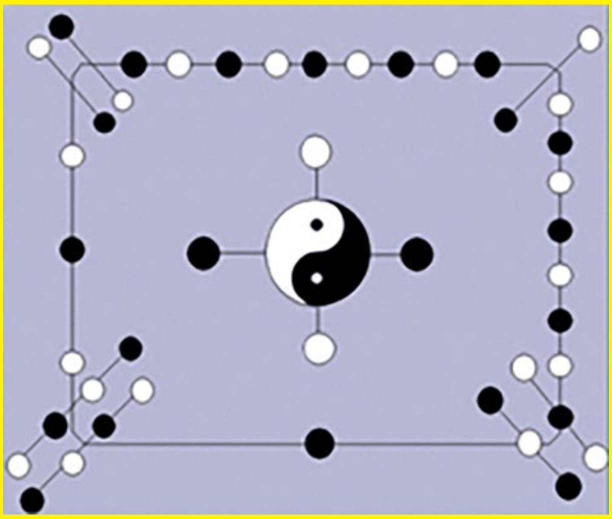
ISSN 1937 - 1055
VOLUME 1, 2026

INTERNATIONAL JOURNAL OF
MATHEMATICAL COMBINATORICS



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March, 2026

Vol.1, 2026 **ISSN 1937-1055**

International Journal of
Mathematical Combinatorics

(www.mathcombin.com)

Edited By

The Madis of Chinese Academy of Sciences and
Academy of Mathematical Combinatorics & Applications, USA

March, 2026

Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.L.F. MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

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Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

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Famous Words:

The science of mathematics presents the most brilliant example of how pure reason may successfully enlarge its domain without the aid of experience.

By *Immanuel Kant*, a German philosopher and writer.

International J.Math. Combin. Vol.1(2026), 01-26

New Mathematics over Combinatorial Structure – to Reductionism of Human

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Abstract: Usually, one holds the reality of thing T by reductionism, namely subdividing T into elements by measuring their characters with solution of state equation to model its evolution. So, *can we understand the state of thing T by its one element or partial characters?* The answer is certainly Not because such an idea is only a hasty generalization unless there is only one element or all elements are in synchronizing, not being the case of self-organized system or biological system with number≥ 2 of elements in general, namely we lack a mathematics for adapting the recognition of reductionism. Notice that the subdividing of thing T naturally inherits a topological graph $\overrightarrow{G}$ in space, which is equivalently to transform thing T to a labelled graph $\overrightarrow{G}^L$, not the solution but the union of solutions of all state equations of elements, i.e., a Smarandache multuspace. Thus, we should establish a mathematics over $\overrightarrow{G}$ for characterizing the evolution of T, oblivious in the classic. Now, *could we regard $\overrightarrow{G}^L$ not only as a labelled graph but a mathematical element and establish such a mathematics that on $\overrightarrow{G}^L$?* The answer is certainly Yes, i.e., mathematical combinatorics following the CC conjecture, a combinatorial notion on classic mathematical extensions by reconstructed from or made by combinatorialization over topological graph $\overrightarrow{G}$. In this paper, we introduce the non-harmonious group corresponding to the reductionism and show how to establish a mathematical system over $\overrightarrow{G}$ with Smarandache multispace, including two types, i.e., $\overrightarrow{G}$ or its supergraph $\widehat{G}$ invariant for extending classic mathematics largely, including a few interesting results in functional analysis and Euler-Lagrange equation. All of these works constitute a mathematics to reductionism on combinatorial structure $\overrightarrow{G}$.

Key Words: Combinatorial notion, Smarandache multispace, CC conjecture, mathematical combinatorics, non-harmonious group, solution manifold, G-solution, continuity flow, Euler-Lagrange equation.

AMS(2010): 05C10, 05C21, 35A08, 46B25, 51D20, 51H20, 51P05.

§1. Introduction

As we all know, the developing of human depends on the recognition of thing T in nature, particularly by reductionism and the mathematics is such a formal system in logical consistency

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[1] Reported at the *International Conference on New Age Mathematical Sciences*, Kolkata, India, December 11-13, 2025

[2] Received September 16, 2025, Accepted January 12, 2026

that can quantitatively model the evolution of thing T by its element so that the behavior of thing is consistent with that of its elements. Now, *could we really understand things by mathematics?* The answer is certainly Not by Godel's incompleteness theorem, namely there exist always statements in a formal system S that can neither be proved nor disproved so long as S contains the Peano's axioms of arithmetic. *Why is this case so happening?* Because all classic mathematics are incompleteness for adopting the Peano's axioms of arithmetic in default and ignores the inherited combinatorial structure $\overrightarrow{G}$ in reductionism.

1.1 Reductionism

The *reductionism* is a recognitive mechanism of human on things in the universe over years, i.e., subdividing thing T into the smallest elements of recognition.

Example 1.1 Matter M

For holding on the reality of a matter M, one subdivides M into molecule, atom, proton, neutron and finally, the elementary particles for recognition [32] such as the shown in Figure 1,

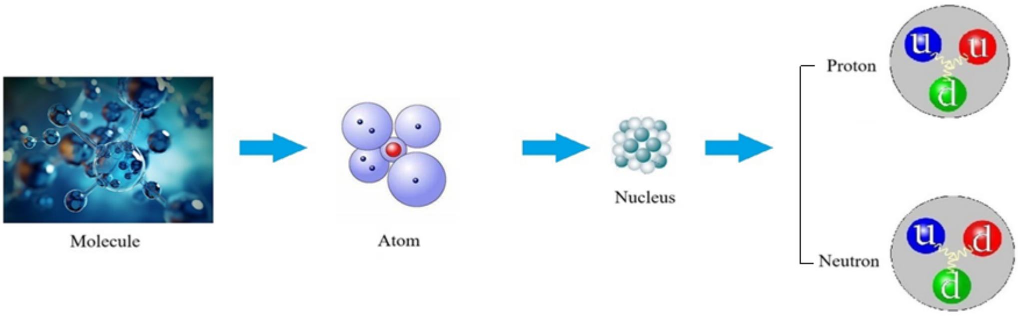

Figure 1.

i.e., subdividing

Matter $M \to$ *Molecules* $\to$ *Atoms* $\to$ *Nucleus and Electronics* $\to$ *elementary particles.*

Example 1.2 Living things L

There are two ways for recognition of living things, i.e., the microscopic and macroscopic recognition [39].

Microscopic Recognition. In this case, one subdivides L into cells and genes.

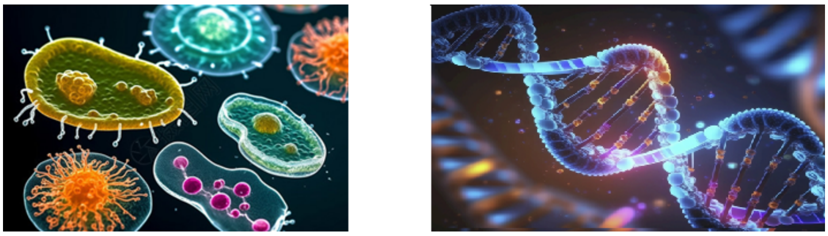

Figure 2.

i.e., subdividing

Living thing $L \to$ *Biological Macromolecule* $\to$ *Cells and Genes.*

Macroscopic Recognition. In this case, one views each one in its macroscopic state as an element. For example, the biological population.

Accordingly, the reductionism is equivalent matter T or living L to a complex network [2] underlying a 1-dimensional topological structure $\overrightarrow{G}$. Now, *could one hold on all behavior of things by such a complex network?* The answer is certainly Not. For example, the complex networks of male has 3.6×10^{13} cells and female has 2.8×10^{13} cells. If so, it will greatly increase the recognitive complexity, not the nature of thing but the artificial.

There is an assumption in reductionism without proof, namely the reality of thing T can be held on if the behavior of all elements are recognized. So, *could we conclude this assumption is right?* The answer is certainly Not because we can't assert T is the same of any one of its elements unless they are moving in synchronization such as those shown in Figure 3, where (a) fishes or (b) birds move from disorder to order.


(a) (b)

Figure 3.

However, elements in a self-organized system are usually not so, we can not assert it is the same of all elements in thing T because it inherits naturally a 1-dimensional topological structure $\overrightarrow{G}$ in reductionism.

Different from the subdividing on matter, living into smallest recognitive units or particles, the five elements [29], [40], i.e., the wood, fire, earth, metal and water of Chinese are five moving typing trends in nature. They are not particles or elements in periodic table of elements but five moving trends in nature. For examples, the character of fire is fierce, intense and rising, and the water is nourish, moisten and descend, i.e., the five elements provided the evolution mechanisms of things in the universe by the ancient Chinese.

Figure 4.

Furthermore, the five elements are not exist alone as particles but with interaction, i.e., generating and overcoming form 2 cycles as follows:

$$Wood \dashrightarrow Fire \dashrightarrow Earth \dashrightarrow Metal \dashrightarrow Water \dashrightarrow Wood \cdots, \text{ a cycle}$$

and

$$Wood \to Earth \to Water \to Fire \to Metal \to Wood \cdots, \text{ another cycle,}$$

where $\dashrightarrow$ denotes the generating action, $\to$ denotes the overcoming action, namely the whole evolution of thing T is under the mechanisms of 5 elements as shown in Figure 5.

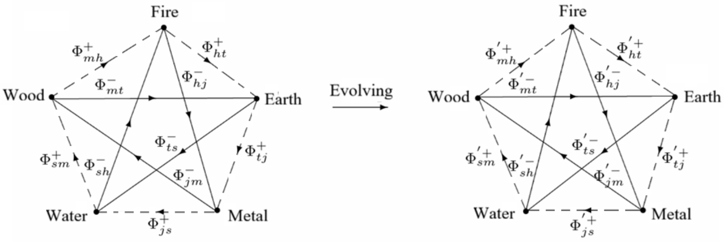

Figure 5.

1.2. Smarandache Multispace

Smarandache multispace is a recognitive notion on thing T from the local to the whole, against the limitation of human implied in the fable of six blind men with an elephant [28]. In this fable, each of the blind touched different part locally and claimed the different shape of an elephant, namely the 1st one claimed it is like a big smooth radish, the 2nd one claimed it is like a tube, the 3rd one claimed it is like a big fan, the 4th one claimed it is like a wall, the 5th one claimed it is like a big pillar and the 6th one claimed it is like a piece of rope. They stuck to their own views and then fell into an endless argument! So, *why are they arguing? why this case happens?* A sophist explained the reason to the blinds: *Why you are thinking about the shape of elephant different is because each of you touches the different part of the elephant's body. Essentially, an elephant has all characters that you are talking about!* Then, *what is an elephant shape in eyes of the sophist by locally recognition of the blind men?* The answer is the union of characters recognized by the 6 blind, i.e., the union of all the local to form a whole

Figure 6.

$$\begin{aligned} \text{An elephant} = &\{\textit{4 big pillars}\}\bigcup\{\textit{1 gross rope}\}\bigcup\{\textit{1 tube}\}\\ &\bigcup\{\textit{2 big fans}\}\bigcup\{\textit{1 big wall}\}\bigcup\{\textit{2 big raddishes}\}, \end{aligned}$$

i.e., recognized by the sophist with the known character on that of the six parts of the six blinds. such as those shown in Figure 7

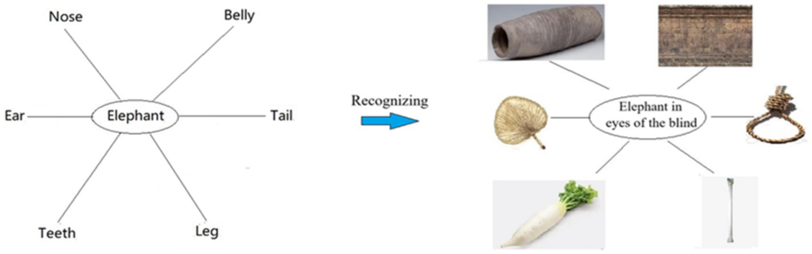

Figure 7.

with a tree structure

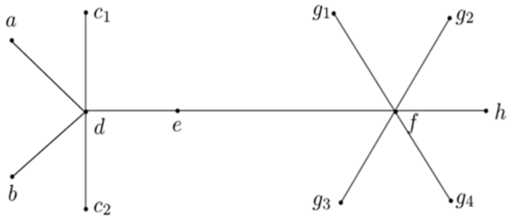

Figure 8

where a = {teeth}, b = {nose}, c_1, c_2 = {ears}, d = {head}, e = {neck}, f = {belly}, g_1, g_2, g_3, g_4 = {legs}, h = {tail}. Then, *what is a Smarandache multispace or multisystem?* Formally a Smarandache multispace or multisystem is defined by

Definition 1.3([7],[33]) *Let* $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ *be* m *mathematical spaces or systems, different two by two, i.e., for any two spaces or systems* $(\Sigma_i;\mathcal{R}_i)$ *and* $(\Sigma_j;\mathcal{R}_j)$, $\Sigma_i \neq \Sigma_j$ *or* $\Sigma_i = \Sigma_j$ *but* $\mathcal{R}_i \neq \mathcal{R}_j$. *Then, a Smarandache multispace or Smarandache multisystem* $\widetilde{\Sigma}$ *is a union* $\bigcup\limits_{i=1}^{m} \Sigma_i$ *with rules* $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$ *on* $\widetilde{\Sigma}$, *i.e., the union of rules* $\mathcal{R}_i$ *on* Σ_i *for integers* $1 \leq i \leq m$, *denoted by* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$.

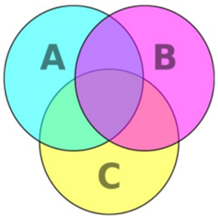

Figure 9

Such as the multisets that shown in Figure 9. In classical mathematics, algebraic rings, fields topological groups, continuous groups are all cases of Smarandache multispace with $m = 2$.

Now, *why Smarandache multispace or multisystem is important in recognition of human?* Because we are all blind in face of the unknown, as those of blind in fable of six blind men with an elephant. Generally, let the elements of thing T be $e_1, e_2, \cdots, e_m$ in reductionism and denote the mathematical reality of thing T by $T_{\mathcal{M}}$. So, one recognizes the mathematical reality of thing T by a union

$$T_{\mathcal{M}} = \bigcup_{i=1}^{m} \mathcal{R}(e_i)$$

of local recognitions of $e_1, e_2, \cdots, e_m$ by human, where $\mathcal{R}(e_i)$ is the recognition of e_i for integers $1 \leq i \leq m$. Then, *how to characterize* $\mathcal{R}(e_i)$ *by mathematics?* One applies mathematical system $(\Sigma_i, \mathcal{R}_i)$ for characterizing the evolving of element e_i. In this case, $T_{\mathcal{M}}$ is nothing else but a Smarandache multispace or multisystem.

Notice that any thing T inherits a topological structure of 1-dimension in its reductionism, which holds also for Smarandache multispace or multisystem, i.e., the intersection graph G.

Definition 1.4([7],[10]) *For an integer* $m \geq 1$, *let* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *be a Smarandache multispace or Smarandache systems consisting of* m *mathematical systems* $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$. *An inherited topological structure* $G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]$ *of* $\left(\widetilde{\Sigma};\widetilde{\mathcal{R}}\right)$ *is a labeled topological graph defined following:*

$$V\left(G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]\right) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\},$$

$$E\left(G^L\left[\widetilde{\Sigma};\widetilde{\mathcal{R}}\right]\right)=\{(\Sigma_i,\Sigma_j)\,|\Sigma_i\bigcap\Sigma_j\neq\emptyset,\ 1\leq i\neq j\leq m\}$$

with labeling

$$L:\ \Sigma_i\to L(\Sigma_i)=\Sigma_i,$$

$$L:\ (\Sigma_i,\Sigma_j)\to L(\Sigma_i,\Sigma_j)=\Sigma_i\bigcap\Sigma_j$$

for integers $1\leq i\neq j\leq m$.

For example, a K_4-e system is shown in Figure 10.

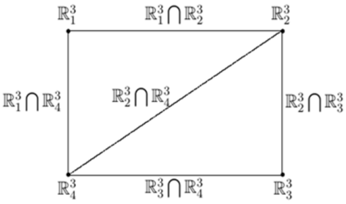

Figure 10

1.3. CC Conjecture

CC conjecture is a philosophical notion for extending mathematics over combinatorial structure G. We know the subdivision of compact surface in surface topology following.

Theorem 1.5(Radó,1925, [30]) *Any compact surface S admits a triangulation.*

Theorem 1.5 implies that a 1-dimensional skeleton in a 2-cell partition of closed surface corresponds to a topological graph G and a 2-cell embedded of graph G on surface S is nothing else but a 2-cell partition of S which enables one getting the classification theorem following.

Theorem 1.6([30]]) *Any connected compact surface S is either homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes, i.e., its surface presentation $\mathcal{S}$ is elementary equivalent to one of the standard surface presentations following.*

(1) *the sphere* $S^2=\left\langle a|aa^{-1}\right\rangle$;

(2) *the connected sum of p tori*

$$\underbrace{T^2\#T^2\#\cdots\#T^2}_{p}=\left\langle\ a_i,b_i,1\leq i\leq p\mid\prod_{i=1}^{p}a_ib_ia_i^{-1}b_i^{-1}\right\rangle;$$

(3) *the connected sum of q projective planes*

$$\underbrace{P^2\#P^2\cdots\#P^2}_{q}=\left\langle\ a_i,1\leq i\leq q\mid\prod_{i=1}^{q}a_i\right\rangle.$$

Theorem 1.7([30]) *Let S be a connected compact surface with a presentation $\mathcal{S}$. Then*

$$\chi(S)=\begin{cases}2, & \text{if } \mathcal{S}\sim_{El}S^2,\\ 2-2p, & \text{if } \mathcal{S}\sim_{El}\underbrace{T^2\#T^2\#\cdots\#T^2}_{p},\\ 2-q, & \text{if } \mathcal{S}\sim_{El}\underbrace{P^2\#P^2\#\cdots\#P^2}_{q}.\end{cases}$$

Notice that the classification theorem of compact surface is just in one vertex, one face case, i.e. surface over a bouquet B_n as shown in Figure 11.

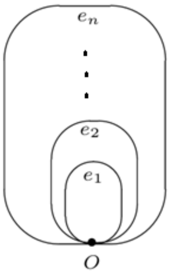

Figure 11

Generally, let G be a connected graph. Then, we have the results following.

Theorem 1.8 (Duke 1966, Stahl 1978, [4],[31]) *Let G be a connected graph and let $GR(G)$, $CR(G)$ be the respective genus range of G on orientable or non-orientable surfaces. Then, $GR(G)$ and $CR(G)$ both are unbroken interval of integers*

Theorem 1.8 has another form, namely any surface is 2-dimensional spaces $\mathbb{R}^2$ and generally, an n-dimensional manifold M is an n-dimensional space $\mathbb{R}^n$ over graph G. Generally, let $\overrightarrow{G}$ be a directed graph. It is this notion and Smarandache multispace inherited with a combinatorial structure that motivated me to make CC conjecture for extending mathematics widely in 2006.

Conjecture 1.9 (CC conjecture, Mao 2006, [6],[8]) *Any mathematical science can be reconstructed from or made by combinatorialization.*

Essentially, this CC conjecture is not a conjecture on mathematical problem but a philosophical notion for extending classic mathematics.

(1) It assumes that one can select finite combinatorial rulers with axioms to reconstruct or make a generalization for classical mathematics, and a mathematics is not complete if its combinatorialization be not completed.

(2) One can extend different mathematical systems over combinatorial structure $\overrightarrow{G}$ and find new mathematics corresponding to the theory of return to origin.reductionism, i.e.,

$$\textit{Mathematics over combinatorial structure } \overrightarrow{G} \ \Leftrightarrow \ \textit{Reductionism}$$

Then, *could we really establish such mathematics over combinatorial structure $\overrightarrow{G}^L$?* The answer is certainly Yes, which needs to view $\overrightarrow{G}^L$ as an element of mathematics entirely by labeled graph $\overrightarrow{G}^L$. For examples, [12] on classification of n-dimensional manifold by listing graph, [9] and [10] on combinatorial manifold with differential, [11] on combinatorial fields and generally, mathematics on non-mathematics by combinatorics, all of them are motivated by CC conjecture.

The main purpose of this paper is to show how to establish such mathematics over combinatorial structure $\overrightarrow{G}$, i.e., mathematical combinatorics following the CC conjecture, including the mathematical model of non-harmonious group or equation's combinatorics corresponding to reductionism and how to establish a mathematical system over $\overrightarrow{G}$ with Smarandache multispace, particularly the algebra and calculus over an invariant $\overrightarrow{G}$ for extending algebra and calculus largely and generally, the continuity flow theory on a closed family $\mathscr{G}$ of graph union with Banach space $\mathscr{B}$ and G-isomorphic operators, i.e., $\overrightarrow{G}$ or its supergraph $\widehat{G}$ invariant for generalizing a few interesting results in functional analysis such as the closed graph theorem, the Hahn-Banach theorem and also the Euler-Lagrange equation. All of these works constitute a mathematics to reductionism on combinatorial structure $\overrightarrow{G}$.

For terminologies and notations not mentioned here, we follow references [1] for algebra, [2] for complex network, [3] for functional analysis, [30] for algebraic topology, [4],[5] and [31] for topological graphs, [32] for elementary particles, [7] and [33]-[35] for Smarandache multisystems and multispaces.

§2. Non-Harmonious Groups

Let T be a thing in the universe. *How do we characterize the physical state* $\psi(t,\mathbf{x})$ *of thing* T *by the reductionism, particularly a self-organized system?* Usually, a physical state of thing T is characterized by differential equation

$$\mathscr{F}\left(t, x_1, x_2, x_3, \psi_t, \psi_{x_1}, \psi_{x_2}, \psi_{x_3}, \cdots\right) = 0 \tag{2.1}$$

in physics ([15]), but *is it a solvable equation for self-organized system in mathematics?* The answer is certainly Not by the Godel's incompleteness theorem, i.e., it maybe characterized by a solvable system or a non-solvable system of solvable equations, namely, the solution of equation should be generalized. Then, *what is a non-harmonious group?* Let $\{e_1, e_2, \cdots, e_m\}$ be all elements of thing T in reductionism. Then, a non-harmonious group is formally defined in the following.

Definition 2.1([25],[28]) *A non-harmonious group* $\mathcal{S}$ *is such a mathematical system* $\mathcal{S}$ *consisting of elements* $\mathbf{e}_i$, $1 \leq i \leq m, m \geq 2$ *with interactions constrained on a system of equations*

$$(ES_m)\left\{\begin{array}{ll} \mathscr{F}_{\mathbf{e}_1}(\mathbf{x},\mathbf{y}) = 0, & (\mathbf{e}_1) \\ \mathscr{F}_{\mathbf{e}_2}(\mathbf{x},\mathbf{y}) = 0, & (\mathbf{e}_2) \\ \cdots\cdots\cdots\cdots, & \cdots \\ \mathscr{F}_{\mathbf{e}_m}(\mathbf{x},\mathbf{y}) = 0, & (\mathbf{e}_m) \end{array}\right. \tag{2.2}$$

at time t*, where* $\mathscr{F}_{\mathbf{e}_i}(\mathbf{x}^0,\mathbf{y}^0) = 0$ *and* $\mathscr{F}_{\mathbf{e}_i}$ *satisfies the existence condition of implicit function theorem in a neighborhood* U *of point* $(\mathbf{x}^0,\mathbf{y}^0)$ *in Euclidean space* $\mathbb{R}^n$ *for integers* $1 \leq i \leq m$.

Now, let $S_{\mathscr{F}_{\mathbf{e}_i}} \subset \mathbb{R}^n$ with $\mathscr{F}_{\mathbf{e}_i} : S_{\mathscr{F}_{\mathbf{e}_i}} \to 0$ be a solution manifold of the ith equation in system (ES_m). Then, *what is the condition of non-solvable or solvable of system* (ES_m)*?* The answer is

$$\bigcap_{i=1}^{m} S_{\mathscr{F}_{\mathbf{e}_i}} = \emptyset \quad \text{or} \quad \bigcap_{i=1}^{m} S_{\mathscr{F}_{\mathbf{e}_i}} \neq \emptyset, \tag{2.3}$$

i.e., the condition of non-solvable or solvable in geometrical space ([19]). So, *what is meaning of system* (ES_m) *has or has no solution? Is it a meaningless of thing* T *in the non-solvable case?* The answer is certainly not because the non-solvable of (ES_m) indicates only that there is no overlap state in elements $e_1, e_2, \cdots, e_m$ at time t, not implies the state of existing or not and solvable if the state of all elements have overlap, it is a serious matter characterizing the group behavior of $\{e_1, e_2, \cdots, e_m\}$. Then, *how to characterize the group behavior of system* $\{e_1, e_2, \cdots, e_m\}$ *or thing* T*?* The answer is nothing else but the Smarandache multispace $\bigcup\limits_{i=1}^{m} S_{\mathscr{F}_{\mathbf{e}_i}}$, not the intersection or solution $\bigcap\limits_{i=1}^{m} S_{\mathscr{F}_{\mathbf{e}_i}}$ of system (ES_m).

Notice that the group behavior $\bigcup\limits_{i=1}^{m} S_{\mathscr{F}_{\mathbf{e}_i}}$ of system (ES_m) has combinatorial character, i.e., underlying a topological graph defined following.

Definition 2.2([9],[11]) *For any integer $m \geq 1$, the G-solution of system (ES_m) on non-harmonious group S is a labeled graph $G^L[ES_m]$ with vertex and edge sets defined by*

$$V\left(G^L[ES_m]\right) = \left\{S_{\mathscr{F}_{\mathbf{e}_i}},\ 1 \leq i \leq m\right\},$$

$$E\left(G^L[ES_m]\right) = \left\{\ (S_{\mathscr{F}_{\mathbf{e}_i}}, S_{\mathscr{F}_{\mathbf{e}_j}}) \mid if\ S_{\mathscr{F}_{\mathbf{e}_i}} \bigcap S_{\mathscr{F}_{\mathbf{e}_j}} \neq \emptyset\ for\ integers\ 1 \leq i, j \leq m\ \right\}$$

and labels on vertices and edges of G by

$$L:\ S_{\mathscr{F}_{\mathbf{e}_i}} \to S_{\mathscr{F}_{\mathbf{e}_i}}, \quad (S_{\mathscr{F}_{\mathbf{e}_i}}, S_{\mathscr{F}_{\mathbf{e}_j}}) \to S_{\mathscr{F}_{\mathbf{e}_i}} \bigcap S_{\mathscr{F}_{\mathbf{e}_j}}, \quad 1 \leq i \neq j \leq m.$$

Different from the usual, one can always conclude the existence of (ES_m) in this case.

Theorem 2.3([25],[28]) *For any integer $m \geq 1$, a G-solution $G^L[ES_m]$ of system (ES_m) on a non-harmonious group S is always existing.*

For example, let $(LDES_6^1)$ be a system of homogeneous differential equations $(\mathbf{e}_1) - (\mathbf{e}_6)$ as follows:

$$(LDES_6^1)\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (\mathbf{e}_1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (\mathbf{e}_2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (\mathbf{e}_3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (\mathbf{e}_4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (\mathbf{e}_5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (\mathbf{e}_6) \end{cases}$$

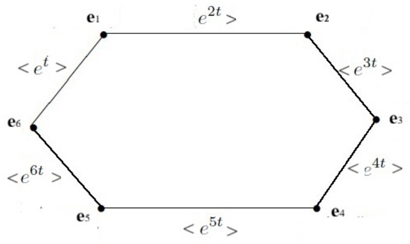

Figure 12

It is easily to know the solution space of (1) – (6) are respectively

$$\left\langle e^t, e^{2t}\right\rangle, \left\langle e^{2t}, e^{3t}\right\rangle, \left\langle e^{3t}, e^{4t}\right\rangle, \left\langle e^{4t}, e^{5t}\right\rangle, \left\langle e^{5t}, e^{6t}\right\rangle, \left\langle e^{6t}, e^t\right\rangle$$

and

$$\begin{aligned} \left\langle e^t, e^{2t}\right\rangle \bigcap \left\langle e^{2t}, e^{3t}\right\rangle &= \left\langle e^{2t}\right\rangle, & \left\langle e^{2t}, e^{3t}\right\rangle \bigcap \left\langle e^{3t}, e^{4t}\right\rangle &= \left\langle e^{3t}\right\rangle, \\ \left\langle e^{3t}, e^{4t}\right\rangle \bigcap \left\langle e^{4t}, e^{5t}\right\rangle &= \left\langle e^{4t}\right\rangle, & \left\langle e^{4t}, e^{5t}\right\rangle \bigcap \left\langle e^{5t}, e^{6t}\right\rangle &= \left\langle e^{5t}\right\rangle, \\ \left\langle e^{5t}, e^{6t}\right\rangle \bigcap \left\langle e^{6t}, e^t\right\rangle &= \left\langle e^{6t}\right\rangle, & \left\langle e^{6t}, e^t\right\rangle \bigcap \left\langle e^t, e^{2t}\right\rangle &= \left\langle e^t\right\rangle. \end{aligned}$$

Thus, if we remove the origin point O, the G-solution of $(LDES_6^1)$ can be shown in Figure 12.

So, *what is the role of the existence of G-solution on non-harmonious groups?* Its significance is to determine the global stability of non-harmonious group [13]. Let $G^L(t)$ be a G-solution of differential system (ES_m) with initial values $G^L(t_0)$ and let $\omega : V\left(G^L[ES_m]\right) \to \mathbb{R}$ be a functional. A system (ES_m) is said to be *ω-stable* if there exists a number $\delta(\varepsilon)$ for any number $\varepsilon > 0$ such that

$$\left\|\omega\left(G^{L_1(t)-L_2(t)}\right)\right\| < \varepsilon \tag{2.4}$$

or furthermore, *asymptotically ω-stable* if

$$\lim_{t\to\infty}\left\|\omega\left(G^{L_1(t)-L_2(t)}\right)\right\|=0 \tag{2.5}$$

if the initial values hold with $\|L_1(t_0)(v)-L_2(t_0)(v)\|<\delta(\varepsilon)$ for $\forall v\in V(\overrightarrow{G})$ ([18]). In this case, if there is a Liapunov ω-function $L(\omega(t)):\mathscr{O}\to\mathbb{R}, n\geq 1$ on $\overrightarrow{G}$ with open $\mathscr{O}\subset\mathbb{R}^n$ such that $L(\omega(t))\geq 0$ with equality hold only if $(x_1,x_2,\cdots,x_n)=(0,0,\cdots,0)$ and $\dot{L}(\omega(t))<0$ if $t\geq t_0$.

Denoted by $\mathbf{O}$ the zero G-solution of system (ES_m), i.e., all vertices and edges on $G^L[ES_m]$ are labeled by $\mathbf{0}$, we get a result on ω-stability of (ES_m) following.

Theorem 2.4([18]) *If there is a Liapunov ω-function $L(\omega(t)):\mathscr{O}\to\mathbf{R}$ on $G^L[ES_m]$ of system (ES_m), then it is ω-stable, and furthermore, if $\dot{L}(\omega(t))<0$ for $G^L[ES_m]\neq\mathbf{O}$, then it is asymptotically ω-stable.*

Particularly, let

$$\omega\ =\sum_{v\in V(G)}X_v(t)\quad\text{or}\quad\prod_{v\in V(G)}X_v(t) \tag{2.6}$$

on a linear system of differential equations (ES_m) [13]. Then, we have the sum-stability and prod-stability of linear system of differential equations following.

Definition 2.5([13]) *Let the inherited graph $G^L[ES_m]$ of linear systems (ES_m) of differential equation with be G initial value $X_v(0), v\in V(G)$. Then, G-solution $G^L[ES_m]$ is called sum-stable or prod-stable if for all solutions $Y_v(t)$, $v\in V(G)$ of the linear differential equations of (ES_m) with $|Y_v(0)-X_v(0)|<\delta_v$ exists for all $t\geq 0$,*

$$\left|\sum_{v\in V(G)}Y_v(t)-\sum_{v\in V(G)}X_v(t)\right|<\varepsilon\quad\text{or}\quad\left|\prod_{v\in V(G)}Y_v(t)-\prod_{v\in V(G)}X_v(t)\right|<\varepsilon \tag{2.7}$$

or furthermore, it is called asymptotically sum-stable or asymptotically prod-stable if

$$\lim_{t\to 0}\left|\sum_{v\in V(G)}Y_v(t)-\sum_{v\in V(G)}X_v(t)\right|=0\quad\text{or}\quad\lim_{t\to 0}\left|\prod_{v\in V(G)}Y_v(t)-\prod_{v\in V(G)}X_v(t)\right|=0. \tag{2.8}$$

We know the result on sum-stability and prod-stability of linear system of differential equations following.

Theorem 2.6([13]) *A zero $\mathbf{O}$-solution of system (ES_m) of linear homogenous differential equation is asymptotically sum-stable if and only if* $\mathrm{Re}\alpha_v<0$ *for each $\overline{\beta}_v(t)e^{\alpha_v t}\in\mathscr{B}_v$ with vertex $v\in G^L$ and a zero $\mathbf{O}$-solution of systems (ES_m) of linear homogenous differential equation is asymptotically prod-stable if and only if*

$$\sum_{v\in V(G)}\mathrm{Re}\alpha_v\ <\ 0 \tag{2.9}$$

for each $\overline{\beta}_v(t)e^{\alpha_v t} \in \mathscr{B}_v$ *with vertex* $v \in G^L[ES_m]$.

Certainly, different functionals ω on different combinatorial structure G will lead to different ω-stabilities of (ES_m) and there are open problems following.

Problem 2.7 *Construct different functional for characterizing* ω*-stability of non-harmonious groups* (ES_m).

Problem 2.8 *For a specific non-harmonious groups* (ES_m)*, construct functional* ω *on typical subgraphs of* $G^L[ES_m]$ *such as* P_2, P_3, C_3, K_4 *of small order.*

Meanwhile, characterizing the non-solvable system of equations of non-harmonious groups (ES_m) by combinatorics is a meaningful thing for holding on the reality of things T. For example, the non-solvable system of ordinary differential equations and the first order partial differential equations are respectively characterized in [13] and [18].

§3. Mathematics over an Invariant Graph

Let $\{e_1, e_2, \cdots, e_m\}$ be elements of thing T in reductionism and the evolving of elements $e_i, 1 \leq i \leq m$ are respectively characterized by mathematical systems $(\Sigma_1, \mathcal{R}_1), (\Sigma_2, \mathcal{R}_2), \cdots, (\Sigma_m, \mathcal{R}_m)$, namely, the pair $(\bigcup\limits_{i=1}^{m} \Sigma_i; \bigcup\limits_{i=1}^{m} \mathcal{R}_i)$ is a Smarandache multispace inherited with a topological structure G^L by Definition 1.4. Generally, $(\bigcup\limits_{i=1}^{m} \Sigma_i; \bigcup\limits_{i=1}^{m} \mathcal{R}_i)$ is not a mathematical system unless a union of mathematical systems, characterized by algebraic operations in $\bigcup\limits_{i=1}^{m} \mathcal{R}_i$ with operation diagram of elements [7], [36]-[37].

Now, *could we establish new mathematics over topological graph* G^L*?* The answer is certainly Yes by combinatorics! Generally, let $V(G^L) = \{v_1, v_2, \cdots, v_m\}$. Define labeled graphs G^{L*}, $\mathbf{o}_G$ respectively by elements in $\bigcup\limits_{i=1}^{m} \Sigma_i$ and operations in $\bigcup\limits_{i=1}^{m} \mathcal{R}_i$ on G^L with

$$L^* : v_i \to \alpha_i \in \Sigma_i, \quad \mathbf{o}_G : v_i \to \circ_i \in \mathcal{R}_i, \quad 1 \leq i \leq m \tag{3.1}$$

and

$$G_1^{L^*} \mathbf{o}_G G_2^{L^*} \ = \ G^{L_1^* \mathbf{o}_G L_2^*}. \tag{3.2}$$

Particularly, if $\mathcal{R}_i = \mathcal{R}$ for integers $1 \leq i \leq m$, i.e., each mathematical system $(\Sigma_i, \mathcal{R}_i)$ is with the same operation set $\mathcal{R}$ then the eq.(3.2) is simplified to

$$G_1^{L^*} \mathbf{o}_G G_2^{L^*} \ = \ G^{L_1^* \circ L_2^*}, \quad \circ \in \mathcal{R}. \tag{3.3}$$

Denoted by $\mathscr{G}^L$, $\mathbf{o}_{\mathscr{G}}$ all labeled graphs G^{L*} and all operations $\mathbf{o}_G$, respectively. Then, a Smarandache multispace $(\bigcup\limits_{i=1}^{m} \Sigma_i; \bigcup\limits_{i=1}^{m} \mathcal{R}_i)$ is convert to a system $(\mathscr{G}^L, \mathbf{o}_{\mathscr{G}})$ over combinatorial structure G^L. In this case, if we view G^{L^*} as a mathematical element in whole, we can establish new mathematics, particularly the algebra and calculus over G^L.

3.1. Algebra over an Invariant G

Let $(G_1;\circ_1),(G_2;\circ_2),\cdots,(G_m;\circ_m)$ be respectively groups with identities $1_{\circ_1},1_{\circ_2},\cdots,1_{\circ_m}$ ([1]). Then, a Smarandache multigroup is defined by

$$\left(\widetilde{G};\mathcal{O}\right)=\left(\bigcup_{i=1}^{m}G_i;\mathcal{O}\right) \tag{3.4}$$

with a combinatorial structure G^L, where $\mathcal{O}=\{\circ_i,1\le i\le m\}$ and two multigroups $(\widetilde{G};\mathcal{O})$, $(\widetilde{G}';\mathcal{O}')$ are isomorphic if there is a $1-1$ isomorphism $\phi:\widetilde{G}\to\widetilde{G}'$ such that $\phi(G_i)=G_i'$ and $\phi(G_i\bigcap G_j)=\phi(G_i)\bigcap\phi(G_j)$, $1\le i\neq j\le m$.

Generally, a multigroup $(\widetilde{G};\mathcal{O})$ is not a group, even an algebraic system if it is not closed, i.e., $a\circ_i b\notin\widetilde{G}$ for some $\forall a,b\in\widetilde{G}$ and an integer $1\le i\le m$ in general. But it is a multigroup over combinatorial structure G^L. *Could we characterizes the multigroup by groups over combinatorial structure?* The answer is certainly Yes by group over combinatorial structure G^L.

Now, let $V(G^L)=\{v_1,v_2,\cdots,v_m\}$ be the vertex set of G^L. In this case, the labelling $L_1^*\mathbf{o}_G L_2^*$ in eq.(3.2)

$$G^{L_1^*}\mathbf{o}_G G^{L_2^*}=G^{L_1^*\mathbf{o}_G L_2^*}$$

is $L_1^*\mathbf{o}_G L_2^*:v_i\to L_1^*(v_i)\circ_i L_2^*(v_i)$ for integers $1\le i\le m$ is shown in Figure 13.

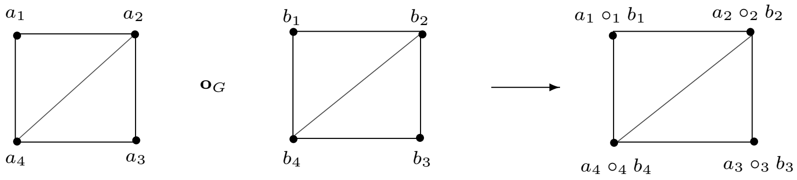

Figure 13

Clearly, the element $G^{L_I^*}$ with $L_I^*:v_i\to 1_{\circ_i}$ is the identity of $(G^L;\mathbf{o}_G)$ and for an element $G^{L^*}\in G^L$, $G^{L^{-*}}$ is its inverse, where $L^{-*}:L^*(v_i)\to L^{-*}(v_i)$ for integers $1\le i\le m$, and the associative

$$\left(G^{L_1^*}\mathbf{o}_G G^{L_2^*}\right)\mathbf{o}_G G^{L_3^*}=G^{L_1^*}\mathbf{o}_G\left(G^{L_2^*}\mathbf{o}_G G^{L_3^*}\right). \tag{3.5}$$

is implied by the associative law of group. Thus, $(G^L;\mathbf{o}_G)$ is a group by definition.

Theorem 3.1 *$(G^L;\mathbf{o}_G)$ is a group, i.e., a multigroup $(\widetilde{G};\mathcal{O})$ is a group over a combinatorial structure G^L of groups.*

For two isomorphic multigroups G^{L_1},G^{L_2}, if G^{L_1},G^{L_2} are isomorphic then there is a $1-1$ isomorphism $\phi:\widetilde{G}\to\widetilde{G}'$ such that $\phi(G_i)=G_i'$ and $\phi(G_i\bigcap G_j)=\phi(G_i)\bigcap\phi G_j$, $1\le i\neq j\le m$ by definition, namely the labeled graphs G^{L_1} and G^{L_2} are isomorphic with an isomorphism $\phi|_{v\in V(G)}$ of group. Thus, we have the following conclusion.

Theorem 3.2 *Two multigroups G^{L_1},G^{L_2} are isomorphic if and only if there is an isomorphism*

$\phi : G^{L_1} \to G^{L_2}$ *of graph such that*

$$\phi(L_1)(v) = L_2(\phi(v)), \quad v \in V(G) \tag{3.6}$$

and $\phi|_{v\in V(G)}$ *is an isomorphism of group.*

Problem 3.3 *Establish a multigroup* $(\widetilde{G}; \mathcal{O})$ *theory of commutative or non-commutative groups over a topological graph* G^L *such as* $G = K_m, P_m, C_m, \cdots$ *with small* $m = 2, 3, 4, \cdots$.

Particularly, if $\circ_1 = \circ_2 = \cdots = \circ_m = \circ$, namely all groups $(G_1; \circ_1), (G_2; \circ_2), \cdots, (G_m; \circ_m)$ are all subgroups of a group, i.e., the group is shown as in Figure 14, we get a corollary by Theorem 3.1 following.

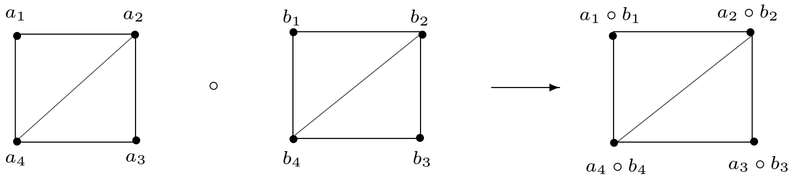

Figure 14

Corollary 3.4 *If* $\circ_1 = \circ_2 = \cdots = \circ_m = \circ$, *a multigroup* $(\widetilde{G}; \circ)$ *is a group, i.e., a group over a combinatorial structure* G *of subgroups.*

Problem 3.5 *Establish a multigroup* $(\widetilde{G}; \circ)$ *theory, determine all multigroups of a commutative or non-commutative group over a topological graph* G^L *such as* $G = K_m, P_m, C_m$ *with small* $m = 2, 3, 4, \cdots$.

Similarly, we can also establish multialgebra by rings, skew field and field. Let $(R_1; \{+_1, \cdot_1\})$, $(R_2; \{+_2, \cdot_m\}), \cdots, (R_m; \{+_2, \cdot_m\}))$ be respectively rings, skew fields or field with "+" identities $0_{+_1}, 0_{+_2}, \cdots, 0_{+_m}$ and "$\cdot$" identities $1_{\cdot_1}, 1_{\cdot_2}, \cdots, 1_{\cdot_m}$. Then, a Smarandache multiring, skew multifield or a multifeld is defined by

$$\left(\widetilde{R}; \mathcal{O}^2\right) = \left(\bigcup_{i=1}^{m} R_i; \mathcal{O}^2\right)$$

with a combinatorial structure G^L by Definition 1.4, where $\mathcal{O}^2 = \{+_G, \cdot_G\}$ and

$$+_G = \bigcup_{i=1}^{m}\{+_i\}, \ \cdot_G = \bigcup_{i=1}^{m}\{\cdot_i\}.$$

Two multirings, skew multifields or multifields $(\widetilde{R}; \mathcal{O}^2), (\widetilde{R}'; \mathcal{O}'^2)$ are isomorphic if there is a $1-1$ isomorphism $\phi : \widetilde{R} \to \widetilde{R}'^2$ such that $\phi(R_i) = R_i'$ and $\phi(R_i \bigcap R_j) = \phi(R_i) \bigcap \phi R_j$, $1 \le i \ne j \le m$. In this case, the operation (3.2) on G^{L^*} is

$$G^{L_1^*} \mathbf{o}_G G^{L_2^*} = G^{L_1^* \mathbf{o}_G^2 L_2^*}, \tag{3.7}$$

where the labelling $L_1^*\mathbf{o}_G^2L_2^* : v_i \to \{L_1^*(v_i) +_i L_2^*(v_i), L_1^*(v_i) \cdot_i L_2^*(v_i)\}$ for integers $1 \le i \le m$ shown in Figure 15.

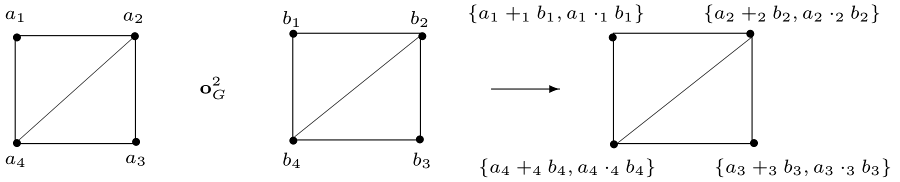

Figure 15

Clearly, the elements $G^{L_I^+}$ and $G^{L_I^{\cdot}}$ with $L_I^+ : v_i \to 1_{+_i}$, $L_I^{\cdot} : v_i \to 1_{\cdot_i}$ are respectively the identity of $(G^L; \mathbf{o}_G^2)$ of operations $+$ and $\cdot$, and for an element $G^{L^*} \in G^L$, G^{-L^*} and $G^{L^{-\cdot}}$ are respectively the inverses of G^{L^*} in $(R_i; +_i)$, $(R; \cdot_i)$. Furthermore, the associative

$$\left(G^{L_1^*}\mathbf{o} * 2_G G^{L_2^*}\right)\mathbf{o}_G^2 G^{L_3^*} \ = \ G^{L_1^*}\mathbf{o}_G^2\left(G^{L_2^*}\mathbf{o}_G^2 G^{L_3^*}\right), \tag{3.8}$$

and distributive

$$\begin{aligned} G^{L_1^*} \cdot_G (G^{L_2^*} +_G G^{L_3^*}) \quad &= \quad G^{L_1^*} \cdot_G G^{L_2^*} + G^{L_1^*} \cdot_G G^{L_3^*}, \\ (G^{L_2^*} +_G G^{L_3^*}) \cdot_G \ G^{L_1^*} \quad &= \quad G^{L_2^*} \cdot_G G^{L_1^*} +_G G^{L_3^*} \cdot_G G^{L_1^*} \end{aligned} \tag{3.9}$$

are respectively implied in the associative and distributive laws of ring, skew field or field by definition.

Theorem 3.6 *$(G^L; \mathbf{o}_G^2)$ is a ring, skew field or field, i.e., a multiring, skew multifield or multifield $(\widetilde{R}; \mathcal{O}^2)$ is a ring, skew field or field over a combinatorial structure G^L of rings, skew fields or fields.*

Similar to Theorem 3.2, we have the criterion for isomorphic multirings, skew multifields or multifields following.

Theorem 3.7 *Two multirings, skew multifields or multifields G^{L_1}, G^{L_2} are isomorphic if and only if there is an isomorphism $\phi : R^{L_1} \to R^{L_2}$ of graph such that*

$$\phi(L_1)(v) = L_2(\phi(v)), \quad v \in V(G) \tag{3.10}$$

and $\phi|_{v \in V(G)}$ is an isomorphism of ring, skew field or field.

Particularly, if $+_1 = +_2 = \cdots = +_m = +$ and $\cdot_1 = \cdot_2 = \cdots = \cdot_m = \cdot$, namely all rings, skew fields or fields $(R_1; \{+1, \cdot_1\}), (R_2; \{+2, \cdot_2\}), \cdots, (R_m; \{+m, \cdot_m\})$ are subrings, skew subfields or subfield of a ring, a skew field or a field, respectively.

Corollary 3.8 *If $+_1 = +_2 = \cdots = +_m = +$ and $\cdot_1 = \cdot_2 = \cdots = \cdot_m = \cdot$, a multiring, skew multifield or multifield $(\widetilde{G}; \circ)$ is a ring, skew field or field, i.e., a multiring, skew multifield or multifield $(\widetilde{R}; \mathcal{O}^2)$ is a ring, skew field or field over a combinatorial structure G of subrings,* *skew subfields or subfields.*

Problem 3.9 *Establish a multiring, skew multifield and multifield* $(\widetilde{R};\mathbf{o}_G^2)$ *theory over a topological graph* G^L *such as* $G=K_m,P_m,C_m,\cdots$ *with small* $m=2,3,4,\cdots$.

3.2. Calculus over an invariant G

Certainly, all elements of thing T in reductionism are not separated but interactive, i.e., one should characterize simultaneously the elements with interaction for holding on the evolving of T but the algebra over an invariant G^L can not take on this responsibility because it fails to characterize the interaction between elements. For this objective, a useful way is by G-flows on edges of G^L holding with conservative laws at each vertex of G^L, i.e., replacing the interaction by substance flow in microscopic. For example, the generating or overcoming interaction with wood inadequacy or excessiveness of Chinese 5 elements are shown in Figure 16.

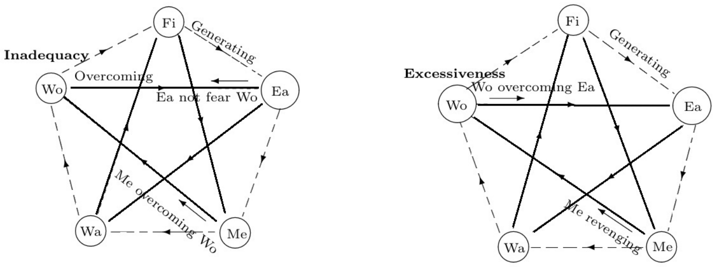

Figure 16

where Wo = Wood, Fi = Fire, Ea = Earth, Me = Metal and Wa = Water.

So, *what is a G-flow?* Generally, let $\mathscr{V}$ be a vector space over field $\mathscr{F}$. A *G-flow* is a labeled graph $\overrightarrow{G}^L$ ([16]) with a labeling mapping

$$L: v\to L(v)\in\mathscr{V},\ \forall v\in V(G)\ \text{ and }\ (v,u)\to L(v,u)\in\mathscr{V},\ \forall(v,u)\in E(G),$$

holding with $L(v,u)=-L(u,v)$, called the *vertex flow* and *edge flow* such as those shown in Figure 17,

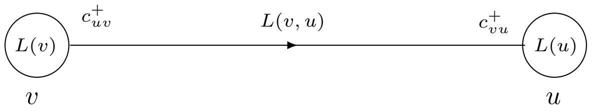

Figure 17

where the action of end-scalars $c_{vu}^+, c_{uv}^+\in\mathscr{F}$ on flow $L(v,u)$ is

$$c_{vu}^+: L(v,u)\to c_{vu}^+L(v,u),\quad c_{uv}^+: L(u,v)\to c_{uv}^+L^{A_{uv}^+}(u,v)$$

and meanwhile, for any vertex $v\in V(G^L)$ holding with conservative law

$$\sum_{u\in N_G^-(v)} c_{uv}^+L\left(u,v\right)-\sum_{u\in N_G^+(v)} c_{vu}^+L\left(v,u\right)\ =\ \mathbf{c}(v), \tag{3.11}$$

where $N_G^-(v)$ and $N_G^+(v)$ are respectively the in-neighborhood and out-neighborhood of vertex $v \in V(G)$, i.e., adjacent vertices $N_G^-(v) \subset N_G(v)$ flow into v and adjacent vertices $N_G^+(v) \subset N_G(v)$ flow out v with $N_G^-(v) \cup N_G^+(v) = N_G(v)$, and $\mathbf{c}(v)$ is the conservative vector at vertex v.

So, *are there really G-flows in the universe?* The answer is certainly Yes! For example, the 12 meridians with 671 acupoints on human body in Chinese medicine shown in Figure 18, which is nothing else but a G-flow of 671 vertices ([29]). Generally, we establish calculus on G-flows for vector space $\mathscr{V}$ over a field $\mathscr{F}$ to measure the microscopic changing of G-flow ([25],[28]).

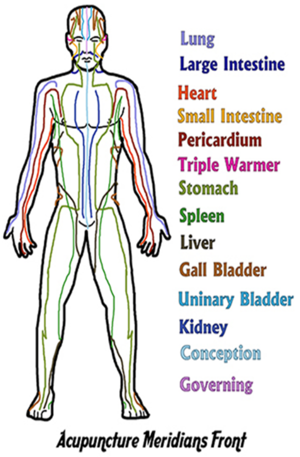

Figure 18

(1) Operations

Define the addition "+" and multiplication "·" by

$$\overrightarrow{G}^L + \overrightarrow{G}^{L'} = \overrightarrow{G}^{L+L'}, \qquad \overrightarrow{G}^L \cdot \overrightarrow{G}^{L'} = \overrightarrow{G}^{L\cdot L'}, \tag{3.12}$$

where the mappings $L + L'$ and $L \cdot L'$ are respectively defined by

$$L + L'(v,u) = L(v,u) + L'(v,u), \qquad L' \cdot L(v,u) = L'(v,u)L(v,u) \tag{3.13}$$

for any edge $(v,u) \in E(G)$.

(2) Distance

$$\rho\left(\overrightarrow{G}^L, \overrightarrow{G}^{L'}\right) = \sqrt{\sum_{(v,u)\in E(G)} \left(L(v,u) - L'(v,u)\right)^2} = \rho\left(\mathbf{O}, \overrightarrow{G}^{L-L'}\right), \tag{3.14}$$

and particularly, the *norm* $\left|\overrightarrow{G}^L\right|$ of G-flow $\overrightarrow{G}^L$ in $\overrightarrow{G}^{L'} = \mathbf{O}$, i.e.,

$$\left|\overrightarrow{G}^L\right| = \rho\left(\mathbf{O}, \overrightarrow{G}^L\right) = \sqrt{\sum_{(v,u)\in E(G)} L^2(v,u)}. \tag{3.15}$$

(3) Continuous Mapping

A continuous mapping on G-flow is designed to characterize the changing caused by a slight changing of the G-flow. For two continuous evolving $\overrightarrow{G}^L[\mathbf{x}], \overrightarrow{G}^{L'}[\mathbf{x}]$ of G-flow, let f be a mapping of $\overrightarrow{G}^L[\mathbf{x}]$. If for any positive number ε there is a real number δ such that

$$\rho\left(\overrightarrow{G}^L[\mathbf{x}], \overrightarrow{G}^{L'}[\mathbf{x}_0]\right) < \delta \quad \Rightarrow \quad \rho\left(f\left(\overrightarrow{G}^L[\mathbf{x}]\right), \overrightarrow{G}^{L'}[\mathbf{x}_0]\right) < \varepsilon, \tag{3.16}$$

then f is said to be *continuous* at $\overrightarrow{G}^{L'}[\mathbf{x}_0]$, denoted by

$$\lim_{\overrightarrow{G}^L \to \overrightarrow{G}^{L'}} f\left(\overrightarrow{G}^L[\mathbf{x}]\right) = \overrightarrow{G}^{L'}[\mathbf{x}_0]. \tag{3.17}$$

(4) Differential

Let $\overrightarrow{G}^L[t]$ be a continuous evolving of G-flow on t with a continuous mapping f and $f(\overrightarrow{G}^L[t+\Delta t]) \to f\left(\overrightarrow{G}^L[t]\right)$ if $\Delta t \to 0$. In this case, if the limitation

$$\lim_{\Delta t\to 0} \frac{f\left(\overrightarrow{G}^L[t+\Delta t]\right) - f\left(G^L[t]\right)}{\overrightarrow{G}^{\Delta t}} = \overrightarrow{G}^{\lim\limits_{\Delta t\to 0} \frac{f(L)[t+\Delta t]-f(L)[t]}{\Delta t}}$$

exists, it is said that f is *differential* at G-flow $\overrightarrow{G}^L[t]$, denoted by

$$\frac{df}{dt} = \lim_{\Delta t\to 0} \frac{f\left(\overrightarrow{G}^L[t+\Delta t]\right) - f\left(\overrightarrow{G}^L[t]\right)}{\overrightarrow{G}^{\Delta t}}, \tag{3.18}$$

where $\Delta : (v,u) \in E(G) \to \Delta t$ in $\overrightarrow{G}^{\Delta t}$. Now, if the mapping $C : (v,u) \to c_{vu}$ is a constant for any edge $(v,u) \in E(G)$, $\overrightarrow{G}^C$ is called a *definitive constant flow* and abbreviated $\overrightarrow{G}^C$ to C.

(5) Integral

By definition, we know that

$$\frac{d}{dt} f\left(\overrightarrow{G}^L\right)[t] = F\left(\overrightarrow{G}^L[t]\right) \quad \Rightarrow \quad \frac{d}{dt} f\left(\overrightarrow{G}^L[t] + C\right) = F\left(\overrightarrow{G}^L[t]\right).$$

Thus, we generally define the *indefinite integral* of f on G-flow $\overrightarrow{G}^L[t]$ to be

$$\int F\left(\overrightarrow{G}^L[t]\right) dt = f\left(\overrightarrow{G}^L[t]\right) + C = \overrightarrow{G}^{f(L)+C}. \tag{3.19}$$

So, there is a relation of differential with the integral as follows:

$$\begin{aligned} \int \left(\frac{df}{dt}\left(\overrightarrow{G}^L[t]\right)\right) dt &= f\left(\overrightarrow{G}^L[t]\right) + C, \\ \frac{df}{dt}\left(\int \left(f\left(\overrightarrow{G}^L[t]\right)\right) dt\right) &= f\left(\overrightarrow{G}^L[t]\right). \end{aligned} \tag{3.20}$$

Define the definite integral of f by

$$\int\limits_a^b f\left(\overrightarrow{G}^L[t]\right) dt = \lim_{\mu\to 0} \sum_{k=1}^n f\left(\overrightarrow{G}^L[\xi_i]\right) \cdot G^{\Delta t_k} \tag{3.21}$$

with

$$\int\limits_a^b f\left(\overrightarrow{G}^L[t]\right) dt = F\left(\overrightarrow{G}^L[t]\right)\Bigg|_{t=b} - F\left(\overrightarrow{G}^L[t]\right)\Bigg|_{t=a}, \tag{3.22}$$

i.e., the fundamental theorem of calculus if

$$\frac{d}{dt}F\left(\overrightarrow{G}^L[t]\right)=f\left(\overrightarrow{G}^L[t]\right).$$

Notice that the evolving of G-flow is always keep G invariant, namely it is evolving with a formula

$$f\left(\overrightarrow{G}^L[\mathbf{x}]\right) = \overrightarrow{G}^{f(L)[\mathbf{x}]} \tag{3.23}$$

and get the differential and integral, i.e.,

$$\frac{df}{dt} = \lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G}^L[t+\Delta t]\right)-f\left(\overrightarrow{G}^L[t]\right)}{\overrightarrow{G}^{\Delta t}} = \overrightarrow{G}^{\lim\limits_{\Delta t\to 0}\frac{f(L)[t+\Delta t]-f(L)[t]}{\Delta t}} = \overrightarrow{G}^{\frac{df(L)}{dt}}$$

and

$$\overrightarrow{G}^{\frac{df(L)}{dL}} = \overrightarrow{G}^{\lim\limits_{\Delta t\to 0}\frac{f(L)[t+\Delta t]-f(L)[t]}{\Delta t}} = \lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G}^L[t+\Delta t]\right)-f\left(\overrightarrow{G}^L[t]\right)}{\overrightarrow{G}^{\Delta t}} = \frac{df}{dt}.$$

with more interesting formulas. For example, the exponential identity

$$e^{\overrightarrow{G}^L[\mathbf{x}]} = \mathbf{I}+\frac{\overrightarrow{G}^L[\mathbf{x}]}{1!}+\frac{\overrightarrow{G}^{2L}[\mathbf{x}]}{2!}+\cdots+\frac{\overrightarrow{G}^{nL}[\mathbf{x}]}{n!}+\cdots \tag{3.24}$$

on G-flow, which is a further generalization of identities

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots \tag{3.25}$$

and

$$e^{tA} = I_n+A+\frac{A^2}{2!}+\frac{A^3}{3!}+\cdots+\frac{A^n}{n!}+\cdots. \tag{3.26}$$

on a matrix A.

§4. Banach and Hilbert Space over a Graph Family

Notice that elements in the evolving of thing T are not always invariant but variant in a closed space. We should generalize G-flows to a more generalization, called *continuity flow* G^L with labeling in a Banach space, also with linear operators in case, called G-isomorphic operators, which have both the characters of graphs and vectors with metric, and can be applied to model the behavior of things T in the macroscopic and the microscopic, see [18]-[28], particularly [28] for details.

Definition 4.1([20],[28]) *A continuity flow* $\left(\overrightarrow{G};L,\mathscr{A}\right)$ *is an oriented topological graph* $\overrightarrow{G}^L$ *in space* $\mathscr{S}$ *associated with a mapping* $L: v\to L(v),\ (v,u)\to L(v,u)$, *2 end-operators* $A^+_{vu}\in\mathscr{A}: L(v,u)\to L^{A^+_{vu}}(v,u)$ *and* $A^+_{uv}\in\mathscr{A}: L(u,v)\to L^{A^+_{uv}}(u,v)$ *on a Banach space* $\mathscr{B}$ *over a field*

$\mathscr{F}$ such as those shown in Figure 19 with $L(v,u) = -L(u,v)$, $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ for $\forall (v,u) \in E(\overrightarrow{G}^L)$.

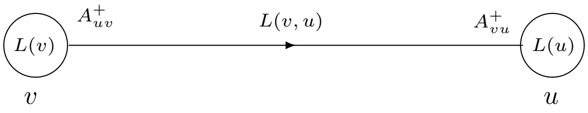

Figure 19

And meanwhile, holding with the continuity equation

$$\sum_{u \in N^-_G(v)} L^{A^+_{uv}}(u,v) - \sum_{u \in N^+_G(v)} L^{A^+_{uv}}(u,v) \ = \ L(v) \tag{4.1}$$

at any vertex $v \in V(\overrightarrow{G}^L)$ of topological graph $\overrightarrow{G}^L$, where $N^-_G(v), N^+_G(v)$ are respectively the in-neighborhood and out-neighborhood of vertex $v \in V(\overrightarrow{G}^L)$, namely all vertices in $N^-_G(v) \subset N_G(v)$ or $N^+_G(v) \subset N_G(v)$ flow into or out of the vertex v and $N^-_G(v) \cup N^+_G(v) = N_G(v)$.

Different from G-flow with an invariant G, by applying the operation of graph we consider $\overrightarrow{G}^L$ as a family of vectors underlying a topological graph $\overrightarrow{G}$ with addition, multiplication and scalar multiplication defined by

$$G^L + G'^{L'} \ = \ (G \setminus G')^L \bigcup \left(G \bigcap G'\right)^{L+L'} \bigcup (G' \setminus G)^{L'}, \tag{4.2}$$

$$G^L \cdot G'^{L'} \ = \ (G \setminus G')^L \bigcup \left(G \bigcap G'\right)^{L \cdot L'} \bigcup (G' \setminus G)^{L'}, \tag{4.3}$$

$$\lambda \cdot G^L \ = \ G^{\lambda \cdot L}. \tag{4.4}$$

where, for any vertex $v \in V(G)$ and edge $(v,u) \in E(G)$, $L(v), L'(v), L(v,u), L'(v,u) \in \mathscr{B} L+L' :$ $v \to L(v) + L'(v)$, $(v,u) \to L(v,u) + L'(v,u)$, $L \cdot L' : v \to L(v) \cdot L'(v)$, $(v,u) \to L(v,u) \cdot L'(v,u)$ $\lambda \cdot L : v \to \lambda \cdot L(v)$, $(v,u) \to \lambda \cdot L(v,u)$, $L(v) \cdot L'(v)$ and $L(v,u) \cdot L'(v,u)$ denotes the *Hadamard product* of vectors in Banach space $\mathscr{B}$, namely

$$(x_1, x_2, \cdots, x_n) \cdot (y_1, y_2, \cdots, y_n) \ = \ (x_1 y_1, x_2 y_2, \cdots, x_n y_n). \tag{4.5}$$

Let $\mathscr{G}$ be a graph family closed under the union operation of graph, $\mathscr{B}$ be a Banach space over field $\mathscr{F}$ and denoted by $\mathscr{G}_{\mathscr{B}}$ all continuity flows $\overrightarrow{G}^L$ with $\overrightarrow{G} \in \mathscr{G}$, $L : V(\overrightarrow{G}) \bigcup E(\overrightarrow{G}) \to \mathscr{B}$. Then, we have the result following.

Theorem 4.1([24],[25]) *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a linear space $(\mathscr{B}; +, \cdot)$, then, all continuity flows $(\mathscr{G}_{\mathscr{B}}; +, \cdot)$ is a linear space, and furthermore, a commutative ring if $\mathscr{B}$ is a commutative ring $(\mathscr{B}; +, \cdot)$ over a field $\mathscr{F}$.*

Now, assume all end-operators are continuous linear operators in $\mathscr{A}$ and define the norm of a continuity flow $\overrightarrow{G}^L$ by

$$\left\|\overrightarrow{G}^L\right\| = \sum_{(v,u)\in E\left(\overrightarrow{G}\right)} \left\|L^{A^+_{vu}}(v,u)\right\|, \tag{4.6}$$

where $\|\cdot\|$ is the norm on Banach space $\mathscr{B}$. Then, we can verify the non-negative, homogeneity and the triangle inequality hold with $\mathscr{G}_{\mathscr{B}}$ and the non-negative, conjugacy and the linearity if $\mathscr{B}$ is further a Hilbert space, i.e.,

Theorem 4.2([24],[25]) *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a Banach space $(\mathscr{B};+,\cdot)$, then, $\mathscr{G}_{\mathscr{B}}$ with linear operators A^+_{vu}, A^+_{uv} for $\forall(v,u)\in E\left(\bigcup\limits_{G\in\mathscr{G}}\overrightarrow{G}\right)$ is a Banach space, and furthermore, $\mathscr{G}_{\mathscr{B}}$ is a Hilbert space if $\mathscr{B}$ is a Hilbert space.*

Notice that the G-invariant property is the principle of mathematics over combinatorial structure (3.19), i.e.,

$$f\left(\overrightarrow{G}^L[\mathbf{x}]\right) = \overrightarrow{G}^{f(L[\mathbf{x}])},$$

But *how it performs on the graph family $\mathscr{G}$?* Answering this question need a convention on equivalent continuity flows following.

Convention 4.3([25]) *If $L(v,u)=\mathbf{0}$ for an edge $(v,u)\in E\left(\overrightarrow{G}^L\right)$, we always identify $\overrightarrow{G}^L$ with $\left(\overrightarrow{G}\setminus(v,u)\right)^L$, i.e., $\overrightarrow{G}^L=\left(\overrightarrow{G}\setminus(v,u)\right)^L$.*

By Convention 4.3, we can always choose the supgraph $\widehat{G}=\bigcup\limits_{G\in\mathscr{G}} G$ and the G-invariant property

$$f\left(G^L[\mathbf{x}]\right) = G^{f(L)[\mathbf{x}]}$$

to be $\widehat{G}$-invariant, i.e.,

$$f\left(\widehat{G}^L[\mathbf{x}]\right) = \widehat{G}^{f(L)[\mathbf{x}]} \tag{4.7}$$

and then, introduce the linear $\widehat{G}$-invariant operator on Banach flow space $\mathscr{G}_{\mathscr{B}}$. In this case, a G-isomorphic operator can be generally defined by

Definition 4.4([25],[28]) *A mapping $f: G_1^{L_1}\to G_2^{L_2}$ is a G-isomorphic operator between continuity flows $G_1^{L_1}$ and $G_2^{L_2}$ if*

(1) there is an isomorphism $\varphi:\widehat{G}\to\widehat{G}$ with $\widehat{G}\supset G_1,G_2$ in graph;

(2) for $\forall(v,u)\in E\left(G_1\right)$ there is $L_2=f\circ\varphi\circ L_1$ but for $\forall(v,u)\in E\left(G_2\setminus G_1\right)$, $f:\mathbf{0}\to L_2(v,u)$ and for $\forall(v,u)\in E(G_1\setminus G_2)$ and $\forall(v,u)\in E(\widehat{G}\setminus(G_1\bigcup G_2))$, $f:L(v,u)\to\mathbf{0}$

and a G-isomorphic operator $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}'}$ is linear if

$$f\left(\lambda\overrightarrow{G}_1^{L_1}+\mu\overrightarrow{G}_2^{L_2}\right) = f\left(\lambda\overrightarrow{G}_1^{L_1}\right)+f\left(\mu\overrightarrow{G}_2^{L_2}\right), \tag{4.8}$$

is continuous if

$$\|\overrightarrow{G}_1^{L_1}-\overrightarrow{G}_0^{L_0}\|<\delta(\varepsilon)\ \Rightarrow\ \|f\left(\overrightarrow{G}_1^{L_1}\right)-f\left(\overrightarrow{G}_0^{L_0}\right)\|<\varepsilon, \tag{4.9}$$

is bounded if there exists a constant $\xi\in[0,+\infty)$ *such that*

$$\|f\left(\overrightarrow{G}_1^{L_1}\right)-f\left(\overrightarrow{G}_2^{L_2}\right)\|<\xi\|\overrightarrow{G}_1^{L_1}-\overrightarrow{G}_2^{L_2}\|. \tag{4.10}$$

By definition, $\mathscr{G}_{\mathscr{B}}^{\pm}$ is all harmonic flows G^L with $\overrightarrow{G}\in\mathscr{G}$, $L:v\to(L(v),-L(v))$ for $v\in E(\overrightarrow{G})$ and, $L:(v,u)\to(L(v,u),-L(v,u))$, the action of end-operators $A_{vu}^{+}:(L(v,u),-L(v,u))\to(L^{A_{vu}^{+}}(v,u),-L^{A_{vu}^{+}}(v,u))$, $A_{uv}^{+}:(L(v,u),-L(v,u))\to(L^{A_{uv}^{+}}(v,u),-L^{A_{uv}^{+}}(v,u))$. Now, if we do not label $\overrightarrow{G}$ with 2-pairs $(L(v),-L(v)),(L(v,u),-L(v,u))$ but only $L(v),L(v,u)$ for $v\in V(G)$ and $(v,u)\in E(G)$, then $\mathscr{G}_{\mathscr{B}}^{\pm}$ is nothing else but continuity flow space $\mathscr{G}_{\mathscr{B}}$, and a linear operator $\mathbf{T}:\mathscr{G}_{\mathscr{B}}^{\pm}\to\mathscr{G}_{\mathscr{B}}^{\pm}$ to be $\mathbf{T}:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}}$ with $\widehat{G}$ invariant, i.e., a G-isomorphic operator on on continuity flow space $\mathscr{G}_{\mathscr{B}}$, and all proofs in [22] hold on $\mathscr{G}_{\mathscr{B}}$ also.

Theorem 4.5(Fixed Flow Theorem) *For a continuous G-isomorphic contractor $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}'}$ there is only one continuity flow $G^L\in\mathscr{G}_{\mathscr{B}}$ such that $f\left(G^L\right)=G^L$.*

Proof It is inferred from the proof of Theorem 2.8 on harmonic flows in [22]. □

Theorem 4.6(Banach Inverse Theorem) *A G-isomorphic linear operator $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}}$ is continuous if and only if it is bounded and furthermore, if f is $1-1$ then the inverse operator f^{-1} of f is also a G-isomorphic continuous operator.*

Proof It is inferred from the proof of Theorem 2.10 on harmonic flows in [22]. □

For a G-isomorphic operator $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}}$, its image Grap$f$ of is defined by

$$\mathrm{Grap}f\ =\ \left\{\left(\overrightarrow{G}^L,f\left(\overrightarrow{G}^L\right)\right)\middle|\ \overrightarrow{G}^L\in\mathscr{G}_{\mathscr{B}}\right\} \tag{4.11}$$

and f is *closed* if the image Grap$f$ of f is closed.

Theorem 4.7(Closed Graph Theorem) *If $\mathbf{T}:\mathscr{G}_{\mathscr{B}_1}\to\mathscr{G}_{\mathscr{B}_2}$ is a closed linear operator with Banach spaces $\mathscr{B}_1$, $\mathscr{B}_2$, then $\mathbf{T}$ is continuous.*

Proof It is inferred from the proof of Theorem 2.13 on harmonic flows in [22]. □

Particularly, a G-isomorphic linear operator $f:\mathscr{G}_{\mathscr{B}}\to\mathbb{R}$ or $\mathbb{C}$ is called a *flow functional*, which can be applied to generalize the Hahn-Banach theorem to Banach flow space $\mathscr{G}_{\mathscr{B}}$.

Theorem 4.8(Hahn-Banach Theorem) *Let $\mathscr{H}_{\mathscr{B}}$ be a subspace of Banach flow space $\mathscr{G}_{\mathscr{B}}$ and let $F:\mathscr{H}_{\mathscr{B}}\to\mathbb{C}$ be a continuous linear flow functional on $\mathscr{H}_{\mathscr{B}}$. Then, there is a continuous linear flow functional $\widetilde{F}:\mathscr{G}_{\mathscr{B}}\to\mathbb{C}$ satisfies the conditions that if $\overrightarrow{G}^L\in\mathscr{H}_{\mathscr{B}}$ then $\widetilde{F}(\overrightarrow{G}^L)=F(\overrightarrow{G}^L)$ and $\|\widetilde{F}\|=\|F\|$.* Particularly, *if $\mathbf{O}\neq\overrightarrow{G}_0^{L_0}\in\mathscr{G}_{\mathscr{B}}$, there is a continuous linear flow functional*

F such that $\|F\|=1$ *and* $\|F(\overrightarrow{G}_0^{L_0})\|=\|\overrightarrow{G}_0^{L_0}\|$.

Proof It is inferred from the proof of Theorem 2.23 on harmonic flows in [22]. □

Particularly, let $\mathscr{G}_{\mathscr{B}}$ be a Hilbert flow space. Then, we know

Theorem 4.9(Fréchet-Riesz Theorem]) *Let* $f:\mathscr{G}_{\mathscr{B}}\to\mathbb{C}$ *be a continuous linear flow functional. For any continuous flow* $\overrightarrow{G}^L\in\mathscr{G}_{\mathscr{B}}$*, there uniquely exists a continuous flow of* $\overrightarrow{G'}^{L'}\in\mathscr{G}_{\mathscr{B}}$ *holding with*

$$f\left(\overrightarrow{G}^L\right)=\left\langle\overrightarrow{G}^L,\overrightarrow{G'}^{L'}\right\rangle. \tag{4.12}$$

Proof It is proved on G-flows by the proof of Theorem 3.5 in [16], which can be naturally generalized to continuity flow similarly. Firstly, let

$$\mathscr{G}_{\mathscr{B}} \ = \ \mathscr{N}(f)+\mathscr{N}^+(f)$$

be an orthogonal decomposition of $\mathscr{G}_{\mathscr{B}}$ and $\overrightarrow{G}^{L'}=\lambda\overrightarrow{G}^L$ with $\lambda=\overline{f(\overrightarrow{G}^L)}/\|\overrightarrow{G}^L\|$ such that

$$f\left(\overrightarrow{G}^L\right)=\left\langle\overrightarrow{G}^L,\overrightarrow{G'}^{L'}\right\rangle$$

and if there is another $\overrightarrow{G}^*\in\mathscr{G}_{\mathscr{B}}$ holding with (4.10), then there must be

$$\begin{aligned}\left\langle\overrightarrow{G}^L,\overrightarrow{G'}^{L'}\right\rangle \ &= \ \left\langle\overrightarrow{G}^L,\overrightarrow{G*}^{L*}\right\rangle\\ \Rightarrow \ \left\langle\overrightarrow{G}^L,\overrightarrow{G'}^{L'}-\overrightarrow{G*}^{L*}\right\rangle \ &= \ \mathbf{O}\end{aligned}$$

i.e., $\overrightarrow{G'}^{L'}=\overrightarrow{G*}^{L*}$, the uniqueness of $\overrightarrow{G'}^{L'}$. □

Furthermore, we can establish differential theory on Hilbert flow space $\mathscr{G}_{\mathscr{B}}$ and obtain the dynamic Euler-Lagrange equations on a thing T, i.e., let $\overrightarrow{G}^L\in\mathscr{G}_{\mathscr{B}}$ with $L:(v,u)\to L(v,u)$ for $(v,u)\in E(G^L)$ and all end-operators in $\mathscr{A}$ satisfying $[A,\frac{\partial}{\partial x_i}]=\mathbf{0}$ for $\forall A\in\mathscr{A}$. In this case, we can also define differential and integral operators by

$$\partial_i:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}} \quad \text{and} \quad \int_C\overrightarrow{G}^{L^2}dz=\overrightarrow{G}^{\int_C L^2dz} \tag{4.13}$$

for integers $1\le i\le n$. Certainly, they are all $\widehat{G}$-isomorphic operators.

Now, let $\mathscr{L}[L(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))]:(v,u)\in E(G)\to\mathscr{L}[L(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))](v,u)$ be a differentiable functional with $[\mathscr{L},A]=\mathbf{0}$ for all end-operators $A\in\mathscr{A}$. Then, we know that

$$\frac{\partial\overrightarrow{G}^{\mathscr{L}}}{\partial x_i}-\frac{d}{dt}\frac{\partial\overrightarrow{G}^{\mathscr{L}}}{\partial\dot{x}_i}=\mathbf{O}, \qquad 1\le i\le n \tag{4.14}$$

by the proof of Theorem 3.4 on harmonic flows L^2 in [22]. Particularly, if $\mathscr{L}\left[\overrightarrow{G}^L[t]\right]$ of continuity flow $\overrightarrow{G}^L[t]$ is independent on (v,u), i.e., they are synchronized then the dynamic behavior of $\overrightarrow{G}^L[t]$ can be characterized by n equations

$$\frac{\partial L}{\partial x_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}=0, \qquad 1\leq i\leq n, \tag{4.15}$$

i.e., the Euler-Lagrange equations, which establishes graph dynamics suggested in [2].

So, *what is the importance of Frechet-Riesz theorem in Hilbert flow space?* Certainly, the subdividing on thing T can be kept going with the technological progress. For example, the models on proton, neutron and meson by quarks are shown in Figure 20.

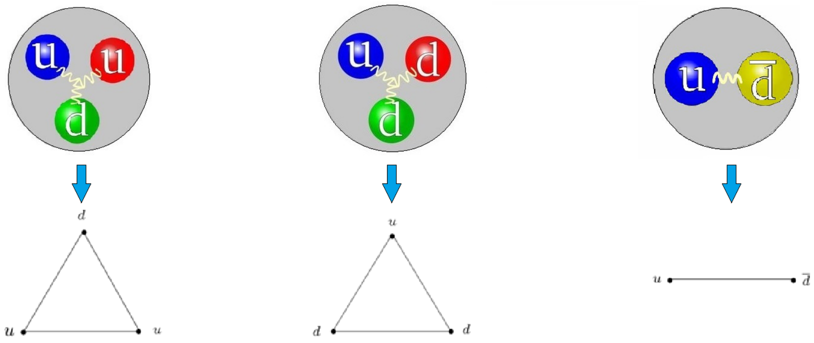

Figure 20

Now, *could we conclude the* 3 *hypothesis in quantum mechanics inherited with combinatorial structure are also true?* The answer is certainly Yes because the Frechet-Riesz theorem in Hilbert flow space $\mathscr{G}_{\mathscr{B}}$ concludes that the existence of Hermitian operator H in a quantum Q with quark structure $\overrightarrow{G}^L$, i.e., the correctness of the 3 hypothesis in quantum mechanics with combinatorial structure in reductionism.

§5. **Conclusion**

There are 2 motivations for the development of mathematics, i.e., one is the need of human recognition on the universe and another is solving problems within mathematics itself such as the Hilbert's 23 problems. In fact, most mathematicians spend their whole life on the second but few one on the first, i.e., the original intention of mathematics, which resulted in Einstein's complaint words on mathematics, i.e.,

> *As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*

Certainly, Einstein's complaint on mathematics implies that the developing of mathematics can not only rely on solving problems in mathematics itself, which is only the mathematical perfection and rather than the game pastime of our mathematicians but also the return to the original needs of recognition of human because mathematics is providing the recognitive way of human on the nature and promoting the sustainable development for human, i.e., the harmony of human with the nature, including the correctly understanding the nature and following the nature. Among the recognitive ways, the reductionism is a systemic mechanism. However, we lack mathematics to reductionism in the classic. In fact, the harmony of human with the nature needs the human to refrain from disturbing nature, which results in the continuity flow or simply $\overrightarrow{G}$-flow $\overrightarrow{G}^L$ in human activities, i.e., substance flow with conservative laws of nature in microscopic ([27]), corresponding to the reductionism and the establishing of technological systems for cyclical use of non-renewable resources ([29]) in order to achieve the conservation of substance flow in local region or the global.

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Applications and Comparative Analysis of Smarandache Weak, Strong and Weak-Strong Structures in Any Field of Knowledge (Second Version)

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Abstract: As a particular case, the Smarandache Algebraic Structures, a fascinating generalization concept meant to explore "hybrid" algebraic systems that generalize or extend classical structures (like groups, rings, fields, etc.). Let's unpack and compare the Smarandache weak/strong/weak-strong structures with classical and other generalized algebraic structures, and also consider their possible applications.

Key Words: Algebraic structures, Smarandache groupoids/semigroups/semirings/semifields/semivector spaces/loops/rings/near-rings/non-associative rings/bialgebraic structures/fuzzy algebra/linear algebra/special definite algebraic structures, Smarandache weak structure, Smarandache strong structure, Smarandache weak structure, hyperStructures.

AMS(2010): 08A05, 08A68.

§1. Overview of Smarandache Structures

Smarandache algebraic structures arise when a classical mathematical structure (say a group, ring, semigroup, lattice, etc.) contains within it a proper subset that satisfies a contradictory or opposite property.

This paper presents a comparative and applicative study of the Smarandache Weak, Strong, and WeakCStrong structures in algebra, exploring their theoretical foundations and relationships with several generalized algebraic systems such as fuzzy, neutrosophic, and hyperstructures. These Smarandache systems extend classical algebraic structures by embedding subsets with differing structural strengths or properties, thereby enabling the modeling of heterogeneous, partially consistent systems. The research further highlights applications of these structures across mathematics, computer science, physics, systems theory, and social sciences, showcasing their value in hybrid and multi-domain problem modeling.

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[1] Received October 22, 2025. Accepted January 18, 2026

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Traditional algebraic systems such as groups, rings, and fields are characterized by uniformity: all elements and operations strictly obey well-defined axioms. However, real-world systems-whether in mathematics, computation, physics, or society-rarely exhibit such total homogeneity. To bridge this gap, Florentin Smarandache introduced the concept of Smarandache structures, where internal subsets of an algebraic system follow stronger or weaker properties than the system as a whole.

In particular, Smarandache weak, strong and weak-strong structures enable formal reasoning about heterogeneous systems containing regions or components with varying scientific or logical strength. This flexibility has turned Smarandache structures into robust modeling tools for hybrid, fractal, or partially consistent systems.

The goal is to generalize algebraic (and in general any scientific) ideas to settings where inconsistencies, partial operations, or gradations of properties can coexist.

§2. Smarandache Weak, Strong and Weak-Strong Structures A Summary

Type	Defining Idea	Key Example	Nature
Smarandache Weak Structure (Sw)	A structure where a proper subset is endowed with a stronger property than the whole structure	A semigroup that contains a proper subset that's a group	A "weak" version because the stronger property holds only on part of the system
Smarandache Strong Structure (Ss)	A structure where every part or subset satisfies at least the property defining the structure	A ring where every subring is itself a field (theoretically extreme)	Everything locally satisfies or strengthens the global definition
Smarandache Weak-Strong Structure (Sws)	A hybrid system where some parts exhibit a stronger property and some a weaker one relative to the parent structure	A semigroup containing both subsets forming groups and subsets forming weaker monoids	Captures interaction between differing degrees of structure-openness

§3. Comparison with Related Algebraic/Nonclassical Systems

Category	Classical Analog	Comparison
Fuzzy Structures	Fuzzy groups, fuzzy rings	Fuzzy structures generalize algebraic operations via membership grades. Smarandache structures instead generalize by logical inclusion of contradictorysub-structures
Partial Algebraic Structures	Partial groups, semigroups	Smarandache structures can host both total and partial operations inside one main structure
Paraconsistent or Neutrosophic Systems	Neutrosophic logic-based algebra	These allow elements to be true, false and indeterminate. Smarandache systems can similarly include opposite algebraic properties within the same structure
Fractal Algebraic Structures	Nested algebraic hierarchy	Smarandache weak-strong structures resemble fractally self-similar systems where subsets have intensified versions of the global property
Hyperstructures (Hypergroups, Hyperrings)	Many-to-one operation systems	Smarandache structures retain deterministic operations but allow contradictory local laws to coexist

§4. Applications

4.1. Algebraic and Theoretical Mathematics

- Generalization of algebraic hierarchies: Smarandache ideas broaden the scope beyond fixed algebraic laws;

- Useful in constructing large hybrid systems where subunits follow different but related axioms;
- Enable the study of partial validity of algebraic rules linking to non-integrable algebraic systems.

4.2. Logic and Computer Science

- Direct connections to neutrosophic logic modeling inconsistent or changing data structures;
- Can model distributed systems with different local rule sets;
- In AI reasoning, can represent systems where some agents or nodes obey weaker or stronger consistency rules.

4.3. Physics and Complex Systems

- Smarandache strong and weak structures can represent multi-scale symmetries e.g., local vs. global conservation laws;
- Used to explore non-homogeneous field theories or crystalline structures with hierarchical subgroup symmetries.

4.4. Information and Social Sciences

- Smarandache weak-strong concepts capture social networks or decision processes where local substructures (e.g., subcommunities) obey stricter or looser rules than the global network.

§5. Conceptual Summary of Relationship

Classical Structures: Uniform property everywhere; Smarandache weak: Stronger property somewhere inside Smarandache wtrong: Strong or stronger property everywhere locally S-marandache weak-strong: Mixed regions of stronger and weaker lawfulness.

This generalization enables study of heterogeneous algebraic (and in general scientific) universes where rigidity (classical) and flexibility (nonclassical) coexist a bridge between traditional algebra and generalized, logical, or neutrosophic mathematics.

Let's create a diagram or concept map showing the relationships among the weak, strong and weakCstrong Smarandache structures and their analogs (e.g., fuzzy and neutrosophic systems). It could help visualize these relationships intuitively.

Let's dive deeper into the applications of Smarandache weak / strong / weak-strong structures, organized by field.

These structures have found theoretical and practical uses in mathematics, computer science, physics, engineering, logic and social sciences, particularly in systems with heterogeneous, partially consistent, or hierarchical properties.

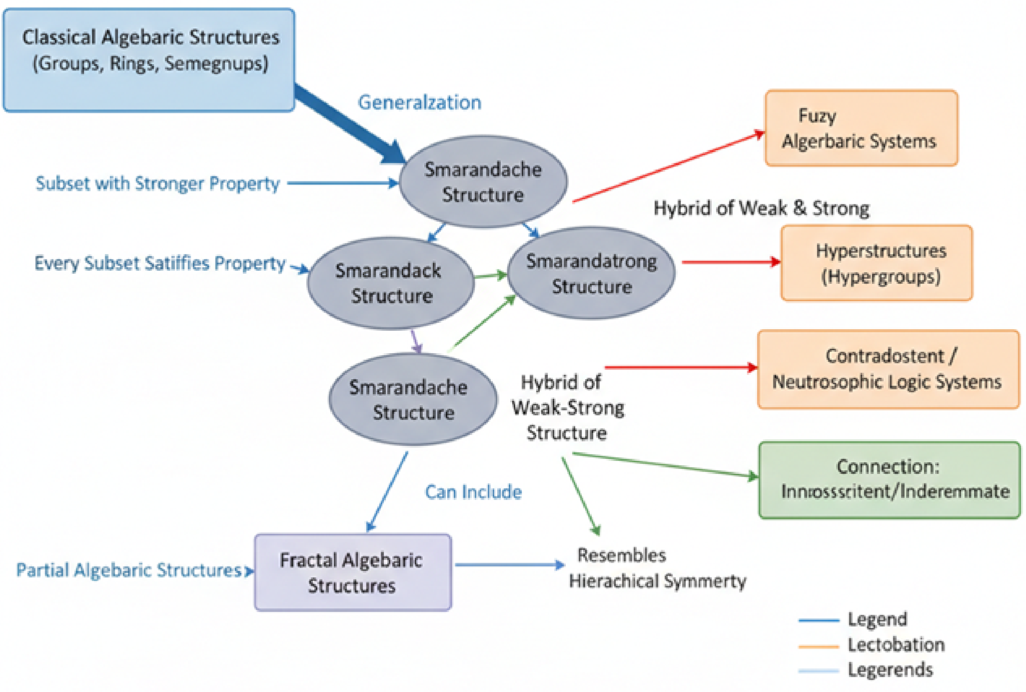


Figure 1.

5.1. Applications in Mathematics and Algebra

a. Generalized Algebraic Systems

- Extension of Groups, Rings, Fields

Smarandache structures allow new kinds of algebraic systems where subsets exhibit stronger or weaker properties. For eExample, a Smarandache semigroup with a subgroup inside that forms a group.

These help study partial satisfaction of algebraic axioms.

b. Bridge between Structures

- Used to relate non-associative structures (loops, quasigroups, near-rings) with associative ones.

This hybrid modeling is particularly useful in ring theory and semigroup theory.

c. Structural Decomposition

- Smarandache weak-strong structures are handy in decomposing algebraic entities identifying inner subregions of higher algebraic strength, symmetry, or order.

d. Algebraic Stability and Resistance Studies

- They describe systems where some components are resistant to structural breakdown, akin to algebraic robustnessimportant in abstract error-tolerant systems.

5.2. Applications in Computer Science and Artificial Intelligence

a. Logical Systems & Knowledge Representation

- Smarandache systems underpin Neutrosophic Logica three-valued logic (True, False, Indeterminate). They model incomplete, inconsistent, and uncertain data, extending fuzzy logic.

b. Multi-Agent Systems

- Useful in defining agent-based systems where,

(1) Some agents follow strict rules (strong);

(2) Others follow looser, heuristic rules (weak);

(3) And mixed agents coexist (weak-strong).

c. Fault-Tolerant Computing

- Model heterogeneous networks where nodes have different operational reliability-critical in distributed systems, error correction, and adaptive computation.

d. Databases and Knowledge Networks

- Applied to heterogeneous database systems where sub-databases hold different integrity constraints-Smarandache structures let these coexist logically.

5.3. Applications in Engineering and Systems Theory

a. Control Systems

- Hybrid and adaptive control systems often exhibit zones with stricter control laws and others with flexible tolerances.

Smarandache WeakCStrong models describe such multi-domain dynamics.

b. Signal Processing

- Can represent filters or transformations that exhibit strong behavior under certain frequency domains but weak elsewhere-useful in piecewise or adaptive filtering.

c. Robotics

- In cooperative robotics, clusters of robots might operate under varying algorithmssome strictly synchronized, others independently adaptivemodeled with weak/strong Smarandache frameworks.

5.4. Applications in Physics and Theoretical Sciences

a. Hierarchical Symmetries

- Represent systems with local and global symmetries, e.g. crystals, particle fields, and energy domains. For example, a quantum field that behaves like a strong group at small scales and a weak semigroup globally.

b. Non-Homogeneous and Anisotropic Media

- Used to model materials or fields where specific subregions have stronger correlations or physical laws-like superconductors, magnetic domains, or composite materials.sed to model materials or fields where specific subregions have stronger correlations or physical laws-like superconductors, magnetic domains, or composite materials.

c. Quantum Mechanics and String Theory

- In quantum systems, local subsystems might respect stricter conservation laws than global ensembles; Smarandache weak-strong formulations capture this non-uniform coherence.

5.5. Applications in Social Sciences, Economics, and Humanities

a. Decision-Making Models

- Represent societies or organizations where some subgroups follow stricter policies and others freer or looser onestypifying partial consistency.

b. Economics and Game Theory

- Used in hybrid markets or negotiation frameworks where some sectors show stable (strong) behavior while others fluctuate or act flexibly (weak).

c. Network Theory

- Social or informational networks exhibit heterogeneous structure strength-clusters (nodes or communities) may have tighter intra-links and weaker inter-links, fitting a Smarandache weakCstrong pattern.

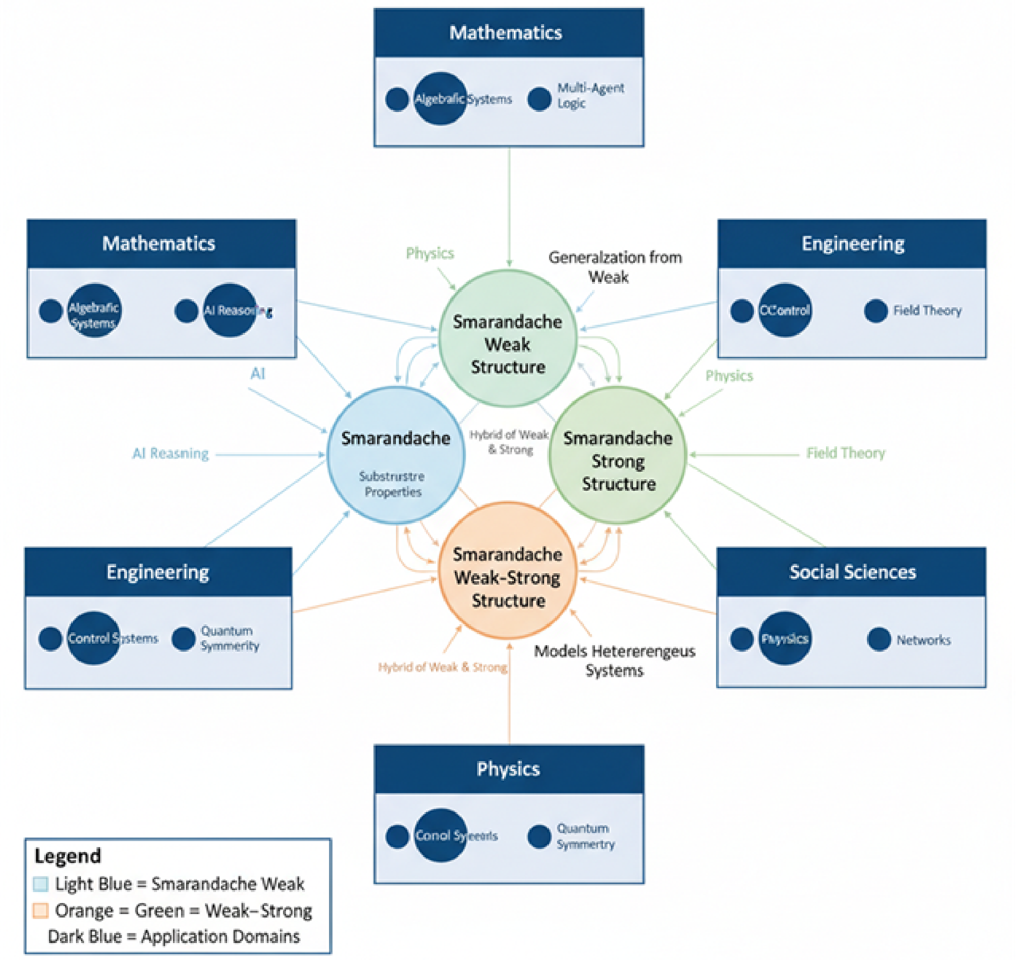
5.6. **Cross-Disciplinary Applications and Emerging Areas**

Field	Smarandache Structure Role	Example/Context
Cryptography	Hybrid algebraic groups for cryptographic key evolution	Key systems with partial group behavior
Artificial General Intelligence (AGI)	Modeling reasoning systems with varying logical strength	Blends strict symbolic reasoning with heuristic inference
Systems Biology	Modeling networks with zones of strong regulation and weak regulation	Gene regulatory networks with variable interaction strength
Linguistics	Rules that apply strictly in formal grammar, loosely in natural speech	Smarandache linguistic systems (formal+informal coexistence)

5.7. **Summary of Application Scope**

Structure Type	Core Domain of Use	Application Essence
Smarandache weak	Structural discovery	Find strong laws hidden in weak global systems
Smarandache strong	Stability modeling	Systems maintaining local structural integrity
Smarandache weakCstrong	Hybrid systems	Model multi-level or mixed-consistency phenomena

The conceptual mind map of Smarandache structures and their applications are shown in Figures 2-12 following.



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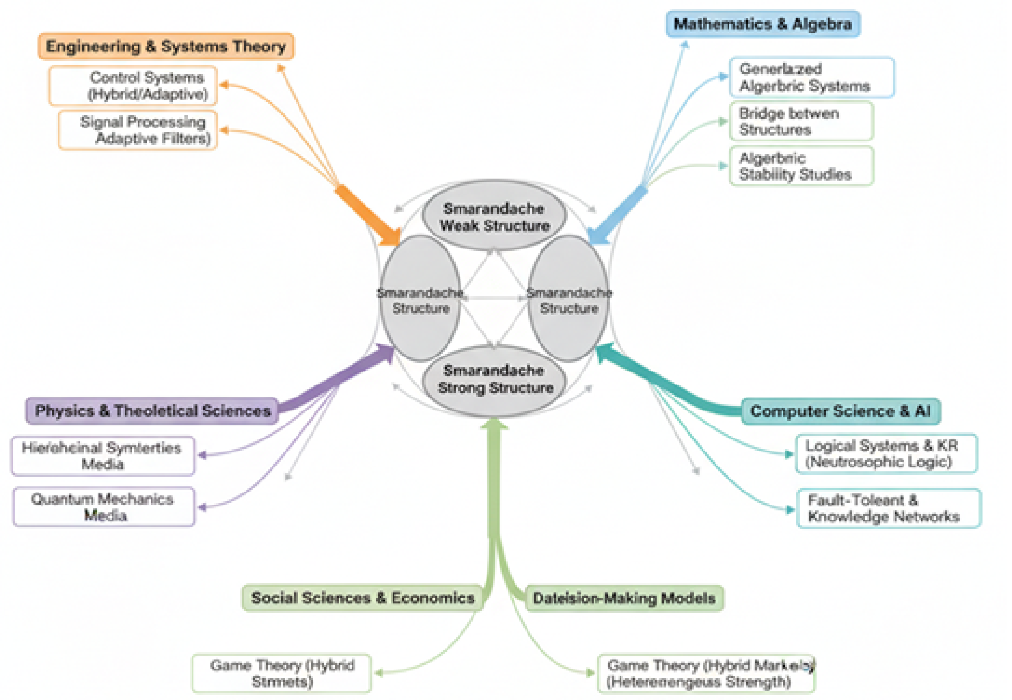

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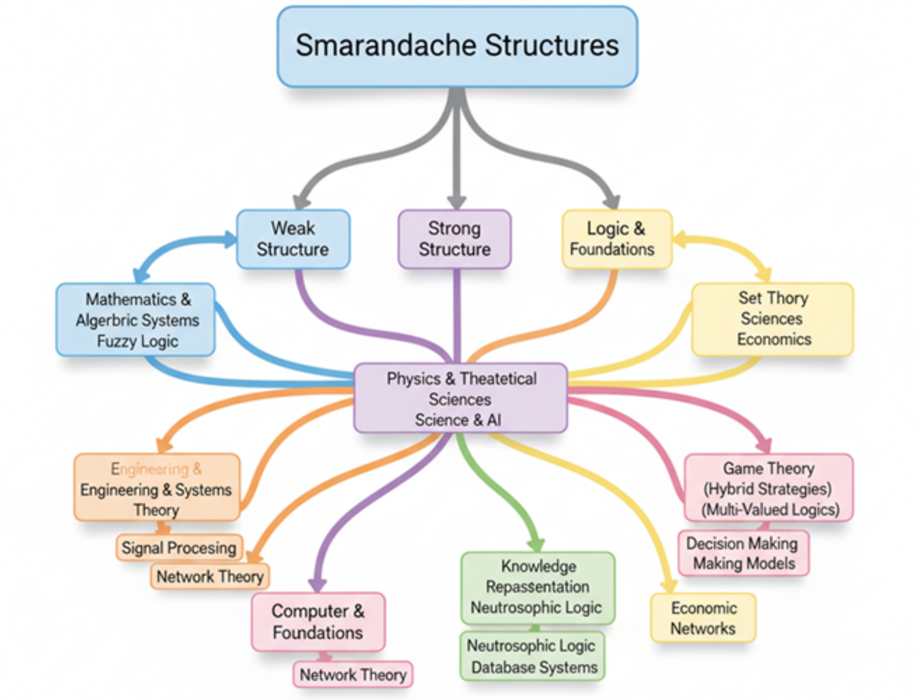

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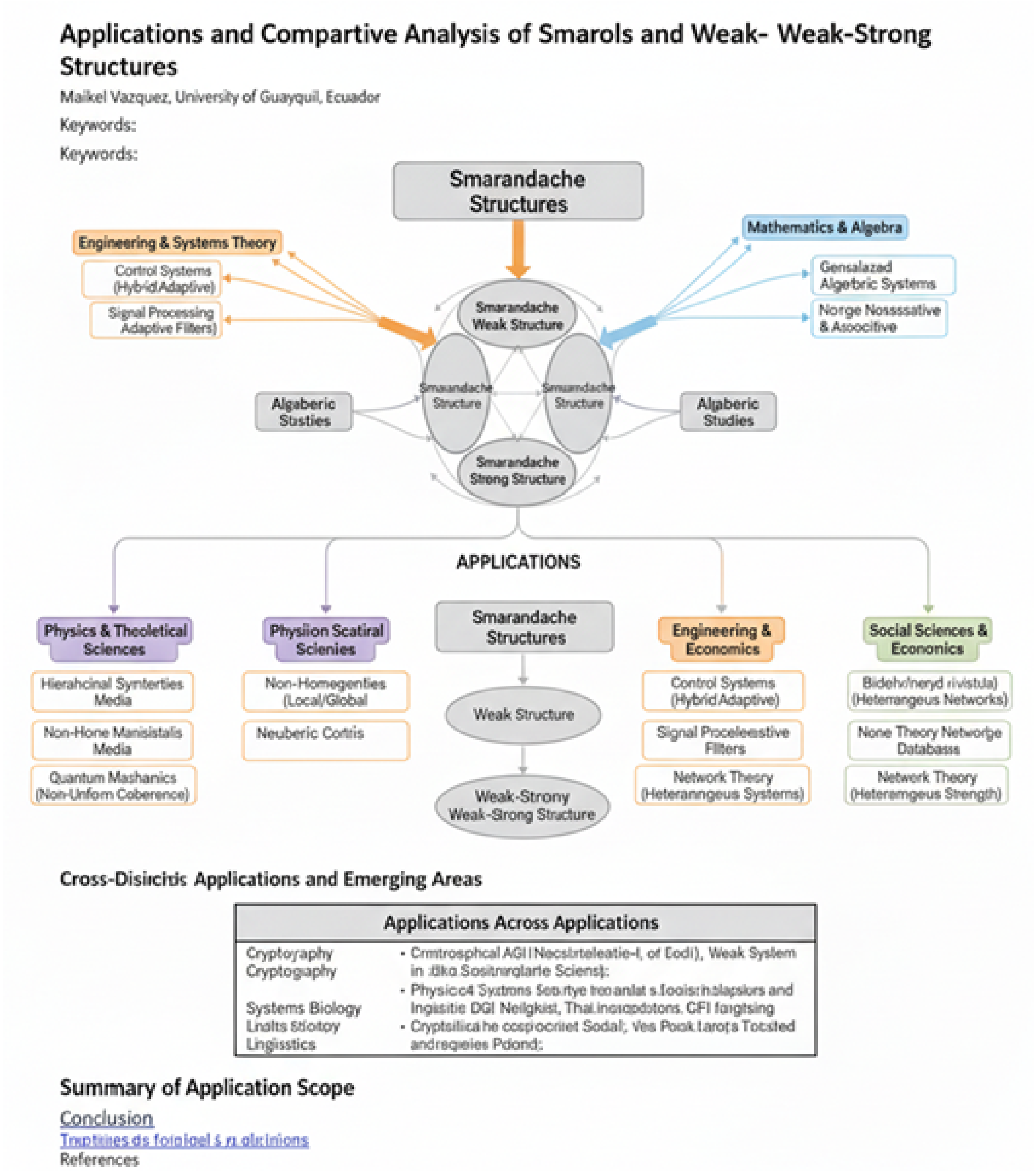

Applications Across Applications	
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Figure 5.

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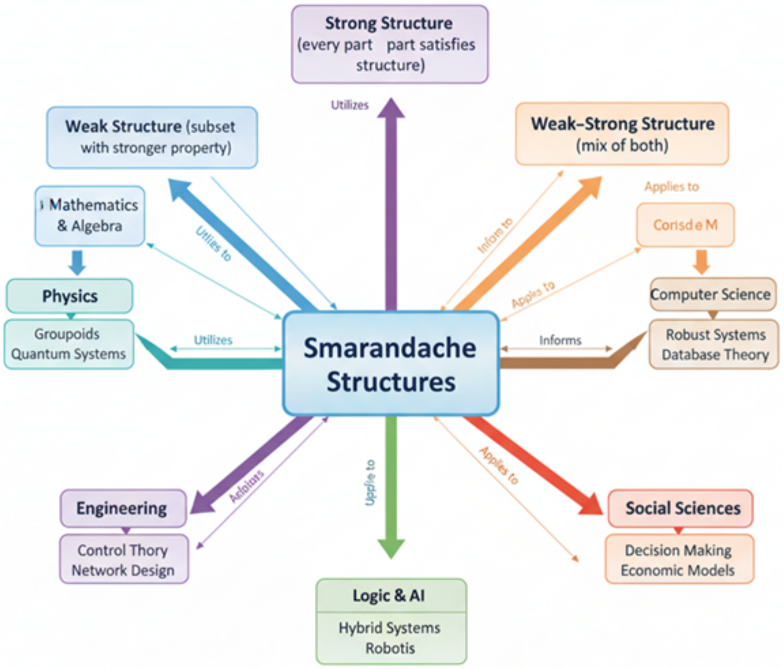

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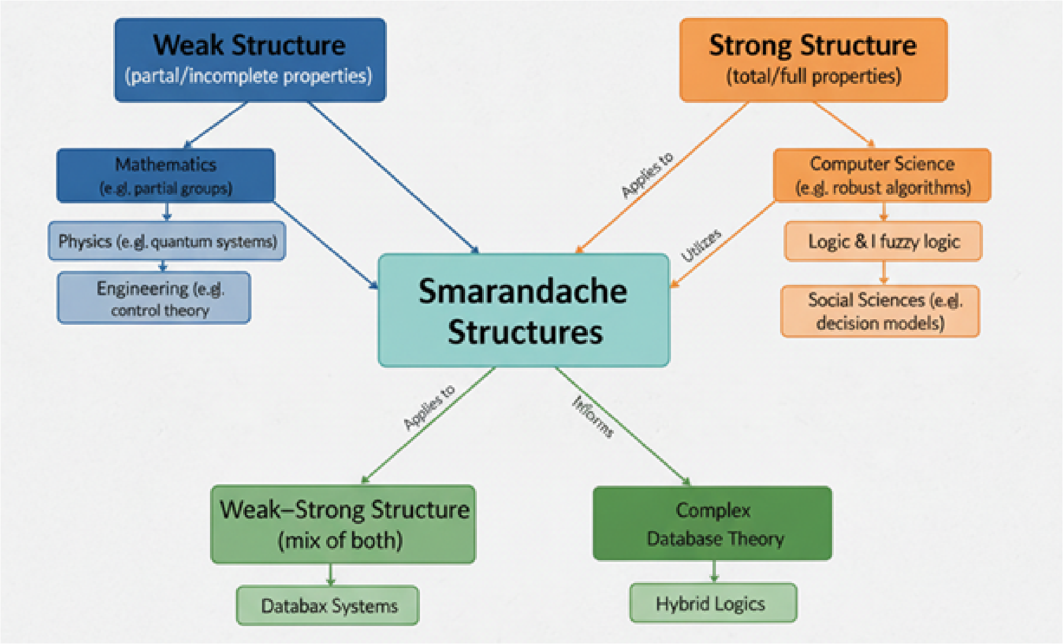

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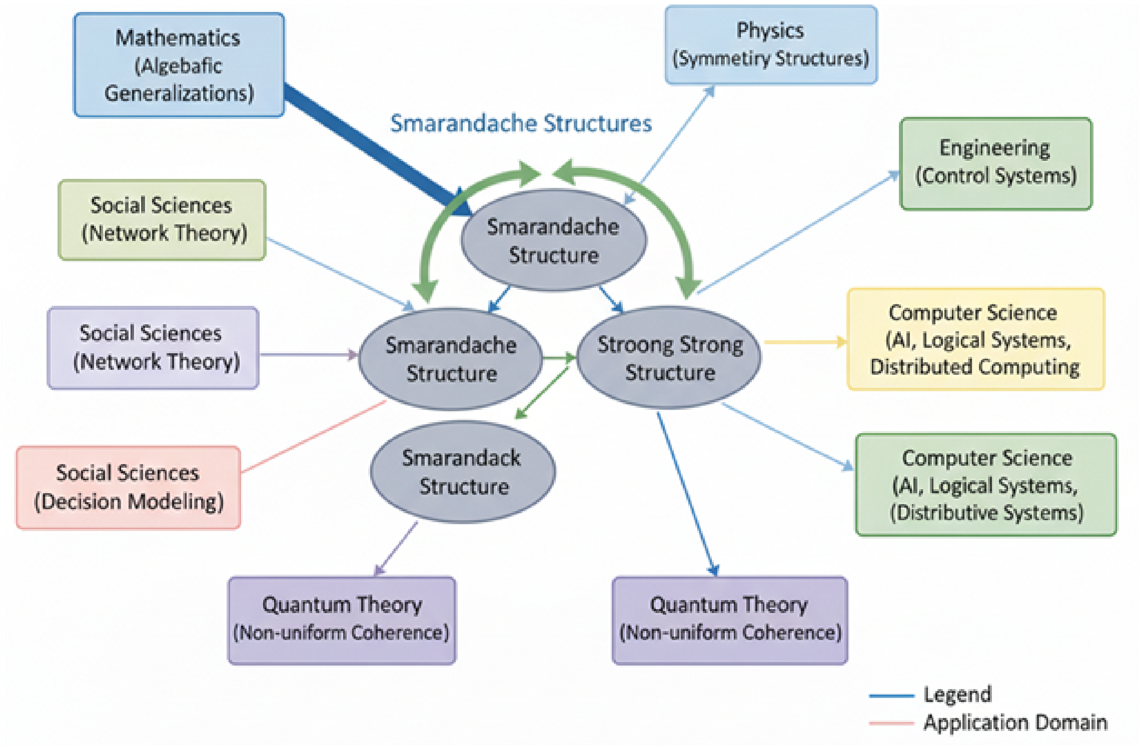

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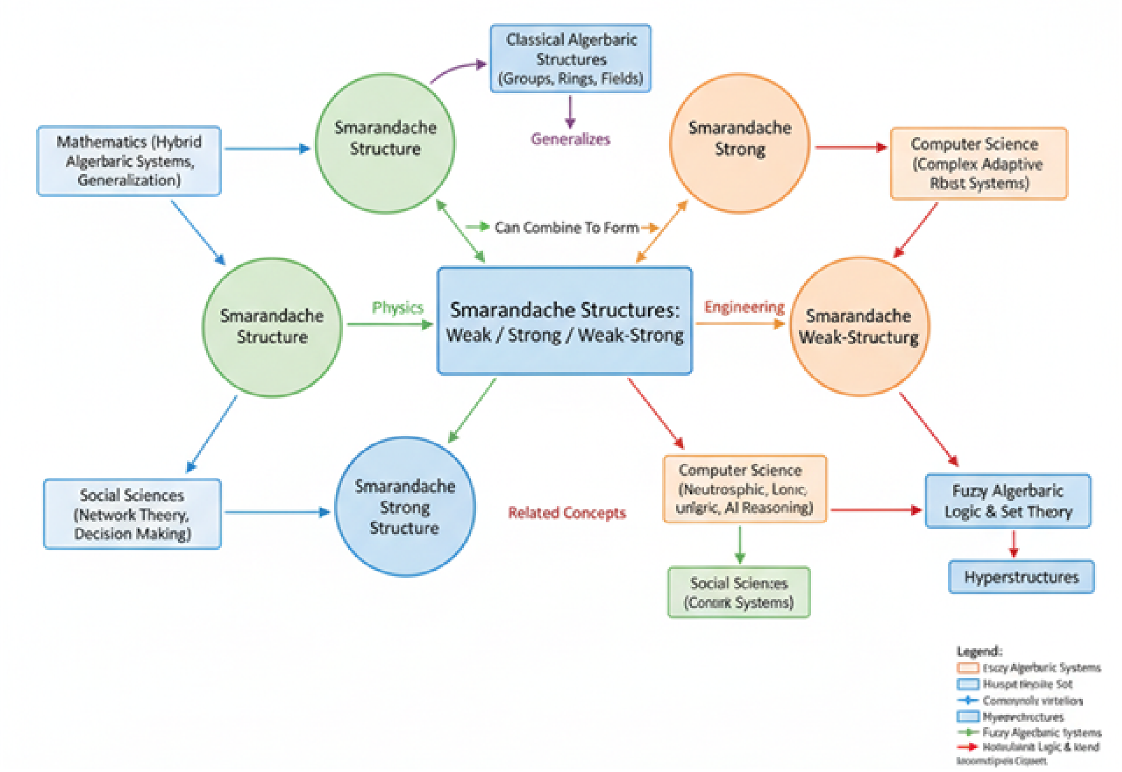

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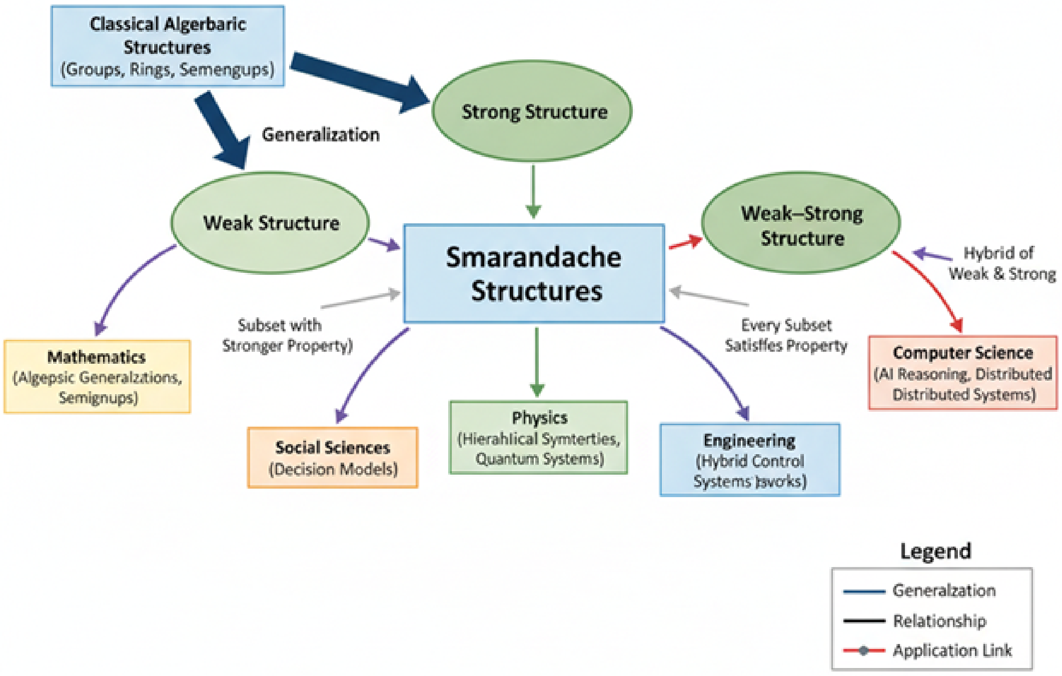

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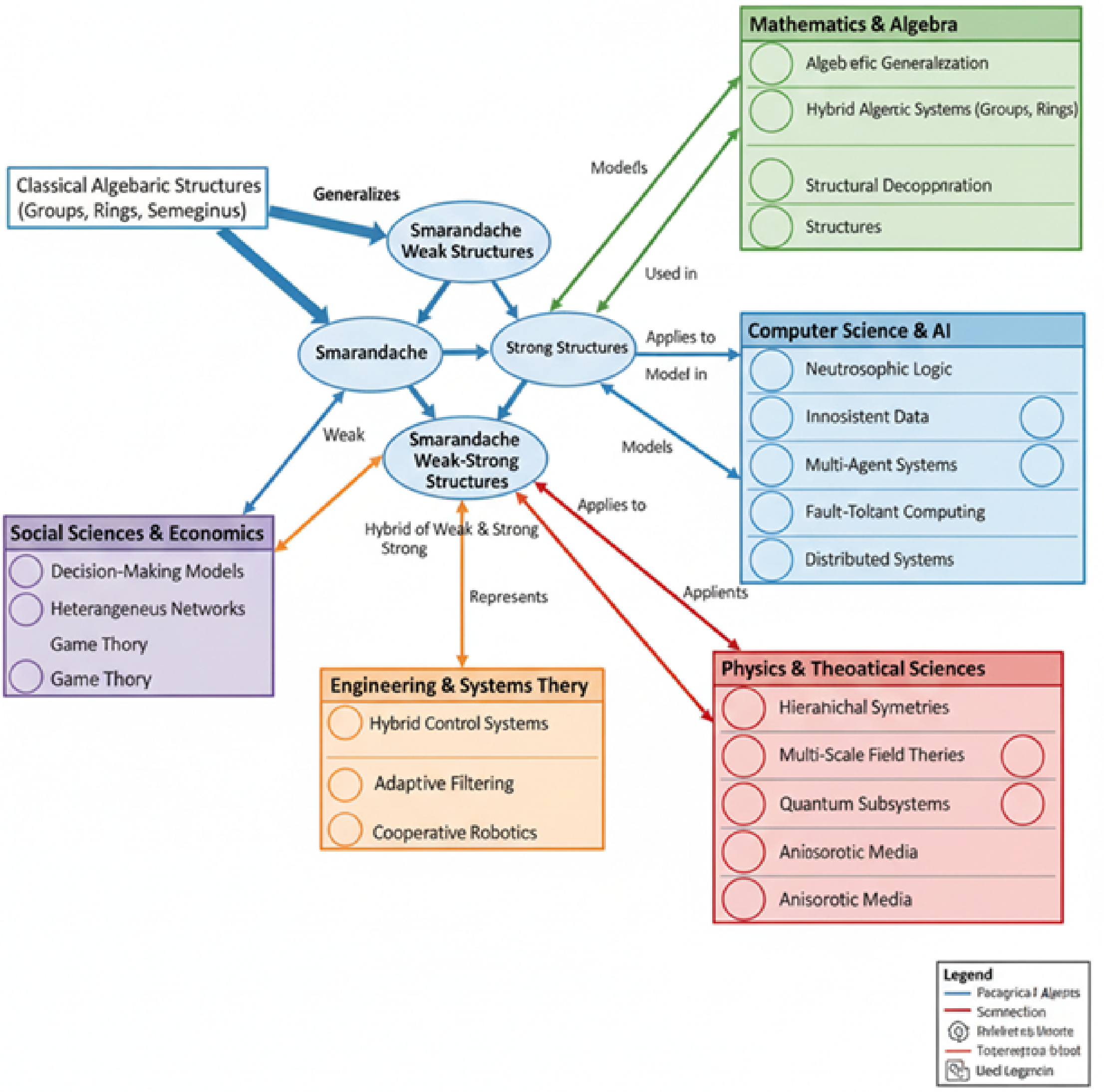

Figure 12.

- The mind map visualize how relationships between Smarandache weak, strong and weakCstrong structures and their connections to mathematics, logic, computer science, physics, engineering, and social sciences.
- The flowchart will illustrate the hierarchical and comparative structure showing how Weak, Strong, and WeakCStrong types differ and link to applications.
- Both will use rich color coding for clarity (not just grayscale academic diagrams).

§6. Conclusion

Smarandache weak, strong and weakCstrong structures provide a flexible scientific basis for modeling systems that combine contradictory, partial, or layered properties.

Their abstract generality translates effectively across disciplinesbe it for algebraists designing new generalized structures, computer scientists creating hybrid logical reasoning systems, physicists studying hierarchical symmetries, or sociologists modeling organizational diversity.

Future exploration may include computational implementations, category-theory reformulations, and physical analogs of weakCstrong algebraic coupling.

Generally, in any field of knowledge (not only mathematics), one has:

A structure (S) is said to have a Smarandache weak structure if it contains a proper subset that possesses a stronger property than (S) itself:

https://fs.unm.edu/SmarandacheWeakStructures.htm

A structure is Smarandache strong if every component or subset satisfies the property defining the structure, or in some cases, a stronger version of it:

https://fs.unm.edu/SmarandacheStrongStructures.htm

The weakCstrong system thus captures graded algebraicity, essential for hybrid modeling of multi-domain or multi-phase systems:

https://fs.unm.edu/SmarandacheStrong-WeakStructures.htm

This hybrid model combines both conditions: some subsets exhibit a stronger property, while others exhibit weaker or partial structural properties relative to the main structure.

The Smarandache approach is unique because it generalizes by inclusion, not by relaxation: subsets can contradict the global structure, allowing non-homogeneous algebraic environments.

The Smarandache frameworks provide a unifying algebraic language for describing inconsistencywithout invalidating structure.

They bridge purely mathematical constructs and real-world systemic heterogeneity, serving both theoretical and modeling purposes.

The weak type emphasizes emergence (stronger local order in weaker global settings).

The strong type ensures stability (uniform local lawfulness).

The weakCstrong type captures dynamic equilibrium (hierarchies of structural strength).

Smarandache structures are meta-mathematical toolsnot just algebraic/scientific curiosities. They enable:

- Modeling of mixed-consistency systems;
- Analysis of local/global structure relationships;
- And generalization of mathematical frameworks across disciplines from pure algebra to data science and physics.

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International J.Math. Combin. Vol.1(2026), 43-65

Multiplicative A-Metric Spaces

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Abstract: In this paper, we introduce multiplicative A-metric spaces (MA-spaces), a multi-point distance structure that unifies A-metric and multiplicative metric frameworks. The distance is defined on ordered n-tuples via a positive multiplicative function satisfying an A-type triangle inequality in product form. This setting includes multiplicative metric spaces ($n = 2$) and multiplicative S-metric spaces ($n = 3$) as special cases. We develop the basic topology of MA-spaces through multiplicative open balls, convergence, Cauchy sequences, and completeness. Furthermore, we establish fixed point results for multiplicative contractive and expansive mappings, extending classical theorems of Banach, Kannan, and Wang to a general multiplicative multi-point framework.

Key Words: Multiplicative metric, S-metric, A-metric, multiplicative S-metric, convergence, fixed point.

AMS(2010): 54H25, 47H10.

§1. Introduction

Metric spaces have been a cornerstone of modern mathematics since their formal introduction by Fréchet in the early twentieth century. In a classical metric space (X, d), each pair of points $\alpha, \beta \in X$ is assigned a nonnegative real number $d(\alpha, \beta)$ satisfying nonnegativity, identity of indiscernibles, symmetry, and the triangle inequality.However, the increasing complexity of applications in nonlinear analysis, optimization, and computer science has motivated the development of generalized distance structures, including multi-point distances and multiplicative (product-type) geometries.

One notable line of development began with Gähler's 2-metric space [6] in 1963 and D-hage's D-metric space [7] in 1984. Mustafa and Sims [3] addressed several structural issues by introducing the G-metric space, obtaining a well-behaved topology and fixed point theory.

Sedghi et al. [1] proposed the concept of an S-metric space, assigning a nonnegative real number $S(\alpha, \beta, \gamma)$ to each triple. Abbas et al. [2] extended S-metrics to an n-tuple structure called an A-metric space.

Parallel to these developments, Bashirov et al. [5] and later works such as [4] studied multiplicative metric spaces, where the usual additive triangle inequality is replaced by a multiplicative inequality and the neutral element is 1. A standard bridge between additive and

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[1] Received October 11, 2025, Accepted February 2, 2026

multiplicative settings is provided by logarithmic/exponential transforms.

Fixed point theory has been central since Banach's contraction principle [9] and many contractive conditions exist, including Kannan-type mappings [10] and comparisons by Rhoades [11]. Expansive mappings also play an essential role (e.g. [12]-[21]).

The aim of the present paper is to unify the multi-point flexibility of A-metrics with multiplicative distance geometry into a single framework, called a *multiplicative A-metric space* (MA-space), and to develop both contractive and expansive fixed point principles in this setting.

§2. **Preliminaries**

In this section, we recall several generalized distance structures that are essential for the development of multiplicative A-metric spaces. These concepts extend the classical notion of metric distance by allowing multi-variable distance functions and, in some cases, multiplicative distance aggregation. Such generalizations play an important role in nonlinear analysis, fixed point theory, and generalized topology.

Throughout this paper, X denotes a nonempty set unless otherwise specified.

Definition 2.1(S-metric space, [1]) *Let X be a nonempty set. A mapping*

$$S : X^3 \to [0, \infty)$$

*is called an S-*metric *on X if for all $\alpha, \beta, \gamma, \mu \in X$ the following conditions hold:*

(1) **Nonnegativity and identity property.**

$$S(\alpha, \beta, \gamma) \geqslant 0, \quad S(\alpha, \beta, \gamma) = 0 \iff \alpha = \beta = \gamma.$$

(2) **Permutation invariance.** *The value of $S(\alpha, \beta, \gamma)$ remains unchanged under any permutation of its arguments.*

(3)**S-triangle inequality.**

$$S(\alpha, \beta, \gamma) \leqslant S(\alpha, \alpha, \mu) + S(\beta, \beta, \mu) + S(\gamma, \gamma, \mu).$$

In this case, the pair (X, S) is called an S-metric space.

Remark 2.2 Every classical metric space (X, d) generates an S-metric by defining

$$S(\alpha, \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma), \qquad \alpha, \beta, \gamma \in X.$$

Thus, ordinary metric spaces are naturally embedded in the class of S-metric spaces.

Definition 2.3(Multiplicative S-metric, [8]) *Let X be a nonempty set. A mapping*

$$S^* : X^3 \to [1, \infty)$$

is called a multiplicative S-metric *on* X *if for all* $\alpha, \beta, \gamma, \mu \in X$*:*

(1) **Positivity and identity condition.**

$$S^*(\alpha, \beta, \gamma) \geqslant 1, \quad S^*(\alpha, \beta, \gamma) = 1 \iff \alpha = \beta = \gamma.$$

(2) **Permutation invariance.** $S^*(\alpha, \beta, \gamma)$ is invariant under any permutation of its arguments.

(3) **Multiplicative triangle inequality.**

$$S^*(\alpha, \beta, \gamma) \leqslant S^*(\alpha, \alpha, \mu)\, S^*(\beta, \beta, \mu)\, S^*(\gamma, \gamma, \mu).$$

Definition 2.4(A-metric space, [2]) *Let* $n \geqslant 2$ *be fixed. A mapping*

$$A : X^n \to [0, \infty)$$

is called an A-metric *on* X *if for all* $\alpha_1, \ldots, \alpha_n, \mu \in X$ *the following properties hold:*

(1) **Nonnegativity and identity condition.**

$$A(\alpha_1, \ldots, \alpha_n) \geqslant 0, \quad A(\alpha_1, \ldots, \alpha_n) = 0 \iff \alpha_1 = \cdots = \alpha_n.$$

(2) **Permutation symmetry.** *The value of* $A(\alpha_1, \cdots, \alpha_n)$ *is invariant under any permutation of its arguments.*

(3) A**-type triangle inequality.**

$$A(\alpha_1, \ldots, \alpha_n) \leqslant \sum_{i=1}^{n} A(\underbrace{\alpha_i, \ldots, \alpha_i}_{n-1}, \mu).$$

In this case, (X, A) *is called an* A*-metric space.*

§3. Multiplicative A-Metric Spaces

In this section, we introduce the concept of multiplicative A-metric spaces, which can be viewed as a multiplicative multi-point extension of classical A-metric spaces. This structure combines the multi-variable distance mechanism of A-metrics with the product-based geometry of multiplicative metrics. The resulting framework allows distances to be measured simultaneously among multiple points while preserving multiplicative consistency. Throughout this section, X denotes a nonempty set and $n \geqslant 3$ is a fixed integer.

Definition 3.1(Multiplicative A-metric space) *Let* $n \geqslant 3$ *be fixed. A mapping*

$$\mathcal{A}^* : X^n \longrightarrow [1, \infty)$$

is called a multiplicative A-metric *on* X *if for all* $\alpha_1, \ldots, \alpha_n, \mu \in X$ *the following conditions* *hold:*

$(MA1)$ **Positivity and identity condition.**

$$\mathcal{A}^*(\alpha_1,\ldots,\alpha_n)\geqslant 1,$$

and

$$\mathcal{A}^*(\alpha_1,\ldots,\alpha_n)=1 \quad \text{if and only if} \quad \alpha_1=\cdots=\alpha_n.$$

$(MA2)$ **Permutation invariance.** *The value of* $\mathcal{A}^*(\alpha_1,\cdots,\alpha_n)$ *remains unchanged under any permutation of its arguments.*

$(MA3)$ **Multiplicative A-triangle inequality.**

$$\mathcal{A}^*(\alpha_1,\ldots,\alpha_n)\leqslant \prod_{i=1}^{n}\mathcal{A}^*(\underbrace{\alpha_i,\ldots,\alpha_i}_{n-1},\mu).$$

The pair $(X,\mathcal{A}^*)$ *is called a multiplicative* A*-metric space.*

Remark 3.2 In multiplicative distance settings, the neutral element is 1, which plays the role analogous to zero in additive metric spaces. Consequently, closeness between points is measured by values approaching 1, and convergence of sequences is characterized by multiplicative distances tending to 1.

Definition 3.3(Induced two-point distance) *Let* $(X,\mathcal{A}^*)$ *be a multiplicative* A*-metric space. Define a two-point distance function*

$$D: X\times X\to[1,\infty)$$

by

$$D(\alpha,\beta):=\mathcal{A}^*(\underbrace{\alpha,\ldots,\alpha}_{n-1},\beta), \qquad \alpha,\beta\in X.$$

Lemma 3.4 *For all* $\alpha,\beta\in X$*,*

(1) $D(\alpha,\beta)\geqslant 1$*, and* $D(\alpha,\beta)=1$ *if and only if* $\alpha=\beta$*;*

(2) $D(\alpha,\beta)=D(\beta,\alpha)$*.*

Proof From Definition 3.3,

$$D(\alpha,\beta)=\mathcal{A}^*(\alpha,\ldots,\alpha,\beta).$$

Applying condition $(MA1)$ to this n-tuple gives $D(\alpha,\beta)\geqslant 1$, and equality holds only when all entries are equal, which is equivalent to $\alpha=\beta$. The symmetry property follows immediately from (MA2), since permutation of arguments does not change the value of $\mathcal{A}^*$. □

Lemma 3.5(Two-point multiplicative inequality) *For all* $\alpha, \beta, \gamma \in X$*,*

$$D(\alpha,\gamma) \leqslant D(\alpha,\beta)^{\,n-1}\, D(\beta,\gamma).$$

Proof Applying $(MA3)$ to the n-tuple $(\underbrace{\alpha,\ldots,\alpha}_{n-1},\gamma)$ with $\mu=\beta$ yields

$$\mathcal{A}^*(\underbrace{\alpha,\cdots,\alpha}_{n-1},\gamma) \leqslant \prod_{i=1}^{n} \mathcal{A}^*(\underbrace{\alpha_i,\cdots,\alpha_i}_{n-1},\beta).$$

The first $n-1$ factors equal $\mathcal{A}^*(\underbrace{\alpha,\cdots,\alpha}_{n-1},\beta) = D(\alpha,\beta)$, while the last factor equals $\mathcal{A}^*(\underbrace{\gamma,\ldots,\gamma}_{n-1},\beta)$, which by permutation invariance equals $\mathcal{A}^*(\underbrace{\beta,\ldots,\beta}_{n-1},\gamma) = D(\beta,\gamma)$. Substituting these identities gives the desired inequality. □

Definition 3.6(Multiplicative open and closed balls) *Let* $\alpha \in X$ *and* $r>1$*. Define*

$$B(\alpha,r) := \{\beta \in X :\ D(\alpha,\beta) < r\}, \qquad \overline{B}(\alpha,r) := \{\beta \in X :\ D(\alpha,\beta) \leqslant r\}.$$

Lemma 3.7 *Let* $\alpha \in X$*,* $r>1$*, and* $\xi \in B(\alpha,r)$*. Then there exists* $\varepsilon>1$ *such that*

$$B(\xi,\varepsilon) \subset B(\alpha,r).$$

Consequently, the family $\{B(\alpha,r) : \alpha \in X,\ r>1\}$ *forms a base for a topology on* X*.*

Proof Let $C := D(\alpha,\xi)$ so that $1 \leqslant C < r$. From Lemma 3.5, for any $\gamma \in X$ we have

$$D(\alpha,\gamma) \leqslant C^{\,n-1} D(\xi,\gamma).$$

Choose $\varepsilon = r/C^{\,n-1} > 1$. If $\gamma \in B(\xi,\varepsilon)$ then $D(\xi,\gamma) < \varepsilon$, which implies

$$D(\alpha,\gamma) < C^{\,n-1}\varepsilon = r,$$

hence $\gamma \in B(\alpha,r)$. Therefore $B(\xi,\varepsilon) \subset B(\alpha,r)$, proving the base property. □

Remark 3.8 The balls $B(\alpha,r)$ generate a topology $\tau_{\mathcal{A}^*}$ on X. A sequence $\{\alpha_k\}$ converges to α if and only if $D(\alpha,\alpha_k) \to 1$.

Definition 3.9(Convergence) *A sequence* $\{\alpha_k\} \subset X$ *converges to* $\alpha \in X$ *if*

$$\lim_{k\to\infty} D(\alpha,\alpha_k) = 1.$$

Definition 3.10(Multiplicative A-Cauchy sequence) *A sequence* $\{\alpha_k\} \subset X$ *is called multiplicative A-Cauchy if for every* $\varepsilon > 1$ *there exists* $N \in \mathbb{N}$ *such that*

$$D(\alpha_k, \alpha_m) < \varepsilon \quad \textit{for all } k, m \geqslant N.$$

Definition 3.11(Completeness) *A multiplicative A-metric space* $(X, \mathcal{A}^*)$ *is said to be complete if every multiplicative A-Cauchy sequence converges to a point in X in the sense of Definition* 3.9.

Proposition 3.12 *Let* (X, d) *be a metric space and let* $n \geqslant 3$. *Define*

$$\mathcal{A}^*(\alpha_1, \cdots, \alpha_n) := \exp\left(\sum_{1 \leqslant i < j \leqslant n} d(\alpha_i, \alpha_j) \right).$$

Then $\mathcal{A}^*$ *is a multiplicative A-metric on X.*

Proof Since $d(\alpha_i, \alpha_j) \geqslant 0$, the sum $\sum_{i<j} d(\alpha_i, \alpha_j)$ is nonnegative, and therefore $\mathcal{A}^*(\alpha_1, \dots, \alpha_n) \geqslant 1$. Moreover, the sum equals zero if and only if $d(\alpha_i, \alpha_j) = 0$ for all i, j, which is equivalent to $\alpha_1 = \cdots = \alpha_n$ by the metric identity property. Hence condition (MA1) holds.

The definition of $\mathcal{A}^*$ depends only on pairwise distances and the summation is taken over all unordered pairs, hence invariance under permutation of arguments is immediate, so (MA2) holds.

Let $\mu \in X$ be arbitrary. By the triangle inequality,

$$d(\alpha_i, \alpha_j) \leqslant d(\alpha_i, \mu) + d(\alpha_j, \mu) \quad \text{for all } i < j.$$

Summing over all pairs $1 \leqslant i < j \leqslant n$ yields

$$\sum_{i<j} d(\alpha_i, \alpha_j) \leqslant \sum_{i<j} \big(d(\alpha_i, \mu) + d(\alpha_j, \mu) \big).$$

Each term $d(\alpha_k, \mu)$ appears exactly $(n-1)$ times in the right-hand double sum, hence

$$\sum_{i<j} d(\alpha_i, \alpha_j) \leqslant (n-1) \sum_{k=1}^{n} d(\alpha_k, \mu).$$

Now observe that

$$\mathcal{A}^*(\underbrace{\alpha_i, \cdots, \alpha_i}_{n-1}, \mu) = \exp\big((n-1) d(\alpha_i, \mu) \big).$$

Therefore

$$\prod_{i=1}^{n} \mathcal{A}^*(\underbrace{\alpha_i, \cdots, \alpha_i}_{n-1}, \mu) = \exp\Big((n-1) \sum_{i=1}^{n} d(\alpha_i, \mu) \Big).$$

Exponentiating the earlier inequality gives (MA3). Hence, $\mathcal{A}^*$ is nothing else but a multiplicative A-metric. □

Proposition 3.13 *Let* (X, d) *be a metric space and let* $n \geqslant 3$*. Define*

$$\mathcal{A}^*(\alpha_1, \cdots, \alpha_n) := \prod_{1 \leqslant i < j \leqslant n} \big(1 + d(\alpha_i, \alpha_j)\big).$$

Then $\mathcal{A}^*$ *is a multiplicative* A*-metric on* X*.*

Proof Since each factor satisfies $1 + d(\alpha_i, \alpha_j) \geqslant 1$, their product is at least 1. The product equals 1 if and only if each factor equals 1, which holds precisely when $d(\alpha_i, \alpha_j) = 0$ for all i, j, implying $\alpha_1 = \cdots = \alpha_n$. Hence (MA1) holds.

Because the definition involves a product over all unordered pairs, permutation of indices leaves the value unchanged, so $(MA2)$ holds.

Fix $\mu \in X$. Using the triangle inequality,

$$d(\alpha_i, \alpha_j) \leqslant d(\alpha_i, \mu) + d(\alpha_j, \mu),$$

and the elementary inequality $1 + a + b \leqslant (1 + a)(1 + b)$ for $a, b \geqslant 0$, we obtain

$$1 + d(\alpha_i, \alpha_j) \leqslant (1 + d(\alpha_i, \mu))(1 + d(\alpha_j, \mu)).$$

Taking products over all pairs $i < j$ gives

$$\prod_{i<j}(1 + d(\alpha_i, \alpha_j)) \leqslant \prod_{i<j}(1 + d(\alpha_i, \mu))(1 + d(\alpha_j, \mu)).$$

Each factor $(1 + d(\alpha_k, \mu))$ appears exactly $(n - 1)$ times in the right-hand product, hence

$$\prod_{i<j}(1 + d(\alpha_i, \alpha_j)) \leqslant \prod_{k=1}^{n}(1 + d(\alpha_k, \mu))^{n-1}.$$

Since

$$\mathcal{A}^*(\underbrace{\alpha_i, \cdots, \alpha_i}_{n-1}, \mu) = (1 + d(\alpha_i, \mu))^{n-1},$$

condition (MA3) follows. □

Proposition 3.14 *Let* (X, d) *be a metric space and let* $n \geqslant 3$*. Define*

$$\mathcal{A}^*(\alpha_1, \cdots, \alpha_n) := \exp\Big(\max_{1 \leqslant i < j \leqslant n} d(\alpha_i, \alpha_j)\Big).$$

Then $\mathcal{A}^*$ *is a multiplicative* A*-metric on* X*.*

Proof Since the maximum of nonnegative numbers is nonnegative, $\mathcal{A}^* \geqslant 1$. Equality holds only when all pairwise distances vanish, which is equivalent to equality of all arguments. Hence (MA1) holds. Symmetry follows from symmetry of the maximum over unordered pairs.

Let $\mu \in X$. For any pair $i < j$, the triangle inequality gives

$$d(\alpha_i, \alpha_j) \leqslant d(\alpha_i, \mu) + d(\alpha_j, \mu) \leqslant \sum_{k=1}^{n} d(\alpha_k, \mu).$$

Therefore

$$\max_{i<j} d(\alpha_i, \alpha_j) \leqslant \sum_{k=1}^{n} d(\alpha_k, \mu).$$

Also,

$$\mathcal{A}^*(\underbrace{\alpha_i, \cdots, \alpha_i}_{n-1}, \mu) = \exp(d(\alpha_i, \mu)).$$

Hence

$$\prod_{i=1}^{n} \mathcal{A}^*(\underbrace{\alpha_i, \cdots, \alpha_i}_{n-1}, \mu) = \exp\Big(\sum_{k=1}^{n} d(\alpha_k, \mu)\Big),$$

which implies $(MA3)$. □

Proposition 3.15 *Let (X, d) be a metric space, $n \geqslant 3$, and $q \in (0, 1]$. Define*

$$\mathcal{A}^*(\alpha_1, \cdots, \alpha_n) := \exp\left(\sum_{1\leqslant i<j\leqslant n} d(\alpha_i, \alpha_j)^q\right).$$

Then $\mathcal{A}^*$ is a multiplicative A-metric on X.

Proof Since $d(\alpha_i, \alpha_j)^q \geqslant 0$, the exponential is at least 1. Equality holds only when all distances are zero, which implies equality of all points, hence (MA1) holds. Symmetry follows from symmetry of the pairwise sum.

Fix $\mu \in X$. Since $q \in (0, 1]$, the function $t \mapsto t^q$ is subadditive on $[0, \infty)$, so for all $a, b \geqslant 0$, $(a + b)^q \leqslant a^q + b^q$. Using triangle inequality,

$$d(\alpha_i, \alpha_j)^q \leqslant (d(\alpha_i, \mu) + d(\alpha_j, \mu))^q \leqslant d(\alpha_i, \mu)^q + d(\alpha_j, \mu)^q.$$

Summing over all pairs gives

$$\sum_{i<j} d(\alpha_i, \alpha_j)^q \leqslant (n-1)\sum_{i=1}^{n} d(\alpha_i, \mu)^q \quad \text{and also} \quad \mathcal{A}^*(\underbrace{\alpha_i, \cdots, \alpha_i}_{n-1}, \mu) = \exp\big((n-1)d(\alpha_i, \mu)^q\big).$$

Exponentiation yields $(MA3)$, completing the proof. □

§4. Topological Properties

In this section, we investigate several fundamental topological characteristics of multiplicative A-metric spaces. In particular, we study convergence behavior, uniqueness of limits, and separation properties of the topology generated by multiplicative open balls. These results confirm that multiplicative A-metric spaces retain many essential structural features analogous to classical metric spaces, while reflecting the multiplicative nature of the underlying distance function.

Lemma 4.1 *A sequence* $\{\alpha_k\} \subset X$ *converges to* $\alpha \in X$ *(that is,* $D(\alpha, \alpha_k) \to 1$*) if and only if for every* $\lambda > 1$ *there exists* $N \in \mathbb{N}$ *such that* $\alpha_k \in B(\alpha, \lambda)$ *for all* $k \geqslant N$*.*

Proof Suppose first that $D(\alpha, \alpha_k) \to 1$. Let $\lambda > 1$ be arbitrary. Since $D(\alpha, \alpha_k)$ converges to 1, by definition of limit there exists $N \in \mathbb{N}$ such that

$$|D(\alpha, \alpha_k) - 1| < \lambda - 1 \quad \text{for all } k \geqslant N.$$

Because $D(\alpha, \alpha_k) \geqslant 1$, this implies $D(\alpha, \alpha_k) < \lambda$ for all $k \geqslant N$. Hence $\alpha_k \in B(\alpha, \lambda)$ for all $k \geqslant N$.

Conversely, suppose that for every $\lambda > 1$ there exists N such that $\alpha_k \in B(\alpha, \lambda)$ whenever $k \geqslant N$. Then for every $\lambda > 1$ there exists N such that $D(\alpha, \alpha_k) < \lambda$ for all $k \geqslant N$. Since $D(\alpha, \alpha_k) \geqslant 1$ for all k, it follows that $D(\alpha, \alpha_k)$ is eventually contained in the interval $[1, \lambda)$. Because this holds for every $\lambda > 1$, the only possible limit value is 1. Hence $D(\alpha, \alpha_k) \to 1$. □

Lemma 4.2 *Limits are unique in a multiplicative A-metric space* $(X, \mathcal{A}^*)$*.*

Proof Suppose $\{\alpha_k\}$ converges to both α and β. Then

$$D(\alpha, \alpha_k) \to 1 \quad \text{and} \quad D(\beta, \alpha_k) \to 1.$$

From Lemma 3.5, for every k,

$$D(\alpha, \beta) \leqslant D(\alpha, \alpha_k)^{n-1} D(\alpha_k, \beta).$$

Since $D(\alpha, \alpha_k) \to 1$ and $D(\alpha_k, \beta) = D(\beta, \alpha_k) \to 1$, the right-hand side converges to 1. Hence

$$D(\alpha, \beta) \leqslant 1.$$

Because $D(\alpha, \beta) \geqslant 1$ always holds, we obtain $D(\alpha, \beta) = 1$, and therefore $\alpha = \beta$. □

Lemma 4.3 *The topology generated by the multiplicative open balls* $\{B(\alpha, r) : \alpha \in X,\, r > 1\}$ *is Hausdorff.*

Proof Let $\alpha, \beta \in X$ with $\alpha \neq \beta$. Then $D(\alpha, \beta) > 1$. Choose $\lambda > 1$ such that

$$\lambda^n < D(\alpha, \beta).$$

Such a choice is possible because the function $\lambda \mapsto \lambda^n$ is continuous and strictly increasing on $(1, \infty)$.

Assume, for contradiction, that there exists $\gamma \in B(\alpha, \lambda) \cap B(\beta, \lambda)$. Then

$$D(\alpha, \gamma) < \lambda \quad \text{and} \quad D(\beta, \gamma) < \lambda.$$

Using Lemma 3.5 with (α, γ, β), we obtain

$$D(\alpha, \beta) \leqslant D(\alpha, \gamma)^{n-1} D(\gamma, \beta).$$

Hence

$$D(\alpha, \beta) < \lambda^{n-1}\lambda = \lambda^n,$$

which contradicts the choice of λ. Therefore

$$B(\alpha, \lambda) \cap B(\beta, \lambda) = \varnothing,$$

and the topology is Hausdorff. □

Remark 4.4 In contrast to classical metric spaces, closed multiplicative balls need not be closed in the induced topology when $n \geqslant 3$. Indeed, let $\{\beta_k\}$ be a sequence such that $\beta_k \to \beta$ and suppose $D(\alpha, \beta_k) \leqslant \lambda$ for all k. From Lemma 3.5, we obtain

$$D(\alpha, \beta) \leqslant D(\alpha, \beta_k)^{n-1} D(\beta_k, \beta).$$

Taking limit as $k \to \infty$ yields

$$D(\alpha, \beta) \leqslant \lambda^{n-1}.$$

If $n \geqslant 3$, then $\lambda^{n-1} > \lambda$ whenever $\lambda > 1$, so this estimate does not imply $D(\alpha, \beta) \leqslant \lambda$. Thus the limit point β need not belong to $\overline{B}(\alpha, \lambda)$, showing that closed balls are not necessarily topologically closed without additional structural assumptions.

§5. Fixed Points for Contractive Mappings

In this section, we establish fixed point results for contractive-type self-mappings defined on multiplicative A-metric spaces.

Throughout this section, let $(X, \mathcal{A}^*)$ be a multiplicative A-metric space and let D denote the associated two-point distance.

Lemma 5.1 *Let $\{\alpha_k\}$ be a finite sequence in X and let $m > k$. Then*

$$D(\alpha_m, \alpha_k) \leqslant \prod_{j=k}^{m-1} D(\alpha_{j+1}, \alpha_j)^{n-1}.$$

Proof Let $m > k$ be fixed. The proof is based on repeated application of the two-point multiplicative inequality stated in Lemma 3.5, which asserts that for all $\alpha, \beta, \gamma \in X$,

$$D(\alpha, \gamma) \leqslant D(\alpha, \beta)^{n-1} D(\beta, \gamma).$$

Applying this inequality with $(\alpha, \beta, \gamma) = (\alpha_m, \alpha_{m-1}, \alpha_k)$ yields

$$D(\alpha_m, \alpha_k) \leqslant D(\alpha_m, \alpha_{m-1})^{n-1} D(\alpha_{m-1}, \alpha_k).$$

The term $D(\alpha_{m-1}, \alpha_k)$ can again be estimated using the same inequality, this time applied with $(\alpha, \beta, \gamma) = (\alpha_{m-1}, \alpha_{m-2}, \alpha_k)$, giving

$$D(\alpha_{m-1}, \alpha_k) \leqslant D(\alpha_{m-1}, \alpha_{m-2})^{n-1} D(\alpha_{m-2}, \alpha_k).$$

Substituting this inequality into the previous bound produces

$$D(\alpha_m, \alpha_k) \leqslant D(\alpha_m, \alpha_{m-1})^{n-1} D(\alpha_{m-1}, \alpha_{m-2})^{n-1} D(\alpha_{m-2}, \alpha_k).$$

Continuing this substitution procedure repeatedly reduces the remaining distance term by shifting the intermediate index downward one position at each stage, while simultaneously introducing an additional multiplicative factor of the form $D(\alpha_{r+1}, \alpha_r)^{n-1}$ for decreasing indices r.

Since the sequence is finite and $m > k$, after finitely many such substitutions the remaining distance term becomes $D(\alpha_{k+1}, \alpha_k)$.

Consequently, after completing all substitutions, the inequality takes the form

$$D(\alpha_m, \alpha_k) \leqslant D(\alpha_m, \alpha_{m-1})^{n-1} D(\alpha_{m-1}, \alpha_{m-2})^{n-1} \cdots D(\alpha_{k+1}, \alpha_k)^{n-1}.$$

Rewriting this product using index notation yields

$$D(\alpha_m, \alpha_k) \leqslant \prod_{j=k}^{m-1} D(\alpha_{j+1}, \alpha_j)^{n-1}.$$

This establishes the desired multiplicative chain estimate. □

Theorem 5.2(Multiplicative Banach type) *Let $(X, \mathcal{A}^*)$ be a complete multiplicative A-metric space and let D be the induced two-point distance. Assume that there exists a constant $q \in (0, 1)$ such that*

$$D(\Phi\alpha, \Phi\beta) \leqslant D(\alpha, \beta)^q \quad \text{for all } \alpha, \beta \in X.$$

Then Φ has a unique fixed point $\xi \in X$. Moreover, for every $\alpha_0 \in X$, the Picard iteration

$$\alpha_{k+1} = \Phi(\alpha_k), \qquad k \geqslant 0,$$

converges to ξ in the multiplicative sense, i.e. $D(\alpha_k, \xi) \to 1$.

Proof Fix $\alpha_0 \in X$ and define the Picard sequence $\alpha_{k+1} = \Phi(\alpha_k)$ for $k \geqslant 0$. Set

$$\delta_k := D(\alpha_{k+1}, \alpha_k) \qquad (k \geqslant 0).$$

By Lemma 3.4, each $\delta_k \geqslant 1$. If $\delta_0 = 1$, then $\alpha_1 = \alpha_0$ and hence α_0 is a fixed point, so the conclusion holds. Assume therefore that $\delta_0 > 1$.

Applying the contractive hypothesis to (α_k, α_{k-1}) gives, for every $k \geqslant 1$,

$$\delta_k = D(\Phi\alpha_k, \Phi\alpha_{k-1}) \leqslant D(\alpha_k, \alpha_{k-1})^q = \delta_{k-1}^q.$$

Iterating yields

$$\delta_k \leqslant \delta_0^{q^k} \qquad (k \geqslant 0).$$

Since $q^k \to 0$ and $\delta_0 > 0$, we have $\delta_0^{q^k} = \exp(q^k \ln \delta_0) \to \exp(0) = 1$, hence $\delta_k \to 1$.

To prove that $\{\alpha_k\}$ is multiplicative A-Cauchy, fix integers $m > k$. Lemma 5.1 implies

$$D(\alpha_m, \alpha_k) \leqslant \prod_{j=k}^{m-1} \delta_j^{n-1}.$$

Taking logarithms (valid since each factor is $\geqslant 1$) yields

$$\ln D(\alpha_m, \alpha_k) \leqslant (n-1) \sum_{j=k}^{m-1} \ln \delta_j.$$

Using $\delta_j \leqslant \delta_0^{q^j}$ and monotonicity of ln on $(0, \infty)$ gives

$$\ln \delta_j \leqslant \ln(\delta_0^{q^j}) = q^j \ln \delta_0.$$

Therefore,

$$\ln D(\alpha_m, \alpha_k) \leqslant (n-1) \ln \delta_0 \sum_{j=k}^{m-1} q^j \leqslant (n-1) \ln \delta_0 \sum_{j=k}^{\infty} q^j = \frac{(n-1) \ln \delta_0}{1-q} q^k.$$

The right-hand side tends to 0 as $k \to \infty$, uniformly in $m > k$. Hence

$$\lim_{k\to\infty} \sup_{m>k} \ln D(\alpha_m, \alpha_k) = 0 \qquad \text{equivalently} \qquad \lim_{k\to\infty} \sup_{m>k} D(\alpha_m, \alpha_k) = 1.$$

This is precisely the multiplicative A-Cauchy property (Definition 3.10). By completeness, there exists $\xi \in X$ such that

$$D(\alpha_k, \xi) \to 1.$$

To show that ξ is a fixed point, apply the contractive hypothesis to (α_k, ξ):

$$D(\alpha_{k+1}, \Phi\xi) = D(\Phi\alpha_k, \Phi\xi) \leqslant D(\alpha_k, \xi)^q \longrightarrow 1.$$

Thus $D(\alpha_{k+1}, \Phi\xi) \to 1$ and also $D(\alpha_{k+1}, \xi) \to 1$. By uniqueness of limits in the induced topology (Lemma 4.2), we must have $\Phi\xi = \xi$, so ξ is a fixed point.

Finally, let η be any fixed point of Φ. Then

$$D(\xi, \eta) = D(\Phi\xi, \Phi\eta) \leqslant D(\xi, \eta)^q.$$

If $D(\xi, \eta) > 1$, then raising to the power $q \in (0, 1)$ strictly decreases the value, so $D(\xi, \eta) \leqslant D(\xi, \eta)^q$ is impossible. Hence $D(\xi, \eta) = 1$, and Lemma 3.4 yields $\xi = \eta$. Therefore the fixed point is unique. □

Theorem 5.3(Multiplicative Kannan type) *Let $(X, \mathcal{A}^*)$ be a complete multiplicative A-metric space and let D be the induced two-point distance. Assume that there exists $q \in \left(0, \frac{1}{2}\right)$ such that*

$$D(\Phi\alpha, \Phi\beta) \leqslant \big(D(\alpha, \Phi\alpha)\, D(\beta, \Phi\beta)\big)^q \quad \textit{for all } \alpha, \beta \in X.$$

Then Φ has a unique fixed point $\xi \in X$. Moreover, for every $\alpha_0 \in X$, the Picard iteration $\alpha_{k+1} = \Phi(\alpha_k)$ converges multiplicatively to ξ, i.e. $D(\alpha_k, \xi) \to 1$.

Proof Fix $\alpha_0 \in X$ and define the Picard sequence $\alpha_{k+1} = \Phi(\alpha_k)$ for $k \geqslant 0$. Set

$$\delta_k := D(\alpha_{k+1}, \alpha_k) \qquad (k \geqslant 0).$$

By Lemma 3.4, $\delta_k \geqslant 1$ for every k. If $\delta_0 = 1$, then $\alpha_1 = \alpha_0$ and hence α_0 is a fixed point; uniqueness (proved below) then yields the conclusion. Assume henceforth that $\delta_0 > 1$.

Applying the Kannan-type condition to (α_k, α_{k-1}) gives, for every $k \geqslant 1$,

$$\delta_k = D(\Phi\alpha_k, \Phi\alpha_{k-1}) \leqslant \big(D(\alpha_k, \Phi\alpha_k)\, D(\alpha_{k-1}, \Phi\alpha_{k-1})\big)^q = (\delta_k \delta_{k-1})^q.$$

Since $\delta_k \geqslant 1$, we may divide by δ_k^q to obtain

$$\delta_k^{1-q} \leqslant \delta_{k-1}^q.$$

Taking logarithms (all terms are $\geqslant 1$) yields

$$(1-q)\ln\delta_k \leqslant q\ln\delta_{k-1}, \qquad \text{equivalently} \qquad \ln\delta_k \leqslant c\,\ln\delta_{k-1}, \quad c := \frac{q}{1-q} \in (0, 1),$$

because $q \in (0, \frac{1}{2})$ implies $0 < c < 1$. Iterating the last inequality gives

$$\ln\delta_k \leqslant c^k \ln\delta_0 \quad (k \geqslant 0),$$

hence $\ln\delta_k \to 0$ and therefore $\delta_k \to 1$.

To show that $\{\alpha_k\}$ is multiplicative A-Cauchy, let $m > k$. Lemma 5.1 yields

$$D(\alpha_m, \alpha_k) \leqslant \prod_{j=k}^{m-1} \delta_j^{n-1}.$$

Taking logarithms gives

$$\ln D(\alpha_m, \alpha_k) \leqslant (n-1) \sum_{j=k}^{m-1} \ln \delta_j.$$

Using $\ln \delta_j \leqslant c^j \ln \delta_0$ and summing the geometric series,

$$\sum_{j=k}^{\infty} \ln \delta_j \leqslant \ln \delta_0 \sum_{j=k}^{\infty} c^j = \frac{c^k}{1-c} \ln \delta_0,$$

and thus

$$\ln D(\alpha_m, \alpha_k) \leqslant (n-1) \frac{c^k}{1-c} \ln \delta_0.$$

The right-hand side tends to 0 as $k \to \infty$, uniformly in $m > k$. Hence

$$\lim_{k\to\infty} \sup_{m>k} \ln D(\alpha_m, \alpha_k) = 0, \qquad \text{equivalently} \qquad \lim_{k\to\infty} \sup_{m>k} D(\alpha_m, \alpha_k) = 1,$$

so $\{\alpha_k\}$ is multiplicative A-Cauchy. Completeness provides $\xi \in X$ such that

$$D(\alpha_k, \xi) \to 1.$$

To verify that ξ is a fixed point, apply the Kannan inequality to (α_k, ξ):

$$D(\alpha_{k+1}, \Phi\xi) = D(\Phi\alpha_k, \Phi\xi) \leqslant \big(D(\alpha_k, \Phi\alpha_k)\, D(\xi, \Phi\xi)\big)^q = (\delta_k\, D(\xi, \Phi\xi))^q.$$

Letting $k \to \infty$ and using $\delta_k \to 1$ gives

$$\limsup_{k\to\infty} D(\alpha_{k+1}, \Phi\xi) \leqslant D(\xi, \Phi\xi)^q.$$

On the other hand, since $D(\alpha_{k+1}, \xi) \to 1$, Lemma 3.5 implies

$$D(\xi, \Phi\xi) \leqslant D(\xi, \alpha_{k+1})^{n-1} D(\alpha_{k+1}, \Phi\xi),$$

and letting $k \to \infty$ yields

$$D(\xi, \Phi\xi) \leqslant \liminf_{k\to\infty} D(\alpha_{k+1}, \Phi\xi).$$

Combining the last two estimates gives

$$D(\xi, \Phi\xi) \leqslant D(\xi, \Phi\xi)^q.$$

Since $D(\xi, \Phi\xi) \geqslant 1$ and $q \in (0, 1)$, the inequality forces $D(\xi, \Phi\xi) = 1$, hence $\Phi\xi = \xi$.

Finally, to prove uniqueness, let η be another fixed point. Then the Kannan condition gives

$$D(\xi, \eta) = D(\Phi\xi, \Phi\eta) \leqslant \big(D(\xi, \Phi\xi)\, D(\eta, \Phi\eta)\big)^q = (1 \cdot 1)^q = 1.$$

Because $D(\xi, \eta) \geqslant 1$, we obtain $D(\xi, \eta) = 1$, and Lemma 3.4 implies $\xi = \eta$. $\square$

Example 5.4 Let $X = \mathbb{R}$ and fix an integer $n \geqslant 3$. Consider the standard metric $d(x, y) = |x - y|$ on $\mathbb{R}$ and define $\mathcal{A}^*$ by the construction in Proposition 3.12, namely

$$\mathcal{A}^*(\alpha_1, \ldots, \alpha_n) = \exp\left(\sum_{1 \leqslant i < j \leqslant n} |\alpha_i - \alpha_j| \right).$$

Then $(\mathbb{R}, \mathcal{A}^*)$ is a multiplicative A-metric space. The induced two-point distance $D(\alpha, \beta) = \mathcal{A}^*(\underbrace{\alpha, \ldots, \alpha}_{n-1}, \beta)$ is explicitly given by

$$D(\alpha, \beta) = \exp\big((n-1)|\alpha - \beta|\big).$$

Indeed, for the n-tuple $(\alpha, \ldots, \alpha, \beta)$ the only nonzero pairwise distances are those involving β, and there are exactly $(n-1)$ such pairs, each equal to $|\alpha - \beta|$, hence the sum of all pairwise distances equals $(n-1)|\alpha - \beta|$, which yields the above formula.

Define $\Phi : \mathbb{R} \to \mathbb{R}$ by $\Phi(\alpha) = c\alpha$ where $c \in (0, 1)$. For any $\alpha, \beta \in \mathbb{R}$,

$$|\Phi(\alpha) - \Phi(\beta)| = |c\alpha - c\beta| = c\,|\alpha - \beta|.$$

Substituting into the expression for D gives

$$\begin{aligned} D(\Phi\alpha, \Phi\beta) &= \exp\big((n-1)|\Phi(\alpha) - \Phi(\beta)|\big) = \exp\big((n-1)c|\alpha - \beta|\big) \\ &= \Big(\exp\big((n-1)|\alpha - \beta|\big)\Big)^c = \big(D(\alpha, \beta)\big)^c. \end{aligned}$$

Thus the multiplicative Banach-type inequality

$$D(\Phi\alpha, \Phi\beta) \leqslant D(\alpha, \beta)^q$$

holds with $q = c \in (0, 1)$. Since $(\mathbb{R}, D)$ is complete (because $D(\alpha, \beta) = \exp((n-1)|\alpha - \beta|)$ is equivalent to the usual metric via the logarithmic transform), Theorem 5.2 applies and yields a unique fixed point.

Finally, the fixed point equation $\Phi(\xi) = \xi$ reduces to $c\xi = \xi$, i.e. $(c-1)\xi = 0$. Since $c \neq 1$, it follows that $\xi = 0$. Therefore 0 is the unique fixed point, and for every $\alpha_0 \in \mathbb{R}$ the Picard iteration $\alpha_{k+1} = c\alpha_k$ converges multiplicatively to 0, equivalently $D(\alpha_k, 0) \to 1$.

Example 5.5 Let $X = [0, \infty)$, fix $n \geqslant 3$, and equip X with the usual metric $d(x, y) = |x - y|$. Define $\mathcal{A}^*$ as in Proposition 3.12 by

$$\mathcal{A}^*(\alpha_1, \ldots, \alpha_n) := \exp\left(\sum_{1 \leqslant i < j \leqslant n} |\alpha_i - \alpha_j| \right).$$

Then $(X, \mathcal{A}^*)$ is a multiplicative A-metric space and the induced two-point distance is

$$D(\alpha, \beta) = \mathcal{A}^*(\underbrace{\alpha, \ldots, \alpha}_{n-1}, \beta) = \exp\big((n-1)|\alpha - \beta|\big), \qquad \alpha, \beta \in X.$$

Fix constants $c \in (0,1)$ and $b \geqslant 0$, and define $\Phi : X \to X$ by

$$\Phi(\alpha) = c\alpha + b, \qquad \alpha \in X.$$

Then $\Phi(X) \subset X$ because $c\alpha + b \geqslant 0$ for all $\alpha \geqslant 0$.

For any $\alpha, \beta \in X$, the translation term cancels and

$$|\Phi(\alpha) - \Phi(\beta)| = |c\alpha + b - (c\beta + b)| = c|\alpha - \beta|.$$

Substituting this into the explicit formula for D gives

$$\begin{aligned} D(\Phi\alpha, \Phi\beta) &= \exp\big((n-1)|\Phi(\alpha) - \Phi(\beta)|\big) = \exp\big((n-1)c|\alpha - \beta|\big) \\ &= \big(\exp((n-1)|\alpha - \beta|)\big)^c = \big(D(\alpha, \beta)\big)^c. \end{aligned}$$

Hence Φ satisfies the multiplicative Banach-type inequality

$$D(\Phi\alpha, \Phi\beta) \leqslant D(\alpha, \beta)^q \quad \text{for all } \alpha, \beta \in X$$

with $q = c \in (0,1)$. Moreover, (X, D) is complete because $D(\alpha, \beta) = \exp((n-1)|\alpha - \beta|)$ is equivalent to the usual metric on $[0, \infty)$ via the logarithmic transform.

Therefore Theorem 5.2 applies and yields existence and uniqueness of a fixed point $\xi \in X$ and multiplicative convergence of the Picard iteration for every initial value.

Finally, the fixed point equation $\Phi(\xi) = \xi$ gives

$$\xi = c\xi + b, \qquad \text{hence} \qquad (1-c)\xi = b,$$

so

$$\xi = \frac{b}{1-c} \in [0, \infty).$$

Consequently, Φ has the unique fixed point $\xi = \frac{b}{1-c}$, and for every $\alpha_0 \in X$ the iteration $\alpha_{k+1} = c\alpha_k + b$ converges multiplicatively to ξ, i.e. $D(\alpha_k, \xi) \to 1$.

§6. Fixed Points for Expansive Mappings

Throughout this section let $(X, \mathcal{A}^*)$ be a multiplicative A-metric space with parameter $n \geqslant 3$ and let D denote the induced two-point multiplicative distance.

Definition 6.1(k-expansive mapping) *Let $k > 1$. A mapping $\Phi : X \to X$ is called* k-expansive *if*

$$D(\Phi(\alpha), \Phi(\beta)) \geqslant (D(\alpha, \beta))^k \quad \forall \alpha, \beta \in X.$$

Lemma 6.2 *If Φ is k-expansive with $k > 1$, then Φ has at most one fixed point.*

Proof Assume ξ, η are fixed points. Then

$$D(\xi, \eta) = D(\Phi\xi, \Phi\eta) \geqslant D(\xi, \eta)^k.$$

Since $D(\xi, \eta) \geqslant 1$ and $k > 1$, if $D(\xi, \eta) > 1$ then

$$D(\xi, \eta)^k > D(\xi, \eta),$$

which contradicts the above inequality. Hence $D(\xi, \eta) = 1$, and therefore $\xi = \eta$. ☐

Definition 6.3(Multiplicative diameter)

$$\operatorname{diam}(X) := \sup_{\alpha,\beta \in X} D(\alpha, \beta) \in [1, \infty].$$

Theorem 6.4 *If* $\operatorname{diam}(X) < \infty$ *and there exists a* k*-expansive mapping* $\Phi : X \to X$ *with* $k > 1$*, then* X *is a singleton.*

Proof Fix arbitrary $\alpha, \beta \in X$. Repeated application of expansiveness gives

$$D(\Phi^m(\alpha), \Phi^m(\beta)) \geqslant D(\alpha, \beta)^{k^m} \qquad (m \geqslant 0).$$

Since $\Phi^m(\alpha), \Phi^m(\beta) \in X$, we have

$$D(\Phi^m(\alpha), \Phi^m(\beta)) \leqslant \operatorname{diam}(X).$$

Hence

$$D(\alpha, \beta)^{k^m} \leqslant \operatorname{diam}(X), \quad \forall m.$$

If $D(\alpha, \beta) > 1$, then since $k^m \to \infty$, the left-hand side diverges to infinity, which contradicts bounded diameter. Thus $D(\alpha, \beta) = 1$, hence $\alpha = \beta$. Since α, β were arbitrary, X is a singleton, completing the proof. ☐

Definition 6.5(D-bounded set) *A subset* $B \subset X$ *is called* D-bounded *if*

$$\sup_{u,v \in B} D(u, v) < \infty.$$

Theorem 6.6 *Let* Φ *be* k*-expansive with* $k > 1$*. If there exists a nonempty* D*-bounded set* B *such that* $\Phi(B) \subset B$*, then every point of* B *is a fixed point. Consequently,* B *is a singleton.*

Proof Fix $\gamma \in B$ and set $\delta = \Phi(\gamma)$. Define $\gamma_m = \Phi^m(\gamma)$. Since $\Phi(B) \subset B$, it follows that $\gamma_m \in B$ for all m.

Let

$$M := \sup_{u,v \in B} D(u, v) < \infty.$$

By repeated expansiveness,

$$D(\gamma_m, \gamma_{m+1}) = D(\Phi^m \gamma, \Phi^m \delta) \geqslant D(\gamma, \delta)^{k^m}.$$

Since $\gamma_m, \gamma_{m+1} \in B$, we must have

$$D(\gamma_m, \gamma_{m+1}) \leqslant M.$$

If $D(\gamma, \delta) > 1$, then $D(\gamma, \delta)^{k^m} \to \infty$, contradicting boundedness. Hence $D(\gamma, \delta) = 1$ and therefore

$$\Phi(\gamma) = \gamma.$$

Since γ was arbitrary in B, every point of B is a fixed point. By Lemma 6.2, there is at most one fixed point, hence B is a singleton. □

Theorem 6.7 *Assume D is continuous with respect to the multiplicative topology. If X is compact and $\Phi : X \to X$ is continuous and k-expansive for some $k > 1$, then X is a singleton.*

Proof Since X is compact, the product space $X \times X$ is compact. By continuity of D, the image

$$D(X \times X) \subset [1, \infty)$$

is compact and hence bounded. Therefore $\operatorname{diam}(X) < \infty$. The conclusion now follows directly from Theorem 6.4. □

Example 6.8 Let $X = \mathbb{R}$ and let $\mathcal{A}^*$ be defined as in Proposition 3.12 with $d(x, y) = |x - y|$. Then

$$D(\alpha, \beta) = \exp((n-1)|\alpha - \beta|).$$

Let $\Phi(\alpha) = c\alpha$ with $|c| \geqslant k > 1$. Then

$$\begin{aligned} D(\Phi\alpha, \Phi\beta) &= \exp((n-1)|c|\,|\alpha - \beta|) \\ &\geqslant \exp((n-1)k|\alpha - \beta|) = D(\alpha, \beta)^k, \end{aligned}$$

so Φ is k-expansive. The fixed point equation $c\xi = \xi$ gives $(c-1)\xi = 0$, hence the unique fixed point is $\xi = 0$.

Proposition 6.9 *For $\Phi(\alpha) = c\alpha$ with $|c| > 1$, the forward orbit $\{\Phi^m(\alpha)\} = \{c^m \alpha\}$ is D-bounded if and only if $\alpha = 0$.*

Proof If $\alpha = 0$, the orbit is constant and therefore bounded. If $\alpha \neq 0$, then $|c^m \alpha| \to \infty$ and hence

$$D(0, c^m \alpha) = \exp((n-1)|c^m \alpha|) \to \infty.$$

So the orbit is not D-bounded. □

§7. Applications

Throughout this section let

$$X := C([0,1]), \qquad \|u\|_\infty := \sup_{t\in[0,1]} |u(t)|.$$

Then $(X, \|\cdot\|_\infty)$ is a Banach space.

Fix $n \geqslant 3$ and define the multiplicative A-metric

$$\mathcal{A}^*(u_1,\ldots,u_n) := \exp\left(\sum_{1\leqslant i<j\leqslant n} \|u_i - u_j\|_\infty\right). \tag{1}$$

The induced two-point multiplicative distance is therefore

$$D(u,v) = \exp\big((n-1)\|u-v\|_\infty\big). \tag{2}$$

Lemma 7.1 *The multiplicative A-metric space* $(C([0,1]), \mathcal{A}^*)$ *is complete. Equivalently,* (X, D) *is multiplicatively complete.*

Proof Let $\{u_m\} \subset X$ be multiplicative A-Cauchy. By definition, for every $\varepsilon > 1$ there exists N such that

$$D(u_m, u_\ell) < \varepsilon \quad \text{for all } m, \ell \geqslant N.$$

Using (2), this is equivalent to

$$\exp((n-1)\|u_m - u_\ell\|_\infty) < \varepsilon,$$

which implies

$$\|u_m - u_\ell\|_\infty < \frac{\ln\varepsilon}{n-1}.$$

Hence $\{u_m\}$ is Cauchy in the Banach space $(X, \|\cdot\|_\infty)$, and therefore there exists $u \in X$ such that $u_m \to u$ uniformly. Passing back to the multiplicative distance,

$$D(u_m, u) = \exp((n-1)\|u_m - u\|_\infty) \to 1,$$

which shows multiplicative convergence. Hence $(X, \mathcal{A}^*)$ is complete. □

7.1 Nonlinear Volterra Integral Equation

Define the operator $\Phi : X \to X$ by

$$(\Phi u)(t) := g(t) + \int_0^t K(t,s,u(s))\,ds, \qquad t \in [0,1], \tag{3}$$

where $g \in C([0,1])$ and K is continuous and satisfies the Lipschitz condition

$$|K(t,s,x) - K(t,s,y)| \leqslant L|x-y|, \qquad L \in [0,1). \tag{4}$$

Theorem 7.2 *Under condition (4) with $L \in [0,1)$, the nonlinear Volterra equation*

$$u(t) = g(t) + \int_0^t K(t,s,u(s))\,ds$$

admits a unique solution $\xi \in C([0,1])$. Moreover, for every $u_0 \in C([0,1])$, the Picard sequence $u_{m+1} = \Phi(u_m)$ converges to ξ in the multiplicative A-sense and hence uniformly.

Proof Let $u, v \in X$. Using (3) and (4),

$$|(\Phi u)(t) - (\Phi v)(t)| \leqslant \int_0^t L|u(s) - v(s)|\,ds \leqslant L\|u-v\|_\infty.$$

Taking supremum over $t \in [0,1]$ yields

$$\|\Phi u - \Phi v\|_\infty \leqslant L\|u-v\|_\infty.$$

Using (2),

$$D(\Phi u, \Phi v) = \exp((n-1)\|\Phi u - \Phi v\|_\infty) \leqslant \exp((n-1)L\|u-v\|_\infty) = D(u,v)^L.$$

Since $L \in (0,1)$, Φ is multiplicative Banach-type contractive. Completeness follows from Lemma 7.1, hence Theorem 5.2 yields existence, uniqueness, and multiplicative convergence.□

7.2 Second-order Boundary Value Problem via Green Operator

Consider the boundary value problem

$$u''(t) = F(t,u(t)), \quad u(0) = \alpha, \quad u(1) = \beta,$$

where F is continuous and satisfies

$$|F(t,x) - F(t,y)| \leqslant L|x-y|.$$

Let the Green kernel be

$$G(t,s) = \begin{cases} s(1-t), & s \leqslant t, \\ t(1-s), & t \leqslant s, \end{cases}$$

which satisfies

$$\sup_{t\in[0,1]} \int_0^1 G(t,s)\,ds = \frac{1}{8}.$$

Define

$$(\Phi u)(t) = \alpha(1-t) + \beta t + \int_0^1 G(t,s)F(s,u(s))ds.$$

Theorem 7.3 *If $L < 8$, then the boundary value problem admits a unique solution $\xi \in C^2([0,1])$. Moreover, Picard iteration converges multiplicatively to ξ.*

Proof For $u, v \in X$,

$$\|\Phi u - \Phi v\|_\infty \leqslant \sup_t \int_0^1 G(t,s)L|u(s)-v(s)|ds \leqslant \frac{L}{8}\|u-v\|_\infty.$$

Hence

$$D(\Phi u, \Phi v) \leqslant D(u,v)^{L/8}.$$

Since $L/8 < 1$, Φ is multiplicative Banach-type contractive. Existence and uniqueness follow. Standard elliptic regularity of Green operators implies $\xi \in C^2([0,1])$. □

7.3 Expansive Linear Operator and Bounded Orbits

Let $(E, \|\cdot\|)$ be a Banach space and define

$$D(x,y) = \exp((n-1)\|x-y\|).$$

Assume $\Phi : E \to E$ is linear and satisfies

$$\|\Phi x - \Phi y\| \geqslant c\|x-y\|, \quad c > 1.$$

Theorem 7.4 *Φ is k-expansive with $k = c$. The only point having a D-bounded forward orbit is 0, and 0 is the unique fixed point.*

Proof Using monotonicity of the exponential function,

$$D(\Phi x, \Phi y) = \exp((n-1)\|\Phi x - \Phi y\|) \geqslant \exp((n-1)c\|x-y\|) = D(x,y)^c.$$

Hence Φ is k-expansive.

Taking $y = 0$ gives $\|\Phi x\| \geqslant c\|x\|$. Iterating yields $\|\Phi^m x\| \geqslant c^m\|x\|$. If $x \neq 0$, then $\|\Phi^m x\| \to \infty$ and therefore $D(0, \Phi^m x) \to \infty$. Thus only $x = 0$ has bounded orbit. Fixed point uniqueness follows from expansiveness. □

§8. Conclusion

In this work, we introduced the framework of multiplicative A-metric spaces (MA-spaces), which provides a natural synthesis of the multi-point structure of A-metrics and the product-type geometry inherent in multiplicative distance models. This construction allows distances to be measured simultaneously across multiple points while preserving the multiplicative structure that frequently arises in nonlinear growth processes, stability theory, and exponential-type transformations.

Within this setting, we developed the fundamental topological structure induced by the multiplicative distance, including multiplicative open balls, neighborhood systems, and the associated multiplicative topology. We established basic sequential properties of MA-spaces, including characterization of convergence via the induced two-point multiplicative distance, uniqueness of limits, and Hausdorff separation. These results confirm that MA-spaces provide a robust analytical environment comparable to classical metric frameworks while offering additional structural flexibility.

On the analytical side, we established fixed point results for multiplicative contractive mappings, including Banach-type and Kannan-type principles. These theorems extend classical fixed point theory to a multi-point multiplicative setting and demonstrate that iterative schemes converge under natural multiplicative contractive conditions. In contrast, for strictly expansive mappings, we identified strong rigidity phenomena, showing that the presence of bounded invariant sets or finite multiplicative diameter forces structural triviality of the space. These results highlight a fundamental dichotomy between multiplicative contraction and expansion behaviors in MA-spaces.

To demonstrate applicability, we constructed a natural exponential embedding of MA-geometry into function spaces, particularly $C([0,1])$ equipped with the supremum norm. Using this embedding, we established existence and uniqueness results for nonlinear Volterra integral equations and second-order boundary value problems via Green operator methods. These applications show that MA-spaces provide an effective analytical framework for studying nonlinear functional equations in infinite-dimensional settings.

Overall, multiplicative A-metric spaces offer a unified platform connecting multi-point distance geometry, multiplicative analysis, and nonlinear fixed point theory. The framework opens several directions for further investigation, including extensions to fractional operators, stochastic functional equations, and generalized multi-point multiplicative structures.

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International J.Math. Combin. Vol.1(2026), 66-74

Ξ-cyclic Codes over $\mathbb{M}^l$ and Their Applications

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Abstract: In this article, for $\Xi \in Aut(\mathbb{M}^l)$, we give an algebraic characterization of a Ξ-cyclic code over $\mathbb{M}^l$, where $\mathbb{M} = \mathbb{F}_q + v\mathbb{F}_q$ with $v^2 = v$, $\mathbb{F}_q$ is a finite field with q elements, $q = p^m$ for prime p, positive integer m and $l \in \mathbb{N}$. We determine its generator polynomial and find its decomposition over $\mathbb{F}_q$. A necessary and sufficient condition for a Ξ-cyclic code over $\mathbb{M}^l$ to be Euclidean dual containing is given. We define an orthogonality preserving Gray map. By using CSS construction, we have the parameters of quantum codes from Ξ-cyclic codes over $\mathbb{M}^l$.

Key Words: Linear code, cyclic code, quantum code, generator polynomial, Gray map.

AMS(2010): 94B15.

§1. Introduction

As quantum error correcting codes protect quantum information, getting optimal quantum error correcting codes is also important in quantum communication and quantum computation. The first quantum codes were introduced by Shor and independently Steane in [7], [8]. Some authors constructed quantum codes by using the connection between classical error correcting codes and quantum codes [2]. In [1], Bhagat and Sarma studied $(\Theta, \Delta_\Theta, \mathbf{a})$ -cyclic codes on $R = \mathbb{F}_q^l$, for an automorphism Θ of R, a $\Theta-$ derivation Δ_Θ of R and $\mathbf{a} \in R^*$. They obtained quantum error correcting codes from them, by using CSS constructions.

In this article, we study Ξ -cyclic codes on $\mathbb{M}^l$, where $\mathbb{M} = \mathbb{F}_q + v\mathbb{F}_q$ with $v^2 = v$ and $l \geq 1$ constructed using an automorphism. It is worth mentioning that the automorphism class that we consider is much larger than what is considered in previous work in [5]. We take a more general form of an automorphism of $\mathbb{M}^l$, namely $\xi_1 \times \xi_2 ... \times \xi_l$, where each ξ_i is an automorphism of $\mathbb{M}$, for $i = 1, \cdots, l$. Later, motivated by the previous work in [1], we obtain the parameters of quantum codes from Ξ-cyclic codes over $\mathbb{M}^l, l \geq 1$.

The paper is organized as follows. Section 2 gives essential preliminaries on skew cyclic codes over $\mathbb{F}_q$ and $\mathbb{F}_q^l$. Section 3 investigates linear codes over $\mathbb{M}^l$, where $\mathbb{M} = \mathbb{F}_q + v\mathbb{F}_q$ with $v^2 = v$, $l \geq 1$. In section 4, we investigate Ξ-cyclic codes over $\mathbb{M}^l$ and establish a decomposition of Ξ-cyclic codes over $\mathbb{M}^l$. We find generator polynomial of Ξ-cyclic codes over $\mathbb{M}^l$. In section 5,

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[1] Received October 17, 2025, Accepted February 16, 2026

we define orthogonality preserving Gray map from $(\mathbb{M}^l)^s$ to $(\mathbb{F}_q)^{2sl}$ and we construct quantum codes from Ξ-cyclic codes over $\mathbb{M}^l$ using CSS construction.

§2. **Preliminaries**

The set $\mathbb{F}_q^n = \{(r_1, \cdots, r_n) | r_u \in \mathbb{F}_q, u = 1, \cdots, n\}$ is a vector space over $\mathbb{F}_q$ with the usual component-wise addition and multiplication by scalars, where $\mathbb{F}_q$ is a finite field with $q = p^m$ elements, for prime p and positive integer m. A code $C_{\mathbb{F}_q}$ of length n over $\mathbb{F}_q$ is non-empty subset of $\mathbb{F}_q^n$ and a code $C_{\mathbb{F}_q}$ is a linear code over $\mathbb{F}_q$, if it is a subspace of $\mathbb{F}_q^n$. Let $\mathbf{c} = (c_1, \cdots, c_n) \in C_{\mathbb{F}_q}$, then the Hamming weight of $\mathbf{c}$ is defined as the number of non-zero components of $\mathbf{c}$ and denoted by $w_H(\mathbf{c})$. The Hamming distance between two codewords $\mathbf{c}, \mathbf{c}^{'} \in C_{\mathbb{F}_q}$ is given by $d_H(\mathbf{c}, \mathbf{c}^{'}) = w_H(\mathbf{c} - \mathbf{c}^{'})$. The minimum distance of $C_{\mathbb{F}_q}$ is defined as $d_H(C_{\mathbb{F}_q}) = min\{d_H(\mathbf{c}, \mathbf{c}^{'}) | \mathbf{c} \neq \mathbf{c}^{'}, \forall \mathbf{c}, \mathbf{c}^{'} \in C_{\mathbb{F}_q}\}$.

In [6], it was stated that the distinct automorphisms of $\mathbb{F}_{p^m}$ over $\mathbb{F}_p$ are exactly the mapping $\theta_1, \cdots, \theta_{m-1}$ defined by $\theta_z(\alpha) = \alpha^{p^z}$ for $\alpha \in \mathbb{F}_{p^m}^*$ and $0 \leq z \leq m-1$. The automorphisms of $\mathbb{F}_{p^m}$ over $\mathbb{F}_p$ construct a cyclic group of order m generated by θ_1.

In [1], Bhagat and Sarma investigated $(\theta, \delta_\theta, \alpha)$ -cyclic codes on $\mathbb{F}_q$, where $\theta \in Aut(\mathbb{F}_q)$, δ_θ is a θ-derivation, $\alpha \in \mathbb{F}_q^*$. By taking $\delta_\theta = 0, \alpha = 1$ in Section 4, [1], we can write the followings.

An $\mathbb{F}_q$-subspace $C_{\mathbb{F}_q}$ of $\mathbb{F}_q^n$ is called θ-cyclic code of length n over $\mathbb{F}_q$ if $T_{\theta, S_{\mathbb{F}_q}}(C_{\mathbb{F}_q}) \subseteq C_{\mathbb{F}_q}$, where $T_{\theta, S_{\mathbb{F}_q}}$ is a map of the form $T_{\theta, S_{\mathbb{F}_q}}(\mathbf{c}) = \theta(\mathbf{c}) S_{\mathbb{F}_q}$, where $\mathbf{c} = (c_0, \cdots, c_{n-1}) \in \mathbb{F}_q$, $\theta \in Aut(\mathbb{F}_q)$, $\theta(\mathbf{c}) = (\theta(c_0), \cdots, \theta(c_{n-1}))$ and

$$S_{\mathbb{F}_q} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 1 & 0 & \dots & 0 \end{pmatrix} \in M_{n \times n}(\mathbb{F}_q)$$

For the polynomial representation,

$$\begin{aligned} \Lambda_{\mathbb{F}_q} \quad &: \quad \mathbb{F}_q^n \longrightarrow \mathbb{F}_q[x, \theta] / < x^n - 1 > \\ \mathbf{c} \quad &= \quad (c_0, \cdots, c_{n-1}) \longmapsto c(x) = c_0 + c_1 x + ... + c_{n-1} x^{n-1} \end{aligned}$$

Let $C_{\mathbb{F}_q}$ be a subset of $\mathbb{F}_q^n$. Then $C_{\mathbb{F}_q}$ is a θ-cyclic code of length n over $\mathbb{F}_q$ if and only if $\Lambda_{\mathbb{F}_q}(C_{\mathbb{F}_q})$ is a left $\mathbb{F}_q[x, \theta]$-submodule of $\mathbb{F}_q[x, \theta] / < x^n - 1 >$. Moreover $\Lambda_{\mathbb{F}_q}(C_{\mathbb{F}_q})$ is generated by a unique monic polynomial $g(x) \in \mathbb{F}_q[x, \theta]$ and $g(x)$ is a right divisor of $x^n - 1$ in $\mathbb{F}_q[x, \theta]$.

In [1], by taking a more general form automorphism Θ of $\mathbb{F}_q^l$, they investigated $(\Theta, \Delta_\Theta, \mathbf{a})$-cyclic codes over $\mathbb{F}_q^l$, where Δ_Θ is a Θ-derivation and $\mathbf{a} \in (\mathbb{F}_q^l)^*, l \in \mathbb{N}$. In [1], the automorphisms of $\mathbb{F}_q^l$ were defined as

$$\begin{aligned} \theta_1 \times \cdots \times \theta_l \quad &: \quad \mathbb{F}_q^l \longrightarrow \mathbb{F}_q^l \\ (r_1, \cdots, r_l) \quad &\longmapsto \quad (\theta_1(r_1), \cdots, \theta_l(r_l)) \end{aligned}$$

where $\theta_i \in Aut(\mathbb{F}_q)$, for $i = 1, \cdots, l$. The set of automorphisms of $\mathbb{F}_q^l$ like that was presented as

$$\Omega_{\mathbb{F}_q^l} = \{\theta_1 \times \cdots \times \theta_l | \theta_i \in Aut(\mathbb{F}_q)\} \subset Aut(\mathbb{F}_q^l)$$

By taking $\Delta_\Theta = 0, \mathbf{a} = \mathbf{1}$, we can express the Definition 4.6 and Theorem 4.8 in [1], as follows;

Let $\Theta \in \Omega_{\mathbb{F}_q^l}$. An $\mathbb{F}_q^l$-submodule $C_{\mathbb{F}_q^l}$ of $(\mathbb{F}_q^l)^s$ is called Θ-cyclic code of length s over $\mathbb{F}_q^l$ if $T_{\Theta, \boldsymbol{S}_{\mathbb{F}_q^l}}(C_{\mathbb{F}_q^l}) \subseteq C_{\mathbb{F}_q^l}$, where $T_{\Theta, \boldsymbol{S}_{\mathbb{F}_q^l}}$ is a map of the form $T_{\Theta, \boldsymbol{S}_{\mathbb{F}_q^l}}(\mathbf{d}) = \Theta(\mathbf{d})\boldsymbol{S}_{\mathbb{F}_q^l}$ and

$$\begin{aligned} \mathbf{d} &= (\mathbf{d_0}, \cdots, \mathbf{d_{s-1}}) \in (\mathbb{F}_q^l)^s, \quad \Theta(\mathbf{d}) = (\Theta(\mathbf{d_0}), \cdots, \Theta(\mathbf{d_{s-1}})), \\ \mathbf{d_t} &= (d_{t,1}, \cdots, d_{t,l}), \quad \Theta(\mathbf{d_t}) = (\theta_1(d_{t,1}), \theta_2(d_{t,2}), \cdots, \theta_l(d_{t,l})) \end{aligned}$$

for $t = 0, \cdots, s-1$ and

$$\boldsymbol{S}_{\mathbb{F}_q^l} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \dots \\ \vdots & \vdots & \vdots & \ddots \\ \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix} \in M_{s \times s}(\mathbb{F}_q^l)$$

where $\mathbf{1} = (1, \cdots, 1), \mathbf{0} = (0, \cdots, 0) \in \mathbb{F}_q^l$.

For the polynomial representation,

$$\begin{aligned} \Lambda_{\mathbb{F}_q^l} &: (\mathbb{F}_q^l)^s \longrightarrow \mathbb{F}_q^l[x, \Theta]/ < x^s - \mathbf{1} > \\ \mathbf{d} &= (\mathbf{d_0}, \cdots, \mathbf{d_{s-1}}) \longmapsto d(x) = \mathbf{d_0} + \mathbf{d_1}x + \cdots + \mathbf{d_{s-1}}x^{s-1} \end{aligned}$$

Let $C_{\mathbb{F}_q^l}$ be a subset of $(\mathbb{F}_q^l)^s$. Then $C_{\mathbb{F}_q^l}$ is a Θ-cyclic code of length s over $\mathbb{F}_q^l$ if and only if $\Lambda_{\mathbb{F}_q^l}(C_{\mathbb{F}_q^l})$ is a left $\mathbb{F}_q^l[x, \Theta]$-submodule of $\mathbb{F}_q^l[x, \Theta]/\langle x^s - \mathbf{1}\rangle$.

In [1], a linear code $C_{\mathbb{F}_q^l}$ of length s over $\mathbb{F}_q^l$ was uniquely written as

$$C_{\mathbb{F}_q^l} = \mathbf{e_1}C_{\mathbb{F}_q,1} \oplus \cdots \oplus \mathbf{e}_l C_{\mathbb{F}_q,l} \tag{$*$}$$

where

$$C_{\mathbb{F}_q,i} = \widetilde{\Pi_i}(C_{\mathbb{F}_q^l}) = \{(\Pi_i(\mathbf{d_0}), \cdots, \Pi_i(\mathbf{d_{s-1}})) \in (\mathbb{F}_q)^s | (\mathbf{d_0}, \cdots, \mathbf{d_{s-1}}) \in C_{\mathbb{F}_q^l}\}$$

are linear codes of length s over $\mathbb{F}_q$,

$$\begin{aligned} \widetilde{\Pi_i} &: (\mathbb{F}_q^l)^s \longrightarrow \mathbb{F}_q^s, \\ (\mathbf{r_0}, \cdots, \mathbf{r_{s-1}}) &\longmapsto (\Pi_i(\mathbf{r_0}), \cdots, \Pi_i(\mathbf{r_{s-1}})), \\ \Pi_i &: \mathbb{F}_q^l \longrightarrow \mathbb{F}_q, \\ \mathbf{r_t} = (r_{t,1}, \cdots, r_{t,l}) &\longmapsto r_{t,i} \end{aligned}$$

where $t = 0, 1, \cdots, s-1$, for $i = 1, \cdots, l$ and $\mathbf{e_1}, \cdots, \mathbf{e}_l$ standart ordered $\mathbb{F}_q$-basis for $\mathbb{F}_q^l$.

§3. Linear Codes over $\mathbb{M}^l$

For $l \in \mathbb{N}$, consider the product ring $\mathbb{M}^l$, where $\mathbb{M} = \mathbb{F}_q + v\mathbb{F}_q, v^2 = v$. Let

$$\begin{aligned} \mathbf{\Pi_1} \ &: \ \mathbb{M}^l \longrightarrow \mathbb{F}_q^l, \\ &\quad (k_1 + vn_1, k_2 + vn_2, \cdots, k_l + vn_l) \longmapsto (k_1, k_2 \cdots, k_l), \\ \mathbf{\Pi_2} \ &: \ \mathbb{M}^l \longrightarrow \mathbb{F}_q^l \\ &\quad (k_1 + vn_1, k_2 + vn_2, \cdots, k_l + vn_l) \longmapsto (k_1 + n_1, k_2 + n_2, \cdots, k_l + n_l). \end{aligned}$$

Extend each $\mathbf{\Pi_j}$ to $\widetilde{\mathbf{\Pi_j}}$ for $j = 1, 2$ as follows:

$$\begin{aligned} \widetilde{\mathbf{\Pi_j}} \ &: \ (\mathbb{M}^l)^s \longrightarrow (\mathbb{F}_q^l)^s, \\ &\quad (\mathbf{m}_0, \cdots, \mathbf{m}_{s-1}) \longmapsto (\mathbf{\Pi_j}(\mathbf{m_0}), \cdots, \mathbf{\Pi_j}(\mathbf{m_{s-1}})) \end{aligned}$$

where $\mathbf{m_t} = (m_{t,1}, \cdots, m_{t,l}) = (k_1^t + vn_1^t, \cdots, k_l^t + vn_l^t) \in \mathbb{M}^l, t = 0, \cdots, s-1$, for $j = 1, 2$.

Let $\boldsymbol{C}_{\mathbb{M}^l}$ be a linear code of length s over $\mathbb{M}^l$. For $j = 1, 2$, define

$$C_{j,\mathbb{F}_q^l} = \mathbf{\Pi}_j(\boldsymbol{C}_{\mathbb{M}^l}) = \widetilde{\{(\mathbf{\Pi}_j}(p_0), \cdots, \Pi_j(p_{s-1})) \in (\mathbb{F}_q^l)^s | (\mathbf{p}_0, \cdots, \mathbf{p}_{s-1}) \in \boldsymbol{C}_{\mathbb{M}^l}\}$$

then each $C_{j,\mathbb{F}_q^l}$ is a linear code of length s over $\mathbb{F}_q^l$, for $j = 1, 2$ and every linear code $\boldsymbol{C}_{\mathbb{M}^l}$ can be uniquely written as

$$\boldsymbol{C}_{\mathbb{M}^l} = (1-v)C_{1,\mathbb{F}_q^l} \oplus vC_{2,\mathbb{F}_q^l}$$

From (*), we have

$$\boldsymbol{C}_{\mathbb{M}^l} = (1-v)[\mathbf{e_1}C_{1,\mathbb{F}_q,1} \oplus \cdots \oplus \mathbf{e}_l C_{1,\mathbb{F}_q,l}] \oplus v[\mathbf{e_1}C_{2,\mathbb{F}_q,1} \oplus ... \oplus \mathbf{e}_l C_{2,\mathbb{F}_q,l}]$$

where $C_{1,\mathbb{F}_q,i}, C_{2,\mathbb{F}_q,i}$ are linear codes of length s over $\mathbb{F}_q$, for $i = 1, \cdots, l$.

§4. Ξ-Cyclic Codes over $\mathbb{M}^l$

The map

$$\begin{aligned} \xi_1 \times \xi_2 \times ... \times \xi_l \ &: \ \mathbb{M}^l \longrightarrow \mathbb{M}^l \\ &\quad (k_1 + vn_1, \cdots, k_l + vn_l) \longmapsto (\xi_1(k_1 + vn_1), \cdots, \xi_l(k_l + vn_l)) \end{aligned}$$

is an automorphism of $\mathbb{M}^l$ where $\xi_i \in Aut(\mathbb{M})$, $\xi_i(k_i + vn_i) = \theta_i(k_i) + v\theta_i(n_i)$, $\theta_i \in Aut(\mathbb{F}_q)$ for $i = 1, \cdots, l$.

The set of automorphisms of $\mathbb{M}^l$ like that is presented as

$$\Omega_{\mathbb{M}^l} = \{\xi_1 \times \cdots \times \xi_l | \xi_i \in Aut(\mathbb{M})\} \subset Aut(\mathbb{M}^l)$$

Let $\Xi \in \Omega_{\mathbb{M}^l}$. An $\mathbb{M}^l$-submodule $\boldsymbol{C}_{\mathbb{M}^l}$ of $(\mathbb{M}^l)^s$ is called Ξ-cyclic code of length s over $\mathbb{M}^l$ if $T_{\Xi,\boldsymbol{S}_{\mathbb{M}^l}}(\boldsymbol{C}_{\mathbb{M}^l}) \subseteq \boldsymbol{C}_{\mathbb{M}^l}$, where $T_{\Xi,\boldsymbol{S}_{\mathbb{M}^l}}$ is a map of the form $T_{\Xi,\boldsymbol{S}_{\mathbb{M}^l}}(\mathbf{p}) = \Xi(\mathbf{p})\boldsymbol{S}_{\mathbb{M}^l}, \mathbf{p} = (\mathbf{p}_0,\cdots,\mathbf{p}_{s-1}) \in (\mathbb{M}^l)^s$, $\Xi(\mathbf{p}) = (\Xi(\mathbf{p}_0),\cdots,\Xi(\mathbf{p}_{s-1}))$, $\mathbf{p_t} = (\mathbf{p}_{t,1},\cdots,\mathbf{p}_{t,l}) = (a_1^t+vb_1^t,\cdots,a_l^t+vb_l^t)$, $\Xi(\mathbf{p_t}) = (\xi_1(p_{t,1}),\xi_2(p_{t,2}),\cdots,\xi_l(p_{t,l})) = (\theta_1(a_1^t)+v\theta_1(b_1^t),\cdots,\theta_l(a_l^t)+v\theta_l(b_l^t))$ for $t = 0,\cdots,s-1$ and

$$\boldsymbol{S}_{\mathbb{M}^l} = \begin{pmatrix} \mathbf{0} & \mathbf{1} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ \mathbf{1} & \mathbf{0} & \ldots & \mathbf{0} \end{pmatrix} \in M_{s\times s}(\mathbb{M}^l)$$

where $\mathbf{1} = (1,\cdots,1), \mathbf{0} = (0,\cdots,0) \in \mathbb{M}^l$.

$$\begin{aligned} \Lambda_{\mathbb{M}^l} \quad &: \quad (\mathbb{M}^l)^s \longrightarrow \mathbb{M}^l[x,\Xi]/<x^s-\mathbf{1}> \\ \mathbf{p} \quad &= \quad (\mathbf{p_0},\cdots,\mathbf{p_{s-1}}) \longmapsto p(x) = \mathbf{p_0}+\mathbf{p_1}x+\cdots+\mathbf{p_{s-1}}x^{s-1} \end{aligned}$$

Let $\boldsymbol{C}_{\mathbb{M}^l}$ be a subset of $(\mathbb{M}^l)^s$. Then $\boldsymbol{C}_{\mathbb{M}^l}$ is a Ξ-cyclic code of length s over $\mathbb{M}^l$ if and only if $\Lambda_{\mathbb{M}^l}(\boldsymbol{C}_{\mathbb{M}^l})$ is a left $\mathbb{M}^l[x,\Xi]$-submodule of $\mathbb{M}^l[x,\Xi]/<x^s-\mathbf{1}>$.

Theorem 4.1 *Let* $\boldsymbol{C}_{\mathbb{M}^l} = (1-v)C_{1,\mathbb{F}_q^l} \oplus vC_{2,\mathbb{F}_q^l}$ *be a linear code of length* s *over* $\mathbb{M}^l$*, where* $C_{1,\mathbb{F}_q^l}, C_{2,\mathbb{F}_q^l}$ *are linear codes of length* s *over* $\mathbb{F}_q^l$*. Then* $\boldsymbol{C}_{\mathbb{M}^l}$ *is a* Ξ*-cyclic code over* $\mathbb{M}^l$ *if and only if each* $C_{1,\mathbb{F}_q^l}, C_{2,\mathbb{F}_q^l}$ *are* Θ*-cyclic codes of length* s *over* $\mathbb{F}_q^l$*, where* $\Xi = \xi_1\times\cdots\times\xi_l \in \Omega_{\mathbb{M}^l}$*,* $\xi_i(a_i+vb_i) = \theta_i(a_i)+v\theta_i(b_i)$ *for every* $a_i+vb_i \in \mathbb{M}$*, for* $i = 1,\cdots,l$ *and* $\Theta = \theta_1\times\cdots\times\theta_l \in \Omega_{\mathbb{F}_q^l}$*,* $\Theta(\mathbf{r}) = (\theta_1(r_1),\cdots,\theta_l(r_l))$ *for every* $\mathbf{r} = (r_1,\cdots,r_l) \in \mathbb{F}_q^l$*.*

Proof Let $\boldsymbol{C}_{\mathbb{M}^l}$ be a Ξ-cyclic code over $\mathbb{M}^l$. Let $\mathbf{\Pi}_j(\mathbf{p}_0),\cdots,\mathbf{\Pi}_j(\mathbf{p}_{s-1})) \in C_{j,\mathbb{F}_q^l}$, for $j = 1,2$ where $\mathbf{p} = (\mathbf{p}_0,\cdots,\mathbf{p}_{s-1}) \in \boldsymbol{C}_{\mathbb{M}^l}$. Let

$$\begin{aligned} \mathbf{p_t} \quad &= \quad (p_{t,1},\cdots,p_{t,l}) = (a_1^t+vb_1^t,\cdots,a_l^t+vb_l^t), \\ \Xi(\mathbf{p_t}) \quad &= \quad (\xi_1(p_{t,1}),\xi_2(p_{t,2}),\cdots,\xi_l(p_{t,l})) = (\theta_1(a_1^t)+v\theta_1(b_1^t),\cdots,\theta_l(a_l^t)+v\theta_l(b_l^t)) \end{aligned}$$

for $t = 0,\cdots,s-1$. Since $\boldsymbol{C}_{\mathbb{M}^l}$ is a Ξ-cyclic code over $\mathbb{M}^l$, then

$$\begin{aligned} T_{\Xi,\boldsymbol{S}_{\mathbb{M}^l}}(\mathbf{p}) \quad &= \quad \Xi(\mathbf{p})\boldsymbol{S}_{\mathbb{M}^l} \\ &= \quad (\mathbf{1}\Xi(\mathbf{p}_{s-1}),\Xi(\mathbf{p}_0),\cdots,\Xi(\mathbf{p}_{s-2})) = ((\theta_1(a_1^{s-1}) \\ &\qquad +v\theta_1(b_1^{s-1}),\theta_2(a_2^{s-1})+v\theta_2(b_2^{s-1}),\cdots,\theta_l(a_l^{s-1}) \\ &\qquad +v\theta_l(b_l^{s-1})),(\theta_1(a_1^0)+v\theta_1(b_1^0),\theta_2(a_2^0) \\ &\qquad +v\theta_2(b_2^0)),\cdots,\theta_l(a_l^0)+v\theta_l(b_l^0)),\cdots,(\theta_1(a_1^{s-2}) \\ &\qquad +v\theta_1(b_1^{s-2}),\theta_2(a_2^{s-2})+v\theta_2(b_2^{s-2}),\cdots,\theta_l(a_l^{s-2})+v\theta_l(b_l^{s-2})) \\ &= \quad ((\theta_1(a_1^{s-1}),\theta_2(a_2^{s-1}),\cdots,\theta_l(a_l^{s-1})) \\ &\qquad +v(\theta_1(b_1^{s-1}),\theta_2(b_2^{s-1}),\cdots,\theta_l(b_l^{s-1})),(\theta_1(a_1^0),\theta_2(a_2^0),\cdots,\theta_l(a_l^0)) \\ &\qquad +v(\theta_1(b_1^0),\theta_2(b_2^0),\cdots,\theta_l(b_l^0)),\cdots,(\theta_1(a_1^{s-2}),\theta_2(a_2^{s-2}),\cdots,\theta_l(a_l^{s-2})) \end{aligned}$$

$$\begin{aligned}
& +v(\theta_1(b_1^{s-2}),\theta_2(b_2^{s-2}),\cdots,\theta_l(b_l^{s-2}))\\
= \ & ((\theta_1(a_1^{s-1}),\theta_2(a_2^{s-1}),\cdots,\theta_l(a_l^{s-1})),(\theta_1(a_1^{0}),\theta_2(a_2^{0}),\\
& \cdots,\theta_l(a_l^{0})),\cdots,(\theta_1(a_1^{s-2}),\theta_2(a_2^{s-2}),\cdots,\theta_l(a_l^{s-2})))\\
& +v((\theta_1(b_1^{s-1}),\theta_2(b_2^{s-1}),\cdots,\theta_l(b_l^{s-1})),(\theta_1(b_1^{0}),\theta_2(b_2^{0}),\cdots,\theta_l(b_l^{0})),\cdots,\\
& (\theta_1(b_1^{s-2}),\theta_2(b_2^{s-2}),\cdots,\theta_l(b_l^{s-2})))\\
= \ & (1-v)((\theta_1(a_1^{s-1}),\cdots,\theta_l(a_l^{s-1})),\cdots,(\theta_1(a_1^{s-2}),\cdots,\theta_l(a_l^{s-2}))\\
& +v((\theta_1(a_1^{s-1})+\theta_1(b_1^{s-1}),\cdots,\theta_l(a_l^{s-1})+\theta_1(b_1^{s-1})),\cdots,(\theta_1(a_1^{s-2})\\
& +\theta_1(b_1^{s-2}),\cdots,\theta_l(a_l^{s-2})+\theta_l(b_l^{s-2}))\in \boldsymbol{C}_{\mathbb{M}^l}.
\end{aligned}$$

So for every $(\boldsymbol{\Pi_j}(\mathbf{p}_0),\cdots,\boldsymbol{\Pi_j}(\mathbf{p}_{s-1}))\in C_{j,\mathbb{F}_q^l}$, we have

$$(\Theta(\boldsymbol{\Pi_j}(\mathbf{p}_{s-1})),\Theta(\boldsymbol{\Pi_j}(\mathbf{p}_0)),\cdots,\Theta(\boldsymbol{\Pi_j}(\mathbf{p}_{s-2})))\in C_{j,\mathbb{F}_q^l}$$

for $j=1,2$. Therefore $C_{j,\mathbb{F}_q^l}$ are Θ-cyclic codes over $\mathbb{F}_q^l$, for $j=1,2$. The other way can be easily seen that. □

By using Theorem 4.10 in [1], we can obtain:

Corollary 4.2 *Let* $\boldsymbol{C}_{\mathbb{M}^l}=(1-v)[\mathbf{e_1}C_{1,\mathbb{F}_q,1}\oplus\cdots\oplus\mathbf{e}_lC_{1,\mathbb{F}_q,l}]\oplus v[\mathbf{e_1}C_{2,\mathbb{F}_q,1}\oplus ...\oplus\mathbf{e}_lC_{2,\mathbb{F}_q,l}]$ *be a linear code of length* s *over* $\mathbb{M}^l$. *Then* $\boldsymbol{C}_{\mathbb{M}^l}$ *is a* Ξ*-cyclic code if and only if each* $C_{j,\mathbb{F}_q,i}$ *is a* θ_i*-cyclic code for* $j=1,2$ *and* $i=1,\cdots,l$.

Theorem 4.3 *Let* $\boldsymbol{C}_{\mathbb{M}^l}=(1-v)C_{1,\mathbb{F}_q^l}\oplus vC_{2,\mathbb{F}_q^l}$ *be a* Ξ*-cyclic code of length* s *over* $\mathbb{M}^l$, *where* $C_{j,\mathbb{F}_q^l}$ *is a linear code of length* s *over* $\mathbb{F}_q^l$, *for* $j=1,2$. *Then there exists* $g_{j,i}(x)\in\mathbb{F}_q[x,\theta_i]$, *for* $j=1,2$ *and* $i=1,\cdots,l$ *such that* $\boldsymbol{C}_{\mathbb{M}^l}=<g(x)>=<(1-v)g_1(x)+vg_2(x)>$, *where* $g_j(x)=(g_{j,1}(x),g_{j,2}(x),\cdots,g_{j,l}(x))$, *for* $j=1,2$. *Moreover* $g(x)$ *is a right divisor of* $x^s-\mathbf{1}$ *in* $\mathbb{M}^l[x,\Xi]$.

Proof It is easily seen from the proof of Theorem 4.11 in [1]. □

§5. Gray Map

We define the Gray map by

$$\begin{aligned}
\boldsymbol{\Psi} \quad &: \quad (\mathbb{M}^l)^s\longrightarrow(\mathbb{F}_q^l)^{2s}\\
\mathbf{p}=(\mathbf{p}_0,\cdots,\mathbf{p}_{s-1}) \quad &\longmapsto \quad (\boldsymbol{\Pi}_1(\mathbf{p}_0),\cdots,\boldsymbol{\Pi}_1(\mathbf{p}_{s-1}),\boldsymbol{\Pi}_2(\mathbf{p}_0),\cdots,\boldsymbol{\Pi}_2(\mathbf{p}_{s-1}))
\end{aligned}$$

where $\mathbf{p_t}=(a_1^t+vb_1^t,\cdots,a_l^t+vb_l^t)$, for $t=0,\cdots,s-1$ and

$$\begin{aligned}
\Psi \quad : \quad & (\mathbb{F}_q^l)^{2s}\longrightarrow\mathbb{F}_q^{2sl}\\
& (\boldsymbol{\Pi}_1(\mathbf{p}_0),\cdots,\boldsymbol{\Pi}_1(\mathbf{p}_{s-1}),\boldsymbol{\Pi}_2(\mathbf{p}_0),\cdots,\boldsymbol{\Pi}_2(\mathbf{p}_{s-1}))\longmapsto o
\end{aligned}$$

where $o=(\boldsymbol{\Pi}_1(\mathbf{p}_0)S,\cdots,\boldsymbol{\Pi}_1(\mathbf{p}_{s-1})S,\boldsymbol{\Pi}_2(\mathbf{p}_0)S,\cdots,\boldsymbol{\Pi}_2(\mathbf{p}_{s-1}S)$ and $S\in GL(l,\mathbb{F}_q)$.

The Gray weight of an element $\mathbf{p} \in (\mathbb{M}^l)^s$ is defined as

$$w_G(\mathbf{p}) = \sum_{t=0}^{s-1}\sum_{j=1}^{2} w_H(\mathbf{\Pi}_j(\mathbf{p}_t)S)$$

For any two distinct codewords $\mathbf{p}, \mathbf{p}^{'} \in \boldsymbol{C}_{\mathbb{M}^l}$, the Gray distance is defined as $d_G(\mathbf{p}, \mathbf{p}^{'}) = w_G(\mathbf{p} - \mathbf{p}^{'})$. The minimum Gray distance of the linear code $\boldsymbol{C}_{\mathbb{M}^l}$ over $\mathbb{M}^l$ denoted by $d_G(\boldsymbol{C}_{\mathbb{M}^l}) = d_G$ is equal to $min\{d_G(\mathbf{p}, \mathbf{p}^{'}) | \forall \mathbf{p}, \mathbf{p}^{'} \in \boldsymbol{C}_{\mathbb{M}^l}, \mathbf{p} \neq \mathbf{p}^{'}\}$.

Theorem 5.1 *The Gray map* $\Psi \circ \boldsymbol{\Psi} = \Phi$ *is a linear and distance preserving map from* $(\mathbb{M}^l)^s$ *(Gray Distance) to* $\mathbb{F}_q^{2sl}$ *(Hamming Distance).*

Theorem 5.2 *If* $\boldsymbol{C}_{\mathbb{M}^l}$ *is a linear code of length* s *over* $\mathbb{M}^l$ *with minimum Gray distance* d_G *and size* M*, then* $\Phi(\boldsymbol{C}_{\mathbb{M}^l})$ *is an* $\mathbb{F}_q$ *linear code with parameters* $(2sl, M, d_H)$ *with* $d_G = d_H$*.*

Theorem 5.3 *Let* $\boldsymbol{C}_{\mathbb{M}^l}$ *be a linear code of length* s *over* $\mathbb{M}^l$*. Then* $\Phi(\boldsymbol{C}_{\mathbb{M}^l})$ *is Euclidean dual containing if* $\boldsymbol{C}_{\mathbb{M}^l}$ *is Euclidean dual containing provided* $S.S^{\mathbf{T}} = \lambda I_l$*, where* $S \in GL(l, \mathbb{F}_q)$*,* $\mathbf{T}$ *denotes transpose of a matrix and* $\lambda \in \mathbb{F}_q^*$*.*

Proof Let $\mathbf{n} = (\mathbf{n}_0, \cdots, \mathbf{n}_{s-1}), f = (f_0, \cdots, f_{s-1}) \in \boldsymbol{C}_{\mathbb{M}^l}$, where

$$\mathbf{n_t} = (a_1^t + b_1^t v, \cdots, a_l^t + b_l^t v), \quad \mathbf{f_t} = (c_1^t + d_1^t v, \cdots, c_l^t + d_l^t v)$$

for $t = 0, \cdots, s-1$. Then

$$\langle \mathbf{n}, \mathbf{f} \rangle_E = \sum_{i=1}^{l}\sum_{t=0}^{s-1} a_i^t c_i^t + v \sum_{i=1}^{l}\sum_{t=0}^{s-1} a_i^t d_i^t + b_i^t c_i^t + b_i^t d_i^t.$$

Since $\langle \mathbf{n}, \mathbf{f} \rangle_E = 0$, then we have

$$\sum_{i=1}^{l}\sum_{t=0}^{s-1} a_i^t c_i^t = 0 \quad \text{and} \quad \sum_{i=1}^{l}\sum_{t=0}^{s-1} a_i^t d_i^t + b_i^t c_i^t + b_i^t d_i^t = 0.$$

Now

$$\langle \Phi(\mathbf{n}), \Phi(\mathbf{f}) \rangle_E = \sum_{t=0}^{s-1} \mathbf{\Pi}_1(\mathbf{n}_t) S S^{\mathbf{T}} (\mathbf{\Pi}_1(\mathbf{f}_t))^T + \sum_{t=0}^{s-1} \mathbf{\Pi}_2(\mathbf{n}_t) S S^{\mathbf{T}} (\mathbf{\Pi}_2(\mathbf{f}_t))^T,$$

where $S \in GL(l, \mathbb{F}_q)$. So

$$\lambda \sum_{i=1}^{l}\sum_{t=0}^{s-1} a_i^t c_i^t + \lambda \sum_{i=1}^{l}\sum_{t=0}^{s-1} a_i^t c_i^t + \lambda \sum_{i=1}^{l}\sum_{t=0}^{s-1} a_i^t d_i^t + b_i^t c_i^t + b_i^t d_i^t = 0$$

The desired result is obtained as the proof of Theorem 5.4 in [1]. □

Theorem 5.4([4]) *Let* $C_{\mathbb{F}_q} =< g(x) >$ *be a* θ*-cyclic code of length* n *over* $\mathbb{F}_q$ *such that* n *is* *a multiple of* $ord(\theta)$ *and* $\theta \in Aut(\mathbb{F}_q)$*.Then* $C_{\mathbb{F}_q}$ *contains its dual if and only if* $h^*(x)h(x)$ *is divisible by* $x^n - 1$ *on the right, where* $x^n - 1 = h(x)g(x)$ *and* $h^*(x) = \beta_{n-1} + \theta(\beta_{n-s-1})x + ... + \theta^{n-s}(\beta_0)x^{n-s}$*, for* $h(x) = \beta_0 + \beta_1 x + ... + \beta_{n-1}x^{n-1}$*.*

Theorem 5.5 *Let* $\boldsymbol{C}_{\mathbb{M}^l} = (1-v)[\mathbf{e_1}C_{1,\mathbb{F}_q,1} \oplus \cdots \oplus \mathbf{e}_l C_{1,\mathbb{F}_q,l}] \oplus v[\mathbf{e_1}C_{2,\mathbb{F}_q,1} \oplus \cdots \oplus \mathbf{e}_l C_{2,\mathbb{F}_q,l}]$ *be a* Ξ*-cyclic code of length* s *over* $\mathbb{M}^l$ *such that* s *is a multiple of* $ord(\Xi) = lcm[ord(\xi_1), \cdots, ord(\xi_l)]$ *and* $\Xi \in \Omega_{\mathbb{M}^l}$*. Let* $C_{j,\mathbb{F}_q,i} =< g_{j,i}(x) >$ *and* $x^s - 1 = h_{j,i}(x)g_{j,i}(x) = g_{j,i}(x)h_{j,i}(x)$*, for* $h_{j,i}(x), g_{j,i}(x) \in \mathbb{F}_q[x, \theta_i]$*. Then* $\boldsymbol{C}_{\mathbb{M}^l}^{\perp} \subseteq \boldsymbol{C}_{\mathbb{M}^l}$ *if and only if* $h_{j,i}^*(x)h_{j,i}(x)$ *is divisible by* $x^s - 1$*, for* $j = 1, 2$ *and* $i = 1, \cdots, l$*.*

Corollary 5.6 *Let* $\boldsymbol{C}_{\mathbb{M}^l} = (1-v)[\mathbf{e_1}C_{1,\mathbb{F}_q,1} \oplus \cdots \oplus \mathbf{e}_l C_{1,\mathbb{F}_q,l}] \oplus v[\mathbf{e_1}C_{2,\mathbb{F}_q,1} \oplus \cdots \oplus \mathbf{e}_l C_{2,\mathbb{F}_q,l}]$ *be a* Ξ*-cyclic code of length* s *over* $\mathbb{M}^l$ *such that* s *is a multiple of* $lcm[ord(\xi_1), \cdots, ord(\xi_l)]$ *and* $\Xi \in \Omega_{\mathbb{M}^l}$*. Then* $\boldsymbol{C}_{\mathbb{M}^l}^{\perp} \subseteq \boldsymbol{C}_{\mathbb{M}^l}$ *if and only if* $C_{j,\mathbb{F}_q,i}^{\perp} \subseteq C_{j,\mathbb{F}_q,i}$*, for* $j = 1, 2$ *and* $i = 1, \cdots, l$*.*

Theorem 5.7(CSS Construction, [2]) *Let* C_1 *and* C_2 *be* $[n, k_1, d_1]$ *and* $[n, k_2, d_2]$ *linear codes over* $\mathbb{F}_q$ *respectively with* $C_2^{\perp} \subset C_1$*. Futhermore, let* $d = \{d_1, d_2\}$*. Then there exists a QECC, with parameters* $[[n, k_1 + k_2 - n, d]]_q$*. In particular, if* $C_1^{\perp} \subseteq C_1$*, then there exists a QECC with parameters* $[[n, 2k_1 - n, d]]_q$*.*

Theorem 5.8 *Let* $\boldsymbol{C}_{\mathbb{M}^l} = (1-v)[\mathbf{e_1}C_{1,\mathbb{F}_q,1} \oplus \cdots \oplus \mathbf{e}_l C_{1,\mathbb{F}_q,l}] \oplus v[\mathbf{e_1}C_{2,\mathbb{F}_q,1} \oplus \cdots \oplus \mathbf{e}_l C_{2,\mathbb{F}_q,l}]$ *be a* Ξ*-cyclic code of length* s *over* $\mathbb{M}^l$ *such that* s *is a multiple of* $lcm[ord(\xi_1), \cdots, ord(\xi_l)]$*. If* $C_{j,\mathbb{F}_q,i}^{\perp} \subseteq C_{j,\mathbb{F}_q,i}$*, for* $j = 1, 2$ *and* $i = 1, \cdots, l$*, then* $\boldsymbol{C}_{\mathbb{M}^l}^{\perp} \subseteq \boldsymbol{C}_{\mathbb{M}^l}$ *and there exists a quantum error correcting code with parameters* $[[(2sl, \sum_{i=1}^{l} \sum_{j=1}^{2} k_{j,i} - 2sl, d_H]]_q$*, where* d_H *denotes the Hamming distance of the code* $\Phi(\boldsymbol{C}_{\mathbb{M}^l})$ *and* $k_{j,i} = s-deg\ g_{j,i}(x)$*, for* $i = 1, \cdots, l$ *and* $j = 1, 2$*.*

Example 5.9 Let $\mathbb{F}_8 = \mathbb{F}_2[\alpha]$ be the field of order 8, where $\alpha^3 = \alpha + 1$. Let $l = 2, s = 6$ and $\Theta = \theta \times \theta^2 \in \Omega_{\mathbb{F}_8^2}$, where θ is the Frobenius automorphism of $\mathbb{F}_8$. If

$$\begin{aligned} C_{1,\mathbb{F}_8,i} &= \left\langle g_{1,i}(x) = \left(\alpha + \alpha^2\right)(1 + x) + \alpha\left(x^2 + x^3\right)\right\rangle, \\ C_{2,\mathbb{F}_8,i} &= \left\langle g_{2,i}(x) = \left(1 + \alpha^2\right)(1 + x) + \left(\alpha + \alpha^2\right)\left(x^2 + x^3\right)\right\rangle \end{aligned}$$

for $i = 1, 2$, then $C_{1,\mathbb{F}_8^2}, C_{2,\mathbb{F}_8^2}$ are Θ-cyclic codes of length 6 over $\mathbb{F}_8^2$. So, $\boldsymbol{C}_{\mathbb{M}^2}$ is an Ξ-cyclic codes of length 6 over $\mathbb{M}^2$. Moreover, we have

$$\begin{aligned} h_{1,i}(x) &= (1 + \alpha)\left(1 + x^2\right) + \left(\alpha^2 + 1\right)\left(x + x^3\right) \\ h_{1,i}^*(x) &= \left(1 + \alpha^2\right)(1 + x) + (1 + \alpha)\left(x^2 + x^3\right) \\ h_{2,i}(x) &= \alpha + \left(\alpha + \alpha^2\right)x + \left(1 + \alpha^2\right)x^2 + (1 + \alpha)x^3 \\ h_{2,i}^*(x) &= (1 + \alpha)(1 + x) + \alpha\left(x^2 + x^3\right) \end{aligned}$$

for $i = 1, 2$. So $x^6 - 1$ is divisible by $g_{j,i}(x)\, h_{j,i}^*(x)$ for $i = 1, 2$,$j = 1, 2$. Then $\boldsymbol{C}_{\mathbb{M}^2}^{\perp} \subseteq \boldsymbol{C}_{\mathbb{M}^2}$. If

$$S = \begin{pmatrix} 1 & 1 + \alpha + \alpha^2 \\ 1 + \alpha + \alpha^2 & 1 \end{pmatrix}$$

then $\Psi(C_{i,\mathbb{F}_8^2})$ is a $[12, 6, 6]$ linear code over $\mathbb{F}_8$ for $i = 1, 2$. Hence $\Phi(\boldsymbol{C}_{\mathbb{M}^2})$ is a $[24, 12, 6]$ linear code over $\mathbb{F}_8$. Hence, by Theorem 11, we have a quantum code with $[[24, 0, 6]]$.

§6. Conclusion

We take a more general form of an automorphism of $\mathbb{M}^l$, namely $\xi_1 \times \xi_2 \cdots \times \xi_l$, where each ξ_i is an automorphism of $\mathbb{M}$, for $i = 1, \cdots, l$. We introduce the algebraic structure of Ξ-cyclic codes over $\mathbb{M}^l$ and obtain the parameters of quantum codes from Ξ-cyclic codes over $\mathbb{M}^l$, $l \geq 1$, by using CSS constructions.

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Smarandache Curves, Surfaces and Geometries as Frameworks in Differential Geometry and Fundamental Mathematics

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Abstract: This paper presents the definition and distinctions between the Smarandache curves and all other types of curves, the definition and distinctions between Smarandache surfaces and all other types of surfaces, the definition and distinctions between Smarandache geometries and all other types of geometries, and includes nearly all research results so far.

Key Words: Smarandache curve, Smarandache surface, Smarandache geometry.

AMS(2010): 14A15, 14H10, 14J29.

§1. Introduction

The concepts of Smarandache curves, surfaces and geometries are frameworks in differential geometry and foundational mathematics, developed by Florentin Smarandache. Their primary distinction from classical counterparts lies in their hybrid structure and the partial negation of axioms.

1.1. Smarandache Curve. A Smarandache curve (or S-curve) is a regular curve in a geometric space (like Euclidean or Minkowski space) whose position vector is defined as a linear combination of the vectors of a moving frame (such as the Frenet-Serret frame, Bishop frame, or Darboux frame) of another base curve, first discussed in [28] and then, more and more research papers on Smarandache curves, surfaces [27]-[130] were published.

Definition Example: A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve:

https://fs.unm.edu/SG/NCSmarandacheCurvesOfMannheimCurve.pdf

Distinctions from Other Curves:

- *Classical Curves:* Curves like helices or planar curves are typically defined by constant relationships between their intrinsic properties (e.g., constant curvature and torsion for a general helix).
- *Smarandache Curves:* They are derived curves whose geometric properties are intrinsi-

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[1] Received October 23, 2025. Accepted February 16, 2026

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cally linked to, and defined by, the moving frame of a different base curve. This gives them a more flexible and general structure, allowing for complex geometric analysis and applications in fields like robotics and physics. They are considered hybrid curves because they can satisfy properties of different types of curves simultaneously.

A Smarandache curve is a derived curve whose position vector is constructed by combining some of the Frenet frame vectors (the tangent T, normal N and binormal B) of another given regular space curve.

Formally, let $\alpha(s)$ be a regular space curve parameterized by arc length s with Frenet frame $\{T(s), N(s), B(s)\}$. Then, a Smarandache curve $\beta(s)$ is defined by

$$\beta(s) = aT(s) + bN(s) + cB(s),$$

where a, b, c are constants (or sometimes simple functions of s). Depending on which Frenet vectors are used, we have

Type	Definition
TN Smarandache curve	$\beta(s) = T(s) + N(s)$
TB Smarandache curve	$\beta(s) = T(s) + B(s)$
TNB Smarandache curve	$\beta(s) = T(s) + N(s) + B(s)$
etc.	Other combinations possible

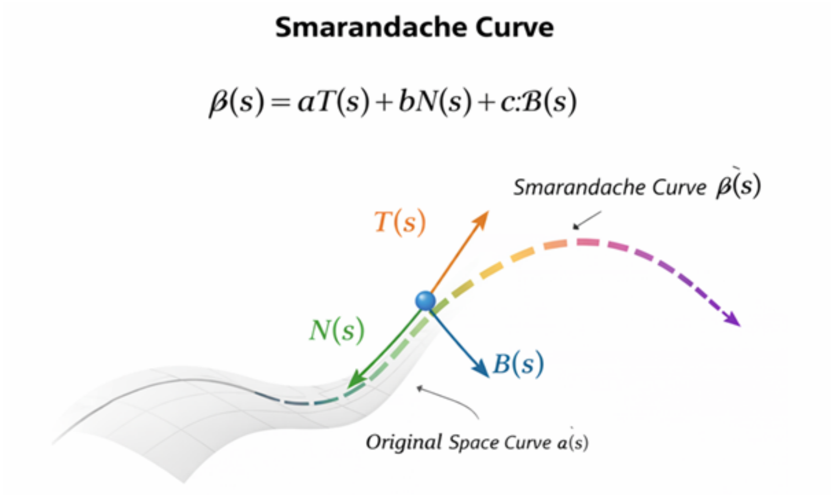

Figure 1.

1.2. Smarandache Surfaces. A Smarandache surface (or S-surface) is generally a surface, often a ruled surface, whose base curve and/or ruling direction vectors are themselves Smarandache curves.

Definition Example: A ruled surface is a curved surface which can be generated by the continuous motion of a straight line in space along a space curve called a directrix. This straight line is called a generator, or ruling, of the surface.

- A Smarandache ruled surface uses a Smarandache curve for the base curve (or the ruling vector). For instance, a type-2 Smarandache ruled surface uses a Smarandache curve as its base curve.

Distinctions from Other Surfaces:

- *Classical Surfaces:* Surfaces like cylinders, cones, or minimal surfaces are defined by standard properties (e.g., zero Gaussian curvature for developable surfaces, zero mean curvature for minimal surfaces).
- *Smarandache Surfaces:* Their construction is based on the hybrid nature of S-curves, meaning S-surfaces themselves are often surfaces of hybrid geometrical structures. This makes them a more complex and general category, incorporating the unique geometric characteristics inherited from the S-Curves used in their definition.

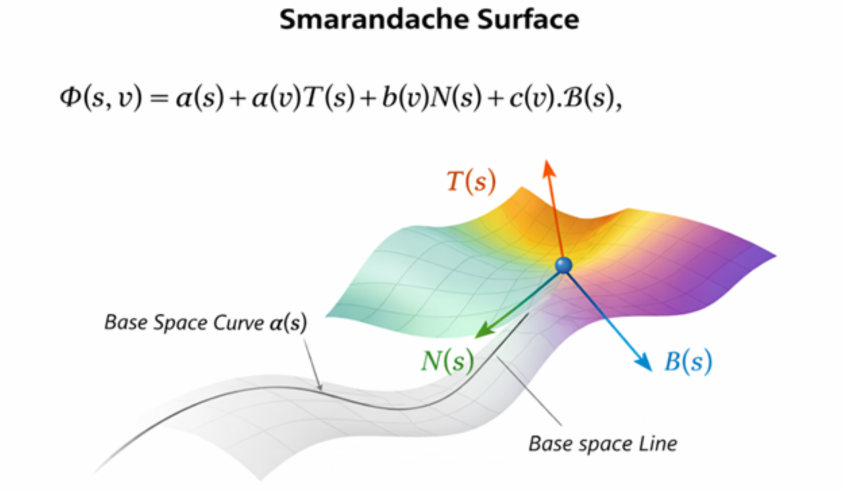

Figure 2.

1.3. Smarandache Geometries. A Smarandache geometry (or S-geometry) is a geometry in which at least one axiom is Smarandachely denied (https://fs.unm.edu/SG/). A Smarandache geometry (or S-geometry) is a geometry first discussed in [10] on in which at least one axiom is Smarandachely denied (https://fs.unm.edu/SG/) and then, more books and articles [1]-[26] were published.

Definition: An axiom is Smarandachely denied if it behaves in at least two different ways within the same space. This can mean the axiom is:

(1) Validated and Invalided (partially true and partially false);
(2) Only Invalided but in at least two distinct ways.

Distinctions from Other Geometries:

- *Classical geometries* (Euclidean, Riemannian, Hyperbolic, Elliptic): These are homogeneous spaces where a given axiom (like the parallel postulate) holds uniformly throughout the space (e.g., in Euclidean geometry, there is exactly one parallel line through an external point).

- *Smarandache Geometries:* They are hybrid or heterogeneous multispaces that combine structures from different classical geometries into a single space. For example, a single S-Geometry can be partially Euclidean and partially Non-Euclidean. This concept introduces the degree of negation of an axiom, which is analogous to the degrees of truth and falsehood in fuzzy logic or neutrosophic logic.

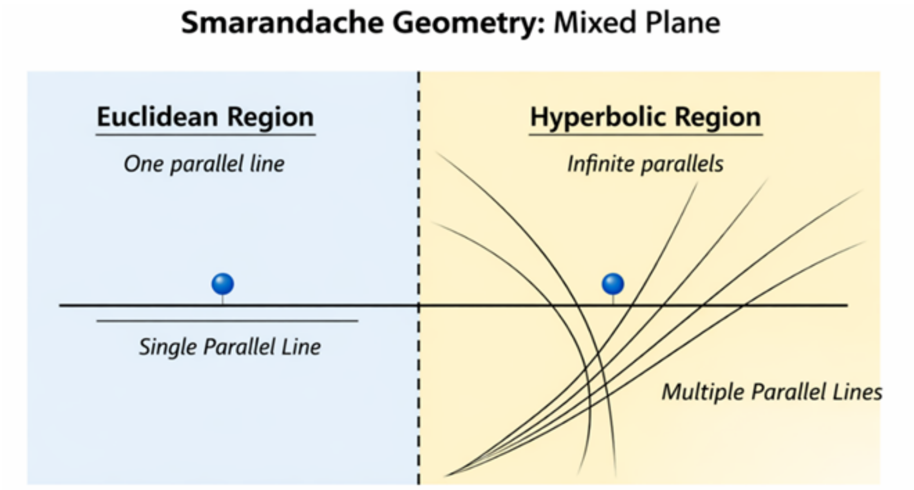

Figure 3.

The overall significance of these Smarandache notions is the creation of hybrid structures that allow for the simultaneous existence of multiple, sometimes contradictory, properties within a single defined entity, providing a more general and flexible framework for theoretical and applied mathematics. The Smarandache curve example, the most elementary type is an involute-evolute curve pair or a curve defined by a simple linear combination of the base curve's moving frame.The Smarandache surfaces are often constructed as ruled surfaces where the base curve or ruling vector is a Smarandache curve. The Smarandache geometries involve the partial denial of an axiom within a single space.

§2. Smarandache Euclidean-Hyperbolic Geometry

Axiom Smarandachely Denied: The Euclid's fifth postulate (the parallel postulate).

- *Space (S):* Consider a single geometric plane.
- *Subspace (Euclidean region):* This is the standard Euclidean plane, where for a given line ($L1$) and an external point ($P1$), there is exactly one line through $P1$ parallel to ($L1$).
- *Subspace (Hyperbolic Region):* This is a region (e.g., a disk or portion of the plane) where for a given line ($L2$) and an external point ($P2$), there are infinitely many lines through $P2$ parallel to $L2$.
- *Smarandache Geometry:* The space (S) is the union of the two distinct regions ($R1$ and $R2$). The parallel postulate is *Smarandachely denied* because it holds true (validated) in the R1 part, and is false (invalided) in the $R2$ part, all within the same space (S).

Distinction: Classical geometries are homogeneous (e.g., Euclidean or hyperbolic, but not both). Smarandache geometry is a heterogeneous multispace, where two (or more) types of geometry coexist and influence properties based on location (https://fs.unm.edu/SG/).

Connecting Smarandache geometries with neutrosophic logic helps explain the foundational philosophy behind these hybrid structures.

§3. Smarandache Geometries and Neutrosophic Logic

Neutrosophy is a branch of philosophy and logic, developed by Florentin Smarandache, that studies the origin, nature, and scope of neutralities. It is characterized by the representation of any idea, concept, or proposition as a triplet.

Neutrosophic logic (NL) is the corresponding mathematical tool where a proposition is defined by three independent components in the real unit interval:

- (*Truth*): The degree to which it is true;
- (*Indeterminacy*): The degree to which it is indeterminate or neutral;
- (*Falsehood*): The degree to which it is false.

Conceptual Relationship. The fundamental idea of Smarandache geometry (S-geometry) is a direct application of Neutrosophic logic to the structure of axioms:

(1) *Classical Geometry:* An axiom is either true ($T = 1, I = 0, F = 0$), as in Euclidean geometry where the parallel postulate is true, or false ($T = 0, I = 0, F = 1$), as in Hyperbolic geometry where it is false.

(2) *Smarandache Geometry:* An S-geometry exists when a single axiom is simultaneously true in one part of the space and false in another part of the space. This hybrid structure is the geometric manifestation of an Indeterminate or Neutral state.

Concept	Geometric Example (Parallel Postulate)	Neutrosophic Logic Equivalent
Axiom is Validated (True)	Exactly one parallel line exists (Euclidean region)	–
Exactly one parallel line exists (Euclidean region)	Infinitely many parallel lines exist (Hyperbolic region) Infinitely many parallel lines exist (Hyperbolic region)	–
Axiom is Smarandachely Denied	Both the Euclidean and hyperbolic cases exist within the same space	and simultaneously, which implies indeterminacy

In essence, S-geometries are the geometric spaces that model the philosophical concepts of Neutrosophy, allowing for the rigorous study of hybrid systems and contradictions in mathematics. This is further explored in fields like:

(*a*) NeutroGeometrywhich is a geometry that has at least one axiom that is partially false ($0 < F < 1$) and no axiom that is totally false;

(*b*) AntiGeometry, which is a geometry that has at least one axiom that is totally false ($F = 1$), for example the Non-Euclidean geometries are particular cases of the AntiGeometry, as a newer development (https://fs.unm.edu/NG/);

(*c*) unlike the classical geometry, where all axioms are totally true, $T = 1$.

Update books, research papers on Smarandache geometries, Smarandache curves and Smarandache surfaces with resources on the internet can be found in references [1]-[130] following.

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International J.Math. Combin. Vol.1(2026), 90-103

Cyclic Contraction on Supermetric Spaces

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Abstract: In this paper, we investigate fixed point properties of cyclic contractive mappings in the framework of supermetric spaces. After recalling the fundamental axioms of supermetrics together with the corresponding notions of convergence and completeness, we introduce a cyclic contraction scheme formulated on mixed pairs of subsets (A, B) and establish a fixed point theorem ensuring existence, uniqueness, and convergence of Picard iterates. Several constructive examples are presented to demonstrate the applicability of the theory. As an application, a nonlinear integral equation is reformulated as a fixed point problem in an appropriate function space, yielding existence and uniqueness of continuous solutions.

Key Words: Cyclic contraction, supermetric space, fixed point theory, nonlinear integral equation.

AMS(2010): 47H10, 54H25, 54E50.

§1. Introduction

Banach's contraction principle (Banach, 1922) is a cornerstone of nonlinear analysis and remains a basic tool for proving existence and uniqueness of solutions to nonlinear problems. A classical and still active direction is to enlarge either (i) the underlying distance framework or (ii) the class of admissible contractive conditions, while preserving completeness–convergence mechanisms needed for Picard iteration.

On the side of generalized distances, several models are now standard. Partial metric spaces, introduced by Matthews [11], allow nonzero self-distance and are useful in computer-science motivated fixed point problems. The b-metric of Czerwik [5] relaxes the triangle inequality by a coefficient and has become a basic platform for nonlinear operator theory. Multi-point geometries such as S-metric spaces were proposed by Sedghi–Shobe–Aliouche [15]. In a different direction, fuzzification of distance and its applications originate from Zadeh's fuzzy set theory [17].

In parallel, many authors have developed nonlinear contractive conditions that extend the Banach inequality. Representative frameworks include rational-type contractions [1], the α–ψ contractive scheme of Samet–Vetro–Vetro [14], and Wardowski's F-contractions [16]. Fixed point techniques have also been combined with order structures; see, for example, Ran–Reurings

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[1] Received September 24, 2025, Accepted February 20, 2026

[12], Bhaskar–Lakshmikantham [4], Altun–Simsek [2], and Jleli–Samet [7].

More recently, Karapınar and Khojasteh introduced the concept of a *supermetric space* [9]. A supermetric replaces triangle-type inequalities by a sequence-based control axiom, yet still supports a Banach-type fixed point principle. Subsequent work in this environment includes contractive models of rational form [8].

Another influential line is the *cyclic* approach, where a mapping alternates between two (or more) sets and the contractive requirement is enforced along the cyclic pattern. This viewpoint was developed systematically by Kirk–Srinivasan–Veeramani [10] (see also [13]) and has proved useful in settings where contraction is naturally available only on mixed pairs. Motivated by these developments, we adapt cyclic contractive ideas to the supermetric framework and derive fixed point results under completeness and natural closure hypotheses.

§2. **Preliminaries**

In this section, we recall the basic notions of supermetric spaces together with the associated concepts of convergence, Cauchy sequences, and completeness. These structures were introduced by Karapınar and Khojasteh as a sequenceCcontrolled generalization of classical metric spaces and several metricCtype extensions [9] and [8]. Unlike metric and bCmetric spaces, the supermetric framework does not rely on a triangle-type inequality; instead, it uses a limsup sequence control condition that is sufficient to develop fixed point theory and completeness results [9]. Related generalized distance frameworks include partial metric spaces [11], b-metric spaces [5], and S-metric spaces [15], all of which motivated the search for more flexible convergence structures.

Definition 2.1(Supermetric space, [9]) *Let X be a nonempty set. A function $m : X \times X \to [0, \infty)$ is called a supermetric if*

($SM1$) *if $m(x, y) = 0$, then $x = y$ for all $x, y \in X$;*

($SM2$) *$m(x, y) = m(y, x)$ for all $x, y \in X$;*

($SM3$) *there exists $s \geqslant 1$ such that for every $y \in X$ there exist* distinct *sequences $\{x_n\}, \{y_n\} \subset X$ with $m(x_n, y_n) \to 0$ and*

$$\limsup_{n\to\infty} m(y_n, y) \leqslant s \limsup_{n\to\infty} m(x_n, y).$$

Then (X, m) is called a supermetric space.

Remark 2.1(Interpretation of the axioms)

(i) Condition (SM1) is a weak identity property. Unlike classical metrics, the converse $m(x, x) = 0$ is not required, which aligns the model with partial metric philosophy, [11];

(ii) Condition (SM2) ensures symmetric distance behavior as in classical metric theory;

(iii) Condition (SM3) replaces the triangle inequality by a sequence comparison mechanism. It guarantees that if two sequences become mutually close, then their relative proximity to any fixed point is controlled. This property is fundamental in establishing convergence of Picard iteration sequences [9].

Definition 2.2(Convergence and Cauchy sequence, [9]) *Let (X, m) be a supermetric space.*

1) *A sequence $\{x_n\}$ converges to $x \in X$ if*

$$\lim_{n\to\infty} m(x_n, x) = 0;$$

(2) *A sequence $\{x_n\}$ is called Cauchy if*

$$\lim_{n\to\infty} \sup\{m(x_n, x_k) : k > n\} = 0;$$

(3) *The space (X, m) is complete if every Cauchy sequence converges to some point in X.*

Remark 2.2 If (X, d) is a usual metric space, then (X, d) is automatically a supermetric space. Indeed, (SM1)C(SM2) hold trivially and (SM3) holds with $s = 1$ by choosing two distinct sequences converging to the same limit. Hence supermetric spaces extend metric spaces while preserving fixed point tools based on completeness [3] and [9].

Remark 2.3 Supermetric convergence generates a natural topology on X. In many applications, supermetric convergence is equivalent to convergence under an associated metric on bounded subsets. This feature makes supermetric spaces suitable for nonlinear operator analysis and fixed point theory, including rational and nonlinear contractions [1] and [8].

Remark 2.4 The development of supermetric spaces is consistent with the broader program of extending distance geometry to accommodate nonlinear phenomena and computational models. For instance, partial metrics allow nonzero self-distance [11], b-metrics relax the triangle inequality constant [5], and multi-point distance structures such as S-metrics extend pairwise distance notions [15]. Supermetrics unify several of these behaviors under a sequence-control framework [9].

§3. New Examples of Supermetric Spaces

Proposition 3.1 *Let $X = \mathbb{R}$ and fix $\gamma \in [0, 1)$. Define*

$$m(x, y) := |x - y| + \gamma\big(|x| + |y|\big), \qquad x, y \in \mathbb{R}.$$

Then $(\mathbb{R}, m)$ is a supermetric space in the sense of Definition 2.1. *In particular, axiom* (SM3) *holds with the constant $s = 2$.*

Proof The function m is nonnegative and symmetric. If $m(x, y) = 0$, then $|x-y|+\gamma(|x|+|y|) = 0$ forces $|x - y| = 0$, hence $x = y$, so (SM1) holds.

Fix any $y \in \mathbb{R}$ and choose the distinct sequences $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n+1}$. Then

$$m(x_n, y_n) = \left|\frac{1}{n} - \frac{1}{n+1}\right| + \gamma\left(\frac{1}{n} + \frac{1}{n+1}\right) \to 0.$$

Moreover,

$$\begin{aligned} m(x_n, y) &= \left|y - \frac{1}{n}\right| + \gamma\left(|y| + \frac{1}{n}\right) \to |y| + \gamma|y|, \\ m(y_n, y) &= \left|y - \frac{1}{n+1}\right| + \gamma\left(|y| + \frac{1}{n+1}\right) \to |y| + \gamma|y|. \end{aligned}$$

Hence

$$\limsup_{n\to\infty} m(y_n, y) = \limsup_{n\to\infty} m(x_n, y).$$

Consequently,

$$\limsup_{n\to\infty} m(y_n, y) \leqslant 2 \limsup_{n\to\infty} m(x_n, y),$$

so (SM3) holds with $s = 2$. Therefore $(\mathbb{R}, m)$ is a supermetric space. □

Proposition 3.2 *Let $(E, \|\cdot\|)$ be a real normed linear space and fix $\gamma \in (0,1)$. Define*

$$m(x,y) := \|x - y\| + \gamma\frac{\|x\| + \|y\|}{1 + \|x\| + \|y\|} \qquad (x, y \in E).$$

Then (E, m) is a supermetric space. In particular, (SM3) *holds with $s = 2$.*

Proof The function m is well-defined, nonnegative, and symmetric. If $m(x,y) = 0$, then $\|x - y\| = 0$, hence $x = y$, so (SM1) holds.

Fix any $y \in E$ and choose a nonzero $u \in E$. Define the distinct sequences $x_n = \frac{u}{n}$ and $y_n = \frac{u}{n+1}$. Then $\|x_n - y_n\| = \frac{\|u\|}{n(n+1)} \to 0$ and $\|x_n\| + \|y_n\| \to 0$, which yields $m(x_n, y_n) \to 0$.

Since $x_n \to 0$ and $y_n \to 0$ in norm, we have $\|x_n - y\| \to \|y\|$ and $\|y_n - y\| \to \|y\|$, and also

$$\frac{\|x_n\| + \|y\|}{1 + \|x_n\| + \|y\|} \to \frac{\|y\|}{1 + \|y\|}, \qquad \frac{\|y_n\| + \|y\|}{1 + \|y_n\| + \|y\|} \to \frac{\|y\|}{1 + \|y\|}.$$

Hence both sequences $m(x_n, y)$ and $m(y_n, y)$ converge to $\|y\| + \gamma\frac{\|y\|}{1+\|y\|}$. Therefore their limsups are equal and so

$$\limsup_{n\to\infty} m(y_n, y) \leqslant 2 \limsup_{n\to\infty} m(x_n, y).$$

Thus (SM3) holds with $s = 2$, and (E, m) is a supermetric space. □

Proposition 3.3 *Let $X = \mathbb{R}$ and fix $\gamma \in (0,1)$. Define*

$$m(x,y) := |x - y| + \gamma|\arctan x - \arctan y| \qquad (x, y \in \mathbb{R}).$$

Then $(\mathbb{R}, m)$ is a supermetric space, and (SM3) *holds with $s = 2$.*

Proof The function m is nonnegative and symmetric. If $m(x,y)=0$, then $|x-y|=0$, hence $x=y$, so (SM1) holds.

Fix any $y\in\mathbb{R}$ and take distinct sequences $x_n=\frac{1}{n}$ and $y_n=\frac{1}{n+1}$. Then $|x_n-y_n|\to 0$ and continuity of arctan at 0 gives $|\arctan x_n-\arctan y_n|\to 0$, hence $m(x_n,y_n)\to 0$.

Also, since $x_n\to 0$ and $y_n\to 0$, continuity of $t\mapsto|t-y|$ and $t\mapsto|\arctan t-\arctan y|$ implies

$$m(x_n,y)\to|y|+\gamma|\arctan y|,\qquad m(y_n,y)\to|y|+\gamma|\arctan y|.$$

Thus their limsups coincide and therefore

$$\limsup_{n\to\infty} m(y_n,y)\leqslant 2\limsup_{n\to\infty} m(x_n,y).$$

Hence (SM3) holds with $s=2$, so $(\mathbb{R},m)$ is a supermetric space [9]. □

§4. Cyclic Contraction Mappings

In this section, we introduce cyclic self-mappings and cyclic contraction mappings in the framework of supermetric spaces and establish the fundamental structural properties that will be used in the main results. The concept of cyclic mappings originates from the idea of decomposing the domain into two interacting subsets and studying contractive behavior only across these subsets rather than globally. Such a formulation is particularly useful in generalized distance structures where classical triangle-type estimates may not be available.

Let (X,m) be a supermetric space and let A,B be nonempty subsets of X. Throughout this section, the union $A\cup B$ will serve as the effective domain of the mapping under consideration.

Definition 4.1(Cyclic map and cyclic contraction) *Let (X,m) be a supermetric space and let $A,B\subseteq X$ be nonempty sets.*

A mapping $T:A\cup B\to A\cup B$ is called a cyclic map *if*

$$T(A)\subseteq B\quad\textit{and}\quad T(B)\subseteq A.$$

A cyclic map T is called a cyclic contraction *if there exists a constant $\alpha\in(0,1)$ such that*

$$m(Tx,Ty)\leqslant\alpha\, m(x,y)\qquad\textit{for all } x\in A,\ y\in B. \tag{1}$$

Remark 4.1 The contractive condition is imposed only for pairs $(x,y)\in A\times B$. This is natural because the cyclic structure forces the orbit

$$x_0\in A\ \Rightarrow\ x_1=Tx_0\in B\ \Rightarrow\ x_2=Tx_1\in A\ \Rightarrow\ \cdots$$

to alternate between the two sets. Therefore, all successive distances that appear in Picard iteration are of mixed type.

Requiring the inequality on the whole set $A\cup B$ would reduce the model to a classical global contraction and would not capture the cyclic geometry.

Remark 4.2 Unlike metric or b-metric spaces, supermetric spaces do not assume any triangle inequality. Therefore, the formulation of cyclic contractions is chosen so that convergence analysis relies mainly on geometric decay of successive distances rather than summation-type estimates. This makes cyclic contractions particularly well suited for supermetric settings.

Example 4.1 Let m be the supermetric defined in Proposition 3.1, namely

$$m(x,y) = |x-y| + \gamma(|x|+|y|), \quad \gamma \in [0,1).$$

Define

$$A = [0,\infty), \qquad B = (-\infty, 0],$$

and define the mapping $T : A \cup B \to A \cup B$ by

$$T(x) = -\frac{1}{2}x.$$

We first verify cyclicity. If $x \in A$, then $x \geqslant 0$ and hence $Tx = -\frac{1}{2}x \leqslant 0$, so $Tx \in B$. Similarly, if $x \in B$, then $x \leqslant 0$ and hence $Tx = -\frac{1}{2}x \geqslant 0$, so $Tx \in A$. Thus T is cyclic.

Next, let $x \in A$ and $y \in B$. Then

$$Tx = -\frac{1}{2}x, \qquad Ty = -\frac{1}{2}y.$$

Hence

$$|Tx - Ty| = \left| -\frac{1}{2}x + \frac{1}{2}y \right| = \frac{1}{2}|x-y|.$$

Also,

$$|Tx| + |Ty| = \frac{1}{2}|x| + \frac{1}{2}|y| = \frac{1}{2}(|x|+|y|).$$

Therefore,

$$m(Tx,Ty) = |Tx-Ty| + \gamma(|Tx|+|Ty|) = \frac{1}{2}|x-y| + \frac{\gamma}{2}(|x|+|y|) = \frac{1}{2}m(x,y).$$

Hence condition (1) holds with $\alpha = \frac{1}{2}$.

§5. **Main Result**

In this section, we establish existence and uniqueness of fixed points for cyclic contraction mappings in complete supermetric spaces.

Theorem 5.1 *Let (X,m) be a complete supermetric space and let $A, B \subseteq X$ be nonempty closed subsets. Assume that $T : A \cup B \to A \cup B$ is a cyclic contraction in the sense of Definition 4.1. Then T has a unique fixed point $x^* \in A \cap B$. Moreover, for any initial point $x_0 \in A \cup B$, the Picard iteration $x_{n+1} = Tx_n$ converges to x^* with respect to m.*

Proof Fix $x_0 \in A \cup B$ and define the Picard sequence $x_{n+1} = Tx_n$ for $n \geqslant 0$. If $x_n = x_{n+1}$ for some n, then x_n is a fixed point and the sequence is constant. Hence we assume $x_n \neq x_{n+1}$ for all n.

Since T is cyclic, the orbit alternates between the two sets; specifically, if $x_0 \in A$ then $x_{2n} \in A$ and $x_{2n+1} \in B$ for all n, and the reverse holds if $x_0 \in B$. Consequently, each successive pair (x_n, x_{n+1}) is a mixed pair, so the cyclic contraction inequality applies repeatedly along the orbit.

From the cyclic contraction property we obtain

$$m(x_{n+1}, x_{n+2}) = m(Tx_n, Tx_{n+1}) \leqslant \alpha\, m(x_n, x_{n+1}),$$

and an immediate induction gives

$$m(x_n, x_{n+1}) \leqslant \alpha^n m(x_0, x_1), \qquad n \geqslant 0.$$

Since $0 < \alpha < 1$, it follows that $m(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

The above geometric decay implies that the sequence satisfies the contraction-orbit condition known in supermetric spaces. In particular, contraction-generated sequences in complete supermetric spaces are Cauchy in the sense that

$$\lim_{n\to\infty} \sup\{m(x_n, x_k) : k > n\} = 0$$

(see Theorem 2.6, [9]). Hence $\{x_n\}$ is Cauchy. By completeness of (X, m), there exists $x^* \in X$ such that $m(x_n, x^*) \to 0$.

To verify that x^* is a fixed point, we use the contraction property again. For each n,

$$m(x_{n+1}, Tx^*) = m(Tx_n, Tx^*) \leqslant \alpha m(x_n, x^*).$$

Since $m(x_n, x^*) \to 0$, we obtain $m(x_{n+1}, Tx^*) \to 0$. By the convergence principle of supermetric spaces [9], this implies $m(x^*, Tx^*) = 0$. By the identity axiom (SM1), it follows that $Tx^* = x^*$.

Next, because the subsequence $\{x_{2n}\}$ lies in A and converges to x^*, and A is closed, we obtain $x^* \in A$. Similarly, $x^* \in B$ because $\{x_{2n+1}\} \subset B$ and B is closed. Hence $x^* \in A \cap B$.

Finally, suppose $u, v \in A \cap B$ are fixed points. Then

$$m(u, v) = m(Tu, Tv) \leqslant \alpha m(u, v).$$

Since $\alpha \in (0, 1)$, this implies $m(u, v) = 0$, and hence $u = v$ by (SM1). Thus the fixed point is unique.

Therefore, for every initial point $x_0 \in A \cup B$, the Picard iteration converges to the unique fixed point $x^* \in A \cap B$. □

Theorem 5.1 has several immediate consequences that clarify the behavior of cyclic contractions and connect the supermetric framework with classical fixed point theory.

Corollary 5.1(Banach-type contraction as a special case) *Let (X, m) be a complete supermetric* *space and let* $T : X \to X$ *satisfy*

$$m(Tx, Ty) \leqslant \alpha\, m(x, y) \qquad \text{for all } x, y \in X,$$

for some $\alpha \in (0, 1)$*. Then* T *has a unique fixed point* $x^* \in X$ *and the Picard iteration* $x_{n+1} = Tx_n$ *converges to* x^* *for every* $x_0 \in X$*.*

Proof Apply Theorem 5.1 with $A = B = X$. Then T is trivially cyclic and the mixed-pair inequality reduces to the global contraction inequality. Hence the conclusion follows. ☐

Corollary 5.2(Uniqueness of the best proximity point when $A \cap B = \varnothing$) *Let* (X, m) *be a complete supermetric space and* $A, B \subseteq X$ *be nonempty closed sets. Assume* $T : A \cup B \to A \cup B$ *is cyclic and satisfies*

$$m(Tx, Ty) \leqslant \alpha\, m(x, y) \qquad \text{for all } x \in A,\ y \in B,$$

for some $\alpha \in (0, 1)$*. If* $A \cap B = \varnothing$*, then* T *has no fixed point in* $A \cup B$*. Nevertheless, any orbit* $\{x_n\}$ *generated by Picard iteration satisfies*

$$m(x_n, x_{n+1}) \to 0,$$

and every cluster point (if it exists) must lie in $\overline{A} \cap \overline{B}$*.*

Proof If $A \cap B = \varnothing$ and $x = Tx$, then $x \in A$ implies $Tx \in B$, contradicting $Tx = x$, and similarly if $x \in B$. Thus no fixed point exists. The decay $m(x_n, x_{n+1}) \leqslant \alpha^n m(x_0, x_1)$ follows exactly as in the proof of Theorem 5.1, hence $m(x_n, x_{n+1}) \to 0$. If a subsequence converges to z, then even and odd subsequences converge to the same z, forcing z to belong to both closures. Since A, B are closed, this means $z \in A \cap B$, so in the disjoint case there can be no such limit point in X. ☐

Corollary 5.3(Error estimate for Picard iteration) *Under the hypotheses of Theorem* 5.1*, let* x^* *be the unique fixed point and* $x_{n+1} = Tx_n$ *be the Picard iteration. Then for every* $n \geqslant 0$*,*

$$m(x_{n+1}, x^*) \leqslant \alpha\, m(x_n, x^*) \leqslant \alpha^{n+1} m(x_0, x^*).$$

In particular, the convergence is at least geometric with ratio α*.*

Proof Since $x^* = Tx^*$ and each pair (x_n, x^*) is admissible in the cyclic estimate (because $x^* \in A \cap B$), we have

$$m(x_{n+1}, x^*) = m(Tx_n, Tx^*) \leqslant \alpha\, m(x_n, x^*),$$

and the bound follows by iteration. ☐

Corollary 5.4 *Under the hypotheses of Theorem* 5.1*, let* $\{x_n\}$ *and* $\{y_n\}$ *be Picard iterates* *generated from two initial points* $x_0, y_0 \in A \cup B$*. Then for all* $n \geqslant 0$*,*

$$m(x_n, y_n) \leqslant \alpha^n \, m(x_0, y_0)$$

whenever $(x_0, y_0) \in A \times B$ *or* $(x_0, y_0) \in B \times A$*. Consequently, the iteration is asymptotically stable:* $m(x_n, y_n) \to 0$*.*

Proof If (x_0, y_0) is a mixed pair, then by cyclicity each (x_n, y_n) remains a mixed pair. Hence the contraction inequality yields

$$m(x_{n+1}, y_{n+1}) = m(Tx_n, Ty_n) \leqslant \alpha \, m(x_n, y_n),$$

and iteration gives the estimate. □

Remark 5.1 Theorem 5.1 provides three key practical outputs: existence and uniqueness of an equilibrium $x^* \in A \cap B$, global convergence of alternating Picard orbits to x^*, and a quantitative geometric convergence rate governed by the cyclic contraction constant α. These features are the supermetric analogue of the classical Banach contraction mechanism [3] and [9].

Proposition 5.1 *Let* $X = \mathbb{R}^2$ *and write* $x = (x_1, x_2)$*,* $y = (y_1, y_2)$*. Equip* X *with the* ℓ^1 *Cnorm* $\|x\|_1 := |x_1| + |x_2|$ *and fix* $\gamma \in [0, 1)$*. Define*

$$m(x, y) := \|x - y\|_1 + \gamma\big(\|x\|_1 + \|y\|_1\big), \qquad x, y \in X.$$

Let

$$A := \{x \in X : \ x_1 \geqslant 0\}, \qquad B := \{x \in X : \ x_1 \leqslant 0\}.$$

For a fixed $\beta \in (0, 1)$ *define* $T : A \cup B \to A \cup B$ *by*

$$T(x_1, x_2) := (-\beta x_1, \ \beta x_2).$$

Then T *is a cyclic contraction on* $A \cup B$ *(in the sense of Definition* 4.1*,* A *and* B *are closed in* (X, m)*, and the unique fixed point of* T *is* $(0, 0) \in A \cap B$*. Consequently, Theorem* 5.1 *applies and every Picard orbit converges to* $(0, 0)$*.*

Proof We first note that m is nonnegative and symmetric because $\| \cdot \|_1$ is a norm. If $m(x, y) = 0$, then $\|x - y\|_1 = 0$, hence $x = y$; therefore (SM1) holds. To verify the supermetric control axiom, fix an arbitrary $z \in X$ and choose a nonzero vector $u \in X$. Define distinct sequences $x_n := \frac{u}{n}$ and $y_n := \frac{u}{n+1}$. Then

$$\|x_n - y_n\|_1 = \left\| \frac{u}{n} - \frac{u}{n+1} \right\|_1 = \frac{\|u\|_1}{n(n+1)} \to 0, \qquad \|x_n\|_1 + \|y_n\|_1 \to 0,$$

so $m(x_n, y_n) \to 0$. Moreover, $x_n \to 0$ and $y_n \to 0$ in $\| \cdot \|_1$, hence by continuity of the norm,

$$m(x_n, z) = \|x_n - z\|_1 + \gamma(\|x_n\|_1 + \|z\|_1) \to \|z\|_1 + \gamma\|z\|_1,$$

and similarly $m(y_n, z) \to \|z\|_1 + \gamma\|z\|_1$. Thus

$$\limsup_{n\to\infty} m(y_n, z) = \limsup_{n\to\infty} m(x_n, z),$$

and therefore the defining inequality in (SM3) holds for any constant $s > 1$ (for instance $s = 2$). Hence (X, m) is a supermetric space in the sense of Definition 2.1; in particular, since $\mathbb{R}^2$ is complete in $\|\cdot\|_1$, the induced Cauchy notion from m is complete as well (equivalently, one may invoke the completeness mechanism established for supermetric contractions in [9].

The sets A and B are closed because the coordinate map $x \mapsto x_1$ is continuous under $\|\cdot\|_1$, hence also under m on X, and $A = \{x : \ x_1 \geqslant 0\}$, $B = \{x : \ x_1 \leqslant 0\}$ are inverse images of closed rays.

We now check the cyclic property. If $x \in A$, then $x_1 \geqslant 0$, so $T(x)_1 = -\beta x_1 \leqslant 0$ and thus $T(x) \in B$. If $x \in B$, then $x_1 \leqslant 0$, so $T(x)_1 = -\beta x_1 \geqslant 0$ and thus $T(x) \in A$. Hence $T(A) \subseteq B$ and $T(B) \subseteq A$.

For the cyclic contraction inequality, let $x \in A$ and $y \in B$. Then

$$\|Tx - Ty\|_1 = \|(-\beta x_1, \beta x_2) - (-\beta y_1, \beta y_2)\|_1 = \|\big(-\beta(x_1 - y_1), \beta(x_2 - y_2)\big)\|_1 = \beta\|x - y\|_1,$$

and also

$$\|Tx\|_1 + \|Ty\|_1 = \beta\|x\|_1 + \beta\|y\|_1 = \beta(\|x\|_1 + \|y\|_1).$$

Substituting into the definition of m yields the exact scaling identity

$$m(Tx, Ty) = \beta\|x - y\|_1 + \gamma\beta(\|x\|_1 + \|y\|_1) = \beta\, m(x, y).$$

Thus (1) holds with $\alpha = \beta \in (0, 1)$, so T is a cyclic contraction.

Finally, if $x^* = (x_1^*, x_2^*)$ is a fixed point, then

$$(x_1^*, x_2^*) = T(x_1^*, x_2^*) = (-\beta x_1^*, \beta x_2^*),$$

which implies $(1 + \beta)x_1^* = 0$ and $(1 - \beta)x_2^* = 0$. Since $\beta \in (0, 1)$, we conclude $x_1^* = x_2^* = 0$. Hence the fixed point is unique and equals $(0, 0) \in A \cap B$.

All hypotheses of Theorem 5.1 are satisfied; therefore, for every $x_0 \in A \cup B$ the Picard iteration $x_{n+1} = Tx_n$ converges to $(0, 0)$ in the supermetric m. $\square$

§6. Application to a Nonlinear Integral Equation

Let $X = C([0, 1], \mathbb{R})$ be the Banach space of all real-valued continuous functions on $[0, 1]$ equipped with the supremum norm

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|.$$

Define the supermetric

$$m(f,g) := \|f-g\|_\infty \qquad (f,g \in X).$$

Then (X,m) is a complete supermetric space, since every metric space is a supermetric space and completeness follows from completeness of the sup-norm space.

Consider the nonlinear integral equation

$$u(x) = \int_0^1 K(x,t,u(t))\,dt, \qquad x \in [0,1], \tag{2}$$

where $K : [0,1]\times[0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous and satisfies the uniform Lipschitz condition: there exists $L \in (0,1)$ such that

$$|K(x,t,r)-K(x,t,q)| \leqslant L|r-q| \quad \text{for all } x,t \in [0,1],\ r,q \in \mathbb{R}. \tag{H}$$

Theorem 6.1 *Assume hypothesis* (H) *holds. Then the nonlinear integral equation* (2) *admits a unique solution* $u^* \in C([0,1],\mathbb{R})$*. Moreover, for any initial function* $u_0 \in X$*, the successive approximation sequence defined by*

$$u_{n+1} = Tu_n$$

converges uniformly on $[0,1]$ *to* u^**.*

Proof Define the operator $T : X \to X$ by

$$(Tf)(x) = \int_0^1 K(x,t,f(t))\,dt.$$

We first show that T is well defined. Since K is continuous and f is continuous, the mapping $t \mapsto K(x,t,f(t))$ is continuous on $[0,1]$ for each fixed x. Hence $(Tf)(x)$ exists for all $x \in [0,1]$. Standard parameter continuity results for integrals imply that Tf is continuous on $[0,1]$, so $T(X) \subseteq X$.

Next we establish the contraction property. Let $f,g \in X$. Then for each $x \in [0,1]$, using the Lipschitz condition (H),

$$\begin{aligned}
|(Tf)(x)-(Tg)(x)| &= \left|\int_0^1 \big(K(x,t,f(t))-K(x,t,g(t))\big)\,dt\right| \\
&\leqslant \int_0^1 |K(x,t,f(t))-K(x,t,g(t))|\,dt \\
&\leqslant \int_0^1 L|f(t)-g(t)|\,dt \\
&\leqslant L\|f-g\|_\infty.
\end{aligned}$$

Taking supremum over $x \in [0,1]$ yields

$$\|Tf-Tg\|_\infty \leqslant L\|f-g\|_\infty.$$

Hence

$$m(Tf, Tg) \leqslant L\, m(f, g).$$

Thus T is a contraction mapping on (X, m) with contraction constant $L \in (0, 1)$. Since (X, m) is complete, Theorem 5.1 (with $A = B = X$) guarantees that T admits a unique fixed point $u^* \in X$.

Finally, if u^* is a fixed point of T, then

$$u^*(x) = (Tu^*)(x) = \int_0^1 K(x, t, u^*(t))\, dt,$$

so u^* is a solution of the integral equation (2). Conversely, any solution of (2) is a fixed point of T. Hence the solution is unique.

The convergence of the Picard iteration $u_{n+1} = Tu_n$ to u^* follows directly from the contraction property.

For related integral-equation applications involving rational-type nonlinear contractions, see [6]. □

§7. Conclusion

In this work, we developed a cyclic contraction framework in the setting of complete supermetric spaces and established a fixed point theorem ensuring existence, uniqueness, and convergence of Picard iterates. The obtained results extend classical contraction principles to a generalized distance structure where no triangle-type inequality is assumed, thereby remaining fully consistent with the intrinsic axiomatic structure of supermetric spaces.

The cyclic formulation allows contractive behavior to be imposed only across interacting subsets rather than globally, which makes the theory suitable for problems possessing alternating or decomposed domain structures. The convergence analysis is based on the geometric decay of successive orbit distances together with the sequence-control mechanism that characterizes supermetric spaces.

The applicability of the theoretical results was demonstrated through an application to a nonlinear integral equation, where existence and uniqueness of continuous solutions were obtained via the supermetric contraction framework. This confirms that cyclic supermetric methods can be effectively applied to nonlinear functional problems.

Future research directions include extensions to multi-cyclic structures, rational and implicit cyclic contractions, stochastic supermetric models, and applications to optimization, differential equations, and computational mathematics.

Acknowledgement

The authors express their sincere gratitude to the Institute for Excellence in Higher Education, Bhopal, for providing academic support and a conducive research environment for carrying out this research work. The authors are also deeply thankful to the anonymous referees and the editors for their constructive comments, valuable suggestions, and careful review, which significantly improved the quality and presentation of this manuscript.

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International J.Math. Combin. Vol.1(2026), 104-118

A Note on Grill and Primal:
Relationship to Graph-Width Parameters

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Abstract: Graph width parameterssuch as tree-width and branch-widthare fundamental tools for measuring the structural complexity of graphs and for enabling efficient algorithms on restricted graph classes. Filters, grills, and primals are classical set-theoretic constructs that formalize notions of *largeness* and primality in set theory, topology, and algebra. Research on the relationship between grills and primals and graph width parameters has not been extensively developed. Therefore, in this paper, we extend the definitions of grills and primals to the setting of symmetric submodular connectivity systems and show that branch-width naturally arises as their dual notion. This duality unifies concepts from graph decomposition and topological set theory, offering new insights into the interplay between connectivity and combinatorial obstructions.

Key Words: Grill, filter, primal, weak filter, branch-width.

AMS(2010): 05C62, 05C83.

§1. Introduction

1.1 Filter, Grill and Primal

In set theory and related fields, the notions of *filter*, *ideal*, *grill* and *primal* play significant roles. Filters are families of sets used to identify *large* subsets of a universal set. They serve as fundamental tools in set theory [1, 2], topology [3], graph theory [4, 5], neutrosophic theory [6, 7], and fuzzy theory [8]. Filters underpin key concepts such as convergence and compactness [9, 10]. Grills [11-13] and primals [14] are closely related structures that, under dual closure properties, serve complementary purposes in topology and lattice theory. A maximal filter is called an *ultrafilter*, and owing to its theoretical importance, it has been extensively studied in a manner similar to filters.

1.2 Graph Width Parameters

Graph theory is the study of graphs, mathematical structures consisting of vertices connected by edges, modeling pairwise relationships [15, 16]. A graph parameter is a numerical invariant assigned to a graph, quantifying specific structural, combinatorial, or algorithmic prop-

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[1] Received August 11, 2025, Accepted February 25, 2026

erties. Graph width parameters measure the *width* of a graph under various hierarchical decompositions, offering insight into its structural complexity [15]. Prominent examples include: tree-width [17], path-width [18-19], branch-width [20], Boolean-width [21], linear-width [22], twin-width [23-25], path-distance-width [26, 27], clique-width [28, 29], hypertree-width [30-32], and superhypertree-width [33, 34]. Many computationally hard problems become tractable on graph classes of bounded width. Among these, branch-widthdefined via a branch decomposition that associates edges of the graph with the leaves of a treeholds a central position [35-37].

1.3 Connectivity Systems

A symmetric submodular function is a set function invariant under complement and satisfying diminishing returns across set unions and intersections. A *connectivity system* is a pair (X, f), where X is a finite set and $f : 2^X \to \mathbb{N}$ is a symmetric submodular function [20]. Connectivity systems generalize graph cut functions and provide a unifying framework for studying width parameters such as tree-width and branch-width [38, 39]. It is known that ultrafilters and branch-width are in a dual relationship, meaning that the existence of an ultrafilter determines the value of the branch-width.

1.4 Contribution of this Paper

From the above discussion, it is clear that research on ultrafilters and graph width parameters is important. However, this line of study is still in its early stages, and it cannot yet be said that a wide range of related research has been conducted. In particular, grills in the context of connectivity systems have received little attention. Motivated by this gap, in this paper we extend the classical notions of *grills* and *primals* to connectivity systems and establish a duality theorem linking these structures to the graph width parameter branch-width. This unexpected correspondence not only deepens our theoretical understanding of graph parameters but also suggests new directions for algorithmic and structural investigations in discrete mathematics.

§2. Preliminaries

In this section, we briefly present the definitions and notations used throughout this paper. Unless otherwise stated, all concepts discussed herein are assumed to be defined over finite sets.

A *partition* of a set X is a collection of nonempty, pairwise disjoint subsets $\{X_i\}_{i\in I} \subseteq 2^X$ such that $\bigcup_{i\in I} X_i = X$. Each subset X_i is referred to as a *block* (or *part*) of the partition. For additional background on the fundamentals of set theory, the reader is referred to standard texts such as [40, 41].

2.1 Grills and Primals in Set Theory

This subsection presents the formal definitions of *primals* and *grills*, which are dual combinatorial structures defined over a finite set. These notions arise naturally in topology and lattice theory and play a key role in the duality analysis of set systems.

Definition 2.1(Primal and Grill) *Let X be a finite nonempty set.*

- *A primal $P \subseteq \mathcal{P}(X)$ is a nonempty family of subsets of X satisfying the following conditions:*

(1) *$X \notin P$;*

(2) *If $A \in P$ and $B \subseteq A$, then $B \in P$ (downward closed);*

(3) *If $A \cap B \in P$, then $A \in P$ or $B \in P$ (prime under intersection).*

- *A grill $G \subseteq \mathcal{P}(X)$ is a nonempty family of subsets of X satisfying the following conditions:*

(1) *$\emptyset \notin G$;*

(2) *If $A \in G$ and $A \subseteq B$, then $B \in G$ (upward closed);*

(3) *If $A \cup B \in G$, then $A \in G$ or $B \in G$ (prime under union).*

Example 2.2(Examples of Primal and Grill) Let $X = \{1, 2, 3\}$, and consider the following two families of subsets of X:

(1) A primal:

$$P = \big\{\{1\}, \{2\}, \{1, 2\}\big\}.$$

(2) A grill:

$$G = \big\{\{1, 2\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

We verify that these families satisfy the respective axioms:

(1) Primal P:

- $X = \{1, 2, 3\} \notin P$.
- The family is downward closed: since $\{1, 2\} \in P$, both $\{1\}$ and $\{2\} \subseteq \{1, 2\}$ are also in P.
- The family is prime under intersection: for any $A, B \in \mathcal{P}(X)$, if $A \cap B \in P$, then at least one of A or B belongs to P.

(2) Grill G:

- $\emptyset \notin G$.
- The family is upward closed: since $\{1, 2\} \in G$ and $\{1, 2\} \subseteq \{1, 2, 3\}$, it follows that $\{1, 2, 3\} \in G$.
- The family is prime under union: for instance, $\{1, 2\}, \{2, 3\} \in G$, and their union $\{1, 2, 3\} \in G$ implies that at least one of the components is in G.

2.2 **Connectivity Systems**

The concept of a *symmetric submodular function* plays a fundamental role in various areas of discrete mathematics and combinatorial optimization [42-44]. While such functions are typically defined over the real numbers, we focus in this paper on the subclass of symmetric submodular functions that take values in the set of natural numbers $\mathbb{N}$.

Definition 2.3(Symmetric Submodular Function) *Let X be a finite set. A function $f : 2^X \to \mathbb{N}$*

is called symmetric submodular if it satisfies the following two conditions:

(1) (Symmetry) *For all $A \subseteq X$,*

$$f(A) = f(X \setminus A).$$

(2) (Submodularity) *For all $A, B \subseteq X$,*

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B).$$

Definition 2.4(Connectivity System) *A connectivity system is a pair (X, f), where X is a finite set and $f : 2^X \to \mathbb{N}$ is a symmetric submodular function.*

Connectivity systems serve as a general framework for modeling connectivity in graphs and matroids. They are particularly important in the study of graph width parameterssuch as branch-width and tree-widthwhich rely on symmetric submodular functions to capture structural complexity [20].

Example 2.5(Graph-Induced Connectivity Function) Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, and let $X = E$. Define the function $f : 2^X \to \mathbb{N}$ by

$$f(A) := \text{number of vertices incident with both an edge in } A \text{ and an edge in } X \setminus A.$$

In other words, $f(A)$ counts the number of vertices that are *shared* between edges in A and those in $X \setminus A$.

Claim. f is symmetric and submodular and hence (X, f) forms a connectivity system:

- (*Symmetry.*) By definition, the number of shared vertices between A and $X \setminus A$ is equal to that between $X \setminus A$ and A, so $f(A) = f(X \setminus A)$.
- (*Submodularity.*) For any $A, B \subseteq X$, the inequality

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

holds. This follows from standard results on cut-rank and edge boundary functions in graphs.

Hence, (X, f) defines a valid connectivity system derived from a graph structure.

2.3 **Filters on Connectivity Systems**

We now introduce the notion of a *filter* on a connectivity system. This concept generalizes the classical idea of a filter in set theory by incorporating a submodular constraint [45, 46]. Filters on connectivity systems are known to serve as *obstructions* that constrain the values of graph width parameters. Other prominent examples of obstructions include *tangles* [38, 47-50] and *loose tangles* [20, 51]. In particular, it has been shown that filters are *complementarily equivalent* to loose tangles [45], highlighting their structural duality.

Definition 2.6(Filter of Order $k+1$, [46]) *Let X be a finite set, and let $f : 2^X \to \mathbb{N}$ be a symmetric submodular function. A nonempty family of subsets $\mathcal{F} \subseteq 2^X$ is called a* filter of order $k+1$ *on the connectivity system (X, f) if it satisfies the following axioms:*

$(F0)$ For all $A \in \mathcal{F}$, we have $f(A) \leq k$;

$(F1)$ If $A, B \in \mathcal{F}$ and $f(A \cap B) \leq k$, then $A \cap B \in \mathcal{F}$ (closed under intersection when small enough);

$(F2)$ If $A \in \mathcal{F}$, $A \subseteq B \subseteq X$, and $f(B) \leq k$, then $B \in \mathcal{F}$. (upward closed under bounded f);

$(F3)$ $\emptyset \notin \mathcal{F}$ (nontriviality).

Example 2.7(Example of a Filter of Order $k+1$) Let $G = (V, E)$ be the undirected graph with vertex set $V = \{1, 2, 3, 4\}$ and edge set

$$X = E = \{e_1 = \{1,2\},\ e_2 = \{2,3\},\ e_3 = \{3,4\},\ e_4 = \{4,1\}\},$$

i.e., G is a 4-cycle.

Define the function $f : 2^X \to \mathbb{N}$ by

$$f(A) := \text{number of vertices incident to both an edge in } A \text{ and an edge in } X \setminus A.$$

It is well known that f is symmetric and submodular, hence (X, f) forms a connectivity system.

Now fix $k = 1$, and define the subset family

$$\mathcal{F} = \{A \subseteq X \mid f(A) \leq 1,\ A \neq \emptyset\}.$$

We verify that $\mathcal{F}$ satisfies the axioms of a filter of order $k + 1 = 2$:

$(F0)$ By definition, all $A \in \mathcal{F}$ satisfy $f(A) \leq 1$;

$(F1)$ Let $A, B \in \mathcal{F}$, and suppose $f(A \cap B) \leq 1$. Then $A \cap B \in \mathcal{F}$ by construction;

$(F2)$ If $A \in \mathcal{F}$, $A \subseteq B \subseteq X$, and $f(B) \leq 1$, then $B \in \mathcal{F}$;

$(F3)$ $\emptyset \notin \mathcal{F}$ by definition.

Hence, $\mathcal{F}$ is a filter of order 2 on the connectivity system (X, f).

§3. **Results: Extension to Connectivity Systems**

In this section, we extend the classical notions of *primal* and *grill* from set theory to the setting of connectivity systems. The following definitions introduce the respective generalizations.

Definition 3.1(Primal and Grill of Order $k+1$ on a Connectivity System) *Let (X, f) be a connectivity system, where X is a finite set and $f : 2^X \to \mathbb{N}$ is a symmetric submodular function. Let $k \in \mathbb{N}$. Define*

- *A nonempty family $\mathcal{P} \subseteq 2^X$ is called a **primal of order** $k+1$ if*

(1) $X \notin \mathcal{P}$;

(2) $\forall A \in \mathcal{P},\ f(A) \leq k$;
(3) *If* $A \in \mathcal{P}$, $B \subseteq A$, *and* $f(B) \leq k$, *then* $B \in \mathcal{P}$;
(4) *If* $A \cap B \in \mathcal{P}$ *and* $f(A) \leq k$, $f(B) \leq k$, *then* $A \in \mathcal{P}$ *or* $B \in \mathcal{P}$.
A nonempty family $\mathcal{G} \subseteq 2^X$ *is called a* ***grill of order*** $k+1$ *if*

(1) $\emptyset \notin \mathcal{G}$;
(2) $\forall A \in \mathcal{G},\ f(A) \leq k$;
(3) *If* $A \in \mathcal{G}$, $A \subseteq B \subseteq X$, *and* $f(B) \leq k$, *then* $B \in \mathcal{G}$;
(4) *If* $A \cup B \in \mathcal{G}$ *and* $f(A) \leq k$, $f(B) \leq k$, *then* $A \in \mathcal{G}$ *or* $B \in \mathcal{G}$.

Example 3.2(Primal and Grill of Order $k+1=2$ on a Toy Connectivity System) Let $X = \{1,2,3\}$, and define

$$f\colon 2^X \to \mathbb{N}, \quad f(A) \ = \ \min\bigl(|A|,\ |X \setminus A|\bigr).$$

One checks easily that f is symmetric and submodular, so (X,f) is a connectivity system. Take $k=1$, so we consider order $k+1=2$.

(1) The Primal of Order 2. Define

$$\mathcal{P} = \bigl\{\, A \subseteq X \mid f(A) \leqslant 1,\ A \neq X \bigr\} = \bigl\{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\bigr\}.$$

- $X \notin \mathcal{P}$ by construction;
- Every $A \in \mathcal{P}$ satisfies $f(A) \leqslant 1$;
- Downward closure: if $A \in \mathcal{P}$ and $B \subseteq A$ with $f(B) \leqslant 1$, then $B \in \mathcal{P}$;
- Primality under intersection: whenever $A, B \subseteq X$ both satisfy $f(\cdot) \leqslant 1$ and $A \cap B \in \mathcal{P}$, then at least one of A, B lies in $\mathcal{P}$.

(2) The Grill of Order 2. Define

$$\mathcal{G} = \bigl\{\, A \subseteq X \mid f(A) \leqslant 1,\ A \neq \varnothing \bigr\} = \bigl\{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\bigr\}.$$

- $\varnothing \notin \mathcal{G}$ by construction;
- Every $A \in \mathcal{G}$ satisfies $f(A) \leqslant 1$;
- Upward closure: if $A \in \mathcal{G}$ and $A \subseteq B \subseteq X$ with $f(B) \leqslant 1$, then $B \in \mathcal{G}$;
- Grill- primality under union: whenever $A, B \subseteq X$ both satisfy $f(\cdot) \leqslant 1$ and $A \cup B \in \mathcal{G}$, then at least one of A, B lies in $\mathcal{G}$.

One checks readily that $\mathcal{P}$ and $\mathcal{G}$ satisfy all the axioms of a primal and a grill of order 2, respectively.

Theorem 3.3 *Let* (X,f) *be any connectivity system and fix* $k \in \mathbb{N}$. *Suppose*

$$f(A) \ \leqslant \ k \quad \textit{for every proper subset } A \subsetneq X.$$

Then,

(1) *The* primal of order $k+1$ *on* (X,f) *is*

$$\mathcal{P} \;=\; \bigl\{ A \subseteq X \mid A \neq X \bigr\},$$

which coincides exactly with the maximal classical primal on X*.*

(2) *The* grill of order $k+1$ *on* (X,f) *is*

$$\mathcal{G} \;=\; \bigl\{ A \subseteq X \mid A \neq \varnothing \bigr\},$$

which coincides exactly with the maximal classical grill on X*.*

In particular, in the toy connectivity system of the example (where $X = \{1,2,3\}$*,* $f(A) = \min\{|A|, |X \setminus A|\}$*, and* $k = 1$*), these families reduce to* $\mathcal{P} = \{ A \subseteq X : A \neq X\}$ *and* $\mathcal{G} = \{ A \subseteq X : A \neq \varnothing\}$*, which are precisely the classical primal and grill of* X*.*

Proof We prove the two assertions in turn.

(1) **Primal case.** By definition the primal of order $k+1$ is

$$\{ A \subseteq X \mid X \notin \mathcal{P},\ f(A) \leqslant k,\ \text{downward closure \& primality under intersections}\}.$$

Since $f(A) \leqslant k$ for every proper $A \subsetneq X$ by hypothesis, the condition $f(A) \leqslant k$ is vacuous on all $A \neq X$. Thus the connectivity primal axioms reduce exactly to:

$$\mathcal{P} = \{ A \subseteq X \mid A \neq X\},$$

and one checks immediately that this family satisfies

$$X \notin \mathcal{P}, \quad (\forall A \in \mathcal{P}, B \subseteq A \implies B \in \mathcal{P}), \quad \text{and } (A \cap B \in \mathcal{P} \implies A \in \mathcal{P} \text{ or } B \in \mathcal{P}),$$

which are exactly the axioms of a classical primal in set theory.

(2) **Grill case.** Similarly, the grill of order $k+1$ is by definition

$$\{ A \subseteq X \mid \varnothing \notin \mathcal{G},\ f(A) \leqslant k,\ \text{upward closure \& primality under unions}\}.$$

Again $f(A) \leqslant k$ for every nonempty $A \subseteq X$, so the boundedness condition is automatic on all $A \neq \varnothing$. Hence

$$\mathcal{G} = \{ A \subseteq X \mid A \neq \varnothing\},$$

and one checks at once that

$$\varnothing \notin \mathcal{G}, \quad (\forall A \in \mathcal{G}, A \subseteq B \implies B \in \mathcal{G}), \quad \text{and } (A \cup B \in \mathcal{G} \implies A \in \mathcal{G} \text{ or } B \in \mathcal{G}),$$

which are exactly the axioms of a classical grill in set theory.

Thus under the stated condition on f, the connectivity-system notions of primal and grill coincide with their classical set-theoretic counterparts. In particular, in the toy example with $f(A) = \min\{|A|, |X \setminus A|\}$ and $k = 1$, one sees immediately that $f(A) \leqslant 1$ for every proper

$A \subsetneq X$, so the families become all non-X subsets (primal) and all non-empty subsets (grill), as claimed. □

Theorem 3.4 *Let (X, f) be a connectivity system with f a symmetric submodular function and let $k \in \mathbb{N}$. Then, any grill or primal of order $k+1$ on (X, f) corresponds to a filter of order $k+1$ on the same system.*

Proof The proof follows from the observation that both the primal and grill satisfy the axioms of a filter of order $k+1$, namely boundedness under f, closure under intersection or union under certain conditions, and monotonicity with respect to inclusion. The primal corresponds to the dual of a filter under complement, and the grill represents a filter closed under unions rather than intersections. Hence, each structure captures the essence of a filter under appropriate dual notions. □

§4. Results: Weak Grill in Set Theory and Logic

The notion of a *weak filter* has been widely studied in the context of logic and knowledge representation [52-55]. More recently, this concept has been generalized to the framework of connectivity systems [56]. In this section, we further introduce the notions of *weak grill* and *weak primal* within connectivity systems.

Definition 4.1(Weak Filter of Order $k+1$,[56]) *Let (X, f) be a connectivity system, where X is a finite set and $f : 2^X \to \mathbb{N}$ is a symmetric submodular function. A nonempty family $W \subseteq 2^X$ is called a* weak filter of order $k+1$ *if the following conditions hold:*

(FB) $\forall A \in W,\ f(A) \le k$ (bounded by k)*;*
(FH) *If $A \in W$, $A \subsetneq B \subseteq X$, and $f(B) \le k$, then $B \in W$* (hereditarily upward closed)*;*
(WIS) *If $A, B \in W$ and $f(A \cap B) \le k$, then $A \cap B \neq \emptyset$* (weak intersection condition)*;*
(FW) $\emptyset \notin W$ (nontriviality).

Example 4.2(Weak Filter of Order $k+1 = 2$ on the Toy Connectivity System) Let $X = \{1, 2, 3\}$, and define

$$f\colon 2^X \to \mathbb{N}, \quad f(A) = \min\bigl(|A|,\ |X \setminus A|\bigr).$$

Then (X, f) is a connectivity system. Take $k = 1$, so we consider order $k+1 = 2$.

Define the family

$$W \;=\; \bigl\{\, A \subseteq X \mid 1 \in A,\ f(A) \leqslant 1 \bigr\} \;=\; \bigl\{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\bigr\}.$$

We verify that W satisfies the axioms of a weak filter of order 2:

(FB) Boundedness: For each $A \in W$,

$$f(A) \;\leqslant\; 1,$$

since $\min(|A|, |X \setminus A|) \leqslant 1$ for all these subsets.

(FH) Hereditary upward closure. Whenever $A \in W$ and $A \subsetneq B \subseteq X$ with $f(B) \leqslant 1$, then $1 \in A \subseteq B$ and $f(B) \leqslant 1$ imply $B \in W$. For example, $\{1\} \subsetneq \{1,2\}$ and $f(\{1,2\}) = 1$, so $\{1,2\} \in W$.

(WIS) Weak intersection condition. If $A, B \in W$ and $f(A \cap B) \leqslant 1$, then $A \cap B$ must be nonempty. Indeed, every set in W contains 1, so $A \cap B \supseteq \{1\} \neq \varnothing$.

(FW) Nontriviality. $\varnothing \notin W$ by construction.

Hence W is a weak filter of order 2 on (X, f).

Definition 4.3(Weak Primal and Weak Grill) *Let (X, f) be a connectivity system, where X is a finite set and $f : 2^X \to \mathbb{N}$ is a symmetric submodular function. We define the following:*

- *A family $\mathcal{P} \subseteq 2^X$ is called a* weak primal of order $k+1$ *if*

(1) $X \notin \mathcal{P}$;
(2) $\forall A \in \mathcal{P},\ f(A) \leq k$;
(3) *If $A \in \mathcal{P}$, $B \subseteq A$, and $f(B) \leq k$, then $B \in \mathcal{P}$;*
(4) *If $f(A) \leq k$, $f(B) \leq k$, and $A \cap B \neq \emptyset$, then $A \in \mathcal{P}$ or $B \in \mathcal{P}$.*

- *A family $\mathcal{G} \subseteq 2^X$ is called a* weak grill of order $k+1$ *if*

(1) $\emptyset \notin \mathcal{G}$;
(2) $\forall A \in \mathcal{G},\ f(A) \leq k$;
(3) *If $A \in \mathcal{G}$, $A \subseteq B \subseteq X$, and $f(B) \leq k$, then $B \in \mathcal{G}$;*
(4) *If $f(A) \leq k$, $f(B) \leq k$, and $A \cup B \neq X$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$.*

Example 4.4(Weak Primal and Weak Grill of Order $k+1 = 2$ on a Toy Connectivity System) Let $X = \{1, 2, 3\}$ and define

$$f(A) \ = \ \min\bigl(|A|,\ |X \setminus A|\bigr).$$

One checks easily that f is symmetric and submodular. In particular,

$$f(\emptyset) = 0,\ f(\{i\}) = 1\ (i = 1, 2, 3),\ f(\{i, j\}) = 1,\ f(X) = 0.$$

Set $k = 1$, so we consider order $k + 1 = 2$.

(1) Weak Primal. Define

$$\mathcal{P} = \bigl\{\, A \subseteq X \mid A \neq X,\ f(A) \leqslant 1 \bigr\} = \bigl\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\bigr\}.$$

We verify the weak primal axioms:

(a) $X \notin \mathcal{P}$ by construction.

(b) For every $A \in \mathcal{P}$, one has $f(A) \leqslant 1$.

(c) Downward closure: if $A \in \mathcal{P}$ and $B \subseteq A$ with $f(B) \leqslant 1$, then $B \subsetneq X$ and $f(B) \leqslant 1$, so $B \in \mathcal{P}$.

(d) Weak intersection. If $f(A) \leqslant 1$, $f(B) \leqslant 1$, and $A \cap B \neq \emptyset$, then both A and B are proper subsets with $f \leqslant 1$, hence lie in $\mathcal{P}$; in particular at least one of them does.

(2) Weak Grill. Define

$$\mathcal{G} = \{\, A \subseteq X \mid A \neq \emptyset,\ f(A) \leqslant 1 \} = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

We verify the weak grill axioms:

(a) $\emptyset \notin \mathcal{G}$ by construction.

(b) For every $A \in \mathcal{G}$, one has $f(A) \leqslant 1$.

(c) Hereditary upward closure: if $A \in \mathcal{G}$ and $A \subseteq B \subseteq X$ with $f(B) \leqslant 1$, then $B \neq \emptyset$ and $f(B) \leqslant 1$, so $B \in \mathcal{G}$.

(d) Weak union. If $f(A) \leqslant 1$, $f(B) \leqslant 1$, and $A \cup B \neq X$, then $A \cup B$ is a nonempty proper subset with $f \leqslant 1$, so $A \cup B \in \mathcal{G}$; hence at least one of A or B must already lie in $\mathcal{G}$.

Thus $\mathcal{P}$ and $\mathcal{G}$ indeed form a weak primal and a weak grill of order 2 on (X, f).

Theorem 4.5 *Let (X, f) be a connectivity system with symmetric submodular function f. Then every grill of order $k+1$ is a weak grill of order $k+1$, and every primal of order $k+1$ is a weak primal of order $k+1$.*

Proof This follows directly from the fact that the axioms of a grill (resp. primal) imply those of a weak grill (resp. weak primal). In particular, the conditions involving unions and intersections are weakened in the latter definitions. □

Theorem 4.6 *Let (X, f) be a connectivity system. Then every weak grill or weak primal of order $k+1$ is equivalent to a weak filter of order $k+1$.*

Proof Each definition encodes the same bounding condition $f(A) \leq k$, the monotonicity with respect to inclusion, and a form of weak consistency under union or intersection. Thus, the structures are logically equivalent under these criteria. □

Remark 4.7 We note that the concepts of *weak grill* and *weak primal* can also be defined in standard set theory without reference to submodularity. However, the symmetric submodular structure allows these objects to reflect more nuanced combinatorial constraints, particularly in graph-theoretic applications.

§5. Conclusion and Future Work

This paper introduced and analysed the notions of *grill* and *primal* in the setting of connectivity systems. Our results clarify how these two structures behave under symmetric submodular connectivity functions and highlight their dual relationship with branch-width.

Future research will pursue two main directions. First, we will extend grills and primals to uncertainty frameworks such as fuzzy sets [57, 58], intuitionistic fuzzy sets [59, 60], hyperfuzzy Sets [61C63], neutrosophic Sets [64,65], and plithogenic sets [66C68]. Second, we aim to investigate how the concepts developed here interact with hypergraphs [69,70] and superhypergraphs [71C73], with the goal of enriching both theories and uncovering new applications.

Acknowledgments

We sincerely thank all those who offered insights, inspiration, and assistance during the course of this research. Special appreciation is extended to our readers for their interest and to the authors of cited works, whose contributions laid the groundwork for this study. We also acknowledge the individuals and institutions that provided the resources and infrastructure needed to produce and disseminate this paper. Finally, we are grateful to everyone who supported us in various capacities throughout this project.

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International J.Math. Combin. Vol.1(2026), 119-131

On (m, s)-Support Regular Graphs

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Abstract: We define $m-$support of vertex in a graph and (m, s)-support regular graph. Then it provides some results on $(2, s)-$support regular graphs and method to construct such graphs as in [6]. Graphs which are induced subgraphs of $(2, s)-$ support regular graphs are constructed and illustrated. The smallest order of $(2, s)-$ support regular graph containing G and G^c as induced subgraphs are constructed.

Key Words: Induced graph, distance degree regular graph, support regular, $(2, s)-$support regular graphs.

AMS(2010): 05C12.

§1. Introduction

Only simple connected graphs are considered. Distance d-regular graphs was introduced by [3]. The concept of 2-regular graphs was introduced by [2]. [15] proved that for any graph with the highest degree as r, G is contained as an induced subgraph (IS) of a r-regular graph. [4] found the least required new vertices which when given to G, we get the r-regular graph. [5], [6] dealt in detail about $(r, m, k)-$regular graphs. [7] constructed minimal $(r, 2, k)-$regular graphs containing G and G^c. With these ideas, we define m-support of a vertex, and detail about $(2, s)-$support regular graphs and its construction similar to [5]-[7].

§2. Preliminaries

Definition 2.1 *G is (k, l) regular, when all node of G is at distance k from exactly l nodes.*

Definition 2.2 *A graph is $(2, l)$ regular when $d_2(v_i) = 1$, for all nodes v_i.*

Definition 2.3 *A graph is (r, k, l) regular whenever $d(v) = r$, and $d_k(v) = l$, for all $v \in V$.*

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[1] Received October 10, 2025, Accepted February 26, 2026

Definition 2.4 *The support of a node is defined as the* $\sum$ *of degrees of its adjacent nodes. That is* $s(z) = \sum\limits_{zx \in E(G)} d(x)$.

Definition 2.5 *G is support regular, if the support of all the vertices are alike.*

§3. m-Support of a Vertex

Definition 3.1 *The m-support of a node x in a graph G is given as* $s_m(x) = \sum\limits_{xy \in E(G)} d_m(y)$, *where* $d_m(y)$ *represent the counting of nodes at a distance m from y.*

Definition 3.2 *In G, 1-support of vertex* $s_1(v) = \sum\limits_{uv \in E(G)} d_1(u)$, *where* $d_1(u)$ *is count of vertices at distance 1 from u.*

Definition 3.3 *The* $2-$*support of a node is* $s_2(x) = \sum\limits_{xy \in E(G)} d_2(y)$ $d_2(y) =$ *count of nodes at distance 2 from y.*

Definition 3.4 *The* $3-$*support of node is* $s_3(v) = \sum\limits_{uv \in E(G)} d_3(u)$, $d_3(u) =$ *count of vertices at distance 3 from u.*

Example 3.5 Consider the following graph.

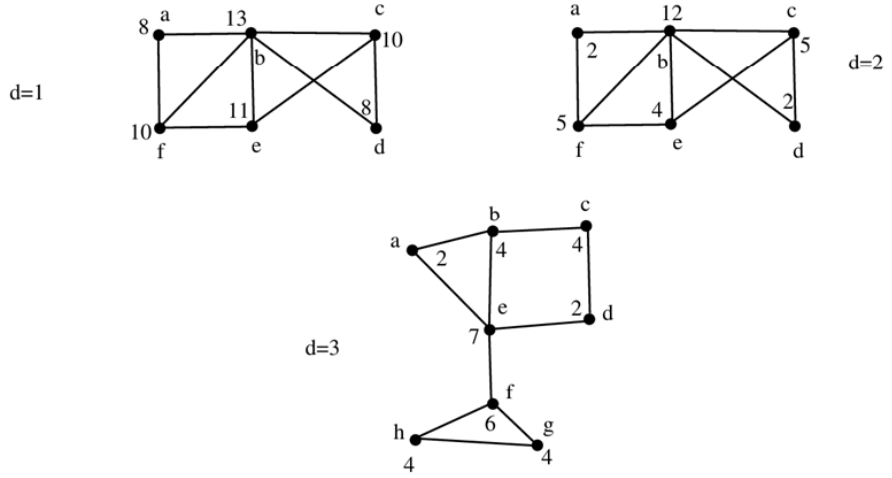


Figure 1. The support of vertices are denoted near them at respective distances

§4. (m, s)-Support Regular Graphs

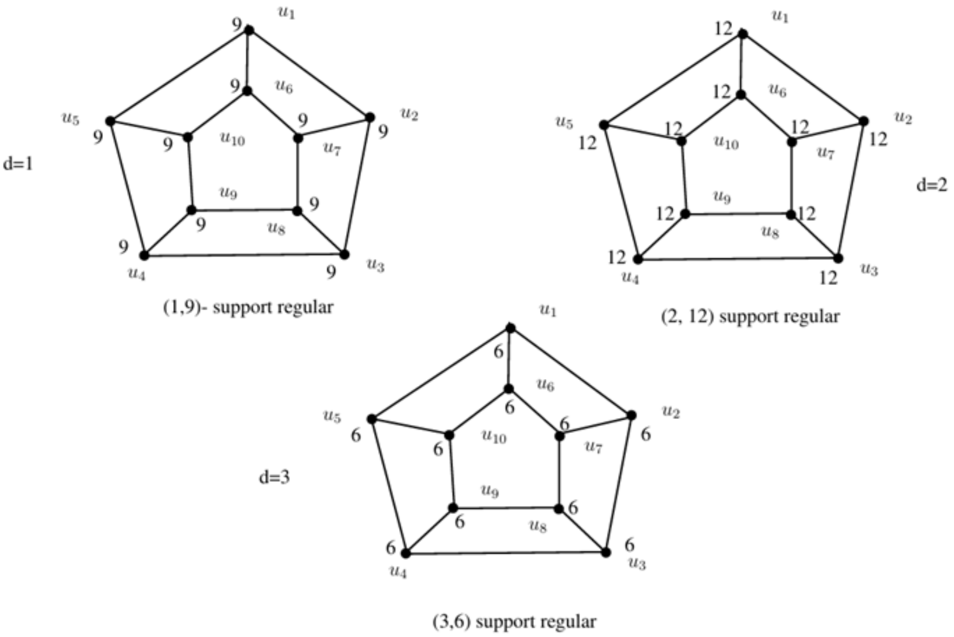
Definition 4.1 *G is said to be (m,s)-support regular, if* $s_m(v_i)$ *is same for all* $v_i \in V(G)$.

Definition 4.2 *G is (1,s)-support regular, if* $s_1(v_i)$ *are alike for all* $v_i \in V(G)$.

Definition 4.3 *G is $(2, s)$ support regular, if $s_2(v_i)$ is same $\forall$ nodes in G.*

Definition 4.4 *A graph is (3,s)-support regular, if $s_3(v_i)$ is same for all $v_i \in V(G)$.*

Example 4.5 Some examples to represent the definitions.



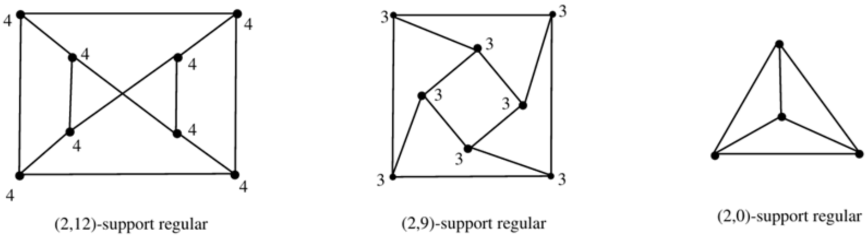
(1,9)- support regular

(2, 12) support regular

(3,6) support regular

Figure 2.

Example 4.6 All regular graphs which are $(2, s)$-support regular.



(2,12)-support regular

(2,9)-support regular

(2,0)-support regular

Figure 3.

Remark 4.7 There are no non-regular $(2, k)$ regular graphs which are $(2, s)$-support regular.

Observation 4.9 A connected graph is (2,0) support regular, iff it is complete graph $K_n, n \geq 1$.

Proposition 4.9 *Every (r, m, k)-regular graphs are (m, rk)-support regular.*

Proof Since K is (r,m,k) regular, i.e., $d(x) = r, d_m(x) = k, \forall$ x in G.

Now, for $v \in V$, $s_m(v) = \sum\limits_{uv \in E} d_m(u) = k + k + \cdots + k$ (r times) $= rk$. This is true for all vertices of G, i.e., $s_m(v_i) = rk$, $\forall v \in V(G)$.

Theorem 4.10 *A (m, rk)-support regular graph consist of atleast $r + k + 1$ vertices.*

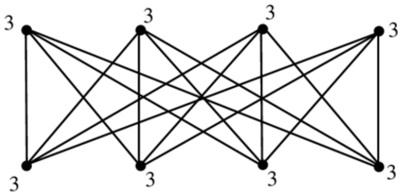
Proof From Result 4.9, (r, m, k) regular graphs are $(m, rk)-$support regular. And the number of vertices in (r, m, k) regular graph is at least r+k+1. $\Rightarrow$ the number of vertices in (m, rk)-support regular graphs is at least $r + k + 1$. □

Observation 4.11 The following are some (2,s)-support regular graphs.

(1) $K_{l,l}$ is (2, l(l-1))-support regular;
(2) Cycle C_{2m} is (2,2)-support regular;
(3) Cycle C_{2m+1} is (2,4)-support regular;
(4) Peterson graph is (2,18)-support regular;
(5) Any r-regular graph with diameter $< a$, is $(a, 0)-$support regular.

Theorem 4.12 *For $n \geq 1$, $\exists$ a $(2,(n-1))-$support regular graph with $|G| = 2n$.*

Proof For $r \geq 1$, a complete bipartite graph $K_{r,r}$ exist, which is $(2, r(r-1))$-support regular of order $2r$. □



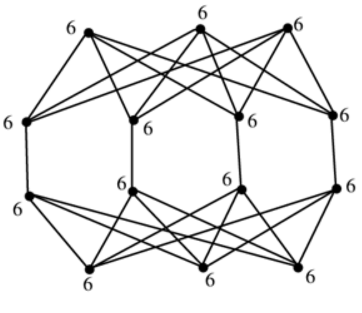
(2,12)-support regular

Figure 4.

Theorem 4.13 *For $n \geq 1$, $\exists$ a $(2, 2n(n-1))-$support regular graph, $|G| = 4n - 2$.*

Proof For any $n \geq 1$, let F_n be formed by 2 disjoint replicas of $K_{n-1,n}$ by adding edge between the nodes of the two replica sets of size n. The new graph is $(2, 2n(n-1))-$support regular and $G = |4r - 2|$. □

Example 4.14 Putting $n = 4$ in above theorem we obtain a $(2, 24)$ support regular graph of order 14 shown in Figure 5.



(2,24) support regular

Figure 5.

Observation 4.15 For $n \geq 1$, the minimal cardinality of $(2, n(n-1))-$ support regular graph, say H, which contains $K_{n,n}$ of cardinality 2n is H.

Theorem 4.16 *Graphs of order* $n \geq 2$ *are induced subgraphs of* $(2, 2n(n+1))-$*support regular graphs of order* $5n$.

Proof Take G and $|G| = n \geq 2$, and $V(G) = \{y_1, y_2, \cdots, y_n\}$. Let G_l denote replica of G with $V(G_l) = \left\{y_1^l, y_2^l, \cdots, y_n^l : 1 \leq l \leq 5\right\}$. H_1 as the graph whose vertex set is such as

$$V(H_1) = \bigcup_{l=1}^{5} V(G_l) = \left\{y_a^l : 1 \leq a \leq n, 1 \leq l \leq 5\right\}.$$

The edge set is given as

$$\begin{aligned} E(H_1) \quad = \quad & \bigcup_{l=1}^{5} E(G_l) \bigcup_{l=1}^{4} \left\{y_b^l y_a^{l+1}, y_b^5 y_a^1 : y_b^1 y_a^1 \notin E(G_1)\right\} \text{(where } 1 \leq b \leq n, b+1 \leq a \leq n \leq b) \\ & \bigcup_{f=1}^{n} \left\{y_f^a y_f^{a+1}, y_f^5 y_f^1 : 1 \leq a \leq 4\right\}. \end{aligned}$$

So, H_1 thus obtained contains G as IS. Now, in H_1, $d(y_a^l) = n+1, 1 \leq a \leq n$ and $d_2(y_a^l) = 2n, 1 \leq l \leq 5$. This implies $s_2(y_a) = \sum\limits_{y_a^l y_b^l \in E(H_1)} d_2(y_b) = 2n + 2n + \cdots$ $((n+1)$ times $= 2n(n+1))$. Thus, H_1 is $(2, 2n(n+1))-$support regular graph and $|H| = 5n$ containing G as an IS. □

Example 4.17 A graph constructed based on above theorem for $n = 4$ is shown in Figure 6..

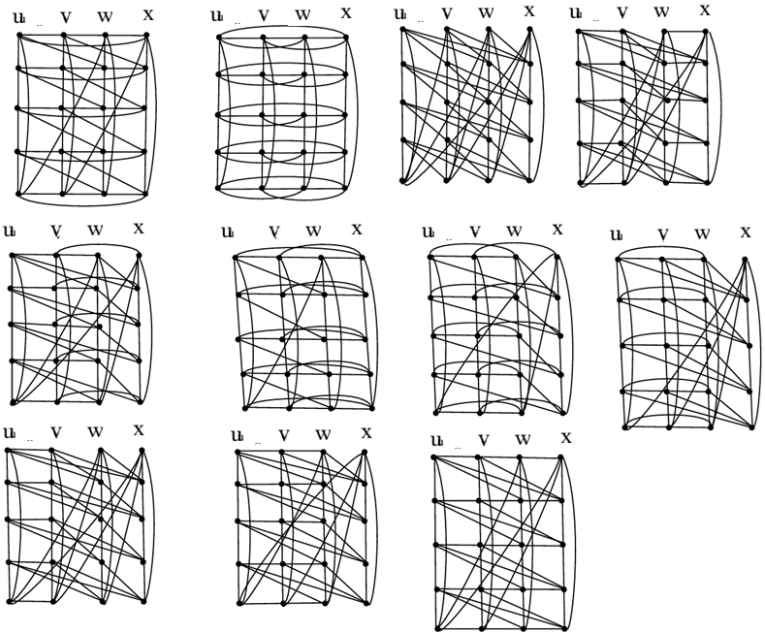


Figure 6.

Corollary 4.18 *Every G of cardinality* $n \geq 2$ *are IS of* $(2,(2n-2)\,n+1)$*-support regular graphs with cardinality* $3n$*.*

Proof We set the value of l as $1 \leq l \leq 3$ in theorem 4.16, we obtain H containing G as an IS. □

Example 4.19 An illustration of above corollary, here we have $n = 4$.

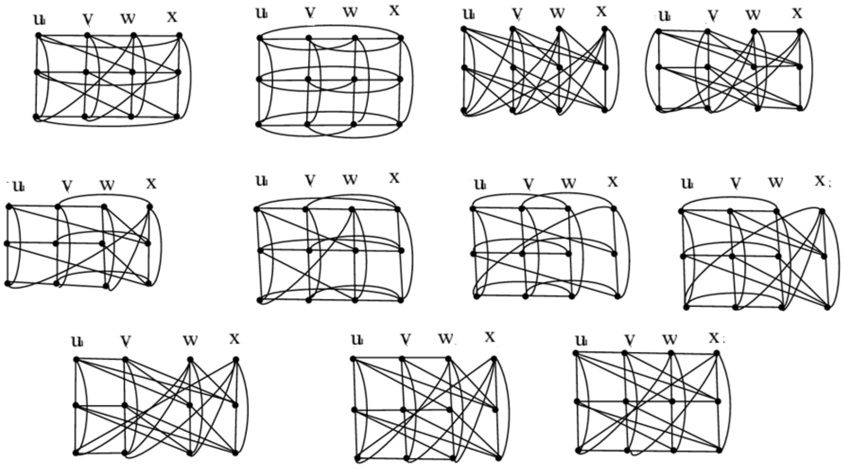


Figure 7.

Corollary 4.20 *Every G with cardinality* $n \geq 2$ *are IS of* $(2,(n+1)(2n-1))-$ *support regular graphs with cardinality* $4n$*.*

Proof Taking 4 replica of G in Theorem 4.16, we get the desired result. □

Example 4.21 An illustration of Corollary 4.20, for $n = 3$, we obtain the following $(2,20)-$support regular graph, say H and $|H| = 12$ which contains G as an IS.

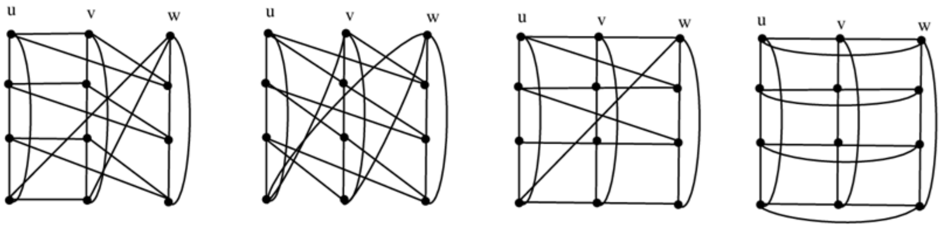


Figure 8.

Theorem 4.22 *For* $r \geq 1$*, every G with* $|G| = n \geq 2$ *is an IS of* $(2,(n+r-1)(nr-1))-$*support regular graph of order* $2nr$*.*

Proof Let there be G and $|G| = n \geq 2$, whose nodes are $V(G) = \{x_1, x_2, \cdots, x_n\}$. And we denotes the copies of G as G_l, whose vertex set is such that $V(G_l) = \{x_1^l, x_2^l, \cdots, x_n^l : 1 \leq l \leq r.\}$.

And G_{k+r} be another copy of given graph, where the vertex set is as

$$V(G_{k+r}) = \left\{y_1^k, y_2^k, \cdots, y_n^k : 1 \le k \le r\right\}.$$

Let H denote the graph with vertex set as $V(H) = \left\{x_a^l, y_a^l : 1 \le a \le n, 1 \le l \le r\right\}$. Then, the lines of H are given as

$$\begin{aligned} E(H) &= \bigcup_{l=1}^{2r} E(G_l) \bigcup_{l=1}^{r} \left\{x_b^l y_a^l, y_b^l x_a^l : x_b^1 x_a^1 \notin E(G_1), 1 \le b \le n, b+1 \le a \le n\right\} \\ & \bigcup_{f=1}^{n} \left\{x_f^a y_f^{a+b} : 1 \le a \le r, 0 \le b \le r-1\right\} \text{(superscripts are taken over modulo r).} \end{aligned}$$

The graph H contains G as an IS. Also, we have $d_H(x_i^l) = d_H(y_i^l) = n + r - 1, (1 \le l \le r)$ and $d_2(x_i^l) = d_2(y_i^l) = nr - 1, (1 \le a \le n)$ in H. Also, $s_2(x_a^l) = s_2(y_a^l) = (n + r - 1)(nr - 1)$ in H. Therefore H is a $(2, (n + r - 1)(nr - 1))-$support regular graph. Hence the proof. □

Example 6.23 For representation of above theorem, here we have $r = 3, n = 3$ in Figure 9.

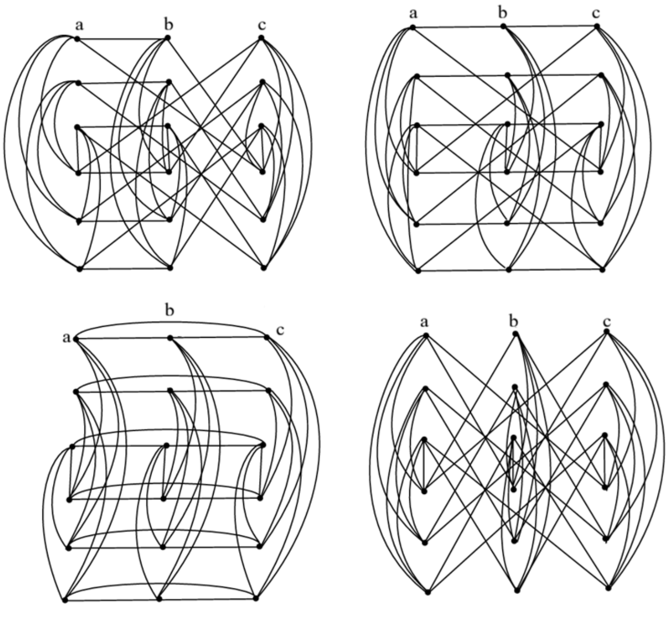


Figure 9.

Corollary 4.24 *Every graph with* $|G| = n \ge 2$*, is an IS of* $(2, n-1)-$*support regular graph with cardinality* $2n$*.*

Proof The case of theorem 4.22, where r=1. Take G such that $|G| = n \ge 2$ with $V(G) = \{z_1, z_2, \cdots, z_n\}$. G_1 be copy of given graph with and $V(G_1) = \left\{z_1^1, z_2^1, \cdots, z_n^1\right\}$ and G_2 be another copy such that $V(G_2) = \left\{x_1^1, x_2^1, ...x_n^1\right\}$. Let H be graph whose vertex set is $V(H) =$

$V(G_1)\bigcup V(G_2)$ and the edge set is

$$\begin{aligned} E(H) &= E(G_1)\bigcup E(G_2) \\ &\bigcup \left\{z_b^1x_a^1, x_b^1z_a^1 : z_b^1z_a^1 \notin E(G_1), 1 \le b \le n, b+1 \le a \le n\right\} \bigcup_{f=1}^{n} \left\{z_f^1x_f^1\right\}. \end{aligned}$$

Then, the graph H contains given G as an IS. Also in H, $d_H(z_a^1) = d_H(x_a^1) = n$ and $d_2(z_a^1) = d_2(x_a^1) = n-1$, (i=1 to n) and $s_2(z_a^1) = s_2(x_a^1) = n(n-1)$. H is therefore $(2, n(n-1))-$support regular graph with $|H| = 2n$ containing G as an IS. □

Example 4.25 The following explains the above corollary for $n = 4$.

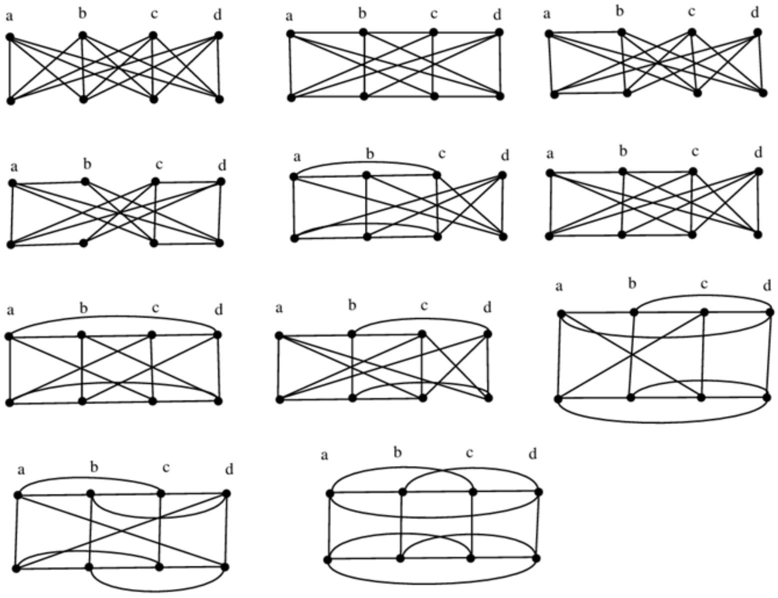


Figure 10.

Corollary 4.26 *All graphs with* $|G| = n \ge 2$ *are IS of a* $(2, (n+1)(2n-1))-$ *support regular graphs.*

Corollary 4.27 *All graphs of order* $n \ge 2$ *are IS of a* $(2, (n+2)(3n-1))-$ *support regular graphs.*

Remark 4.28 We can conclude that if $r = 1, 2, 3, \cdots n$, then the given graph of cardinality $n \ge 2$, is contained as IS in $(2, n(n-1))$, $(2, (n+1)(2n-1)), \cdots, (2, 2n(n^2-1))-$ support regular graphs with respective cardinality as $2n, 4n, \cdots, 2n^2$.

§5. Minimal $(2, s)$-Support Regular Graph containing G as an IS

We construct the minimal $(2, s)-$support regular graph containing a G as an IS, taking inspirations from [5].

Theorem 5.1 *For $r > 1$, every G with $|G| = n \geq 2$ is an IS of $(2, (n+r-2)(r-1)(n-1))-$ support regular graph of cardinality nr.*

Proof Consider G with $|G| = n \geq 2$ and $V(G) = \{y_1, y_2, y_3, \cdots, y_n\}$. G_l be replicas of given graph with $V(G_l) = \left\{y_1^l, y_2^l, \cdots, y_n^l\right\}$ such that l= 1 to r. Let H be graph such that $V(H) = \bigcup\limits_{l=1}^{r} V(G_l)$. The lines of H are

$$\begin{aligned} E(H) &= \bigcup_{l=1}^{r} E(G_l) \bigcup_{l=1}^{r-1} \left\{y_f^l y_e^{l+1}, y_f^r y_e^1 : y_f^1 y_e^1 \notin E(G_1)\right\} \ (f = 1 \text{ to } n, e = f+1\text{to } n) \\ & \bigcup_{g=1}^{n} \left\{y_g^e y_g^{e+f}\right\} \end{aligned}$$

where $e = 1$ to $r-1$, $f = 1$ to $r-e$. The resulting is a graph which contains the given graph as an IS. In H, $d(y_e^l) = n+r-2$, (e = 1 to n, l= 1 to r) and of order nr.
The following cases are examined to find the d_2 degree and 2-support of the vertices.

Case 1. $l = 1$.

Suppose $y \in V(G_1)$, then $y = y_f^1$, for some f. Let $y_f^1 \notin$ [1-neighbourhood of y_e^1] but in $V(H)$. Then by construction join y_f^1 to y_e^2 and y_e^2 with y_e^1. Then we have y_f^1 in [2-neighbourhood of y_e^1]. We can infer that $V(H)-$ [1-neighbourhood of y_e^1] $\subseteq$ [2- neighbourhood of y_e^1].

Consider $y_f^1 \in$ [2$-$neighbourhood of y_e^1], then y_f^1 not connected to y_e^1 in [1-neighbourhood], thus $y_f^1 \in V(H)-$ [1-neighbourhood of y_e^1], thus [2-neighbourhood of y_e^1] $\subseteq V(H)$- [1-neighbourhood of y_e^1], thus [2-neighbourhood of y_e^1] $= V(H)$-[1-neighbourhood of y_e^1]. Now, $d_2(y_e^1) = (r-1)(n-1)$, where a= 1 to n. The support of vertices are $s_2(y_a^1) = (n+r-2)(n-1)(r-1)$.

Case 2. $2 \leq l \leq r-1$.

If $y \in V(G_l)$, then $y = y_f^l$, for some f. In this also we obtain $d_2(y_e^l) = (r-1)(n-1)$, where $e = 1$ to n. The support of vertices are $s_2(y_e^l) = (n+r-2)(n-1)(r-1)$.

Case 3. $l = r$.

If $y \in V(G_l)$, then $y = y_f^r$, for some f. Similar to above cases, we get $d_2(v_e^r) = (m-1)(n-1), 1 \leq e \leq n$. The support of the vertices is $s_2(v_e^r) = (n+r-2)(r-1)(n-1)$.

Similarly for $2 \leq l \leq r$, $d_2(y_e^l) = (r-1)(n-1)$, for $1 \leq e \leq n$ and $s_2(y_e^l) = (n+r-2)(r-1)(n-1)$. Therefore H is $(2, (n+r-2)(r-1)(n-1))-$support regular graph containing G as an IS. This completes the proof. □

Corollary 5.2 *For $r > 1$, the smallest order of a $(2, (n+r-2)(r-1)(n-1))-$support regular graph containing G with $|G| = n \geq 2$ as an IS, is nr.*

Proof Consider H constructed in the above theorem which is $(2, (n+r-2)(r-1)(n-1))-$support regular and $|H| = nr$. Suppose it is of order nr-1, i.e., for each $y_e \in H$, $d_2(y_e) = (r-1)(n-1)$ and $s_2(y_e) = (n+r-2)(r-1)(n-1)$. Then H has at least $n+r-2+(r-1)(n-1)+1 = nr$ vertices, which contradicts. Hence the proof. □

Corollary 5.3 *Every graph where* $n \geq 2$ *is an IS of a* $(2, n(n-1))-$*support regular graph of minimal cardinality* $2n$.

Corollary 5.4 *All graphs with order* $n \geq 2$ *is IS of a* $(2, (n+1)2(n-1))-$*support regular graph of minimal cardinality* $3n$.

Example 5.5 The above corollary is explained as follows, for $n = 2, m = 3$.

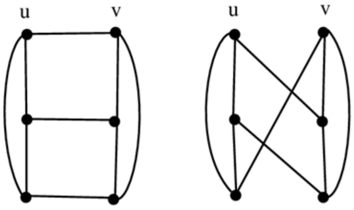


Figure 11.

Corollary 5.6 *Every* G *such that* $|G| = n \geq 2$ *is an IS of a* $(2, (n+2)3(n-1))-$*support regular graph of minimal order* $4n$.

Remark 5.7 When m takes values as $2, 3, 4, 5, \cdots, n$ there exist $(2, n(n-1))$, $(2, (n+1)2(n-1)), \cdots, (2, (2n-2)(n-1)^2)-$support regular graphs of minimalistic order $2n, 3n, 4n, \cdots, nn^2, \cdots$, respectively containing G of order $n \geq 2$ as IS.

§6. Minimal $(2, s)$-Support Regular Graph Containing G and G^c as IS

In this section, as in [7] we construct the minimally existing $(2, s)$ support regular graph containing G and G^c as IS.

Theorem 6.1 *For any* G*, where* $|G| = n \geq 2$, $\exists$ $(2, (r+2n-2)(r-1)(2n-1))-$ *support regular graph, say* H *with* $|H| = 2nr$ *such that it contains* G *and* G^c *as IS.*

Proof Take G, such that $|G| = n \geq 2$, then node set of G and G^c are alike and let them be $\{x_a^1\}$ where a=1 to n. Let $\breve{G}$, isomorphic to G^c and $V(\breve{G}) = \{y_a^1\}$ and a= 1 to n, also y_a^1 correspond to x_a^1 ($a = 1$ to n). Consider $G_1 = G \bigcup \grave{G}$. Now, $V(G_1) = \{x_a^1, y_a^1\}$ and $a = 1$ to n. G_l, $t = 2$ to l denote $(l-1)$ copies of G_1 and $V(G_l) = \{x_a^l, y_a^l\}$ and $a = 1$ to n, $l = 2$ to r and x_a^l, y_a^l, ($l = 2$ to r) correspond to $x_a^1 y_a^1$ ($a = 1$ to n), respectively. The new graph H has the vertex set $V(H) = \bigcup\limits_{l=1}^{r} V(G_l)$, and

$$E(H) = \bigcup_{l=1}^{r} E(G_l) \bigcup_{l=1}^{r-1} \{x_b^l x_a^{r+1}, x_l^r x_a^1 : x_b^1 x_a^1 \notin E(G_1); y_b^l y_a^{l+1}, y_b^r y_a^1 : y_b^1 y_a^1 \notin E(G_1)\}$$
$$(b = 1 \text{ to } n, a = b+1 \text{ to } n)$$

$$\bigcup_{f=1}^{n}\left\{y_f^a y_f^{a+b}; y_f^a y_f^{a+b}\right\}\ (a=1 \text{ to } r-1, b=1 \text{ to } r-a)$$

$$\bigcup_{l=1}^{r-1}\left\{x_b^l y_a^{l+1}, x_b^l y_a^1\right\}\ (a,b=1 \text{ to } n).$$

The resulting graph H contains G_1 as IS and in H, $d(x_a^l)=r+2(n-1)$, a= 1 to n, l= 1 to r and H is graph of order 2nr and it contains G and G^c as IS.

The calculation of d_2 and s_2 of nodes is in the below.

Case 1. $l=1$, if $x \in V(G_1)$ then $x \in V(G)$ or $x \in V(\breve{G})$.

Subcase 1.1 If $v \in V(G)$, then $x=x_b^1$, for any b. Let $x_b^1 \notin$ [1-neighbourhood of x_a^1] but in $V(H)$. Then by construction join x_b^1 to x_a^2 and x_a^2 with x_a^1. Then we have x_b^1 in [2-neighbourhood of x_a^1]. We can infer that $V(H)-$ (1-neighbourhood of x_a^1) $\subseteq$ [2-neighbourhood of x_a^1].

Consider $x_b^1 \in$ [2-neighbourhood of x_a^1], then x_b^1 not connected to x_a^1 in [1-neighbourhood], thus $x_b^1 \in V(H)$- [1-neighbourhood of x_a^1], thus [2-neighbourhood of x_a^1] $\in V(H)$-[1-neighbourhood of x_a^1], thus [2-neighbourhood of x_a^1] $= V(H)$-[1-neighbourhood of x_a^1]. Now, $d_2(x_a^1)=(r-1)(n-1)$, where $a=1$ to n. The support of vertices are

$$s_2(x_a^1)=(n+r-2)(n-1)(r-1).$$

Subcase 1.2 If $v \in V(\breve{G})$, then $x=x_b^1$, for any b. As in above case, we get $d_2(x_a^1) = (r-1)(2n-1), 1 \le a \le n$. The support of the vertices is

$$s_2(x_a^1)=(r+2n-2)(r-1)(2n-1).$$

The similar case follows for vertices when $2 \le l \le r-1$ and $x \in V(G_l)$ and also when $l=r$ and $x \in V(G_l)$. In all these cases we have H is a $(2,(r+2n-2)(r-1)(2n-1))-$support regular graph with cardinality 2nr containing G and G^c as IS. □

Corollary 6.2 *For all $m \ge 1$, the minimal order of $(2,(m+2n-2)(m-1)(2n-1))-$support regular graph which contains G and G^c is 2nr.*

Proof Let us take the graph H as constructed in above theorem, such that $|H|=$ 2nr. Suppose $|H|=2mn-1$. For each node, $s_2(v_i)=(m+2n-2)(m-1)(2n-1) \Rightarrow d_2(v_i) = (m-1)(2n-1)$ and $d(v_i)=m+2n-2$. Therefore, H has at least

$$(m+2n-2)+(m-1)(2n-1)+1=2mn$$

vertices, which contradicts. □

Corollary 6.3 *Any G with $|G|=n \ge 2$, then G and G^c are IS of $(2,2n(2n-1))-$ support regular graph of cardinality $4n$.*

Example 6.4 An illustration of above corollary, for $n = 2$ is shown in Figure 12.

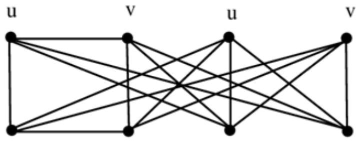


Figure 12.

Corollary 6.5 *Any G with $|G| = n \geq 2$, then G and G^c are IS of* $(2, 2n+1(2n-1))-$ *support regular graphs of minimalistic order* $6n$.

Example 6.6 An representation of above corollary is shown in Figure 13.

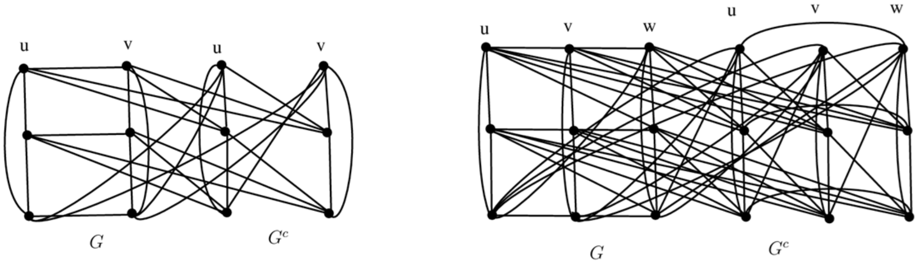


Figure 13.

Remark 6.7 Setting values for m as, $m = 4, \cdots$ then, there exist $(2, (2n+3)4(2n-1)), (2, (2n+4)5(2n-1)) \cdots -$support regular graphs of minimalistic order $8n, 10n, \cdots$ respectively containing G, $(|G| = n \geq 2)$ and G^c as IS.

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International J.Math. Combin. Vol.1(2026), 132-143

PMC-Labeling of Subdivision of Path and Cycle Related Graphs

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Abstract: The graph $G=(V,E)$ consists of p vertices and q edges. Let

$$\rho=\begin{cases} \frac{p}{2}, & p \text{ is even} \\ \frac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $\Gamma=\{\pm 1, \pm 2, \ldots, \pm\rho\}$. Consider a function $\lambda: V \rightarrow \Gamma$ that allocates unique labels from Γ to the various vertices of V when p is even and allocates a unique labels in Γ to $p-1$ vertices of V, repeating a label for the remaining one vertex when p is odd. Then the labeling as mentioned above is called a pair mean cordial labeling (PMC-labeling) if for every edge uv of G, there is a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u)+\lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u)+\lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1}-\bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ are denoted the number of edges labelled with 1 and the number of edges not labelled with 1, respectively. A graph G that has a pair mean cordial labeling is called a pair mean cordial graph (PMC-Graph). In this paper we prove that the subdivision of path, cycle, wheel, crown, helm, fan graph, friendship graph, coconut tree, double comb graph, jellyfish graph, flower graph, sunflower graph, gear graph and jewel graph are PMC-labeling.

Key Words: PMC-labeling, Smarandachely PMC-graph, path, cycle, wheel, helm and crown.

AMS(2010): 05C38, 05C78.

§1. Introduction

We begin with simple, finite, connected and undirected graph $G=(V(G),E(G))$. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G, denoted by p. The cardinality of the edge set is called the size of G, denoted by q. A graph with p vertices and q edges is called a (p,q) graph. Terms not defined here are used in the sense of Harary [8]. The first half of the 18th century saw the introduction of graph Theory, following the solution of the Konigsberg Bridge problem in 1736. Since then, graph theory has emerged as a powerful tool in the field of mathematical research

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[1] Received August 26, 2025, Accepted February 28, 2026

for its ability to represent any physical problem involving the arrangement of objects. A graph is a collection of vertices and edges between them. Ever since it broke into the mainstream of mathematical research, graph theory has found application in diverse fields ranging from biochemistry, architecture, psychology, economics, linguistics, sociology, electrical engineering and computer science to operations research. Graph labeling is a strong relation between number theory and graph structures. It is the potential area of research in graph theory. If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling. The concept of graph labeling is a frontier between number theory and structure of graphs. A systematic study of various applications of graph labeling is carried out in Bloom and Golomb [1]. It is interesting to note that the labelled graphs serve as useful models for a minor road range of applications. Labeled graphs are used in numerous areas like coding theory, X-ray crystallography, the design of good radar type codes, astronomy, circuit design, communication network addressing, data base management. A detailed study on graph labelling is reported in [7]. The concept of cordial labeling was introduced by Cahit [6]. The study of Zumkeller numbers [10] is a part of number theory which is one of the important branches of mathematics. The concept of k-Zumkeller labeling of graphs has been introduced and investigated in the literature [2,3,4,5,9]. The idea of PMC-labeling of some simple graphs has been previously reported in [11,12,13,14]. In this paper we investigate the PMC-labeling of the subdivision of path, cycle, wheel, crown, helm, fan graph, friendship graph, coconut tree, double comb graph, jellyfish graph, flower graph, sunflower graph, gear graph and jewel graph.

§2. Preliminaries

In this section, we present a few fundamental definitions that are essential for the upcoming section.

Definition 2.1 *A wheel graph W_n is the graph $K_1 + C_n$.*

Definition 2.2 *A crown $C_n \odot K_1$ is obtained by joining a pendent edge to each vertex of the cycle C_n.*

Definition 2.3 *A helm graph H_n is a graph obtained from a wheel by attaching a pendant vertex at each vertex of the cycle C_n.*

Definition 2.4 *A fan f_n, $n \geq 2$ is obtained by joining all vertices of P_n to a further vertex called the center.*

Definition 2.5 *A friendship graph F_n is a graph which consists of n triangles with a common vertex.*

Definition 2.6 *A coconut tree $CT_{m,n}$ is a graph obtained by connecting the center vertex of $K_{1,n}$ with a pendant vertex of the path P_m.*

Definition 2.7 *A double comb graph $P_n \odot 2K_1$ is obtained by joining the two pendant edge to* *each vertices of the path* P_n.

Definition 2.8 *A jelly fish graphs* $J(m,n)$ *obtained from a cycle* $C_4 : u_1u_2u_3u_4u_1$ *by joining* u_1 *and* u_3 *with an edge and appending* m *pendent edges to* u_2 *and* n *pendent edges to* u_4.

Definition 2.9 *A flower graph* Fl_n *is the graph obtained from a helm* H_n *by joining each pendant vertex to the apex of the helm.*

Definition 2.10 *A sunflower graph* S_n *is obtained by taking a wheel with central vertex* u *and the cycle* $C_n : u_1u_2 \dots u_nu_1$ *and new vertices* $v_1v_2 \dots v_n$ *where* v_i *is joined by vertices* $u_i, u_{i+1\,(mod\,n)}$.

Definition 2.11 *A gear graph* G_n *is obtained from the wheel* W_n *by adding a vertex between every pair of adjacent vertices of the cycle* C_n.

Definition 2.12 *A jewel graph* J_n *is the graph with vertex set* $V(J_n) = \{u, v, x, y, u_i \mid 1 \leq i \leq n\}$ *and edge set* $E(J_n) = \{ux, uy, xy, xv, yv, uu_i, vu_i \mid 1 \leq i \leq n\}$.

Definition 2.13 *A subdivision graph* $S(G)$ *of a graph* G *is obtained from* G *by inserting a new vertex of degree 2 on each edge of* G.

Definition 2.14 *Let* $G = (V, E)$ *be a graph consisting of* p *vertices,* q *edges,*

$$\rho = \begin{cases} \frac{p}{2}, & p \text{ is even} \\ \frac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $\Gamma = \{\pm 1, \pm 2, \cdots, \pm \rho\}$. *Consider a function* $\lambda : V \to \Gamma$ *that allocates unique labels from* Γ *to the various vertices of* V *when* p *is even and allocates a unique labels in* Γ *to* $p - 1$ *vertices of* V, *repeating a label for the remaining one vertex when* p *is odd. Then the labeling as mentioned above is called a pair mean cordial labeling (PMC-labeling) if for every edge* uv *of* G, *there is a labeling* $\frac{\lambda(u)+\lambda(v)}{2}$ *if* $\lambda(u) + \lambda(v)$ *is even and* $\frac{\lambda(u)+\lambda(v)+1}{2}$ *if* $\lambda(u) + \lambda(v)$ *is odd such that* $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ *where* $\bar{\mathbb{S}}_{\lambda_1}$ *and* $\bar{\mathbb{S}}_{\lambda_1^c}$ *are denoted the number of edges labelled with* 1 *and the number of edges not labelled with* 1, *respectively. A graph* G *that has a pair mean cordial labeling is called a pair mean cordial graph (PMC-graph).*

Otherwise, if there are $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ *for a graph* G, *it is called a Smarandachely PMC-graph.*

An example of PMC-graph is shown in Figure 1.

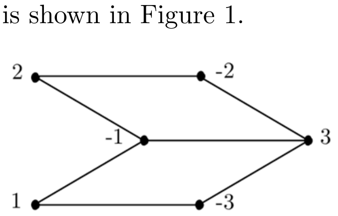

Figure 1. An example of PMC-graph

§3. Main Results

In this section, we investigate the PMC-labeling behavior of the subdivision of path, cycle, wheel, crown, helm, fan graph, friendship graph, coconut tree, double comb graph, jellyfish graph, flower graph, sunflower graph, gear graph and jewel graph.

Theorem 3.1 *The subdivision of path P_n, $S(P_n)$ is a PMC-graph for all $n \geq 1$.*

Proof We have $S(P_n) \simeq P_{2n-1}$ and P_{2n-1} is a PMC-graph [11]. □

Theorem 3.2 *The subdivision of cycle C_n, $S(C_n)$ is a PMC-graph for all $n \geq 3$.*

Proof We obtain $S(C_n) \simeq C_{2n}$ and C_{2n} is a PMC-graph [11]. □

Theorem 3.3 *The subdivision of wheel graph W_n, $S(W_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Consider the subdivision of wheel graph $S(W_n)$, $n \geq 3$. Denote by $V(S(W_n)) = \{v_0, v_i, u_i, w_i \mid 1 \leq i \leq n\}$ and $E(S(W_n)) = \{v_0u_i, u_iv_i, v_iw_i \mid 1 \leq i \leq n\} \cup \{w_iv_{i+1}, w_nv_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(W_n)$ respectively. Then the order and size of $S(W_n)$ respectively are $3n+1$ and $4n$. The theorem is established by discussing two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(W_n)) \to \{\pm 1, \pm 2, \cdots, \pm\frac{3n+1}{2}\}$. Take $\lambda(v_0) = n+2$. Then assign the labels $-1, -2, \cdots, -n$ and $2, 3, \cdots, n+1$ respectively to the vertices $w_1, w_2, \cdots, w_n$ and $v_1, v_2, \cdots, v_n$. Further assign the labels $-n-1, -n-2, \cdots, \frac{-3n-1}{2}$ and $n+3, n+4, \cdots, \frac{3n+1}{2}$ to the vertices $u_1, u_2, \cdots, u_{\frac{n+1}{2}}$ and $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$ respectively. Finally assign the label 1 to the vertex u_n.

Case 2. n is even.

Define the injective function $\lambda : V(S(W_n)) \to \{\pm 1, \pm 2, \cdots, \pm\frac{3n}{2}\}$. Apply the Case 1 labeling to the vertices $v_0, v_i, w_i, 1 \leq i \leq n$. Next assign the labels $-n-1, -n-2, \cdots, \frac{-3n}{2}$ and $n+3, n+4, \cdots, \frac{3n}{2}$ to the vertices $u_1, u_2, \cdots, u_{\frac{n}{2}}$ and $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, \cdots, u_{n-2}$ respectively. Finally assign the labels $1, 1$ to the vertices u_{n-1}, u_n respectively. Hence the labeling in both cases results in $2n$ edges assigned the label 1 and $2n$ edges without the label 1. Therefore vertex labeling λ is a PMC-labeling of the subdivision of wheel graph $S(W_n)$. □

Example 3.4 A PMC-labeling of the subdivision of wheel graph $S(W_4)$ is shown in Figure 2.

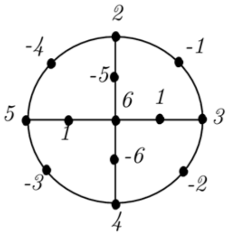

Figure 2. A PMC-labeling of the subdivision of wheel graph $S(W_4)$

Theorem 3.5 *The subdivision of crown graph $C_n \odot K_1$, $S(C_n \odot K_1)$ is a PMC-graph for all $n \geq 3$.*

Proof Let us consider the subdivision of crown graph $S(C_n \odot K_1)$, $n \geq 3$. Denote by $V(S(C_n \odot K_1)) = \{u_i, v_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(S(C_n \odot K_1)) = \{u_iv_i, u_iy_i, y_iw_i \mid 1 \leq i \leq n\} \cup \{v_iu_{i+1}, v_nu_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(C_n \odot K_1)$ respectively. Then the order and size of $S(C_n \odot K_1)$ respectively are $4n$ and $4n$. Define the injective function $\lambda : V(S(C_n \odot K_1)) \to \{\pm 1, \pm 2, \cdots, \pm 2n\}$. First assign the labels $-1, -2, \cdots, -n$ and $2, 3, \cdots, n+1$ respectively to the vertices $y_1, y_2, \cdots, y_n$ and $x_1, x_2, \cdots, x_n$. Further assign the labels $-n-1, -n-2$ and $n+4, n+5, \cdots, 2n$ to the vertices u_1, u_2 and $u_3, u_4, \cdots, u_{n-1}$ respectively. Fix the label $n+2$ to the vertex u_n. Next assign the labels $n+3$ and $-n-3, -n-4, \cdots, -2n$ to the vertices v_1 and $v_2, v_3, \cdots, v_{n-1}$ respectively. Finally assign the label 1 to the vertex v_n. From the above labeling technique, we obtain $2n$ edges labelled with 1 and $2n$ edges not labelled with 1. Therefore, the vertex labeling λ is a PMC-labeling of the subdivision of crown graph $S(C_n \odot K_1)$. □

Theorem 3.6 *The subdivision of helm graph H_n, $S(H_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Consider the subdivision of helm graph $S(H_n)$, $n \geq 3$. Denote by $V(S(H_n)) = \{v_0, v_i, u_i, w_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(S(H_n)) = \{v_0u_i, u_iv_i, v_iy_i, y_ix_i, v_iw_i \mid 1 \leq i \leq n\} \cup \{w_iv_{i+1}, w_nv_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(H_n)$ respectively. Then the order and size of $S(H_n)$ respectively are $5n+1$ and $6n$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(H_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{5n+1}{2}\}$. Take $\lambda(v_0) = 2n+2$. Then assign the labels $-1, -2, \cdots, -n$ and $2, 3, \ldots, n+1$ respectively to the vertices $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$. Moreover assign the labels $-n-1, -n-2, \cdots, -2n$ and $n+2, n+3, \ldots, 2n+1$ to the vertices $w_1, w_2, \cdots, w_n$ and $v_1, v_2, \cdots, v_n$ respectively. Thereafter assign the labels $-2n-1, -2n-2, \cdots, \frac{-5n-1}{2}$ and $2n+3, 2n+4, \cdots, \frac{5n+1}{2}$ respectively to the vertices $u_1, u_2, \cdots, u_{\frac{n+1}{2}}$ and $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$. Finally, fix the label 1 to vertex u_n.

Case 2. n is even.

Define the injective function $\lambda : V(S(H_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{5n}{2}\}$. Apply the Case 1 labeling to the vertices $v_0, v_i, w_i, x_i, y_i 1 \leq i \leq n$. Then assign the labels $-2n-1, -2n-2, \cdots, \frac{-5n}{2}$ and $2n+3, 2n+4, \cdots, \frac{5n}{2}$ respectively to the vertices $u_1, u_2, \cdots, u_{\frac{n}{2}}$ and $u_{\frac{n+2}{2}}, u_{\frac{n+4}{2}}, \cdots, u_{n-2}$. Finally assign the labels $1, 1$ to the vertices u_{n-1}, u_n respectively. Hence the labeling in both cases results in $3n$ edges assigned the label 1 and $3n$ edges without the label 1. Therefore, the vertex labeling λ is a PMC-labeling of the subdivision of helm graph $S(H_n)$. □

Theorem 3.7 *The subdivision of fan graph f_n, $S(f_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Let us consider the subdivision of fan graph $S(f_n)$, $n \geq 3$. Denote by $V(S(f_n)) = \{v_0, v_i, u_i, w_j \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq n-1\}$ and $E(S(f_n)) = \{v_0u_i, u_iv_i \mid 1 \leq i \leq n\} \cup$

$\{v_iw_i, w_iv_{i+1} \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(f_n)$ respectively. Then the order and size of $S(f_n)$ respectively are $3n$ and $4n-2$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(f_n)) \rightarrow \{\pm 1, \pm 2, \cdots, \pm\frac{3n-1}{2}\}$. Let $\lambda(v_0) = 1$ and $\lambda(u_1) = 2$. Then assign the labels $-1, -2, \cdots, -n$ and $3, 4, \cdots, n+1$ respectively to the vertices $v_1, v_2, \cdots, v_n$ and $w_1, w_2, \cdots, w_{n-1}$. Moreover assign the labels $-n-1, -n-2, \cdots, \frac{-3n+1}{2}$ and $n+2, n+3, \cdots, \frac{3n-1}{2}$ to the vertices $u_2, u_3, \cdots, u_{\frac{n+1}{2}}$ and $u_{\frac{n+3}{2}}, u_{\frac{n+5}{2}}, \cdots, u_{n-1}$ respectively. Finally, fix the label $\frac{-3n+1}{2}$ to the vertex u_n.

Case 2. n is even.

Define the injective function $\lambda : V(S(f_n)) \rightarrow \{\pm 1, \pm 2, \cdots, \pm\frac{3n}{2}\}$. Apply the Case 1 labeling to the vertices $v_0, v_i, w_i, u_1, 1 \leq i \leq n$. Then assign the labels $-n-1, -n-2, \cdots, \frac{-3n}{2}$ and $n+2, n+3, \cdots, \frac{3n}{2}$ respectively to the vertices $u_1, u_2, \cdots, u_{\frac{n+2}{2}}$ and $u_{\frac{n+4}{2}}, u_{\frac{n+6}{2}}, \cdots, u_n$. Hence the labeling in both cases results in $2n-1$ edges assigned the label 1 and $2n-1$ edges without the label 1. Therefore, the vertex labeling λ is a PMC-labeling of the subdivision of fan graph $S(f_n)$. □

Theorem 3.8 *The subdivision of friendship graph F_n, $S(F_n)$ is a PMC-graph for all $n \geq 1$.*

Proof Let us consider the subdivision of friendship graph $S(F_n)$, $n \geq 1$. Denote by $V(S(F_n)) = \{v_0, v_i, u_i, x_i, y_i \mid 1 \leq i \leq n\}$ and $E(S(F_n)) = \{v_0x_i, v_0y_i, x_iv_i, y_iu_i, v_iz_i, z_iu_i \mid 1 \leq i \leq n\}$ the vertex set and edge set of $S(F_n)$ respectively. Then the order and size of $S(F_n)$ respectively are $5n+1$ and $6n$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function

$$\lambda : V(S(F_n)) \rightarrow \{\pm 1, \pm 2, \ldots, \pm\frac{5n+1}{2}\}.$$

Let $\lambda(v_0) = 1$. Then assign the labels $-1, -3, \cdots, -2n+1$ and $-2, -4, \cdots, -2n$ respectively to the vertices $v_1, v_2, \cdots, v_n$ and $u_1, u_2, \cdots, u_n$. Moreover assign the labels $3, 5, \cdots, 2n+1$ and $2, 4, \cdots, 2n$ to the vertices $z_1, z_2, \cdots, z_n$ and $x_1, x_2, \cdots, x_n$ respectively. Finally, assign the labels $-2n-1, -2n-2, \cdots, \frac{-5n-1}{2}$ and $2n+2, 2n+3, \cdots, \frac{5n+1}{2}$ respectively to the vertices $y_1, y_3, \cdots, y_n$ and $y_2, y_4, \cdots, y_{n-1}$.

Case 2. n is even.

Define the injective function

$$\lambda : V(S(F_n)) \rightarrow \{\pm 1, \pm 2, \cdots, \pm\frac{5n}{2}\}.$$

Apply the Case 1 labeling to the vertices $v_i, u_i, z_i, x_i 1 \leq i \leq n$. Then assign the labels $-2n-1, -2n-2, \cdots, \frac{-5n}{2}$ and $2n+2, 2n+3, \cdots, \frac{5n}{2}$ respectively to the vertices $y_1, y_3, \cdots, y_{n-1}$

and $y_2, y_4, \cdots, y_{n-2}$. Hence the labeling in both cases results in $3n$ edges assigned the label 1 and $3n$ edges without the label 1. Therefore, the vertex labeling λ is a PMC-labeling of the subdivision of friendship graph $S(F_n)$. $\square$

Example 3.9 A PMC-labeling of the subdivision of friendship graph $S(F_4)$ is shown in Figure 3.

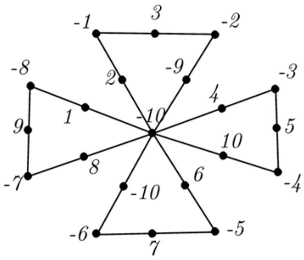

Figure 3. A PMC-labeling of the subdivision of friendship graph $S(F_4)$

Theorem 3.10 *The subdivision of coconut tree $CT_{n,m}$, $S(CT_{n,m})$ is a PMC-graph for all $n, m \geq 1$.*

Proof Let us consider the subdivision of coconut tree $S(CT_{n,m})$, $n, m \geq 1$. Denote by $V(S(CT_{n,m})) = \{u_i, v_j, y_j \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq m\} \cup \{x_i \mid 1 \leq i \leq n-1\}$ and $E(S(CT_{n,m})) = \{u_n y_j, y_j v_j \mid 1 \leq i \leq m\} \cup \{u_i x_i, x_i u_{i+1} \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(CT_n)$ respectively. Then the order and size of $S(CT_n)$ respectively are $2n + 2m - 1$ and $2n + 2m - 2$. Define the injective function $\lambda : V(S(CT_{n,m})) \to \{\pm 1, \pm 2, \cdots, \pm(n + m - 1)\}$. First assign the labels $-1, -2, \cdots, -m$ and $2, 3, \cdots, m + 1$ respectively to the vertices $y_1, y_2, \cdots, y_m$ and $v_1, v_2, \cdots, v_m$. Then assign the labels $-m - 1, -m - 2, \cdots, -m - n + 1$ and 1 to the vertices $u_1, u_2, \cdots, u_{n-1}$ and u_n respectively. Finally assign the labels $m + 2, m + 3, \cdots, m + n - 1$ and 1 to the vertices $x_1, x_2, \cdots, x_{n-2}$ and x_{n-1} respectively. Hence the number of edges labeled with 1 is $n + m - 1$ while the number of the edges not labeled with 1 is $n + m - 1$. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of coconut tree $S(CT_{n,m})$. $\square$

Theorem 3.11 *The subdivision of double comb graph DC_n, $S(DC_n)$ is a PMC-graph for all $n \geq 2$.*

Proof Consider the subdivision of double comb graph $S(DC_n)$, $n \geq 2$. Denote by $V(S(DC_n)) = \{u_i, v_i, w_i, x_j, y_i, z_i \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq n - 1\}$ and $E(S(DC_n)) = \{u_i y_i, y_i v_i, u_i z_i, z_i w_i, \mid 1 \leq i \leq n\} \cup \{u_i x_i, x_i u_{i+1} \mid 1 \leq i \leq n - 1\}$ the vertex set and edge set of $S(DC_n)$ respectively. Then the order and size of $S(DC_n)$ respectively are $6n - 1$ and $6n - 2$. Define the injective function $\lambda : V(S(DC_n)) \to \{\pm 1, \pm 2, \cdots, \pm(3n - 1)\}$. First assign the labels $-1, -2, \cdots, -n$ and $2, 3, \cdots, n+1$ respectively to the vertices $y_1, y_2, \cdots, y_n$ and $v_1, v_2, \cdots, v_n$. Then assign the labels $-n - 1, -n - 2, \cdots, -2n$ and $n + 2, n + 3, \cdots, 2n + 1$ to the vertices $z_1, z_2, \cdots, z_n$ and $w_1, w_2, \cdots, w_n$ respectively. Moreover assign the labels $-2n - 1, -2n - 2, \cdots, -3n + 1$ and 1 respectively to the vertices $u_1, u_2, \cdots, u_{n-1}$ and u_n. Finally assign the labels $2n + 2, 2n + 3, \cdots, 3n - 1$ and 1 to the vertices $x_1, x_2, \cdots, x_{n-2}$ and x_{n-1}

respectively. Hence the number of edges labeled with 1 is $3n-1$ while the number of the edges not labeled with 1 is $3n-1$. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of double comb graph $S(DC_n)$. □

Theorem 3.12 *The subdivision of jelly fish graph $J_{n,m}$, $S(J_{n,m})$ is a PMC-graph for all $n, m \geq 1$.*

Proof Let us consider the subdivision of jelly fish graph $S(J_{n,m})$, $n, m \geq 1$. Denote by $V(S(J_{n,m})) = \{a, b, c, d, w, a', b', c', d', u_i, x_i, v_j, y_j \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq m\}$ and $E(S(J_{n,m})) = \{aa', a'b, bb', b'c, cc', c'd, dd', d'a, aw, wc, dx_i, x_iu_i, by_j, y_jv_j, \mid 1 \leq i \leq n \,\text{and}\, 1 \leq j \leq m\}$ the vertex set and edge set of $S(J_{n,m})$ respectively. Then the order and size of $S(J_{n,m})$ respectively are $2n+2m+9$ and $2n+2m+10$. Define the injective function $\lambda : V(S(J_{n,m})) \to \{\pm 1, \pm 2, \ldots, \pm(n+m+4)\}$. First assign the labels $-3, -4, -2, -1, 4$ and $6, 5, 2, 3$ respectively to the vertices a, b, c, d, w and a', b', c', d'. Then assign the labels $-5, -6, \cdots, -n-4$ and $6, 7, \cdots, n+5$ to the vertices $x_1, x_2, \cdots, x_n$ and $u_1, u_2, \cdots, u_n$ respectively. Moreover assign the labels $-n-5, -n-6, \cdots, -n-m-4$ and $n+6, n+7, \cdots, n+m+4$ respectively to the vertices $y_1, y_2, \cdots, y_m$ and $v_1, v_2, \cdots, v_{m-1}$. Finally assign the label 1 to the vertex v_m respectively. Hence the number of edges labeled with 1 is $n+m+5$ while the number of the edges not labeled with 1 is $n+m+5$. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of jelly fish graph $S(J_{n,m})$. □

Example 3.13 A PMC-labeling of the subdivision of jelly fish graph $S(J_{3,4})$ is shown in Figure 4.

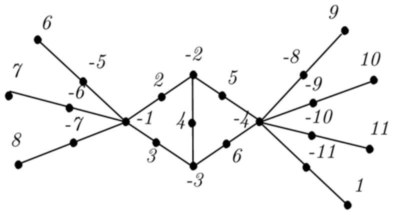

Figure 4. A PMC-labeling of the subdivision of jelly fish graph $S(J_{3,4})$

Theorem 3.14 *The subdivision of flower graph fl_n, $S(fl_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Consider the subdivision of flower graph $S(fl_n)$, $n \geq 3$. Denote by $V(S(fl_n)) = \{v_0, v_i, u_i, x_i, y_i, z_i, w_i \mid 1 \leq i \leq n\}$ and $E(S(fl_n)) = \{v_0x_i, x_iv_i, v_iy_i, y_iu_i, v_0z_i, z_iu_i, v_iw_i \mid 1 \leq i \leq n\} \cup \{w_iv_{i+1}, w_nv_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(fl_n)$ respectively. Then the order and size of $S(fl_n)$ respectively are $6n+1$ and $8n$. This theorem is analyzed by considering two cases.

Case 1. n is even.

Define the injective function $\lambda : V(S(fl_n)) \to \{\pm 1, \pm 2, \cdots, \pm 3n\}$. Let $\lambda(v_0) = 1$. First assign the labels $-1, -4, \cdots, \frac{-3n+4}{2}$ and $4, 7, \cdots, \frac{3n+2}{2}$ respectively to the vertices $u_1, u_3, \cdots, u_{n-1}$ and $u_2, u_4, \cdots, u_n$. Moreover assign the labels $2, 5, \cdots, \frac{3n-2}{2}$ and $-2, -5, \cdots, \frac{-3n+2}{2}$ to the vertices $y_1, y_3, \cdots, y_{n-1}$ and $y_2, y_4, \cdots, y_n$ respectively. Now assign the labels $3, 6, \cdots, \frac{3n}{2}$

and $-3, -6, \cdots, \frac{-3n}{2}$ respectively to the vertices $z_1, z_3, \cdots, z_{n-1}$ and $z_2, z_4, \cdots, z_n$. Assign the labels $\frac{3n+4}{2}, \frac{3n+6}{2}, \cdots, \frac{5n+2}{2}$ and $\frac{-3n-2}{2}, \frac{-3n-4}{2}, \cdots, \frac{-5n}{2}$ to the vertices $v_1, v_2, \cdots, v_n$ and $w_1, w_2, \cdots, w_n$ respectively. Further assign the labels $\frac{5n+4}{2}, \frac{5n+6}{2}, \cdots, 3n$ and $\frac{-5n-2}{2}, \frac{-5n-4}{2}, \cdots, -3n$ to the vertices $x_1, x_3, \cdots, x_{n-3}$ and $x_2, x_4, \cdots, x_n$ respectively. Finally fix the label 1 to the vertex x_{n-1}.

Case 2. n is odd.

Let us define the injective function $\lambda : V(S(fl_n)) \to \{\pm 1, \pm 2, \cdots, \pm 3n\}$. Let $\lambda(v_0) = 1$. First assign the labels $-1, -4, \cdots, \frac{-3n+1}{2}$ and $4, 7, \cdots, \frac{3n-1}{2}$ respectively to the vertices $u_1, u_3, \cdots, u_n$ and $u_2, u_4, \cdots, u_{n-1}$. Moreover, assign the labels $2, 5, \cdots, \frac{3n+1}{2}$ and $-2, -5, \cdots, \frac{-3n+5}{2}$ to the vertices $y_1, y_3, \cdots, y_n$ and $y_2, y_4, \cdots, y_{n-1}$, respectively. Assign the labels $3, 6, \cdots, \frac{3n+3}{2}$ and $-3, -6, \cdots, \frac{-3n+3}{2}$ respectively to the vertices $z_1, z_3, \cdots, z_n$ and $z_2, z_4, \cdots, z_{n-1}$. Assign the labels $\frac{3n+5}{2}, \frac{3n+7}{2}, \cdots, \frac{5n+3}{2}$ and $\frac{-3n-1}{2}, \frac{-3n-3}{2}, \cdots, \frac{-5n+1}{2}$ to the vertices $w_1, w_2, \cdots, w_n$ and $v_1, v_2, \cdots, v_n$ respectively. Further assign the labels $\frac{5n+5}{2}, \frac{5n+7}{2}, \cdots, 3n$ and $\frac{-5n-1}{2}, \frac{-5n-3}{2}, \cdots, -3n$ to the vertices $x_2, x_4, \cdots, x_{n-3}$ and $x_1, x_3, \cdots, x_n$ respectively. Finally fix the label 1 to the vertex x_{n-1}. Hence the labeling in both cases results in $4n$ edges assigned the label 1 and $4n$ edges without the label 1. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of flower graph $S(fl_n)$. $\square$

Example 3.15 A PMC-labeling of the subdivision of flower graph $S(fl_4)$ is shown in Figure 5.

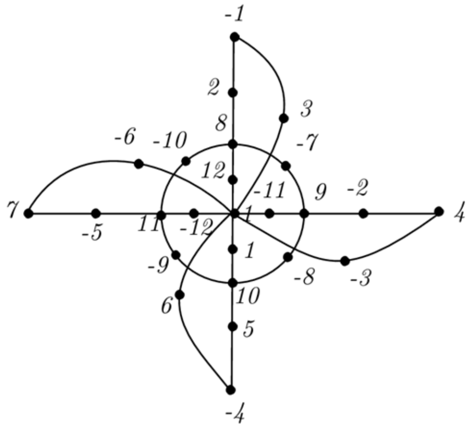

Figure 5. A PMC-labeling of the subdivision of flower graph $S(fl_4)$

Theorem 3.16 *The subdivision of sun flower graph SF_n, $S(SF_n)$ is a PMC-graph for all $n \geq 3$.*

Proof Consider the subdivision of sun flower graph $S(SF_n)$, $n \geq 3$. The vertex set and edge set of $S(SF_n)$ are denoted by $V(S(SF_n)) = \{v_0, v_i, u_i, x_i, y_i, z_i, w_i \mid 1 \leq i \leq n\}$ and $E(S(SF_n)) = \{v_0x_i, x_iv_i, v_iw_i, v_iy_i, y_iu_i, z_iu_i \mid 1 \leq i \leq n\} \cup \{w_iv_{i+1}, w_nv_1, z_iv_{i+1}, z_nv_1 \mid 1 \leq i \leq n-1\}$ respectively. Then the order and size of $S(SF_n)$ respectively are $6n+1$ and $8n$. This theorem is analyzed by considering two cases.Let us define the injective function $\lambda : V(S(SF_n)) \to \{\pm 1, \pm 2, \cdots, \pm 3n\}$. Let $\lambda(v_0) = -3n$.

Case 1. n is even.

In this case, apply the Case 1 of Theorem 3.14 labeling to the vertices $u_i, y_i, z_i, v_i, w_i 1 \leq i \leq n$. After that assign the labels $\frac{-5n-2}{2}, \frac{-5n-4}{2}, \cdots, -3n$ and $\frac{5n+4}{2}, \frac{5n+6}{2}, \cdots, 3n$ to the vertices $x_1, x_3, \cdots, x_{n-1}$ and $x_2, x_4, \cdots, x_{n-2}$ respectively. Finally fix the label 1 to the vertex x_n.

Case 2. n is odd.

Apply the Case 2 of Theorem 3.14 labeling to the vertices $u_i, y_i, z_i, v_i, w_i 1 \leq i \leq n$. Thereafter assign the labels $\frac{-5n-1}{2}, \frac{-5n-3}{2}, \cdots, -3n$ and $\frac{5n+5}{2}, \frac{5n+7}{2}, \cdots, 3n$ to the vertices $x_1, x_3, \cdots, x_n$ and $x_2, x_4, \cdots, x_{n-3}$ respectively. Finally fix the label 1 to the vertex x_{n-1}. Hence the labeling in both cases results in $4n$ edges assigned the label 1 and $4n$ edges without the label 1. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of sun flower graph $S(SF_n)$. □

Example 3.17 A PMC-labeling of the subdivision of sun flower graph $S(SF_4)$ is shown in Figure 6.

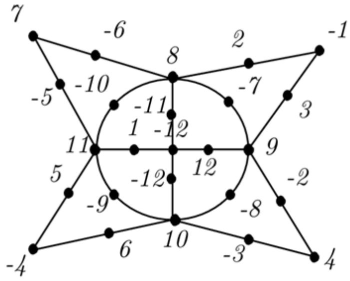

Figure 6. A PMC-labeling of the subdivision of sun flower graph $S(SF_4)$

Theorem 3.18 *The subdivision gear graph* G_n, $S(G_n)$ *is a PMC-graph for all* $n \geq 3$.

Proof Consider the subdivision of gear graph $S(G_n)$, $n \geq 3$. Denote by $V(S(G_n)) = \{v_0, v_i, u_i, x_i, y_i, z_i \mid 1 \leq i \leq n\}$ and $E(S(G_n)) = \{v_0x_i, x_iv_i, v_iy_i, y_iu_i, z_iu_i \mid 1 \leq i \leq n\} \cup \{z_iv_{i+1}, z_nv_1 \mid 1 \leq i \leq n-1\}$ the vertex set and edge set of $S(G_n)$ respectively. Then the order and size of $S(G_n)$ respectively are $5n+1$ and $6n$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(G_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{5n+1}{2}\}$. Let $\lambda(v_0) = \frac{-5n-1}{2}$. In this case, apply the Case 2 of Theorem 3.14 labeling to the vertices $u_i, y_i, z_i, v_i 1 \leq i \leq n$. After that assign the labels $\frac{3n+5}{2}, \frac{3n+7}{2}, \cdots, \frac{5n+1}{2}$ to the vertices $x_1, x_2, \cdots, x_{n-1}$ respectively. Finally fix the label 1 to the vertex x_n.

Case 2. n is even.

Let us define the injective function $\lambda : V(S(G_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{5n+1}{2}\}$. Let $\lambda(v_0) = 1$. Apply the Case 1 of Theorem 3.14 labeling to the vertices $u_i, y_i, z_i, v_i 1 \leq i \leq n$. After that assign the labels $\frac{3n+4}{2}, \frac{3n+6}{2}, \cdots, \frac{5n}{2}$ to the vertices $x_1, x_2, \cdots, x_{n-1}$ respectively. Finally fix the label 1 to the vertex x_n. Hence the labeling in both cases results in $3n$ edges assigned the label 1 and $3n$ edges without the label 1. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of gear graph $S(G_n)$. $\square$

Theorem 3.19 *The subdivision of jewel graph JL_n, $S(JL_n)$ is a PMC-graph for all $n \geq 1$.*

Proof Consider the subdivision of jewel graph $S(JL_n)$, $n \geq 1$. The vertex set and edge set of $S(JL_n)$ are denoted by $V(S(JL_n)) = \{a, b, c, d, x, a', b', c', d', u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(S(JL_n)) = \{aa', a'b, bb', b'c, cc', c'd, dd', d'a, ax, xc, dv_i, v_iu_i, bw_i, w_iu_i \mid 1 \leq i \leq n\}$ respectively. Then the order and size of $S(JL_n)$ respectively are $3n + 9$ and $4n + 10$. This theorem is analyzed by considering two cases.

Case 1. n is odd.

Define the injective function $\lambda : V(S(JL_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+9}{2}\}$. First assign the labels $2, 4, 3, -4, -1$ and $-3, -2, 5, 6$ respectively to the vertices a, b, c, d, x and a', b', c', d'. Assign the labels $7, 10, \cdots, \frac{3n+5}{2}$ and $-7, -10, \cdots, \frac{-3n-5}{2}$ to the vertices $u_1, u_3, \cdots, u_{n-2}$ and $u_2, u_4, \cdots, u_{n-1}$ respectively. Assign the label 1 to the vertex u_n. Next assign the labels $-5, -8, \cdots, \frac{-3n-7}{2}$ and $8, 11, \cdots, \frac{3n+7}{2}$ respectively to the vertices $v_1, v_3, \cdots, v_n$ and $v_2, v_4, \cdots,$ v_{n-1}. Also assign the labels $-6, -9, \cdots, \frac{-3n-9}{2}$ and $9, 12, \cdots, \frac{3n+9}{2}$ respectively to the vertices $w_1, w_3, \cdots, w_n$ and $w_2, w_4, \cdots, w_{n-1}$.

Case 2. n is even.

Let us define the injective function $\lambda : V(S(JL_n)) \to \{\pm 1, \pm 2, \cdots, \pm \frac{3n+8}{2}\}$. First assign the labels to the vertices a, b, c, d, x and a', b', c', d' as in Case (1). Then assign the labels $7, 10, \cdots, \frac{3n+8}{2}$ and $-7, -10, \cdots, \frac{-3n-8}{2}$ to the vertices $u_1, u_3, \cdots, u_{n-1}$ and $u_2, u_4, \cdots, u_n$ respectively. Next assign the labels $-5, -8, \cdots, \frac{-3n-4}{2}$ and $8, 11, \cdots, \frac{3n+4}{2}$ respectively to the vertices $v_1, v_3, \cdots, v_{n-1}$ and $v_2, v_4, \cdots, v_{n-2}$. Then fix the label 1 to the vertex v_n. Also assign the labels $-6, -9, \cdots, \frac{-3n-6}{2}$ and $9, 12, \cdots, \frac{3n+6}{2}$ respectively to the vertices $w_1, w_3, \cdots, w_{n-1}$ and $w_2, w_4, \cdots, w_{n-2}$. Finally fix the label 1 to the vertex w_n. Hence the labeling in both cases results in $2n + 5$ edges assigned the label 1 and $2n + 5$ edges without the label 1. Therefore the vertex labeling λ is a PMC-labeling of the subdivision of jewel graph $S(JL_n)$. $\square$

§4. Conclusion

In this paper, we have examined the PMC-labeling behavior of the subdivision of path, cycle, wheel, crown, helm, fan graph, friendship graph, coconut tree, double comb graph, jellyfish graph, flower graph, sunflower graph, gear graph and jewel graph. The exploration of other classes of graphs and the investigation of more complex PMC-labeling properties remain open challenges.

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An International Journal on Mathematical Combinatorics

ISSN 1937-1055