil.$$

Conversely, there 3-sets constructed in Case 3 show that there indeed exist 3-sets $X_1, X_2, \cdots, X_m$ whose intersection graph is K_m, where

$$m = l - 2 + 2\begin{pmatrix} l-2 \\ 2 \end{pmatrix}.$$

Therefore, we get that

$$\dim_{min}\mathscr{E}_{K_m}(3) = 2 + \lceil\sqrt{m}\rceil$$

if $m \geq 11$. This completes the proof. $\square$

§3. A Combinatorial Model of Euclidean Spaces $\mathbf{R}^n$ with $n \geq 4$

A *combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ is the combinatorial space $\mathscr{E}_{K_m}(n_1, \cdots, n_m)$ of $\mathbf{R}^{n_1}$, $\mathbf{R}^{n_2}$, $\cdots$, $\mathbf{R}^{n_m}$ such that for any integers $i, j,\ 1 \leq i \neq j \leq m$,

$$\mathbf{R}^{n_i}\bigcap\mathbf{R}^{n_j} = \bigcap_{k=1}^{m}\mathbf{R}^{n_k},$$

which is in fact a *p-brane* with $p = \dim \bigcap\limits_{k=1}^{m} \mathbf{R}^{n_k}$ in string theory ([15]-[17]), seeing Fig.3.1 for details.

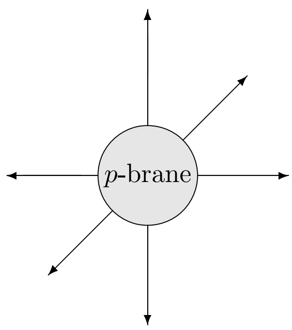

Fig.3.1

For $\forall p \in \widetilde{\mathbf{R}}(n_1, \cdots, n_m)$ we can present it by an $m \times n_m$ coordinate matrix $[\overline{x}]$ following with $x_{il} = \frac{x_l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$,

$$[\overline{x}] = \begin{bmatrix} x_{11} & \cdots & x_{1\widehat{m}} & x_{1(\widehat{m})+1)} & \cdots & x_{1n_1} & \cdots & 0 \\ x_{21} & \cdots & x_{2\widehat{m}} & x_{2(\widehat{m}+1)} & \cdots & x_{2n_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ x_{m1} & \cdots & x_{m\widehat{m}} & x_{m(\widehat{m}+1)} & \cdots & \cdots & x_{mn_m-1} & x_{mn_m} \end{bmatrix}.$$

By definition, we know the following result.

Theorem 3.1 *Let* $\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$ *be a fan-space. Then*

$$\dim\widetilde{\mathbf{R}}(n_1,\cdots,n_m)=\widehat{m}+\sum_{i=1}^{m}(n_i-\widehat{m}),$$

where

$$\widehat{m}=\dim(\bigcap_{k=1}^{m}\mathbf{R}^{n_k}).\qquad\square$$

The *inner product* $\langle(A),(B)\rangle$ of (A) and (B) is defined by

$$\langle(A),(B)\rangle=\sum_{i,j}a_{ij}b_{ij}.$$

Then we know the next result by definition.

Theorem 3.2 *Let* $(A),(B),(C)$ *be* $m\times n$ *matrixes and* α *a constant. Then*

(1) $\langle\alpha A,B\rangle=\alpha\langle B,A\rangle$;

(2) $\langle A+B,C\rangle=\langle A,C\rangle+\langle B,C\rangle$;

(3) $\langle A,A\rangle\geq 0$ with equality hold if and only if $(A)=O_{m\times n}$.

Theorem 3.3 *Let* $(A),(B)$ *be* $m\times n$ *matrixes. Then*

$$\langle(A),(B)\rangle^2\leq\langle(A),(A)\rangle\langle(B),(B)\rangle$$

and with equality hold only if $(A)=\lambda(B)$, *where* λ *is a real constant.*

Proof If $(A)=\lambda(B)$, then $\langle A,B\rangle^2=\lambda^2\langle B,B\rangle^2=\langle A,A\rangle\langle B,B\rangle$. Now if there are no constant λ enabling $(A)=\lambda(B)$, then $(A)-\lambda(B)\neq O_{m\times n}$ for any real number λ. According to Theorem 3.2, we know that

$$\langle(A)-\lambda(B),(A)-\lambda(B)\rangle>0,$$

i.e.,

$$\langle(A),(A)\rangle-2\lambda\langle(A),(B)\rangle+\lambda^2\langle(B),(B)\rangle>0.$$

Therefore, we find that

$$\Delta=(-2\langle(A),(B)\rangle)^2-4\langle(A),(A)\rangle\langle(B),(B)\rangle<0,$$

namely,

$$\langle(A),(B)\rangle^2<\langle(A),(A)\rangle\langle(B),(B)\rangle.\qquad\square$$

Theorem 3.4 *For a given integer sequence* $n_1,n_2,\cdots,n_m,m\geq 1$ *with* $0<n_1<n_2<\cdots<n_m$, $(\widetilde{\mathbf{R}}(n_1,\cdots,n_m);d)$ *is a metric space.*

Proof We only need to verify that each condition for a metric space is hold in $(\widetilde{\mathbf{R}}(n_1,\cdots,n_m);d)$. For two point $p,q\in\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$, by definition we know that

$$d(p,q)=\sqrt{\langle [p]-[q],[p]-[q]\rangle}\geq 0$$

with equality hold if and only if $[p]=[q]$, namely, $p=q$ and

$$d(p,q)=\sqrt{\langle [p]-[q],[p]-[q]\rangle}=\sqrt{\langle [q]-[p],[q]-[p]\rangle}=d(q,p).$$

Now let $u\in\widetilde{\mathbf{R}}(n_1,\cdots,n_m)$. By Theorem 3.3, we then find that

$$\begin{aligned}
&(d(p,u)+d(u,p))^2\\
&=\langle [p]-[u],[p]-[u]\rangle+2\sqrt{\langle [p]-[u],[p]-[u]\rangle\,\langle [u]-[q],[u]-[q]\rangle}\\
&+\langle [u]-[q],[u]-[q]\rangle\\
&\geq \langle [p]-[u],[p]-[u]\rangle+2\,\langle [p]-[u],[u]-[q]\rangle+\langle [u]-[q],[u]-[q]\rangle\\
&=\langle [p]-[q],[p]-[q]\rangle=d^2(p,q).
\end{aligned}$$

Whence, $d(p,u)+d(u,p)\geq d(p,q)$ and $(\widetilde{\mathbf{R}}(n_1,\cdots,n_m);d)$ is a metric space. □

According to Theorem 3.1, a combinatorial fan-space $\widetilde{R}(n_1,n_2,\cdots,n_m)$ can be turned into a Euclidean space $\mathbf{R}^n$ with $n=\widehat{m}+\sum\limits_{i=1}^{m}(n_i-\widehat{m})$. Now the inverse question is that *for a Euclidean space* $\mathbf{R}^n$*, weather there exist a combinatorial Euclidean space* $\mathscr{E}_G(n_1,\cdots,n_m)$ *of Euclidean spaces* $\mathbf{R}^{n_1}$*,* $\mathbf{R}^{n_2}$*,* $\cdots$*,* $\mathbf{R}^{n_m}$ *such that* $\dim\mathbf{R}^{n_1}\cup\mathbf{R}^{n_2}\cup\cdots\cup\mathbf{R}^{n_m}=n$? We get the following decomposition result of Euclidean spaces.

Theorem 3.5 *Let* $\mathbf{R}^n$ *be a Euclidean space,* $n_1,n_2,\cdots,n_m$ *integers with* $\widehat{m}<n_i<n$ *for* $1\leq i\leq m$ *and the equation*

$$\widehat{m}+\sum_{i=1}^{m}(n_i-\widehat{m})=n$$

hold for an integer $\widehat{m}, 1\leq\widehat{m}\leq n$*. Then there is a combinatorial fan-space* $\widetilde{\mathbf{R}}(n_1,n_2,\cdots,n_m)$ *such that*

$$\mathbf{R}^n\cong\widetilde{\mathbf{R}}(n_1,n_2,\cdots,n_m).$$

Proof Not loss of generality, assume the normal basis of $\mathbf{R}^n$ is $\overline{\epsilon}_1=(1,0,\cdots,0)$, $\overline{\epsilon}_2=(0,1,0\cdots,0)$, $\cdots$, $\overline{\epsilon}_n=(0,\cdots,0,1)$. Then its coordinate system of $\mathbf{R}^n$ is $(x_1,x_2,\cdots,x_n)$. Since

$$n-\widehat{m}=\sum_{i=1}^{m}(n_i-\widehat{m}),$$

choose

$$\begin{aligned}
&\mathbf{R}_1=\langle\overline{\epsilon}_1,\overline{\epsilon}_2,\cdots,\overline{\epsilon}_{\widehat{m}},\overline{\epsilon}_{\widehat{m}+1},\cdots,\overline{\epsilon}_{n_1}\rangle;\\
&\mathbf{R}_2=\langle\overline{\epsilon}_1,\overline{\epsilon}_2,\cdots,\overline{\epsilon}_{\widehat{m}},\overline{\epsilon}_{n_1+1},\overline{\epsilon}_{n_1+2},\cdots,\overline{\epsilon}_{n_2}\rangle;\\
&\mathbf{R}_3=\langle\overline{\epsilon}_1,\overline{\epsilon}_2,\cdots,\overline{\epsilon}_{\widehat{m}},\overline{\epsilon}_{n_2+1},\overline{\epsilon}_{n_2+2},\cdots,\overline{\epsilon}_{n_3}\rangle;\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;
\end{aligned}$$

$$\mathbf{R}_m = \left\langle \bar{\epsilon}_1, \bar{\epsilon}_2, \cdots, \bar{\epsilon}_{\widehat{m}}, \bar{\epsilon}_{n_{m-1}+1}, \bar{\epsilon}_{n_{m-1}+2}, \cdots, \bar{\epsilon}_{n_m} \right\rangle.$$

Calculation shows that $\dim\mathbf{R}_i = n_i$ and $\dim(\bigcap\limits_{i=1}^{m} \mathbf{R}_i) = \widehat{m}$. Whence $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ is a combinatorial fan-space. Whence,

$$\mathbf{R}^n \cong \widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m). \qquad \square$$

Notice that a combinatorial fan-space $\widetilde{\mathbf{R}}(n_1, n_2, \cdots, n_m)$ is in fact $\mathscr{E}_{K_m}(n_1, n_2, \cdots, n_m)$. Let $n_i = 3$ for $1 \leq i \leq m$. We get a result following by Theorem 3.5.

Corollary3.1 *Let $\mathbf{R}^n$ be a Euclidean space with $n \geq 4$. Then there is a combinatorial Euclidean space $\mathscr{E}_{K_m}(3)$ such that*

$$\mathbf{R}^n \cong \mathscr{E}_{K_m}(3)$$

with $m = \frac{n-1}{2}$ or $m = n-2$.

§4. A Particle in Euclidean Spaces $\mathbf{R}^n$ with $n \geq 4$

Corollary 3.1 asserts that an Euclidean space $\mathbf{R}^n$ can be really decomposed into 3-dimensional Euclidean spaces $\mathbf{R}^3$ underlying a complete graph K_m with $m = \frac{n-1}{2}$ or $m = n-2$. This suggests that we can visualize a particle in Euclidean space $\mathbf{R}^n$ by detecting its partially behavior in each $\mathbf{R}^3$. That is to say, we are needed to establish a *parallel probe* for Euclidean space $\mathbf{R}^n$ if $n \geq 4$.

Generally, a *parallel probe* on a combinatorial Euclidean space $\mathscr{E}_G(n_1, n_2, \cdots, n_m)$ is the set of probes established on each Euclidean space $\mathbf{R}^{n_i}$ for integers $1 \leq i \leq m$, particularly for $\mathscr{E}_G(3)$ which one can detects a particle in its each space $\mathbf{R}^3$ such as those shown in Fig.4.1 in where $G = K_4$ and there are four probes P_1, P_2, P_3, P_4.

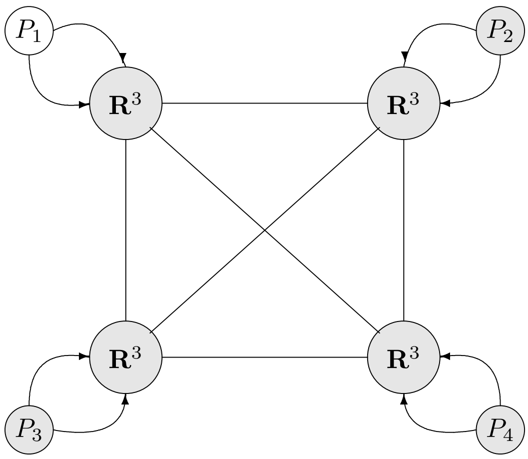

Fig.4.1

Notice that data obtained by such parallel probe is a set of local data $F(x_{i1}, x_{i2}, x_{i3})$ for $1 \leq i \leq m$ underlying G, i.e., the detecting data in a spatial $\bar{\epsilon}$ should be same if $\bar{\epsilon} \in \mathbf{R}^3_u \cap \mathbf{R}^3_v$, where $\mathbf{R}^3_u$ denotes the $\mathbf{R}^3$ at $u \in V(G)$ and $(\mathbf{R}^3_u, \mathbf{R}^3_v) \in E(G)$.

For data not in the $\mathbf{R}^3$ we lived, it is reasonable that we can conclude that all are the same as we obtained. Then we can analyze the global behavior of a particle in Euclidean space $\mathbf{R}^n$ with $n \geq 4$.

Then *how to apply this speculation*? Let us consider the gravitational field with dimensional$\geq$ 4. We know the Einstein's gravitation field equations in $\mathbf{R}^3$ are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu},$$

where $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$, $R = g^{\mu\nu} R_{\mu\nu}$ are the respective *Ricci tensor*, *Ricci scalar curvature* and

$$\kappa = \frac{8\pi G}{c^4} = 2.08 \times 10^{-48} cm^{-1} \cdot g^{-1} \cdot s^2$$

Now for a gravitational field $\mathbf{R}^n$ with $n \geq 4$, we decompose it into dimensional 3 Euclidean spaces $\mathbf{R}^3_u$, $\mathbf{R}^3_v$, $\cdots$, $\mathbf{R}^3_w$. Then we find Einstein's gravitational equations shown in [4] as follows:

$$R_{\mu_u\nu_u} - \frac{1}{2} g_{\mu_u\nu_u} R = -8\pi G \mathscr{E}_{\mu_u\nu_u},$$

$$R_{\mu_v\nu_v} - \frac{1}{2} g_{\mu_v\nu_v} R = -8\pi G \mathscr{E}_{\mu_v\nu_v},$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$R_{\mu_w\nu_w} - \frac{1}{2} g_{\mu_w\nu_w} R = -8\pi G \mathscr{E}_{\mu_w\nu_w}$$

for each $\mathbf{R}^3_u$, $\mathbf{R}^3_v, \cdots$, $\mathbf{R}^3_w$. If we decompose $\mathbf{R}^n$ into a combinatorial Euclidean fan-space $\widetilde{R}(\underbrace{3, 3, \cdots, 3}_{m})$, then $u, v, \cdots, w$ can be abbreviated to $1, 2 \cdots, m$. In this case, these gravitational equations can be represented by

$$R_{(\mu\nu)(\sigma\tau)} - \frac{1}{2} g_{(\mu\nu)(\sigma\tau)} R = -8\pi G \mathscr{E}_{(\mu\nu)(\sigma\tau)}$$

with a coordinate matrix

$$[\overline{x}_p] = \begin{bmatrix} x^{11} & \cdots & x^{1\widehat{m}} & \cdots & x^{13} \\ x^{21} & \cdots & x^{2\widehat{m}} & \cdots & x^{23} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x^{m1} & \cdots & x^{m\widehat{m}} & \cdots & x^{m3} \end{bmatrix}$$

for a point $p \in \mathbf{R}^n$, where $\widehat{m} = \dim(\bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i})$ a constant for $\forall p \in \bigcap\limits_{i=1}^{m} \mathbf{R}^{n_i}$ and $x^{il} = \frac{x^l}{m}$ for $1 \leq i \leq m, 1 \leq l \leq \widehat{m}$. Because the local behavior is that of the projection of the global. Whence, the following principle for determining behavior of particles in $\mathbf{R}^n$, $n \geq 4$ hold.

Projective Principle *A physics law in a Euclidean space* $\mathbf{R}^n \cong \widetilde{R}(\underbrace{3,3,\cdots,3}_{m})$ *with* $n \geq 4$ *is invariant under a projection on* $\mathbf{R}^3$ *in* $\widetilde{R}(\underbrace{3,3,\cdots,3}_{m})$.

Applying this principe enables us to find a spherically symmetric solution of Einstein's gravitational equations in Euclidean space $\mathbf{R}^n$.

§5. Discussions

A simple calculation shows that the dimension of the combinatorial Euclidean fan-space $\widetilde{R}(\underbrace{3,3,\cdots,3}_{m})$ in Section 3 is

$$\dim\widetilde{R}(\underbrace{3,3,\cdots,3}_{m}) = 3m + (1-m)\widehat{m}, \qquad (4-1)$$

for example, $\dim\widetilde{R}(\underbrace{3,3,\cdots,3}_{m}) = 6,\ 9,\ 12$ if $\widehat{m} = 0$ and $5,\ 7,\ 9$ if $\widehat{m} = 1$ and $m = 2,\ 3,\ 4$ with an additional time dimension t.

We have discussed in Section 1 that the visible geometry is the spherical geometry of dimensional 3. That is why the sky looks like a spherical surface. In these geometrical elements, such as those of point, line, ray, block, body,$\cdots$, etc., we can only see the image of bodies on our spherical surface, i.e., surface blocks.

Then *what is the geometry of transferring information*? Here, the term *information* includes information known or not known by human beings. So the geometry of transferring information consists of all possible transferring routes. In other words, a combinatorial geometry of dimensional≥ 1. Therefore, not all information transferring can be seen by our eyes. But some of them can be felt by our six organs with the helps of apparatus if needed. For example, the *magnetism* or *electromagnetism* can be only detected by apparatus. Consider $\widehat{m}$ the discussion is divided into two cases, which lead to two opposite conclusions following.

Case 1. $\widehat{m} = 3$.

In this case, by the formula $(4-1)$ we get that $\dim\widetilde{R}(\underbrace{3,3,\cdots,3}_{m}) = 3$, i.e., all Euclidean spaces $\mathbf{R}_1^3, \mathbf{R}_2^3, \cdots, \mathbf{R}_m^3$ are in one $\mathbf{R}^3$, which is the most enjoyed case by human beings. If it is so, all the behavior of Universe can be realized finally by human beings, particularly, the observed interval is ds and all natural things can be come true by experiments. This also means that the discover of science will be ended, i.e., we can find an ultimate theory for the Universe - the *Theory of Everything.* This is the earnest wish of Einstein himself beginning, and then more physicists devoted all their lifetime to do so in last century.

Case 2. $\widehat{m} \leq 2$.

If the Universe is so, then $\dim \widetilde{R}(\underbrace{3,3,\cdots,3}_{m}) \geq 4$. In this case, the observed interval in the field $\mathbf{R}^3_{human}$ where human beings live is

$$ds^2_{human} = a(t,r,\theta,\phi)dt^2 - b(t,r,\theta,\phi)dr^2 - c(t,r,\theta,\phi)d\theta^2 - d(t,r,\theta,\phi)d\phi^2.$$

by Schwarzschild metrics in R^3. But we know the metric in $\widetilde{R}(\underbrace{3,3,\cdots,3}_{m})$ should be $ds_{\widetilde{R}}$. Then

how to we explain the differences $(ds_{\widetilde{R}} - ds_{human})$ *in physics?*

Notice that one can only observes the line element ds_{human}, i.e.,, a projection of $ds_{\widetilde{R}}$ on $\mathbf{R}^3_{human}$ by the projective principle. Whence, all contributions in $(ds_{\widetilde{R}} - ds_{human})$ come from the spatial direction not observable by human beings. In this case, it is difficult to determine the exact behavior and sometimes only partial information of the Universe, which means that each law on the Universe determined by human beings is an approximate result and hold with conditions.

Furthermore, if $\widehat{m} \leq 2$ holds, because there are infinite underlying connected graphs, i.e., there are infinite combinations of $\mathbf{R}^3$, one can not find an ultimate theory for the Universe, which means the discover of science for human beings will endless forever, i.e., there are no a *Theory of Everything*.

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Counting Rooted Eulerian Planar Maps

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Abstract: In this paper a new method for establishing generating equations of rooted Eulerian planar maps will be provided. It is an algebraic method instead of the constructional one used as before and plays an important role in finding the kind of equations. Some equations of rooted loopless Eulerian planar maps will be obtained by using the method and some results will be corrected and simplified here.

Keywords: Eulerian map, generating function, enumerating equation, Smarandache multi-embedding, multi-surface.

MSC(2000): 05A15, 05C30

§1. Introduction

A *Smarandache multi-embedding of a graph G on a multi-surface $\widetilde{S}$* is a continuous mapping $\varsigma : G \to \widetilde{S}$ such that there are no intersections between any two edges unless its endpoints, where $\widetilde{S}$ is an unions of surfaces underlying a graph H. Particularly, if $|V(H)| = 1$, i.e., $\widetilde{S}$ is just a surface, such multi-embedding is the common embedding of G.

With respect to the enumeration of rooted Eulerian planar maps the first result for enumerating rooted general Eulerian planar maps with vertex partition was achieved by Tutte [10] in the early 1960's. In 1986 the enumeration of rooted non-separable Eulerian planar maps with vertex partition was studied by Liu [4]. In 1992 the enumeration of rooted loopless Eulerian planar maps with vertex partition and other variables as parameters were investigated by Liu [5,6,7] too. From then on some new results were obtained [1,2,8,9], but the method used there was so difficult that one can not understand them easily. In 2004 the enumeration of unrooted Eulerian and unicursl planar maps with the number of edges was resulted by Liskovets [3] based on the rooted results. In present article we will provide an algebraic method instead of that used in the past for counting this kind of planar maps. It will paly an important role in establishing the equations of all kinds of rooted Eulerian planar maps. As examples, some equations of rooted loopless Eulerian planar maps can be derived by using the method. The procedure

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and some results in [5,6,8] will be reduced greatly and updated properly.

In general, *rooting* a map means distinguishing one edge on the boundary of *the outer face* as the *root-edge*, and one end of that edge as the *root-vertex*. In diagrams we usually represent the root-edge as an edge with an arrow in the outer face, the arrow being drawn from the root-vertex to the other end. So the outer face is also called the *root-face*. A planar map with a rooting is said to be a *rooted planar map*. We say that two rooted planar maps are *combinatorially equivalent* or up to *root-preserving isomorphism* if they are related by one to one correspondence of their elements, which maps vertices onto vertices, edges onto edges and faces onto faces,and which preserves incidence relations and the rooted elements. Otherwise, *combinatorially inequivalent* or *nonisomorphic* here.

Let $\mathcal{M}$ be any set of maps. For a map $M \in \mathcal{M}$ let $M - R$ and $M \bullet R$ be the resultant maps of deleting the root-edge $R(M)$ from M and contracting $R(M)$ into a vertex as the new root-vertex, respectively. For a vertex v of M let $val(v)$ be the valency of the vertex v. Moreover, the valency of the root-vertex of M is denoted by $val(M)$.

Terminologies and notations not explained here refer to [9].

§2. Relations on Maps

In order to set up the enumerating equation satisfied by some generating functions we have to introduce the operations on maps in $\mathcal{M}$.

Let

$$\mathcal{M}\langle R\rangle = \{M - R \mid M \in \mathcal{M}\}; \quad \mathcal{M}(R) = \{M \bullet R \mid M \in \mathcal{M}\}, \tag{31}$$

and let

$$\begin{cases} \widetilde{\bigtriangledown}\mathcal{M} = \sum\limits_{M\in\mathcal{M}} \{\bigtriangledown_i M \mid i = 1, 2, \cdots, l(M) - 1\}; \\ \bigtriangledown\mathcal{M} = \sum\limits_{M\in\mathcal{M}} \{\bigtriangledown_i M \mid i = 0, 1, 2, \cdots, l(M)\}, \end{cases} \tag{32}$$

where $\bigtriangledown_i M$ is the resultant map of splitting the root-vertex of M into two vertices v'_r and v''_r with a new edge $\langle v'_r, v''_r\rangle$ as the root-edge of the new map $\bigtriangledown_i M$ such that the valency of its root-vertex $val(\bigtriangledown_i M) = i + 1$.

Further,write that

$$\begin{cases} \mathcal{M}^{(e)} = \{M \in \mathcal{M} \mid val(M) \equiv 0(\mathrm{mod}2)\}; \\ \mathcal{M}^{(o)} = \{M \in \mathcal{M} \mid val(M) \equiv 1(\mathrm{mod}2)\}. \end{cases} \tag{33}$$

It is clear that $\mathcal{M}^{(e)}$ and $\mathcal{M}^{(o)}$ stand for maps in $\mathcal{M}$ with the valency of root-vertex of the maps being even and odd, respectively.

Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two sets of maps. For two maps $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$, let $M_1\dot{+}M_2$ be the map $M_1 \bigcup M_2$ such that

(i) $M_1 \bigcap M_2$ is only a vertex as the root-vertex of $M_1\dot{+}M_2$;

(ii) M_1 is inside one of the faces incident with the root-vertex of M_2;

(iii) The root-edge of $M_1\dot{+}M_2$ is the same as that of M_2;

(iv) The first occurrence of the edges in M_1 incident with the root-vertex of $M_1\dot{+}M_2$ is the root-edge of M_1 when one moves around the root-vertex of $M_1\dot{+}M_2$ in the rotational direction starting from the root-edge of $M_1\dot{+}M_2$.

For the maps $M_i \in \mathcal{M}_i$, $i = 1, 2, \cdots, k$, we define that

$$\begin{cases} M_1\dot{+}M_2\dot{+}\cdots\dot{+}M_k = \left(M_1\dot{+}M_2\dot{+}\cdots\dot{+}M_{k-1}\right)\dot{+}M_k; \\ \mathcal{M}_1 \odot \mathcal{M}_2 \odot \cdots \odot \mathcal{M}_k = \left\{M_1\dot{+}M_2\dot{+}\cdots\dot{+}M_k \mid M_i \in \mathcal{M}_i, 1 \leqslant i \leqslant k\right\}, \\ \mathcal{M}^{\odot k} = \mathcal{M}_1 \odot \mathcal{M}_2 \odot \cdots \odot \mathcal{M}_k|_{\mathcal{M}_1=\mathcal{M}_2=\cdots=\mathcal{M}_k=\mathcal{M}}. \end{cases} \tag{34}$$

Now, we have to introduce another kind operation in order to finish the construction of the sets of maps as follows.

For two maps $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$, let $M_1\hat{+}M_2$ be the resultant map of identifying the two root-edges of M_1 and M_2 such that M_1 is inside the non-root-face incident with the root-edge of M_2, or onto the non-root-side of M_2 if the root-edge of M_2 is a cut-edge. Of course, the root-edge of $M_1\hat{+}M_2$ has to be the identified edge and the non-root-face incident with the root-edge of $M_1\hat{+}M_2$ is the same as in M_1.

For the maps $M \in \mathcal{M}$ and $M_i \in \mathcal{M}_i$, $i = 1, 2, \cdots, k$, we define that

$$\begin{cases} M_1\hat{+}M_2\hat{+}\cdots\hat{+}M_k = \left(M_1\hat{+}M_2\hat{+}\cdots\hat{+}M_{k-1}\right)\hat{+}M_k; \\ \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_k = \left\{M_1\hat{+}M_2\hat{+}\cdots\hat{+}M_k \mid M_i \in \mathcal{M}_i, 1 \leqslant i \leqslant k\right\}; \\ \mathcal{M}^{\oplus k} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_k|_{\mathcal{M}_1=\mathcal{M}_2=\cdots=\mathcal{M}_k=\mathcal{M}}. \end{cases} \tag{35}$$

A map is called *Eulerian* if all its vertices are of even valency. It is well-known that a map is Eularian if and only if it has an Eularian circuit, a circuit containing each of the edges exactly once. A map is called *loopless* if there is no any *loop* in the map.

Let $\mathcal{E}_{nl}$ be *the set of all rooted loopless Eulerian planar maps* with the *vertex map* ϑ in $\mathcal{E}_{nl}$ as a special case. Of course, the loop map O is not in $\mathcal{E}_{nl}$. It is easily checked that no Eulerian maps has a separable edge.

The enumerating problems of rooted loopless Eulerian planar maps will be discussed here by using a new method witch is much simpler than that used in the past [4,5,6,9].

Let $\mathcal{E}_{\mathrm{nl}_0} = \{\vartheta\}$ and $\mathcal{E}_{\mathrm{nl}_i} = \{M \in \mathcal{E}_{\mathrm{nl}} - \vartheta \mid \mathrm{R(M)}$ is i multi-edges in $M\}$, for $i \geqslant 1$. Then the set $\mathcal{E}_{\mathrm{nl}}$ can be partitioned into the following form

$$\mathcal{E}_{\mathrm{nl}} = \sum_{i\geqslant 0} \mathcal{E}_{\mathrm{nl}_i}, \quad \text{and} \quad \mathcal{E}_{\mathrm{nl}}(R) = \sum_{i\geqslant 0} \mathcal{E}_{\mathrm{nl}_i}(R), \tag{36}$$

where $\mathcal{E}_{\mathrm{nl}_0}(R) = \mathcal{E}_{\mathrm{nl}_0} = \{\vartheta\}$ and $\mathcal{E}_{\mathrm{nl}_1}(R) = \mathcal{E}_{\mathrm{nl}} - \mathcal{E}_{\mathrm{nl}_0}$. Let $\mathcal{E}_{in} = \mathcal{E}_{\mathrm{nl}}\dot{+}\{O\}$ be the set of all rooted *inner Eulerian planar maps* [9], then from (4) we have

$$\mathcal{E}_{\mathrm{nl}_i}(R) = \mathcal{E}_{in}^{\odot i-1} \odot \mathcal{E}_{\mathrm{nl}},$$

for $i \geqslant 2$.

If we write $\mathcal{E}_{\rm in}^{\odot 0}=\mathcal{E}_{\rm nl_0}=\{\vartheta\}$, then from (6) we have

$$\begin{aligned}\mathcal{E}_{\rm nl}(R) &= \sum_{i\geqslant 0}\mathcal{E}_{{\rm nl}_i}(R)=\mathcal{E}_{\rm nl_0}(R)+\mathcal{E}_{\rm nl_1}(R)+\sum_{i\geqslant 2}\mathcal{E}_{{\rm nl}_i}(R)\\ &= \mathcal{E}_{\rm nl_0}+(\mathcal{E}_{\rm nl}-\mathcal{E}_{\rm nl_0})+\sum_{i\geqslant 2}(\mathcal{E}_{\rm in}^{\odot i-1}\odot\mathcal{E}_{\rm nl})\\ &= \mathcal{E}_{\rm nl}+\sum_{i\geqslant 1}(\mathcal{E}_{\rm in}^{\odot i}\odot\mathcal{E}_{\rm nl})=\mathcal{E}_{\rm nl_0}\odot\mathcal{E}_{\rm nl}+\sum_{i\geqslant 1}(\mathcal{E}_{\rm in}^{\odot i}\odot\mathcal{E}_{\rm nl})\\ &= \mathcal{E}_{\rm in}^{\odot 0}\odot\mathcal{E}_{\rm nl}+\sum_{i\geqslant 1}(\mathcal{E}_{\rm in}^{\odot i}\odot\mathcal{E}_{\rm nl}).\end{aligned}$$

i.e.,

$$\mathcal{E}_{\rm nl}(R)=\sum_{i\geqslant 0}(\mathcal{E}_{\rm in}^{\odot i}\odot\mathcal{E}_{\rm nl}). \tag{37}$$

Now, In order to enumerate the maps in $\mathcal{E}_{\rm nl}$ conveniently, we need to reconstruct the set $\mathcal{E}_{\rm nl}$ according to the construction of $\mathcal{E}_{\rm nl}(R)$ in (7). Hence, we suppose that

$$\mathcal{F}=\sum_{i\geqslant 0}\left[\left(\widetilde{\bigtriangledown}\mathcal{E}_{\rm in}\right)^{\oplus i}\oplus(\bigtriangledown\mathcal{E}_{\rm nl})\right], \tag{38}$$

where $\left(\widetilde{\bigtriangledown}\mathcal{E}_{\rm in}\right)^{\oplus 0}$ is defined as $\mathcal{E}_{\rm nl_0}$.

In general, a map in $\mathcal{F}$ may be not Eulerian. It is obvious that $\mathcal{F}$ can be classified into two classes $\mathcal{F}^{(e)}$ and $\mathcal{F}^{(o)}$ where $\mathcal{F}^{(e)}$ is just what we need because the maps in it are all Eulerian, i.e., $\mathcal{F}^{(e)}\subseteq\mathcal{E}_{\rm nl}-\mathcal{E}_{\rm nl_0}$. Conversely, for any map $M\in\mathcal{E}_{\rm nl}-\mathcal{E}_{\rm nl_0}$, there is a set $\mathcal{E}_{{\rm nl}_i}$, $i\geqslant 1$ such that $M\in\mathcal{E}_{{\rm nl}_i}$, thus $M\bullet R\in\mathcal{E}_{{\rm nl}_i}(R)=\mathcal{E}_{\rm in}^{\odot i-1}\odot\mathcal{E}_{\rm nl}$. So we have $M\in\mathcal{E}_{{\rm nl}_i}=\left[(\widetilde{\bigtriangledown}\mathcal{E}_{\rm in})^{\oplus i-1}\oplus(\bigtriangledown\mathcal{E}_{\rm nl})\right]^{(e)}\subset\mathcal{F}^{(e)}$, i.e., $\mathcal{E}_{\rm nl}-\mathcal{E}_{\rm nl_0}\subseteq\mathcal{F}^{(e)}$. In the other words, we have

$$\mathcal{E}_{\rm nl}=\mathcal{E}_{\rm nl_0}+\mathcal{F}^{(e)}\quad\text{and}\quad\mathcal{F}^{(e)}=\mathcal{F}-\mathcal{F}^{(o)}. \tag{39}$$

In addition, it is not difficult to see that

$$\mathcal{E}_{\rm in}\langle R\rangle=\mathcal{E}_{\rm nl}. \tag{40}$$

§3. Equations with Vertex Partition

In this section we want to discuss the following generating function for the set $\mathcal{M}$ of some maps.

$$g_{\mathcal{M}}(x:\underline{y})=\sum_{M\in\mathcal{M}}x^{l(M)}\underline{y}^{\underline{n}(M)}, \tag{41}$$

in which $\underline{y}(M)$ and $\underline{n}(M)$ stand for infinite vectors, and

$$\underline{y}^{\underline{n}(M)}=\prod_{i\geqslant 1}y_i^{n_i(M)};\quad \underline{y}=(y_1,y_2,\cdots);\quad \underline{n}(M)=(n_1(M),n_2(M),\cdots),$$

where $l(M)=val(M)$ and $n_i(M)$ is the number of the non-root vertices of valency i, $i\geqslant 1$. The function (11) is said to be the *vertex partition function* of $\mathcal{M}$. Naturally, for a Eulerian planar map $M\in\mathcal{E}_{\rm nl}$, we may let $l(M)=val(M)=2m(M)$ and $n_{2j+1}(M)\equiv 0$ for $j\geqslant 0$.

For this reason we need to introduce the following *Blisard* -operator in y

$$\int_y y^i = y_i, \quad i \geqslant 1 \quad \text{and} \quad \int_y y^0 = 1,$$

which is a *linear operator* and for a function $f(z)$ we define that

$$\delta_{x,y} f = \frac{f(x) - f(y)}{x^2 - y^2}. \tag{42}$$

They are said to be (x, y)-deference of $f(z)$.

In the following, the new algebraic method is used for enumerating the set of maps in $\mathcal{F}^{(e)}$.

Lemma 3.1 *For the set $\mathcal{F}^{(e)}$, we have*

$$g_{\mathcal{F}^{(e)}}(x : \underline{y}) = \int_y \frac{x^2 y^2 \delta_{x,y}(f + z^2 f^2)}{1 - x^2 y^2 \delta_{x,y}(2f + z^2 f^2)}, \tag{43}$$

where $f = f(z) = g_{\mathcal{E}_{\mathrm{nl}}}(z : \underline{y})$.

Proof From the definitions (3), (8) and (11), we have

$$\begin{aligned}
g_{\mathcal{F}}(x : \underline{y}) &= \sum_{i \geqslant 0} x \int_y y \left(\sum_{M \in \mathcal{E}_{\mathrm{in}}} \sum_{j=1}^{l(M)-1} x^j y^{l(M)-j} \underline{y}^{\underline{n}(M)} \right)^i \sum_{M \in \mathcal{E}_{\mathrm{nl}}} \sum_{j=0}^{l(M)} x^j y^{l(M)-j} \underline{y}^{\underline{n}(M)} \\
&= x \int_y y \sum_{i \geqslant 0} \left(xy^{-1} \frac{g_{\mathcal{E}_{\mathrm{in}}}(y) - x^{-1} y g_{\mathcal{E}_{\mathrm{in}}}(x)}{1 - xy^{-1}} \right)^i \frac{f(y) - xy^{-1} f(x)}{1 - xy^{-1}} \\
&= \int_y \sum_{i \geqslant 0} \left(xy \frac{yf(y) - xf(x)}{y - x} \right)^{i+1} \\
&= \int_y \frac{xy(yf(y) - xf(x))}{y\,(1 + x^2 f(x)) - x\,(1 + y^2 f(y))} \\
&= x^2 \int_y y^2 \frac{\left(1 + y^2 f(y)\right) f(y) - \left(1 + x^2 f(x)\right) f(x)}{y^2\,(1 + x^2 f(x))^2 - x^2\,(1 + y^2 f(y))^2} + g_{\mathcal{F}^{(o)}}(x : \underline{y})
\end{aligned}$$

i.e.,

$$g_{\mathcal{F}}(x : \underline{y}) = \int_y \frac{x^2 y^2 \delta_{x,y}(f + z^2 f^2)}{1 - x^2 y^2 \delta_{x,y}(2f + z^2 f^2)} + g_{\mathcal{F}^{(o)}}(x : \underline{y}),$$

where $f = f(z) = g_{\mathcal{E}_{\mathrm{nl}}}(z : \underline{y})$ and

$$g_{\mathcal{F}^{(o)}}(x : \underline{y}) = \int_y \frac{xy \delta_{x,y}(z^2 f)}{1 - x^2 y^2 \delta_{x,y}(2f + z^2 f^2)}. \tag{44}$$

This lemma can be derived from (9) immediately. □

Theorem 3.1 *The generating function $f = f(z) = g_{\mathcal{E}_{\mathrm{nl}}}(z : \underline{y})$ with vertex partition satisfies the following enumerating equation*

$$f = \int_y \frac{1 - x^2 y^2 \delta_{x,y} f}{1 - x^2 y^2 \delta_{x,y}(2f + z^2 f^2)}, \tag{45}$$

This is a modification and simplification to the result (3.13) in [5].

Proof It is clear that $g_{\mathcal{E}_{\mathrm{nl}_0}}(x : \underline{y}) = 1$. So from (9) and (13), Eq.(15) is obtained by grouping the terms. □

§4. Equations with the Numbers of Vertices and Faces

In what following we want to study the following generating function for the set $\mathcal{M}$ of some maps.

$$f_{\mathcal{M}}(x,y,z) = \sum_{M\in\mathcal{M}} x^{l(M)}y^{n(M)}z^{q(M)}, \tag{46}$$

where $l(M) = val(M)$ and $n(M)$ and $q(M)$ are the numbers of non-root vertices and inner faces of $M \in \mathcal{M}$, respectively. It is clear that we may write $l(M) = val(M) = 2m(M)$ if $M \in \mathcal{E}_{\mathrm{nl}}$ is an Eulerian map.

In fact, this section will provide a functional equation satisfied by the generating function $f = f_{\mathcal{E}_{\mathrm{nl}}}(x,y,z)$ with the valency of root-vertex, the numbers of non-root vertices and inner faces of the maps in $\mathcal{E}_{\mathrm{nl}}$, respectively, as three parameters.

Summing the results as above, we can obtain the following results.

Lemma 4.1 *For the set $\mathcal{E}_{\mathrm{in}}$, we have*

$$f_{\mathcal{E}_{\mathrm{in}}}(x,y,z) = x^2zf, \tag{47}$$

where $f = f_{\mathcal{E}_{\mathrm{nl}}}(x,y,z)$.

Proof The Lemma is obtained directly from (10) and (16). □

In the following, the algebraic method is used again for enumerating the set of maps in $\mathcal{F}^{(e)}$.

Lemma 4.2 *For the set $\mathcal{F}^{(e)}$, we have*

$$f_{\mathcal{F}^{(e)}}(x,y,z) = x^2y\frac{(1+zf^*)\,f^* - \left(1+x^2zf\right)f}{\left(1+x^2zf\right)^2 - x^2\left(1+zf^*\right)^2}, \tag{48}$$

where $f = f_{\mathcal{E}_{\mathrm{nl}}}(x,y,z)$ and $f^* = f_{\mathcal{E}_{\mathrm{nl}}}(1,y,z)$.

Proof By (8),(9),(12) and (16) we have

$$\begin{aligned}
f_{\mathcal{F}}(x,y,z) &= xy\sum_{i\geqslant 0}\left(\sum_{M\in\mathcal{E}_{\mathrm{in}}}\sum_{j=1}^{2m(M)-1} x^jy^{n(M)}z^{q(M)}\right)^i \sum_{M\in\mathcal{E}_{\mathrm{nl}}}\sum_{j=0}^{2m(M)} x^jy^{n(M)}z^{q(M)} \\
&= xy\sum_{i\geqslant 0}\left(\frac{xf^*_{\mathcal{E}_{\mathrm{in}}} - f_{\mathcal{E}_{\mathrm{in}}}}{1-x}\right)^i\frac{f^*-xf}{1-x} = yz^{-1}\sum_{i\geqslant 0}\left(xz\frac{f^*-xf}{1-x}\right)^{i+1} \\
&= \frac{xy(f^*-xf)}{1-x\left(1+zf^*\right)+x^2zf},
\end{aligned}$$

i.e.,

$$f_{\mathcal{F}}(x,y,z) = x^2y\frac{(1+zf^*)\,f^* - \left(1+x^2zf\right)f}{\left(1+x^2zf\right)^2 - x^2\left(1+zf^*\right)^2} + f_{\mathcal{F}^{(o)}}(x,y,z),$$

where $f^*_{\mathcal{E}_{\text{in}}} = f_{\mathcal{E}_{\text{in}}}(1,y,z)$ and

$$f_{\mathcal{F}^{(o)}}(x,y,z) = \frac{xy\left(f^* - x^2f\right)}{\left(1+x^2zf\right)^2 - x^2\left(1+zf^*\right)^2}. \tag{49}$$

The Lemma is obtained directly from the definition of $\mathcal{F}^{(e)}$ in (9). □

Theorem 4.1 *The generating function $f = f_{\mathcal{E}_{\text{nl}}}(x,y,z)$ with the valency of root-vertex, the numbers of non-root vertices and inner faces of the maps in $\mathcal{E}_{\text{nl}}$, respectively, as three parameters satisfies the following cubic equation*

$$f = 1 + x^2y\frac{(1+zf^*)\,f^* - \left(1+x^2zf\right)f}{\left(1+x^2zf\right)^2 - x^2\left(1+zf^*\right)^2}, \tag{50}$$

where $f^* = f(1,y,z)$.

Proof From (9) we have

$$f = f_{\mathcal{E}_{\text{nl}_0}}(x,y,z) + f_{\mathcal{F}^{(e)}}(x,y,z),$$

where $f_{\mathcal{E}_{\text{nl}_0}}(x,y,z) = 1$. By substituting (18) into the above formula Eq(20) holds. □

§5. Equations with the Edge Number and the Root-Face Valency

In this section we study the following generating function for the set $\mathcal{M}$ of some maps.

$$f_{\mathcal{M}}(x,y,z) = \sum_{M\in\mathcal{M}} x^{l(M)}y^{s(M)}z^{p(M)}, \tag{51}$$

where $l(M) = val(M)$ and $s(M)$ and $p(M)$ are the number of edges and the valency of root-face of $M \in \mathcal{M}$, respectively. we may also write $l(M) = val(M) = 2m(M)$ if $M \in \mathcal{E}_{\text{nl}}$ is an Eulerian map.

In this section we provide a functional equation satisfied by the generating function $f = f_{\mathcal{E}_{\text{nl}}}(x,y,z)$ with the valency of root-vertex, the number of edges the valency of the root-face of the maps in $\mathcal{E}_{\text{nl}}$, respectively, as three parameters. Write that

$$h_{\mathcal{E}_{\text{nl}}}(x,y) = f_{\mathcal{E}_{\text{nl}}}(x,y,1), \quad F_{\mathcal{E}_{\text{nl}}}(y,z) = f_{\mathcal{E}_{\text{nl}}}(1,y,z), \quad H_{\mathcal{E}_{\text{nl}}}(y) = f_{\mathcal{E}_{\text{nl}}}(1,y,1).$$

Lemma 5.1 *For the set $\mathcal{E}_{\text{in}}$, we have*

$$f_{\mathcal{E}_{\text{in}}}(x,y,z) = x^2yzf, \tag{52}$$

where $f = f_{\mathcal{E}_{\text{nl}}}(x,y,z)$.

Proof The Lemma is obtained directly from (10) and (21). □

Lemma 5.2 *For the set $\mathcal{F}^{(e)}$, we have*

$$f_{\mathcal{F}^{(e)}}(x,y,z)=\frac{x^2yz[FH_0-(1+x^2yh)f]}{(1+x^2yh)^2-x^2H_0^2}-\frac{(1-z)x^2y^2zHFhf}{1-x^2y^2H^2h^2}, \tag{53}$$

where $h=h_{\mathcal{E}_{\rm nl}}(x,y)$, $F=F_{\mathcal{E}_{\rm nl}}(y,z)$, $H=H_{\mathcal{E}_{\rm nl}}(y)$ and $H_0=1+yH$.

Proof By (8),(9),(12) and (21) we have

$$\begin{aligned} f_{\mathcal{F}}(x,y,z) &= xyz\sum_{i\geqslant 0}\left(\sum_{M\in\mathcal{E}_{\rm in}}\sum_{j=1}^{2m(M)-1}x^jy^{s(M)}\right)^i\sum_{M\in\mathcal{E}_{\rm nl}}\sum_{j=0}^{2m(M)}x^jy^{s(M)}z^{p(M)} \\ &- \sum_{k\geqslant 1}x^ky^k(z-z^2)fh^{k-1}H^{k-1}F \\ &= xyz\sum_{i\geqslant 0}\left(\frac{xH_{\mathcal{E}_{\rm in}}-h_{\mathcal{E}_{\rm in}}}{1-x}\right)^i\frac{F-xf}{1-x}-xyz(1-z)Ff\sum_{k\geqslant 1}(xyHh)^{k-1} \\ &= \frac{xyz(F-xf)}{1-x-xH_{\mathcal{E}_{\rm in}}+h_{\mathcal{E}_{\rm in}}}-\frac{(1-z)xyzFf}{1-xyHh}, \end{aligned}$$

where $H_{\mathcal{E}_{\rm in}}=yH$,$h_{\mathcal{E}_{\rm in}}=x^2yh$, i.e.,

$$f_{\mathcal{F}}(x,y,z)=\frac{x^2yz[FH_0-(1+x^2yh)f]}{(1+x^2yh)^2-x^2H_0^2}-\frac{(1-z)x^2y^2zHFhf}{1-x^2y^2H^2h^2}+f_{\mathcal{F}^{(o)}}(x,y,z),$$

where $H_0=1+yH$ and

$$f_{\mathcal{F}^{(o)}}(x,y,z)=\frac{xyz[(1+x^2yh)F-x^2H_0f]}{(1+x^2yh)^2-x^2H_0^2}-\frac{(1-z)xyzFf}{1-x^2y^2H^2h^2}. \tag{54}$$

The Lemma is obtained directly from the definition of $\mathcal{F}^{(e)}$ in (9). □

Theorem 5.1 *The generating function $f=f_{\mathcal{E}_{\rm nl}}(x,y,z)$ with the valency of root-vertex, the numbers of non-root vertices and inner faces of the maps in $\mathcal{E}_{\rm nl}$, respectively, as three parameters satisfies the following cubic equation*

$$f=1+\frac{x^2yz[H_0F-(1+x^2yh)f]}{(1+x^2yh)^2-x^2H_0^2}-\frac{(1-z)x^2y^2zHFhf}{1-x^2y^2H^2h^2}, \tag{55}$$

where $h=h_{\mathcal{E}_{\rm nl}}(x,y)$, $F=F_{\mathcal{E}_{\rm nl}}(y,z)$, $H=H_{\mathcal{E}_{\rm nl}}(y)$ and $H_0=1+yH$.

Proof From (9) we have

$$f=f_{\mathcal{E}_{\rm nl_0}}(x,y,z)+f_{\mathcal{F}^{(e)}}(x,y,z),$$

where $f_{\mathcal{E}_{\rm nl_0}}(x,y,z)=1$. By substituting (23) into the above formula Eq(25) holds. □

Theorem 5.2 *The generating function $h=h_{\mathcal{E}_{\rm nl}}(x,y)$ with the valency of root-vertex and the number of edges of the maps in $\mathcal{E}_{\rm nl}$, respectively, as two parameters satisfies the following cubic equation*

$$h_0^3-h_0^2-(y+H_0^2)x^2h_0+x^2H_0^2+x^4yH_0=0, \tag{56}$$

where $H_0 = 1 + yH_{\mathcal{E}_{\rm nl}}(y)$ *and* $h_0 = 1 + x^2yh_{\mathcal{E}_{\rm nl}}(x,y)$.

This is a modification and simplification to the result (4.11) in [5].

Proof For any map $M \in \mathcal{E}_{\rm nl}$, since the number of vertices of M is $n(M)+1$ and the number of faces of M is $q(M)+1$, the number $s(M)$ of edges of M is $n(M)+q(M)$ by Eulerian formula. It follows from (16) and (21) that $h = h_{\mathcal{E}_{\rm nl}}(x,y) = f_{\mathcal{E}_{\rm nl}}(x,y,y)$. So if we take $z=y$, then Eq(20) becomes Eq(26) by grouping the terms where $H = f^*_{\mathcal{E}_{\rm nl}}(y,y) = f_{\mathcal{E}_{\rm nl}}(1,y,y)$.

Of course, Eq(26) may be also derived by substituting $y_{2i} = y^i$ into Eq(15) and replacing x^2 in it with x^2y since $s(M) = \sum_{i\geqslant 0} in_{2i}(M)$, or by substituting $z=1$ into Eq(25). □

Note that Eq(20) and Eq(26) have been solved in the forms of parametric expressions or explicit formulae in [2] and [7], respectively.

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The n^{th} Power Signed Graphs-II

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. In this paper, we present solutions of some signed graph switching equations involving the line signed graph, complement and n^{th} power signed graph operations.

Keywords: Smarandachely k-signed graphs, Smarandachely k-marked graphs, signed graphs, marked graphs, balance, switching, line signed graph, complementary signed graph, n^{th} power signed graph.

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§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [6]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of S* and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (See [7]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

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A *marking* of S is a function $\mu : V(G) \to \{+,-\}$; A signed graph S together with a marking μ by S_μ.

The following characterization of balanced signed graphs is well known.

Proposition 1.1(E. Sampathkumar [8]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exist a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Given a marking μ of S, by *switching* S with respect to μ we mean reversing the sign of every edge of S whenever the end vertices have opposite signs in $\mathcal{S}_\mu$ [1]. We denote the signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and this signed graph is called the *μ-switched signed graph* or just *switched signed graph.* A signed graph S_1 switches to a signed graph S_2 (that is, they are *switching equivalent* to each other), written $S_1 \sim S_2$, whenever there exists a marking μ such that $\mathcal{S}_\mu(S_1) \cong S_2$.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [13]) or *cycle isomorphic* (see [14]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [14]):

Proposition 1.2(T. Zaslavsky [14]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

Behzad and Chartrand [4] introduced the notion of line signed graph $L(S)$ of a given signed graph S as follows: Given a signed graph $S = (G, \sigma)$ its *line signed graph* $L(S) = (L(G), \sigma')$ is the signed graph whose underlying graph is $L(G)$, the line graph of G, where for any edge e_ie_j in $L(S)$, $\sigma'(e_ie_j)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [5], is as follows:

The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [5] (See also E. Sampathkumar et al. [9]).

Proposition 1.3(**M. Acharya** [2]) *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

For any positive integer k, the k^{th} iterated line signed graph, $L^k(S)$ of S is defined as follows:

$$L^0(S) = S,\ L^k(S) = L(L^{k-1}(S)).$$

Corollary 1.4 *For any signed graph $S = (G, \sigma)$ and for any positive integer k, $L^k(S)$ is balanced.*

Let $S = (G, \sigma)$ be a signed graph. Consider the marking μ on vertices of S defined as follows: for each vertex $v \in V$, $\mu(v)$ is the product of the signs on the edges incident with v. *The complement* of S is a signed graph $\overline{S} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$,

$\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S}$ as defined here is a balanced signed graph due to Proposition 1.1.

§2. n^{th} Power signed graph

The n^{th} *power graph* G^n of G is defined in [3] as follows:

The n^{th} power has same vertex set as G, and has two vertices u and v adjacent if their distance in G is n or less.

In [12], we introduced a natural extension of the notion of n^{th} power graphs to the realm of signed graphs: Consider the marking μ on vertices of S defined as follows: for each vertex $v \in V$, $\mu(v)$ is the product of the signs on the edges incident at v. The n^{th} *power signed graph* of S is a signed graph $S^n = (G^n, \sigma')$, where G^n is the underlying graph of S^n, where for any edge $e = uv \in G^n$, $\sigma'(uv) = \mu(u)\mu(v)$.

The following result indicates the limitations of the notion of n^{th} power signed graphs as introduced above, since the entire class of unbalanced signed graphs is forbidden to n^{th} power signed graphs.

proposition 2.1(P. Siva Kota Reddy et al.[12]) *For any signed graph $S = (G, \sigma)$, its n^{th} power signed graph S^n is balanced.*

For any positive integer k, the k^{th} iterated n^{th} power signed graph, $(S^n)^k$ of S is defined as follows:

$$(S^n)^0 = S,\ (S^n)^k = S^n((S^n)^{k-1}).$$

Corollary 2.2 *For any signed graph $S = (G, \sigma)$ and any positive integer k, $(S^n)^k$ is balanced.*

The *degree* of a signed graph switching equation is then the maximum number of operations on either side of an equation in standard form. For example, the degree of the equation $S \sim \overline{L(S)}$ is one, since in standard form it is $L(S) \sim \overline{S}$, and there is one operation on each side of the equation. In [12], the following signed graph switching equations are solved:

- $\overline{S} \sim (L(S))^n$ (1)
- $L(\overline{S}) \sim (L(S))^n$ (2)
- $\overline{L(S)} \sim \overline{S}^n$, where $n \geq 2$ (3)
- $L^2(S) \sim S^n$, where $n \geq 2$ (4)
- $L^2(S) \sim \overline{S^n}$, where $n \geq 2$, and (5)
- $L^2(S) \sim \overline{S}^n$, where $n \geq 2$. (6)

Recall that $L^2(S)$ is the second iterated line signed graph S.

Several of these signed graph switching equations can be viewed as generalized of earlier work [11]. For example, equation (1) is a generalization of $L(S) \sim \overline{S}$, which was solved by Siva Kota Reddy and Subramanya [11]. When $n = 1$ in equations (3) and (4), we get $L(S) \sim S$ and $L^2(S) \sim S^2$, which was solved in [11]. If $n = 1$ in (5) and (6), the resulting signed graph switching equation was solved by Siva Kota Reddy and Subramanya [11].

Further, in this paper we shall solve the following three signed graph switching equations:

- $L(S) \sim S^n$ (7)
- $\overline{L(S)} \sim S^n \quad (or L(S) \sim \overline{S^n})$ (8)
- $L(S) \sim (\overline{S})^n$ (9)

In the above expressions, the equivalence (i.e, $\sim$) means the switching equivalent between corresponding graphs.

Note that for $n = 1$, the equation (7) is reduced to the following result of E. Sampathkumar et al. [10].

Proposition 2.3(E. Sampathkumar et al. [10]) *For any signed graph $S = (G, \sigma)$, $L(S) \sim S$ if, and only if, S is a balanced signed graph and G is 2-regular.*

Note that for $n = 1$, the equations (8) and (9) are reduced to the signed graph switching equation which is solved by Siva Kota Reddy and Subramanya [11].

Proposition 2.4 (P. Siva Kota Reddy and M. S. Subramanya [11]) *For any signed graph $S = (G, \sigma)$, $L(S) \sim \overline{S}$ if, and only if, G is either C_5 or $K_3 \circ K_1$.*

§3. The Solution of $L(S) \sim S^n$

We now characterize signed graphs whose line signed graphs and its n^{th} power line signed graphs are switching equivalent. In the case of graphs the following result is due to J. Akiyama et. al [3].

Proposition 3.1(J. Akiyama et al. [3]) *For any $n \geq 2$, the solutions to the equation $L(G) \cong G^n$ are graphs $G = mK_3$, where m is an arbitrary integer.*

Proposition 3.2 *For any signed graph $S = (G, \sigma)$, $L(S) \sim S^n$, where $n \geq 2$ if, and only if, G is mK_3, where m is an arbitrary integer.*

Proof Suppose $L(S) \sim S^n$. This implies, $L(G) \cong G^n$ and hence by Proposition 3.1, we see that the graph G must be isomorphic to mK_3.

Conversely, suppose that G is mK_3. Then $L(G) \cong G^n$ by Proposition 3.1. Now, if S is a signed graph with underlying graph as mK_3, by Propositions 1.3 and 2.1, $L(S)$ and S^n are balanced and hence, the result follows from Proposition 1.2. $\square$

§4. Solutions of $\overline{L(S)} \sim S^n$

In the case of graphs the following result is due to J. Akiyama et al. [3].

Proposition 4.1(J. Akiyama et al. [3]) *For any $n \geq 2$, $G = C_{2n+3}$ is the only solution to the equation $\overline{L(G)} \cong G^n$.*

Proposition 4.2 *For any signed graph $S = (G, \sigma)$, $\overline{L(S)} \sim S^n$, where $n \geq 2$ if, and only if, G is C_{2n+3}.*

Proof Suppose $\overline{L(S)} \sim S^n$. This implies, $\overline{L(G)} \cong G^n$ and hence by Proposition 4.1, we see that the graph G must be isomorphic to C_{2n+3}.

Conversely, suppose that G is C_{2n+3}. Then $\overline{L(G)} \cong G^n$ by Proposition 4.1. Now, if S is a signed graph with underlying graph as C_{2n+3}, by definition of complementary signed graph and Proposition 2.1, $\overline{L(S)}$ and S^n are balanced and hence, the result follows from Proposition 1.2. □

In [3], the authors proved there are no solutions to the equation $L(G) \cong (\overline{G})^n, n \geq 2$. So its very difficult, in fact, impossible to construct switching equivalence relation of $L(S) \sim (\overline{S})^n$.

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Dynamical Knot and Their Fundamental Group

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Abstract: In this article, we introduce the fundamental group of the dynamical trefoil knot. Also the fundamental group of the limit dynamical trefoil knot will be achieved. Some types of conditional dynamical manifold restricted on the elements of a free group and their fundamental groups are presented. The dynamical trefoil knot of variation curvature and torsion of manifolds on their fundamental group are deduced .Theorems governing these relations are obtained.

Keywords: Dynamical trefoil knot, fundamental group, knot group, Smarandache multi-space.

AMS(2010): 51H20, 57N10, 57M05, 14F35, 20F34

§1. Introduction

A means of describing how one state develops into another state over the course of time. Technically, a dynamical system is a smooth action of the reals or the integers on another object (usually a manifold). When the reals are acting, the system is called a continuous dynamical system, and when the integers are acting, the system is called a discrete dynamical system. If f is any continuous function, then the evolution of a variable x can be given by the formula $x_{n+1} = f(x_n)$. This equation can also be viewed as a difference equation $x_{n+1}-x_n = f(x_n)-x_n$, so defining $g(x) \equiv f(x) - x$ gives $x_{n+1} - x_n = g(x_n) * 1$,which can be read "as n changes by 1 unit, x changes by $g(x)$. This is the discrete analog of the differential equation $x'(n) = g(x(n))$.

In other words; a dynamic system is a set of equations specifying how certain variables change over time. The equations specify how to determine (compute) the new values as a function of their current values and control parameters. The functions, when explicit, are either difference equations or differential equations. Dynamic systems may be stochastic or deterministic. In a stochastic system, new values come from a probability distribution. In a deterministic system, a single new value is associated with any current value $[1, 11]$.

The dynamical systems were discussed in $[1, 9, 11]$. The fundamental groups of some types of a manifold were studied in $[2, 6-8, 10]$.

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§1. Definitions

1. The set of homotopy classes of loops based at the point x_0 with the product operation $[f][g] = [f.g]$ is called the fundamental group and denoted by $\pi_1(X, x_0)$ [3].

2. Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained by identifying x_0 and y_0 to a single point [5].

3. A knot is a subset of 3-space that is homeomorphic to the unit circle and a trefoil knot is the simplest nontrivial knot, it can be obtained by joining the loose ends of an overhand knot [5].

4. A Smarandache multi-space is a union of n spaces equipped with some different structures for an integer $n \geq 2$, which can be used for discrete or connected space [4].

5. Given a knot k, the fundamental group $\pi_1(R^3 - k)$ is called the knot group of k [5].

6. A dynamical system in the space X is a function $q = f(p, t)$ which assigns to each point p of the space X and to each real number$t$, $\infty < t < \infty$ a definite point $q \in X$ and possesses the following three properties :

a- Initial condition : $f(p, 0) = p$ for any point$p \in X$.

b- Property of continuity in both arguments simultaneously:

$$\lim_{\substack{p \to p_0 \\ t \to t_0}} f(p,t) = f(p_0, t_0)$$

c- Group property $f(f(p, t_1), t_2) = f(p, t_1 + t_2)$[11].

§2. The Main Results

Aiming to our study, we will introduce the following:

Theorem 3.1 *Let K be a trefoil knot then there are two types of dynamical trefoil knot $D_i : K \to \overline{K}$, $i = 1, 2$, $D_i(K) \neq K$,which induces dynamical trefoil knot $\bar{D}_i$: $\pi_1(K) \to \pi_1(\overline{K})$ such that $\bar{D}_i(\pi_1(K))$ is a free group of rank ≤ 4 or identity group.*

Proof Let $D_1 : K \to \overline{K}$ be a dynamical trefoil knot such that $D_1(K)$ is dynamical crossing i.e. the point of upper arc crossing touch the point of lower crossing, where $D_1(c) = p_1$ as in Figure 1(a) then we have the induced dynamical trefoil knot $\bar{D}_1 : \pi_1(K) \to \pi_1(\overline{K})$ such that $\bar{D}_1(\pi_1(K)) = \pi_1(D_1(K)) \approx \pi_1(S_1^1) * \pi_1(S_2^1)$, thus $\bar{D}_1(\pi_1(K)) \approx Z * Z$, so $\bar{D}_1(\pi_1(K))$is a free group of rank $= 2$. Also, if $D_1 : K \to \overline{K}$ such that $D_1(c) = p_1,\ D_1(b) = p_2$ then $D_1(K)$ is space as in Figure 1(b) and so $\bar{D}_1(\pi_1(K)) = \pi_1(D_1(K)) \approx \pi_1(S_1^1) * \pi_1(S_2^1) * \pi_1(S_3^1)$, thus $\bar{D}_1(\pi_1(K))$ is a free group of rank $= 3$. Moreover, if $D_1 : K \to \overline{K}$ such that $D_1(c) = p_1,\ D_1(b) = p_2, D_1(a) = p_3$, then $D_1(K)$ is space as in Figure 1(c) and so $\bar{D}_1(\pi_1(K)) =$

$\pi_1(D_1(K)) \approx \pi_1(S_1^1) * \pi_1(S_2^1) * \pi_1(S_3^1) * \pi_1(S_4^1)$, hence $\bar{D}_1(\pi_1(K))$is a free group of rank $= 4$. There is another type $D_2 : K \to \overline{K}$ such that $D_2(K)$ is dynamical trefoil knot with singularity as in FIGURE 1(d) then we obtain the induced dynamical trefoil knot $\bar{D}_2 : \pi_1(K) \to \pi_1(\overline{K})$ such that $\bar{D}_2(\pi_1(K)) = \pi_1(D_2(K)) = 0$. Therefore, $\bar{D}_i(\pi_1(K))$ is a free group of rank≤ 4 or identity group. $\square$

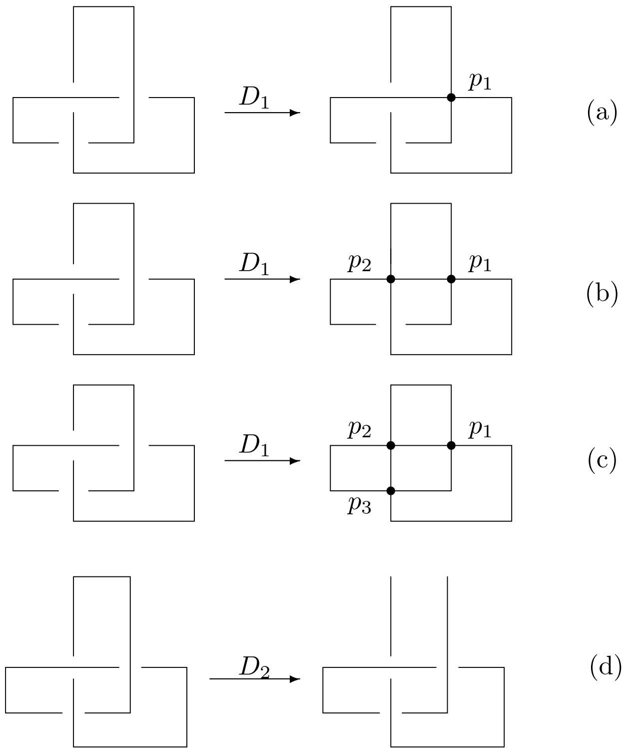

FIGURE 1

Theorem 3.2 *The fundamental group of the limit dynamical trefoil knot is the identity group.*

Proof Let $D_1 : K \to K_1$, $D_2 : D_1(K) \to D_1(K_2)$ $,\ldots,$ $D_n : D_{n-1}(D_{n-2})\ldots(D_1(K) \to D_{n-1}(D_{n-2})\ldots(D_1(K_n)$ such that $\lim\limits_{n\to\infty}(D_n(D_{n-1})\ldots(D_1(K)\ldots)$ is a point as in FIGURE 2 (a,b) ,then $\pi_1(\lim\limits_{n\to\infty}(D_n(D_{n-1})\ldots(D_1(K)\ldots)) = 0$. $\square$

Theorem 3.3 *There are different types of dynamical link graph L which represent a trefoil knot ,where $D(L) \neq L$ such that $\pi_1(D(L))$ is a free group of rank ≤ 3.*

Proof Let L be a link graph which represent a trefoil knot and consider the following dynamical edges $D(e) = a, D(f) = c, D(g) = b$ as in FIGURE 3(a) then $\pi_1(D(L)) \approx \pi_1(S^1)$ and so $\pi_1(D(L))$ is a free group of rank 1. Now, if $D(e) \neq e, D(f) \neq f, D(g) \neq g$ as in FIGURE 3(b)we get the same result. Also, if $D(e) = e, D(f) = f, D(g) \neq g$ as in FIGURE 3(c) then, $\pi_1(D(L)) \approx \pi_1(S_1^1) * \pi_1(S_2^1) * \pi_1(S_3^1)$, thus $\pi_1(D(L))$ is a free group of rank 3.Moreover, if

$D(e) = e, D(f) \neq f, D(g) \neq g$ as in FIGURE 3(d)then $\pi_1(D(L)) \approx \pi_1(S_1^1) * \pi_1(S_2^1)$ .Hence $\pi_1(D(L))$ is a free group of rank 2. Therefore $\pi_1(D(L))$ is a free group of rank ≤ 3. □

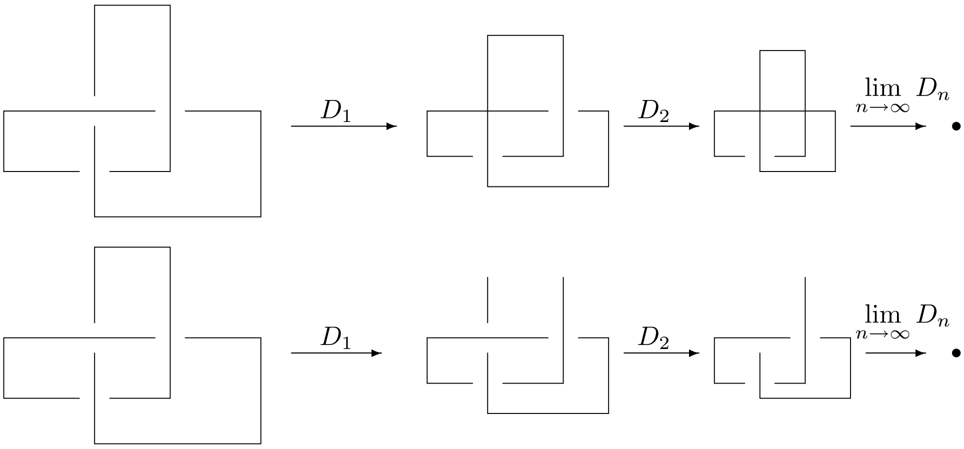

FIGURE 2

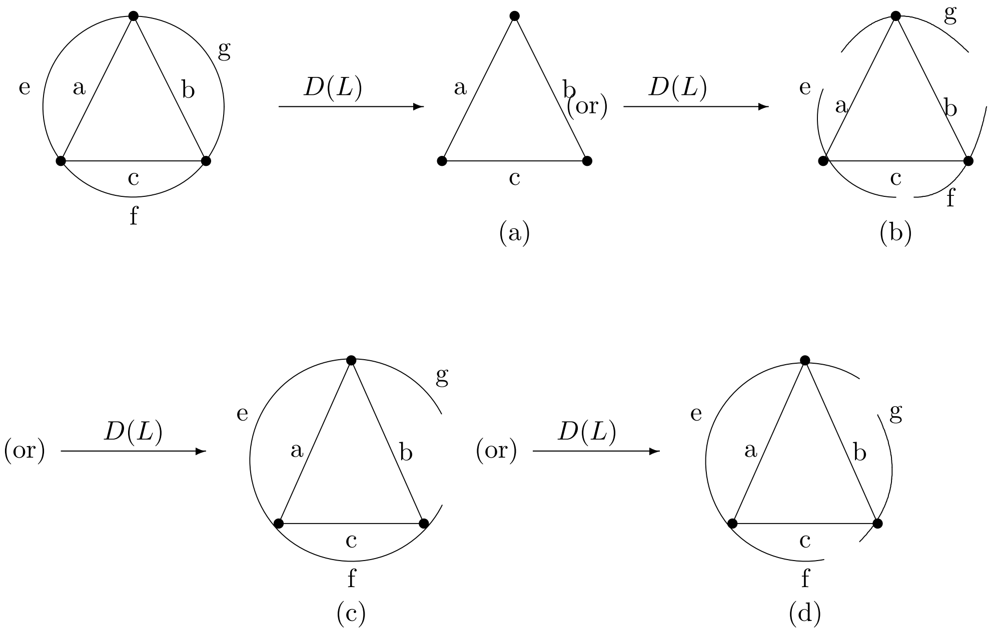

FIGURE 3

Theorem 3.4 *The fundamental group of limit dynamical link graph of n vertices is a free group of rank n.*

Proof Let K be link graph of n vertices , then $\lim_{n\to\infty}(D(K))$ is a graph with only one vertex and n-loops as in FIGURE 4 ,for n=3 and so $\pi_1(\lim_{n\to\infty}(D(K))) = \pi_1(\overset{n}{\underset{i\ =1}{\vee}} S_i^1) \approx \underbrace{Z * Z * ... * Z}_{n \text{ terms}}$. Hence, $\pi_1(\lim_{n\to\infty}(D(K)))$ is a free group of rank n. □

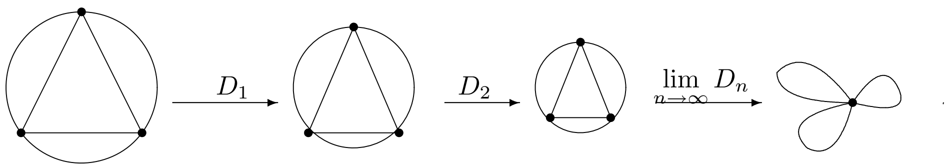

FIGURE 4

Theorem 3.5 *Let I be the closed interval $[0,1]$. Then there is a sequence of dynamical manifolds $D_i : I \to I_i,\ i = 1,2,...,n$ with variation curvature and torsion such that $\lim_{n\to\infty} D_n(I)$ is trefoil knot and $\pi_1(R^3 - \lim_{n\to\infty} D_n(I)) \approx Z$.*

Proof Consider the sequence of dynamical manifolds with variation curvature and torsion : $D_1 : I \to I_1, D_2 : I_1 \to I_2, ..., D_n : I_{n-1} \to I_n$ such that $\lim_{n\to\infty} D_n(I)$ is a trefoil knot as in FIGURE 5, Therefore, $\pi_1(R^3 - \lim_{n\to\infty} D_n(I)) \approx Z$. □

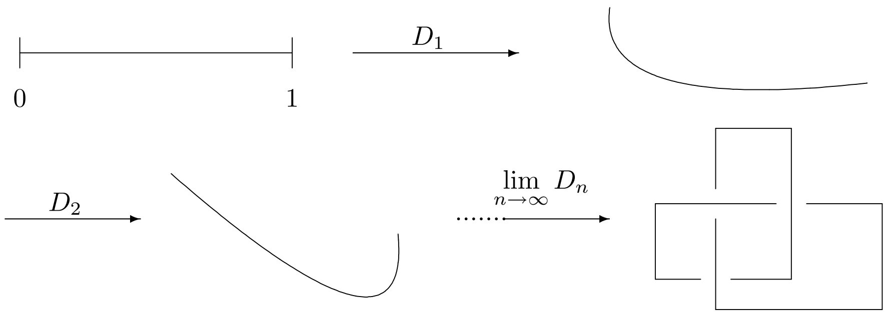

FIGURE 5

Theorem 3.6 *The knot group of the limit dynamical sheeted trefoil knot is either isomorphic to Z or identity group.*

Proof Let $\overline{K}$ be a sheet trefoil knot with boundary $\{A, B\}$as in FIGURE 6 and $D : \overline{K} \to \overline{K}$ is dynamical sheeted trefoil knot of $\overline{K}$ into itself,then we get the following sequence: $D_1 : \overline{K} \to \overline{K}, D_2 : D_1(\overline{K}) \to D_1(\overline{K})$, . . . ,
$D_n : (D_{n-1}) \dots (D_1(\overline{K}) \dots) \to (D_{n-1}) \dots (D_1(\overline{K}) \dots)$
such that $\lim_{n\to\infty}(D_n(D_{n-1}) \dots (D_1(\overline{K}) \dots) = k$ where, k is a trefoil knot as in FIGURE 6(a) then $\pi_1(R^3 - k) \approx Z$..Also, if $\lim_{n\to\infty}(D_n(D_{n-1}) \dots (D_1(\overline{K}) \dots)$ =point as in FIGURE 6(b,c) then$\pi_1(R^3 - \lim_{n\to\infty}(D_n(D_{n-1}) \dots (D_1(\overline{K}) \dots)) = \pi_1(R^3 - \textit{one point})$. Hence,

$$\pi_1(R^3 - \lim_{n\to\infty}(D_n(D_{n-1})\ldots(D_1(\overline{K})\ldots)) = 0.$$

Therefore, the knot group of the limit dynamical sheeted trefoil knot is either isomorphic to Z or identity group.

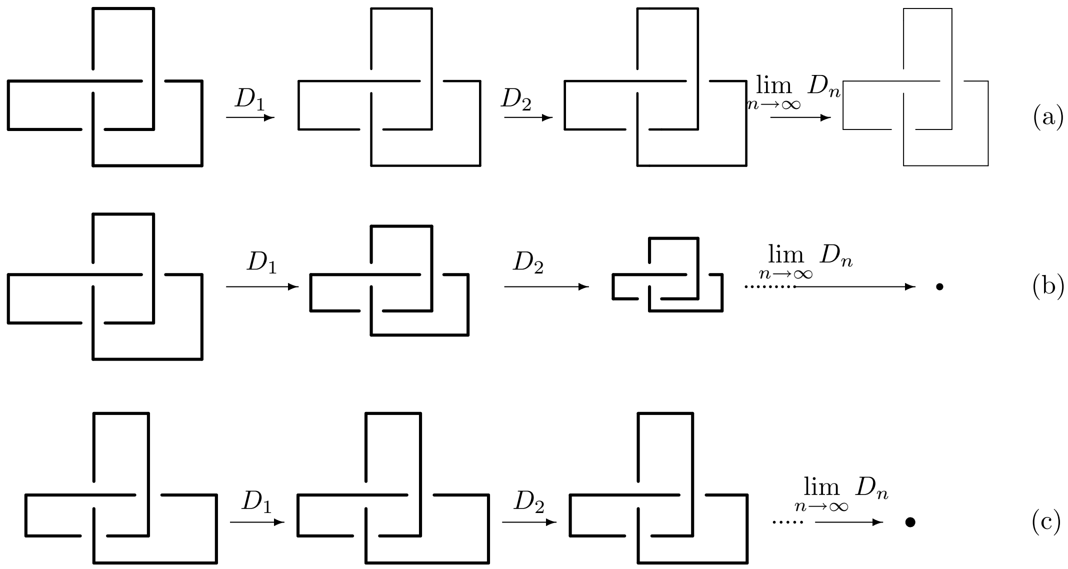

FIGURE 6

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The Crossing Number of the Cartesian Product of Star S_n with a 6-Vertex Graph

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Abstract: Calculating the crossing number of a given graph is in general an elusive problem and only the crossing numbers of few families of graphs are known. Most of them are the Cartesian product of special graphs. This paper determines the crossing number of the Cartesian product of star S_n with a 6-vertex graph.

Keywords: Smarandache $\mathscr{P}$-drawing, crossing number, Cartesian product, Star.

AMS(2010): 05C, 05C62

§1. Introduction

For definitions not explained here, readers are referred to [1]. Let G be a simple graph with vertex set V and edge set E. By a *drawing* of G on the plane Π, we mean a collection of points P in Π and open arcs A in $\Pi - P$ for which there are correspondences between V and P and between E and A such that the vertices of an edge correspond to the endpoints of the open arcs. A drawing is called *good,* if for all arcs in A, no two with a common endpoint meet, no two meet in more than one point, and no three have a common point. A *crossing* in a good drawing is a point of intersection of two arcs in A. A *Smarandache $\mathscr{P}$-drawing* of a graph G for a graphical property $\mathscr{P}$ is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$. A Smarandache $\mathscr{P}$-drawing is said to be *optimal* if $\mathscr{P} = G$ and it minimizes the number of crossings. The *crossing number $cr(G)$* of a graph G is the number of crossings in any optimal drawing of G in the plane. Let D be a good drawing of the graph G, we denote by $cr(D)$ the number of crossings in D.

Let P_n and C_n be the path and cycle of length n, respectively, and the *star* S_n be the complete bipartite graph $K_{1,n}$.

Given two vertex disjoint graphs G_1 and G_2, the *Cartesian product* $G_1 \times G_2$ of G_1 and G_2

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is defined by

$$\begin{cases} V(G_1 \times G_2) = & V(G_1) \times V(G_2) \\ \\ E(G_1 \times G_2) = & \{(u_1,\, u_2)\,(v_1, v_2)\,|\, u_1 = v_1 \text{ and } u_2\, v_2 \in E(G_2), \\ & \qquad\qquad \text{or } u_2 = v_2 \text{ and } u_1\, v_1 \in E(G_1)\} \end{cases}$$

Let G_1 be a graph homeomorphic to G_2, then $cr(G_1) = cr(G_2)$. And if G_1 is a subgraph of G_2, it is easy to see that $cr(G_1) \leq cr(G_2)$.

Calculating the crossing number of a given graph is in general an elusive problem [2] and only the crossing numbers of few families of graphs are known. Most of them are Cartesian products of special graphs, partly because of the richness of their repetitive patterns. The already known results on the crossing number of $G \times H$ fit into three categories:

(i) *G and H are two small graphs.* Harary, et al. obtained the crossing number of $C_3 \times C_3$ in 1973 [3]; Dean and Richter [4] investigated the crossing number of $C_4 \times C_4$; Richter and Thomassen [5] determined the crossing number of $C_5 \times C_5$; in [6] Anderson, et al. obtained the crossing number of $C_6 \times C_6$; Klešč [7]studied the crossing number of $K_{2,3} \times C_3$. These results are usually used as the induction basis for establishing the results of type (ii):

(ii) *G is a small graph and H is a graph from some infinite family.* In [8], the crossing numbers of $G \times C_n$ for any graph G of order four except S_3 were studied by Beineke and Ringeisen, this gap was bridged by Jendrol' et al. in [9]. The crossing numbers of Cartesian products of 4-vertex graphs with P_n and S_n are determined by Klešč in [10], he also determined the crossing numbers of $G \times P_n$ for any graph G of order five [11-13]. For several special graphs of order five, the crossing numbers of their products with C_n or S_n are also known, most of which are due to Klešč [14-17]. For special graphs G of order six, Peng et al. determined the crossing number of the Cartesian product of the Petersen graph $P(3,1)$ with P_n in [18], Zheng et al. gave the bound for the crossing number of $K_m \times P_n$ for $m \geqslant 3, n \geqslant 1$, and they determined the exact value for $cr(K_6 \times P_n)$, see [19], and the authors [20] established the crossing number of the Cartesian product of P_n with the complete bipartite graph $K_{2,4}$.

(iii) *Both G and H belong to some infinite family.* One very long attention-getting problem of this type is to determine the crossing number of the Cartesian product of two cycles, C_m and C_n, which was put forward by Harary et al. [3], and they conjectured that $cr(C_m \times C_n) = (m-2)n$ for $n \geq m$. In the next three decades, many authors were devoted to this problem and the conjecture has been proved true for $m = 3, 4, 5, 6, 7$, see [8,21-24]. In 2004, the problem was progressed by Glebsky and Salazar, who proved that the crossing number of $C_m \times C_n$ equals its long-conjectured value for $n \geq m(m+1)$ [25]. Besides the Cartesian product of two cycles, there are several other results. D.Bokal [26] determined the crossing number of the Cartesian product $S_m \times P_n$ for any $m \geq 3$ and $n \geq 1$ used a quite newly introduced operation: the zip product. Tang, et al. [27] and Zheng, et al. [28] independently proved that the crossing number of $K_{2,m} \times P_n$ is $2n\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ for arbitrary $m \geq 2$ and $n \geq 1$.

Stimulated by these results, we begin to investigate the crossing number of the Cartesian product of star S_n with a 6-vertex graph G_2 shown in Figure 1, and get its crossing number is $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2n + 2\lfloor \frac{n}{2} \rfloor$, for $n \geq 1$.

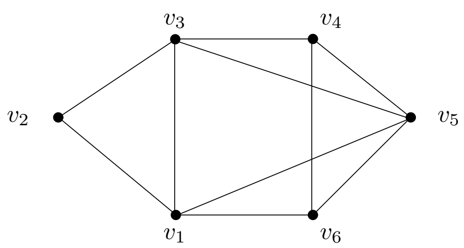


Figure1: The graph G_2

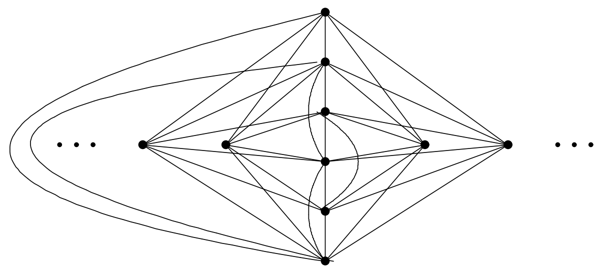


Figure 2: The graph H_n

§2. Some Basic Lemmas and the Main Result

Let A and B be two disjoint subsets of E. In a drawing D, the number of crossings made by an edge in A and another edge in B is denoted by $cr_D(A, B)$. The number of crossings made by two edges in A is denoted by $cr_D(A)$. So $cr(D) = cr_D(E)$. By counting the number of crossings in D, we have Lemma 1.

Lemma 1 *Let A, B, C be mutually disjoint subsets of E. Then*

$$\begin{aligned} cr_D(A \cup B, C) &= cr_D(A, C) + cr_D(B, C); \\ cr_D(A \cup B) &= cr_D(A) + cr_D(B) + cr_D(A, B). \end{aligned} \tag{1}$$

The crossing numbers of the complete bipartite graph $K_{m,n}$ were determined by Kleitman [29] for the case $m \leq 6$. More precisely, he proved that

$$cr(K_{m,n}) = \lfloor \tfrac{m}{2} \rfloor \lfloor \tfrac{m-1}{2} \rfloor \lfloor \tfrac{n}{2} \rfloor \lfloor \tfrac{n-1}{2} \rfloor, \quad \text{if } m \leq 6 \tag{2}$$

For convenience, $\lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ is often denoted by $Z(m, n)$ in our paper. To obtain the main result of the paper, first we construct a graph H_n which is shown in Figure 2. Let $V(H_n) = \{v_1, v_2, v_3, v_4, v_5, v_6; t_1, t_2, \cdots, t_n\}$, $E(H_n) = \{v_i t_j | \ 1 \leqslant i \leqslant 6; 1 \leqslant j \leqslant n\} \cup \{v_1v_2, v_1v_3, v_1v_5, v_1v_6, v_2v_3, v_3v_4, v_3v_5, v_4v_5, v_4v_6, v_5v_6\}$. Let T^i be the subgraph of H_n induced by the edge set $\{v_i t_j | 1 \leqslant j \leqslant n\}$, and let t_i be the vertex of T^i of degree six. Clearly, the induced subgraph $[v_1, v_2, \cdots, v_6] \cong G_2$. Thus, we have

$$H_n = G_2 \cup K_{6,n} = G_2 \cup (\bigcup_{i=1}^{n} T^i) \tag{3}$$

For a graph G, the *removal number* $r(G)$ of G is the smallest nonnegative integer r such that the removal of some r edges from G results in a planar subgraph of G. By removing an edge from each crossing of a drawing of G in the plane we get a set of edges whose removal leaves a planar graph. Thus we have the following.

Lemma 2 *For any drawing D of G, $cr(D) \geqslant r(G)$.*

Lemma 3 $cr(H_1) = 1$, $cr(H_2) = 4$.

Proof A good drawing of H_1 in Figure 3 shows that $cr(H_1) \leqslant 1$, and a good drawing of H_2 in Figure 4 shows that $cr(H_2) \leqslant 4$. By Lemma 2, we only need to prove that $r(H_1) \geqslant 1$ and $r(H_2) \geqslant 4$.

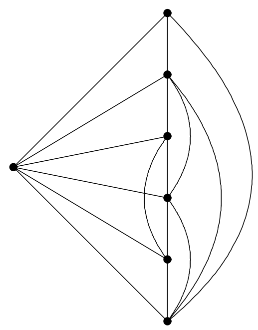


Figure 3: A good drawing of H_1

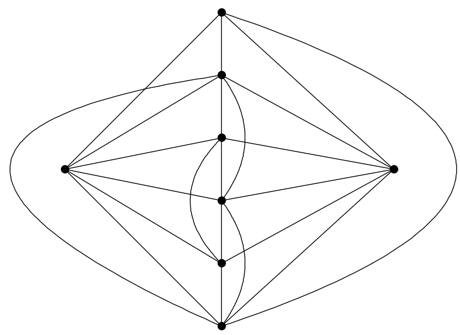


Figure 4: A good drawing of H_2

Let $r = r(H_1)$ and let H_1' be a planar subgraph of H_1 having $16 - r$ edges. It is easy to see that H_1' is a connected spanning subgraph of H_1. By Euler's formula, in any planar drawing of H_1', there are $11 - r$ faces. Since H_1' has girth at least 3, $2(16 - r) \geqslant 3(11 - r)$, so $r \geqslant 1$, that is $r(H_1) \geqslant 1$. Similarly, we can have $r(H_2) \geqslant 4$. $\square$

In a drawing D, if an edge is not crossed by any other edge, we say that it is *clean* in D; if it is crossed by at least one edge, we say that it is *crossed* in D.

Lemma 4 *Let D be a good drawing of H_n. If there are two different subgraphs T^i and T^j such that $cr_D(T^i, T^j) = 0$, then $cr_D(G_2, T^i \cup T^j) \geqslant 4$.*

Proof We label the vertices of G_2, see Figure 1. Since the two subgraphs T^i and T^j do not cross each other in D, the induced drawing $D|_{T^i \cup T^j}$ of $T^i \cup T^j$ divides the plane into six regions that there are exactly two vertices of G_2 on the boundary of each region.

Assume to the contrary that $cr_D(G_2, T^i \cup T^j) \leqslant 3$. The degrees of vertices v_1, v_3 and v_5 in G_2 are all 4, so there are at least two crossings on the edges incident to v_1, v_3 and v_5, respectively. We can assert that edges v_1v_3, v_3v_5 and v_1v_5 must be crossed. Otherwise, without loss of generality, we may assume that the edge v_1v_3 is clean, then the vertices v_1 and v_3 must lie on the boundary of the same region, and there are at least two crossings on the edges (except the edge v_1v_3) incident to vertices v_1 and v_3, respectively, a contradiction. Since the degree of vertex v_4 in G_2 is 3, one can easily see that there is at least one more crossing on the edges incident to v_4, contradicts to our assumption and completes the proof. $\square$

To obtain our main result, the following theorem is introduced.

Theorem 1 *$cr(H_n) = Z(6, n) + n + 2\lfloor \frac{n}{2} \rfloor$, for $n \geqslant 1$.*

Proof A good drawing in Figure 2 shows that $cr(H_n) \leq Z(6, n) + n + 2\lfloor \frac{n}{2} \rfloor$. Now we prove the reverse inequality by induction on n. By Lemma 3, the cases hold for $n = 1$ and $n = 2$. Now suppose that $n \geqslant 3$, and for all $l < n$, there is

$$cr(H_l) \geq Z(6, l) + l + 2\lfloor \tfrac{l}{2} \rfloor \tag{4}$$

and for a certain good drawing D of H_n, assume that

$$cr_D(H_n) < Z(6,n) + n + 2\lfloor \frac{n}{2} \rfloor \tag{5}$$

The following two cases are discussed:

Case 1. Suppose that there are at least two different subgraphs T^i and T^j that do not cross each other in D. Without loss of generality, assume that $cr_D(T^{n-1}, T^n) = 0$. By Lemma 4, $cr_D(G_2, T^{n-1} \cup T^n) \geq 4$. As $cr(K_{3,6}) = 6$, for all i, $i = 1, 2, \cdots, n-2$, $cr_D(T^i, T^{n-1} \cup T^n) \geqslant 6$. Using (1), (2), (3) and (4), we have

$$\begin{aligned} cr_D(H_n) &= cr_D(G_2 \cup \bigcup_{i=1}^{n-2} T^i \cup T^{n-1} \cup T^n) \\ &= cr_D(G_2 \cup \bigcup_{i=1}^{n-2} T^i) + cr_D(T^{n-1} \cup T^n) + cr_D(G_2, T^{n-1} \cup T^n) \\ &\quad + \sum_{i=1}^{n-2} cr_D(T^i, T^{n-1} \cup T^n) \\ &\geqslant Z(6, n-2) + (n-2) + 2\lfloor \frac{n-2}{2} \rfloor + 4 + 6(n-2) \\ &= Z(6,n) + n + 2\lfloor \frac{n}{2} \rfloor \end{aligned}$$

This contradicts (5).

Case 2. Suppose that $cr_D(T^i, T^j) \geqslant 1$ for any two different subgraphs T^i and T^j, $1 \leqslant i \neq j \leqslant n$. Using (1), (2) and (3), we have

$$\begin{aligned} cr_D(H_n) &= cr_D(G_2) + cr_D(\bigcup_{i=1}^{n} T^i) + cr_D(G_2, \bigcup_{i=1}^{n} T^i) \\ &\geqslant cr_D(G_2) + Z(6,n) + \sum_{i=1}^{n} cr_D(G_2, T^i) \end{aligned} \tag{6}$$

This, together with (5) implies that

$$cr_D(G_2) + \sum_{i=1}^{n} cr_D(G_2, T^i) < n + 2\lfloor \frac{n}{2} \rfloor$$

So, there is at least one subgraph T^i that $cr_D(G_2, T^i) \leqslant 1$.

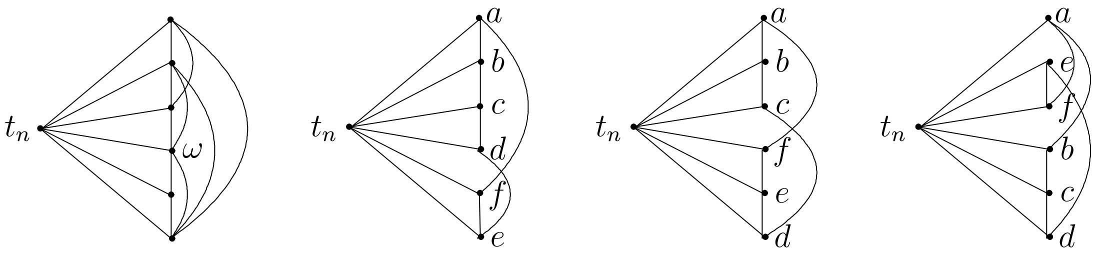
Subcase 2.1 Suppose that there is at least one subgraph T^i that do not cross the edges of G_2. Without loss of generality, we may assume that $cr_D(G_2, T^n) = 0$. Let us consider the 6-cycle C_6 of the graph G_2. Hence G_2 consists of C_6 and four additional edges.

Subcase 2.1.1 Suppose that the edges of C_6 do not cross each other in D. Since $cr_D(G_2, T^n) = 0$, then the possibility of $C_6 \cup T^n$ must be as shown in Figure 5(1). Consider the four additional edges of G_2, they cannot cross the edges of T^n and the edges of C_6 either, so the unique possibility is $cr_D(G_2 \cup T^n) = 2$, see Figure 5(1). Consider now a subdrawing of $G_2 \cup T^n \cup T^i$ of the drawing D for some $i \in \{1, 2, \ldots, n-1\}$. If t_i locates in the region labeled ω, then we have

$cr_D(G_2, T^i) \geqslant 4$, using $cr_D(T^n, T^i) \geqslant 1$, we get $cr_D(G_2 \cup T^n, T^i) \geqslant 5$. If t_i locates in the other regions, one can see that on the boundary of these regions there are at most three vertices of G_2, and there are at least two vertices of G_2 are in a region having no common edge with it, in this case we have $cr_D(G_2 \cup T^n, T^i) \geqslant 5$. Using (1), (2) and (3), we can get

$$\begin{aligned}
cr_D(H_n) &= cr_D(G_2 \cup T^n \cup \bigcup_{i=1}^{n-1} T^i) \\
&= cr_D(G_2 \cup T^n) + cr_D(\bigcup_{i=1}^{n-1} T^i) + \sum_{i=1}^{n-1} cr_D(G_2 \cup T^n, T^i) \\
&\geqslant 1 + Z(6, n-1) + 5(n-1) \\
&\geqslant Z(6,n) + n + 2\lfloor \frac{n}{2} \rfloor
\end{aligned}$$

which contradicts (5).



(1) (2) (3) (4)

Figure 5

Subcase 2.1.2 Suppose that the edges of C_6 cross each other in D. By the above arguments in Subcase 2.1.1, we can assert that in D there must exist a subgraph T^i, $i \in \{1, 2, \cdots, n-1\}$, such that $cr_D(G_2 \cup T^n, T^i) \leqslant 4$. The condition $cr_D(G_2, T^n) = 0$ implies that $cr_D(C_6, T^n) = 0$. In this case the vertex t_n of T^n lies in the region with all six vertices of C_6 on its boundary, and the condition $cr_D(G_2 \cup T^n, T^i) \leqslant 4$ enforces that in the subdrawing of $C_6 \cup T^n$ there is a region with at least three vertices of C_6 on its boundary. In this case C_6 cannot have more than two internal crossings. If C_6 has only one internal crossing, then the possibilities of $C_6 \cup T^n$ are shown in Figure 5(2) and Figure 5(3). If C_6 has two internal crossings, then the possibility of $C_6 \cup T^n$ is shown in Figure 5(4). The vertices of G_2 are labeled by a, b, c, d, e, f, respectively. Since $cr_D(G_2, T^n) = 0$, the four edges of G_2 not in C_6 do not cross the edges of T^n.

Consider the case shown in Figure 5(2). The three possible edges ac, ce, ae and the fourth possible edge bd or bf or df separate the subdrawing of $G_2 \cup T^n$ into several regions with at most three vertices of G_2 on each boundary. The three possible edges bd, bf, df and the fourth possible edge ac or ce or ae separate the subdrawing of $G_2 \cup T^n$ into several regions with at most three vertices of G_2 on each boundary. If the vertex t_i of T^i locates in the region with three vertices of G_2 on its boundary, one can note that there are at least 2 vertices of G_2 do not on the boundary of its neighborhood region, then $cr_D(G_2 \cup T^n, T^i) \geqslant 5$; if the vertex t_i of T^i locates in the region with at most two vertices of G_2 on its boundary, one can see that there is at least one vertex of G_2 is in a region having no common edge with it, then $cr_D(G_2 \cup T^n, T^i) \geqslant 5$, a contradiction. If the possibility of $C_6 \cup T^n$ is as shown in Figure 5(3) or Figure 5(4), then a similar contradiction can be made by the analogous arguments.

Subcase 2.2 Suppose that $cr_D(G_2, T^i) \geqslant 1$ for $1 \leqslant i \leqslant n$. Together with our former assumption, there is at least one subgraph T^i that $cr_D(G_2, T^i) = 1$. Without loss of generality, assume that $cr_D(G_2, T^n) = 1$.

Subcase 2.2.1 Suppose that $cr_D(C_6, T^n) = 0$. Then the possibilities of $C_6 \cup T^n$ are shown in Figure 5. It is clear that, in each region whose boundary composed of segments of edges that incident with t_n, there are at most two vertices of G_2. Adding the four additional possible edges of G_2 that have one crossing with the edges of T^n, then there are at most three vertices of G_2 on the boundary of each region. Consider now a subdrawing of $G_2 \cup T^n \cup T^i$ of the drawing D for some $i \in \{1, 2, \ldots, n-1\}$. If t_i locates in one of the regions with three vertices of G_2 on its boundary, then then we have $cr_D(G_2, T^i) \geqslant 3$, using $cr_D(T^n, T^i) \geqslant 1$, we have $cr_D(G_2 \cup T^n, T^i) \geqslant 4$. If t_i locates in one of the regions with at most two vertices of G_2 on its boundary, then one can see that there are at least two vertices of G_2 are in a region having no common edge with it, in this case we have $cr_D(G_2 \cup T^n, T^i) \geqslant 6$. Let

$$M = \{T^i | t_i \text{ lies in the region with three vertices of } \mathrm{G}_2 \text{ on its boundary}\}$$

Using (1), (2) and (3), we have

$$\begin{aligned}
cr_D(H_n) &= cr_D(G_2 \cup T^n \cup \bigcup_{i=1}^{n-1} T^i) \\
&= cr_D(G_2 \cup T^n) + cr_D(\bigcup_{i=1}^{n-1} T^i) + \sum_{T^i \in M} cr_D(G_2 \cup T^n, T^i) \\
&\quad + \sum_{T^i \notin M} cr_D(G_2 \cup T^n, T^i) \\
&\geqslant 1 + Z(6, n-1) + 4|M| + 6(n - 1 - |M|)
\end{aligned}$$

Together with (5), we can get

$$2|M| \geqslant 5n - 5 - 2\lfloor \frac{n}{2} \rfloor - 6\lfloor \frac{n-1}{2} \rfloor \geqslant 2\lfloor \frac{n}{2} \rfloor \tag{7}$$

Combined with (6) and (7), we can get

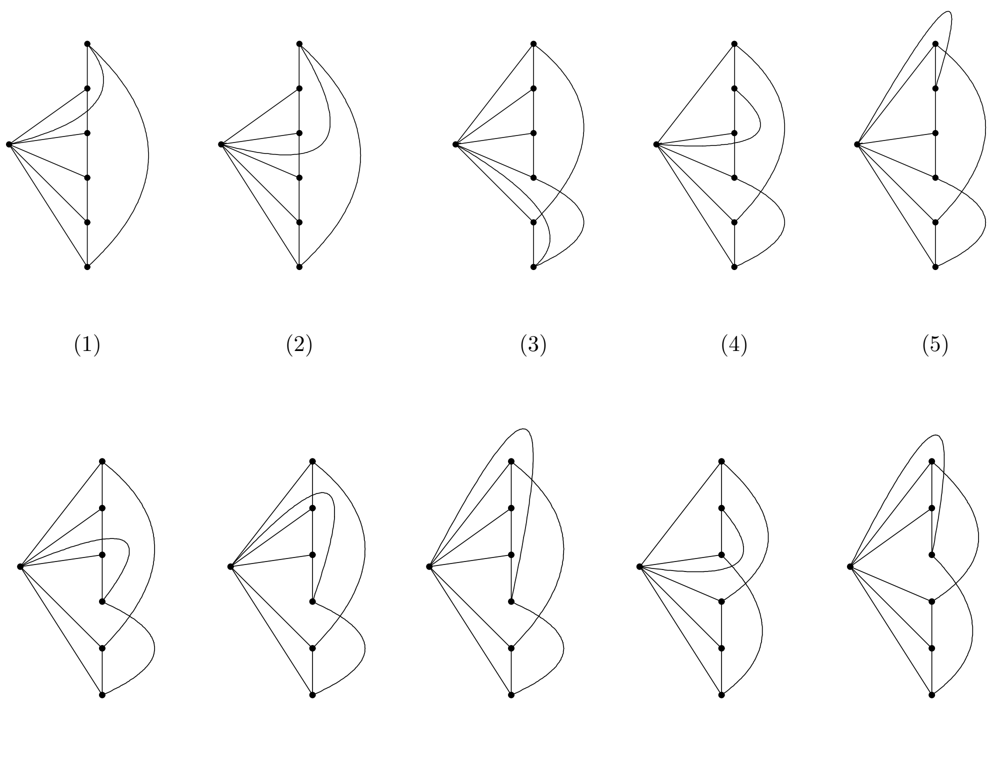
$$\begin{aligned}
cr_D(H_n) &= cr_D(G_2) + cr_D(\bigcup_{i=1}^{n} T^i) + cr_D(G_2, \bigcup_{i=1}^{n} T^i) \\
&= cr_D(G_2) + cr_D(\bigcup_{i=1}^{n} T^i) + \sum_{T^i \in M} cr_D(G_2, T^i) + \sum_{T^i \notin M} cr_D(G_2, T^i) \\
&\geqslant Z(6, n) + 3|M| + (n - |M|) \\
&\geqslant Z(6, n) + n + 2\lfloor \frac{n}{2} \rfloor
\end{aligned}$$

which contradicts (5).

Subcase 2.2.2 Suppose that $cr_D(C_6, T^n) = 1$, then the subdrawing of $C_6 \cup T^n$ must be one of the ten possibilities shown in Figure 6. Adding the four additional possible edges of G_2 that do not cross T^n, it is not difficult to see that there are at most three vertices of G_2 on the boundary of every region. Consider now a subdrawing of $G_2 \cup T^n \cup T^i$ of the drawing D for some $i \in \{1, 2, \ldots, n-1\}$. One can see that the number of crossings between the edges of $G_2 \cup T^n$ and the edges of T^i are divided into two classes:

(1) In the subdrawing of $G_2 \cup T^n$, we have $cr_D(G_2 \cup T^n, T^i) \geqslant 5$ no matter which region does t_i locate in, then a contradiction can be made by the similarly arguments in Subcase 2.1.1.

(2) In the subdrawing of $G_2 \cup T^n$, $cr_D(G_2 \cup T^n, T^i) = 4$ when t_i locates in the region with three vertices of G_2 on its boundary (and $cr_D(G_2 \cup T^n, T^i) = 4$ if and only if $cr_D(G_2, T^i) = 3$ and $cr_D(T^n, T^i) = 1$), and $cr_D(G_2 \cup T^n, T^i) \geqslant 6$ when t_i locates in the other regions, then a contradiction can be made by the similarly arguments in Subcase 2.2.1. That completes the proof of the theorem. $\square$



(1) (2) (3) (4) (5)

(6) (7) (8) (9) (10)

Figure 6: Ten possibilities of $C_6 \cup T^n$

Lemmas 5 and 6 are trivial observations.

Lemma 5 *If there exists a crossed edge e in a drawing D and deleting it results in a new drawing D^*, then $cr(D) \geqslant cr(D^*) + 1$.*

Lemma 6 *If there exists a clean edge $e = uv$ in a drawing D and contracting it into a vertex*

$u = v$ results in a new drawing D^*, then $cr(D) \geqslant cr(D^*)$.

Let H be a graph isomorphic to G_2. Consider a graph G_H obtained by joining all vertices of H to six vertices of a connected graph G such that every vertex of H will only be adjacent to exactly one vertex of G. Let G_H^* be the graph obtained from G_H by contracting the edges of H.

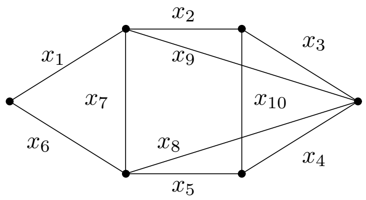


Figure 7

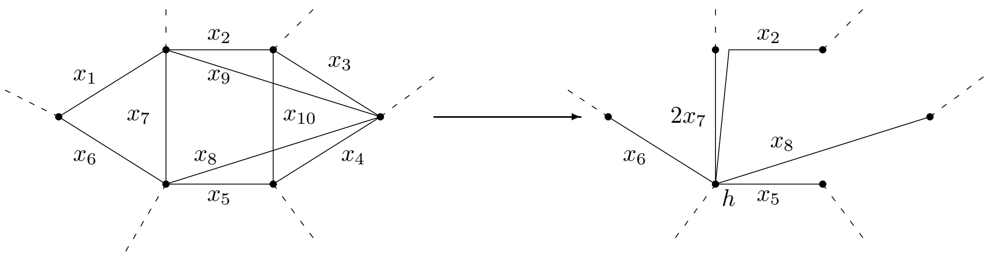


Figure 8

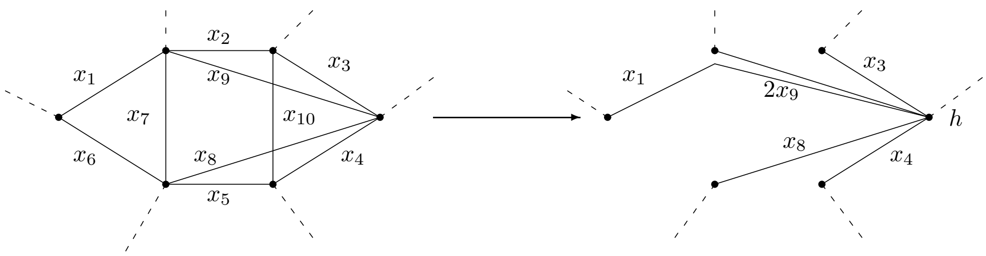


Figure 9

Lemma 7 $cr(G_H^*) \leqslant cr(G_H) - 1$.

Proof Let D be an optimal drawing of G_H. The subgraph H has ten edges and let $x_1, x_2, \ldots, x_9, x_{10}$ denote the numbers of crossings on the edges of H, see Figure 7. The following two cases are distinguished.

Case 1. Suppose that at least one of $x_1, x_2, \ldots, x_6, x_{10}$ is greater than 0, then either $x_7 < x_1 + x_3 + x_4 + x_9 + x_{10}$ or $x_9 < x_2 + x_5 + x_6 + x_7 + x_{10}$ holds. Figure 8 shows that H can be contracted to the vertex h with at least one crossing decreased if $x_7 < x_1 + x_3 + x_4 + x_9 + x_{10}$. Figure 9 shows that H can be contracted to the vertex h with at least one crossing decreased if $x_9 < x_2 + x_5 + x_6 + x_7 + x_{10}$. That means $cr(G_H^*) \leqslant cr_D(G_H) - 1 = cr(G_H) - 1$.

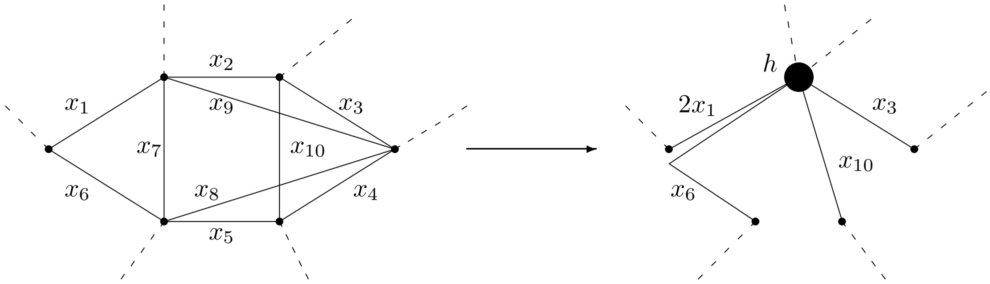


Figure 10

Case 2. Suppose that $x_1 = x_2 = \cdots = x_6 = x_{10} = 0$, then we have $x_7 + x_8 + x_9 \geqslant 1$ since $cr(H_1) = 1$. Figure 10 shows that H can be contracted to the vertex h in the following way: first, delete the edges x_7, x_8 and x_9, (for convenience, here we use x_i to denote the respective edge with x_i crossings), then redraw the former edge x_7 closely enough to edges x_1 and x_6, at last, contract the edge x_2 into a vertex h. By Lemma 5, the first step decreases at least one crossing. And by Lemma 6, the second and last steps do not increase the number of crossings. That means $cr(G_H^*) \leqslant cr_D(G_H) - 1 = cr(G_H) - 1$. This completes the proof. $\square$

Consider now the graph $G_2 \times S_n$. For $n \geqslant 1$ it has $6(n+1)$ vertices and edges that are the edges in $n+1$ copies G_2^i, $i = 0, 1, \cdots, n$, and in the six stars S_n, see Figure 11.

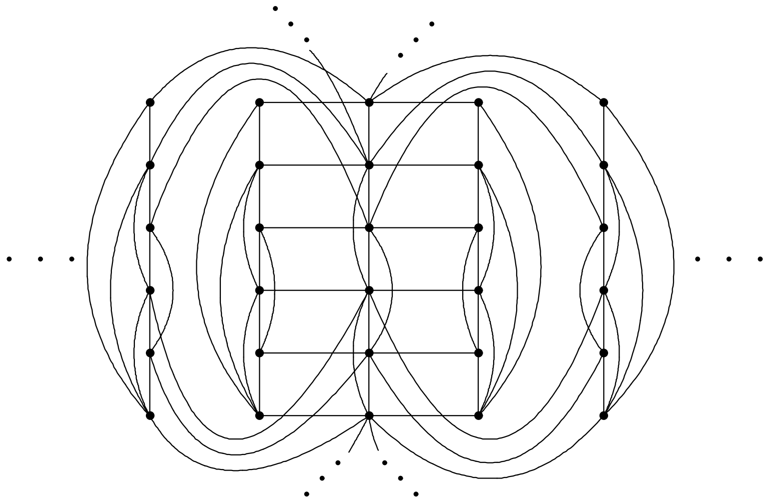


Figure 11: A good drawing of $G_2 \times S_n$

Now, we can get the main theorem.

Theorem 2 *$cr(G_2 \times S_n) = Z(6, n) + 2n + 2\lfloor \frac{n}{2} \rfloor$, for $n \geqslant 1$.*

Proof A drawing in Figure 11 shows that $cr(G_2 \times S_n) \leqslant Z(6, n) + 2n + 2\lfloor \frac{n}{2} \rfloor$. Assume that there is an optimal drawing D of $G_2 \times S_n$ with fewer than $Z(6, n) + 2n + 2\lfloor \frac{n}{2} \rfloor$ crossings. Contracting the edges of each G_2^i to a vertex t_i for all $i = 1, 2, \cdots, n$ in D results in a graph homeomorphic to H_n, and using Lemma 7 repeatedly, we have $cr(H_n) \leq cr(G_2 \times S_n) - n =$

$cr_D(G_2 \times S_n) - n < Z(6,n) + n + 2\lfloor \frac{n}{2} \rfloor$, a contradiction with Theorem 1. Therefore, $cr(G_2 \times S_n) = Z(6,n) + 2n + 2\lfloor \frac{n}{2} \rfloor$. $\square$

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Smarandache's Pedal Polygon Theorem in the Poincaré Disc Model of Hyperbolic Geometry

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Abstract: In this note, we present a proof of the hyperbolic a Smarandache's pedal polygon theorem in the Poincaré disc model of hyperbolic geometry.

Keywords: Hyperbolic geometry, hyperbolic triangle, gyro-vector, hyperbolic Pytagorean theorem.

AMS(2010): 30F45, 20N99, 51B10, 51M10

§1. Introduction

Hyperbolic Geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of Geometry. It is also known as a type of non-Euclidean Geometry, being in many respects similar to Euclidean Geometry. Hyperbolic Geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different.

There are known many models for Hyperbolic Geometry, such as: Poincaré disc model, Poincaré half-plane, Klein model, Einstein relativistic velocity model, etc. In this note we choose the Poincaré disc model in order to present the hyperbolic version of the Smarandache's pedal polygon theorem. The Euclidean version of this well-known theorem states that if the points $M_i, i = \overline{1,n}$ are the projections of a point M on the sides $A_iA_{i+1}, i = \overline{1,n}$, where $A_{n+1} = A_1$, of the polygon $A_1A_2...A_n$, then $M_1A_1^2+M_2A_2^2+...+M_nA_n^2 = M_1A_2^2+M_2A_3^2+...+M_{n-1}A_n^2+M_nA_1^2$ [1]. This result has a simple statement but it is of great interest.

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

The most general Möbius transformation of D is

$$z \to e^{i\theta}\frac{z_0 + z}{1 + \overline{z_0}z} = e^{i\theta}(z_0 \oplus z),$$

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which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the groupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a} \cdot gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$;

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$;

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$;

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$;

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$;

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$;

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$;

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$.

Definition 1.1 *The hyperbolic distance function in D is defined by the equation*

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

Theorem 1.2(The Möbius Hyperbolic Pythagorean Theorem) *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$, with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V_s}$ and side gyrolenghts $a, b, c \in (-s, s)$, $\mathbf{a} = -B \oplus C, \mathbf{b} = -C \oplus A, \mathbf{c} = -A \oplus B, a = \|\mathbf{a}\|, b = \|\mathbf{b}\|, c = \|\mathbf{c}\|$ and with gyroangles $\alpha, \beta,$ and γ at the vertices $A, B,$ and C. If $\gamma = \pi/2$, then*

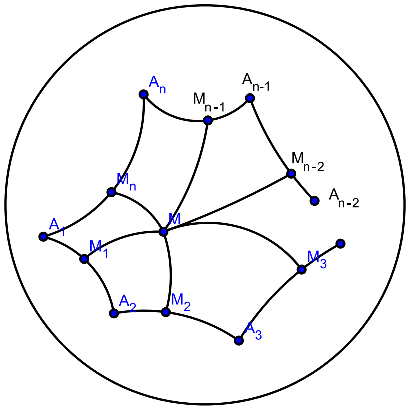
$$\frac{c^2}{s} = \frac{a^2}{s} \oplus \frac{b^2}{s}$$

(see [2, p 290])

For further details we refer to the recent book of A.Ungar [2].

§2. Main Result

In this sections, we present a proof of the hyperbolic a Smarandache's pedal polygon theorem in the Poincaré disc model of hyperbolic geometry.



Figure

Theorem 2.1 *Let $A_1A_2...A_n$ be a hyperbolic convex polygon in the Poincaré disc, whose vertices are the points $A_1, A_2, ..., A_n$ of the disc and whose sides (directed counterclockwise) are $\mathbf{a}_1 = -A_1 \oplus A_2$, $\mathbf{a}_2 = -A_2 \oplus A_3$, ..., $\mathbf{a}_n = -A_n \oplus A_1$. Let the points $M_i, i = \overline{1,n}$ be located on the sides $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ of the hyperbolic convex polygon $A_1A_2...A_n$ respectively. If the perpendiculars to the sides of the hyperbolic polygon at the points $M_1, M_2, ...,$ and M_n are concurrent, then the following equality holds:*

$$|-A_1 \oplus M_1|^2 \ominus |-M_1 \oplus A_2|^2 \oplus |-A_2 \oplus M_2|^2 \ominus |-M_2 \oplus A_3|^2 \oplus ... \oplus |-A_n \oplus M_n|^2 \ominus |-M_n \oplus A_1|^2 = 0.$$

Proof We assume that perpendiculars meet at a point of $A_1A_2...A_n$ and let denote this point by M (see Figure). The geodesic segments $-A_1 \oplus M$, $-A_2 \oplus M$, ..., $-A_n \oplus M$, $-M_1 \oplus M$, $-M_2 \oplus M, ...,$ $-M_n \oplus M$ split the hyperbolic polygon into $2n$ right-angled hyperbolic triangles. We apply the Theorem 1.2 to these $2n$ right-angled hyperbolic triangles one by one, and we easily obtain:

$$|-M \oplus A_k|^2 = |-A_k \oplus M_k|^2 \oplus |-M_k \oplus M|^2 \ ,$$

for all k from 1 to n, and $M_0 = M_n$.Using equalities

$$|-M \oplus A_k|^2 = |-A_k \oplus M|^2 \, , k = \overline{1,n},$$

we have

$$\alpha_k = |-A_k \oplus M_k|^2 \oplus |-M_k \oplus M|^2 = |-M \oplus M_{k-1}|^2 \oplus |-M_{k-1} \oplus A_k|^2 = \alpha_k'$$

for all k from 1 to n, and $M_0 = M_n$. This implies

$$\alpha_1 \oplus \alpha_2 \oplus ... \oplus \alpha_n = \alpha_1^{'} \oplus \alpha_2^{'} \oplus ... \oplus \alpha_n^{'}.$$

Since $((-1,1),\oplus)$ is a commutative group, we immediately obtain

$$|{-A_1} \oplus M_1|^2 \oplus |{-A_2} \oplus M_2|^2 \oplus ... \oplus |{-A_n} \oplus M_n|^2 = |{-M_1} \oplus A_2|^2 \oplus |{-M_2} \oplus A_3|^2 \oplus ... \oplus |{-M_n} \oplus A_1|^2,$$

i.e.

$$|{-A_1} \oplus M_1|^2 \ominus |{-M_1} \oplus A_2|^2 \oplus |{-A_2} \oplus M_2|^2 \ominus |{-M_2} \oplus A_3|^2 \oplus ... \oplus |{-A_n} \oplus M_n|^2 \ominus |{-M_n} \oplus A_1|^2 = 0.$$

□

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The Arc Energy of Digraph

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Abstract: We study the energy of the arc-adjacency matrix of a directed graph D, which is simply called the arc energy of D. In particular, we give upper and lower bounds for the arc energy of D. We show that arc energy of a directed tree is independent of its orientation. We also compute arc energies of directed cycles and some unitary cayley digraphs.

Keywords: Smarandache arc k-energy, digraph, arc adjacency matrix, arc energy.

AMS(2010): 05C50

§1. Introduction

Let D be a simple digraph with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$. Let $|\Gamma(D)| = m$. The arc adjacency matrix of D is the $n \times n$ matrix $A = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & \text{if } \; i < j \quad \text{and} \quad (v_i, v_j) \in \Gamma(D) \\ -1 & \text{if } \; i < j \quad \text{and} \quad (v_j, v_i) \in \Gamma(D) \\ 0 & \text{if } \; v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

For $i > j$ we define $a_{ij} = a_{ji}$. A is a symmetric matrix of order n and all its eigenvalues are real. We denote the eigenvalues of A by $\lambda_1, \lambda_2, \cdots, \lambda_n$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The set $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is called the arc spectrum of D. The characteristic polynomial $|xI - A|$ of the arc adjacency matrix A is called the arc characteristic polynomial of D and it is denoted by $\Phi(D; x)$. The arc energy of D is defined by

$$E_a(D) = \sum_{i=1}^{n} |\lambda_i|.$$

For the majority of conjugated hydrocarbons, The total $\pi-$electron energy, E_π satisfies the relation

$$E_\pi(D) = \sum_{i=1}^{n} |\lambda_i|$$

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where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the molecular graph of the conjugated hydrocarbons. In view of this, Gutman [3] introduced the concept of graph energy $E(G)$ of a simple undirected graph G and he defined it as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the adjacency matrix of G. Survey of development of this topic before 2001 can be found in [4]. For recent development, one can consult [2]. The energy of a graph has close links to chemistry [5]. In many situations chemists use digraph rather than graphs. In this paper we are interested in studying mathematical aspects of arc energy of digraphs. The skew energy of a digraph is recently studied in [1].

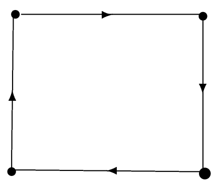
In Section 2 of this paper we study some basic properties of the arc energy and also derive an upper bound for $E_a(D)$. In Section 3 we study arc energy of directed trees. We compute arc energies of directed cycles and some unitary Cayley digraphs in Section 4 and 5 respectively.

§2. Basic Properties of Arc Energy

We begin with the definition of arc energy.

Definition 2.1 *Let A be the arc adjacency matrix of a digraph D. Then its Smarandache arc k-energy $E_a^K(D)$ is defined as $\sum_{i=1}^{n} |\lambda_i|^k$, where n is the order of D and $\lambda_i, 1 \leq i \leq n$ are the eigenvalues of A. Particularly, if $k = 1$, the Smarandache arc k-energy $E_a^1(D)$ is called the arc energy of D and denoted by $E_a(D)$ for abbreviatation.*

Example 2.2 Let D be a directed cycle on four vertices.



Then $A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ and the characteristic polynomial of A is $\lambda^4 - 4\lambda^2 + 4$, and hence the eigenvalues of A are $-\sqrt{2}, \sqrt{2}, -\sqrt{2}, \sqrt{2}$, and the arc energy of D is $4\sqrt{2}$.

Theorem 2.3 *Let D be a digraph with the arc adjacency characteristic polynomial*

$$\Phi(D; x) = b_0 x^n + b_1 x^{n-1} + \cdots + b_n.$$

Then

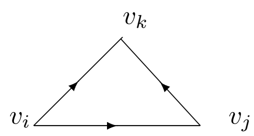
(i) $b_0 = 1$;

(ii) $b_1 = 0$;

(iii) $b_2 = -m$, *the number of arcs of* D;

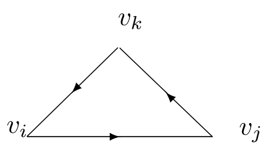
(iv) *For* $i < j < k$, *we define*

$(i, j) =$ *number of triangles of the form*



and

$(i, j, k) =$ *number of triangles of the form*



$$b_3 = -2[(i, j) + (j, k) + (k, i) + (k, j, i) - (j, i) - (k, j) - (i, k) - (i, j, k)].$$

Proof

(i) It follows from the definition, $\Phi(D; x) = \det(xI - A)$, that $b_0 = 1$.

(ii) Since the diagonal elements of A are all zero, the sum of determinants of all 1×1 principal submatrices of $A =$ trace of A $= 0$. So $b_1 = 0$.

(iii) The sum of determinants of all 2×2 principal submatrices of

$$A = \sum_{j<k} \det \begin{bmatrix} 0 & a_{jk} \\ a_{kj} & 0 \end{bmatrix} = \sum_{j<k} -a_{jk}a_{kj} = -\sum_{j<k} a_{jk}^2 = -m.$$

Thus $b_2 = -m$.

(iv) We have

$$b_3 = (-1)^3 \textstyle\sum_{i<j<k} \begin{vmatrix} 0 & a_{ij} & a_{ik} \\ a_{ji} & 0 & a_{jk} \\ a_{ki} & a_{kj} & 0 \end{vmatrix}$$

$$= (-1)^3 \sum_{i<j<k} \begin{vmatrix} 0 & a_{ij} & a_{ik} \\ a_{ij} & 0 & a_{jk} \\ a_{ik} & a_{jk} & 0 \end{vmatrix}$$

$$= -2 \sum_{i<j<k} s_{ij}s_{ik}s_{jk}$$

$$= -2[(i,j)+(j,k)+(k,i)+(k,j,i)-(j,i)-(k,j)-(i,k)-(i,j,k)].$$

□

Theorem 2.4 *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the arc eigenvalues of a digraph D, then*

(i) $\sum_{i=1}^n \lambda_i^2 = 2m$;

(ii) For $1 \leq i \leq n$, $|\lambda_i| \leq \Delta$, *the maximum degree of the underlying graph* G_D.

Proof (i) We have $\sum_{i=1}^n \lambda_i^2 =$ trace of $A^2 = \sum_{i=1}^n \sum_{j=1}^n a_{ij}a_{ji}$

$$= \sum_{i=1}^n \sum_{j=1}^n (a_{ij})^2 = 2m.$$

(ii) Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. The Cauchy-Schwartz inequality state that if $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are real n-vectors then

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right).$$

Let $a_i = 1$ and $b_i = |\lambda_i|$ for $1 \leq i \leq n$, and $i \neq j$. Then

$$\left(\sum_{\substack{i=1\\i\neq j}}^n |\lambda_i|\right)^2 \leq (n-1)\left(\sum_{\substack{i=1\\i\neq j}}^n |\lambda_i|^2\right). \tag{2.1}$$

Since $\sum_{i=1}^n \lambda_i = 0$ we have $\sum_{\substack{i=1,\\i\neq j}}^n \lambda_i = -\lambda_j$. Thus

$$|\sum_{\substack{i=1,\\j\neq i}}^n \lambda_i|^2 = |-\lambda_j|^2.$$

Hence

$$|-\lambda_j|^2 \leq \left(\sum_{\substack{i=1\\i\neq j}}^n |\lambda_i|\right)^2.$$

Using (2.1) in the above inequality we get

$$|-\lambda_j|^2 \leq (n-1)\sum_{i=1}^n \left(|\lambda_i|^2 - |\lambda_j|^2\right).$$

i.e.,

$$n|\lambda_j|^2 \leq 2m(n-1),$$

$$|\lambda_j|^2 \leq (n-1)^2.$$

Hence

$$|\lambda_j| \leq \Delta.$$

$\square$

Corollary 2.5 $E_a(D) \leq n\Delta$.

Theorem 2.6 $\sqrt{2m + n(n-1)p^{2/n}} \leq E_a(D) \leq \sqrt{2mn} \leq n\sqrt{\Delta}$ *where* $p = |det\ A| = \prod_{i=1}^{n} |\lambda_i|$.

Proof We have

$$(E_a(D))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i \neq j} |\lambda_i|\, |\lambda_j|$$

and by the inequality between the arithmetic and geometric means,

$$\frac{1}{n} E_a(D) \geq \left(\prod_{i=1}^{n} |\lambda_i|\right)^{\frac{1}{n}} = |\det\ A|^{\frac{1}{n}}$$

$$\begin{aligned}
\therefore \frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i|\, |\lambda_j| &\geq \left(\prod_{i \neq j} |\lambda_i|\, |\lambda_j|\right)^{\frac{1}{n(n-1)}} \\
&= \left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}} \\
&= \left(\prod_{i=1}^{n} |\lambda_i|\right)^{\frac{2}{n}} = |\prod_{i=1}^{n} \lambda_i|^{\frac{2}{n}} = p^{\frac{2}{n}}.
\end{aligned}$$

Therefore

$$(E_a(D))^2 \geq 2m + n(n-1)p^{\frac{2}{n}}.$$

To prove the right hand side inequality , we apply Schwartz's inequality to the Euclidean vectors $u = (|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|)$ and $v = (1, 1, \cdots, 1)$ to get

$$E_a(D) = \sum_{i=1}^{n} |\lambda_i| \leq \sqrt{\sum_{i=1}^{n} |\lambda_i|^2}\, \sqrt{n} = \sqrt{2mn} \leq \sqrt{n\Delta n} = n\sqrt{\Delta}. \tag{2.2}$$

$\square$

Corollary 2.7 $E_a(D) = n\sqrt{\Delta}$ *if and only if* $A^2 = \Delta I_n$ *where* I_n *is the identity matrix of order* n.

Proof Equality holds in (2.2) if and only if the Schwartz's inequality becomes equality and trace $A^2 = \sum_{i=1}^{n} \lambda_i^2 = 2m = n\Delta$, if and only if, there exists a constant α such that $|\lambda_i|^2 = \alpha$ for all i and G_D is a Δ-regular graph, if and only if, $A^2 = \alpha I_n$ and $\alpha = \Delta$. □

Theorem 2.8 *Each even positive integer $2p$ is the arc energy of a directed star.*

Proof Let $V(K_{1,n}) = \{v_1, \ldots, v_{n+1}\}$. If v_{n+1} is the center of $K_{1,n}$, orient all the edges toward v_{n+1}. Then

$$A = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \ldots & 1 & 0 \end{bmatrix},$$

and its eigenvalues are $\{\sqrt{n}, -\sqrt{n}, 0, 0, \ldots, 0\}$, and so $E_a(K_{1,n}) = 2\sqrt{n}$. Now take $n = p^2$. □

§3. Arc Energies of Trees

We begin with a basic lemma.

Lemma 3.1 *Let D be a simple digraph. and let D' be the digraph obtained from D by reversing the orientations of all the arcs incident with a particular vertex of D. Then $E_a(D) = E_a(D')$.*

Proof Let $A(D)$ be the arc adjacency matrix of D with respect to a labeling of its vertex set. Suppose the orientations of all the arcs incident at vertex v_i of D are reversed. Let the resulting digraph be D'. Then $A(D') = P_i A(D) P_i$ where P_i is the diagonal matrix obtained from the identity matrix by changing the i-th diagonal entry to -1. Hence $A(D)$ and $A(D')$ are orthogonally similar, and so have the same eigenvalues, and hence D and D' have the same arc energy. □

Lemma 3.2 *Let T be a labeled directed tree rooted at vertex v. It is possible, through reversing the orientations of all arcs incident at some vertices other than v, to transform T to a directed tree T' in which the orientations of all the arcs go from low labels to high labels.*

Proof The proof is by induction on n, the order of the tree. For $n = 2$, there is only one arc and the result is true. Assume that any labeled directed tree of order less than n can be transformed in the manner described to a directed tree T' such that the orientations of all the arcs go from low labels to high labels. Consider a labeled directed tree T of order n rooted at v. Let $N(v)$ be the neighbor set of v. For each $w \in N(v)$, reverse the orientations of all the arcs incident at w, if necessary, so that the orientation of the arc between v and w is from low to high labels. Now, by induction assumption, the old-labeled new-orientation subtree T_w rooted at $w \in N(v)$ can be transformed to a directed subtree T'_w such that the orientations of all the arcs go from low labels to high labels. Now combine all the subtrees T'_w and the root v to obtain the required tree T'. □

Theorem 3.3 *The arc energy of a directed tree is independent of its orientation.*

Proof Let T be a labeled directed tree. Since the underlying graph is a tree, it is a bipartite graph, and hence we can label T such that $A(T) = \begin{bmatrix} 0 & Y \\ Y^T & 0 \end{bmatrix}$. By Lemma 3.2, we can transform T to T' such that $A(T') = \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix}$, where X is nonnegative. By applying Lemma 3.1 repeatedly, we conclude that $A(T)$ and $A(T')$ are orthogonally similar, and hence have the same eigenvalues and so the same arc energy. Consequently, T has the same arc energy as the special directed tree T' in which the orientations of all the arcs go from low labels to high labels. $\square$

Corollary 3.4 *The arc energy of a directed tree is the same as the energy of its underlying tree.*

Proof From the proof of Theorem 3.3, the arc energy of a directed tree is equal to the sum of the singular values of $\begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix}$, which is nothing but the adjacency matrix of underlying undirected tree and so the arc energy of a directed tree is the same as the energy of its underlying undirected tree. $\square$

Corollary 3.5 *Energy of a special tournament of order n with vertex set $\{1, 2, \ldots, n\}$ in which all its arcs point from low labels to high labels is same as its underlying tournament.*

§4. Computation of Arc Energies of Cycles

In this section, we compute the arc energies of cycles under different orientations. Given a directed cycle, fix a vertex and label the vertices consecutively. Reversing the arcs incident at a vertex if necessary, we obtain a new directed cycle with arcs going from low labels to high labels with a possible exception of one arc. Hence the arc adjacency matrix of a directed cycle is orthogonally similar to either A^+ or A^- where,

$$A^+ = \begin{bmatrix} 0 & 1 & 0 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \text{ and } A^- = \begin{bmatrix} 0 & 1 & 0 & \ldots & -1 \\ 1 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Case (i): Let C_n^+ be the directed cycle with arc adjacency matrix A^+. We have $A^+ = Z + Z^{n-1}$

where

$$Z = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

which is a circulant matrix. Since $Z^n = I$, the characteristic polynomial of Z is $x^n - 1$. Hence we have $Sp(Z) = \{1, w, w^2, \cdots, w^{n-1}\}$ where $w = e^{\frac{2\pi i}{n}}$ and so

$$\begin{aligned} Sp(C_n^+) &= \{w^j + w^{j(n-1)} \ : \ j = 0, 1, 2, \cdots, n-1\} \\ &= \{w^j + w^{-j} \ : \ j = 0, 1, 2, \cdots, n-1\} \\ &= \{2\cos(\frac{2j\pi}{n}) \ : \ j = 0, 1, 2, \cdots, n-1\}. \end{aligned}$$

For $n = 2k+1$, we have

$$\begin{aligned} E_a(C_n^+) &= \sum_{j=0}^{n-1} 2|\cos(\frac{2j\pi}{n})| = 2 + 4\sum_{j=1}^{k} |\cos(\frac{2j\pi}{(2k+1)})| \\ &= 2 + 4\sum_{j=1}^{k} \cos(\frac{j\pi}{(2k+1)}) = 2 + 4\left(\frac{\sin\frac{(2k+1)\pi}{2(2k+1)}}{2\sin\frac{\pi}{2(2k+1)}} - \frac{1}{2}\right) \\ &= 2\csc(\frac{\pi}{2(2k+1)}) = 2\csc(\frac{\pi}{2n}). \end{aligned}$$

For $n = 4k$,

$$\begin{aligned} E_a(C_n^+) &= \sum_{j=0}^{n-1} 2|\cos(\frac{2j\pi}{n})| = 4 + 8\sum_{j=1}^{k-1} \cos(\frac{j\pi}{2k}) \\ &= 4 + 8\left(\frac{\sin\frac{(2k-1)\pi}{4k}}{2\sin\frac{\pi}{4k}} - \frac{1}{2}\right) = 4\cot(\frac{\pi}{4k}) = 4\cot(\frac{\pi}{n}). \end{aligned}$$

Similarly for $n = 4k+2$

$$E_a(C_n^+) = 4\csc(\frac{\pi}{n}).$$

Putting together the results above, we obtain the following formulas for arc energy of C_n^+ :

$$E_a(C_n^+) = \begin{cases} 2\csc\frac{\pi}{2n} & \text{if} \quad n \equiv 1(\text{mod}2), \\ 4\cot\frac{\pi}{n} & \text{if} \quad n \equiv 0(\text{mod}4), \\ 4\csc\frac{\pi}{n} & \text{if} \quad n \equiv 2(\text{mod}4). \end{cases}$$

Case (ii): Let C_n^- be the directed cycle with arc adjacency matrix A^-. We have $A^- = Z - Z^{n-1}$ where

$$Z = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Since $Z^n = -I$, the characteristic polynomial of Z is $x^n + 1$. Hence we have $Sp(Z) = \{e^{\frac{(2j+1)\pi i}{n}} \mid j = 0, 1, \cdots, (n-1)\}$. So $Sp(A^-) = \{z - z^{n-1} \mid z \in Sp(Z)\}$.

For $n = 2k+1$, we have

$$\begin{aligned}
E_a(C_n^-) &= \sum_{j=0}^{n-1} 2|\cos(\frac{(2j+1)\pi}{2k+1})| = 2\left(\sum_{m=0}^{k}\cos(\frac{m\pi}{2k+1}) - \sum_{m=k+1}^{2k}\cos(\frac{m\pi}{2k+1})\right) \\
&= 2\left(1 + \sum_{m=1}^{k}\cos(\frac{m\pi}{2k+1}) - \sum_{m=k+1}^{2k}\cos\left(\pi - \frac{2k+1-m}{2k+1}\right)\right) \\
&= 2\left(1 + \sum_{m=1}^{k}\cos(\frac{m\pi}{2k+1}) + \sum_{m=k+1}^{2k}\cos\left(\frac{2k+1-m}{2k+1}\right)\pi\right) \\
&= 2\left(1 + 2\sum_{m=1}^{k}\cos(\frac{m\pi}{2k+1})\right) = 2 + 4\sum_{m=1}^{k}\cos(\frac{m\pi}{2k+1}) \\
&= 2\csc(\frac{\pi}{2n}).
\end{aligned}$$

For $n = 4k$, we have

$$\begin{aligned}
E_a(C_n^-) &= \sum_{j=0}^{n-1}|\cos(\frac{(2j+1)\pi}{4k})| = 8\sum_{j=0}^{k-1}\cos(\frac{(2j+1)\pi}{4k}) \\
&= 8\sum_{j=1}^{k}\cos(\frac{(2j-1)\pi}{4k}) = 8\left(\frac{\sin\frac{(k+1)\pi}{4k}\cos(\frac{\pi}{4} - \frac{\pi}{4k})}{\sin\frac{\pi}{4k}}\right).
\end{aligned}$$

Similarly for $n = 4k+2$, we get

$$E_a(C_n^-) = \frac{\sin(\frac{(k+1)\pi}{2(2k+1)})\cos(\frac{k\pi}{2(2k+1)} - \frac{\pi}{2(2k+1)})}{\sin\frac{\pi}{2(2k+1)}}.$$

Putting together the results above, we obtain the following formulas for arc energy of C_n^- :

$$E_a(C_n^-) = \begin{cases} 2\csc(\frac{\pi}{2n}) & \text{if} \quad n \equiv 1(\text{mod}2), \\ 8\left(\frac{\sin\frac{(k+1)\pi}{4k}\cos(\frac{\pi}{4} - \frac{\pi}{4k})}{\sin\frac{\pi}{4k}}\right) & \text{if} \quad n \equiv 0(\text{mod}4), \\ \frac{\sin(\frac{(k+1)\pi}{2(2k+1)})\cos(\frac{k\pi}{2(2k+1)} - \frac{\pi}{2(2k+1)})}{\sin\frac{\pi}{2(2k+1)}} & \text{if} \quad n \equiv 2(\text{mod}4). \end{cases}$$

§4. On the Arc Energies of Some Unitary Cayley Digraphs

We now define the unit Cayley digraph D_n, $n > 1$. The vertex set of D_n is $V(D_n) = \{0, 1, 2, \cdots, (n-1)\}$ and the arc set of D_n is $\Gamma(D_n)$ and is defined as follows:

For $i, j \in \{0, 1, 2, \cdots, (n-1)\}$ with $i < j$ and $(j-i, n) = 1$, $(i, j) \in \Gamma(D_n)$ or $(j, i) \in \Gamma(D_n)$ according as $j - i$ is a quadratic residue or a quadratic non-residue modulo n. In this section we compute arc energies of unitary Cayley digraphs D_n for $n = 2^{\alpha_0}p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, $\alpha_0 = 0$ or 1,

$p_i \equiv 1 \pmod 4$, $i = 1, 2, 3, \cdots, r$. We make use of the following well-known result to establish a formula for arc energy of D_n for certain values of n.

Theorem 5.1 *Let $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $n > 1$ and $(a, n) = 1$. Then $x^2 \equiv a \pmod n$ is solvable if and only if*

(i) $\left(\frac{a}{p_i}\right) = 1$ *for* $i = 1, 2, \cdots, r$

and

(ii) $a \equiv 1 \pmod 4$ *if* $4 \mid n$ *but* $8 \nmid n$ *;* $a \equiv 1 \pmod 8$ *if* $8 \mid n$.

Here $\left(\frac{a}{p_i}\right)$ *is the Legendre symbol.*

Theorem 5.2 *For $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $\alpha_0 = 0$ or 1, $p_i \equiv 1 \pmod 4$, $i = 1, 2, 3, \cdots, r$, the arc adjacency eigenvalues of the unitary Cayley digraph D_n are the Gauss sums $G(r, \chi_f)$, $r = 0, 1, 2, \cdots, n-1$, associated with quadratic character f.*

Proof The arc adjacency matrix of D_n with respect to the natural order of the vertices $0, 1, \cdots, n-1$ is

$$A_n = \begin{pmatrix} (\frac{0}{n}) & (\frac{1}{n}) & (\frac{2}{n}) & \cdots & (\frac{i-1}{n}) & \cdots & (\frac{n-1}{n}) \\ \\ (\frac{1}{n}) & (\frac{0}{n}) & (\frac{1}{n}) & \cdots & (\frac{i-2}{n}) & \cdots & (\frac{n-2}{n}) \\ \vdots & & & & & & \\ (\frac{i-1}{n}) & (\frac{i-2}{n}) & (\frac{i-3}{n}) & \cdots & (\frac{0}{n}) & \cdots & (\frac{n-i}{n}) \\ \vdots & & & & & & \\ (\frac{n-1}{n}) & (\frac{n-2}{n}) & (\frac{n-3}{n}) & \cdots & (\frac{n-i}{n}) & \cdots & (\frac{0}{n}) \end{pmatrix}$$

where

$$\left(\frac{a}{n}\right) = \begin{cases} 1 \text{ if } & (a,n) = 1 \text{ and } x^2 \equiv a (\text{mod } n) \text{ is solvable,} \\ -1 \text{ if } & (a,n) = 1 \text{ and } x^2 \equiv a (\text{mod } n) \text{ is not solvable,} \\ 0 \text{ otherwise.} \end{cases}$$

Since $n = 2^{\alpha_0} p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $n > 1$, where $\alpha_0 = 0$ or 1 and $p_i \equiv 1 \pmod 4$, $i = 1, 2, 3, \cdots, r$, it follows from Theorem 5.1 that $x^2 \equiv -1$ (mod n) is solvable. Thus

$$\left(-\frac{1}{n}\right) = 1. \tag{5.1}$$

Moreover, if $(a, n) = 1$ then

$$\left(\frac{n-a}{n}\right) = \left(\frac{-a}{n}\right) = \left(\frac{-1}{n}\right)\left(\frac{a}{n}\right) = \left(\frac{a}{n}\right) \qquad (\text{ using } (5.1)).$$

Hence the arc adjacency matrix A_n of D_n is circulant. Consequently the eigenvalues of A_n are given by

$$\begin{aligned}\lambda_r &= \sum_{m=0}^{n-1}\left(\frac{m}{n}\right)w^{rm}, \quad r=0,1,\cdots,n-1, \quad w=e^{\frac{2\pi i}{n}} \\ &= \sum_{m=1}^{n-1}\left(\frac{m}{n}\right)w^{rm} = G(r,\chi_f)\end{aligned}$$

where χ_f is the Dirichlet quadratic character mod n. □

Theorem 5.3 *If $n = 2^{\alpha_0}p_1^{\alpha_1}\cdots p_r^{\alpha_r}$, $n>1$, where $\alpha_0 = 0$ or 1 and $p_i \equiv 1$ (mod 4), $i = 1,2,\cdots,r$ then the arc energy of D_n is*

$$E_a(D_n) = \sqrt{n}\ \ \phi(n).$$

Proof By Theorem 5.2, the eigenvalues of D_n are

$$\lambda_r = G(r,\chi_f), \ \ 0 \le r \le n-1.$$

Hence the arc energy of D_n is given by

$$\begin{aligned}E_a(D_n) &= \sum_{r=0}^{n-1}|\lambda_r| = \sum_{r=0}^{n-1}|G(r,\chi_f)| \\ &= \sum_{r=1}^{n-1}|\overline{\chi}_f(r)||G(1,\chi_f)| = |G(1,\chi_f)|\ \phi(n).\end{aligned}$$

Therefore, to complete the proof, we need to compute $|G(1,\chi_f)|$. We have

$$\begin{aligned}|G(1,\chi_f)|^2 &= G(1,\chi_f)\overline{G(1,\chi_f)} = G(1,\chi_f)\sum_{m=1}^{n}\overline{\chi}_f(m)\ e^{\frac{-2\pi i m}{n}} \\ &= \sum_{m=1}^{n}G(m,\chi_f)\ e^{\frac{-2\pi i m}{n}} = \sum_{m=1}^{n}\sum_{j=1}^{n}\left(\frac{j}{n}\right)e^{\frac{2\pi i j m}{n}}e^{\frac{-2\pi i m}{n}} \\ &= \sum_{j=1}^{n}\left(\frac{j}{n}\right)\sum_{m=1}^{n}w^{m(j-1)}, \ \text{ where } \ w=e^{\frac{2\pi i}{n}} \\ &= \left(\frac{1}{n}\right)\sum_{m=1}^{n}1, \ \text{ since} \sum_{m=1}^{n}w^{m(j-1)}=0, \ \text{ if } \ j>1 \\ &= n.\end{aligned}$$

Hence $|G(1,\chi_f)| = \sqrt{n}$ and $E(D_n) = \sqrt{n}\ \ \phi(n)$.

Conclusion The arc spectrum and arc energy of D_n when $n \equiv 1$ or 2(mod 4) was computed (Theorems 5.2 and 5.3.) using fact that the associated arc adjacency matrix A_n was circulant. Since in general A_n is not circulant, we leave open the problem of computing the arc spectrum and arc energy of D_n for any natural number n.

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Euler-Savary's Formula for the Planar Curves in Two Dimensional Lightlike Cone

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Abstract: In this paper, we study the Euler-Savary's formula for the planar curves in the lightlike cone. We first define the associated curve of a curve in the two dimensional lightlike cone Q^2.Then we give the relation between the curvatures of a base curve, a rolling curve and a roulette which lie on two dimensional lightlike cone Q^2.

Keywords: Lightlike cone, Euler Savary's formula, Smarandache geometry, Smarandachely denied-free.

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§1. Introduction

The Euler-Savary's Theorem is well known theorem which is used in serious fields of study in engineering and mathematics.

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways. So the Euclidean geometry is just a Smarandachely denied-free geometry.

In the Euclidean plane E^2,let c_B and c_R be two curves and P be a point relative to c_R. When c_R roles without splitting along c_B, the locus of the point P makes a curve $c_{L.}$ The curves c_B, c_R and c_L are called the base curve, rolling curve and roulette, respectively. For instance, if c_B is a straight line, c_R is a quadratic curve and P is a focus of c_R, then c_L is the Delaunay curve that are used to study surfaces of revolution with the constant mean curvature, (see [1]).The relation between the curvatures of this curves is called as the Euler-Savary's formula.

Many studies on Euler-Savary's formula have been done by many mathematicians. For example, in [4], the author gave Euler-Savary's formula in Minkowski plane. In [5], they expressed the Euler-Savary's formula for the trajectory curves of the 1-parameter Lorentzian spherical motions.

On the other hand, there exists spacelike curves, timelike curves and lightlike(null) curves in semi-Riemannian manifolds. Geometry of null curves and its applications to general reletivity in semi-Riemannian manifolds has been constructed, (see [2]). The set of all lightlike(null)

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vectors in semi-Riemannian manifold is called the lightlike cone. We know that it is important to study submanifolds of the lightlike cone because of the relations between the conformal transformation group and the Lorentzian group of the n-dimensional Minkowski space E_1^n and the submanifolds of the n-dimensional Riemannian sphere S^n and the submanifolds of the (n+1)-dimensional lightlike cone Q^{n+1}. For the studies on lightlike cone, we refer [3].

In this paper, we have done a study on Euler-Savary's formula for the planar curves in two dimensional lightlike cone Q^2. However, to the best of author's knowledge, Euler-Savary's formula has not been presented in two dimensional lightlike cone Q^2. Thus, the study is proposed to serve such a need. Thus, we get a short contribution about Smarandache geometries.

This paper is organized as follows. In Section2, the curves in the lightlike cone are reviewed. In Section3, we define the associated curve that is the key concept to study the roulette, since the roulette is one of associated curves of the base curve. Finally, we give the Euler-Savary's formula in two dimensional cone Q^2.

We hope that, these study will contribute to the study of space kinematics, mathematical physics and physical applications.

§2. Euler-Savary's Formula in the Lightlike Cone $\mathbf{Q}^2$

Let E_1^3 be the 3−dimensional Lorentzian space with the metric

$$g(x,y) = \langle x,y\rangle = x_1y_1 + x_2y_2 - x_3y_3,$$

where $x = (x_1,x_2,x_3)$, $y = (y_1,y_2,y_3) \in E_1^3$.

The lightlike cone Q^2 is defined by

$$Q^2 = \{x \in E_1^3 : g(x,x) = 0\}.$$

Let $x : I \to Q^2 \subset E_1^3$ be a curve, we have the following Frenet formulas (see [3])

$$\begin{aligned} x'(s) &= \alpha(s) \\ \alpha'(s) &= \kappa(s)x(s) - y(s) \\ y'(s) &= -\kappa(s)\alpha(s), \end{aligned} \tag{2.1}$$

where s is an arclength parameter of the curve $x(s)$. $\kappa(s)$ is cone curvature function of the curve $x(s)$,and $x(s)$, $y(s)$, $\alpha(s)$ satisfy

$$\begin{aligned} \langle x,x\rangle &= \langle y,y\rangle = \langle x,\alpha\rangle = \langle y,\alpha\rangle = 0, \\ \langle x,y\rangle &= \langle \alpha,\alpha\rangle = 1. \end{aligned}$$

For an arbitrary parameter t of the curve $x(t)$, the cone curvature function κ is given by

$$\kappa(t) = \frac{\left\langle \frac{dx}{dt}, \frac{d^2x}{dt^2}\right\rangle^2 - \left\langle \frac{d^2x}{dt^2}, \frac{d^2x}{dt^2}\right\rangle \left\langle \frac{dx}{dt}, \frac{dx}{dt}\right\rangle}{2\left\langle \frac{dx}{dt}, \frac{dx}{dt}\right\rangle^5} \tag{2.2}$$

Using an orthonormal frame on the curve $x(s)$ and denoting by $\overline{\kappa}$, $\overline{\tau}$, β and γ the curvature, the torsion, the principal normal and the binormal of the curve $x(s)$ in E_1^3, respectively, we have

$$\begin{aligned} x' &= \alpha \\ \alpha' &= \kappa x - y = \overline{\kappa}\beta, \end{aligned}$$

where $\kappa \neq 0$, $\langle \beta, \beta \rangle = \varepsilon = \pm 1$, $\langle \alpha, \beta \rangle = 0$, $\langle \alpha, \alpha \rangle = 1$, $\varepsilon\kappa < 0$. Then we get

$$\beta = \varepsilon \frac{\kappa x - y}{\sqrt{-2\varepsilon\kappa}}, \quad \varepsilon\overline{\tau}\gamma = \frac{\kappa'}{2\sqrt{-2\varepsilon\kappa}}(x + \frac{1}{\kappa}y). \tag{2.3}$$

Choosing

$$\gamma = \sqrt{\frac{-\varepsilon\kappa}{2}}(x + \frac{1}{\kappa}y), \tag{2.4}$$

we obtain

$$\overline{\kappa} = \sqrt{-2\varepsilon\kappa}, \quad \overline{\tau} = -\frac{1}{2}(\frac{\kappa'}{\kappa}). \tag{2.5}$$

Theorem 2.1 *The curve $x : I \to Q^2$ is a planar curve if and only if the cone curvature function κ of the curve $x(s)$ is constant [3].*

If the curve $x : I \to Q^2 \subset E_1^3$ is a planar curve, then we have following Frenet formulas

$$\begin{aligned} x' &= \alpha, \\ \alpha' &= \varepsilon\sqrt{-2\varepsilon\kappa}\beta, \\ \beta' &= -\sqrt{-2\varepsilon\kappa}\alpha. \end{aligned} \tag{2.6}$$

Definition 2.2 *Let $x : I \to Q^2 \subset E_1^3$ be a curve with constant cone curvature κ (which means that x is a conic section) and arclength parameter s. Then the curve*

$$x_A = x(s) + u_1(s)\alpha + u_2(s)\beta \tag{2.7}$$

is called the associated curve of $x(s)$ in the Q^2, where $\{\alpha, \beta\}$ is the Frenet frame of the curve $x(s)$ and $\{u_1(s), u_2(s)\}$ is a relative coordinate of $x_A(s)$ with respect to $\{x(s), \alpha, \beta\}$.

Now we put

$$\frac{dx_A}{ds} = \frac{\delta u_1}{ds}\alpha + \frac{\delta u_2}{ds}\beta. \tag{2.8}$$

Using the equation (2.2) and (2.6), we get

$$\frac{dx_A}{ds} = (1 + \frac{du_1}{ds} - \sqrt{-2\varepsilon\kappa}u_2)\alpha + (u_1\varepsilon\sqrt{-2\varepsilon\kappa} + \frac{du_2}{ds})\beta. \tag{2.9}$$

Considering the (2.8) and (2.9), we have

$$\begin{aligned} \frac{\delta u_1}{ds} &= (1 + \frac{du_1}{ds} - \sqrt{-2\varepsilon\kappa}u_2) \\ \frac{\delta u_2}{ds} &= (u_1\varepsilon\sqrt{-2\varepsilon\kappa} + \frac{du_2}{ds}) \end{aligned} \tag{2.10}$$

Let s_A be the arclength parameter of x_A. Then we write

$$\frac{dx_A}{ds} = \frac{dx_A}{ds_A}.\frac{ds_A}{ds} = v_1\alpha + v_2\beta \tag{2.11}$$

and using (2.8) and (2.10), we get

$$\begin{aligned} v_1 &= 1 + \frac{du_1}{ds} - \sqrt{-2\varepsilon\kappa}u_2 \\ v_2 &= u_1\varepsilon\sqrt{-2\varepsilon\kappa} + \frac{du_2}{ds}. \end{aligned} \tag{2.12}$$

The Frenet formulas of the curve x_A can be written as follows:

$$\begin{aligned} \frac{d\alpha_A}{ds_A} &= \varepsilon_A\sqrt{-2\varepsilon_A\kappa_A}\beta_A \\ \frac{d\beta_A}{ds_A} &= -\sqrt{-2\varepsilon_A\kappa_A}\alpha_A, \end{aligned} \tag{2.13}$$

where κ_A is the cone curvature function of x_A and $\varepsilon_A = \langle\beta_A, \beta_A\rangle = \pm 1$ and $\langle\alpha_A, \alpha_A\rangle = 1$.

Let θ and ω be the slope angles of x and x_A respectively. Then

$$\overline{\kappa}_A = \frac{d\omega}{ds_A} = (\overline{\kappa} + \frac{d\phi}{ds})\frac{1}{\sqrt{|v_1^2 + \varepsilon v_2^2|}}, \tag{2.14}$$

where $\phi = \omega - \theta$.

If β is spacelike vector, then we can write

$$\cos\phi = \frac{v_1}{\sqrt{v_1^2 + v_2^2}} \text{ and } \sin\phi = \frac{v_2}{\sqrt{v_1^2 + v_2^2}}.$$

Thus, we get

$$\frac{d\phi}{ds} = \frac{d}{ds}(\cos^{-1}\frac{v_1}{\sqrt{v_1^2 + v_2^2}})$$

and (2.14) reduces to

$$\overline{\kappa}_A = (\overline{\kappa} + \frac{v_1 v_2^{'} - v_1^{'} v_2}{v_1^2 + v_2^2})\frac{1}{\sqrt{v_1^2 + v_2^2}}.$$

If β is timelike vector, then we can write

$$\cosh\phi = \frac{v_1}{\sqrt{v_1^2 - v_2^2}} \text{ and } \sinh\phi = \frac{v_2}{\sqrt{v_1^2 - v_2^2}}$$

and we get

$$\frac{d\phi}{ds} = \frac{d}{ds}(\cosh^{-1}\frac{v_1}{\sqrt{v_1^2 - v_2^2}}).$$

Thus, we have

$$\overline{\kappa}_A = (\overline{\kappa} + \frac{v_1 v_2^{'} - v_1^{'} v_2}{v_1^2 - v_2^2})\frac{1}{\sqrt{v_1^2 - v_2^2}}.$$

Let x_B and x_R be the base curve and rolling curve with constant cone curvature κ_B and κ_R in Q^2, respectively. Let P be a point relative to x_R and x_L be the roulette of the locus of P.

We can consider that x_L is an associated curve of x_B such that x_L is a planar curve in Q^2, then the relative coordinate $\{w_{1,}w_2\}$ of x_L with respect to x_B satisfies

$$\begin{aligned}\frac{\delta w_1}{ds_B} &= 1+\frac{dw_1}{ds_B}-\sqrt{-2\varepsilon_B\kappa_B}w_2\\ \frac{\delta w_2}{ds_B} &= w_1\varepsilon_B\sqrt{-2\varepsilon_B\kappa_B}+\frac{dw_2}{ds_B}\end{aligned}\tag{2.15}$$

by virtue of (2.10).

Since x_R roles without splitting along x_B at each point of contact, we can consider that $\{w_{1,}w_2\}$ is a relative coordinate of x_L with respect to x_R for a suitable parameter s_R. In this case, the associated curve is reduced to a point P. Hence it follows that

$$\begin{aligned}\frac{\delta w_1}{ds_R} &= 1+\frac{dw_1}{ds_R}-\sqrt{-2\varepsilon_R\kappa_R}w_2=0\\ \frac{\delta w_2}{ds_R} &= w_1\varepsilon_R\sqrt{-2\varepsilon_R\kappa_R}+\frac{dw_2}{ds_R}=0.\end{aligned}\tag{2.16}$$

Substituting these equations into (2.15), we get

$$\begin{aligned}\frac{\delta w_1}{ds_B} &= (\sqrt{-2\varepsilon_R\kappa_R}-\sqrt{-2\varepsilon_B\kappa_B})w_2\\ \frac{\delta w_2}{ds_B} &= (\varepsilon_B\sqrt{-2\varepsilon_B\kappa_B}-\varepsilon_R\sqrt{-2\varepsilon_R\kappa_R})w_1.\end{aligned}\tag{2.17}$$

If we choose $\varepsilon_B=\varepsilon_R=-1$, then

$$0<(\frac{\delta w_1}{ds_B})^2-(\frac{\delta w_2}{ds_B})^2=(\sqrt{2\kappa_R}-\sqrt{2\kappa_B})^2(w_2^2-w_1^2).\tag{2.18}$$

Hence, we can put

$$w_1=r\sinh\phi\ ,\quad w_2=r\cosh\phi.$$

Differentiating this equations, we get

$$\begin{aligned}\frac{dw_1}{ds_R} &= \frac{dr}{ds_R}\sinh\phi+r\cosh\phi\frac{d\phi}{ds_R}\\ \frac{dw_2}{ds_R} &= \frac{dr}{ds_R}\cosh\phi+r\sinh\phi\frac{d\phi}{ds_R}\end{aligned}\tag{2.19}$$

Providing that we use (2.16), then we have

$$\begin{aligned}\frac{dw_1}{ds_R} &= r\sqrt{2\kappa_R}\cosh\phi-1\\ \frac{dw_2}{ds_R} &= r\sinh\phi\sqrt{2\kappa_R}\end{aligned}\tag{2.20}$$

If we consider (2.19) and (2.20), then we get

$$r\frac{d\phi}{ds_R}=-r\sqrt{2\kappa_R}+\cosh\phi\tag{2.21}$$

Therefore, substituting this equation into (2.14), we have

$$r\overline{\kappa}_L=\pm1+\frac{\cosh\phi}{r\left|\sqrt{2\kappa_R}-\sqrt{2\kappa_B}\right|}\tag{2.22}$$

If we choose $\varepsilon_B = \varepsilon_R = +1$, then from (2.17)

$$0 < (\frac{\delta w_1}{ds_B})^2 + (\frac{\delta w_2}{ds_B})^2 = (\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B})^2 (w_1^2 + w_2^2) \tag{2.23}$$

Hence we can put

$$w_1 = r \sin\phi \ , \quad w_2 = r\cos\phi.$$

Differentiating this equations, we get

$$\begin{aligned} \frac{dw_1}{ds_R} &= \frac{dr}{ds_R}\sin\phi + r\cos\phi\frac{d\phi}{ds_R} = r\sqrt{-2\kappa_R}\cos\phi - 1 \\ \frac{dw_2}{ds_R} &= \frac{dr}{ds_R}\cos\phi - r\sin\phi\frac{d\phi}{ds_R} = -r\sin\phi\sqrt{-2\kappa_R} \end{aligned} \tag{2.24}$$

and

$$r\frac{d\phi}{ds_R} = r\sqrt{-2\kappa_R} - \cos\phi \tag{2.25}$$

Therefore, substituting this equation into (2.14), we have

$$r\overline{\kappa}_L = \frac{\sqrt{-2\kappa_B} + \sqrt{-2\kappa_R}}{\left|\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B}\right|} - \frac{\cos\phi}{r\left|\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B}\right|}, \tag{2.26}$$

where $\overline{\kappa}_L = \sqrt{-2\varepsilon_L \kappa_L}$.

Thus we have the following Euler-Savary's Theorem for the planar curves in two dimensional lightlike cone Q^2.

Theorem 2.3 *Let x_R be a planar curve on the lightlike cone Q^2 such that it rolles without splitting along a curve x_B. Let x_L be a locus of a point P that is relative to x_R. Let Q be a point on x_L and R a point of contact of x_B and x_R corresponds to Q relative to the rolling relation. By (r, ϕ) , we denote a polar coordinate of Q with respect to the origin R and the base line $x_B^{'} \mid_R$. Then curvatures κ_B , κ_R and κ_L of x_B , x_R and x_L respectively, satisfies*

$$\begin{aligned} r\overline{\kappa}_L &= \pm 1 + \frac{\cosh\phi}{r\left|\sqrt{2\kappa_R} - \sqrt{2\kappa_B}\right|}, \quad \text{if } \varepsilon_B = \varepsilon_R = -1, \\ r\overline{\kappa}_L &= \frac{\sqrt{-2\kappa_B} + \sqrt{-2\kappa_R}}{\left|\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B}\right|} - \frac{\cos\phi}{r\left|\sqrt{-2\kappa_R} - \sqrt{-2\kappa_B}\right|} \quad \text{if } \varepsilon_B = \varepsilon_R = +1. \end{aligned}$$

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The man with a new idea is a crank until the idea succeeds.

By Mark Twain, an American writer.

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[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).
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