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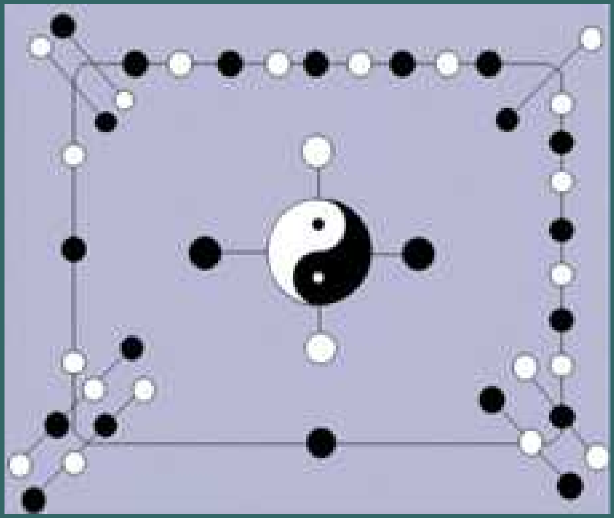

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By Law Vernon, a British writer.

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On the Crypto-Automorphism of the Buchsteiner Loops

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Abstract: In this study, New identities of Buchsteiner loops were obtained via the principal isotopes. It was also shown that the middle inner mapping T_v^{-1} is a crypto-automorphism with companions v and v^{λ}. Our results which are new in a way, complement and extend existing results in literatures.

Key Words: Buchsteiner loop, WWIP-inverse loop, automorphism group, crypto-automorphism.

AMS(2000): 08A05

§1. Introduction

A binary system $(Q,\cdot)$ is called a loop if $a\cdot 1 = a = 1\cdot a$, $\forall a \in Q$, and if the equations $ax = b$ and $ya = b$ have respectively unique solutions $x = a\backslash b$ and $y = b/a, \forall a, b \in Q$. The mappings R_x and L_x for each $x \in Q$, called respectively the right and left translation mappings, are defined as $yx = yR_x$ and $xy = yL_x, \forall y \in Q$, they are one-to-one mapping of Q onto Q. It is important to know that the group generated by all these mappings are called multiplication group $MlpQ$, readers should please see [1,10].

Therefore, a loop $(Q,\cdot)$ is called Buchsteiner loop, if $\forall\ x, y, z \in Q$, the identity

$$x\backslash(xy\cdot z) = (y\cdot zx)/x \tag{1.1}$$

is obeyed. This loop was first noticed by Buchsteiner [3] in 1976. Thereafter much is not heard of it until 2004, when Piroska Csögo, et al came up with a comprehensive study on this loop structure [5,6]. In fact, they presented for the first time, an example of Buchsteiner loop which is conjugacy closed.

A Buchsteiner loop is isomorphic to all its loops isotopes, hence it is a G-loop. It is not an inverse property loop, however it satisfies a kind of inverse known as *doubly weak inverse property*(WWIP) [5].

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A loop $(Q,\cdot)$ is called *doubly weak inverse property* (WWIP) if the identity

$$(x\cdot y)J_\rho\cdot xJ_\rho^2=yJ_\rho \tag{1.2}$$

holds $\forall\ x,y\in Q$. Buchsteiner loop is a is a class of G-loop which is defined concisely by an equation. This makes the study of Buchsteiner loop interesting since G-loop is not known to be described by a first order sentence [5].

These facts, provided the background to obtain some new identities for Buchsteiner loops. These identities, were in turn used to show that T_v^{-1} is a crypto-automorphism with companion v and v^λ.

Definition 1.1 *(1) An isotopism of loops $(Q,\circ)$ and $(,\cdot)$ with same underlying set, is a triple (α,β,γ) of permutation of Q satisfying*

$$x\alpha\cdot y\beta=(x\circ y)\gamma,\forall\ x,y\in Q. \tag{1.3}$$

In this case $(Q,\circ)$ and $(Q,\cdot)$ are said to be isotopic.

*(2)An isotopism (α,β,γ) is called principal if $\gamma=Id_Q$. In such a case $1\in Q$ is identity of $(Q,\circ)$, and if we set $1\alpha=u$ and $1\beta=v$, then (**??**) becomes $x\circ y=x/v\cdot u\backslash y=xR_v^{-1}\cdot yL_u^{-1}$, $\forall x,y\in Q$. Here $\backslash$ and / are left and right division operation in $(Q,\cdot)$. Then the loop $(Q,\circ)$ is called principal isotope of $(Q,\cdot)$.*

(3) An isotopism (α,β,γ) of a loop $(Q,\cdot)$ onto itself is called autotopism. The set $Atp(Q)$ of all autotopisms of a loop Q is a group.

(4) A permutation α of Q is an automorphism if $\alpha\in Aut(Q)$ or if and only if $(\alpha,\alpha,\alpha)\in Atp(Q)$.

Definition 1.2([4]) *Let $(Q,\cdot)$ be any loop. A permutation C on symmetric group of Q is called crypto-automorphism of Q if there exist m, t in Q, such that for every x, y in Q, we have*

$$(x\cdot m)C\cdot(t\cdot y)C=(x\cdot y)C. \tag{1.4}$$

§2. Preliminaries

Lemma 2.1([5]) *A loop Q satisfy the identity (1.1) if and only if*

$$(L_x^{-1},R_x,L_x^{-1}R_x) \tag{2.1}$$

is an autotopism $\forall\ x\in Q$.

Lemma 2.2([5]) *A loop $(Q,\cdot)$ satisfies the Buchsteiner identity $x\backslash(xy\cdot z)=(y\cdot zx)/x$, if and only if $(L_x^{-1},R_x,L_x^{-1}R_x)\in Atp(Q),\forall\ x,y,z\in Q$.*

Theorem 2.1([5]) *Let Q be a Buchsteiner loop. Then $\forall\ x,y\in Q$, $R_{(x,y)}=[L_x,R_y]=L_{(y,x)}^{-1}$.*

Note also that, the commutator $[L_x,R_y]$, is defined as $L_xR_y=R_yL_x[L_x,R_y]\Rightarrow\ L_x^{-1}R_y^{-1}L_xR_y=[L_x,R_y]\Rightarrow\ L_x^{-1}L_y^{-1}L_{yx}=[L_x,R_y]$, since from Lemma 2.2, $R_y^{-1}L_xR_y=L_y^{-1}L_{yx}$.

Theorem 2.2([2]) *Let $(Q, \cdot, \backslash, /)$ be a quasigroup. If $Q(a, b, \circ) \overset{\theta}{\cong} Q(c, d, *)$, then $Q(f, g, \triangle) \overset{\theta}{\cong} Q((f \cdot b)\theta/d, c\backslash(a \cdot g)\theta, \Box)$. If $(Q, \cdot)$ is a loop, then $(f \cdot b)\theta/d = [f \cdot (a\backslash c\theta^{-1})]\theta$ and $c\backslash(a \cdot g)\theta = [(d\theta^{-1}/b) \cdot g]\theta$, where $a, b, c, d, f, g \in Q$.*

§3. Main Results

Our first main result reads:

Theorem 3.1 *A loop $(Q, \cdot, \backslash, /)$ is a Buchsteiner loop if and only if the identity*

$$u\{x\backslash[(xy)/v \cdot z]\} = \{[(uy)/v \cdot u\backslash\{(uz)/v \cdot u\backslash(xv)\}]/(u\backslash(xv))\}v \tag{3.1}$$

holds $\forall u, v, x, y, z \in Q$.

Proof Suppose $(Q, \cdot, \backslash, /)$ is a Buchsteiner loop with any arbitrary principal isotope $(Q, \circ)$ such that $x \circ y = xR_v^{-1} \cdot yL_u^{-1} = x/v \cdot u\backslash y, \forall\ u, v \in Q$. Buchsteiner loops are G-loops [5]. Now choose $u, v \in Q$ such that $(Q, \circ)$ is loop isotope of $(Q, \cdot)$. Therefore, we have $x\backslash[(x \circ y) \circ z] = [y \circ (z \circ x)]/x \Rightarrow x\backslash[(xR_v^{-1} \cdot yL_u^{-1})R_v^{-1} \cdot zL_u^{-1}] = [yR_v^{-1} \cdot (zR_v^{-1} \cdot xL_u^{-1})L_u^{-1}]/x$. Now choose p such that $x\backslash[(xR_v^{-1} \cdot yL_u^{-1})R_v^{-1} \cdot zL_u^{-1}] = p = [yR_v^{-1} \cdot (zR_v^{-1} \cdot xL_u^{-1})L_u^{-1}]/x$, then $[(xR_v^{-1} \cdot yL_u^{-1})R_v^{-1} \cdot zL_u^{-1}] = x \circ p \Leftrightarrow [yR_v^{-1} \cdot (zR_v^{-1} \cdot xL_u^{-1})L_u^{-1}] = p \circ x$. Solving these two separately and equating the answers give

$$u[(x/v)\backslash\{([(x/v) \cdot (u\backslash y)]/v) \cdot (u\backslash z)\}] = [\{(y/v) \cdot (u\backslash[(z/v) \cdot (u\backslash x)])\}/(u\backslash x)]v$$

Setting $x' = x/v \Rightarrow x'v = x$, $y' = u\backslash y \Rightarrow uy' = y$ and $z' = u\backslash z \Rightarrow uz' = z$ in the last expression gives

$$u\{x'\backslash[(x'y')/v \cdot z']\} = \{[(uy')/v \cdot u\backslash\{(uz')/v \cdot u\backslash(x'v)\}]/(u\backslash(x'v))\}v$$

which is the required identity if $x', y'$and $z'$are respectively replaced by x, y and z
Conversely, let $(Q, \cdot)$ be a loop which obeys equation (3.1), working upward the process of the proof of necessary condition, we obtain the Buchsteiner identity relation for any arbitrary u, v-principal isotope $(Q, \circ)$ of $(Q, \cdot)$. $\Box$

Lemma 3.1 *Let $(Q, \cdot)$ be a loop. Then*

(1) Q is a Buchsteiner loop if and only if, $\forall\ x, u, v \in Q$, the triple

$$(R_vL_x^{-1}L_uR_v^{-1}, L_uR_v^{-1}R_{\{u\backslash(xv)\}}L_u^{-1}, L_x^{-1}L_uR_v^{-1}R_{\{u\backslash(xv)\}}) \in Atp(Q). \tag{3.2}$$

(2) In particular, Q is a Buchsteiner loop if $\forall u, v \in Q$, the triple

$$(R_vL_uR_v^{-1}, L_uR_v^{-1}R_{(u\backslash v)}L_u^{-1}, L_uR_v^{-1}R_{(u\backslash v)}) \in Atp(Q). \tag{3.3}$$

Proof (1) Suppose the Q is a Buchsteiner loop, then equation (3.1) of Theorem 3.1 holds in $(Q, \cdot)$. Expressing the equation in term of autotopism gives (3.2). Conversely, suppose the autotopism (3.2) holds in Q, $\forall\ u,v\in Q$, taking any $y,z\in Q$ it implies that, $yR_vL_x^{-1}L_uR_v^{-1}\cdot zL_uR_v^{-1}R_{\{u\backslash(xv)\}}L_u^{-1}=(yz)L_x^{-1}L_uR_v^{-1}R_{\{u\backslash(xv)\}}$, the rest is simple.

(2) Suppose the Q is a Buchsteiner loop, then equation (3.1) of Theorem 3.1 holds in $(Q,\cdot)$, hence the autotopism (3.2) holds in Q. The required result is obtained if we set $x=1$ in this autotopism. □

Theorem 3.2 *Let $(Q,\cdot)$ be a loop, $(Q,\circ)$ an arbitrary principal isotope of $(Q,\cdot)$ and $(Q,*)$ some isotopes of $(Q,\cdot)$. Then $(Q,\cdot)$ is a Buchsteiner loop if and only if the commutative diagram*

$$(Q,\cdot)\xrightarrow[\textit{left principal isotopism}]{(R_v,I,I)}(Q,*)\xrightarrow[\textit{isomorphism}]{(\eta,\eta,\eta)}(Q,\circ)\xrightarrow[\textit{principal isotopism}]{(R^{-1}_{(u\backslash v)},L_u^{-1},I)}(Q,\cdot)$$

holds, where $\eta=L_uR_v^{-1}R_{(u\backslash v)},\forall u,v\in Q$.

Proof Suppose $(Q,\cdot)$ is a Buchsteiner loop, by Lemma 3.1(2) the autotopism (3.3) holds in $(Q,\cdot)$. Thus, $(R_vL_uR_v^{-1},L_uR_v^{-1}R_{(u\backslash v)}L_u^{-1},L_uR_v^{-1}R_{(u\backslash v)})=(R_v,I,I)(\eta,\eta,\eta)(R^{-1}_{(u\backslash v)},L_u^{-1},I)$, where $\eta=L_uR_v^{-1}R_{(u\backslash v)}$. Expressing this in terms of composition supplies the prove of the necessity. Conversely, suppose the commutative diagram holds in Q, we only need to show that the autotopism (3.3) holds in $(Q,\cdot)$. This is obtained by component multiplication of the compositions of the commutative diagram. □

Theorem 3.3 *A Buchsteiner loop $(Q,\cdot,\backslash,/)$ obeys the identities: $((uz)/v)\cdot(u\backslash v)=u\{(u[(u\backslash v)/v\cdot z])/v\cdot(u\backslash v)\}$ and $u\{[u\backslash(yv)]/v\}=\{(y\cdot u\backslash[(u/v)\cdot(u\backslash v)])/(u\backslash v)\}v$.*

Proof From Theorem 3.2, observed that $(Q,\circ)$ and $(Q,*)$ are principal and left principal isotopes of $(Q,\cdot)$ respectively and $\eta=L_uR_v^{-1}R_{(u\backslash v)}$ is an isomorphism. Therefore $(Q,1,v,\circ)\overset{\eta}{\cong}(Q,u,u\backslash v,*)$. Let $(Q,y,z,\triangle)$ be an arbitrary principal isotope of $(Q,\cdot)$, comparing these with the statement of Theorem 2.2, we have $a=1$, $b=v$, $c=u$, $d=u\backslash v$, $f=y$, $g=z$ and $\theta=\eta=L_uR_v^{-1}R_{(u\backslash v)}$. Using these we can compute: $c\backslash(a\cdot g)\theta=u\backslash(1\cdot z)L_uR_v^{-1}R_{(u\backslash v)}=u\backslash\{((uz)/v)\cdot(u\backslash v)\}$ and $[(d\theta^{-1}/b)\cdot g]\theta=[\{(u\backslash v)(L_uR_v^{-1}R_{(u\backslash v)})^{-1}\}/v\cdot z]L_uR_v^{-1}R_{(u\backslash v)}=\{(u[(u\backslash v)/v\cdot z])/v\}(u\backslash v)$. Hence $c\backslash(a\cdot g)\theta=[(d\theta^{-1}/b)\cdot g]\theta\Leftrightarrow u\backslash\{((uz)/v)\cdot(u\backslash v)\}=\{(u[(u\backslash v)/v\cdot z])/v\}(u\backslash v)\Leftrightarrow((uz)/v)\cdot(u\backslash v)=u\{(u[(u\backslash v)/v\cdot z])/v\cdot(u\backslash v)\}$, which proved the first identity. The second is similarly obtained, using appropriate arrangement. □.

Corollary 3.1 *Let $(Q,\cdot)$ be a Buchsteiner loop. Then the identities $(vz)/v=v[(v\cdot v^{\lambda}z)/v]$ and $v\{(v\backslash(yv))/v\}=yv^{\rho}\cdot v$ hold $\forall\ v,y,z\in Q$.*

Proof All of these identities are obtained respectively by identities of Theorem 3.3 by setting $u=v$. □

Corollary 3.2 *If $(Q,\cdot)$ is a Buchsteiner loop, then*

(1) $(vz)/v=v[(v\cdot v^{\lambda}z)/v]$ if and only if $L_v^{-1}=T_vL_{v^{\lambda}}T_v^{-1}$, $\forall v,z\in Q$;

(2) $v\{(v\backslash(yv))/v\}=yv^{\rho}\cdot v$ if and only if $R_v=T_v^{-1}R^{-1}_{v^{\rho}}T_v$, $\forall v,y\in Q$.

Proof Setting $u=v$ in the identities of Theorem 3.3, we obtained $(vz)/v=v[(v\cdot v^{\lambda}z)/v]\ \Rightarrow\ L_v^{-1}=T_vL_{v^{\lambda}}T_v^{-1}$ from the first one. Conversely, suppose $L_v^{-1}=T_vL_{v^{\lambda}}T_v^{-1}$

holds in Q, now for any $z \in Q$ $zL_v^{-1} = zT_vL_{v^\lambda}T_v^{-1} \Leftrightarrow v\backslash z = \{v[v^\lambda(v\backslash(zv))]\}/v$, now set $z = v\backslash(zv)$ and the first identity is obtained. The second assertion is similarly obtained. □

Corollary 3.3 *Let Q be a Buchsteiner loop, then $(T_vL_{v^\lambda}T_v^{-1}, T_v^{-1}R_{v^\rho}^{-1}T_v, T_vL_{v^\lambda}T_v^{-1}T_v^{-1}R_{v^\rho}^{-1}T_v) \in Atp(Q), \forall v \in Q$.*

Proof This is obtained by substituting the assertion of Corollary 3.2 into the autotopism (2.1). □

Lemma 3.2 *A permutation C on symmetric group of a loop Q is called crypto-automorphism, if and only if $(R_mC, L_tC, C) \in Atp(Q)$, where $m, t \in Q$.*

Proof Suppose C is a crypto-automorphism, then by Definition 1.2 equation (1.4) holds in Q, ie $(x \cdot m)C \cdot (t \cdot y)C = (x \cdot y)C \Leftrightarrow xR_mC \cdot yL_tC = (xy)C \Leftrightarrow (R_mC, L_tC, C) \in Atp(Q)$. Thus the result follows. □

Theorem 3.4 *Let $(Q, \cdot)$ be a Buchsteiner loop. Then*

(1) $T_vL_{(v^\lambda,v)}$ is a crypto-automorphism with companions $v\backslash(v^\rho v)$ and v.

(2) T_v is a crypto-automorphism with companions $v\backslash(v^\rho v)$ and v.

Proof (1)Using the autotopism $A = (T_vL_{v^\lambda}T_v^{-1}, T_v^{-1}R_{v^\rho}^{-1}T_v, T_vL_{v^\lambda}T_v^{-1}T_v^{-1}R_{v^\rho}^{-1}T_v)$ in Corollary 3.3 such that for any $y, z \in Q$, we have

$$yT_vL_{v^\lambda}T_v^{-1} \cdot zT_v^{-1}R_{v^\rho}^{-1}T_v = (yz)T_vL_{v^\lambda}T_v^{-1}T_v^{-1}R_{v^\rho}^{-1}T_v.$$

If we set $z = 1$, we obtain

$$\begin{aligned}
&yT_vL_{v^\lambda}T_v^{-1}R_v = yT_vL_{v^\lambda}T_v^{-1}T_v^{-1}R_{v^\rho}^{-1}T_v \\
&\Leftrightarrow yT_vL_{v^\lambda}L_v = yT_vL_{v^\lambda}T_v^{-1}T_v^{-1}R_{v^\rho}^{-1}T_v \\
&\Leftrightarrow yT_v(L_v^{-1}L_{v^\lambda}^{-1})^{-1} = yT_vL_{v^\lambda}T_v^{-1}T_v^{-1}R_{v^\rho}^{-1}T_v \\
&\Leftrightarrow yT_v(L_v^{-1}L_{v^\lambda}^{-1}L_{v^\lambda v})^{-1} = yT_vL_{v^\lambda}T_v^{-1}T_v^{-1}R_{v^\rho}^{-1}T_v.
\end{aligned}$$

From Theorem 2.1, we have $yT_vL_{(v,v^\lambda)} = yT_vL_{v^\lambda}T_v^{-1}T_v^{-1}R_{v^\rho}^{-1}T_v$. Thus we substitute to get $A = (T_vL_{v^\lambda}T_v^{-1}, T_v^{-1}R_{v^\rho}^{-1}T_v, T_vL_{(v,v^\lambda)})$.

Furthermore, $A^{-1} = (T_vL_{v^\lambda}^{-1}T_v^{-1}, T_v^{-1}R_{v^\rho}T_v, L_{(v,v^\lambda)}^{-1}T_v^{-1})$, thus for any $y, z \in Q$, applying A^{-1} we obtain, $yT_vL_{v^\lambda}^{-1}T_v^{-1} \cdot zT_v^{-1}R_{v^\rho}T_v = (yz)L_{(v,v^\lambda)}^{-1}T_v^{-1}$. Now by appropriate calculation, we can re-write $A^{-1} = (L_{(v,v^\lambda)}^{-1}T_v^{-1}R_{(v\backslash(v^\rho v))}^{-1}, L_{(v^\lambda,v)}^{-1}T_v^{-1}L_v^{-1}, L_{(v^\lambda,v)}^{-1}T_v^{-1}) \Leftrightarrow A = (R_{(v\backslash(v^\rho v))}T_vL_{(v^\lambda,v)}, L_vT_vL_{(v^\lambda,v)}, T_vL_{(v^\lambda,v)})$, which proved (1).

(2)$L_{(v^\lambda,v)}$ has been observed to be an automorphism in Q ([5]). Thus taking any $a, b \in Q$, we can write from (1) that

$$\begin{aligned}
A &= (R_{(v\backslash(v^\rho v))}T_vL_{(v^\lambda,v)}, L_vT_vL_{(v^\lambda,v)}, T_vL_{(v^\lambda,v)}) \\
&= (R_{(v\backslash(v^\rho v))}T_v, L_vT_v, T_v)(L_{(v^\lambda,v)}, L_{(v^\lambda,v)}, L_{(v^\lambda,v)})
\end{aligned}$$

and the result follows immediately. □

Theorem 3.5 *Let Q be a Buchsteiner loop, then T_v^{-1} is a crypto-automorphism with companions v and $v^\lambda, \forall\ v \in Q$.*

Proof From Theorem 3.4(2), we observed that T_v is a crypto-automorphism with companions $(v\backslash(v^\rho v)$ and v, thus by definition it implies that, for any a and b in Q, we have $aR_{(v\backslash(v^\rho v))}T_v \cdot bL_vT_v = (ab)T_v$. Setting $b = a^\rho$, we obtain $aR_{(v\backslash(v^\rho v))}T_v \cdot a^\rho L_vT_v = 1 \Rightarrow R_{(v\backslash(v^\rho v))}T_v = J_\rho L_vT_vJ_\lambda$, using the fact that Q is WWIP loop ([5]). This in terms of autotopism, implies $B = (J_\rho L_vT_vJ_\lambda, L_vT_v, T_v) \in Atp(Q)$, finally by appropriate calculation we have $J_\lambda L_vT_vJ_\rho = T_vR_v^{-1}$, and $L_vT_v = T_vL_{v^\lambda}^{-1}$, re-writing we have $B = (T_vR_v^{-1}, T_vL_{v^\lambda}^{-1}, T_v) \in Atp(Q), \forall\ v \in Q$. The result follows by taking the inverse of B. □

Corollary 3.4 *Any Buchsteiner loop Q is an $A-$loop.*

Proof It is straight forward from Corollary 5.4 in [5] and the preceding theorem. □

Remark 3.1 Since all the inner mappings, i.e. $L_{(u,v)}, R_{(u,v)}$ and T_v have been established to exhibit one form of automorphism or the other, then $(Q, \cdot)$ is an A-loop.

References

[1] R. H. Bruck (1966), *A survey of binary systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 185pp.

[2] B.F. Bryant and H. Schneider, Principal loop-isotopes of quasigroups, *Canad. J. Math.* 18(1966), 120-125.

[3] H.H. Buchsteiner, O nekotorom klasse binarnych lup, *Mat. Issled.* 39(1976), 54-66.

[4] R.Capodaglio Di Cocco, On the Isotopism and the Pseudo-Automorphisms of the Loops, *Bollettino U.M.I.* 7(1993), 199-205.

[5] P.Csörgö, A. Drápal and M.K. Kinyon, Buchsteiner loops, Submitted.

[6] P.Csörgö and A. Drápal, Buchsteiner loops and conjugacy closedness, Preprint.

[6] E. Falconer, *Quasigroup Identities Invariant under Isotopy*, Ph.D thesis, Emory University (1969).

[7] T.G. Jaiyeola, *The study of universality of Osborn loops*, Ph.D. thesis, University of Agriculture, Abeokuta, (2008) 256pp.

[8] A.D. Keedwell, The existence of Buchsteiner and conjugacy-closed quasigroups, *European Journal of Combinatorics*, 30(2009) 1382-1385.

[9] H.O. Pflugfelder, Quasigroups and loops: Introduction, *Sigma Series in Pure Math.* 7(1990), Heldermann Verlag, Berlin.

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Generalizations of Poly-Bernoulli Numbers and Polynomials

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Abstract: The concepts of poly-Bernoulli numbers $B_n^{(k)}$, poly-Bernoulli polynomials $B_n^k(t)$ and the generalized poly-Bernoulli numbers $B_n^{(k)}(a,b)$ are generalized to $B_n^{(k)}(t,a,b,c)$ which is called the generalized poly-Bernoulli polynomials depending on real parameters *a,b,c*. Some properties of these polynomials and some relationships between B_n^k , $B_n^{(k)}(t)$, $B_n^{(k)}(a,b)$ and $B_n^{(k)}(t,a,b,c)$ are established.

Key Words: Poly-Bernoulli polynomial, Euler number, Euler polynomial.

AMS(2000): 11B68, 11B73

§1. Introduction

In this paper we shall develop a number of generalizations of the poly-Bernoulli numbers and polynomials, and obtain some results about these generalizations. They are fundamental objects in the theory of special functions.

Euler numbers are denoted with B_k and are the coefficients of Taylor expansion of the function $\dfrac{t}{e^t-1}$ as following:

$$\frac{t}{e^t-1}=\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}.$$

The Euler polynomials $E_n(x)$ are expressed in the following series

$$\frac{2e^{xt}}{e^t+1}=\sum_{k=0}^{\infty}E_k(x)\frac{t^k}{k!}.$$

for more details, see [1]-[4].

In [10], Q.M.Luo, F.Oi and L.Debnath defined the generalization of Euler polynomials $E_k(x,a,b,c)$ which are expressed in the following series:

$$\frac{2c^{xt}}{b^t+a^t}=\sum_{k=0}^{\infty}E_k(x,a,b,c)\frac{t^k}{k!}.$$

where $a,b,c\in\mathcal{Z}^+$. They proved that

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I) for $a = 1$ and $b = c = e$

$$E_k(x+1) = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} E_j(x) \tag{1}$$

and

$$E_k(x+1) + E_k(x) = 2x^k. \tag{2}$$

II) for $a = 1$ and $b = c$,

$$E_k(x+1,1,b,b) + E_k(x,1,b,b) = 2x^k(\ln b)^k. \tag{3}$$

In[5], Kaneko introduced and studied poly-Bernoulli numbers which generalize the classical Bernoulli numbers. Poly-Bernoulli numbers $B_n^{(k)}$ with $k \in \mathcal{Z}$ and $n \in \mathcal{N}$ appear in the following power series:

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \qquad (*)$$

where $k \in \mathcal{Z}$ and

$$Li_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}. \qquad |z| < 1.$$

So for $k \leq 1$,

$$Li_1(z) = -\ln(1-z), Li_0(z) = \frac{z}{1-z}, Li_{-1} = \frac{z}{(1-z)^2}, \dots \ .$$

Moreover when $k \geq 1$, the left hand side of $(*)$ can be written in the form of "interated integrals"

$$\begin{aligned} e^t \frac{1}{e^t - 1} &= \int_0^t \frac{1}{e^t-1} \int_0^t \dots \frac{1}{e^t-1} \int_0^t \frac{t}{e^t-1} dtdt\dots dt \\ &= \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}. \end{aligned}$$

In the special case, one can see $B_n^{(1)} = B_n$.

Definition 1.1 *These poly-Bernoulli polynomials $B_n^{(k)}(t)$ are appeared in the expansion of $\frac{Li_k(1-e^{-x})}{1-e^{-x}} e^{xt}$ as follows:*

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}} e^{xt} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}(t)}{n!} x^n \tag{4}$$

for more details, see $[6] - [11]$.

Proposition 1.1 (Kaneko theorem [6]) *The Poly-Bernoulli numbers of non-negative index k, satisfy the following*

$$B_n^{(k)} = (-1)^n \sum_{m=1}^{n+1} \frac{(-1)^{m-1}(m-1)! \left\{ \begin{matrix} n \\ m-1 \end{matrix} \right\}}{m^k}, \tag{5}$$

and for negative index $-k$, we have

$$B_n^{(-k)} = \sum_{j=0}^{min(n,k)} (j!)^2 \left\{ \begin{array}{c} n+1 \\ j+1 \end{array} \right\} \left\{ \begin{array}{c} k+1 \\ j+1 \end{array} \right\}, \tag{6}$$

where

$$\left\{ \begin{array}{c} n \\ m \end{array} \right\} = \frac{(-1)^m}{m!} \sum_{l=0}^{m} (-1)^l \left(\begin{array}{c} m \\ l \end{array} \right) l^n \quad m, n \geq 0 \tag{7}$$

Definition 1.2 *Let $a, b > 0$ and $a \neq b$. The generalized poly-Bernoulli numbers $B_n^{(k)}(a, b)$, the generalized poly-Bernoulli polynomials $B_n^{(k)}(t, a, b)$ and the polynomial $B_n^{(k)}(t, a, b, c)$ are appeared in the following series respectively.*

$$\frac{Li_k(1-(ab)^{-t})}{b^t - a^{-t}} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}(a, b)}{n!} t^n \quad |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{8}$$

$$\frac{Li_k(1-(ab)^{-t})}{b^t - a^{-t}} e^{xt} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}(x, a, b)}{n!} t^n \quad |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{9}$$

$$\frac{Li_k(1-(ab)^{-t})}{b^t - a^{-t}} c^{xt} = \sum_{n=0}^{\infty} \frac{B_n^{(k)}(x, a, b, c)}{n!} t^n \quad |t| < \frac{2\pi}{|\ln a + \ln b|}, \tag{10}$$

§2. Main Theorems

We present some recurrence formulae for generalized poly-Bernoulli polynomials.

Theorem 2.1 *Let $x \in \mathbb{R}$ and $n \geq 0$. For every positive real numbers a, b and c such that $a \neq b$ and $b > a$, we have*

$$B_n^{(k)}(a, b) = B_n^{(k)} \left(\frac{-\ln b}{\ln a + \ln b} \right) (\ln a + \ln b)^n, \tag{11}$$

$$B_j^{(k)}(a, b) = \sum_{i=1}^{j} (-1)^{j-i} (\ln a + \ln b)^i (\ln b)^{j-i} \left(\begin{array}{c} j \\ i \end{array} \right) B_j^{(k)}, \tag{12}$$

$$B_n^{(k)}(x; a, b, c) = \sum_{l=0}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) (\ln c)^{n-l} B_l^{(k)}(a, b) x^{n-l}, \tag{13}$$

$$B_n^{(k)}(x+1; a, b, c) = B_n^{(k)}(x; ac, \frac{b}{c}, c), \tag{14}$$

$$B_n^{(k)}(t) = B_n^{(k)}(e^{t+1}, e^{-t}), \tag{15}$$

$$B_n^{(k)}(x, a, b, c) = (\ln a + \ln b)^n B_n^{(k)}(\frac{-\ln b + x \ln c}{\ln a + \ln b}). \tag{16}$$

Proof Applying Definition 1.2, we prove formulae (11)-(16) as follows.

(1) For formula (11), we note that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}} &= \sum_{n=0}^{\infty}\frac{B_n^{(k)}(a,b)}{n!}t^n = \frac{1}{b^t}\left(\frac{Li_k(1-e^{-t\ln ab})}{1-e^{-t\ln ab}}\right)\\ &= e^{-t\ln b}\left(\frac{Li_k(1-e^{-t\ln ab})}{1-e^{-t(\ln ab)}}\right)\\ &= \sum_{n=0}^{\infty}B_n^{(k)}\left(\frac{-\ln b}{\ln a+\ln b}\right)(\ln a+\ln b)^n\frac{t^n}{n!}\end{aligned}$$

Therefore

$$B_n^{(k)}(a,b) = B_n^{(k)}\left(\frac{-\ln b}{\ln a+\ln b}\right)(\ln a+\ln b)^n.$$

(2) For formula (12), notice that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}} &= \frac{1}{b^t}\left(\frac{Li_k(1-(ab)^{-t}\ )}{1-e^{-t\ln ab}}\right)\\ &= \left(\sum_{k=0}^{\infty}\frac{(\ln b)^k}{k!}(-1)^k t^k\right)\left(\sum_{n=0}^{\infty}B_n^{(k)}\frac{(\ln a+\ln b)^n}{n!}t^n\right)\\ &= \sum_{j=0}^{\infty}\left(\sum_{i=0}^{j}(-1)^{j-i}B_i^{(k)}\frac{(\ln a+\ln b)^i}{i!(j-i)!}(\ln b)^{j-i}\right)t^j.\end{aligned}$$

We have

$$B_j^{(k)}(a,b) = \sum_{i=0}^{j}(-1)^{j-i}(\ln a+\ln b)^i(\ln b)^{j-i}\begin{pmatrix} j\\ i\end{pmatrix}B_i^{(k)}.$$

(3) For formula (13), by calculation we know that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt} &= \sum_{n=0}^{\infty}B_n^{(k)}(x,a,b,c)\frac{t^n}{n!}\\ &= \left(\sum_{l=0}^{\infty}B_l^{(k)}(a,b)\frac{t^l}{l!}\right)\left(\sum_{i=0}^{\infty}\frac{(\ln c)^i t^i}{i!}x^i\right)\\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\frac{(\ln c)^{l-i}}{i!(l-i)!}B_i^{(k)}(a,b)x^{l-i}t^l\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}(\ln c)^{n-l}B_l^{(k)}(a,b)x^{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

(4) For formula (14), calculation shows that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+1)t} &= \frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}.c^t\\ &= \frac{Li_k(1-(ab)^{-t})}{\left(\frac{b}{c}\right)^t-(ac)^{-t}}c^{xt} = \sum_{n=0}^{\infty}B_n^{(k)}(x;ac,\frac{b}{c},c)\frac{t^n}{n!}.\end{aligned}$$

(5) For formula (15), because of

$$\frac{Li_k(1-e^{-x})}{1-e^{-x}}e^{xt} = \frac{Li_k(1-e^{-x})}{e^{-xt}-e^{-x-xt}} = \frac{Li_k(1-e^{-x})}{(e^{-t})^x-(e^{1+t})^{-x}},$$

so we get that

$$B_n^{(k)}(t) = B_n^{(k)}(e^{t+1}, e^{-t}).$$

(6) For formula (16), write

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(k)}(x,a,b,c)\frac{t^n}{n!} &= \frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt} = \frac{1}{b^t}\frac{Li_k(1-(ab)^{-t})}{(1-(ab)^{-t})}c^{xt}\\ &= e^{t(-\ln b + x\ln c)}\left(\frac{Li_k(1-e^{-t\ln ab})}{1-e^{-t(\ln ab)}}\right)\\ &= \sum_{n=0}^{\infty}(\ln a+\ln b)^n B_n^{(k)}\left(\frac{-\ln b + x\ln c}{\ln a+\ln b}\right)\frac{t^n}{n!}.\end{aligned}$$

So

$$B_n^{(k)}(x,a,b,c) = (\ln a+\ln b)^n B_n^{(k)}\left(\frac{-\ln b+x\ln c}{\ln a+\ln b}\right).$$

□

Theorem 2.2 *Let $x \in \mathbb{R}$, $n \geq 0$. For every positive real numbers a,b such that $a \neq b$ and $b > a > 0$, we have*

$$\begin{aligned}B_n^{(k)}(x+y,a,b,c) &= \sum_{l=0}^{\infty}\begin{pmatrix} n\\ l\end{pmatrix}(\ln c)^{n-l}B_l^{(k)}(x;a,b,c)y^{n-l}\\ &= \sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}(\ln c)^{n-l}B_l^{(k)}(y,a,b,c)x^{n-l}. \qquad (17)\end{aligned}$$

Proof Calculation shows that

$$\begin{aligned}\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{(x+y)t} &= \sum_{n=0}^{\infty}B_n^{(k)}(x+y;a,b,c)\frac{t^n}{n!} = \frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}.c^{yt}\\ &= \left(\sum_{n=0}^{\infty}B_n^{(k)}(x;a,b,c)\frac{t^n}{n!}\right)\left(\sum_{i=0}^{\infty}\frac{y^i(\ln c)^i}{i!}t^i\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}y^{n-l}(\ln c)^{n-l}B_l^{(k)}(x,a,b,c)\right)\frac{t^n}{n!}.\end{aligned}$$

So we get

$$\begin{aligned}\frac{Li_k(1-(ab)^{t})}{b^t-a^{-t}}c^{(x+y)t} &= \frac{Li_k(1-(ab)^{-t}}{b^t-a^{-t}}c^{yt}c^{xt}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{pmatrix} n\\ l\end{pmatrix}x^{n-l}(\ln c)^{n-l}B_l^{(k)}(y,a,b,c)\right)\frac{t^n}{n!}. \qquad \square\end{aligned}$$

Theorem 2.3 *Let $x \in \mathbb{R}$ and $n \geq 0$.For every positive real numbers a,b and c such that $a \neq b$ and $b > a > 0$, we have*

$$B_n^{(k)}(x;a,b,c) = \sum_{l=0}^{n} \begin{pmatrix} n \\ l \end{pmatrix} (\ln c)^{n-l} B_l^{(k)} \left(\frac{-\ln b}{\ln a + \ln b} \right) (\ln a + \ln b)^l x^{n-l}, \tag{18}$$

$$B_n^{(k)}(x;a,b,c) = \sum_{l=0}^{n} \sum_{j=0}^{l} (-1)^{l-j} \begin{pmatrix} n \\ l \end{pmatrix} \begin{pmatrix} l \\ j \end{pmatrix} (\ln c)^{n-l} (\ln b)^{l-j} (\ln a + \ln b)^j B_j^{(k)} x^{n-k}. \tag{19}$$

Proof Applying Theorems 2.1 and 2.2, we know that

$$B_n^{(k)}(x;a,b,c) = \sum_{l=0}^{n} \begin{pmatrix} n \\ l \end{pmatrix} (\ln c)^{n-l} B_l^{(k)}(a,b) x^{n-l}$$

and

$$B_n^{(k)}(a,b) = B_n^{(k)}(\frac{-lnb}{\ln a + \ln b})(\ln a + \ln b)^n$$

Then the relation (18) follow if we combine these formulae. The proof for (19) is similar. □

Now,we give some results about derivatives and integrals of the generalized poly-Bernoulli polynomials in the following theorem.

Theorem 2.4 *Let $x \in \mathbb{R}$.If a, b and $c > 0$, $a \neq b$ and $b > a > 0$,For any non-negative integer l and real numbers α and β we have*

$$\frac{\partial^l B_n^{(k)}(x,a,b,c)}{\partial x^l} = \frac{n!}{(n-l)!} (\ln c)^l B_{n-l}^{(k)}(x,a,b,c) \tag{20}$$

$$\int_{\alpha}^{\beta} B_n^{(k)}(x,a,b,c)dx = \frac{1}{(n+1)\ln c} [B_{n+1}^{(k)}(\beta,a,b,c) - B_{n+1}^{(k)}(\alpha,a,b,c)] \tag{21}$$

Proof Applying induction on n, these formulae (20) and (21) can be proved. □

In [9], GI-Sang Cheon investigated the classical relationship between Bernoulli and Euler polynomials, in this paper we study the relationship between the generalized poly-Bernoulli and Euler polynomials.

Theorem 2.5 *For $b > 0$ we have*

$$B_n^{(k_1)}(x+y,1,b,b) = \frac{1}{2} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} [B_n^{(k_1)}(y,1,b,b) + B_n^{(k_1)}(y+1,1,b,b)] E_{n-k}(x,1,b,b).$$

Proof We know that

$$B_n^{(k_1)}(x+y,1,b,b) = \sum_{k=0}^{\infty} \binom{n}{k} (\ln b)^{n-k} B_k^{(k_1)}(y;1,b,b) x^{n-k}$$

and

$$E_k(x+y,1,b,b) + E_k(x,1,b,b) = 2x^k (\ln b)^k$$

So, we obtain

$$\begin{aligned}
B_n^{(k_1)}(x+y,1,b,b) &= \frac{1}{2}\sum_{k=0}^{n} \binom{n}{k} (\ln b)^{n-k} B_k^{(k_1)}(y;1,b,b) \\
&\times \left[\frac{1}{(\ln b)^{n-k}}(E_{n-k}(x,1,b,b) + E_{n-k}(x+1,1,b,b))\right] \\
&= \frac{1}{2}\sum_{k=0}^{n} \binom{n}{k} B_k^{(k_1)}(y;1,b,b) \\
&\times \left[E_{n-k}(x,1,b,b) + \sum_{j=0}^{n-k} \binom{n-k}{j} E_j(x,1,b,b)\right] \\
&= \frac{1}{2}\sum_{k=0}^{n} \binom{n}{k} B_k^{(k_1)}(y;1,b,b) E_{n-k}(x,1,b,b) \\
&+ \frac{1}{2}\sum_{j=0}^{n} \binom{n}{j} E_j(x;1,b,b) \sum_{k=0}^{n-j} \binom{n-j}{k} B_k^{(k_1)}(y,1,b,b) \\
&= \frac{1}{2}\sum_{k=0}^{n} \binom{n}{k} B_k^{(k_1)}(y;1,b,b) E_{n-k}(x,1,b,b) \\
&+ \frac{1}{2}\sum_{j=0}^{n} \binom{n}{j} B_{n-j}^{(k_1)}(y+1;1,b,b) E_j(x,1,b,b)
\end{aligned}$$

So we have

$$B_n^{(k_1)}(x+y,1,b,b) = \frac{1}{2}\sum_{k=0}^{n} \binom{n}{k} [B_n^{(k_1)}(y,1,b,b) + B_n^{(k_1)}(y+1,1,b,b)] E_{n-k}(x,1,b,b).$$

□

Corollary 2.1 *In Theorem* 2.5*, if* $k_1 = 1$ *and* $b = e$*, then*

$$B_n(x) = \sum_{(k=0),(k\neq 1)}^{n} \binom{n}{k} B_k E_{n-k}(x).$$

For more details see [7].

References

[1] T.Arakawa and M.Kaneko, On poly-Bernoulli numbers, *Comment.Math.Univ.St.Pauli* 48 (1999), 159-167.

[2] B.N.Oue and F.Qi, Generalization of Bernoulli polynomials, *Internat.J.Math.Ed.Sci.Tech.* 33(2002), No,3, 428-431.

[3] M.S.Kim and T.Kim, An explicit formula on the generalized Bernoulli number with order n, *Indian.J.Pure and Applied Math.* 31(2000), 1455-1466.

[4] Hassan Jolany and M.R.Darafsheh, Some another remarks on the generalization of Bernoulli and Euler polynomials, *Scientia Magna*, Vol.5, No.3.

[5] M.Kaneko, Poly-Bernoulli numbers, *Journal de Theorides Numbers De Bordeaux*, 9(1997), 221-228.

[6] Y.Hamahata,H.Masubuch, Special multi-poly-Bernoulli numbers, *Journal of Integer Sequences*, Vol.10(2007).

[7] H.M.Srivastava and A.Pinter, Remarks on some relationships between the Bernoulli and Euler polynomials, *Applied Math. Letter*, 17(2004), 375-380.

[8] Chad Brewbaker, A combinatorial Interpretation of the poly-Bernoulli numbers and two Fermat analogues, *Integers Journal*, 8 (2008).

[9] GI-Sang Cheon, A note on the Bernoulli and Euler polynomials, *Applied Math.Letter*, 16(2003),365-368.

[10] Q.M.Luo,F.Oi and L.Debnath, Generalization of Euler numbers and polynomials, *Int.J. Math. Sci.* (2003), 3893-3901.

[11] Y.Hamahata,H.Masubuchi, Recurrence formulae for multi-poly-Bernoulli numbers, *Integers Journal*, 7(2007).

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Open Alliance in Graphs

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Abstract: A defensive alliance in a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ satisfying the condition that for every vertex $v \in S$, the number of v's neighbors is at least as large as the number of v's neighbors in $V - S$. For a subset $T \subset V, T \neq S$, a defensive alliance S is called *Smarandachely T-strong*, if for every vertex $v \in S$, $|N[v] \cap S| > |N(v) \cap ((V - S) \cup T)|$. In this case we say that every vertex in S is *Smarandachely T-strongly defended*. Particularly, if we choose $T = \emptyset$, i.e., a Smarandachely $\emptyset$-strong is called strong defend for simplicity. The boundary of a set S is the set $\partial S = \bigcup_{v \in S} N(v) - S$. An offensive alliance in a graph G is a set of vertices $S \subseteq V$ such that for every vertex v in the boundary of S, the number of v's neighbors in S is at least as large as the number of v's neighbors in $V - S$. In this paper we study open alliance problem in graphs which was posted as an open question in [S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, *Alliances in graphs*, J. Combin. Math. Combin. Comput. 48 (2004) 157-177].

Key Words: Smarandachely T-strongly defended, defensive alliance, affensive alliance, strongly defended, open.

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§1. Introduction

In this paper we study open alliance in graphs. For graph theory terminology and notation, we generally follow [3]. For a vertex v in a graph $G = (V, E)$, the *open neighborhood* of v is the set $N(v) = \{u : uv \in E\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *boundary* of S is the set $\partial S = \bigcup_{v \in S} N(v) - S$. We denote the degree of v in S by $d_S(v) = N(v) \cap S$. The *edge connectivity*, $\lambda(G)$, of a graph G is the minimum number of edges in a set, whose removal results in a disconnected graph. A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$, written $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. For $S \subseteq V$, the *subgraph induced* by S is the graph $G[S] = (S, E \cap S \times S)$.

The study of *defensive alliance* problem in graphs, together with a variety of other kinds of alliances, was introduced in [2]. A non-empty set of vertices $S \subseteq V$ is called a *defensive alliance* if for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$. In this case, we say that every vertex in S is defended from possible attack by vertices in $V - S$. A defensive alliance is called *strong* if for every vertex $v \in S$, $|N[v] \cap S| > |N(v) \cap (V - S)|$. In this case we say that every

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vertex in S is strongly defended. An (strong) alliance S is called *critical* if no proper subset of S is an (strong) alliance. The *defensive alliance number* of G, denoted $a(G)$, is the minimum cardinality of any critical defensive alliance in G. Also the *strong defensive alliance number* of G, denoted $\hat{a}(G)$, is the minimum cardinality of any critical strong defensive alliance in G. For a subset $T \subset V, T \neq S$, a defensive alliance S is called *Smarandachely T-strong*, if for every vertex $v \in S$, $|N[v] \cap S| > |N(v) \cap ((V-S) \cup T)|$. In this case we say that every vertex in S is Smarandachely T-strongly defended. Particularly, if we choose $T = \emptyset$, i.e., a Smarandachely $\emptyset$-strong is called strong defend for simplicity.

The study of *offensive alliances* was initiated by Favaron et al in [1]. A non-empty set of vertices $S \subseteq V$ is called an *offensive alliance* if for every $v \in \partial(S)$, $|N(v) \cap S| \geq |N[v] \cap (V-S)|$. In this case we say that every vertex in $\partial(S)$ is *vulnerable* to possible attack by vertices in S. An offensive alliance is called *strong* if for every vertex $v \in \partial(S)$, $|N(v) \cap S| > |N[v] \cap (V-S)|$. In this case we say that every vertex $\partial(S)$ is *very vulnerable*. The *offensive alliance number*, $a_o(G)$ of G, is the minimum cardinality of any critical offensive alliance in G. Also the *strong offensive alliance number*, $\hat{a}_o(G)$ of G, is the minimum cardinality of any critical strong offensive alliance in G.

In [2] the authors left the study of open alliances as an open question. In this paper we study open alliance in graphs. An alliance is called *open* (or *total*) if it is defined completely in terms of open neighborhoods. We study open defensive alliances as well as open offensive alliances in graphs.

Recall that a vertex of degree one in a graph G is called a *leaf* and its neighbor is a *support vertex*. Let $S(G)$ denote the set all support vertexes of a graph G.

§2. **Open Defensive Alliance**

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is an *open defensive alliance* if for every vertex $v \in S$, $|N(v) \cap S| \geq |N(v) \cap (V-S)|$. A set $S \subseteq V$ is an *open strong defensive alliance* if for every vertex $v \in S$, $|N(v) \cap S| > |N(v) \cap (V-S)|$. An open (strong) defensive alliance S is called *critical* if no proper subset of S is an open (strong) defensive alliance. The *open defensive alliance number*, $a_t(G)$ of G, is the minimum cardinality of any critical open defensive alliance in G, and the *strong open defensive alliance number*, $\hat{a}_t(G)$ of G, is the minimum cardinality of any critical open strong defensive alliance in G.

We remark that with this definition, strong defensive alliance is equivalent to open defensive alliance, and so we have the following observation.

Observation 2.1 For any graph G, $a_t(G) = \hat{a}(G)$.

Thus we focus on open strong defensive alliances in G. We refer to an $\hat{a}_t(G)$-set as a minimum open strong defensive alliance in G. By definition we have the following.

Observation 2.2 For any $\hat{a}_t(G)$-set S in a graph G, $G[S]$ is connected.

Observation 2.3 Let S be an $\hat{a}_t(G)$-set in a graph G, and $v \in S$. If $deg_{G[S]}(v) = 1$, then

$deg_G(v) = 1$.

Note that for any graph G of n vertices $2 \leq \hat{a}_t(G) \leq n$. In the following we characterize all graphs of order n having open strong defensive alliance number n. For an integer n let $\mathcal{E}_n$ be the class of all graphs G such that $G \in \mathcal{E}_n$ if and only if one of the following holds:
(1) G is a path on n vertices, (2) G is a cycle on n vertices, (3) G is obtained from a cycle on n vertices by identifying two non adjacent vertices.

Theorem 2.4 *For a connected graph G of n vertices, $\hat{a}_t(G) = n$ if and only if $G \in \mathcal{E}_n$.*

Proof First we show that $\hat{a}_t(P_n) = \hat{a}_t(C_n) = n$. Suppose to the contrary, that $\hat{a}_t(P_n) < n$. Let S be a $\hat{a}_t(P_n)$-set. By Observation 2.2, $G[S]$ is connected. So $G[S]$ is a path. Let $v \in S$ be a vertex such that $deg_{G[S]}(v) = 1$. By Observation 2.3, $deg_G(v) = 1$. Then $G[S] = P_n$, a contradiction. Thus $\hat{a}_t(P_n) = n$. Similarly, for any other graph in $\mathcal{E}_n$, $\hat{a}_t(G) = n$.

For the converse suppose that G is a graph of n vertices and $\hat{a}(G) = n$. If $\Delta(G) \leq 2$, then G is a path or a cycle on n vertices, as desired. Suppose that $\Delta(G) \geq 3$. Let v be a vertex of maximum degree in G. Since $V(G) \setminus \{v\}$ is not an open strong defensive alliance in G, there is a vertex $v_1 \in N(v)$ such that $deg(v_1) \leq 2$. If $deg(v_1) = 1$, then $V(G) \setminus \{v_1\}$ is an open strong defensive alliance, which is a contradiction. So $deg(v_1) = 2$. Since $V(G) \setminus \{v_1\}$ is not an open strong defensive alliance, there is a vertex $v_2 \in N(v_1)$ such that $deg(v_2) \leq 2$. If $deg(v_2) = 1$, then $V(G) \setminus \{v_2\}$ is an open strong defensive alliance, which is a contradiction. So $deg(v_2) = 2$. Since $V(G) \setminus \{v_1, v_2\}$ is not an open strong defensive alliance, there is a vertex $v_3 \in N(v_2)$ such that $deg(v_3) \leq 2$. Continuing this process we obtain a path $v_1 - v_2 - ... - v_k$ for some k such that $deg(v_i) = 2$ for $1 \leq i < k$ and either $deg(v_k) = 1$ or $v_k = v$. If $deg(v_k) = 1$, then $V(G) \setminus \{v_1, ..., v_k\}$ is an open strong defensive alliance for G. This is a contradiction. So $v_k = v$. If $deg(v) \geq 5$, then $V(G) \setminus \{v_1, v_2, ..., v_{k-1}\}$ is an open strong defensive alliance for G, a contradiction. So $deg(v) = \Delta(G) = 4$. Since $V(G) \setminus \{v_1, v_2, ..., v_k\}$ is not an open strong defensive alliance, there is a vertex $w_1 \in N(v) \setminus \{v_1, v_{k-1}\}$ with $deg(w_1) \leq 2$. If $deg(w_1) = 1$ then $V(G) \setminus \{w_1\}$ is an open defensive alliance, a contradiction. So $deg(w_1) = 2$. Since $V(G) \setminus \{v_1, v_2, ..., v_k, w_1\}$ is not an open strong defensive alliance, there is a vertex $w_2 \in N(w_1)$ such that $deg(w_2) = 2$. As before, continuing the process, we deduce that there is a path $w_1 - w_2 - ... - w_l$ for some l such that $deg(v_i) = 2$ for $1 \leq i < l$ and $v_l = v$. Since $\Delta(G) = 4$, we conclude that G is obtained by identifying a vertex of C_k with a vertex of C_l. This completes the result. □

As a consequence we have the following result.

Corollary 2.5 *For a connected graph G, $\hat{a}_t(G) = 2$ if and only if $G = P_2$.*

For a nonempty set S in a graph G and a vertex $x \in S$, we let $deg_S(v) = N(v) \cap S$. So a set $S \subseteq V$ is an open defensive alliance if for every vertex $v \in S$, $deg_S(v) \geq deg_{V-S}(v) + 1$. Notice that this is equivalent to $2deg_S(v) \geq deg(v) + 1$.

Proposition 2.6 *For any graph G, $\hat{a}_t(G) = 3$, if and only if $\hat{a}_t(G) \neq 2$, and G has an induced subgraph isomorphic to either (1) the path $P_3 = u - v - w$, where $deg(u) = deg(w) = 1$ and $2 \leq deg(v) \leq 3$, or (2) the cycle C_3, where each vertex is of degree at most three.*

Proof Let G be a graph. Suppose that $\hat{a}_t(G) \neq 2$. If G has an induced subgraph $P_3 = u - v - w$, where $deg(u) = deg(w) = 1$ and $2 \leq deg(v) \leq 3$, then $\{u, v, w\}$ is an open strong defensive alliance, and so $\hat{a}_t(G) = 3$. Similarly, if (2) holds, we obtain $\hat{a}_t(G) = 3$.

Conversely, suppose that $\hat{a}_t(G) = 3$. So $\hat{a}_t(G) \neq 2$. Let $S = \{u, v, w\}$ be a $\hat{a}_t(G)$-set. By Observation 2.2, $G[S]$ is connected. If $G[S]$ is a path, then we let $deg_{G[S]}(u) = deg_{G[S]}(w) = 1$. By definition $deg_G(u) = deg_G(w) = 1$. If $deg_G(v) \geq 4$, then S is not an open strong defensive alliance, which is a contradiction. So $2 \leq deg_G(v) \leq 3$. It remains to suppose that $G[S]$ is a cycle. If a vertex of S has degree at least four in G, then S is not an open strong defensive alliance, a contradiction. Thus any vertex of S has degree at most three in G. □

Let G_1 be a graph obtained from K_4 by removing two edge such that the resulting graph G has a pendant vertex. Let G_2 be a graph obtained from K_4 by removing an edge, with vertices $\{v_1, v_2, v_3, v_4\}$, where $deg(v_1) = deg(v_2) = 2$.

Proposition 2.7 *For any graph G, $\hat{a}_t(G) = 4$ if and only if $\hat{a}_t(G) \notin \{2, 3\}$, and G has an induced subgraph isomorphic to one of the following:*

(1) P_4, with vertices, in order, v_1, v_2, v_3 and v_4, where $deg(v_1) = deg(v_4) = 1$, and $deg(v_2)$ and $deg(v_3)$ are at most three;

(2) C_4, where each vertex is of degree at most three;

(3) K_4, where each vertex has degree at most five;

(4) $K_{1,3}$, with vertices $\{v_1, v_2, v_3, v_4\}$, where $deg(v_i) = 1$ for $i = 2, 3, 4$, and $deg(v_1) \leq 5$;

(5) G_1, where $deg(v_i) \leq 5$ for $i = 1, 2, 3, 4$;

(6) G_2, where $deg(v_i) \leq 3$ for $i = 1, 2$, and $deg(v_i) \leq 5$ for $i = 3, 4$.

Proof It is a routine matter to see that if $\hat{a}_t(G) \notin \{2, 3\}$, and G has an induced subgraph isomorphic to (i) for some $i \in \{1, 2, ..., 6\}$, then $\hat{a}_t(G) = 4$. Suppose that $\hat{a}_t(G) = 4$. Let $S = \{v_1, v_2, v_3, v_4\}$ be a $\hat{a}_t(G)$-set. By Observation 2.2 $G[S]$ is connected. If $G[S]$ is a path, then we assume that $deg_{G[S]}(v_i) = 1$ for $i = 1, 4$, and $deg_{G[S]}(v_i) = 2$ for $i = 2, 3$. Now by Observation 2.3 $deg(v_i) = 1$ for $i = 1, 4$, and $4 = 2deg_{G[S]}(v_i) \geq deg(v_i) + 1$ which implies that $deg(v_i) \leq 3$ for $i = 2, 3$. We deduce that G has an induced subgraph isomorphic to (1). So suppose that $G[S]$ is not a path. If $G[S]$ is a cycle then $4 = 2deg_{G[S]}(v_i) \geq deg(v_i) + 1$ which implies that $deg(v_i) \leq 3$ for $i = 1, 2, 3, 4$, and so G has an induced subgraph isomorphic to (2). We assume now that $\Delta(G[S]) > 2$. So $\Delta(G[S]) = 3$. Let $deg_{G[S]}(v_1) = 3$. If any vertex of $G[S]$ is of maximum degree then $6 = 2deg_{G[S]}(v_i) \geq deg(v_i) + 1$ which implies that $deg(v_i) \leq 5$ for $i = 1, 2, 3, 4$. So G has an induced subgraph isomorphic to (3). Thus we suppose that $G[S]$ is not complete graph. If $deg_{G[S]}(v_i) = 1$ for $i = 2, 3, 4$, then by Observation 2.3 $deg(v_i) = 1$ for $i = 2, 3, 4$, and $6 = 2deg_{G[S]}(v_1) \geq deg(v_1) + 1$, which implies that $deg(v_1) \leq 5$. In this case G has an induced subgraph isomorphic to (4). The other possibilities are similarly verified. □

Proposition 2.8 *For the complete graph K_n , $\hat{a}_t(K_n) = \lceil \frac{n}{2} \rceil + 1$.*

Proof Let S be a $\hat{a}_t(K_n)$-set and let $v \in S$. It follows that $|N(v) \cap S| \geq \lceil \frac{n}{2} \rceil$. So

$|S| \geq \lceil \frac{n}{2} \rceil + 1$. On the other hand let S be any subset of $\lceil \frac{n}{2} \rceil + 1$ vertices of K_n. For any vertex $v \in S$, $\frac{deg(v)-1}{2} \geq \lfloor \frac{n}{2} \rfloor - 1 \geq deg_{V-S}(v)$. Since $deg(v) = deg_S(v) + deg_{V-S}(v)$, $deg_S(v) - 1 \geq deg_{V-S}(v)$. This means that S is a critical open strong defensive alliance, and the result follows. □

Proposition 2.9 $\hat{a}_t(K_{r,s}) = \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + 2$.

Proof Let V_r and V_s be the partite sets of $K_{r,s}$ with $|V_r| = r$ and $|V_s| = s$. Let $S = S_r \cup S_s$ be a $\hat{a}_t(K_{r,s})$-set, where $S_i \subseteq V_i$ for $i = r, s$. For $i \in \{r, s\}$ and a vertex $v \in S_i$, $deg_S(v) \geq \lfloor \frac{n-i}{2} \rfloor$, where $n = r + s$. This implies that $|S| \geq \lfloor \frac{r}{2} \rfloor + \lfloor \frac{s}{2} \rfloor + 2$. On the other hand any set consisting $\lfloor \frac{r}{2} \rfloor + 1$ vertices in V_r and $\lfloor \frac{s}{2} \rfloor + 1$ vertices in V_s forms an open strong defensive alliance. This completes the proof. □

Similarly the following is verified.

Proposition 2.10

(1) $\hat{a}_t(W_n) = \lceil \frac{n+1}{2} \rceil + 1$;

(2) $\hat{a}_t(P_m \times P_n) = max\{m,n\}$ *if* $min\{m,n\} = 1$, *and* $\hat{a}_t(P_m \times P_n) = min\{m,n\}$ *if* $min\{m,n\} \geq 2$.

Proposition 2.11 *If every vertex of a graph G has odd degree then $a_t(G) = \hat{a}_t(G)$.*

Proof Let G be a graph and every vertex of G has odd degree. First it is obvious that $a_t(G) = \hat{a}(G) \leq \hat{a}_t(G)$. Let S be a $a_t(G)$-set and $v \in S$. By definition $deg_S(v) \geq deg_{V-s}(v)$. Since v is of odd degree, we obtain $deg_S(v) \geq deg_{V-s}(v) + 1$. This means that S is an open strong defensive alliance in G, and so $\hat{a}_t(G) \leq a_t(G)$. □

So if every vertex of a graph G has odd degree then any bound of $a_t(G)$ holds for $\hat{a}_t(G)$. We next obtain some bounds for the open defensive alliance number of a graph G.

Proposition 2.12 *For a connected graph G of order n, $\hat{a}_t(G) \leq n - \left\lfloor \frac{\delta(G)-1}{2} \right\rfloor$.*

Proof Let v be a vertex of minimum degree in a connected graph G. Consider a subset $S \subseteq N[v]$ with $|S| = \lfloor \frac{\delta(G)-1}{2} \rfloor$. It follows that $V(G) \setminus S$ is a critical open strong alliance. □

Proposition 2.13 *For any graph G, $\hat{a}_t(G) \geq \lceil \frac{\delta(G)+3}{2} \rceil$.*

Proof Let S be a $\hat{a}_t(G)$-set in a graph G, and let $v \in S$. By definition $deg_S(v) - 1 \geq deg_{V-S}(v)$. By adding $deg_{V-S}(v)$ to both sides of this inequality we obtain $deg_{V-S}(v) - 1 \leq \frac{deg(v)-1}{2}$. By adding $deg_S(v)$ to both sides of this inequality we obtain $\frac{deg(v)+1}{2} \leq deg_S(v)$. But $deg_S(v) \leq |S| - 1$ and $\delta(G) \leq deg(v)$. We deduce that $\frac{\delta(G)+3}{2} \leq |S|$. □

Proposition 2.14 *For any graph G, $a(G) \leq \hat{a}_t(G) - 1$.*

Proof Let S be a $\hat{a}_t(G)$-set in a graph G, and $w \in S$. Let $S' = S - \{w\}$, and $v \in S'$. It follows that $deg_{S'}(v) = deg_S(v) - deg_{\{w\}}(v) \geq deg_{V-S}(v) + 1 - deg_{\{w\}}(v) = deg_{V-S'}(v) + 1 - 2deg_{\{w\}}(v) \geq deg_{V'_S}(v)$, as desired. $\square$

Let $\Pi = [V_1, V_2]$ be a partition of the vertices of a graph G such that there are $\lambda(G)$ edges between V_1 and V_2. Π is called *singular* $\lambda-bipartite$ if $min\{|V_1|, |V_2|\} = 1$, and $non-singular$ $\lambda - bipartite$ if $min\{|V_1|, |V_2|\} > 1$.

Proposition 2.15 *Let G be a graph such that every vertex of G has odd degree. If $\lambda(G) < \delta(G)$ then $\hat{a}_t(G) \leq \lfloor \frac{n}{2} \rfloor + 1$.*

Proof Let $\Pi = [V_1, V_2]$ be a partition of the vertices of a graph G such that there are $\lambda(G)$ edges between V_1 and V_2. Without loss of generality assume that $|V_1| < |V_2|$. This implies that $|V_1| \leq \lfloor \frac{n}{2} \rfloor$. Since $\lambda(G) < \delta(G)$, we have $|V_i| \geq 2$ for $i = 1, 2$. As a result Π is non-singular λ-bipartite. If V_1 is not an open defensive alliance then there is a vertex $u \in V_1$ such that $|N(u) \cap V_1| < |N(u) \cap V_2|$. Then $\Pi_1 = [V_1 - \{u\}, V_2 \cup \{u\}]$ is a partition of the vertices of G and there are less than $\lambda(G)$ edges between $V_1 - \{u\}$ and $V_2 \cup \{u\}$. But $|\Pi_1| = |\Pi| - deg_{V_2}(u) + deg_{V_1}(u)$. So $|\Pi_1| < |\Pi|$. This contradicts the assumption $|\Pi| = \lambda(G)$. Thus V_1 is an open defensive alliance in G and the result follows. $\square$

§3. Open Offensive Alliance

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is an *open offensive alliance* if for every vertex $v \in \partial(S)$, $|N(v) \cap S| \geq |N(v) \cap (V - S)|$. In other words a set $S \subseteq V$ is an open offensive alliance if for every vertex $v \in \partial(S)$, $deg_S(v) \geq deg_{V-S}(v)$, and this is equivalent to $deg(v) \geq 2deg_{V-S}(v)$. A set $S \subseteq V$ is an *open strong offensive alliance* if for every vertex $v \in \partial(S)$, $|N(v) \cap S| > |N(v) \cap (V - S)|$ or, equivalently, $d_S(v) > d_{V-S}(v)$, where $d_S(v) = N(v) \cap S$. An open (strong) offensive alliance S is called *critical* if no proper subset of S is an open (strong) offensive alliance. The *open offensive alliance number*, $a_{ot}(G)$ of G, is the minimum cardinality of any critical open offensive alliance in G, and the *strong open offensive alliance number*, $\hat{a}_{ot}(G)$ of G, is the minimum cardinality of any critical open strong offensive alliance in G.

If S is a critical open offensive alliance of a graph G and $|S| = a_{ot}(G)$, then we say that S is an $a_{ot} - set$ of G. The next proposition follows from the definitions.

Proposition 3.1 *For all graphs G, $a_o(G) = \hat{a}_{ot}(G)$ and $a_{ot}(G) \leq \hat{a}_{ot}(G)$.*

Thus we focus on open offensive alliances in G.

Theorem 3.2 *For a graph G of order n with $\Delta(G) \leq 2$, $a_{ot}(G) = 1$.*

Proof Suppose $S = \{v\}$, where $deg(v) = \triangle(G) \leq 2$. Since for every $w \in \partial S$, $deg_S(w) = 1$ and $deg_{V-S}(w) \leq 1$. Therefore, $d_S(w) \geq d_{V-S}(w)$. So the result immediately follows. $\square$

Corollary 3.3 *For any cycle C_n and path P_n, $a_{to}(C_n) = a_{to}(P_n) = 1$.*

The following has a straightforward proof and therefore we omit its proof.

Proposition 3.4

(1) $a_{ot}(K_n) = \lfloor \frac{n}{2} \rfloor$*;*

(2) For $1 \leq m \leq n$, $a_{ot}(K_{m,n}) = \lceil \frac{m}{2} \rceil$*;*

(3) For any wheel W_n *with* $n \neq 4$, $a_{ot}(W_n) = \lceil \frac{n}{3} \rceil + 1$*;*

(4) If every vertex of a graph G *has odd degree then* $a_{ot}(G) = a_o(G)$*.*

We next obtain some bounds for the open offensive alliance number of a graph G.

Proposition 3.5 *For all graphs* G, $a_{to}(G) \geq \lfloor \frac{\delta(G)}{2} \rfloor$.

Proof Let S be a $a_{ot} - set$ and $v \in \partial S$. By definition for any vertex v of ∂S, $d_S(v) \geq d_{V-S}(v)$. By adding $d_S(v)$ to both sides of this inequality we obtain $d_S(v) \geq \frac{\delta(v)}{2}$. Also it is clear that $a_{to}(G) \geq d_S(v)$ and $\delta(v) \geq \delta$. This completes the proof. □

Let $\alpha(G)$ denote the *vertex covering number* of G. That is the minimum cardinality of a subset S of vertices of G that contains at least one endpoint of every edge.

Proposition 3.6 *For all graphs* G*,*

(1) $a_{to}(G) \leq \lfloor \frac{n}{2} \rfloor$*;*

(2) $a_{to}(G) \leq \alpha(G)$*.*

Proof (1) Let $f : V \longrightarrow \{a, b\}$ be a vertex coloring of G such that the number of edges whose end vertices have the same color is minimum. Let $O = \{uv : f(u) = f(v)\}$, $A = \{u : f(u) = a\}$ and $B = \{u : f(u) = b\}$. Without loss of generality assume that $|B| \leq |A|$. Suppose that B is not an open offensive alliance in G. So three is a vertex $v \in A$ such that $deg_B(v) < deg_A(v)$. Let $f' : V \longrightarrow \{a, b\}$ be a vertex coloring of G with $f'(v) \neq f(v)$ and $f'(x) = f(x)$ if $x \neq v$. Let $O' = \{uv : f'(u) = f'(v)\}$, $A' = A - \{v\}$ and $B' = B \cup \{v\}$. Then $|O'| = |O| - deg_A(v) + deg_B(v)$. But $deg_B(v) < deg_A(v)$. We deduce that $|O'| < |O|$. This is a contradiction since $|O|$ is minimum. Thus B is an open offensive alliance in G, and so the result follows.
(2) Let S be a $\alpha(G)$-set and let $v \in \partial(S)$. Since S is a vertex covering, $deg_S(v) \geq deg_{V-S}(v) + 1 \geq deg_{V-S}(v)$. This implies that S is an open offensive alliance, and the result follows. □

References

[1] O. Favaron, G. Fricke, W. Goddard, S. M. Hedetniemi, S. T. Hedetniemi, P. Kristiansen, R. C. Laskar and D. R. Skaggs, *Offensive alliances in graphs.* Discuss. Math. Graph Theory 24 (2)(2004), 263-275.

[2] S.M. Hedetniemi, S.T. Hedetniemi, and P. Kristiansen, *Alliances in graphs*, J. Combin. Math. Combin. Comput. 48 (2004) 157-177.

[3] D. B. West, *Introduction to graph theory*, (2nd edition), Prentice all, USA (2001).

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The Forcing Weak Edge Detour Number of a Graph

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Abstract: For two vertices u and v in a graph $G = (V, E)$, the *distance* $d(u, v)$ *and detour distance* $D(u, v)$ are the length of a shortest or longest $u - v$ path in G, respectively, and the *Smarandache distance* $d_S^i(u, v)$ is the length $d(u, v) + i(u, v)$ of a $u - v$ path in G, where $0 \le i(u, v) \le D(u, v) - d(u, v)$. A $u - v$ path of length $d_S^i(u, v)$, if it exists, is called a *Smarandachely* $u - v$ i*-detour*. A set $S \subseteq V$ is called a *Smarandachely* i*-detour set* if every edge in G has both its ends in S or it lies on a Smarandachely i-detour joining a pair of vertices in S. In particular, if $i(u, v) = 0$, then $d_S^i(u, v) = d(u, v)$; and if $i(u, v) = D(u, v) - d(u, v)$, then $d_S^i(u, v) = D(u, v)$. For $i(u, v) = D(u, v) - d(u, v)$, such a Smarandachely i-detour set is called a *weak edge detour set* in G. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is a *weak edge detour basis* of G. For any weak edge detour basis S of G, a subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique weak edge detour basis containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing weak edge detour number* of S, denoted by $fdn_w(S)$, is the cardinality of a minimum forcing subset for S. The *forcing weak edge detour number* of G, denoted by $fdn_w(G)$, is $fdn_w(G) = min\{fdn_w(S)\}$, where the minimum is taken over all weak edge detour bases S in G. The forcing weak edge detour numbers of certain classes of graphs are determined. It is proved that for each pair a, b of integers with $0 \le a \le b$ and $b \ge 2$, there is a connected graph G with $fdn_w(G) = a$ and $dn_w(G) = b$.

Key Words: Smarandache distance, Smarandachely i-detour set, weak edge detour set, weak edge detour number, forcing weak edge detour number.

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§1. Introduction

For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest u–v path in G. A u–v path of length $d(u, v)$ is called a u–v *geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, $radG$ and the maximum eccentricity among the vertices of G is its *diameter*, $diamG$ of G. Two vertices u and v of G are *antipodal* if $d(u, v)$

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$= diamG$. For vertices u and v in a connected graph G, the *detour distance* $D(u,v)$ is the length of a longest u–v path in G. A u–v path of length $D(u,v)$ is called a *u–v detour*. It is known that the distance and the detour distance are metrics on the vertex set $V(G)$. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G. The *detour radius*, $rad_D G$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D G$ of G is the maximum detour eccentricity among the vertices of G. These concepts were studied by Chartrand et al. [2].

A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. A vertex v that belongs to every detour basis of G is a *detour vertex* in G. If G has a unique detour basis S, then every vertex in S is a detour vertex in G. These concepts were studied by Chartrand et al. [3].

In general, there are graphs G for which there exist edges which do not lie on a detour joining any pair of vertices of V. For the graph G given in Figure 1.1, the edge v_1v_2 does not lie on a detour joining any pair of vertices of V. This motivated us to introduce the concept of *weak edge detour set* of a graph [5].

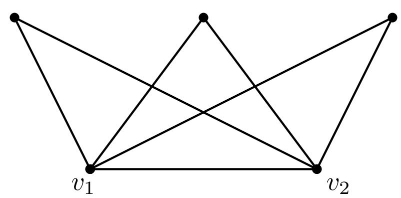

Figure 1: G

The *Smarandache distance* $d_S^i(u,v)$ is the length $d(u,v) + i(u,v)$ of a $u - v$ path in G, where $0 \leq i(u,v) \leq D(u,v) - d(u,v)$. A $u - v$ path of length $d_S^i(u,v)$, if it exists, is called a *Smarandachely $u - v$ i-detour*. A set $S \subseteq V$ is called a *Smarandachely i-detour set* if every edge in G has both its ends in S or it lies on a Smarandachely i-detour joining a pair of vertices in S. In particular, if $i(u,v) = 0$, then $d_S^i(u,v) = d(u,v)$ and if $i(u,v) = D(u,v) - d(u,v)$, then $d_S^i(u,v) = D(u,v)$. For $i(u,v) = D(u,v) - d(u,v)$, such a Smarandachely i-detour set is called a *weak edge detour set* in G. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge detour set of order $dn_w(G)$ is called a *weak edge detour basis* of G. A vertex v in a graph G is a *weak edge detour vertex* if v belongs to every weak edge detour basis of G. If G has a unique weak edge detour basis S, then every vertex in S is a weak edge detour vertex of G. These concepts were studied by A. P. Santhakumaran and S. Athisayanathan [5].

To illustrate these concepts, we consider the graph G given in Figure 1.2. The sets $S_1 = \{u,x\}$, $S_2 = \{u,y\}$ and $S_3 = \{u,z\}$ are the detour bases of G so that $dn(G) = 2$ and the sets $S_4 = \{u,v,y\}$ and $S_5 = \{u,x,z\}$ are the weak edge detour bases of G so that $dn_w(G) = 3$. The vertex u is a detour vertex and also a weak edge detour vertex of G.

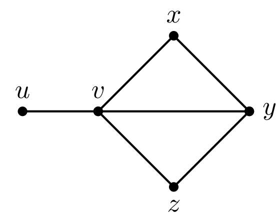

Figure 2: G

The following theorems are used in the sequel.

Theorem 1.1([5]) *For any graph G of order $p \geq 2$, $2 \leq dn_w(G) \leq p$.*

Theorem 1.2([5]) *Every end-vertex of a non-trivial connected graph G belongs to every weak edge detour set of G. Also if the set S of all end-vertices of G is a weak edge detour set, then S is the unique weak edge detour basis for G.*

Theorem 1.3([5]) *If T is a tree with k end-vertices, then $dn_w(T) = k$.*

Theorem 1.4([5]) *Let G be a connected graph with cut-vertices and S a weak edge detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

Throughout this paper G denotes a connected graph with at least two vertices.

§2. Forcing Weak Edge Detour Number of a Graph

First we determine the weak edge detour numbers of some standard classes of graphs so that their forcing weak edge detour numbers will be determined.

Theorem 2.1 *Let G be the complete graph K_p $(p \geq 3)$ or the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$. Then a set $S \subseteq V$ is a weak edge detour basis of G if and only if S consists of any two vertices of G.*

Proof Let G be the complete graph $K_p(p \geq 3)$ and $S = \{u, v\}$ be any set of two vertices of G. It is clear that $D(u, v) = p - 1$. Let $xy \in E$. If $xy = uv$, then both its ends are in S. Let $xy \neq uv$. If $x \neq u$ and $y \neq v$, then the edge xy lies on the u–v detour $P : u, x, y, \ldots, v$ of length $p - 1$. If $x = u$ and $y \neq v$, then the edge xy lies on the u–v detour $P : u = x, y, \ldots, v$ of length $p - 1$. Hence S is a weak edge detour set of G. Since $|S| = 2$, S is a weak edge detour basis of G.

Now, let S be a weak edge detour basis of G. Let S' be any set consisting of two vertices of G. Then as in the first part of this theorem S' is a weak edge detour basis of G. Hence $|S| = |S'| = 2$ and it follows that S consists of any two vertices of G.

Let G be the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$. Let X and Y be the bipartite sets of G with $|X| = m$ and $|Y| = n$. Let $S = \{u, v\}$ be any set of two vertices of G.

Case 1 Let $u \in X$ and $v \in Y$. It is clear that $D(u, v) = 2m - 1$. Let $xy \in E$. If $xy = uv$, then both of its ends are in S. Let $xy \neq uv$ be such that $x \in X$ and $y \in Y$. If $x \neq u$ and $y \neq v$, then the edge xy lies on the u–v detour $P : u, y, x, \ldots, v$ of length $2m - 1$. If $x = u$ and $y \neq v$, then the edge xy lies on the u–v detour $P : u = x, y, \ldots, v$ of length $2m - 1$. Hence S is a weak edge detour set of G.

Case 2 Let $u, v \in X$. It is clear that $D(u, v) = 2m - 2$. Let $xy \in E$ be such that $x \in X$ and $y \in Y$. If $x \neq u$, then the edge xy lies on the u–v detour $P : u, y, x, \ldots, v$ of length $2m - 2$. If $x = u$, then the edge xy lies on the u–v detour $P : u = x, y, \ldots, v$ of length $2m - 2$. Hence S is a weak edge detour set of G.

Case 3 Let $u, v \in Y$. It is clear that $D(u, v) = 2m$. Then, as in Case 2, S is a weak edge detour set of G. Since $|S| = 2$, it follows that S is a weak edge detour basis of G.

Now, let S be a weak edge detour basis of G. Let S' be any set consisting of two vertices of G. Then as in the first part of the proof of $K_{m,n}$, S' is a weak edge detour basis of G. Hence $|S| = |S'| = 2$ and it follows that S consists of any two vertices adjacent or not. □

Theorem 2.2 *Let G be an odd cycle of order $p \geq 3$. Then a set $S \subseteq V$ is a weak edge detour basis of G if and only if S consists of any two adjacent vertices of G.*

Proof Let $S = \{u, v\}$ be any set of two adjacent vertices of G. It is clear that $D(u, v) = p-1$. Then every edge $e \neq uv$ of G lies on the u–v detour and both the ends of the edge uv belong to S so that S is a weak edge detour set of G. Since $|S| = 2$, S is a weak edge detour basis of G.

Now, assume that S is a weak edge detour basis of G. Let S' be any set of two adjacent vertices of G. Then as in the first part of this theorem S' is a weak edge detour basis of G. Hence $|S| = |S'| = 2$. Let $S = \{u, v\}$. If u and v are not adjacent, then since G is an odd cycle, the edges of u–v geodesic do not lie on the u–v detour in G so that S is not a weak edge detour set of G, which is a contradiction. Thus S consists of any two adjacent vertices of G. □

Theorem 2.3 *Let G be an even cycle of order $p \geq 4$. Then a set $S \subseteq V$ is a weak edge detour basis of G if and only if S consists of any two adjacent vertices or two antipodal vertices of G.*

Proof Let $S = \{u, v\}$ be any set of two vertices of G. If u and v are adjacent, then $D(u, v) = p - 1$ and every edge $e \neq uv$ of G lies on the u–v detour and both the ends of the edge uv belong to S. If u and v are antipodal, then $D(u, v) = p/2$ and every edge e of G lies on a u–v detour in G. Thus S is a weak edge detour set of G. Since $|S| = 2$, S is a weak edge detour basis of G.

Now, assume that S is a weak edge detour basis of G. Let S' be any set of two adjacent vertices or two antipodal vertices of G. Then as in the first part of this theorem S' is a weak edge detour basis of G. Hence $|S| = |S'| = 2$. Let $S = \{u, v\}$. If u and v are not adjacent and u and v are not antipodal, then the edges of the u–v geodesic do not lie on the u–v detour in G so that S is not a weak edge detour set of G, which is a contradiction. Thus S consists of any two adjacent vertices or two antipodal vertices of G. □

Corollary 2.4 *If G is the complete graph $K_p\,(p \geq 3)$ or the complete bipartite graph $K_{m,n}\,(2 \leq m \leq n)$ or the cycle $C_p\,(p \geq 3)$, then $dn_w(G) = 2$.*

Proof This follows from Theorems 2.1, 2.2 and 2.3. □

Every connected graph contains a weak edge detour basis and some connected graphs may contain several weak edge detour bases. For each weak edge detour basis S in a connected graph G, there is always some subset T of S that uniquely determines S as the weak edge detour basis containing T. We call such subsets "forcing subsets" and we discuss their properties in this section.

Definition 2.5 *Let G be a connected graph and S a weak edge detour basis of G. A subset $T \subseteq S$ is called a* forcing subset *for S if S is the unique weak edge detour basis containing T. A forcing subset for S of minimum cardinality is a* minimum forcing subset *of S. The* forcing weak edge detour number *of S, denoted by $fdn_w(S)$, is the cardinality of a minimum forcing subset for S. The* forcing weak edge detour number *of G, denoted by $fdn_w(G)$, is $fdn_w(G) = min\ \{fdn_w(S)\}$, where the minimum is taken over all weak edge detour bases S in G.*

Example 2.6 For the graph G given in Figure 2.1(a), $S = \{u, v, w\}$ is the unique weak edge detour basis so that $fdn_w(G) = 0$. For the graph G given in Figure 2.1(b), $S_1 = \{u, v, x\}$, $S_2 = \{u, v, y\}$ and $S_3 = \{u, v, w\}$ are the only weak edge detour bases so that $fdn_w(G) = 1$. For the graph G given in Figure 2.1(c), $S_4 = \{u, w, x\}$, $S_5 = \{u, w, y\}$, $S_6 = \{v, w, x\}$ and $S_7 = \{v, w, y\}$ are the four weak edge detour bases so that $fdn_w(G) = 2$.

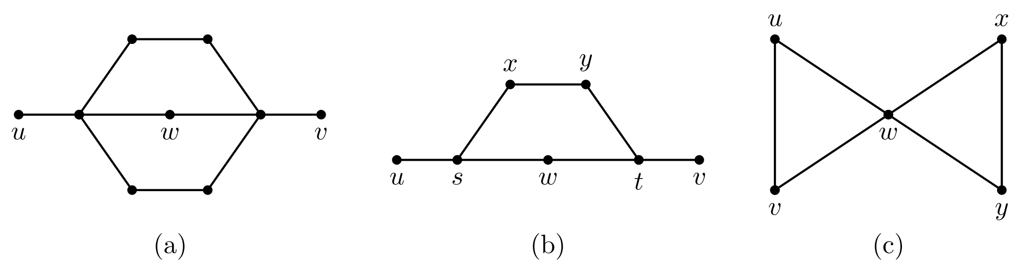

Figure 3: G

The following theorem is clear from the definitions of weak edge detour number and forcing weak edge detour number of a connected graph G.

Theorem 2.7 *For every connected graph G, $0 \leqslant fdn_w(G) \leqslant dn_w(G)$.*

Remark 2.8 The bounds in Theorem 2.7 are sharp. For the graph G given in Figure 2.1(a), $fdn_w(G) = 0$. For the cycle C_3, $fdn_w(C_3) = dn_w(C_3) = 2$. Also, all the inequalities in Theorem 2.7 can be strict. For the graph G given in Figure 2.1(b), $fdn_w(G) = 1$ and $dn_w(G) = 3$ so that $0 < fdn_w(G) < dn_w(G)$.

The following two theorems are easy consequences of the definitions of the weak edge detour number and the forcing weak edge detour number of a connected graph.

Theorem 2.9 *Let G be a connected graph. Then*

a) $fdn_w(G) = 0$ *if and only if G has a unique weak edge detour basis,*

b) $fdn_w(G) = 1$ *if and only if* G *has at least two weak edge detour bases, one of which is a unique weak edge detour basis containing one of its elements, and*

c) $fdn_w(G) = dn_w(G)$ *if and only if no weak edge detour basis of* G *is the unique weak edge detour basis containing any of its proper subsets.*

Theorem 2.10 *Let* G *be a connected graph and let* $\mathscr{F}$ *be the set of relative complements of the minimum forcing subsets in their respective weak edge detour bases in* G*. Then* $\bigcap_{F \in \mathscr{F}} F$ *is the set of weak edge detour vertices of* G*. In particular, if* S *is a weak edge detour basis of* G*, then no weak edge detour vertex of* G *belongs to any minimum forcing subset of* S*.*

Theorem 2.11 *Let* G *be a connected graph and* W *be the set of all weak edge detour vertices of* G*. Then* $fdn_w(G) \leqslant dn_w(G) - |W|$*.*

Proof Let S be any weak edge detour basis S of G. Then $dn_w(G) = |S|$, $W \subseteq S$ and S is the unique weak edge detour basis containing $S - W$. Thus $fdn_w(S) \leqslant |S - W| = |S| - |W| = dn_w(G) - |W|$. $\square$

Remark 2.12 The bound in Theorem 2.11 is sharp. For the graph G given in Figure 2.1(c), $dn_w(G) = 3$, $|W| = 1$ and $fdn_w(G) = 2$ as in Example 2.6. Also, the inequality in Theorem 2.11 can be strict. For the graph G given in Figure 2.2, the sets $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_2, v_3\}$ are the two weak edge detour bases for G and $W = \emptyset$ so that $dn_w(G) = 2$, $|W| = 0$ and $fdn_w(G) = 1$. Thus $fdn_w(G) < dn_w(G) - |W|$.

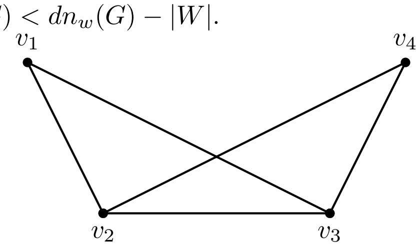

Figure 4: G

In the following we determine $fdn_w(G)$ for certain graphs G.

Theorem 2.13 a) *If* G *is the complete graph* $K_p\,(p \geq 3)$ *or the complete bipartite graph* $K_{m,n}\,(2 \leq m \leq n)$*, then* $dn_w(G) = fdn_w(G) = 2$*.*

b) *If* G *is the cycle* $C_p\,(p \geq 4)$*, then* $dn_w(G) = fdn_w(G) = 2$*.*

c) *If* G *is a tree of order* $p \geq 2$ *with* k *end-vertices, then* $dn_w(G) = k$*,* $fdn_w(G) = 0$*.*

Proof a) By Theorem 2.1, a set S of vertices is a weak edge detour basis if and only if S consists of any two vertices of G. For each vertex v in G there are two or more vertices adjacent with v. Thus the vertex v belongs to more than one weak edge detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any weak edge detour basis of G. Thus the result follows.

b) By Theorems 2.2 and 2.3, a set S of two adjacent vertices of G is a weak edge detour basis of G. For each vertex v in G there are two vertices adjacent with v. Thus the vertex v

belongs to more than one weak edge detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any weak edge detour basis of G. Thus the result follows.

c) By Theorem 1.3, $dn_w(G) = k$. Since the set of all end-vertices of a tree is the unique weak edge detour basis, the result follows from Theorem 2.9(a). □

The following theorem gives a realization result.

Theorem 2.14 *For each pair a, b of integers with $0 \leqslant a \leqslant b$ and $b \geqslant 2$, there is a connected graph G with $fdn_w(G) = a$ and $dn_w(G) = b$.*

Proof The proof is divided into two cases following.

Case 1: $a = 0$. For each $b \geqslant 2$, let G be a tree with b end-vertices. Then $fdn_w(G) = 0$ and $dn_w(G) = b$ by Theorem 2.13(c).

Case 2: $a \geqslant 1$. For each $i\,(1 \leqslant i \leqslant a)$, let $F_i : u_i, v_i, w_i, x_i, u_i$ be the cycle of order 4 and let $H = K_{1,b-a}$ be the star at v whose set of end-vertices is $\{z_1, z_2, \ldots, z_{b-a}\}$. Let G be the graph obtained by joining the central vertex v of H to both vertices u_i, w_i of each $F_i\,(1 \leqslant i \leqslant a)$. Clearly the graph G is connected and is shown in Figure 2.3.

Let $W = \{z_1, z_2, \ldots, z_{b-a}\}$ be the set of all $(b-a)$ end-vertices of G. First, we show that $dn_w(G) = b$. By Theorems 1.2 and 1.4, every weak edge detour basis contains W and at least one vertex from each $F_i\,(1 \leqslant i \leqslant a)$. Thus $dn_w(G) \geqslant (b-a)+a = b$. On the other hand, since the set $S_1 = W \cup \{v_1, v_2, \ldots, v_a\}$ is a weak edge detour set of G, it follows that $dn_w(G) \leqslant |S_1| = b$. Therefore $dn_w(G) = b$.

Next we show that $fdn_w(G) = a$. It is clear that W is the set of all weak edge detour vertices of G. Hence it follows from Theorem 2.11 that $fdn_w(G) \leqslant dn_w(G) - |W| = b-(b-a) = a$. Now, since $dn_w(G) = b$, it is easily seen that a set S is a weak edge detour basis of G if and only if S is of the form $S = W \cup \{y_1, y_2, \ldots, y_a\}$, where $y_i \in \{v_i, x_i\} \subseteq V(F_i)\,(1 \leqslant i \leqslant a)$. Let T be a subset of S with $|T| < a$. Then there is a vertex $y_j\,(1 \leqslant j \leqslant a)$ such that $y_j \notin T$. Let $s_j \in \{v_j, x_j\} \subseteq V(F_j)$ distinct from y_j. Then $S' = (S - \{y_j\}) \cup \{s_j\}$ is a weak edge detour basis that contains T. Thus S is not the unique weak edge detour basis containing T. Thus $fdn_w(S) \geqslant a$. Since this is true for all weak edge detour basis of G, it follows that $fdn_w(G) \geqslant a$ and so $fdn_w(G) = a$. □

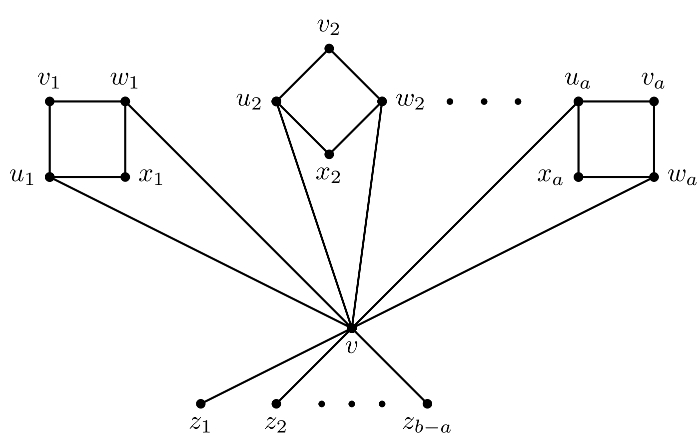

Figure 5: G

References

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Reading MA, 1990.

[2] G. Chartrand, H. Escuadro and P. Zhang, Detour Distance in Graphs, *J. Combin. Math. Combin. Comput.* **53** (2005), 75–94.

[3] G. Chartrand, G. L. Johns and P. Zhang, Detour Number of a Graph, *Util.Math.* **64** (2003), 97–113.

[4] G. Chartrand and P. Zhang, *Introduction to Graph Theory*, Tata McGraw-Hill, 2006.

[5] A. P. Santhakumaran and S. Athisayanathan, *Weak Edge Detour Number of a Graph*, Ars Combin. (To appear)

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Special Smarandache Curves in the Euclidean Space

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Abstract: In this work, we introduce some special Smarandache curves in the Euclidean space. We study Frenet-Serret invariants of a special case. Besides, we illustrate examples of our main results.

Key Words: Smarandache Curves, Frenet-Serret Trihedra, Euclidean Space.

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§1. Introduction

It is safe to report that the many important results in the theory of the curves in E^3 were initiated by G. Monge; and G. Darboux pionnered the moving frame idea. Thereafter, F. Frenet defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry (for more details see [1]).

At the beginning of the 20th century, A. Einstein's theory opened a door to new geometries such as Lorentzian Geometry, which is simultaneously the geometry of special relativity, was established. Thereafter, researchers discovered a bridge between modern differential geometry and the mathematical physics of general relativity by giving an invariant treatment of Lorentzian geometry. They adapted the geometrical models to relativistic motion of charged particles. Consequently, the theory of the curves has been one of the most fascinating topic for such modeling process. As it stands, the Frenet-Serret formalism of a relativistic motion describes the dynamics of the charged particles. The mentioned works are treated in Minkowski space-time.

In the light of the existing literature, in [4] authors introduced special curves by Frenet-Serret frame vector fields in Minkowski space-time. A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a *Smarandache Curve* [4]. In this work, we study special Smarandache Curve in the Euclidean space. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

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§2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of curves in the space E^3 are briefly presented (A more complete elementary treatment can be found in [2].)

The Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $a \in \mathrm{E}^3$ is given by $\|a\| = \sqrt{\langle a, a\rangle}$. φ is called an unit speed curve if velocity vector v of φ satisfies $\|v\| = 1$. For vectors $v, w \in E^3$ it is said to be orthogonal if and only if $\langle v, w\rangle = 0$. Let $\vartheta = \vartheta(s)$ be a regular curve in E^3. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix or an inclined curve. The sphere of radius $r > 0$ and with center in the origin in the space E^3 is defined by

$$S^2 = \left\{p = (p_1, p_2, p_3) \in \mathrm{E}^3 : \langle p, p\rangle = r^2\right\}.$$

Denote by $\{T, N, B\}$ the moving Frenet-Serret frame along the curve φ in the space E^3. For an arbitrary curve $\varphi \in \mathrm{E}^3$, with first and second curvature, κ and τ respectively, the Frenet-Serret formulae is given by [2]

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{1}$$

where

$$\begin{gathered} \langle T, T\rangle = \langle N, N\rangle = \langle B, B\rangle = 1, \\ \langle T, N\rangle = \langle T, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0. \end{gathered}$$

The first and the second curvatures are defined by $\kappa = \kappa(s) = \|T'(s)\|$ and $\tau(s) = -\langle N, B'\rangle$, respectively.

§3. Special Smarandache Curves in E^3

In [4] authors introduced:

Definition 3.1 *A regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve.*

In the light of the above definition, we adapt it to regular curves in the Euclidean space as follows:

Definition 3.2 *Let $\gamma = \gamma(s)$ be a unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Frenet-Serret frame. Smarandache TN curves are defined by*

$$\zeta = \zeta(s_\zeta) = \frac{1}{\sqrt{2}}\left(T + N\right). \tag{2}$$

Let us investigate Frenet-Serret invariants of Smarandache TN curves according to $\gamma = \gamma(s)$. Differentiating (2), we have

$$\zeta' = \frac{d\zeta}{ds_\zeta}\frac{ds_\zeta}{ds} = \frac{1}{\sqrt{2}}\left(-\kappa T + \kappa N + \tau B\right), \tag{3}$$

and hence

$$T_\zeta = \frac{-\kappa T + \kappa N + \tau B}{\sqrt{2\kappa^2 + \tau^2}} \tag{4}$$

where

$$\frac{ds_\zeta}{ds} = \sqrt{\frac{2\kappa^2 + \tau^2}{2}}. \tag{5}$$

In order to determine the first curvature and the principal normal of the curve ζ, we formalize

$$T'_\zeta = \dot{T}_\zeta \frac{ds_\zeta}{ds} = \frac{\delta T + \mu N + \eta B}{(2\kappa^2 + \tau^2)^{\frac{3}{2}}}, \tag{6}$$

where

$$\begin{cases} \delta = -\Big[\kappa^2(2\kappa^2 + \tau^2) + \tau(\tau\kappa' - \kappa\tau')\Big], \\ \mu = -\Big[\kappa^2(2\kappa^2 + 3\tau^2) + \tau(\tau^3 - \tau\kappa' + \kappa\tau')\Big], \\ \eta = \kappa\Big[\tau(2\kappa^2 + \tau^2) - 2(\tau\kappa' - \kappa\tau')\Big]. \end{cases} \tag{7}$$

Then, we have

$$\dot{T}_\zeta = \frac{\sqrt{2}}{(2\kappa^2 + \tau^2)^2}\Big(\delta T + \mu N + \eta B\Big). \tag{8}$$

So, the first curvature and the principal normal vector field are respectively given by

$$\left\|\dot{T}_\zeta\right\| = \frac{\sqrt{2}\sqrt{\delta^2 + \mu^2 + \eta^2}}{(2\kappa^2 + \tau^2)^2} \tag{9}$$

and

$$N_\zeta = \frac{\delta T + \mu N + \eta B}{\sqrt{\delta^2 + \mu^2 + \eta^2}}. \tag{10}$$

On other hand, we express

$$T_\zeta \times N_\zeta = \frac{1}{vl}\begin{vmatrix} T & N & B \\ -\kappa & \kappa & \tau \\ \delta & \mu & \eta \end{vmatrix}, \tag{11}$$

where $v = \sqrt{2\kappa^2 + \tau^2}$ and $l = \sqrt{\delta^2 + \mu^2 + \eta^2}$. So, the binormal vector is

$$B_\zeta = \frac{[\kappa\eta - \tau\mu]\,T + [\kappa\eta + \delta\tau]\,N - \kappa\,[\mu + \delta]\,B}{vl}. \tag{12}$$

In order to calculate the torsion of the curve ζ, we differentiate

$$\zeta'' = \frac{1}{\sqrt{2}}\left\{\begin{array}{c} -(\kappa^2 + \kappa')T + \\ (\kappa' - \kappa^2 - \tau^2)N \\ +(\kappa\tau + \tau')B \end{array}\right\} \tag{13}$$

and thus

$$\zeta''' = \frac{\omega T + \phi N + \sigma B}{\sqrt{2}}, \tag{14}$$

where

$$\begin{cases} \omega = \kappa^3 + \kappa(\tau^2 - 3\kappa') - \kappa'', \\ \phi = -\kappa^3 - \kappa(\tau^2 + 3\kappa') - 3\tau\tau' + \kappa'', \\ \sigma = -\kappa^2\tau - \tau^3 + 2\tau\kappa' + \kappa\tau' + \tau''. \end{cases} \tag{15}$$

The torsion is then given by:

$$\tau_\zeta = \frac{\sqrt{2}\Big[(\kappa^2 + \tau^2 - \kappa')(\kappa\sigma + \tau\omega) + \kappa(\kappa\tau + \tau')(\phi - \omega) + (\kappa^2 + \kappa')(\kappa\sigma - \tau\phi)\Big]}{\big[\tau(2\kappa^2 + \tau^2) + \kappa\tau' - \kappa\tau'\big]^2 + (\kappa'\tau - \kappa\tau')^2 + (2\kappa^3 + \kappa\tau^2)^2}. \tag{16}$$

Definition 3.3 *Let $\gamma = \gamma(s)$ be an unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Frenet-Serret frame. Smarandache NB curves are defined by*

$$\xi = \xi(s_\xi) = \frac{1}{\sqrt{2}}\left(N + B\right). \tag{17}$$

Remark 3.4 The Frenet-Serret invariants of Smarandache NB curves can be easily obtained by the apparatus of the regular curve $\gamma = \gamma(s)$.

Definition 3.5 *Let $\gamma = \gamma(s)$ be an unit speed regular curve in E^3 and $\{T, N, B\}$ be its moving Frenet-Serret frame. Smarandache TNB curves are defined by*

$$\psi = \psi(s_\psi) = \frac{1}{\sqrt{3}}\left(T + N + B\right). \tag{18}$$

Remark 3.6 The Frenet-Serret invariants of Smarandache TNB curves can be easily obtained by the apparatus of the regular curve $\gamma = \gamma(s)$.

§4. **Examples**

Let us consider the following unit speed curve:

$$\begin{cases} \gamma_1 = \frac{9}{208}\sin 16s - \frac{1}{117}\sin 36s \\ \gamma_2 = -\frac{9}{208}\cos 16s + \frac{1}{117}\cos 36s \\ \gamma_3 = \frac{6}{65}\sin 10s \end{cases}. \tag{19}$$

It is rendered in Figure 1.

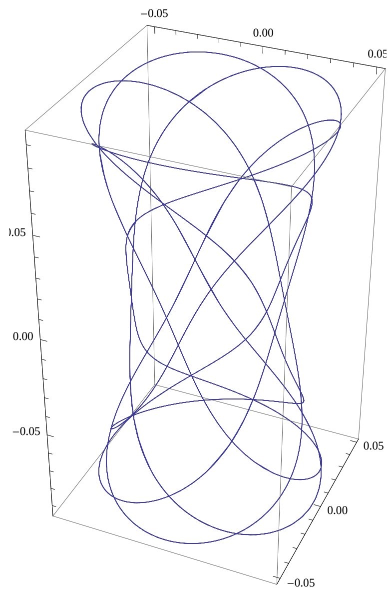

Figure 1: The Curve $\gamma = \gamma(s)$

And, this curve's natural equations are expressed as in [2]:

$$\begin{cases} \kappa(s) = -24 \sin 10s \\ \tau(s) = 24 \cos 10s \end{cases} \tag{20}$$

In terms of definitions, we obtain special Smarandache curves, see Figures $2-4$.

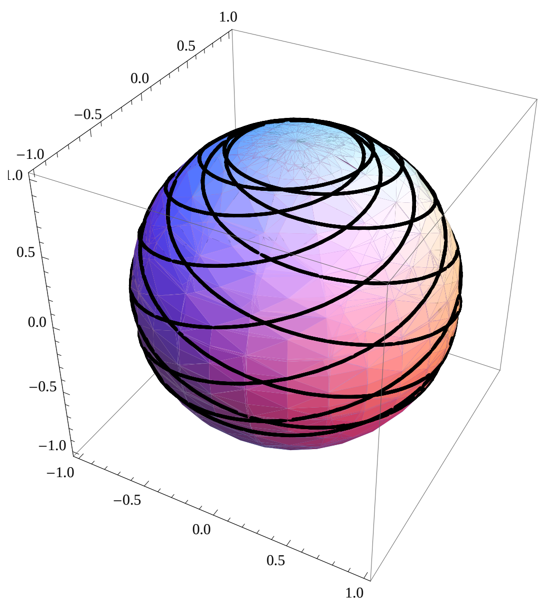

Figure 2: Smarandache TN Curves

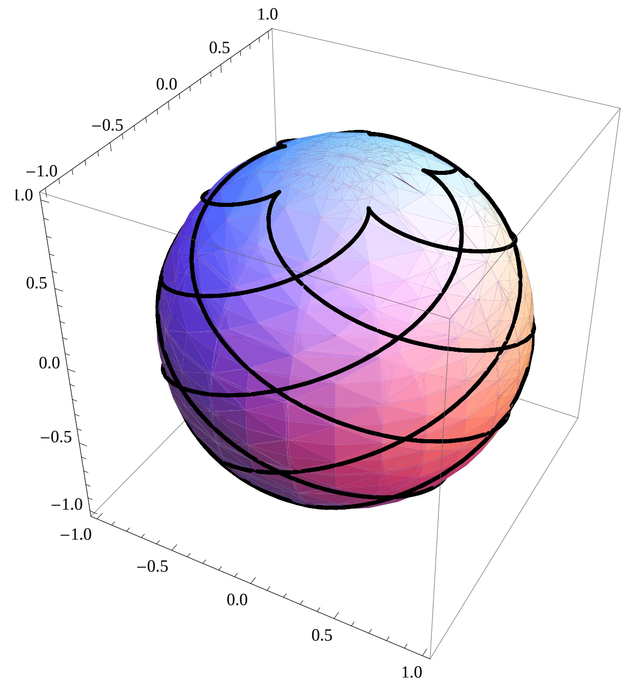

Figure 3: Smarandache NB Curves

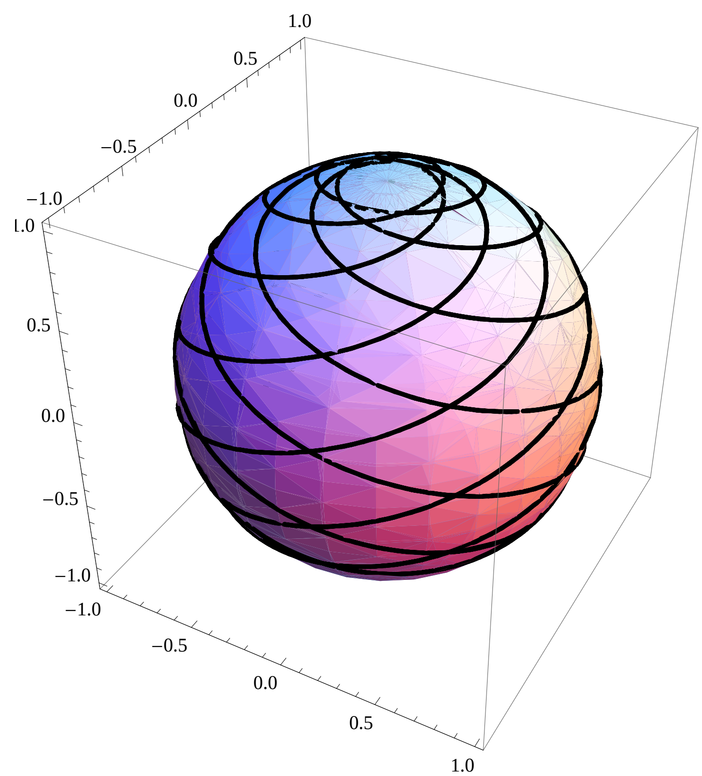

Figure 4: Smarandache TNB Curve

References

[1] Boyer, C.B., *A History of Mathematics,*, John Wiley and Sons Inc., New York, 1968.

[2] Do Carmo, M.P., *Differential Geometry of Curves and Surfaces*, Prentice Hall, Englewood Cliffs, NJ, 1976.

[3] Scofield, P.D., Curves of Constant Precession, *Amer. Math. Monthly,*, 102 (1995) 531–537.

[4] Turgut, M., Yilmaz, S., Smarandache Curves in Minkowski Space-time, *Int. J. Math. Comb.*, 3 (2008) 51–55.

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The H-Line Signed Graph of a Signed Graph

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. Given a connected graph H of order at least 3, the *H-Line Graph* of a graph $G = (V, E)$, denoted by $HL(G)$, is a graph with the vertex set E, the edge set of G where two vertices in $HL(G)$ are adjacent if, and only if, the corresponding edges are adjacent in G and there exists a copy of H in G containing them. Analogously, for a connected graph H of order at lest 3, we define the *H-Line signed graph* $HL(S)$ of a signed graph $S = (G, \sigma)$ as a signed graph, $HL(S) = (HL(G), \sigma')$, and for any edge e_1e_2 in $HL(S)$, $\sigma'(e_1e_2) = \sigma(e_1)\sigma(e_2)$. In this paper, we characterize signed graphs S which are H-line signed graphs and study some properties of H-line graphs as well as H-line signed graphs.

Key Words: Smarandachely k-Signed graphs, Smarandachely k-Marked graphs, Signed graphs, Balance, Switching, H-Line signed graph.

AMS(2000): 05C22

§1. Introduction

For standard terminology and notion in graph theory we refer the reader to Harary [8]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed*

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graph or a *marked graph*. We say that a signed graph is *connected* if its underlying graph is connected. A signed graph $S = (G, \sigma)$ is *balanced* if every cycle in S has an even number of negative edges (See [9]). Equivalently a signed graph is balanced if product of signs of the edges on every cycle of S is positive.

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ.

The following characterization of balanced signed graphs is well known.

Theorem 1.1(E. Sampathkumar [12]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S.

The idea of switching a signed graph was introduced by Abelson and Rosenberg [1] in connection with structural analysis of marking μ of a signed graph S. Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs. The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$. Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [22]) or *cycle isomorphic* (see [23]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [23]):

Theorem 1.2(T. Zaslavsky [23]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. H-Line Signed Graph of a Signed Graph

The line graph $L(G)$ of a nonempty graph $G = (V, E)$ is the graph whose vertices are the edges of G and two vertices are adjacent if and only if the corresponding edges are adjacent. The triangular line graph $\mathcal{T}(G)$ of a nonempty graph was introduced by Jerret [10] as a graph whose vertices are edges of G and two vertices are adjacent if and only if corresponding edges belongs to a common triangle. Triangular graphs were introduced to model a metric space defined on the edge set of a graph. These concepts were generalized in [5] as follows: Let H be a fixed connected graph of order at least 3. For a graph G of size the *H-line graph* of G, denoted by $HL(G)$,is the graph whose vertices are the edges of G and two vertices are adjacent the corresponding edges are adjacent and belong to a copy of H.If $H \cong P_3$ then $HL(G) = L(G)$ and so H-line graph is a generalization of line graphs. Clearly, if a graph is H free, then its H-line graph is trivial.

In [10], the authors introduced the notion of triangular line graph of a graph as follows: The *triangular line graph* of a $G = (V, E)$ denoted by $\mathcal{T}(G) = (V', E')$, whose vertices are the edges of G and two vertices are adjacent the corresponding edges belongs to a triangle in G. Clearly for any graph G, $\mathcal{T}(G) = K_3L(G)$.

Behzad and Chartrand [3] introduced the notion of *line signed graph* $L(S)$ of a given signed graph S as follows: $L(S)$ is a signed graph such that $(L(S))^u \cong L(S^u)$ and an edge e_ie_j in $L(S)$ is negative if, and only if, both e_i and e_j are adjacent negative edges in S. Another notion of line signed graph introduced in [7], is as follows: The *line signed graph* of a signed graph $S = (G, \sigma)$ is a signed graph $L(S) = (L(G), \sigma')$, where for any edge ee' in $L(S)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. In this paper, we follow the notion of line signed graph defined by M. K. Gill [7] (See also E. Sampathkumar et al. [13,14]). For more operations on signed graphs see [15-20].

Proposition 2.1 *For any signed graph $S = (G, \sigma)$, its line signed graph $L(S) = (L(G), \sigma')$ is balanced.*

In [21], the authors extends the notion of triangular line graphs to triangular line signed graphs. We now extend the notion of H-line graph to the realm of signed graph as follows:

Let $S = (G, \sigma)$ ba a signed graph. For any fixed connected graph H of order at least 3, the H-line signed graph of S, denoted by $HL(S)$ is the signed graph $HL(S) = (HL(G), \sigma')$ whose underlying graph is $HL(G)$ and for any edge ee' in $HL(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. Further a signed graph S is said to be H-line signed graph if there exists a signed graph S' such that $HL(S') \cong S$.

We now give a straightforward, yet interesting property of H-line signed graphs.

Theorem 2.2 *For any connected graph H of order at least 3 and for any signed graph $S = (G, \sigma)$, its H-line signed graph $HL(S)$ is balanced.*

Proof Let $\sigma^{'}$ denote the signing of $HL(S)$ and let the signing σ of S be treated as a marking of the vertices of $HL(S)$. Then by definition of $HL(S)$ we see that $\sigma'(e_1, e_2) = \sigma(e_1)\sigma(e_2)$, for every edge (e_1, e_2) of $HL(S)$ and hence, by Theorem 1.1, the result follows. □

Corollary 2.3 *For any two signed graphs S_1 and S_2 with the same underlying graph, $HL(S_1) \sim HL(S_2)$.*

The following result characterizes signed graphs which are H-line signed graphs.

Theorem 2.4 *A signed graph $S = (G, \sigma)$ is a H-line signed graph for some connected graph H of order at least 3 if, and only if, S is balanced signed graph and its underlying graph G is a*

H-line graph.

Proof Suppose that S is H-line signed graph. Then there exists a signed graph $S' = (G', \sigma')$ such that $HL(S') \cong S$. Hence by definition $HL(G) \cong G'$ and by Theorem 2.2, S is balanced.

Conversely, suppose that $S = (G, \sigma)$ is balanced and G is H-line graph. That is there exists a graph G' such that $HL(G') \cong G$. Since S is balanced by Theorem 1.1, there exists a marking μ of vertices of S such that for any edge $uv \in G$, $\sigma(uv) = \mu(u)\mu(v)$. Also since $G \cong HL(G')$, vertices in G are in one-to-one correspondence with the edges of G'. Now consider the signed graph $S' = (G', \sigma')$, where for any edge e' in G' to be the marking on the corresponding vertex in G. Then clearly $HL(S') \cong S$ and so S is H-line graph. □

For any positive integer k, the k^{th} iterated H-line signed graph, $HL^k(S)$ of S is defined as follows:

$$HL^0(S) = S,\ HL^k(S) = HL(HL^{k-1}(S)).$$

Corollary 2.5 *Given a signed graph $S = (G, \sigma)$ and any positive integer k, $HL^k(S)$ is balanced, for any connected graph H of order ≥ 3.*

In [6], the authors proved the following for a graph G its H-line graph $HL(G)$ is isomorphic to G then H is a path or a cycle. Analogously we have the following.

Theorem 2.6 *If a signed graph $S = (G, \sigma)$ satisfies $S \sim HL(S)$ then S is balanced and H is a cycle or a path.*

Theorem 2.7 *For any cycle C_k on $k \geq 3$ vertices, a connected graph G on $n \geq r$ vertices satisfies $C_kL(G) \cong G$ if, and only if, $G = C_k$.*

Proof Suppose that $C_kL(G) \cong G$. Then clearly, G must be unicyclic. Since $C_kL(G) \cong G$, we observe that G must contain a cycle C_k. Next, suppose that G contains a vertex of degree ≥ 3, then the vertex corresponding to the edge not on the cycle in $C_kL(G)$ will be isolated vertex. Hence G must be a cycle C_k.

Conversely, if $G = C_k$, then clearly for any two adjacent edges in C_k belongs to a copy of C_k and so $C_kL(G) \cong L(G)$. Since the line graph of any C_k is C_k itself, we have $C_kL(G) \cong G$.□

Corollary 2.8 *For any cycle C_k on $k \geq 3$ vertices, a graph G on $n \geq r$ vertices satisfies $C_kL(G) \cong G$ if, and only if, G is 2-regular and every component of G is C_k.*

In view of the above theorem we have,

Theorem 2.9 *For any cycle C_k on $k \geq 3$ vertices, a signed graph $S = (G, \sigma)$ connected graph G on $n \geq r$ vertices satisfies $C_kL(S) \sim S$ if, and only if, $G = C_k$.*

Theorem 2.10 *For a path P_k on $k \geq 3$ vertices a connected graph G on $n \geq r$ vertices which contains a cycle of length $r > k$ satisfies $P_kL(G) \cong L(G)$ if, and only if, $G = C_n$ and $n \geq k$.*

Proof The result follows if $k = 3$, since $P_3L(G) = L(G)$. Assume that $k \geq 4$. Clearly G must contain at least k vertices. Suppose that $P_kL(G) \cong L(G)$ and G contains a cycle of length $r \geq k$. Then number of vertices in G and number of edges are equal. Hence G must be unicyclic. Since G contains a cycle of length $r > k$, then any two adjacent edges in C of G belongs to a common P_k. Hence $P_kL(G)$ also contains a cycle of length r. Next, suppose that G contains a vertex of degree ≥ 3, then the vertex corresponding to the edge not on the cycle in $P_kL(G)$ will be adjacent to two adjacent vertices forming a C_3 and so $HL(G)$ is not unicyclic. Hence G must be the cycle C_n .

Conversely, if $G = C_n$ and $n \geq k$, then clearly any two adjacent edges in C_k belongs to a copy of C_k and so $P_kL(G) \cong L(G)$. Since the line graph of C_n is C_n itself, $P_kL(G) \cong L(G)$. $\square$

Corollary 2.11 *For any path P_k on $k \geq 3$ vertices, a graph G on $n \geq r$ vertices satisfies $P_kL(G) \cong G$ if, and only if, G is 2-regular and every component of G is C_r, for some $r \geq k$.*

Analogously, we have the following for signed graphs:

Corollary 2.12 *For any path P_k on $k \geq 3$ vertices, a signed graph $S = (G, \sigma)$ on $n \geq r$ vertices satisfies $P_kL(S) \sim S$ if, and only if, S is balanced and every component of G is C_r, for some $r \geq k$.*

In [10], the authors prove that for any graph G, $\mathcal{T}(G) \cong L(G)$ if, and only if, $G = K_n$. Analogously, we have the following:

Theorem 2.13 *A graph G of order n satisfies $K_rL(G) \cong L(G)$ for some $r \leq n$ if, and only if, $G = K_n$.*

Proof The result is trivial if $k = n$. Suppose that $K_rL(G) \cong L(G)$ and G is not complete for some $r \leq n - 1$. Then there exists at least two nonadjacent vertices u and v in G. Now for any vertex w, the edges uw and vw are adjacent and hence the corresponding vertices are adjacent. But the edges uw and vw can not be adjacent in $K_rL(G)$ since any set of r vertices containing u and v can not induce complete subgraph K_r. Whence, the condition is necessary.

For sufficiency, suppose $G = K_n$ for some $n \geq r$. Then for any two adjacent vertices in L(G), the corresponding edges adjacent edges in G belongs to some K_r. Hence they are also adjacent in $K_rL(G)$ and any two nonadjacent vertices in $L(G)$ remain nonadjacent.This completes the proof. $\square$

Analogously, we have the following result for signed graphs:

Theorem 2.14 *A signed graph $S = (G, \sigma)$ satisfies $K_rL(S) \sim L(S)$, for some $3 \leq k \leq |V(G)|$ if, and only if, S is a balanced on a complete graph.*

§3. Triangular Line Signed Graphs and (0, 1, -1) Matrices

Matrices are very good models to represent a graph. In general given a matrix $A = (a_{ij})$ of order $m \times n$ one can associate many graphs with it (see [11]. On the other hand given any graph G we can associate many matrices such adjacency matric, incidence matrix etc (see [8]). Analogously, given a matrix with entries one can associate many signed graphs (See [11]). In this section, we give a relation between the notion of triangular line graph and some graph associated with $\{0,1\}$-matrices. Also we extend this to triangular signed graphs and some signed graphs associated with matrices whose entries are $-1, 0$, or 1.

Given a $(0,1)$-matrix A, the term graph $T(A)$ of A was defined as follows (See [2]): The vertex set of $T(A)$ consists of m row labels $r_1, r_2, ..., r_m$ and n column labels $c_1, c_2, ..., c_n$ of A and the edge set consists of the unordered pairs $r_i c_j$ for which $a_{ij} \neq 0$.

Given a $(0,1)$-matrix A of order $m \times n$, the graph $G_t(A)$ can be constructed as follows: The vertex set of $G_t(A)$ consists of non-zero entries of A and the edge set consists of distinct pairs of vertices (a_{ij}, a_{kr}) that lie in the same row (i=k) with $a_{ir} \neq 0$ or or same column(j=r) with $a_{kj} \neq 0$. The following result relates the connects the two notions the term graph and G_t graph of a given matrix A:

Theorem 3.1 *For any* $(0,1)$*-matrix A,* $G_t(A) = \mathcal{T}(T(A))$.

Let $A = (a_{ij})$ be any $m \times n$ matrix in which each entry belongs to the set $\{-1, 0, 1\}$; we shall call such a matrix a $(0, \pm 1)$-matrix. The notion of term graph of a $(0,1)$-matrix can be easily extended to term signed graph of a $(0, \pm 1)$-matrix A as follows (see [2]): The vertex set of $T(A)$ consists of m row labels $r_1, r_2, ..., r_m$ and n column labels $c_1, c_2, ..., c_n$ of A, the edge set consists of the unordered pairs $r_i c_j$ for which $a_{ij} \neq 0$ and the sign of the edge $r_i c_j$ is the sign of the nonzero entry a_{ij}.

Next, given any $(0, \pm 1)$-matrix A a *triangular matrix signed graph* $Sg_t(A)$ of A can be constructed as follows: The vertex set of $Sg_t(A)$ is consists of nonzero entries of A and edge set consists of distinct pairs of vertices (a_{ij}, a_{kr}) that lie in the same row $(i = k)$ with $a_{ir} \neq 0$ or same column $(j = r)$ with $a_{kj} \neq 0$; the sign of an edge uv in $Sg(A)$ is defined as the product of sings of the entries of A that correspond to $u = a_{ij}$ and $v = a_{kr}$.

The following is a observation whose proof follows from the definition of triangular line graph and the facts just mentioned above:

Theorem 3.2 *For any* $(0, \pm 1)$ *matrix A,* $Sg_t(A) \cong \mathcal{T}(T_g(A))$.

The *Kronecker product* or *tensor product* of two signed graphs S_1 and S_2, denoted by $S_1 \bigotimes S_2$ is defined (see [2]) as follows:The vertex set of $(S_1 \bigotimes S_2)$ is $V(S_1) \times V(S_2)$, the edge set is $E(S_1 \bigotimes S_2) := \{((u_1, v_1), (u_2, v_2)) : u_1 u_2 \in E(S_1),\ v_1 v_2 \in E(S_2)\}$ and the sign of the edge $(u_1, v_1)(u_2, v_2)$ is the product of the sign of $u_1 u_2$ in S_1 and the sign of $v_1 v_2$ in S_2. In the following result, $A(S)$ will denote the usual adjacency matrix of the given signed graph S and $A \bigotimes B$ denotes the standard tensor product of the given matrices A and B.

Theorem 3.3(M. Acharya [2]) *For any two signed graphs* $S_1 and S_2$*,* $A(S_1 \bigotimes S_2) = A(S_1) \bigotimes A(S_2)$.

Theorem 3.4 *For any signed graph S,* $T(A(S)) = S \bigotimes K_2^+$*, where* K_2^+ *denotes the complete graph* K_2 *with its only edge treated as being positive.*

The *adjacency signed graph* $\eth(S)$ of a given signed graph S is the matrix signed graph $Sg(A(S))$ of the adjacency matrix $A(S)$ of S [2].

Theorem 3.5(M. Acharya [2]) *For any signed graph S,* $\eth(S) = L(S \bigotimes K_2^+)$.

Analogously we define *triangular adjacency signed graph* of $A(S)$, the adjacency matrix of S denoted by $\eth_t(S)$ as the signed graph $Sg_t(A(S))$. We have the following result.

Theorem 3.6 *For any signed graph* S, $\eth_t(S) = \mathcal{T}(S \bigotimes K_2^+)$.

[1] R. P. Abelson and M. J. Rosenberg, Symoblic psychologic: A model of attitudinal cognition, *Behav. Sci.*, 3 (1958), 1-13.

[2] M. Acharya, x-Line sigraph of a sigraph, *J. Combin. Math. Combin. Comput.*, 69(2009), 103-111.

[3] M. Behzad and G. T. Chartrand, Line-coloring of signed graphs, *Elemente der Mathematik*, 24(3) (1969), 49-52.

[4] G. Chartrand, M. A Henning, E. B. Jarrett and C.E. Wall, Quadrilateral Line signed graphs-Preprint.

[5] G. Chartrand, H. Gavlas and M. Schultz, H-distance in graphs, *Congress. Numer*, 101(1994), 3-15.

[6] G. Chartrand, H. Gavlas and M. Schultz, Convergent sequence of iterated H-line graphs, *Discrete Mathematics*, 147(1995), 73-86.

[7] M. K. Gill, *Contributions to some topics in graph theory and its applications*, Ph.D. Thesis, The Indian Institute of Technology, Bombay, 1983.

[8] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., 1969.

[9] F. Harary, On the notion of balance of a signed graph, *Michigan Math. J.*, 2(1953), 143-146.

[10] E. B. Jarrett, *Transformations of graphs and digraphs*, Ph.D. Thesis, Western Michigan University, 1991.

[11] V.Mishra, *Graphs associated with [0, 1] and [0, +1,-1] matrices*, Ph.D. Thesis, The Indian Institute of Technology, Bombay, 1975.

[12] E. Sampathkumar, Point signed and line signed graphs, *Nat. Acad. Sci. Letters*, 7(3) (1984), 91-93.

[13] E. Sampathkumar, P. Siva Kota Reddy, and M. S. Subramanya, The Line n-sigraph of a symmetric n-sigraph, *Southeast Asian Bull. Math.*, to appear.

[14] E. Sampathkumar, M. S. Subramanya and P. Siva Kota Reddy, Characterization of Line Sidigraphs, *Southeast Asian Bull. Math.*(Springer-Verlag), to appear.

[15] E. Sampathkumar, M. S. Subramanya and P. Siva Kota Reddy, Directionally n-signed graphs, *Proc. Int. Conf. ICDM 2008*, 13 (2010), 155162.

[16] E. Sampathkumar, M. S. Subramanya and P. Siva Kota Reddy, Directionally n-signed graphs-II, *International J. Math. Combin.*, 4 (2009), 89-98.

[17] P. Siva Kota Reddy and M. S. Subramanya, Signed graph equation $L^k(S) \sim \overline{S}$, *International J. Math. Combin.*, 4 (2009), 84-88.

[18] P. Siva Kota Reddy, S. Vijay and H. C. Savithri, A Note on Path Sidigraphs, *International J. Math. Combin.*, 1 (2010), 42-46.

[19] P. Siva Kota Reddy, S. Vijay and V. Lokesha, n^{th} Power signed graphs, *Proceedings of the Jangjeon Math. Soc.*, 12(3) (2009), 307-313.

[20] P. Siva Kota Reddy, S. Vijay and V. Lokesha, n^{th} Power signed graphs-II, *International J. Math. Combin.*, 1 (2010), 74-79.

[21] M. S. Subramanya and P. Siva Kota Reddy, Triangular line signed graph of a signed graph, *Advances and Applications in Discrete Mathematics*, 4(1) (2009), 17-23.

[22] T. Sozánsky, Enueration of weak isomorphism classes of signed graphs, *J. Graph Theory*, 4(2)(1980), 127-144.

[23] T. Zaslavsky, Signed Graphs, *Discretes Appl. Math.*, 4(1)(1982), 47-74.

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Min-Max Dom-Saturation Number of a Tree

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Abstract: Let $G = (V, E)$ be a graph and let $v \in V$. Let $\gamma^{min}(v, G)$ denote the minimum cardinality of a minimal dominating set of G containing v. Then $\gamma^{M,m}(G) = max\{\gamma^{min}(v, G) : v \in V(G)\}$ is called the min-max dom-saturation number of G. In this paper we present a dynamic programming algorithm for determining the min-max dom-saturation number of a tree.

Key Words: Domination, Smarandachely k-dominating set, min-max dom-saturation number.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [6].

One of the fastest growing areas in graph theory is the study of domination and related subset problems such as independence, irredundance, covering and matching. An excellent treatment of fundamentals of domination in graphs is given in the book by Haynes et al.[7]. Surveys of several advanced topics in domination are given in the book edited by Haynes et al.[8].

Let $G = (V, E)$ be a graph. A subset S of V is said to be a Smarandachely k-dominating set in G if every vertex in $V - S$ is adjacent to at least k vertices in S. When $k = 1$, the set S is simply called a *dominating set*. A dominating set S is called a minimal dominating set if no proper subset of S is a dominating set of G. The domination number $\gamma(G)$ is the minimum cardinality taken over all minimal dominating sets in G.

Let S be a subset of vertices of a graph G and let $u \in S$. A vertex v is called a private neighbor of u with respect to S if $N[v] \cap S = \{u\}$. A dominating set D of G is a minimal dominating set if and only if every vertex in D has a private neighbor with respect to D.

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In a graph G any vertex of degree 1 is called a leaf and the unique vertex which is adjacent to a leaf is called a support vertex.

Acharya [1] introduced the concept of dom-saturation number $ds(G)$ of a graph, which is defined to be the least positive integer k such that every vertex of G lies in a dominating set of cardinality k. Arumugam and Kala [2] observed that for any graph G, $ds(G) = \gamma(G)$ or $\gamma(G)+1$ and obtained several results on $ds(G)$. Motivated by this concept Arumugam and Subramanian [3] introduced the concept of independence saturation number of a graph and Arumugam et al. [4] introduced the concept of irredundance saturation number of a graph. In [5] we have generalized the concept of min-max and max-min graph saturation parameters for any graph theoretic property P which may be hereditary or super hereditary in the following.

Definition 1.1 *The min-max dom-saturation number* $\gamma^{M,m}(G)$ *is defined as follows. For any* $v \in V(G)$, *let* $\gamma^{min}(v, G) = min\{|S| : S$ *is a minimal dominating set of* G *and* $v \in S\}$ *and let* $\gamma^{M,m}(G) = max\{\gamma^{min}(v, G) : v \in V(G)\}$.

Thus $\gamma^{M,m}(G)$ is the largest positive integer k, with the property that every vertex of G lies in a minimal dominating set of cardinality at least k.

Since the decision problem corresponding to the domination number $\gamma(G)$ is NP-complete, it follows that the decision problem corresponding to $\gamma^{M,m}(G)$ is also NP-complete. Hence developing polynomial time algorithms for determining $\gamma^{M,m}(G)$ for special classes of graphs is an interesting problem.

In this paper we present a dynamic programming algorithm for determining the min-max dom-saturation number of a tree.

§2. Main Results

Let T be a tree rooted at v. For any vertex $u \in V(T)$, let T_u be the subtree of T rooted at u. Let $u_1, \ldots, u_k$ be the children of u in T_u and let $T_i = T_{u_i}$. For any dominating set D of T_u, let $D_i = D \cap V(T_i)$. We now define the following six parameters.

(i) $\gamma^1(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u, u \in D$ and u is isolated in $\langle D\rangle\}$.

(ii) $\gamma^2(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u, u \in D$, u is not isolated in $\langle D\rangle$ and u has a child as its private neighbor$\}$.

(iii) $\gamma^3(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u, u \notin D$ and u is a private neighbor of its child$\}$.

(iv) $\gamma^4(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u - u$ and $u_i \notin D, 1 \le i \le k\}$.

(v) $\gamma^5(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u, u \notin D$ and at least two of its children are in $D\}$.

(vi) $\gamma^{00}(T, u) = min\{|D| : D$ is a minimal dominating set of $T_u - u\}$.

Observation 2.1 If the subtree T_u is a star or if every child of u is a support vertex, then $\gamma^2(T,u)$ is not defined. Also if the vertex u has two leaves as its children then $\gamma^3(T,u)$ is not defined. If u is a support vertex of T_u, then $\gamma^4(T,u)$ is not defined and if the number of children of u is less than two then $\gamma^5(T,u)$ is not defined.

Lemma 2.1 $\gamma^1(T,u) = 1 + \sum\limits_{i=1}^{k} min\{\gamma^4(T_i,u_i), \gamma^5(T_i,u_i), \gamma^{00}(T_i,u_i)\}$.

Proof Let D be a minimal dominating set of T_u, $u \in D, u$ is isolated in $\langle D\rangle$ and $|D| = \gamma^1(T,u)$. Hence $u_i \notin D_i, 1 \leq i \leq k$. If no children of u_i is in D_i, then $|D_i| \geq \gamma^{00}(T_i,u_i)$. If exactly one child of u_i is in D_i, then $|D_i| \geq \gamma^4(T_i,u_i)$. Otherwise $|D_i| \geq \gamma^5(T_i,u_i)$. Thus $|D_i| \geq min\{\gamma^4(T_i,u_i), \gamma^5(T_i,u_i), \gamma^{00}(T_i,u_i)\}$. Hence $|D| \geq 1+\sum\limits_{i=1}^{k} min\{\gamma^4(T_i,u_i), \gamma^5(T_i,u_i), \gamma^{00}(T_i,u_i)\}$. We get the equality. □

The reverse inequality follows from the observation that any minimal dominating set D of T_u having u as an isolated vertex in $\langle D\rangle$ is of the form $D = \left(\bigcup\limits_{i=1}^{k} D_i\right) \cup \{u\}$ where D_i is a minimal dominating set of T_i not containing $u_i, 1 \leq i \leq k$.

Lemma 2.2 *Suppose the subtree T_u of T rooted at u is neither a star nor every child of u is a support vertex. Then* $\gamma^2(T,u) = 1 + \min\limits_{i,j} \{min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\} + \gamma^4(T_j,u_j) + \sum\limits_{r\neq i,j} min\{\gamma^1(T_r,u_r), \gamma^2(T_r,u_r), \gamma^4(T_r,u_r), \gamma^5(T_r,u_r), \gamma^{00}(T_r,u_r)\}\}$ *where the minimum is taken over all i,j such that u_i is not a leaf of T_u and u_j is not a support vertex of T_u.*

Proof Let D be a minimal dominating set of $T_u, u \in D$, u is not isolated in $\langle D\rangle$ and u has one of its children as its private neighbor and $|D| = \gamma^2(T,u)$. Without loss of generality we assume that $u_i \in D$ and u_j is the private neighbor of u with respect to D. Since D is a minimal dominating set it follows that u_i is not a leaf of T_u and u_j is not a support vertex of T_u. Since $u_i \in D, |D_i| \geq min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\}$. Also u_j and all its children are not in D_j, we have $|D_j| \geq \gamma^4(T_j,u_j)$. For $r \neq i,j$,

$$|D_r| \geq min\{\gamma^1(T_r,u_r), \gamma^2(T_r,u_r), \gamma^4(T_r,u_r), \gamma^5(T_r,u_r), \gamma^{00}(T_r,u_r)\}.$$

Hence

$$\begin{aligned} |D| \quad \geq \quad & 1 + \min_{i,j}\{min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\} + \gamma^4(T_j,u_j) \\ & + \sum_{r\neq i,j} min\{\gamma^1(T_r,u_r), \gamma^2(T_r,u_r), \gamma^4(T_r,u_r), \gamma^5(T_r,u_r), \gamma^{00}(T_r,u_r)\}\}, \end{aligned}$$

where the minimum is taken over all i,j such that u_i is not a leaf of T_u and u_j is not a support vertex of T_u.

The reverse inequality is obvious. □

Lemma 2.3 *Let D be a minimal dominating set of T_u such that $u \notin D$. If a child of u, say u_1 is a leaf, then* $\gamma^3(T,u) = 1 + \sum\limits_{i=2}^{k} min\{\gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}$. *If no child of u is a leaf, then* $\gamma^3(T,u) = \min\limits_{1\leq i\leq k}\{min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i)\} + \sum\limits_{j\neq i} min\{\gamma^3(T_j,u_j), \gamma^5(T_j,u_j)\}\}$.

Proof Let D be a minimal dominating set of T_u such that $u \notin D$, u is a private neighbor of a child and $|D| = \gamma^3(T, u)$.

Case 1. Exactly one child, say u_1, of u is a leaf.

Then $u_1 \in D$ and $u_i \notin D$ for all $i > 1$.
Hence $\gamma^3(T, u) \geq 1 + \sum\limits_{i=2}^{k} min\{\gamma^3(T_i, u_i), \gamma^5(T_i, u_i)\}$.

Case 2. No child of u is a leaf.

Without loss of generality we assume that u is the private neighbor of $u_i \in D$. Then $|D_i| \geq min\{\gamma^1(T_i, u_i),\ \gamma^2(T_i, u_i)\}$. Also since u is the private neighbor of u_i, all the other children of u are not in D and hence for all $j \neq i$,

$$|D_j| \geq min\{\gamma^3(T_j, u_j), \gamma^5(T_j, u_j)\}.$$

Thus $|D| \geq \min\limits_{1 \leq i \leq k}\{min\{\gamma^1(T_i, u_i), \gamma^2(T_i, u_i)\} + \sum\limits_{j \neq i} min\{\gamma^3(T_j, u_j), \gamma^5(T_j, u_j)\}\}$.

The reverse inequality is obvious. □

Lemma 2.4 *If u is not a support vertex of T_u, then*

$$\gamma^4(T, u) = \sum_{i=1}^{k} min\{\gamma^3(T_i, u_i), \gamma^5(T_i, u_i)\}.$$

Proof Let D be a minimal dominating set of $T_u - \{u\}$, $u_i \notin D$ and $|D| = \gamma^4(T, u)$. Let $D_i = D \cap V(T_i)$. Since $u_i \notin D_i, |D_i| \geq min\{\gamma^3(T_i, u_i), \gamma^5(T_i, u_i)\}$ and hence $|D| \geq \sum\limits_{i=1}^{k} min\{\gamma^3(T_i, u_i), \gamma^5(T_i, u_i)\}$. The reverse inequality is obvious. □

Lemma 2.5 *If u has more than one child, then*

$$\begin{aligned}\gamma^5(T, u) \quad = \quad & \min_{i,j}\{min\{\gamma^1(T_i, u_i), \gamma^2(T_i, u_i)\} + min\{\gamma^1(T_j, u_j), \gamma^2(T_j, u_j)\} \\ & + \min_{r \neq i,j}\{\gamma^1(T_r, u_r), \gamma^2(T_r, u_r), \gamma^3(T_r, u_r), \gamma^5(T_r, u_r)\}\}.\end{aligned}$$

Proof Let D be a minimal dominating set of T_u such that at least two children of u, say u_i and u_j are in D and $|D| = \gamma^5(T, u)$. Since $u_i, u_j \in D$, $|D_i| \geq min\{\gamma^1(T_i, u_i), \gamma^2(T_i, u_i)\}$ and $|D_j| \geq min\{\gamma^1(T_j, u_j), \gamma^2(T_j, u_j)\}$. For any $r \neq i, j, u_r$ may or may not be in D. Hence

$$|D_r| \geq min\{\gamma^1(T_r, u_r), \gamma^2(T_r, u_r), \gamma^3(T_r, u_r), \gamma^5(T_r, u_r)\}.$$

Thus

$$\begin{aligned}|D| \quad \geq \quad & \min_{i,j}\{min\{\gamma^1(T_i, u_i), \gamma^2(T_i, u_i)\} + min\{\gamma^1(T_j, u_j), \gamma^2(T_j, u_j)\} \\ & + \min_{r \neq i,j}\{\gamma^1(T_r, u_r), \gamma^2(T_r, u_r), \gamma^3(T_r, u_r), \gamma^5(T_r, u_r)\}\}.\end{aligned}$$

The reverse inequality is obvious. □

Lemma 2.6 $\gamma^{00}(T,u) = \sum_{i=1}^{k} min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i), \gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}$.

Proof Let D be a minimal dominating set of $T_u - u$ such that $|D| = \gamma^{00}(T,u)$. Obviously $|D_i| \geq min\{\gamma^1(T_i,u_i),\ \gamma^2(T_i,u_i), \gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}$. Thus

$$|D| \geq \sum_{i=1}^{k} min\{\gamma^1(T_i,u_i), \gamma^2(T_i,u_i), \gamma^3(T_i,u_i), \gamma^5(T_i,u_i)\}.$$

The reverse inequality is obvious. □

Lemma 2.7 $\gamma^{min}(v,T) = min\{\gamma^1(T,v), \gamma^2(T,v)\}$.

Proof Let D be a minimal dominating set of T such that $v \in D$ and $|D| = \gamma^{min}(v,T)$. Since v is either isolated or nonisolated in $\langle D\rangle$, the result follows. □

Based on the above lemmas we have the following dynamic programming algorithm for determining $\gamma^{min}(v,T)$ for trees.

ALGORITHM TO FIND $\gamma^{min}(v,T)$

INPUT: A tree T rooted at v_1, with a BFS ordering of its vertices $\{v_1, v_2, \ldots, v_n\}$.
OUTPUT: Minimum cardinality of a minimal dominating set of T containing v_1.

Step 1. `INITIALIZATION`

for $i = 1$ to n **do**

$\gamma^1(v_i) = 1; \gamma^2(v_i) = \infty; \gamma^3(v_i) = \infty,$

$\gamma^4(v_i) = \infty; \gamma^5(v_i) = \infty; \gamma^{00}(v_i) = 0.$

end for;

Step 2. `COMPUTATION`

for $i = n$ to 1 **do**

Step 2.1: Let $u_{i1}, u_{i2}, \ldots, u_{il}$ be the children of v_i

Step 2.2: `CALCULATE` $\gamma^1(v_i)$

Compute $\gamma^1(v_i) = 1 + \sum_{j=1}^{l} min\{\gamma^4(u_{ij}), \gamma^5(u_{ij}), \gamma^{00}(u_{ij})\}$.

Step 2.3: `CALCULATE` $\gamma^2(v_i)$

If there exists a child of v_i which is not a leaf and there exists a child of v_i which is not a support then compute

$$\gamma^2(v_i) = 1 + \min_{j,k}\{min\{\gamma^1(u_{ij}), \gamma^2(u_{ij})\} + \gamma^4(u_{ik}) + \sum_{r\neq j,k}\{\gamma^1(u_{ir}), \gamma^2(u_{ir}), \gamma^4(u_{ir}), \gamma^5(u_{ir}), \gamma^{00}(u_{ir})\}.$$

where the minimum is taken over all $j,k, j \neq k$ such that u_{ik} is not a support vertex and u_{ij} is not a leaf.

Step 2.4: `CALCULATE` $\gamma^3(v_i)$

If v_i has exactly one child which is a leaf, say u_1, then compute $\gamma^3(v_i) = 1 + \sum_{j=2}^{l} min\{\gamma^3(u_{ij}), \gamma^5(u_{ij})\}$

otherwise

$\gamma^3(v_i) = \min_{1\le j\le l}\{min\{\gamma^1(u_{ij}), \gamma^2(u_{ij})\} + \sum_{k\neq j}\{\gamma^3(u_{ik}), \gamma^5(u_{ik})\}\}$.

Step 2.5: `CALCULATE` $\gamma^4(v_i)$

If v_i is not a support vertex then compute

$\gamma^4(v_i) = \sum_{j=1}^{l} min\{\gamma^3(u_{ij}), \gamma^5(u_{ij})\}$

Step 2.6: `CALCULATE` $\gamma^5(v_i)$

If v_i has more than one child then compute

$\gamma^5(v_i) = \min_{j\neq k}\{\gamma^1(u_{ij}), \gamma^2(u_{ij})\} + min\{\gamma^1(u_{ik}), \gamma^2(u_{ik})\} + \min_{r\neq j,k}\{\gamma^1(u_{ir}), \gamma^2(u_{ir}), \gamma^3(u_{ir}), \gamma^5(u_{ir})\}$

Step 2.7: `CALCULATE` $\gamma^{00}(v_i)$

Compute $\gamma^{00}(v_i) = \sum_{j=1}^{l}\{\gamma^1(u_{ij}), \gamma^2(u_{ij}), \gamma^3(u_{ij}), \gamma^5(u_{ij})\}$

end for;

Step 3. Compute $\gamma^{min}(v_1, T) = min\{\gamma^1(v_1), \gamma^2(v_1)\}$.

Observation 2.2 Using the above algorithm for any given vertex v of T the parameter $\gamma^{min}(v, T)$ can be computed. Applying the above algorithm for each vertex v we compute $\gamma^{min}(v, T)$ for all $v \in V$ and $\gamma^{M,m}(T) = max\{\gamma^{min}(v, T) : v \in V(T)\}$ can be computed.

Example 2.1 A tree rooted at the vertex 1 and the table showing the computations of the above algorithm are given below.

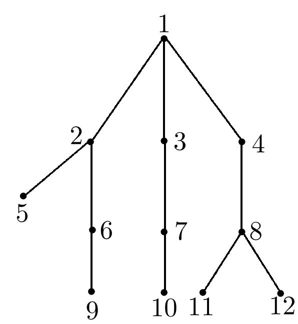

Figure 1

	γ^1	γ^2	γ^3	γ^4	γ^5	γ^{00}
12	1	∞	∞	∞	∞	0
11	1	∞	∞	∞	∞	0
10	1	∞	∞	∞	∞	0
9	1	∞	∞	∞	∞	0

	γ^1	γ^2	γ^3	γ^4	γ^5	γ^{00}
8	1	∞	∞	∞	2	2
7	1	∞	1	∞	∞	1
6	1	∞	1	∞	∞	1
5	1	∞	∞	0	∞	0
4	3	∞	1	2	∞	1
3	2	∞	1	1	∞	1
2	2	2	2	∞	2	2
1	5	5	4	4	5	4

Hence $\gamma^{min}(1,T) = min(\gamma^1(T,1), \gamma^2(T,1)) = 5$.

Repeated application of the algorithm gives $\gamma^{min}(2,T) = 4$, $\gamma^{min}(3,T) = 5$, $\gamma^{min}(4,T) = 5$, $\gamma^{min}(5,T) = 5$, $\gamma^{min}(6,T) = 4$, $\gamma^{min}(7,T) = 4$, $\gamma^{min}(8,T) = 4$, $\gamma^{min}(9,T) = 4$, $\gamma^{min}(10,T) = 5$, $\gamma^{min}(11,T) = 6$, $\gamma^{min}(12,T) = 6$. Hence $\gamma^{M,m}(T) = max\{\gamma^{min}(i,T) : 1 \leq i \leq 12\} = 6$.

§3. Conclusion

Courcelle has proved that if a graph property can be expressed in extended monadic second order logic (EMSO), then for every fixed $w \geq 1$, there is a linear-time algorithm for testing this property on graphs having treewidth at most w. The property of a subset S of V being a minimal dominating set can be expressed in EMSO and hence for families of graphs with bounded treewidth, a linear time algorithm can be developed for computing $\gamma^{min}(v,G)$ for any given vertex v. Hence developing such algorithm for specific families of graphs of bounded treewidth is an interesting problem for further research.

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References

[1] Acharya, B.D., The strong domination number of a graph and related concepts, *J. Math. Phys. Sci.* 14(5), 471–475 (1980)

[2] Arumugam, S., Kala, R., Domsaturation number of a graph, *Indian J. Pure. Appl. Math.* 33(11), 1671–1676 (2002)

[3] Arumugam, S., Subramanian, M., Independence saturation and extended domination chain in graphs, *AKCE J. Graph. Combin.* 4(2), 59–69 (2007)

[4] Arumugam, S., Odile Favaron, Sudha, S., Irredundance saturation number of a graph, *The Australasian Journal of Combinatorics* 46, 37–49 (2010)

[5] Arumugam, S., Sudha, S., Min-Max and Max-Min Graph Saturation Parameters,(communicated)

[6] Chartrand, G., Lesniak, L., *Graphs and Digraphs*, Chapman and Hall, CRC, 4^{th} edition, 2005.

[7] Haynes, T.W., Hedetniemi, S.T., Slater, P.J., *Fundamentals of Domination in Graphs*, Marcel Dekker Inc., 1998.

[8] Haynes, T.W., Hedetniemi, S.T., Slater, P.J., *Domination in Graphs-Advanced Topics*, Marcel Dekker Inc., 1998.

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Embeddings of Circular graph $C(2n+1,2)(n \geq 2)$ on the Projective Plane

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Abstract: Researches on embeddings of graphs on the projective plane have significance to determine the total genus distributions of graphs. Based on the embedding model of joint tree, this paper calculated the embedding number of the circular graph $C(2n+1,2)(n \geq 2)$ on the projective plane. Therefore, embeddings of K_5 on the projective plane is solved.

Key Words: Surface, genus, embeddings, joint tree, Smarandachely k-drawing.

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§1. Introduction

In this paper, a surface is a compact 2-dimensional manifold without boundary. It is orientable or nonorientable. Given a graph G and a surface S, a *Smarandachely k-drawing* of G on S is a homeomorphism ϕ: $G \to S$ such that $\phi(G)$ on S has exactly k intersections in $\phi(E(G))$ for an integer k. If $k = 0$, i.e., there are no intersections between in $\phi(E(G))$, or in another words, each connected component of $S - \phi(G)$ is homeomorphic to an open disc, then G has an 2-cell embedding on S. Two embeddings $h : G \to S$ and $g : G \to S$ of G into a surface S are said to be equivalent if there is a homeomorphism $f : S \to S$ such that $f \circ h = g$.

Given a graph G, how many nonequivalent embeddings of G are there into a given surface is one of important problems in topological graph theory. It can be tracked back to the genus distributions or total genus distributions of graphs. Since Gross and Furst [1] had introduced these concepts, the genus distributions or total genus distributions of a few graph classes had been solved by scholars [2-7]. However, for many other graph classes, we have not solved the related problems temporarily. There are always relationships among the numbers of embeddings of a graph on different genus surfaces. Therefore, researching on embeddings of graphs on sphere,torus,projective plane,Klein bottle has special significance. The embedding model of joint tree [8] is a special method which had promoted the research on genus distributions or total distributions of graphs [9-12].Basing on this model,this paper calculated the embedding number of circular graph $C(2n+1,2)(n \geq 2)$ on the projective plane.

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§2. Related Knowledge and Lemmas

A surface can be represented by a polygon of even edges in the plane whose edges are pairwise identified and directed clockwise or counterclockwise. To distinguish the direction of each edge, we index each edge by "+"(always omitted) and "−" . For example, sphere, torus, projective plane, Klein bottle can be represented by $O_0 = aa^-$, $O_1 = aba^-b^-$, $N_1 = aa$, $N_2 = aabb$ respectively. In general,

$$O_p = \prod_{i=1}^{p} a_i b_i a_i^- b_i^-, N_q = \prod_{i=1}^{q} a_i a_i$$

denote respectively an orientable surface with genus p and a nonorientable surface with genus $q(p \geq 1, q \geq 1)$. Edge a is called a twisted edge if the directions of the identical edges a is the same. Otherwise edge a is called an untwisted edge. A nonorientable surface has at least one twist edge.

The following three operations don't change the type of a surface:

Operation 1 $Aaa^- \Leftrightarrow A$.

Operation 2 $AabBab \Leftrightarrow AcBc$.

Operation 3 $AB \Leftrightarrow (Aa)(a^-B)$.

Among the above three operations, the parentheses stand for cyclic order. A and B stand liner order and they aren't empty except operation 2. Actually the above operations determine a topological equivalence denoted $\sim$. Therefore, They introduce three relations of topological equivalence.

Relation 1 $AxByCx^-Dy^-E \sim ADCBExyx^-y^-$.

Relation 2 $AxBxC \sim AB^-Cxx$.

Relation 3 $Axxyzy^-z^- \sim Axxyyzz$.

Based on the above operations and relations, It is easy to obtain the following lemmas:

Lemma 2.1([8]) *Suppose S_1 is an orientable surface with genus p and S_2 is a nonorientable surface with genus q.*

(1) If $S = S_1xyx^-y^-$, Then S is an orientable surface with genus $p+1$;

(2) If $S = S_2xyx^-y^-$, Then S is a nonorientable surface with genus $q+2$;

(3) If $S = S_1xx$, Then S is a nonorientable surface with genus $2p+1$;

(4) If $S = S_2xx$, Then S is a nonorientable surface with genus $q+1$.

Lemma 2.2 *Suppose surface S is nonorientable and $S = AxByCx^-Dy^-$, then the nonrientable genus of S is not less than 3.*

Proof According to relation1, $S = AxByCx^-Dy^- \sim ADCBxyx^-y^-$. Let$S_2 = ADCB$, then S_2 is nonorientable and its genus is at least 1. Based on Lemma **??**, the nonorientable genus of surface S is not less than 3. □

Lemma 2.3 *Suppose surface S is nonorientable, if $S = AxByCyDx$ or $S = AxByCxDy^-$, then the nonorientable genus is not less than 2.*

Proof If $S = AxByCyDx$, according to relation 2,

$$S = AxByCyDx \sim AxBC^{-}Dxyy \sim AD^{-}CB^{-}yyxx.$$

According to Lemma 2.1,the nonorientable genus of S is not less than 2;

Suppose $S = AxByCxDy^{-}$, according to relation 2,

$$S = AxByCxDy^{-} \sim AC^{-}y^{-}B^{-}Dy^{-}xx \sim AC^{-}D^{-}Bxxy^{-}y^{-}.$$

According to Lemma 2.1, the nonorientable genus of S is not less than 2. □

The embedding model of joint tree may be introduced in the following way: Given a spanning tree T of a graph $G = (V, E)$, we split every cotree edge into two edges and label them by the identical letter. The two edges are called the semi-edges of the original cotree edge. The resulting graph is the joint tree of the original graph G. Suppose the number of cotree edges is β. Given a direction to every semi-edge so that the direction of each pair of semi-edges can be the same or opposite. Beginning with a vertex, we walk all over the edges of the joint tree by its rotation. Writing the letter of semi-edges of the original graph cotree edges by order. we obtain a polygon of 2β edges which is exactly the associated surface of the graph G. There is a 1 to 1 correspondence between the associated surfaces and the embeddings of graph G. Hence an embedding of a graph G on a surface can be exactly represented by an associate surface of the graph G.

§3. Main Conclusions

The first, we investigate the structure character of polygon representation of projective plane.

Definition 3.1 *If surface $S = AxByCxDy$, then x and y are said to be interlaced in S; if surface $S = AxBxCyDy$, then x and y are said to be parallel in S.*

According to Lemmas 2.2 and 2.3, it is easy to obtain the following theorem:

Theorem 3.1 *Suppose S is a projective plane. If two edges in the polygon representation of S are all twisted, then they must be interlaced; otherwise, they must be parallel.*

Definition 3.2 *Circular graph $C(2n+1,2)$ $(n \geq 2)$ is obtained by appending chords $\{u_j u_{j+2} \mid j = 1, 2, \cdots, 2n+1\}$ on the circle $u_1 u_2 \cdots u_{2n+1} u_1$. Figure 1 is the circular graph $C(7,2)$. $a_i = u_{2i-1}u_{2i+1}(i = 1, 2, \cdots, n)$ are called odd chords; $b_i = u_{2i}u_{2i+2}(i = 1, 2, \cdots, n-1)$ are called even chords. Specially, let $c_0 = u_{2n+1}u_1$, $a_0 = u_{2n+1}u_2$, $b_0 = u_{2n}u_1$. Denote the collection of odd chords by E_1, $E_1 = \{a_i \mid i = 1, 2, \cdots, n\}$; Denote the collection of even chords by E_2, $E_2 = \{b_i \mid i = 1, 2, \cdots, n-1\}$. The subscriptions of vertices are the Residue Class Modules of $2n+1$.*

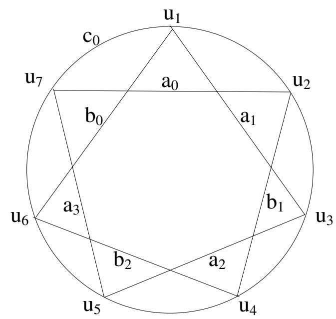

Figure 1: C(7,2)

There are some researches on embeddings of circular graphs in paper [13]. According to it, a circular graph can be embedded on the projective plane. But the embedding number and structure have not been investigated yet. In this paper, we calculated the embedding number of circular graphs on the projective plane.

We choose path $u_1u_2...u_{2n}u_{2n+1}$ as the spanning tree of the circular graph $C(2n+1,2)$ $(n \geq 2)$. Then by splitting each cotree edge, we obtain the joint tree. The two edges by splitting one cotree edge are called semi-edges of the original cotree edge. The upside of the spanning tree is the side which the semi-edge a_0 incident with vertex u_{2n+1} is placed. The other side is called the underside of the spanning tree. Considering the special positions of cotree edges c_0, a_0, b_0, we discuss the embedding of circular graph $C(2n+1,2)(n \geq 2)$ on the projective plane basing on whether the three cotree edges are twisted.

First, according to Lemmas 2.2 and 2.3, if the associated surface of circular graph $C(2n+1,2)(n \geq 2)$ is projective plane, then we have the following claims:

Claim 1 There are at most three twisted edges in $E_1 \cup E_2$.

In fact, if there are more than three twisted edges in $E_1 \cup E_2$, there will exist two twisted edges and they are parallel in the associated surface. It contradicts to Theorem 3.1.

Claim 2 Each semi-edges pair of an untwisted edge must be placed on the same side of the spanning tree.

In fact, if a semi-edges pair of an untwisted edge are placed on the distinct sides of the spanning tree, the untwisted edge and c_0 must be interlaced in the associated surface of graph G. It contradicts to Theorem 3.1.

Claim 3 If a_{i-1}, a_i(or b_{i-1}, b_i) are two untwisted edges in E_1(orE_2) and they are placed on distinct sides of the spanning tree, then b_{i-1}(ora_i) is twisted and its two semi-edges must be placed on distinct side of the spanning tree.

As is shown in Figure 2, a_{i-1}, a_i are two untwisted edges and placed on the two sides of the spanning tree respectively. If b_{i-1} is not twisted, then it must be interlaced with one of the three edges a_{i-1}, a_i, c_0. If b_{i-1} is twisted but its two semi-edges are placed on the same side of the spanning tree, it will be interlaced with a_{i-1} or a_i. The two cases all contradict to Theorem 3.1. Similarly we can prove the case of the two edges b_{i-1}, b_i.

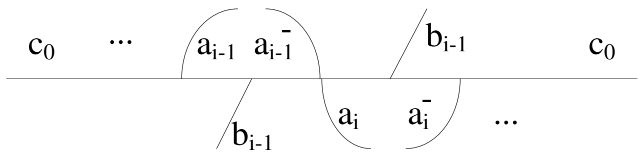

Figure 2: The side-transferring of untwisted neighbor edges

Theorem 3.2 *The embedding number of a circular graph $C(2n+1,2)(n \geq 2)$ on the projective plane is $8n+6$.*

Proof There are two embedding cases when considering whether c_0 is twisted.

Case 1 c_0 is untwisted

Because c_0 is untwisted, each semi-edges pair of a twisted edge must be placed on the same side of the spanning tree. Otherwise, it will be interlaced with c_0 and contract to Theorem 3.1. On the other side, every two twisted edges must be interlaced in the associated surface. all the twisted edges are placed on the same side. According to Claim3, there are no side-transferring case of neighbor untwisted edges in E_1 or E_2. Otherwise, there must exist a twisted edge that its semi-edges pair are placed on the two distinct side of the spanning tree respectively. It contracts to the above discussion. According to whether a_0, b_0 are twisted edges, The embeddings can be classified into four subcases:

Subcase 1.1 a_0 and b_0 are all untwisted

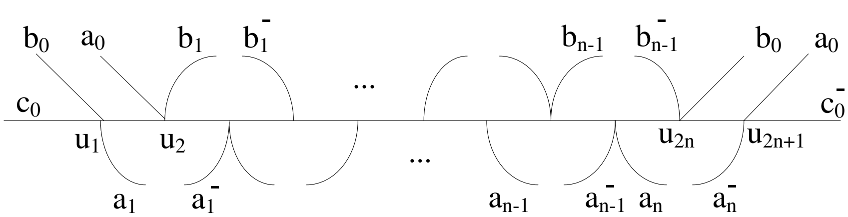

Figure 3: The joint tree of Subcase 1.1

As shown in Figure 3, a_0 and b_0 can only be placed on the same side of the spanning tree. If there are twisted edges in $E_1 \cup E_2$. they can only be a_1 or a_n. Suppose a_1 is twisted, then it must be on the upside of the spanning tree. Furthermore, b_1 can only be placed on the underside and also is a_n. Corresponding b_{n-1} is on the upside. Therefore, the sequence of untwisted edges $b_1 b_2 \cdots b_{n-1}$ will shift sides at one vertex. It contradicts to the above discussion. Then a_1 can't be a twisted edge, so is a_n in the same way. Then there are no twisted edges in $E_1 \cup E_2$. According to claim 3 and the above discussion, the untwisted edges sequence $b_1 b_2 \cdots b_{n-1}$ must be on the upside of the spanning tree while another untwisted edges sequence $a_1 a_2 \cdots a_n$ must be on the underside. Beginning at semi-edge c_0 incident to vertex u_1. Walk along all the joint tree edges by its rotation, we get the associated surface:

$$S = c_0 b_0 a_0 b_1 b_1^- b_2 b_2^- \cdots b_{n-1} b_{n-1}^- b_0 a_0 c_0^- a_n^- a_n a_{n-1}^- a_{n-1} \cdots a_1^- a_1$$
$$\sim b_0 a_0 b_0 a_0 \sim N_1.$$

Considering the symmetry of the two sides of the spanning tree, the embedding number of Subcase 1.1 is 2.

Subcase 1.2 a_0 is twisted, b_0 is untwisted

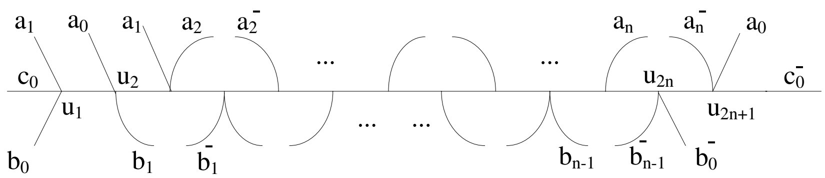

Figure 4: The joint tree of Subcase 1.2

Similarly, according to Theorem 3.1, a_0 and b_0 can only be placed on the distinct side of the spanning tree(as shown in Figure 4. If there is no twisted edge in $E_1 \cup E_2$, then a_n can only be placed on the upside because the untwisted edge can only be placed on the underside of the spanning tree. Then the sequence of untwisted edges $a_1 a_2 \cdots a_n$ will shift sides at one vertex. It contradicts to the above discussion. So there is no twisted edges in $E_1 \cup E_2$.

Each twisted edge in $E_1 \cup E_2$ and a_0 must be interlaced and they are placed on the same side. Then, the twisted edges in $E_1 \cup E_2$ can only be the following edges:a_1, b_1, a_n. a_1 and a_n can't all be twisted edges, otherwise they will be parallel. However there are at least one twisted edge among them, otherwise the sequence of untwisted edges $a_1 a_2 \cdots a_n$ will shift sides. If a_n is twisted, then it can only be placed on the upside and be interlaced with a_0. So b_1 and a_1 must be untwisted. Furthermore, a_1 must be placed on the underside while b_1 must be placed on the upside. Therefore, the untwisted edge b_{n-1} can only be placed on the underside. It indicate that the untwisted edges sequence $b_1 b_2 \cdots b_{n-1}$ shift sides at one vertex. It contradicts to Claim 3. So a_1 must be twisted and a_n is untwisted. If b_1 is also twisted, Then it will be placed on the upside. So a_2 will be placed on the underside while a_n will be placed on the upside. It is to say that the untwisted edges sequence $a_2 a_3 \cdots a_n$ will shift sides and contradicts to Claim 3.

Based on the above discussion, a_1 is the only twisted edge in $E_1 \cup E_2$. According to Theorem 3.1 and Claims 1,2,3, the rotations of the joint tree are only fixed. The associated surface is

$$S = c_0 a_1 a_0 a_1 a_2 a_2^- \cdots a_n a_n^- a_0 c_0^- b_0^- b_{n-1}^- b_{n-1} \cdots b_1 b_1^- b_0$$
$$\sim a_1 a_0 a_1 a_0 \sim N_1.$$

So the embedding number of Subcase 1.2 on the projective plane is also 2.

Subcase 1.3 a_0 is untwisted, b_0 is twisted

Similarly, a_0 and b_0 can only be placed on the distinct side of the spanning tree. discussed in the same way with Subcase 1.2, a_n is the only twisted edges in $E_1 \cup E_2$. The joint tree is shown in Figure 5:

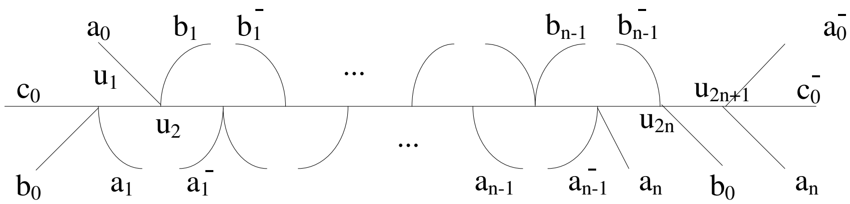

Figure 5; joint tree of subcase 1.3

The associated surface is

$$
\begin{aligned}
S &= c_0 a_0 b_1 b_1^- \cdots b_{n-1} b_{n-1}^- a_0^- c_0^- a_n b_0 a_n a_{n-1}^- a_{n-1} \cdots a_1^- a_1 b_0 \\
&\sim a_n b_0 a_n b_0 \sim N_1.
\end{aligned}
$$

The embedding number of Subcase 1.3 is 2.

Subcase 1.4 a_0, b_0 are all untwisted

As shown in Figure 6, a_0, b_0 can only be placed on the distinct side respectively, otherwise they are interlaced and contradict to Theorem 3.1. Furthermore, a_1 must be placed on the underside and a_n must be placed on the upside. In correspondence, b_1 is on the upside and b_{n-1} is on the underside. Because the associated surface is projective plane, so there are at least one twisted edge in $E_1 \cup E_2$.

If there is only one twisted edge in $E_1 \cup E_2$ and it is $a_i (1 \leq i \leq n)$, then the untwisted sequence $b_1 b_2 \cdots b_{n-1}$ will shift sides at one vertex and contradiction happens. similarly is the case that $b_i (1 \leq i \leq n-1)$ is the only twisted edge. So there are at least two twisted edges in $E_1 \cup E_2$.

If there are two twisted edges in $E_1 \cup E_2$, then the twisted edges pair must be the following combinations: $\{a_i, b_i\}$, $\{a_i, b_{i-1}\}$, $\{a_i, a_{i+1}\}$, $\{b_i, b_{i+1}\}$. If the twisted edge pair are $a_i, a_{i+1} (1 \leq i \leq n-1)$, Then the untwisted edges sequence $b_1 b_2 \cdots b_{n-1}$ will shift sides. Similarly, if the twisted edges pair are $b_i, b_{i+1} (1 \leq i \leq n-2)$, the untwisted edges sequence $a_1 a_2 \cdots a_n$ will shift sides. According to Claim 3, contradiction happens.

If the twisted edges pair is $a_i, b_i (1 \leq i \leq n-1)$, according to Theorem 3.1, they are on the underside. The joint tree is shown in Figure 6.

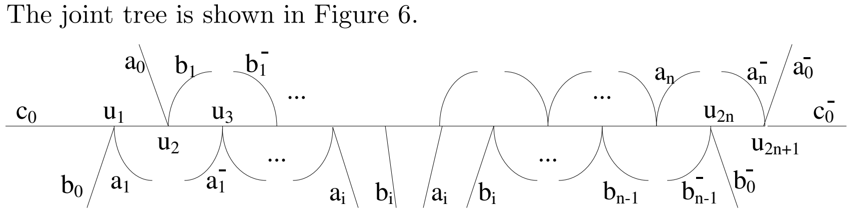

Figure 6: The joint tree of embedding Subcase 1.4 (a_i, b_i is twisted)

The associated surface

$$\begin{aligned}S &= c_0a_0b_1b_1^- \cdots b_{i-1}b_{i-1}^-a_{i+1}a_{i+1}^- \cdots a_na_n^-a_0^- \\ &\quad c_0^-b_0^-b_{n-1}^-b_{n-1} \cdots b_{i+1}^-b_{i+1}b_ia_ib_ia_ia_{i-1}^-a_{i-1} \cdots a_1^-a_1b_0 \\ &\sim b_ia_ib_ia_i \sim N_1\end{aligned}$$

and the embedding number of this case is $2(n-1)$.

If the twisted edges pair is $a_i, b_{i-1}(2 \le i \le n)$, then they are on upside. The associated surface

$$\begin{aligned}S &= c_0a_0b_1b_1^- \cdots b_{i-2}b_{i-2}^-b_{i-1}a_ib_{i-1}a_ia_{i+1}a_{i+1}^- \cdots a_na_n^-a_0^- \\ &\quad c_0^-b_0^-b_{n-1}^-b_{n-1} \cdots b_i^-b_ia_{i-1}^-a_{i-1} \cdots a_1^-a_1b_0 \\ &\sim b_{i-1}a_ib_{i-1}a_i \sim N_1\end{aligned}$$

and the embedding number of this case is also $2(n-1)$.

If there are three twisted edges in $E_1 \cup E_2$, Then the twisted edges must be the following two combinations: $\{a_i, a_{i+1}, b_i\}$ and $\{b_i, b_{i+1}, a_{i+1}\}$. Suppose $a_i, a_{i+1}, b_i(1 \le i \le n-2)$ are twisted edges and placed on the underside of the spanning tree. The untwisted edges a_n must be placed on the upside. According to Claim3, the untwisted edges sequence $a_n \cdots a_{i+2}$ are on the upside. Therefore, the untwisted edge b_{i+1} will be interlaced with a_{i+1} or a_{i+2}. It contradicts Theorem 3.1. Suppose $a_i, a_{i+1}, b_i(2 \le i \le n-1)$ are placed on the upside of the spanning tree, similarly, the untwisted edges sequence $a_1 \cdots a_{i-1}$ must be placed on the underside. Therefore, the untwisted edge b_{i-1} must be interlaced with a_{i-1} or a_i. It contradicts Theorem 3.1. Similarly, If b_i, b_{i+1}, a_{i+1} are twisted edges, contradiction will also happen.

So the embedding number of the Subcase1.4 on the projective plane is $4n-4$. The embedding number of the Case 1 on the projective plane is $4n+2$.

Case 2 c_0 is twisted

In this case, semi-edges pair of each twisted edge can only be placed on the distinct side. Otherwise, the twisted edge and c_0 will be parallel and contradicts to Theorem 3.1. There are at most two twisted edges in $E_1 \cup E_2$, otherwise there will exist two twisted edges and they are parallel in the associated surface. According to whether a_0 and b_0 are twisted, the embedding can be classified into four subcases.

Subcase 2.1 a_0, b_0 are all twisted

If there are twisted edges in $E_1 \cup E_2$, they can only be the following combinations: $a_i, a_{i+1}(1 \le i \le n-1)$ or $b_i, b_{i+1}(1 \le i \le n-2, n > 2)$. In fact, among the untwisted edges sequence $b_1b_2 \cdots b_{n-1}$, b_1, b_{n-1} are all on the underside. If the sequence shift sides, then it will shift sides two times continuously and $a_i, a_{i+1}(1 \le i \le n-1)$ will be twisted edges. similarly, $b_i, b_{i+1}(1 \le i \le n-2, n > 2)$ may be twisted edges in the same way.

If there are no twisted edges in $E_1 \cup E_2$, the untwisted edges sequence $a_1a_2 \cdots a_n$ must be placed on the upside while the the untwisted edges sequence $b_1b_2 \cdots b_{n-1}$ must be placed on the underside. the associated surface

$$S = c_0 b_0 a_1 a_1^- a_2 a_2^- \cdots a_n a_n^- a_0 c_0 b_0^- b_{n-1}^- b_{n-1} \cdots b_1^- b_1 a_0$$
$$\sim c_0 b_0 a_0 c_0 b_0 a_0 \sim N_1.$$

The embedding number of this subcase on the projective plane is 2.

If $a_i, a_{i+1} (1 \leq i \leq n-1)$ are twisted edges, the joint tree is shown in Figure 7.

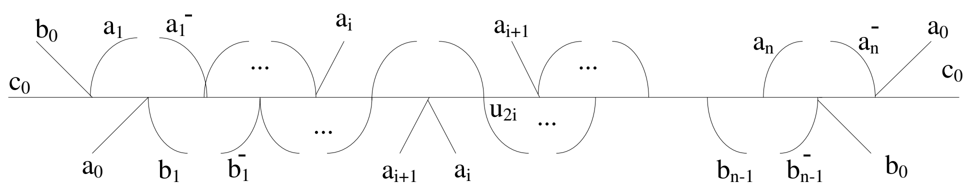

Figure 7: The joint tree of Subcase2.1(a_i, a_{i+1} are twisted)

The associated surface

$$s = c_0 b_0 a_1 a_1^- \cdots a_{i-1} a_{i-1}^- a_i b_i b_i^- a_{i+1} a_{i+2} a_{i+2}^- \cdots a_n a_n^- a_0$$
$$c_0 b_0 b_{n-1}^- b_{n-1} \cdots b_{i+1}^- b_{i+1} a_i a_{i+1} b_{i-1}^- b_{i-1} \cdots b_1^- b_1 a_0$$
$$\sim c_0 b_0 a_i a_{i+1} a_0 c_0 b_0 a_i a_{i+1} a_0 \sim N_1$$

and the embedding number of this subcase on the projective plane is $2(n-1)$.

If $b_i, b_{i+1} (1 \leq i \leq n-2, n > 2)$ are twisted edges, the joint tree is shown in Figure 8.

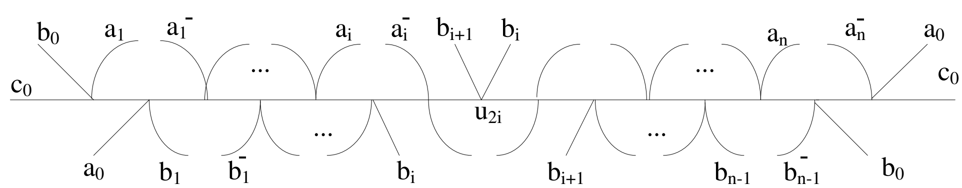

Figure 8: The joint tree of Subcase 2.1(b_i, b_{i+1} is twisted)

The associated surface

$$S = c_0 b_0 a_1 a_1^- \cdots a_i a_i^- b_{i+1} b_i a_{i+2} a_{i+2}^- \cdots a_n a_n^- a_0$$
$$c_0 b_0 b_{n-1}^- b_{n-1} \cdots b_{i+1} a_{i+1} a_{i+1}^- b_i b_{i-1}^- b_{i-1} \cdots b_1 b_1^- a_0$$
$$\sim c_0 b_0 b_{i+1} b_i a_0 c_0 b_0 b_{i+1} b_i a_0 \sim N_1$$

and the embedding number of this subcase on the projective plane is $2(n-2) = 2n-4$.

So The embedding number of subcase 1.2 on the projective plane is $4n-4$.

Subcase 2.2 a_0 is twisted, b_0 is untwisted

As shown in Figure 9, the semi-edges pair of a_0 must be placed on the two distinct sides and b_0 be placed on the upside.

If there is no twisted edges in $E_1 \cup E_2$, then a_1 and a_n can only be placed on the distinct side. Then the untwisted edges sequence $a_1 a_2 \cdots a_n$ will shift sides and contradict to Claim3. So there are twisted edges in $E_1 \cup E_2$. However, the twisted edges in $E_1 \cup E_2$ can only be a_1, a_n, b_{n-1}. Suppose a_n is twisted, then a_1, b_{n-1} are untwisted. Then the untwisted edges sequence $a_1 a_2 \cdots a_{n-1}$ must be placed on the upside of the spanning tree. Therefore b_{n-1} must be on the underside and interlaced with a_n. Contradiction happens.

If a_1 is twisted, then a_n, b_{n-1} are untwisted. The untwisted edges sequences $b_1 b_2 \cdots b_{n-1}$ and $a_2 a_3 \cdots a_n$ are placed on the upside and underside respectively. The joint tree is shown in Figure 9:

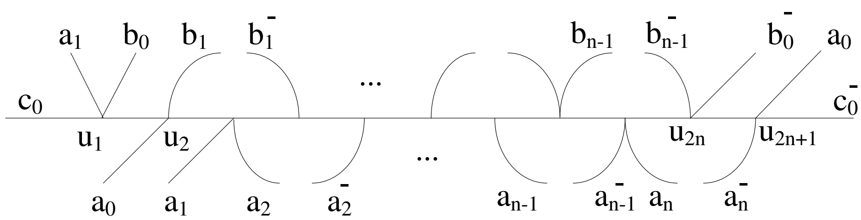

Figure 9: The joint tree of subcase 2.2(a_1 is twisted)

The associated surface

$$
\begin{aligned}
S &= c_0 a_1 b_0 b_1 b_2^- \cdots b_{n-1} b_{n-1}^- b_0^- a_0 c_0 a_n^- a_n a_{n-1}^- a_{n-1} \cdots a_2^- a_2 a_0 \\
&\sim c_0 a_1 a_0 c_0 a_1 a_0 \sim N_1.
\end{aligned}
$$

If b_{n-1} is twisted, then a_1, a_n are untwisted. The joint tree is shown in Figure 10.

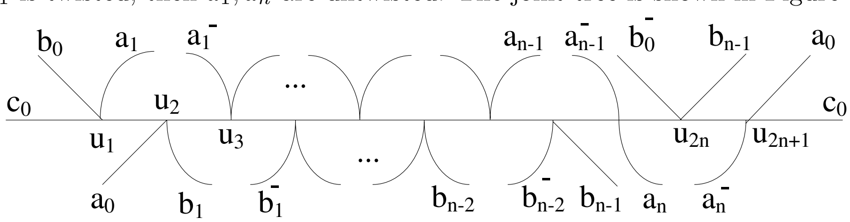

Figure 10: The joint tree of subcase2.2(b_{n-1} is twisted)

The associated surface

$$
\begin{aligned}
S &= c_0 b_0 a_1 a_1^- a_2 a_2^- \cdots a_{n-1} a_{n-1}^- b_0^- b_{n-1} a_0 c_0 a_n^- a_n b_{n-1} b_{n-2}^- b_{n-2} \cdots b_1^- b_1 a_0 \\
&\sim c_0 b_{n-1} a_0 c_0 b_{n-1} a_0 \sim N_1
\end{aligned}
$$

and the embedding number of subcase2.2 on the projective plane is 4.

Subcase 2.3 a_0 is untwisted, b_0 is twisted

Similarly, in this case, the twisted edges in $E_1 \cup E_2$ can only be b_1 or a_n. If b_1 is twisted, the associated surface

$$
\begin{aligned}
S &= c_0 b_0 b_1 a_0 a_2 a_2^- a_3 a_3^- \cdots a_n a_n^- a_0^- c_0 b_0 b_{n-1}^- b_{n-1} b_{n-2}^- b_{n-2} \cdots b_2^- b_2 b_1 \\
&\sim c_0 b_0 b_1 c_0 b_0 b_1 \sim N_1.
\end{aligned}
$$

If a_n is twisted, the associated surface

$$S = c_0 b_0 a_0 b_1 b_1^- b_2 b_2^- \cdots b_{n-1} b_{n-1}^- a_0^- a_n c_0 b_0 a_n a_{n-1}^- a_{n-1} a_{n-2}^- a_{n-2} \cdots a_1^- a_1$$
$$\sim c_0 b_0 a_n c_0 b_0 a_n \sim N_1$$

and the embedding number of Subcase 2.3 on the projective plane is 4.

Subcase 2.4 a_0, b_0 are all untwisted

a_0 and b_0 must be placed on the distinct side of the spanning tree. If there are twisted edges in $E_1 \cup E_2$, then the semi-edges of the twisted edge must be placed on the distinct side. It will be interlaced with a_0 and b_0. So the edges in $E_1 \cup E_2$ are all untwisted. However, the untwisted edges a_1 and a_n can only be placed on the distinct side. Then the untwisted edges sequence $a_1 a_2 \cdots a_n$ will shift sides at one vertex. Contradiction happens. So Subcase 2.4 can't be embedded on the projective plane.

Then the embedding number of Case 2 on the projective plane is $4n+4$.

Based on the above discussion, the embedding number of circular graph $C(2n+1)(n \geq 2)$ on the projective plane is $8n+6$. □

Let $n = 2$, we obtain the following corollary:

Corollary 3.1 *The embedding number of complete graph K_5 on the projective plane is* 22.

References

[1] Gross J. L., Furst M. L., Hierarchy of imbedding distribution invariants of graph, *J. Graph Theory*, 11:205–220, 1987.

[2] Furst M. L., Gross J. L, Stateman R., Genus distributions for two classes of graphs, *J. Combin. Theory Ser. B*, 46:22–36, 1989.

[3] Gross J. L., Robbins D.P., Tucker T. W., Genus distributions for bouquets of circles, *J. Combin. Theory Ser. B*, 47:292–306, 1989.

[4] Kwak J. H, Lee J., Genus polynomials of dippoles of circles, *Disrete Math*, 33:115–125, 1993.

[5] Tear E. H., Genus distributions for ringer ladders, *Discrete Math*, 216:235–252, 2000.

[6] Chen J., Gross J. L., Rieper R. G., Overlap matrics and total imbedding distrbution, *Discrete Math*, 128:73–94, 1994.

[7] Kwak J. H., Shim S. H., Total embedding distributions for buquets of circles, *Disrete Math*, 248:93–108, 2002.

[8] Liu Y. P., *Algebraic Principles of Maps* (in Chinese), Beijing: Higher Education Press, 2006.

[9] Chen Y. C., Liu Y. P., The total embedding distributions of cacti and necklaces, *Acta Math Sinica(English Series)*, 22(5):1583–1590, 2006.

[10] Yang Y., Liu Y. P., Total genus distributions of two classes of 4-regular graphs (in Chinese), *Acta Math Sinica(Chinese Series)*, 50(5):1190–1200, 2007.

[11] Zhao X. M., Liu Y. P., Genus distribution for cirle-like graph (in Chinese), *Acta Mathematica Scientia*, 28:757–767, 2008.

[12] Yang Y., *Classification of graph embeddings on surfaces* (in Chinese), Ph. D. Thesis, Beijing Jiaotong University, 2008.

[13] Ren H., Deng M., *Embeddings of circular graphs* (in Chinese), *Acta Mathematica Scientia*, 27(6):1148-1154, 2007.

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A Note On Jump Symmetric n-Sigraph

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$ where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ $(\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k))$ is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 2-signed graph or Smarandachely 2-marked graph is called abbreviated a *signed graph* or a *marked graph*. In this note, we obtain a structural characterization of jump symmetric n-sigraphs. The notion of jump symmetric n-sigraphs was introduced by E. Sampathkumar, P. Siva Kota Reddy and M. S. Subramanya [Proceedings of the Jangjeon Math. Soc., 11(1) (2008), 89-95].

Key Words: Smarandachely symmetric n-sigraph, Smarandachely symmetric n-marked graph, Balance, Jump symmetric n-sigraph.

AMS(2000): 05C22

§1. Introduction

For standard terminology and notion in graph theory we refer the reader to West [6]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *Smarandachely symmetric n-sigraph (Smarandachely symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function.

A *sigraph (marked graph)* is an ordered pair $S = (G, \sigma)$ $(S = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S and $\sigma : E \to \{+, -\}$ $(\mu : V \to \{+, -\})$ is a function. Thus a Smarandachely symmetric 1-sigraph (Smarandachely symmetric 1-marked graph) is a sigraph (marked graph).

The *line graph* $L(G)$ of graph G has the edges of G as the vertices and two vertices of $L(G)$

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are adjacent if the corresponding edges of G are adjacent.

The *jump graph* $J(G)$ of a graph $G=(V,E)$ is $\overline{L(G)}$, the complement of the line graph $L(G)$ of G (See [1] and [2]).

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/Smarandachely symmetric n-sigraph/Smarandachely symmetric n-marked graph.

An n-tuple $(a_1, a_2, ..., a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [4], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P. Siva Kota Reddy [3]):

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(i) S_n is identity balanced *(or* i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and*

(ii) S_n is balanced*, if every cycle in S_n contains an even number of non-identity edges.*

Note An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [4].

Proposition 1.1(E. Sampathkumar et al. [4]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

The *line n-sigraph* $L(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is defined as follows (See [5]): $L(S_n) = (L(G), \sigma')$, where for any edge ee'in $L(G)$, $\sigma'(ee') = \sigma(e)\sigma(e')$.

The *jump n-sigraph* of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph $J(S_n) = (J(G), \sigma')$, where for any edge ee' in $J(S_n)$, $\sigma'(ee') = \sigma(e)\sigma(e')$. This concept was introduced by E. Sampathkumar et al. [4]. This notion is analogous to the line n-sigraph defined above. Further, an n-sigraph $S_n = (G, \sigma)$ is called *jump n-sigraph*, if $S_n \cong J(S'_n)$ for some signed graph S'. In the following section, we shall present a characterization of jump n-sigraphs. The following result indicates the limitations of the notion of jump n-sigraphs defined above, since the entire class of i-unbalanced n-sigraphs is forbidden to be jump n-sigraphs.

Proposition 1.2(E. Sampathkumar et al. [4]) *For any n-sigraph $S_n = (G, \sigma)$, its jump n-sigraph $J(S_n)$ is i-balanced.*

§2. Characterization of Jump n-Sigraphs

The following result characterize n-sigraphs which are jump n-sigraphs.

Proposition 2.1 *An n-sigraph $S_n = (G, \sigma)$ is a jump n-sigraph if, and only if, S_n is i-balanced*

n-sigraph and its underlying graph G is a jump graph.

Proof Suppose that S_n is i-balanced and G is a jump graph. Then there exists a graph H such that $J(H) \cong G$. Since S_n is i-balanced, by Proposition 1.1, there exists a marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $J(S'_n) \cong S_n$. Hence S_n is a jump n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a jump n-sigraph. Then there exists a n-sigraph $S'_n = (H, \sigma')$ such that $J(S'_n) \cong S_n$. Hence G is the jump graph of H and by Proposition 1.2, S_n is i-balanced. $\square$

References

[1] G. T. Chartrand, H. Hevia, E. B. Jaretre, and M. Schutz, Subgraph distance in graphs defined by edge transfers, *Discrete mathematics,* 170 (1997), 63-79.

[2] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., 1969.

[3] R. Rangarajan and P. Siva Kota Reddy, Notions of balance in symmetric n-sigraphs, *Proceedings of the Jangjeon Math. Soc.*, 11(2) (2008), 145-151.

[4] E. Sampathkumar, P. Siva Kota Reddy, and M. S. Subramanya, Jump symmetric n-sigraph, *Proceedings of the Jangjeon Math. Soc.* 11(1) (2008), 89-95.

[5] E. Sampathkumar, P. Siva Kota Reddy, and M. S. Subramanya, The Line n-sigraph of a symmetric n-sigraph, *Southeast Asian Bull. Math.*(Springer-Verlag), to appear.

[6] D. B. West, *Introduction to Graph Theory*, Prentice-Hall of India Pvt. Ltd., 1996.

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New Families of Mean Graphs

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Abstract: Let $G(V,E)$ be a graph with p vertices and q edges. A vertex labeling of G is an assignment $f : V(G) \to \{1,2,3,\ldots,p+q\}$ be an injection. For a vertex labeling f, the induced *Smarandachely edge m-labeling* f_S^* for an edge $e = uv$, an integer $m \geq 2$ is defined by

$$f_S^*(e) = \left\lceil \frac{f(u)+f(v)}{m} \right\rceil.$$

Then f is called a *Smarandachely super m-mean labeling* if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1,2,3,\ldots,p+q\}$. Particularly, in the case of $m = 2$, we know that

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

Such a labeling is usually called a *super mean labeling*. A graph that admits a Smarandachely super mean m-labeling is called *Smarandachely super m-mean graph*, particularly, *super mean graph* if $m = 2$. In this paper, we discuss two kinds of constructing larger mean graphs. Here we prove that $(P_m; C_n)m \geq 1$, $n \geq 3$, $(P_m; Q_3)m \geq 1$, $(P_{2n}; S_m)m \geq 3$, $n \geq 1$ and for any $n \geq 1$ $(P_n; S_1)$, $(P_n; S_2)$ are mean graphs. Also we establish that $[P_m; C_n]m \geq 1$, $n \geq 3$, $[P_m; Q_3]m \geq 1$ and $[P_m; C_n^{(2)}]m \geq 1$, $n \geq 3$ are mean graphs.

Key Words: Labeling, mean labeling, mean graphs, Smarandachely edge m-labeling, Smarandachely super m-mean labeling, super mean graph.

AMS(2000): 05C78

§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph. Let $G(V,E)$ be a graph with p vertices and q edges. A path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. The graph $P_2 \times P_2 \times P_2$ is called the cube and is denoted by Q_3. For notations and terminology we follow [1].

[1]Received March 26, 2010. Accepted June 18, 2010.

A vertex labeling of G is an assignment $f : V(G) \to \{1, 2, 3, \ldots, p+q\}$ be an injection. For a vertex labeling f, the induced *Smarandachely edge m-labeling* f_S^* for an edge $e = uv$, an integer $m \geq 2$ is defined by

$$f_S^*(e) = \left\lceil \frac{f(u)+f(v)}{m} \right\rceil.$$

Then f is called a *Smarandachely super m-mean labeling* if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. Particularly, in the case of $m = 2$, we know that

$$f^*(e) = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even;} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd.} \end{cases}$$

Such a labeling is usually called a *super mean labeling.* A graph that admits a Smarandachely super mean m-labeling is called *Smarandachely super m-mean graph*, particularly, *super mean graph* if $m = 2$. The mean labeling of the Petersen graph is given in Figure 1.

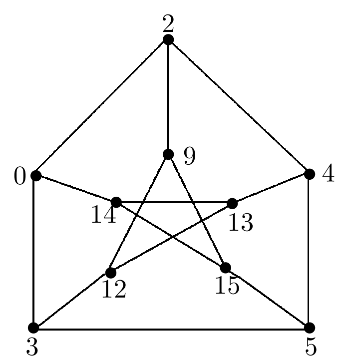

Figure 1

A super mean labeling of the graph $K_{2,4}$ is shown in Figure 2.

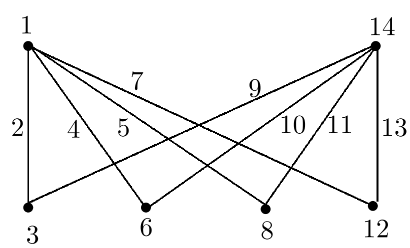

Figure 2

The concept of mean labeling was first introduced by Somasundaram and Ponraj [2] in the year 2003. They have studied in [2-5,8-9], the meanness of many standard graphs like $P_n, C_n, K_n (n \leq 3)$, the ladder, the triangular snake, $K_{1,2}, K_{1,3}, K_{2,n}$, $K_2 + mK_1$, $K_n^c + 2K_2$, $S_m + K_1$, $C_m \cup P_n (m \geq 3, n \geq 2)$, quadrilateral snake, comb, bistars $B(n)$, $B_{n+1,n}, B_{n+2,n}$, the carona of ladder, subdivision of central edge of $B_{n,n}$, subdivision of the star $K_{1,n} (n \leq 4)$, the friendship graph $C_3^{(2)}$, crown $C_n \odot K_1$, $C_n^{(2)}$, the dragon, arbitrary super subdivision of a path etc. In addition, they have proved that the graphs $K_n (n > 3), K_{1,n} (n > 3)$, $B_{m,n} (m > n+2)$, $S(K_{1,n}) n > 4$, $C_3^{(t)} (t > 2)$, the wheel W_n are not mean graphs.

The concept of super mean labeling was first introduced by R. Ponraj and D. Ramya [6]. They have studied in [6-7] the super mean labeling of some standard graphs. Also they determined all super mean graph of order ≤ 5. In [10], the super meanness of the graph C_{2n} for $n \geq 3$, the H-graph, Corona of a H-graph, 2-corona of a H-graph, corona of cycle C_n for $n \geq 3$, mC_n-snake for $m \geq 1, n \geq 3$ and $n \neq 4$, the dragon $P_n(C_m)$ for $m \geq 3$ and $m \neq 4$ and $C_m \times P_n$ for $m = 3, 5$ are proved.

Let C_n be a cycle with fixed vertex v and $(P_m; C_n)$ the graph obtained from m copies of C_n and the path $P_m : u_1u_2 \ldots u_m$ by joining u_i with the vertex v of the i^{th} copy of C_n by means of an edge, for $1 \leq i \leq m$.

Let Q_3 be a cube with fixed vertex v and $(P_m; Q_3)$ the graph obtained from m copies of Q_3 and the path $P_m : u_1u_2 \ldots u_m$ by joining u_i with the vertex v of the i^{th} copy of Q_3 by means of an edge, for $1 \leq i \leq m$.

Let S_m be a star with vertices $v_0, v_1, v_2, \ldots, v_m$. Define $(P_{2n}; S_m)$ to be the graph obtained from $2n$ copies of S_m and the path $P_{2n} : u_1u_2 \ldots u_{2n}$ by joining u_j with the vertex v_0 of the j^{th} copy of S_m by means of an edge, for $1 \leq j \leq 2n$, $(P_n; S_1)$ the graph obtained from n copies of S_1 and the path $P_n : u_1u_2 \ldots u_n$ by joining u_j with the vertex v_0 of the j^{th} copy of S_1 by means of an edge, for $1 \leq j \leq n$, $(P_n; S_2)$ the graph obtained from n copies of S_2 and the path $P_n : u_1u_2 \ldots u_n$ by joining u_j with the vertex v_0 of the j^{th} copy of S_2 by means of an edge, for $1 \leq j \leq n$.

Suppose $C_n : v_1v_2 \ldots v_nv_1$ be a cycle of length n. Let $[P_m; C_n]$ be the graph obtained from m copies of C_n with vertices $v_{1_1}, v_{1_2}, \ldots, v_{1_n}, v_{2_1}, \ldots, v_{2_n}, \ldots, v_{m_1}, \ldots, v_{m_n}$ and joining v_{i_j} and v_{i+1_j} by means of an edge, for some j and $1 \leq i \leq m-1$.

Let Q_3 be a cube and $[P_m; Q_3]$ the graph obtained from m copies of Q_3 with vertices $v_{1_1}, v_{1_2}, \ldots, v_{1_8}, v_{2_1}, v_{2_2}, \ldots, v_{2_8}, \ldots, v_{m_1}, v_{m_2}, \ldots, v_{m_8}$ and the path $P_m : u_1u_2 \ldots u_m$ by adding the edges $v_{1_1}v_{2_1}, v_{2_1}v_{3_1}, \ldots, v_{m-1_1}v_{m_1}$ (i.e) $v_{i_1}v_{i+1_1}, 1 \leq i \leq m-1$.

Let $C_n^{(2)}$ be a friendship graph. Define $[P_m; C_n^{(2)}]$ to be the graph obtained from m copies of $C_n^{(2)}$ and the path $P_m : u_1u_2 \ldots u_m$ by joining u_i with the center vertex of the i^{th} copy of $C_n^{(2)}$ for $1 \leq i \leq m$.

In this paper, we prove that $(P_m; C_n)m \geq 1, n \geq 3, (P_m; Q_3)m \geq 1, (P_{2n}; S_m)m \geq 3, n \geq 1$, and for any $n \geq 1 (P_n; S_1), (P_n; S_2)$ are mean graphs. Also we establish that $[P_m; C_n]m \geq 1$, $n \geq 3$, $[P_m; Q_3]m \geq 1$ and $[P_m; C_n^{(2)}]m \geq 1, n \geq 3$ are mean graphs.

§2. Mean Graphs $(P_m; G)$

Let G be a graph with fixed vertex v and let $(P_m; G)$ be the graph obtained from m copies of G and the path $P_m : u_1u_2 \ldots u_m$ by joining u_i with the vertex v of the i^{th} copy of G by means of an edge, for $1 \leq i \leq m$.

For example $(P_4; C_4)$ is shown in Figure 3.

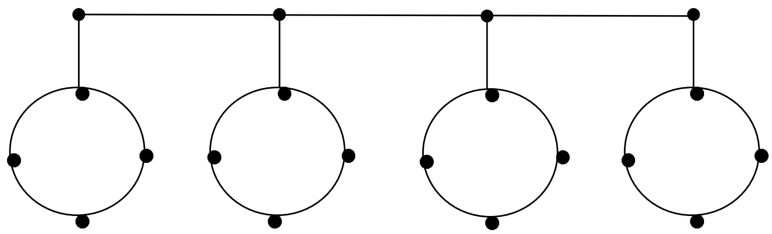

Figure 3

Theorem 2.1 *$(P_m; C_n)$ is a mean graph, $n \geq 3$.*

Proof Let $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$ be the vertices in the i^{th} copy of $C_n, 1 \leq i \leq m$ and $u_1, u_2, \ldots, u_m$ be the vertices of P_m. Then define f on $V(P_m; C_n)$ as follows:

Take $n = \begin{cases} 2k & \text{if } n \text{ is even} \\ 2k+1 & \text{if } n \text{ is odd.} \end{cases}$

Then $f(u_i) = \begin{cases} (n+2)(i-1) & \text{if } i \text{ is odd} \\ (n+2)i - 1 & \text{if } i \text{ is even} \end{cases}$

Label the vertices of v_{i_j} as follows:

Case (i) n is odd

When i is odd,

$$f(v_{i_j}) = (n+2)(i-1) + 2j - 1, 1 \leq j \leq k+1$$
$$f(v_{i_{k+1+j}}) = (n+2)i - 2j + 1, 1 \leq j \leq k, 1 \leq i \leq m.$$

When i is even,

$$f(v_{i_j}) = (n+2)i - 2j, 1 \leq j \leq k,$$
$$f(v_{i_{k+j}}) = (n+2)(i-1) + 2(j-1), 1 \leq j \leq k+1, 1 \leq i \leq m$$

Case (ii) n is even

When i is odd,

$$f(v_{i_j}) = (n+2)(i-1) + 2j - 1, 1 \leq j \leq k+1$$
$$f(v_{i_{k+1+j}}) = (n+2)i - 2j, 1 \leq j \leq k-1, 1 \leq i \leq m$$

When i is even,

$$f(v_{i_j}) = (n+2)i - 2j, 1 \leq j \leq k+1$$
$$f(v_{i_{k+1+j}}) = (n+2)(i-1) + 2j + 1, 1 \leq j \leq k-1, 1 \leq i \leq m$$

The label of the edge $u_i u_{i+1}$ is $(n+2)i, 1 \leq i \leq m-1$.

The label of the edge $u_i v_{i_1}$ is $\begin{cases} (n+2)(i-1)+1 & \text{if } i \text{ is odd,} \\ (n+2)i - 1 & \text{if } i \text{ is even} \end{cases}$

and the label of the edges of the cycle are

$(n+2)i-1, (n+2)i-2, \ldots, (n+2)i-n \quad$ if i is odd,

$(n+2)i-2, (n+2)i-3, \ldots, (n+2)i-(n+1) \quad$ if i is even.

For example, the mean labelings of $(P_6; C_5)$ and $(P_5; C_6)$ are shown in Figure 4. $\square$

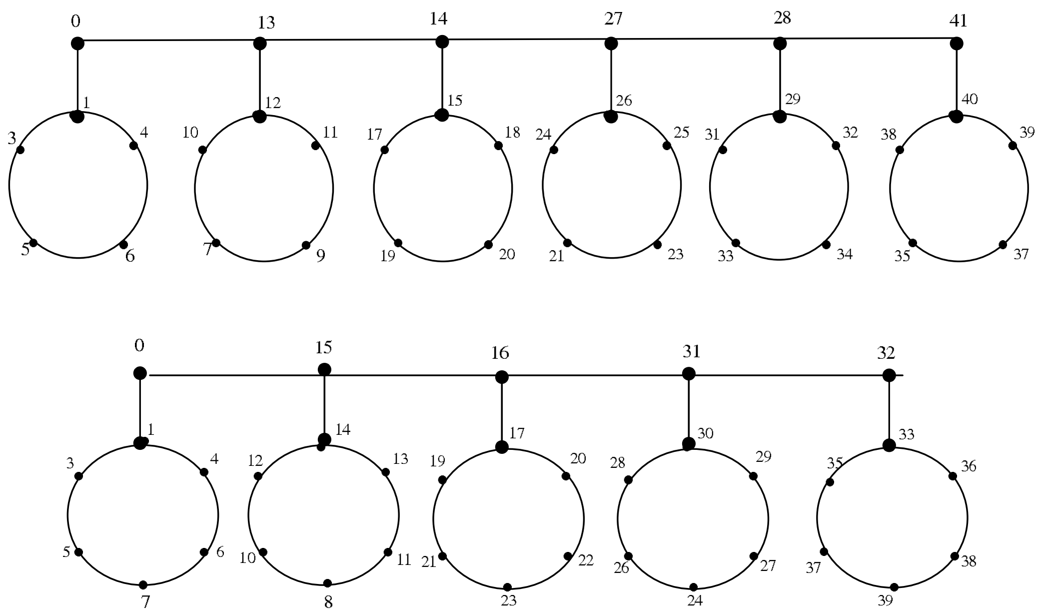

Figure 4

Theorem 2.2 *$(P_m; Q_3)$ is a mean graph.*

Proof For $1 \leq j \leq 8$, let v_{i_j} be the vertices in the i^{th} copy of $Q_3, 1 \leq i \leq m$ and $u_1, u_2, \ldots, u_m$ be the vertices of P_m.

Then define f on $V(P_m; Q_3)$ as follows:

$$f(u_i) = \begin{cases} 14i - 14 & \text{if } i \text{ is odd} \\ 14i - 1 & \text{if } i \text{ is even.} \end{cases}$$

When i is odd,

$$f(v_{i_1}) = 14i - 13, \quad 1 \leq i \leq m$$
$$f(v_{i_j}) = 14i - 13 + j, \quad 2 \leq j \leq 4, 1 \leq i \leq m$$
$$f(v_{i_5}) = 14i - 5, \quad 1 \leq i \leq m$$
$$f(v_{i_j}) = 14i - 9 + j, \quad 6 \leq j \leq 8, 1 \leq i \leq m$$

when i is even,

$$f(v_{i_j}) = 14i - 1 - j, \quad 1 \le j \le 3, 1 \le i \le m$$
$$f(v_{i_4}) = 14i - 6, 1 \le i \le m$$
$$f(v_{i_j}) = 14i - 5 - j, 5 \le j \le 7, 1 \le i \le m$$
$$f(v_{i_8}) = 14i - 14, 1 \le i \le m$$

The label of the edges of P_m are $14i$, $1 \le i \le m - 1$.

The label of the edges of $u_i v_{i_1} = \begin{cases} 14i - 13, & \text{if } i \text{ is odd} \\ 14i - 1, & \text{if } i \text{ is even} \end{cases}$

The label of the edges of the cube are

$14i - 1, 14i - 2, \ldots, 14i - 12$ if i is odd,

$14i - 2, 14i - 3, \ldots, 14i - 13$ if i is even.

For example, the mean labeling of $(P_5; Q_3)$ is shown in Figure 5. □

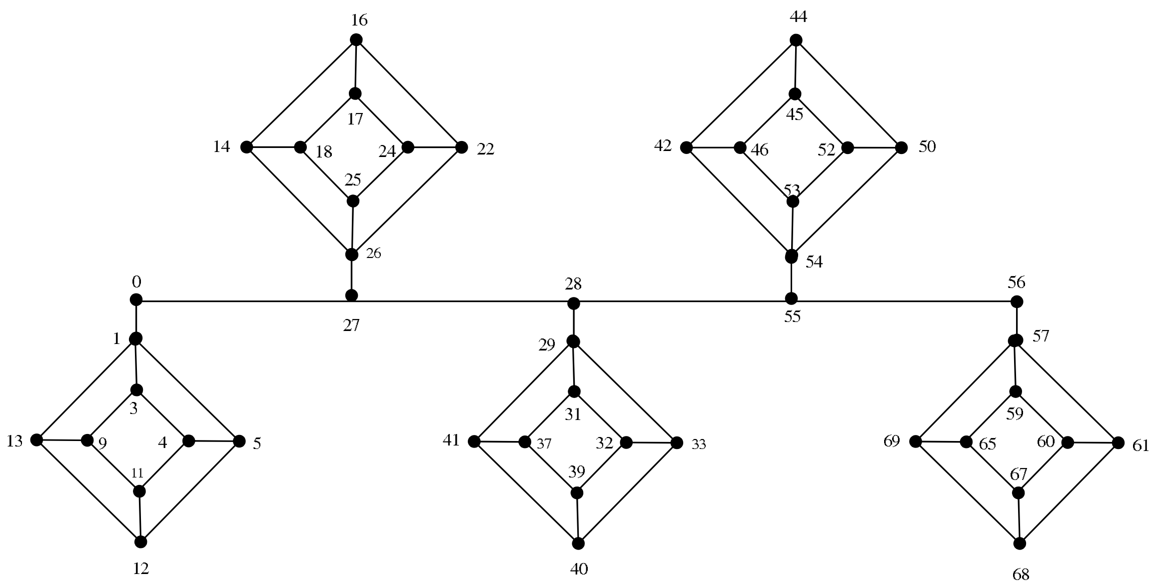

Figure 5

Theorem 2.3 *$(P_{2n}; S_m)$ is a mean graph, $m \ge 3, n \ge 1$.*

Proof Let $u_1, u_2, \ldots, u_{2n}$ be the verfices of P_{2n}. Let $v_{0_j}, v_{1_j}, v_{2_j}, v_{3_j}, \ldots, v_{m_j}$ be the vertices in the j^{th} copy of $S_m, 1 \le j \le 2n$.

Label the vertices of $(P_{2n}; S_m)$ as follows:

$$f(u_{2j+1}) = (2m+4)j, \quad 0 \le j \le n-1,$$
$$f(u_{2j}) = (2m+4)j-1, \quad 1 \le j \le n,$$
$$f(v_{0_{2j+1}}) = (2m+4)j+1, \quad 0 \le j \le n-1,$$
$$f(v_{0_{2j}}) = (2m+4)j-2, \quad 1 \le j \le n,$$
$$f(v_{i_{2j+1}}) = (2m+4)j+2i, \quad 0 \le j \le n-1, 1 \le i \le m$$
$$f(v_{i_{2j}}) = (2m+4)(j-1)+2i+1, \quad 1 \le j \le n, 1 \le i \le m$$

The label of the edge $u_j u_{j+1}$ is $(m+2)j, 1 \le j \le 2n-1$
The label of the edge $u_j v_{0_j}$ is

$$\begin{cases} (m+2)(j-1)+1, & \text{if } j \text{ is odd} \\ (m+2)j-1, & \text{if } j \text{ is even} \end{cases}$$

The label of he edge $v_{0_j} v_{i_j}$ is

$$\begin{cases} (m+2)(j-1)+i+1, & \text{if } j \text{ is odd, } 1 \le i \le m \\ (m+2)(j-1)+i, & \text{if } j \text{ is even, } 1 \le i \le m \end{cases}$$

For example, the mean labeling of $(P_6; S_5)$ is shown in Figure 6. $\square$

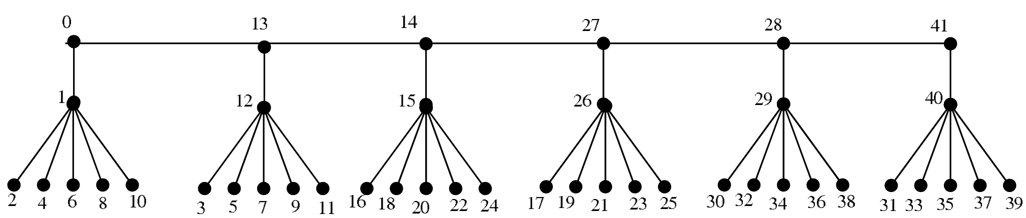

Figure 6

Theorem 2.4 *$(P_n; S_1)$ and $(P_n; S_2)$ are mean graphs for any $n \ge 1$.*

Proof Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n. Let $v_{o_1}, v_{0_2}, \ldots, v_{0_n}$ and $v_{1_1}, v_{1_2}, \ldots, v_{1_n}$ be the vertices of S_1.

Label the vertices of $(P_n; S_1)$ as follows:

$$f(u_j) = \begin{cases} 3j-3 & \text{if } j \text{ is odd, } 1 \le j \le n \\ 3j-1 & \text{if } j \text{ is even, } 1 \le j \le n \end{cases}$$
$$f(v_{0_j}) = 3j-2, \quad 1 \le j \le n$$
$$f(v_{1_j}) = \begin{cases} 3j-1 & \text{if } j \text{ is odd, } 1 \le j \le n \\ 3j-3 & \text{if } j \text{ is even, } 1 \le j \le n \end{cases}$$

The label of the edges of P_n are $3j, 1 \leq j \leq n-1$.

The label of the edges $u_j v_{0_j}$ is $\begin{cases} 3j-2, & \text{if } j \text{ is odd} \\ 3j-1, & \text{if } j \text{ is even} \end{cases}$

The label of the edges $v_{0_j} v_{1_j}$ is $\begin{cases} 3j-1, & \text{if } j \text{ is odd} \\ 3j-2, & \text{if } j \text{ is even} \end{cases}$

Let $v_{0_1}, v_{0_2}, \ldots, v_{0_n}, v_{1_1}, v_{1_2}, \ldots, v_{1_n}$ and $v_{2_1}, v_{2_2}, \ldots, v_{2_n}$ be the vertices of S_2.
Label the vertices of $(P_n; S_2)$ as follows:

$$f(u_j) = \begin{cases} 4j-4 & \text{if } j \text{ is odd, } 1 \leq j \leq n \\ 4j-1 & \text{if } j \text{ is even, } 1 \leq j \leq n \end{cases}$$

$$f(v_{0_j}) = 4j-2, \quad 1 \leq j \leq n$$

$$f(v_{1_j}) = \begin{cases} 4j-3 & \text{if } j \text{ is odd, } 1 \leq j \leq n, \\ 4j-4 & \text{if } j \text{ is even, } 1 \leq j \leq n, \end{cases}$$

$$f(v_{2_j}) = \begin{cases} 4j-1 & \text{if } j \text{ is odd, } 1 \leq j \leq n, \\ 4j-3 & \text{if } j \text{ is even, } 1 \leq j \leq n, \end{cases}$$

The label of the edges of P_n are $4j, 1 \leq j \leq n-1$

The label of the edges $u_j v_{0_j}$ is $\begin{cases} 4j-3, & \text{if } j \text{ is odd} \\ 4j-1 & \text{if } j \text{ is even} \end{cases}$

The label of the edges $v_{0_j} v_{1_j}$ is $\begin{cases} 4j-2, & \text{if } j \text{ is odd} \\ 4j-3, & \text{if } j \text{ is even} \end{cases}$

The label of the edges $v_{0_j} v_{2_j}$ is $\begin{cases} 4j-1, & \text{if } j \text{ is odd} \\ 4j-2, & \text{if } j \text{ is even} \end{cases}$

For example, the mean labelings of $(P_7; S_1)$ and $(P_6; S_2)$ are shown in Figure 7. □

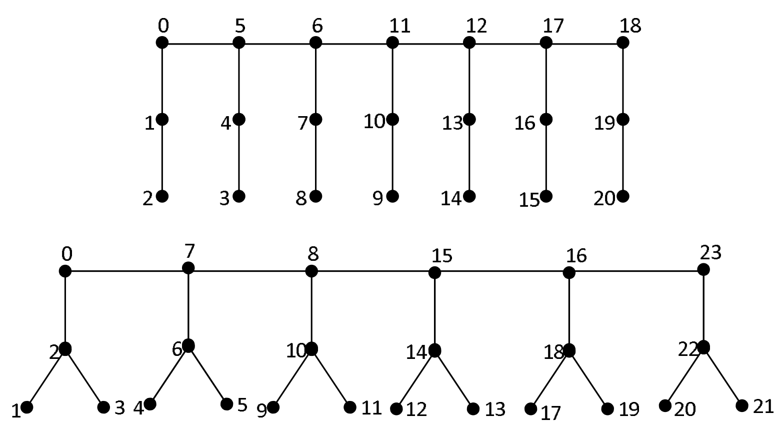

Figure 7

§3. Mean Graphs $[P_m; G]$

Let G be a graph with fixed vertex v and let $[P_m; G]$ be the graph obtained from m copies of G by joining v_{i_j} and v_{i+1_j} by means of an edge, for some j and $1 \leq i \leq m-1$.

For example $[P_5; C_3]$ is shown in Figure 8.

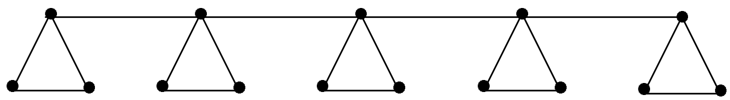

Figure 8

Theorem 3.1 *$[P_m; C_n]$ is a mean graph.*

Proof Let $u_1, u_2, \ldots, u_m$ be the vertices of P_m. Let $v_{i_1}, v_{i_2}, \ldots, v_{i_n}$ be the vertices of the i^{th} copy of $C_n, 1 \leq i \leq m$ and joining $v_{i_j}(= u_i)$ and $v_{i+1_j}(= u_{i+1})$ by means of an edge, for some j.

Case (i) $n = 4t, t = 1, 2, 3, \ldots$

Define $f : V([P_m; C_n]) \to \{0, 1, 2, \ldots, q\}$ by

$$f(v_{i_j}) = (n+1)(i-1) + 2(j-1), 1 \leq j \leq 2t+1$$
$$f(v_{i_{2t+1+j}}) = (n+1)i - 2j, 1 \leq j \leq 2t-1, 1 \leq i \leq m$$

The label of the edge $v_{i_{(t+1)}} v_{i+1_{(t+1)}}$ is $(n+1)i, 1 \leq i \leq m-1$. The label of the edges of the cycle are $(n+1)i-1, (n+1)i-2, \ldots, (n+1)i-n, 1 \leq i \leq m$.

For example, the mean labeling of $[P_4; C_8]$ is shown in Figure 9.

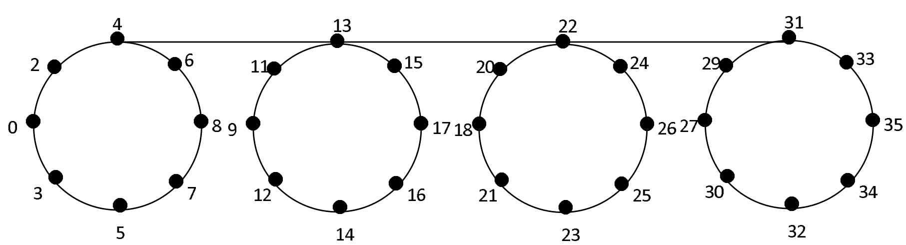

Figure 9

Case (ii) $n = 4t+1, t = 1, 2, 3, \ldots$

Define $f : V([P_m; C_n]) \to \{0, 1, 2, \ldots, q\}$ by

$$f(v_{i_1}) = (n+1)(i-1), 1 \leq i \leq m$$
$$f(v_{i_j}) = (n+1)(i-1) + 2j - 1, 2 \leq j \leq 2t+1, 1 \leq i \leq m$$
$$f(v_{i_{(2t+1+j)}}) = (n+1)i - 2j, 1 \leq j \leq 2t, 1 \leq i \leq m$$

The label of the edge $v_{i_{(t+1)}}v_{i+1_{(t+1)}}$ is $(n+1)i, 1 \le i \le m-1$. The label of the edges of the cycle are $(n+1)i-1, (n+1)i-2, \ldots, (n+1)i-n, 1 \le i \le m$.

For example, the mean labeling of $[P_6; C_5]$ is shown in Figure 10.

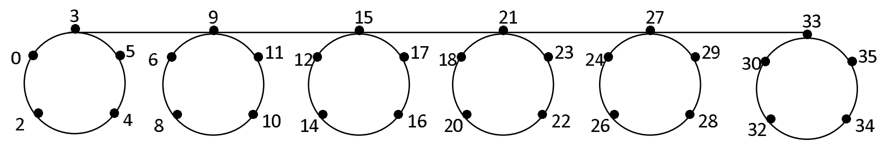

Figure 10

Case (iii) $n = 4t+2, t = 1, 2, 3, \ldots$

Define $f : V([P_m; C_n]) \to \{0, 1, 2, \ldots, q\}$ by

$$f(v_{i_1}) = (n+1)(i-1), 1 \le i \le m$$
$$f(v_{i_j}) = (n+1)(i-1) + 2j - 1, 2 \le j \le 2t+1, 1 \le i \le m$$
$$f(v_{i_{(2t+1+j)}}) = (n+1)i - 2j + 1, 1 \le j \le 2t+1, 1 \le i \le m$$

The label of the edge $v_{i_{(t+1)}}v_{i+1_{(t+1)}}$ is $(n+1)i, 1 \le i \le m-1$. The label of the edges of the cycle are $(n+1)i-1, (n+1)i-2, \ldots, (n+1)i-n, 1 \le i \le m$.

For example, the mean labeling of $[P_5; C_6]$ is shown in Figure 11.

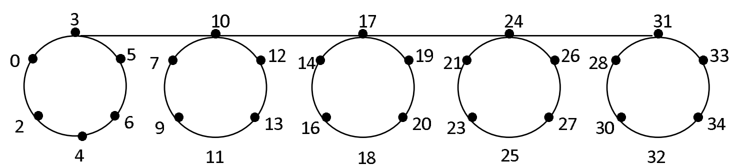

Figure 11

Case (iv) $n = 4t-1, t = 1, 2, 3, \ldots$

Define $f : V([P_m; C_n]) \to \{0, 1, 2, \ldots, q\}$ by

$$f(v_{i_j}) = (n+1)(i-1) + 2(j-1), 1 \le j \le 2t, 1 \le i \le m$$
$$f(v_{i_{(2t+j)}}) = (n+1)i - 2j + 1, 1 \le j \le 2t-1, 1 \le i \le m$$

The label of the edge $v_{i_{(t+1)}}v_{i+1_{(t+1)}}$ is $(n+1)i, 1 \le i \le m-1$. The label of the edges of the cycle are $(n+1)i-1, (n+1)i-2, \ldots, (n+1)i-n, 1 \le i \le m$.

For example, the mean labeling of $[P_7; C_3]$ is shown in Figure 12. □

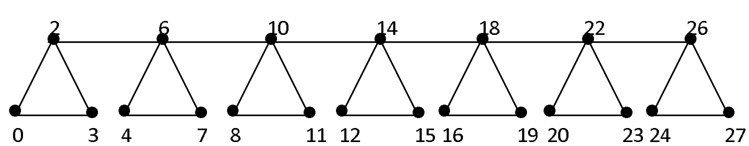

Figure 12

Theorem 3.2 *$[P_m; Q_3]$ is a mean graph.*

Proof For $1 \leq j \leq 8$, let v_{i_j} be the vertices in the i^{th} copy of $Q_3, 1 \leq i \leq m$. Then define f on $V[P_m; Q_3]$ as follows:

When i is odd.

$$f(v_{i_1}) = 13i - 13, 1 \leq i \leq m$$
$$f(v_{i_j}) = 13i - 13 + j, 2 \leq j \leq 4, 1 \leq i \leq m$$
$$f(v_{i_5}) = 13i - 5, 1 \leq i \leq m$$
$$f(v_{i_j}) = 13i - 9 + j, 6 \leq j \leq 8, 1 \leq i \leq m$$

When i is even.

$$f(v_{i_j}) = 13i - j, 1 \leq j \leq 3, 1 \leq i \leq m$$
$$f(v_{i_4}) = 13i - 5, 1 \leq i \leq m$$
$$f(v_{i_j}) = 13i - j - 4, 5 \leq j \leq 7, 1 \leq i \leq m$$
$$f(v_{i_8}) = 13i - 13, 1 \leq i \leq m$$

The label of the edge $v_{i_1}v_{(i+1)_1}$ is $13i, 1 \leq i \leq m-1$. The label of the edges of the cube are $13i-1, 13i-2, \ldots, 13i-12, 1 \leq i \leq m$.

For example the mean labeling of $[P_4; Q_3]$ is shown in Figure 13. □

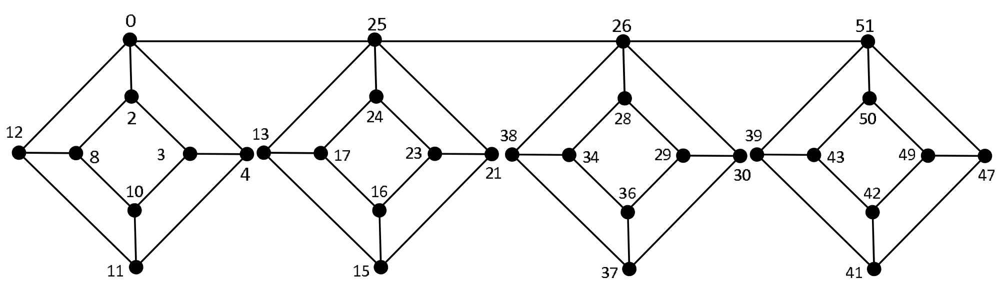

Figure 13

Theorem 3.3 *$[P_m; C_n^{(2)}]$ is a mean graph.*

Proof Let $u_1, u_2, \ldots, u_m$ be the vertices of P_m and the vertices $u_i, 1 \leq i \leq m$ is attached with the center vertex in the i^{th} copy of $C_n^{(2)}$. Let $u_i = v_{i_1}$ (center vertex in the i^{th} copy of $C_n^{(2)}$).

Let v_{i_j} and v'_{i_j} for $1 \leq i \leq m, 2 \leq j \leq n$ be the remaining vertices in the i^{th} copy of $C_n^{(2)}$.

Then define f on $V[P_m, C_n^{(2)}]$ as follows:

$$\text{Take } n = \begin{cases} 2k & \text{if } n \text{ is even} \\ 2k+1 & \text{if } n \text{ is odd.} \end{cases}$$

Label the vertices of v_{i_j} and v'_{i_j} as follows:

Case (i) When n is odd

$$f(v_{i_1}) = (2n+1)i - (n+1), 1 \le i \le m$$
$$f(v_{i_j}) = (2n+1)i - (n+2) - 2(j-2), 2 \le j \le k+2$$
$$f(v_{i_{k+2+j}}) = (2n+1)i - 2(n-1) + 2(j-1), 1 \le j \le k-1, k \ge 2$$
$$f(v'_{i_j}) = (2n+1)i - (n-1) + 2(j-2), 2 \le j \le k+1$$
$$f(v'_{i_{k+1+j}}) = (2n+1)i - 1 - 2(j-1), 1 \le j \le k, 1 \le i \le m$$

Case (ii) When n is even

$$f(v_{i_j}) = (2n+1)i - (n+1) - 2(j-1), 1 \le j \le k+1$$
$$f(v_{i_{k+1+j}}) = (2n+1)i - 2(n-1) + 2(j-1), 1 \le j \le k-1, 1 \le i \le m$$
$$f(v'_{i_j}) = (2n+1)i - (n-1) + 2(j-2), 2 \le j \le k+1$$
$$f(v'_{i_{k+1+j}}) = (2n+1)i - 2 - 2(j-1), 1 \le j \le k-1, 1 \le i \le m$$

The label of the edge u_iu_{i+1} is $(2n+1)i, 1 \le i \le m-1$ and the label of the edges of $C_n^{(2)}$ are $(2n+1)i-1, (2n+1)i-2, \ldots, (2n+1)i-2n$ for $1 \le i \le m$.

For example the mean labelings of $[P_4, C_6^{(2)}]$ and $[P_5, C_3^{(2)}]$ are shown in Figure 14. □

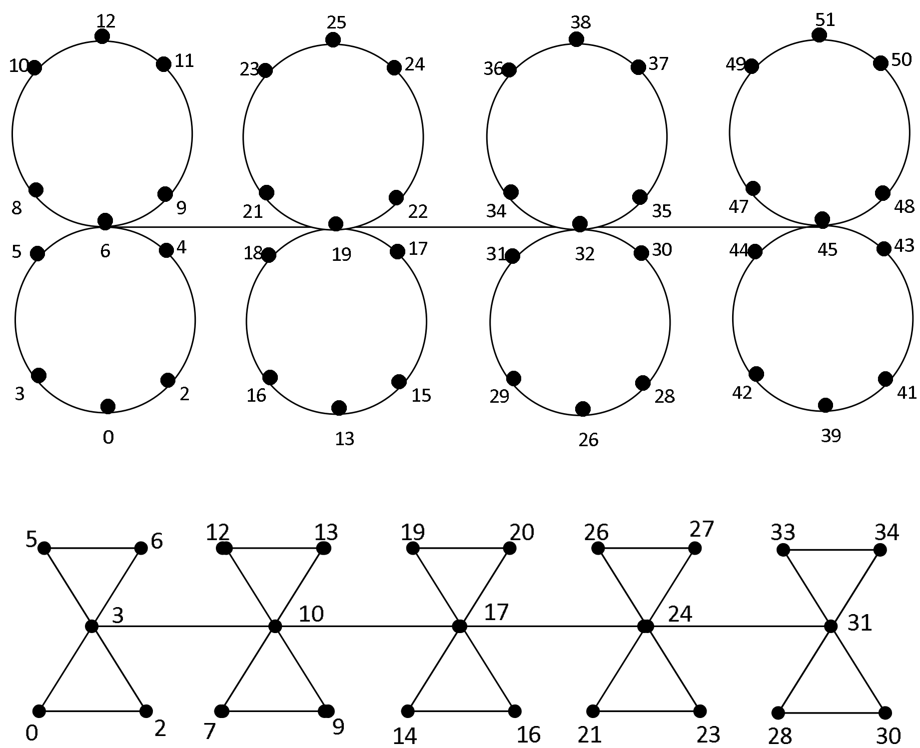

Figure 14

References

[1] R. Balakrishnan and N. Renganathan, *A Text Book on Graph Theory*, Springer Verlag, 2000.

[2] S. Somasundaram and R. Ponraj, Mean labelings of graphs, *National Academy Science letter,* **26** (2003), 210 - 213.

[3] S. Somasundaram and R. Ponraj, Non - existence of mean labeling for a wheel, *Bulletin of pure and Applied Sciences*, (Section E Maths & Statistics) **22E**(2003), 103 - 111.

[4] S. Somasundaram and R. Ponraj, Some results on mean graphs, *Pure and Applied Mathematika Sciences*, **58**(2003), 29 - 35.

[5] S. Somasundaram and R. Ponraj, On Mean graphs of order ≤ 5, *Journal of Decision and Mathematical Sciences*, **9**(1 - 3) (2004), 48 - 58.

[6] R. Ponraj and D. Ramya, Super mean labeling of graphs, Reprint.

[7 R. Ponraj and D. Ramya, On super mean graphs of order ≤ 5, *Bulletin of Pure and Applied Sciences*, (Section E Maths and Statistics) **25E** (2006), 143-148.

[8] R. Ponraj and S. Somasundaram, *Further results on mean graphs*, Proceedings of Sacoeference, August 2005. 443 - 448.

[9] R. Ponraj and S.Somasundaram, Mean labeling of graphs obtained by identifying two graphs, *Journal of Discrete Mathematical Sciences and Cryptography*, **11**(2)(2008), 239-252.

[10] R. Vasuki and A. Nagarajan, Some results on super mean graphs, *International Journal of Mathematical Combinatorics*, **3**(2009), 82-96.

International J.Math. Combin. Vol.2 (2010), 81-94

The (a, d)-Ascending Subgraph Decomposition

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Abstract: Let G be a graph of size q and a, n, d be positive integers for which $\frac{n}{2}[2a + (n-1)d] \leq q < \binom{n+1}{2}[2a + nd]$. Then G is said to have (a, d)-ascending subgraph decomposition ((a, d)-ASD) if the edge set of G can be partitioned into n-non-empty sets generating subgraphs $G_1, G_2, G_3, \ldots, G_n$ with out isolating vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$ and $|E(G_i)| = a + (i-1)d$. In this paper, we find (a, d)-ASD for $K_n, K_{m,n}$ and for product graphs.

Key Words: ASD, (a, d)-ASD, Smarandachely (P, Q)-decomposition, Smarandachely (a, d)-decomposition.

AMS(2000): 05C70

§1. Introduction

By a graph we mean a finite undirected graph without loops or multiple edges. A wheel on p vertices is denoted by W_p. A path of length t is denoted by P_{t+1}. A graph obtained from two graphs G_1 and G_2 by taking one copy of G_1 (which has p-vertices) and p copies of G_2 and then joining the i^{th} vertex of G_1 to every vertex of the i^{th} copy of G_2 is denoted by $G_1 \odot G_2$. Terms not defined here are used in the sense of Harary [4]. Throughout this paper $G \subset H$ means G is a subgraph of H.

Let $G = (V, E)$ be a simple graph of order p and size q. If $G_1, G_2, \ldots, G_n$ are edge disjoint subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup \cdots \cup E(G_n)$, then $\{G_1, G_2, \ldots, G_n\}$ is said to be a decomposition of G.

The concept of ASD was introduced by Alavi et al. [1]. The graph G of size q where $\binom{n+1}{2} \leq q < \binom{n+2}{2}$, is said to have an ascending subgraph decomposition (ASD) if

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G can be decomposed into n-subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$. We generalize the concept of ASD as follows:

Definition 1.1 *A graph G has a Smarandachely (P,Q)-decomposition for graphical properties P and Q, $P \subset Q$ if the edge set $E(G)$ can be partitioned into non-empty sets generating subgraphs $H \in P$ without isolating vertices such that each such H is isomorphic to a proper subgraph of $J \in Q$. In particular, we define a Smarandachely (a,d)- decomposition is a Smarandachely (P,Q)-decomposition, where $P = \{G_j/|E(G_j)| = a + (j-1)d\}$ and $Q = P = \{G_{j+1}/G_j \in P$ and $|E(G_{j+1})| = a + jd\}$ into subgraphs $G_1, G_2, \ldots, G_n$.*

In other words G is a simple graph of size q and a, n, d are positive integers for which $\frac{n}{2}[2a + (n-1)d] \leq q < \left(\frac{n+1}{2}\right)[2a + nd]$. Then (a,d)-ascending subgraph decomposition $((a,d) - ASD)$ of G is the edge disjoint decomposition of G into subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$ and $|E(G_i)| = a + (i-1)d$. The following theorems will be useful in proving certain results in Section 2.

Theorem 1.2([1]) *Let G be a graph of size q, where $\begin{pmatrix} n+1 \\ 2 \end{pmatrix} \leq q < \begin{pmatrix} n+2 \\ 2 \end{pmatrix}$ for some positive integers n, such that G has an ascending subgraph decomposition $G_1, G_2, \ldots, G_n$ such that G_i has size i for $1 \leq i \leq n-1$ and G_n has size $q - \begin{pmatrix} n \\ 2 \end{pmatrix}$.*

Theorem 1.3([2]) *$C_n \times C_n$ is decomposed into two Hamilton cycles if n is odd.*

Theorem 1.4([2]) *K_n is (i) decomposed into $\frac{n}{2}$-Hamilton cycles if n is odd. (ii) decomposed into $\left\lfloor \frac{n+1}{2} \right\rfloor$-Hamilton cycles and a 1-factors if n is even.*

§2. Main Results

Definition 2.1 *Let G be a graph of size q and a, n, d be positive integers for which $\left(\frac{n}{2}\right)[2a+(n-1)d] \leq q < \left(\frac{n+1}{2}\right)[2a+nd]$. Then G is said to have (a,d)- ascending subgraph decomposition $((a,d)-ASD)$ if the edge set of G can be partitioned into n non-empty sets generating subgraphs $G_1, G_2, \ldots, G_n$ without isolated vertices such that each G_i is isomorphic to a proper subgraph of G_{i+1} for $1 \leq i \leq n-1$ and $|E(G_i)| = a + (i-1)d$.*

Remark 2.2 From the above definition, the usual ASD of G coincides with $(1,1)$-ASD of G.

Example 2.3 Consider the Graph G.

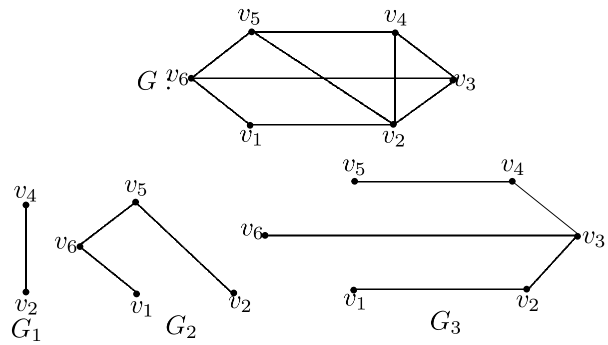


Fig.2.1

Clearly, $\{G_1, G_2, G_3\}$ is a $(1,2)$-ASD of G.

Theorem 2.4 *Let G be a graph of size q, where $\binom{n}{2}[2a+(n-1)d] \leq q < \binom{n+1}{2}[2a+nd]$ for some positive integer n, such that G has (a, d)- ASD, then G has an (a, d)-ASD $G_1, G_2, \ldots, G_n$ such that G_i has size $a+(i-1)d$ for $1 \leq i \leq n-1$ and G_n has size $q - \binom{n-1}{2}[2a+(n-2)d]$.*

The following number theoretical result will be useful for proving further results.

Lemma 2.5 *Given that the numbers $a, a+d, a+2d, \ldots, a+(n-1)d$ are in A.P $(a, d \in Z)$. Then their sum is*

(i) $S_n = (a-d)n + d\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ *if $d \leq a$ and*

(ii) $S_n = a\begin{pmatrix} n+1 \\ 2 \end{pmatrix} + (d-a)\begin{pmatrix} n \\ 2 \end{pmatrix}$ *if $d \geq a$.*

§3. (a, d)-ASD on Complete Graphs and Complete Bipartite Graphs

Theorem 3.1 *K_{n+1} has (a, d)-ASD if and only if $a = 1, d = 1$.*

Proof Suppose the graph K_{n+1} has (a, d)-ASD $G_1, G_2, \ldots, G_n$ with $|E(G_i)| = a+(i-1)d$, for $1 \leq i \leq n$.

By (ii) of Lemma 2.5, $q(K_{n+1}) = a\begin{pmatrix} n+1 \\ 2 \end{pmatrix} + (d-a)\begin{pmatrix} n \\ 2 \end{pmatrix}$. Also since $q(K_{n+1}) = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}$, we have $a = 1$ and $d = 1$. $\square$

As it was mentioned in [3] that the complete graph K_{n+1} with $(n+1)$ vertices could easily be proved to have a star ASD and a path ASD, The converse follows.

Theorem 3.2 *$K_{n,n}$ has (a, d)-ASD, $d \geq a$ if and only if $a = 1$ and $d = 2$.*

Proof Suppose the graph $K_{n,n}$ admits $(a,d)-ASD$, $d \geq a$. If the graph $K_{n,n}$ admits $(a,d)-ASD$ $G_1, G_2, \ldots, Gn$ then by (ii) of Lemma 2.5, we have $|E(K_{n,n})| = a\begin{pmatrix} n+1 \\ 2 \end{pmatrix} + (d-a)\begin{pmatrix} n \\ 2 \end{pmatrix}$.

Also, $|E(K_{n,n})| = n^2 = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} + \begin{pmatrix} n \\ 2 \end{pmatrix}$, so we have $a = 1$ and $d = 2$.

Conversely, suppose $a = 1, d = 2$.

Case (i) When n is even, $n = 2k, k \in Z^+$.

Then $K_{n,n}$ can be decomposed into k-hamilton cycles $H_1, H_2, \ldots, H_k$. Now, decompose the hamilton cycles H_i into paths G_i and $G_{n-(i-1)}$ of length $2i-1$ and $2n-(2i-1)$ for $1 \leq i \leq k$. Clearly, $\{G_1, G_2, \ldots, G_n\}$ is the required (1,2)-ASD of $K_{n,n}$.

Case (ii) When n is odd, $n = 2k+1, k \in Z^+$.

Let (X, Y) be the bipartition of $K_{n,n}$, where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. Define $H_1 = \{(x_n, y_j) : j = n-2\}$. For $2 \leq i \leq n-1$, define H_i by $H_{n+1-i} = \{(x_i, y_j) : j = 2i-2$ to $i+n-2\} \cup \{(x_j, y_{i+j-2}) : j = i+1$ to $n\}$, where addition is taken module n with residues $1, 2, 3, \ldots, n$ instead of the usual residues $0, 1, 2, \ldots, n-1$. $H_n = \{(x_1, y_k) : k = 1, 2, \ldots, n\} \cup \{(x_{j+1}, y_j) : 1 \leq j \leq n-1\}$. Clearly, $\{H_1, H_2, \ldots, H_n\}$ is a $(1,2)-ASD$ of $K_{n,n}$. $\square$

Example 3.3 Consider the graph $K_{7,7}$. Let (X, Y) be the bipartition of $K_{7,7}$ where $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $Y = \{y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$.

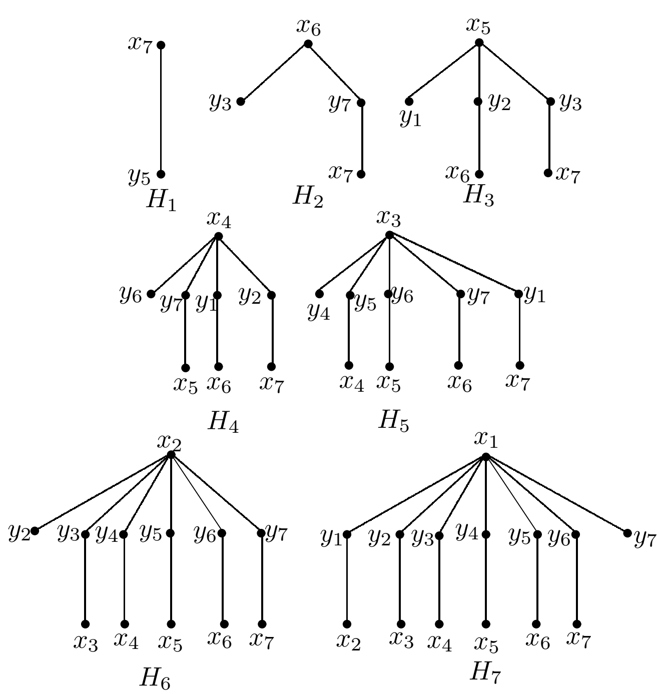


Fig. 3.1

Clearly, $\{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$ is $a(1, 2) - ASD$ of $K_{7,7}$.

Theorem 3.4 *$K_{n,n}(n > 1)$ admits $(a, d)-ASD, d < a$ if and only if $n = 2a-1$ and $d = 1, a > 1$.*

Proof Suppose the graph $K_{n,n}(n > 1)$ admits $(a, d) - ASD$ where $d < a$, then by (i) Lemma 2.5, we have $|E(K_{n,n})| = (a-d)n + d\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$. Also, $|E(K_{n,n})| = n^2$. Therefore, $n^2 = (a-d)n + d\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$ and so $(2-d)n^2 = (2a-d)n$. Then $n = \frac{2a-d}{2-d}$. Since, $n > 1$, $a > d$, we have $2 - d > 0$. Then $d = 1$ and $a > 1$. Hence $n = 2a - 1$.

Conversely, Suppose $n = 2a - 1, d = 1$ and $a > 1$. Let (X, Y) be the bipartition of $K_{n,n}$ where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$.

Define $T_{n-j-1} = \{(x_j, y_i) : 1 \le i \le n\} \cup \{(y_{i-j+1}, x_i) : \frac{n+2j+1}{2} \le i \le n\}$ where $1 \leqslant j \le \frac{n-1}{2}$ and $T_j = \{(x+n-j+1, y_i) : 1 \le i \le \frac{n-1}{2} + j\}$ where $1 \le j \le \frac{n-1}{2}$. Clearly, $\{T_1, T_2, \ldots, T_n\}$ is the required $(a, 1) - ASD$ of $K_{n,n}$. □

Example 3.5 Consider the graph $K_{5,5}$. Let (X, Y) be the bipartition of $K_{5,5}$ where $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$. Clearly, $\{T_1, T_2, T_3, T_4, T_5\}$ is a $(3, 1) - ASD$ of $K_{5,5}$.

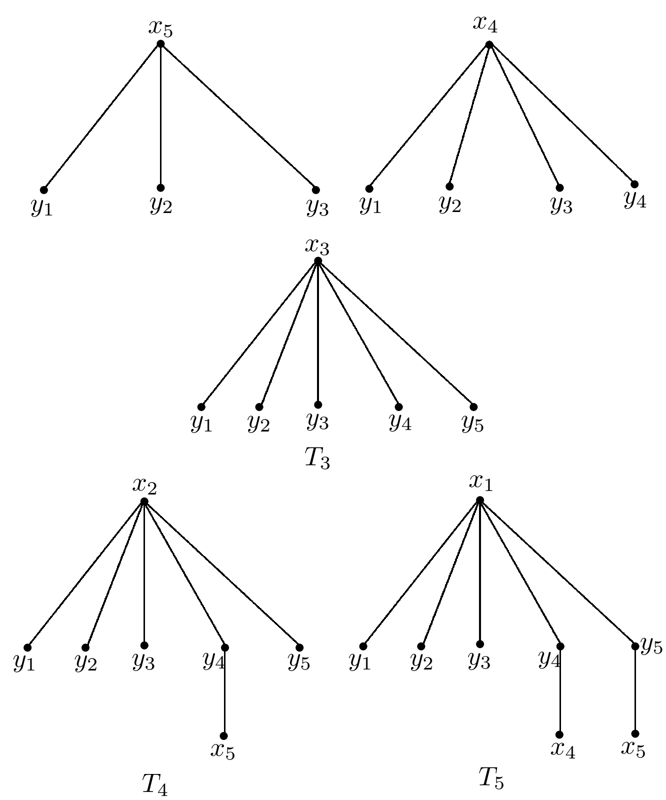


Fig. 3.2

§4. $(a,d)-ASD$ on Product Graphs

In this section, we prove some product graphs admit $(a,d)-ASD$.

Theorem 4.1 *$C_n\times C_n(n>3)$ has $(2,4)-ASD$ when n is odd.*

Proof Note that $|E(C_n\times C_n)|=2n^2$ and $|V(C_n\times C_n)|=n^2$. By Theorem 1.2, The graph $C_n\times C_n$ (n-odd) can be decomposed into two Hamilton cycles C_1 and C_2 of length n^2 respectively.

Case (i) When $n=2k+1, k\equiv 1(mod\ 2)$.

Let $P_1=C_1-(v,x)$ and $P_2=C_2-(v,y)$ where $v,x,y\in V(C_n\times C_n)$ and $x\neq y$. First, define $P_1=(xvy)$ when $k=3$, decompose the path P_1 into paths P_i of length $(4i-2), 6\leq i\leq 7$ and decompose the path P_2 into paths P_i of length $(4i-2), 2\leq i\leq 5$. For, $k>4$, decompose the path P_1 into paths P_i of length $(4i-2)$, where $2\leq i\leq k-\lfloor\frac{k}{2}\rfloor-1$ and $2\left(2-\left\lfloor\frac{k}{2}\right\rfloor\right)+\left\lfloor\frac{k}{2}\right\rfloor+1\leq i\leq n$. Also decompose the path P_2 into paths P_i of length $(4i-2)$, where $\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)\leq i\leq 2\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)+\left\lfloor\frac{k}{2}\right\rfloor$. This is possible because of

$$
\begin{aligned}
\mathcal{L}(P_1^1) &= \sum_{j=1}^{k-\lfloor\frac{k}{2}\rfloor-2}(2+4j)+\sum_{j=2+\left(2\left(k-\lfloor\frac{k}{2}\rfloor\right)+k-\lfloor\frac{k}{2}\rfloor\right)4}^{n-1}(2+4j)\\
&=\frac{\left(k-\lfloor\frac{k}{2}\rfloor-2\right)}{2}\left(12+\left(\left(k-\left\lfloor\frac{k}{2}\right\rfloor-2\right)-1\right)4\right)\\
&\quad+\frac{\left(\lfloor\frac{k}{2}\rfloor+1\right)}{2}\left(2\left(2+\left(2\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)+\left\lfloor\frac{k}{2}\right\rfloor\right)4\right)+4\left\lfloor\frac{k}{2}\right\rfloor\right)\\
&=2\left(k-\left\lfloor\frac{k}{2}\right\rfloor-2\right)\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)+\left(\frac{\lfloor\frac{k}{2}\rfloor+1}{2}\right)\left(4+16k-4\left\lfloor\frac{k}{2}\right\rfloor\right)\\
&=2k^2-4k-4k\left\lfloor\frac{k}{2}\right\rfloor+4\left\lfloor\frac{k}{2}\right\rfloor+2\left\lfloor\frac{k}{2}\right\rfloor^2+2+8k+8k\left\lfloor\frac{k}{2}\right\rfloor-2\left\lfloor\frac{k}{2}\right\rfloor^2\\
&=2k^2+4k+2+4k\left\lfloor\frac{k}{2}\right\rfloor+4\left\lfloor\frac{k}{2}\right\rfloor\\
&=2k^2+4k+2+2k(k-1)+2(k-1)\\
&=4k^2+4k\\
&=(2k+1)^2-1=n^2-1\\
\mathcal{L}(P_2') &= \left(\frac{k+1}{2}\right)\left(2\left(2+\left(k-\left\lfloor\frac{k}{2}\right\rfloor-1\right)4\right)+4k\right)\\
&=(k+1)\left(6k-2\left(2\left\lfloor\frac{k}{2}\right\rfloor+1\right)\right)\\
&=(k+1)(6k-2k)=(2k+1)^2-1=n^2-1.
\end{aligned}
$$

From the above construction, clearly, $\{P_1,P_2,\ldots,P_n\}$ is a $(2,4)-ASD$ of $C_n\times C_n$.

Case (ii) When $n=2k+1, k\equiv 0(mod\ 2)$.

Let $P_1' = C_1 - (v,x)$ and $P_2' = C_2 - (v,y)$ where $v,x,y \in V(C_n \times C_n)$ and $x \neq y$. First define $P_1 = (xvy)$, then decompose the path P_1' into paths P_2 of length 6 and P_j of length $(2+4j), 4 \leq j \leq n-1$ and $j = 0,1(mod\ 4)$ and also decompose the path P_2' into paths P_j of length $(2+4j), 2 \leq j \leq n-1$ and $j = 2,3(mod\ 4)$. This is possible, since

$$\begin{aligned}
\mathcal{L}(P_1') &= 6 + \sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{n-1} (2+4j) \\
&= 6 + 2\sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{n-1} 1 + 4\sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{n-1} j \\
&= 6 + 2\sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{2k} 1 + 4\sum_{\substack{j=4 \\ j\equiv 0,1(mod\ 4)}}^{2k} j \\
&= 6 + 2(k-1) + (4k^2 + 2k - 4) = (2k+1)^2 - 1 = n^2 - 1
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}(P_2') &= \sum_{\substack{j=2 \\ j\equiv 2,3(mod\ 4)}}^{n-1} (2+4j) \\
&= 2\sum_{\substack{j=2 \\ j\equiv 2,3(mod\ 4)}}^{2k} 1 + 4\sum_{\substack{j=2 \\ j\equiv 2,3(mod\ 4)}}^{2k} j \\
&= 2k + 4\sum_{\substack{j=2 \\ j\equiv 2,3(mod\ 4)}}^{2k} j \\
&= 2k + (2k + 4k^2) = (2k+1)^2 - 1 = n^2 - 1.
\end{aligned}$$

As in the case clearly, $\{P_1, P_2, \ldots, P_n\}$ is a $(2,4) - ASD$ of $C_n \times C_n$. □

Theorem 4.2 *$P_{n+1} \times P_{n+1}$ with size $q = 2n(n+1)$ admits $(4,4) - ASD$.*

Proof Let $G = P_{n+1} \times P_{n+1}$. Define $W_{i,j} = (u_i, v_j)$, where $1 \leq i,j \leq n+1$ and also define $V(G) = \{W_{i,j} : 1 \leq i,j \leq n+1\}$, $|E(G)| = 2(n^2+n)$.

Case (i) $n \equiv 3(mod\ 4), n = 4m-1 (m \in Z^+)$.

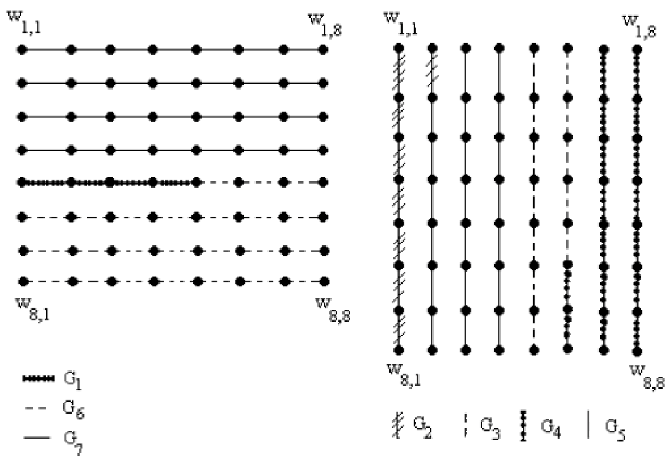
First define, $G_n = \{(W_{i,j}, V_{i,j+1}) : 1 \leq i \leq 4, 1 \leq j \leq n\}$ and define for $1 \leq k \leq \frac{n-3}{4}$.

$$\begin{aligned}
G_k &= \{(W_{i,j}, V_{i,j+1}) : i = 4k+1, 1 \leq j \leq 4k\} \\
G_{n-k} &= \{(W_{i,j}, W_{i,j+1}) : i = 4k+1, 4k+1 \leq j \leq n \text{ and} \\
&\qquad 4k+2 \leq i \leq 4(k+1), 1 \leq j \leq n\}
\end{aligned}$$

Also, define for $1 \leq \mathcal{L} \leq \dfrac{n+1}{4}$ and $k = \dfrac{n-3}{4}$.

$$G_{\mathcal{L}+k} = \{(W_{i,j}, V_{i+1,j}) : j = 4\mathcal{L}-3, 1 \le i \le n \text{ and}$$
$$j = 4\mathcal{L}-2, 1 \le i \le 4\mathcal{L}-3\}$$
$$G_{n-(\mathcal{L}+k)} = \{(W_{i,j}, W_{i+1,}) : 4\mathcal{L}-2 \le i \le n, j = 4\mathcal{L}-2 \text{ and}$$
$$1 \le i \le n, 4\mathcal{L}-1 \le j \le 4\mathcal{L}\}$$

Clearly, $\{G_1, G_2, \ldots, G_n\}$ is a $(4,4)-ASD$ of $P_{n+1} \times P_{n+1}$ (See Fig. 4.1).



$P_8 \times P_8$

Fig. 4.1

Case (ii) $n \equiv 0(mod\ 4), n = 4m(m \in Z^+)$.

First define, $G_n = \{(W_{i,j}, W_{i,j+1}) : 1 \le i \le 4, 1 \le j \le n\}$ and define for $1 \le k \le \frac{n-4}{4}$.

$$G_k = \{(W_{i,j}, W_{i,j+1}) : i = 4k+1, 1 \le j \le 4k\}$$
$$G_{n-k} = \{(W_{i,j}, W_{i,j+1}) : i = 4k+1, 4k+1 \le j \le n \text{ and}$$
$$4k+2 \le i \le 4(k+1), 1 \le j \le n\}$$

Define for $1 \le \mathcal{L} \le \frac{n-4}{4}$ and $p = \frac{n-4}{4}$.

$$G_{\mathcal{L}+p+1} = \{(W_{i,j}, W_{i+1,j}) : j = 4\mathcal{L}, 1 \le i \le n \text{ and}$$
$$j = 4\mathcal{L}+1, 1 \le i \le 4\mathcal{L}\}$$
$$G_{n-(\mathcal{L}+p+1)} = \{(W_{i,j}, W_{i+1,j}) : 4\mathcal{L}+1 \le i \le n, j = 4\mathcal{L}+1 \text{ and}$$
$$1 \le i \le n, 4\mathcal{L}+2 \le j \le 4l+3\}$$
$$G_{(p+1)} = \{(W_{i,j}, W_{i+1,j}) : i = n+1, 1 \le j \le n\} \text{ and}$$
$$G_{n-(p+1)} = \{(W_{i,j}, W_{i+1,j}) : 1 \le i \le n, 1 \le j \le 3\}$$

Finally define $G_{n/2} = \{(W_{i,j}, W_{i+1,j}) : 1 \leq i \leq n, n \leq j \leq n+1\}$. From the above construction clearly, $\{G_1, G_2, \ldots, G_n\}$ is a $(4,4) - ASD$ of $P_{n+1} \times P_{n+1}$ (See Fig. 4.2).

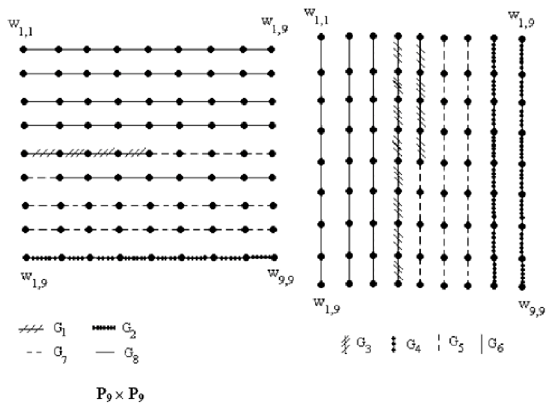


Fig 4.2

Case (iii) $n \equiv 1(mod\ 4), n = 4m+1 (m \in Z^+)$.

First define,

$$\begin{aligned} G_n = &\{(W_{i,j+1}W_{i,j}W_{i+1,j}) : i = 1, j = 1\} \\ &\cup \{(W_{i,i-1}W_{i,i}W_{i,i+1}W_{i-1,i}W_{i,i}W_{i+1,i}) : 2 \leq i \leq n\} \\ &\cup \{(W_{i,j-1}W_{i,j}W_{i-1,j}) : i = n+1, j = n+1\} \end{aligned}$$

Define for $1 \leq r \leq \dfrac{n-5}{2}$

$$\begin{aligned} G_{n-2r} = &\{(W_{i,j+1}W_{i,j}W_{i+1,j}) : i = 1, j = 2r+1\} \\ &\cup \{(W_{i,j-1}W_{i,j}W_{i,j+1}W_{i-1,j}W_{i,j}W_{i+1,j}) : 2 \leq i \leq n-2r \text{ and } j = 2r+i\} \\ &\cup \{(W_{i,j}W_{i+1,j}W_{i+1,j-1}) : i = n-2r \text{ and } j = n+1\} \end{aligned}$$

Also, define for $r = \dfrac{n-3}{2}$,

$$\begin{aligned} G'_2 = &\{(W_{i,j}W_{i,j+1}W_{i,j}W_{i+1,j}) : i = 3, j = 2r+1\} \\ &\cup \{(W_{i,j}W_{i+1,j}W_{i+1,j-1}) : i = n-2r, j = n+1\} \\ G'_3 = &\{(W_{i,j+1}W_{i,j}W_{i+1,j}) : i = 1, j = 2r+1\} \\ &\cup \{(W_{i,j-1}W_{i,j}W_{i,j+1}W_{i-1,j}W_{i,j}W_{i+1,j}) : i = 2, j = 2r+i\} \end{aligned}$$

Define for $1 \leq k \leq \dfrac{n-3}{2}$

$$\begin{aligned} G'_{n-2k-1} = &\{(W_{i+1,j}W_{i,j}W_{i,j+1}) : i = 1, j = 2k+1\} \\ &\cup \{(W_{i-1,j}W_{i,j}W_{i+1,j}W_{i,j-1}W_{i,j}W_{i,j+1}) : i = 2k+j \text{ and } \\ &\quad 2 \leq j \leq n-2k-2\} \\ &\cup \{(W_{i,j}W_{i,j+1}W_{i-1,j+1}) : i = n+1, j = n-2k-2\} \end{aligned}$$

Define

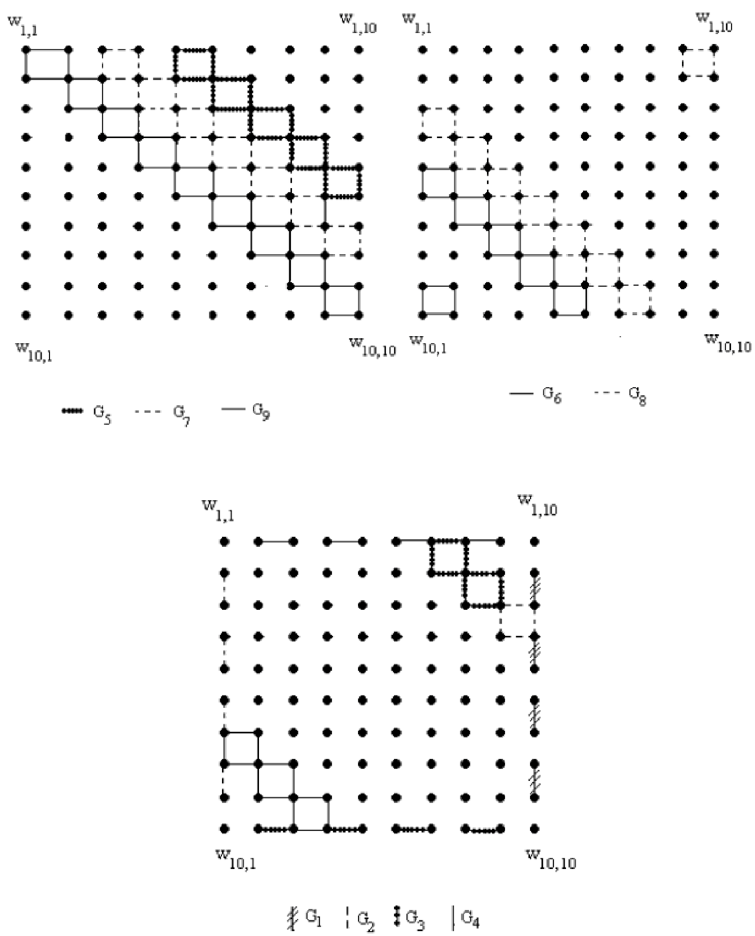
$C_1 = (W_{1,n}, W_{2,n}, W_{2,n+1}, W_{1,n+1}, W_{1,n})$,

$C_2 = (W_{n,1}, W_{n+1,n}, W_{n+1,2}, W_{n,2}, W_{n,1})$ and

$M = \{(W_{i,j}, W_{i,j+1}) : i = 1, n+1 \text{ and } j \equiv 0(mod\ 2)\}$

$\quad \cup\{(W_{i,j}, W_{i+1,j}) : j = 1, n+1 \text{ and } i \equiv 0(mod\ 2)\}$.

Let $G_{n-1} = G'_{n-1} \cup C_1$ and $G_{n-3} = G'_{n-3} \cup C_2$. Define $G_1 = M_0, G_2 = G'_2 \cup M_1, G_3 = G'_3 \cup M_2$ and $G_{n-2k+1} = G'_{n-2k-1} \cup M_k$, where $3 \leq k \leq \frac{n-3}{2}$ and $M_i \cong 4K_2$ are suitably chosen from M in order to form $G_1, G_2, \ldots, G_n$ as $(4,4) - ASD$ (See Fig 4.3).



$\mathbf{P_{10} \times P_{10}}$

Fig. 4.3

Case (iv) $n \equiv 2(mod\ 4), n = 4\mathcal{L} + 2(\mathcal{L} \in Z^+)$.

For $1 \leq m \leq \mathcal{L}$ and $m \equiv 1(mod\ 2)$, define

$$\begin{aligned} G_m &= \{(W_{i,j}, W_{i+1,j}) : 4m-3 \leq i \leq 4m-2, n+2-m \leq j \leq n+1\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 4m+1 \leq i \leq 4m-2, n+2-m \leq j \leq n+1\} \text{ and} \\ G_{n-(m-1)} &= \{(W_{i,j}, W_{i+1,j}) : 4m-3 \leq i \leq 4m-2 \text{ and } 1 \leq j \leq n+1-m\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 4m+1 \leq i \leq 4m+2 \text{ and } 1 \leq j \leq n+1-m\}. \end{aligned}$$

For $1 \leq m \leq \mathcal{L}$ and $m \equiv 0(mod\ 2)$, define

$$\begin{aligned} G_m &= \{(W_{i,j}, W_{i+1,j}) : 4m-5 \leq i \leq 4m-4, n+m-2 \leq j \leq n+1\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 4m-1 \leq i \leq 4m, n+m-2 \leq j \leq n+1\} \text{ and} \\ G_{n-(m-1)} &= \{(W_{i,j}, W_{i+1,j}) : 4m-5 \leq i \leq 4m-4 \text{ and } 1 \leq j \leq n+m-3\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 4m-1 \leq i \leq 4m \text{ and } 1 \leq j \leq n+m-3\}. \end{aligned}$$

For $1 \leq m \leq \mathcal{L}$ and $m \equiv 1(mod\ 2)$, define

$$\begin{aligned} G_{m+\mathcal{L}} &= \{(W_{i,j}, W_{i,j+1}) : n-m-\mathcal{L}+2 \leq i \leq n+1 \text{ and} \\ &\quad 4m-3 \leq j \leq 4m-2\} \\ &\cup \{(W_{i,j}, W_{i,j+1}) : n-m-\mathcal{L}+2 \leq i \leq n+1 \text{ and} \\ &\quad 4m+1 \leq j \leq 4m+2\} \text{ and} \\ G_{n-(m+\mathcal{L}+1)} &= \{(W_{i,j}, W_{i+1,j}) : 1 \leq i \leq n-m-\mathcal{L}+1 \text{ and } 4m-3 \leq j \leq 4m-2\} \\ &\cup \{(W_{i,j}, W_{i+1,j}) : 1 \leq i \leq n-m-\mathcal{L}+1 \text{ and } 4m+1 \leq i \leq 4m+2\} \end{aligned}$$

and for $1 \leq m \leq \mathcal{L}$ and $m \equiv 0(mod\ 2)$,

$$\begin{aligned} G_{m+\mathcal{L}} &= \{(W_{i,j}, W_{i,j+1}) : n-m-\mathcal{L}+3 \leq i \leq n+1 \text{ and } 4m-5 \leq j \leq 4m-4\} \\ &\cup \{(W_{i,j}, W_{i,j+1}) : n-m-\mathcal{L}+3 \leq i \leq n+1 \text{ and } 4m-1 \leq j \leq 4m\} \text{ and} \\ G_{n-(m+\mathcal{L}+1)} &= \{(W_{i,j}, W_{i,j+1}) : 1 \leq i \leq n-m-\mathcal{L}-2 \text{ and } 4m-5 \leq j \leq 4m-4\} \\ &\cup \{(W_{i,j}, W_{i,j+1}) : 1 \leq i \leq n-m-\mathcal{L}+2 \text{ and } 4m-1 \leq j \leq 4m\}. \end{aligned}$$

When $\mathcal{L}$ is even, define

$$\begin{aligned} G_{(n/2)} &= \{(W_{i,j}, W_{i,j+1}) : 2 \leq i \leq n+1, n-1 \leq j \leq n\} \text{ and} \\ G_{(n/2)+1} &= \{(W_{i,j}, W_{i+1,j}) : n-1 \leq i \leq n, 1 \leq j \leq n\} \\ &\cup \{(W_{i,j}, W_{i,j+1}) : i = 1, n-1 \leq j \leq n\}. \end{aligned}$$

When $\mathcal{L}$ is odd, define

$$\begin{aligned} G_{(n/2)} &= \{(W_{i,j}, W_{i,j+1}) : 2 \leq i \leq n+1, n-3 \leq j \leq n-2\} \text{ and} \\ G_{(n/2)+1} &= \{(W_{i,j}, W_{i+1,j}) : n-3 \leq i \leq n-2 \text{ and } 1 \leq j \leq n+1\}. \end{aligned}$$

From the above construction clearly, $\{G_1, G_2, \ldots, G_n\}$ is a $(4,4)-ASD$ of $P_{n+1} \times P_{n+1}$. See Fig. 4.4(a) and Fig. 4.4(b). $\square$

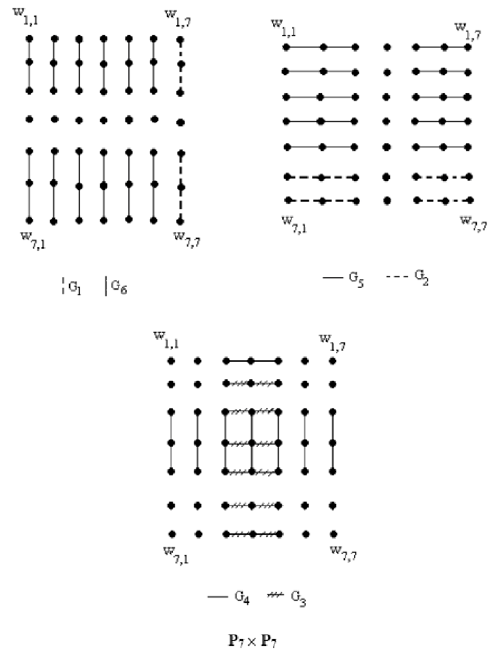


Fig. 4.4(a)

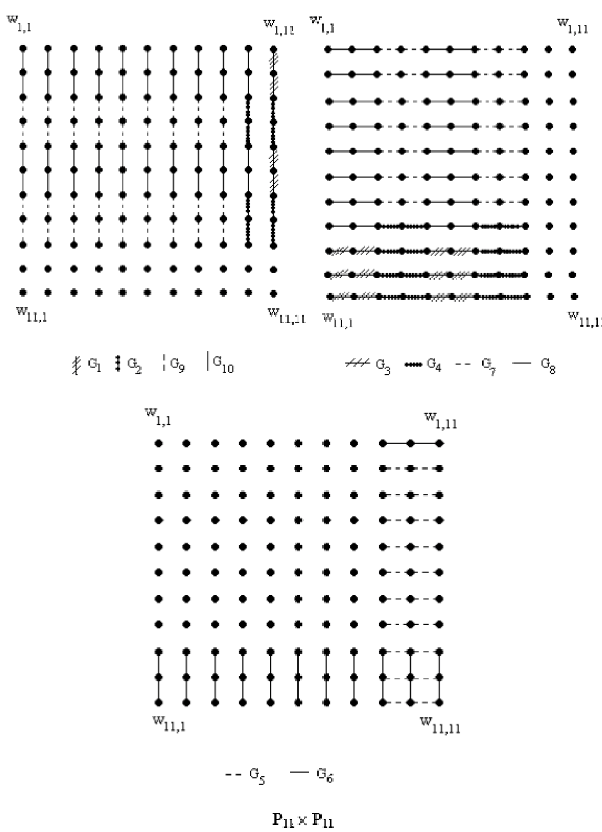


Fig. 4.4(b)

§5. $(a, d) - ASD$ on Some Special Graphs

In this section $(a, d) - ASD$ is established for some special graphs like wheel, Carona and a special type in caterpillar.

Theorem 5.1 *$W_{n^2+1} = K_1 + C_{n^2} (n \geq 3)$ has $(a, d) - ASD$, $d \geq a$ if and only if $a = 2$ and $d = 4$.*

Proof Suppose W_{n^2+1} has $(a, d) - ASD$, $d \geq a$, By (ii) of Lemma 2.5, $|E(W_{n^2+1})| = a\begin{pmatrix} n+1 \\ 2 \end{pmatrix} + (d-a)\begin{pmatrix} n \\ 2 \end{pmatrix}$, also we have $|E(W_{n^2+1})| = 2n^2$.

From the above relations, we have $a = 2$ and $d = 4$. Conversely, let $V(W_{n^2+1}) = \{u_1, v_1, v_2, \ldots, v_{n^2})$. Define $G_1 = (u_1, v_1) \cup (v_1, v_2)$ and

$$G_i = \left\{((u_i, v_j) \cup (v_j, v_{j+1})) : \sum_{k=1}^{i-1}(2k-1) \leq j \leq \sum_{k=1}^{i}(2k-1)\right\}.$$

for $2 \leq i \leq n$. Where addition is taken modulo n^2 with residues $1, 2, 3, \ldots, n^2$ instead of the usual residues $0, 1, 2, \ldots, n^2 - 1$. Then clearly, $G_i \subseteq G_{i+1}, 1 \leq i \leq n-1$ and $|E(Gi)| = 2(2i-1)$ for $1 \leq i \leq n$. Hence, $\{G_1, G_2, \ldots, G_n\}$ is a $(2, 4) - ASD$ of W_{n^2+1}. $\square$

Example 5.2 A decomposition of W_{n^2+1}, where $n = 3$ into $(2, 4) - ASD$ is illustrated in Fig. 5.1. Clearly, $\{G_1, G_2, G_3\}$ is a $(2, 4) - ASD$.

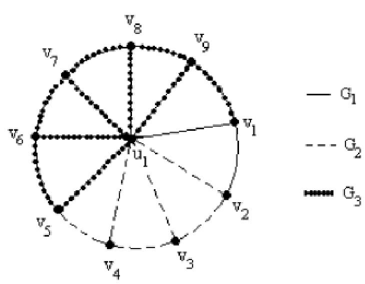


Fig. 5.1

Definition 5.3 *Let $T = S(v_1, v_2, \ldots, v_{n-1}, v_n, v_{n+1})$ be a caterpillar where v_i means n leaves attached to each vertex and v_{n+1} means no leaf attached to the last vertex.*

Theorem 5.4 *The caterpillar $T = S(v_0, v_1, v_2, \ldots, v_{n-1}, v_n)$ has an $(a, d) - ASD$, $(d \geq a)$ if and only if $a = 2$ and $d = 2$.*

Proof Suppose T admits (a,d) - ASD ($d \geq a$) By (ii) of Lemma 2.5, $|E(T)| = a\begin{pmatrix} n+1 \\ 2 \end{pmatrix} + (d-a)\begin{pmatrix} n \\ 2 \end{pmatrix}$. Also, $|E(T)| = (n+1)n = n^2+1 = 2\begin{pmatrix} n+1 \\ 2 \end{pmatrix}$. From the above two relations, we have $a = 2$ and $d = 2$.

Conversely, suppose $a = 2$, $d = 2$. Let

$$V(G) = \{v_1, v_2, \ldots, v_n, v_{n+1}\} \cup \left\{v_1^{(k)}, v_2^{(k)}, \ldots, v_n^{(k)} : 1 \le k \le n\right\},$$

where v_i are vertices on the path P_n and $v_j^{(k)}(1 \le k \le n)$ are the vertices of the star at each $v_j(1 \le j \le n)$. Define for $1 \le k \le n, T_k = \{(v_k, v_{k+1})\} \cup \left\{\left(v_k, v_j^{(k)}\right) : 1 \le j \le n\right\}$.

Case (i) When n is odd, $n = 2m + 1$.

Decompose T_k for $k \equiv 0, 1(mod\ 2)$ into G_m and $G_{n-(m-1)}, 1 \le m \le \frac{n-1}{2}$. Where

$$G_m = \{(v_{2k}, v_{2k+1})\} \bigcup \left\{\left(v_{k+1}, v_j^{(k+1)}\right) : n - (2k - 2) \le j \le n\right\}$$

and

$$G_{n-(m-1)} = \left\{\left(v_{k+1}, v_j^{(k)}\right) : 1 \le j \le n - (2k - 1)\right\} \bigcup \{(v_{2k-1}, v_{2k})\} \bigcup \left\{\left(v_k, v_j^{(k)}\right) : 1 \le j \le n\right\}.$$

Define $G_{\frac{n+1}{2}} = \{(v_n, v_{n+1})\} \cup \left\{\left(v_n, v_j^{(n)}\right) : 1 \le j \le n\right\}$. Clearly $G_i \subseteq G_{i+1}, 1 \le i \le n - 1$ and $|E(G_i)| = 2i, 1 \le i \le n$. Hence $\{G_1, G_2, \ldots, G_n\}$ is a $(2, 2) - ASD$ of T.

Case (ii) When n is even, $n = 2m$.

Decompose T_k for $k \equiv 0, 1(mod\ 4)$ into G_m and $G_{n-(m-1)}, 1 \le m \le \frac{n}{2}$ as in Case (i). Clearly $G_i \subseteq G_{i+1}, 1 \le i \le n - 1$. Hence $\{G_1, G_2, \dot{;} G_n\}$ is a $(2, 2) - ASD$ of T. $\square$

Corollary 5.5 *The corona $C_n \odot nK_1$ has $(a, d) - ASD$, $(d \ge a)$ if and only if $a = 2$ and $d = 2$.*

Proof By taking $v_{n+1} = v_1$ in $T = S(v_1, v_2, \ldots, v_n, v_{n+1})$. We have $T = C_n \odot nK_1$. $\square$

References

[1] Y. Alavi, A.J. Boals, G. Chartrand, P.Erdos and O.R. Oellermann, The ascending Subgraph decomposition problem, *Congressus Numerantium*, **58**(1987), 7-14.

[2] J. Bosak, Decomposition of graphs, *Mathematics and its applications*, (East European series) Vol 47, Kluwer Academic Publications, 1980.

[3] R.J. Faudree, A. Gyarfas and R.H. Schelp, Graphs which have an ascending subgraph decomposition, *Congressus Numerantium*, **59**(1987), 49-54.

[4] F. Harary, *Graph Theory*, Addition - Wesley Publishing company Inc, USA,1969.

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Smarandachely Roman Edge s-Dominating Function

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Abstract: For an integer $n \geq 2$, let $I \subset \{0, 1, 2, \cdots, n\}$. A *Smarandachely Roman s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph G = (V,E) is a function $f : V \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(u) - f(v)| \geq s$ for each edge $uv \in E$ with $f(u)$ or $f(v) \in I$. Similarly, a *Smarandachely Roman edge s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(e) - f(h)| \geq s$ for adjacent edges $e, h \in E$ with $f(e)$ or $f(h) \in I$. Particularly, if we choose $n = s = 2$ and $I = \{0\}$, such a Smarandachely Roman s-dominating function or Smarandachely Roman edge s-dominating function is called *Roman dominating function* or *Roman edge dominating function*. The Roman edge domination number $\gamma_{re}(G)$ of G is the minimum of $f(E) = \sum_{e \in E} f(e)$ over such functions. In this paper, we find lower and upper bounds for Roman edge domination numbers in terms of the diameter and girth of G.

Key Words: Smarandachely Roman s-dominating function, Smarandachely Roman edge s-dominating function, diameter, girth.

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§1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. As usual $|V| = n$ and $|E| = q$ denote the number of vertices and edges of the graph G, respectively. The open neighborhood $N(v)$ of the vertex v is the set $\{u \in V(G)|\ uv \in E(G)\}$ and its closed neighborhood $N[v] = N(v) \cup \{v\}$. Similarly, the open neighborhood of a set $S \subseteq V$ is the set $N[S] = \bigcup_{v \in S} N(v)$, and its closed neighborhood is $N(S) = N(S) \cup S$. The minimum and maximum vertex degrees in G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

The degree of an edge $e = uv$ of G is defined by $deg\ e = deg\ u + deg\ v - 2$ and $\delta'(G)$ ($\Delta'(G)$) is the minimum (maximum) degree among the edges of G (the degree of a edge is the

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number of edges adjacent to it). A vertex of degree one is called a pendant vertex or a leaf and its neighbor is called a support vertex.

A set $D \subseteq V$ is said to be a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. The minimum cardinality of such a set is called the domination number of G and is denoted by $\gamma(G)$. For a complete review on the topic of domination and its related parameters, see [5].

Mitchell and Hedetniemi in [6] introduced the notion of edge domination as follows. A set F of edges in a graph G is an edge dominating set if every edge in $E - F$ is adjacent to at least one edge in F. The minimum numbers of edges in such a set is called the edge domination number of G and is denoted by $\gamma_e(G)$. This concept is also studied in [1].

For an integer $n \geq 2$, let $I \subset \{0, 1, 2, \cdots, n\}$. A *Smarandachely Roman s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph G = (V,E) is a function $f : V \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(u) - f(v)| \geq s$ for each edge $uv \in E$ with $f(u)$ or $f(v) \in I$. Similarly, a *Smarandachely Roman edge s-dominating function* for an integer s, $2 \leq s \leq n$ on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2, \cdots, n\}$ satisfying the condition that $|f(e) - f(h)| \geq s$ for adjacent edges $e, h \in E$ with $f(e)$ or $f(h) \in I$. Particularly, if we choose $n = s = 2$ and $I = \{0\}$, such a Smarandachely Roman s-dominating function or Smarandachely Roman edge s-dominating function is called *Roman dominating function* or *Roman edge dominating function*.

The concept of Roman dominating function (RDF) was introduced by E. J. Cockayne, P. A. Dreyer, S. M. Hedetniemi and S. T. Hedetniemi in [3]. (See also [2,4,7]). A Roman dominating function on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $f(V) = \sum_{u \in V} f(u)$. The Roman domination number of a graph G, denoted by $\gamma_R(G)$, equals the minimum weight of a Roman dominating function on G.

A *Roman edge dominating function* (REDF) on a graph $G = (V, E)$ is a function $f : E \to \{0, 1, 2\}$ satisfying the condition that every edge e for which $f(e) = 0$ is adjacent to at least one edge h for which $f(h) = 2$. The weight of a Roman edge dominating function is the value $f(E) = \sum_{e \in E} f(e)$. The Roman edge domination number of a graph G, denoted by $\gamma_{re}(G)$, equals the minimum weight of a Roman edge dominating function on G. This concept is also studied in Soner et al. in [8]. A $\gamma - set$, $\gamma_r - set$ and γ_{re}-set, can be defined as a minimum dominating set (MDS), a minimum Roman dominating set (MRDS) and a minimum Roman edge dominating set (MREDS), respectively.

The purpose of this paper is to establish sharp lower and upper bounds for Roman edge domination numbers in terms of the diameter and the girth of G.

Soner et al. in [8] proved that:

Theorem A *For a graph G of order p,*

$$\gamma_e(G) \leq \gamma_{re}(G) \leq 2\gamma_e(G).$$

Theorem B *For cycles C_p with $p \geq 3$ vertices,*

$$\gamma_{re}(C_p) = \lceil 2p/3 \rceil.$$

Here we observe the following properties.

Property 1 *For any connected graph G with $p \geq 3$ vertices,*

$$\gamma_{re}(G) = \gamma_r(L(G)).$$

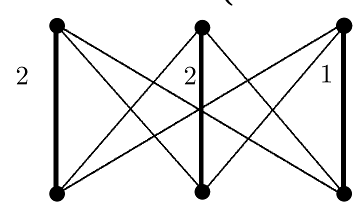
Property 2 *a) If an edge e has degree one and h is adjacent to e, then every such h must be in every REDS of G.*

b) For the path graph P_k with $k \geq 2$ vertices,

$$\gamma_{re}(P_k) = \lfloor 2k/3 \rfloor.$$

c) For the complete bipartite graph $K_{m,n}$ with $m \leq n$ vertices,

$$\gamma_{re}(K_{m,n}) = \begin{cases} \text{2m-1} & \text{if } m = n, \\ \text{2m} & \text{otherwise.} \end{cases}$$

ex: 

$$\gamma_{re}(K_{3,3}) = 5$$

d) $\gamma_{re}(G \cup H) = \gamma_{re}(G) + \gamma_{re}(H)$.

In the following theorem, we establish the result relating to maximum edge degree of G.

Theorem 1 *Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re} - function$ and G has no isolated edges, then*

$$2q/(\Delta'(G) + 1) - |E_1| \leq \gamma_{re}(G) \leq q - \Delta'(G) + 1.$$

Furthermore, equality hold for P_3, P_4, and C_3.

Proof Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re} - function$. Since E_2 dominates the set E_0, so $S = (E_1 \cup E_2)$ is a edge dominating set of G. Then

$$2|S|\Delta'(G) \geq 2\sum_{e\in S} deg(e) = 2\sum_{e\in S} |N(e)| \geq 2|\bigcup_{e\in S} N(e)| \geq 2|E - S| \geq 2q - 2|S|.$$

Thus

$$2q/(\Delta'(G) + 1) \leq 2|S| = 2(|E_1| + |E_2|) = |E_1| + \gamma_{re}(G).$$

Converse, let $deg\ \ e = \Delta'(G)$, if for every edge $x \in N(e)$ is adjacent to an edge h which is not adjacent to e. Then clearly, $E(G) - N(e) \cup h$ is an REDS. Thus $\gamma_{re}(G) \leq q - \Delta'(G) + 1$ follows. $\Box$

Corollary 1 *Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re} - function$ and G has no isolated edges. If $|E_1| = 0$, then*

$$2q/(\Delta'(G)+1) \leq \gamma_{re}(G) \leq q - \Delta'(G) + 1.$$

In this section sharp lower and upper bounds for $\gamma_{re}(G)$ in terms of diam(G) are presented. Recall that the eccentricity of vertex v is $ecc(v) = max\{d(u,v) : u \in V,\ u \neq v\}$ and the diameter of G is $diam(G) = max\{ecc(v) : v \in V\}$. Throughout this section we assume that G is a nontrivial graph of order $n \geq 2$.

Theorem 2 *If a graph G has diameter two, then $\gamma_{re}(G) \leq 2\delta'$. Further, the equality holds if $G = P_3$.*

Proof Since G has diameter two, $N(e)$ dominates $E(G)$ for all edge $e \in E(G)$. Now, let $e \in E(G)$ and $deg\ \ e = \delta'$. Define $f : E(G) \longrightarrow \{0,1,2\}$ by $f(e_i) = 2$ for $e_i \in N(e)$ and $f(e_i) = 0$ otherwise. Obviously f is a Roman edge dominating function of G. Thus $\gamma_{re}(G) \leq 2\delta'$. For P_3, $\gamma_{re}(P_3) = 2 = 2 \times 1$. □

Theorem 3 *For any connected graph G on n vertices,*

$$\lceil (diam(G)+1)/2 \rceil \leq \gamma_{re}(G)$$

With equality for P_n, $(2 \leq n \leq 5)$.

Proof The statement is obviously true for K_2. Let G be a connected graph with vertices $n \geq 3$. Suppose that $P = e_1e_2...e_{diam(G)}$ is a longest diametral path in G. By Theorem B, $\gamma_{re}(P) = \lceil 2diam(G)/3 \rceil$, and $\lceil (diam(G)+1)/2 \rceil < \lceil 2(diam(G)+1)/3 \rceil$, then $\lceil (diam(G)+1)/2 \leq \lceil 2diam(G)/3 \rceil \leq \gamma_{re}(P)$, let $f = (E_0, E_1, E_2)$ be a $\gamma_{re}(P) - function$. Define $g : E(G) \longrightarrow \{0,1,2\}$ by $g(e) = f(e)$ for $e \in E(P)$ and $g(h_i) \leq 1$ for $h_i \in E(G) - E(P)$, then $w(g) = w(f) + \sum_{h_i \in E(G)-E(P)} h_i$. Obviously g is a REDF for G and hence

$$\lceil (diam(G)+1)/2 \rceil \leq \gamma_{re}(G).$$ □

Theorem 4 *For any connected graph G on n vertices,*

$$\gamma_{re}(G) \leq q - \lfloor (diam(G)-1)/3 \rfloor.$$

Furthermore, this bound is sharp for C_n and P_n.

Proof Let $P = e_1e_2...e_{diam(G)}$ be a diametral path in G. Moreover, let $f = (E_0, E_1, E_2)$ be a $\gamma_{re}(P) - function$. By Property 2(b), the weight of f is $\lceil 2diam(G)/3 \rceil$. Define $g : E(G) \longrightarrow \{0,1,2\}$ by $g(e) = f(e)$ for $e \in E(P)$ and $g(e) = 1$ for $e \in E(G) - E(P)$. Obviously g is a REDF for G. Hence,

$$\gamma_{re}(G) \leq w(f) + (q - diam(G)) \leq q - \lfloor (diam(G)-1)/3 \rfloor.$$ □

Theorem 5([8]) *For any connected graph G on n vertices,*

$$\gamma_{re}(G) \leq n - 1$$

and equality holds if G is isomorphic to W_5, P_3, C_4, C_5, K_n and $K_{m,m}$.

Theorem 6 *For any connected graph G on n vertices,*

$$\gamma_{re}(G) \leq n - \lceil diam(G)/3 \rceil.$$

Furthermore, this bound is sharp for P_n. And equality hold for $K_{m,m}$, P_{3k}, $(k > 0)$, K_n, W_5, C_4 and C_5.

Proof The technic proof is same with that of Theorem 3. □

In this section we present bounds on Roman edge domination number of a graph G containing cycle, in terms of its grith. Recall that the grith of G (denoted by $g(G)$) is that length of a smallest cycle in G. Throughout this section, we assume that G is a nontrivial graph with $n \geq 3$ vertices and contains a cycle. The following result is very crucial for this section.

Theorem 7 *For a graph G of order n with $g(G) \geq 3$ we have $\gamma_{re}(G) \geq \lceil 2g(G)/3 \rceil$.*

Proof First note that if G is the n-cycle then $\gamma_{re}(G) = \lceil 2n/3 \rceil$ by Theorem B. Now, let C be a cycle of length $g(G)$ in G. If $g(G) = 3$ or 4, then we need at least 1 or 2 edges, to dominate the edges of C and the statement follows by Theorem A. Let $g(G) \geq 5$. Then an edge not in $E(G)$, can be adjacent to at most one edge of C for otherwise we obtain a cycle of length less than $g(G)$ which is a contradiction. Now the result follows by Theorem A. □

Theorem 8 *For any connected graph with n vertices, $\delta'(G) \geq 2$ and $g(G) \geq 3$. Then $\gamma_{re}(G) \geq n - \lfloor g(G)/3 \rfloor$. Furthermore, the bound is sharp for $K_{m,m}$, C_n, K_n and W_n.*

Proof Let G be a such graph with n-vertices, if we prove the $\gamma_{re}(C_n) \geq n - \lfloor g(C_n)/3 \rfloor$. Then this proof satisfying the any graph of order n. Since $g(C_n) \geq g(G)$ then $n - g(C_n) \leq n - g(G)$. By Theorem B, $\gamma_{re}(C_n) = \lceil 2n/3 \rceil = \lceil 2g(C_n)/3 \rceil = n - \lceil n/3 \rceil \leq n - \lfloor n/3 \rfloor \leq n - \lfloor g(G)/3 \rfloor$. □

Theorem 9 *For a simple connected graph G with n-vertices and $\delta' \leq 2$, if $g(G) \geq 5$, then $\gamma_{re}(G) \geq 2\delta'$. The bound is sharp for C_5 and C_6.*

Proof Let G be such a graph and C be a cycle with $g(G)$ edges. If $n = 5$, then G is a $5 - cycle$ and $\gamma_{re}(G) = 4 = 2\delta'$. For $n \geq 6$, since $\delta' \leq 2$, then $\gamma_{re}(G) \geq \lceil 2g(G)/3 \rceil \geq 2\delta'$ by Theorem 7. □

Theorem 10 *Let T be any tree and let $e = uv$ be an edge of maximum degree Δ'. If $1 < diam(G) \leq 5$ and $degw \leq 2$ for every vertex $w \neq u, v$, then $\gamma_{re}(G) = q - \Delta' + 1$.*

Proof Let T be a tree with $diam(T) \leq 4$ and $degw \leq 2$ for every vertex $w \neq u, v$, where $e = uv$ is an edge of maximum degree in T. If $diam(T) = 2$ or 3, then $\gamma_{re}(G) = q - \Delta' + 1 = 2$. If $diam(T) = 4$ or 5, then each non-pendent edge of T is adjacent to a pendent edge of T and hence the set $E_1 \cup E_2$ of all non-pendent edges of T forms a minimum edge dominating set and $\gamma_{re}(G) = |E_1| + 2|E_2| = q - \Delta' + 1$. □

Theorem 11([8]) *Let G be a tree or a unicyclic graph, then $\gamma_{re}(G) \leq \gamma_r(G)$.*

Theorem 12 *Let T is an $n-vertex$ tree, with $n \geq 2$, then $\gamma_{re}(T) \leq 2n/3$. The bound is sharp for P_n.*

Proof We use induction on n. The statement is obviously true for K_2. If $diamT = 2$ or 3, then T has a dominating edge, and $\gamma_{re}(T) \leq 2 \leq 2n/3$.

Hence we may assume that $diamT \geq 4$. For a subtree T' with n' vertices, where $n' \geq 2$, the induction hypothesis yields an REDF f' of T' with weight at most $2n'/3$. We find a subtree T' such that adding a bit more weight to f' will yield a small enough REDF f for T.

Let P be a longest path in T chosen to maximize the degree of its next-to-last vertex v, and let u be the non-leaf neighbor of v and let $h = uv$.

Case 1. Let $deg_T(v) > 2$. Obtain T' by deleting v and its leaf neighbors. Since $diamT \geq 4$, we have $n' \geq 2$. Define f on $E(T)$ by $f(e) = f'(e)$ except for $f(h) = 2$ and $f(e) = 0$ for each edge e adjacent to h. Not that f is an RDF for T and that $w(f) = w(f') + 2 \leq 2(n-3)/3 + 2 \leq 2n/3$.

Case 2. Let $deg_T(v) = deg_T(u) = 2$. Obtain T' by deleting v and u and the leaf neighbor z of v. Since $diamT \geq 4$, we have $n' \geq 2$. If $n' = 2$, then T is P_5 and has an REDF of weight 3. Otherwise, the induction hypothesis applies. Define f on $E(T)$ by letting $f(e) = f'(e)$ except for $f(h) = 2$ and $f(e) = 0$ for each edge e adjacent to h. Again f is an REDF, and the computation $w(f) < 2n/3$ is the same as in Case 1.

Case 3. Let $deg_T(u) > 2$ and every penultimate neighbor of u has degree 2. Obtain T' by deleting v and its leaf neighbors and u. Define f on $E(T)$ by $f(e) = f'(e)$ except for $f(h) = 2$ and $f(e) = 0$ for each edge e adjacent to h. Not that f is an RDF for T and that $w(f) = w(f') + 2 \leq 2(n-3)/3 + 2 \leq 2n/3$. If some neighbor of u is a leaf. Obtain T' by deleting v and its leaf neighbors and u and its leaf neighbors. Define f on $E(T)$ by $f(e) = f'(e)$ except for $f(h) = 2$ and $f(e) = 0$ for each edge e adjacent to h. Not that f is an RDF for T and that $w(f) = w(f') + 2 \leq 2(n-3)/3 + 2 \leq 2n/3$. From the all cases above $w(f) = w(f') + 2 \leq 2(n-3)/3 + 2 \leq 2n/3$. This completes the proof. □

Corollary 2 *Let T is an $q-edge$ tree, with $q \geq 1$, then $\gamma_{re}(T) \leq 2(q+1)/3$.*

Theorem 13 *Let $f = (E_0, E_1, E_2)$ be any $\gamma_{re}(T)-function$ of a connected graph T of $q \geq 2$. Then*

(1) $1 \leq |E_2| \leq (q+1)/3$;

(2) $0 \leq |E_1| \leq 2q/3 - 4/3$;

(3) $(q+1)/3 \leq |E_0| \leq q - 1$.

Proof By Theorem 12, $|E_1| + 2|E_2| \leq 2(q+1)/3$.

(1) If $E_2 = \varnothing$, then $E_1 = q$ and $E_0 = \varnothing$. The REDF $(0, q, 0)$ is not minimum since $|E_1| + 2|E_2| > 2(q+1)/3$. Hence $|E_2| \geq 1$. On the other hand, $|E_2| \leq (q+1)/3 - |E_1|/2 \leq (q+1)/3$.

(2) Since $|E_2| \geq 1$, then $|E_1| \leq 2(q+1)/3 - 2|E_2| \leq 2(q+1)/3 - 2 = 2q/3 - 4/3$.

(3) The upper bound comes from $|E_0| \leq q - |E_2| \leq q - 1$. For the lower bound, adding on both side $2|E_0| + 2|E_1| + 2|E_2| = 2q$, $-|E_1| - 2|E_2| \geq -2(q+1)/3$ and $-|E_1| \geq -2(q+1)/3 + 2$

gives $2|E_0| \geq (2q+2)/3$. Therefor, $|E_0| \geq (q+1)/3$. $\square$

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References

[1] S. Arumugam and S. Velamal, Edge domination in graphs, *Taiwanese journal of Mathematics*, 2(1998)173-179.

[2] E. W. Chambers, B. Kinnersley, N. Prince and D. B. West, Extremal problems for Roman domination, *Discrete Math.*, 23(2009)1575-1586.

[3] E. J. Cockayne, P. A. Dreyer Jr, S. M. Hedetniemi and S. T. Hedetniemi, Roman domination in graphs, *Discrete Math.*, 278(2004)11-22.

[4] O. Favaron, H. Karami, R. Khoeilar and S. M. Sheikholeslami, On the Roman domination number of a graph, *Discrete Math.*, 309(2009)3447-3451.

[5] T. W. Haynes, S. T Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc, New York,(1998).

[6] S. Mitchell and S.T. Hedetniemi, Edge domination in tree, *Proc* 8^{th} *SE Conference on Combinatorics, Graph Theory and Computing*, 19(1977)489-509.

[7] B. P. Mobaraky and S. M. Sheikholeslami, Bounds on Roman domination numbers of graphs, *Discrete Math.*, 60(2008)247-253.

[8] N. D. Soner, B. Chaluvaraju and J. P. Srivastava, Roman edge domination in graphs, *Proc. Nat. Acad. Sci. India Sect.* A, 79(2009)45-50.

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Euler-Savary Formula
for the Lorentzian Planar Homothetic Motions

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Abstract: One-parameter planar homothetic motion of 3-lorentzian planes, two are moving and one is fixed, have been considered in ref. [19]. In this paper we have given the canonical relative systems of a plane with respect to other planes so that the plane has a curve on it, which is spacelike or timelike under homothetic motion. Therefore, Euler-Savary formula giving the relation between curvatures of the trajectory curves drawn on the points on moving L and fixed plane L' is expressed separately for the cases whether the curves are spacelike or timelike. As a result it has been found that Euler-Savary formula stays the same whether these curves are spacelike or timelike. We have also found that if homothetic scala h is equal to 1 then the Euler-Savary formula becomes an equation which exactly the same is given by ref. [6].

Key Words: Homothetic Motion, Euler-Savary Formula, Lorentz Plane, kinematics, Smarandache Geometry.

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§1. Introduction

We know that the angular velocity vector has an important role in kinematics of two rigid bodies, especially one Rolling on another, [15] and [16]. To investigate to geometry of the motion of a line or a point in the motion of plane is important in the study of planar kinematics or planar mechanisms or in physics. Mathematicians and physicists have interpreted rigid body motions in various ways. K. Nomizu [16] has studied the 1-parameter motions of orientable surface M on tangent space along the pole curves using parallel vector fields at the contact points and he gave some characterizations of the angular velocity vector of rolling without sliding. H.H. Hacısalihoğlu showed some properties of 1-parameter homothetic motions in Euclidean space [8]. The geometry of such a motion of a point or a line has a number of applications in geometric modeling and model-based manufacturing of the mechanical products or in the design of robotic motions. These are specifically used to generate geometric models of shell-type objects and thick surfaces, [4,7,17].

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As a model of spacetimes in physics, various geometries such as those of Euclid, Riemannian and Finsler geometries are established by mathematicians.

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969), i.e., an axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways, [11, 18].

In the Euclidean geometry, also called parabolic geometry, the fifth Euclidean postulate that there is only one parallel to a given line passing through an exterior point, is kept or validated. While in the Riemannian geometry, called elliptic geometry, the fifth Euclidean postulate is also invalidated as follows: there is no parallel to a given line passing through an exterior point [11].

Thus, as a particular case, Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries may be united altogether, in the same space, by some Smarandache geometries. These last geometries can be partially Euclidean and partially Non-Euclidean. Howard Iseri [10] constructed a model for this particular Smarandache geometry, where the Euclidean fifth postulate is replaced by different statements within the same space, i.e. one parallel, no parallel, infinitely many parallels but all lines passing through the given point, all lines passing through the given point are parallel. Linfan Mao [12,13] showed that Smarandache geometries are generalizations of Pseudo-Manifold Geometries, which in their turn are generalizations of Finsler Geometry, and which in its turn is a generalization of Riemann Geometry.

The Euler-Savary theorem is a well-known theorem and studied systematically in two and three dimensional Euclidean space E^2 and E^3 by [2,3,14]. This theorem is used in serious fields of study in engineering and mathematics. For each mechanism type a simple graphical procedure is outlined to determine the circles of inflections and cusps, which are useful to compute the curvature of any point of the mobile plane through the Euler-Savary equation. By taking Lorentzian plane L^2 instead of Euclidean plane E^2, Ergin [5] has introduced 1-parameter planar motion in Lorentzian plane. Furthermore he gave the relation between the velocities, accelerations and pole curves of these motions. In the L^2 Lorentz plane Euler-Savary formula is given in references, [1], [6] and [9].

Let L (moving), L' (fixed) be planes and the coordinate systems of these planes be $\{O;\vec{e}_1, \vec{e}_2(\text{timelike})\}$ and $\{O';\vec{e}'_1,\vec{e}'_2(\text{timelike})\}$, respectively. Therefore, one-parameter Lorentzian planar homothetic motion is defined by the transformation [19]

$$\vec{x}' = h\vec{x} - \vec{u}, \tag{1}$$

where h is homothetic scale, $\overrightarrow{OO'} = \vec{u}$, is vector combining the systems (fixed and moving) initial points and the vectors $\vec{X}$, $\vec{X}'$ show the position vectors of the point $X \in L$ with respect to moving and fixed systems, respectively. In the one-parameter Lorentzian planar homothetic motion the relation

$$\vec{V}_a = \vec{V}_f + h\vec{V}_r$$

holds where $\vec{V}_a$, $\vec{V}_f$ and $\vec{V}_r$ represent to absolute, sliding and relative velocity of the motion, respectively [19].

We have given the canonical relative systems of a plane with respect to others planes so that the plane has a curve on it which is spacelike or timelike under homothetic motions. Thus

Euler-Savary formula, which gives the relation between the curvatures of the trajectory curves drawn an the points of moving plane L and fixed plane L', is expressed separately for the cases whether the curves are spacelike or timelike. Finally it has been observed that Euler-Savary formula does not change whether these curves are spacelike or timelike and if homothetic scale is equal to 1 then the Euler-Savary formula takes the form in reference [6].

§2. Moving Coordinate Systems and Their Velocities

Let L_1, L be the moving planes and L' be the fixed plane. The perpendicular coordinate systems of the planes L_1, L and L' are $\{B;\vec{a}_1,\vec{a}_2\}$, $\{O;\vec{e}_1,\vec{e}_2\}$ and $\{O';\vec{e}_1',\vec{e}_2'\}$, respectively. Suppose that θ and θ' are the rotation angles of one parameter Lorentzian homothetic motions of L_1 with respect to L and L', respectively. Therefore, in one parameter Lorentzian homothetic motions L_1/L and L_1/L' following relations are holds

$$\begin{aligned}\vec{a}_1 &= \cosh\theta\vec{e}_1+\sinh\theta\vec{e}_2\\ \vec{a}_2 &= \sinh\theta\vec{e}_1+\cosh\theta\vec{e}_2\end{aligned} \tag{2}$$

$$\overrightarrow{OB}=\vec{b}=b_1\vec{a}_1+b_2\vec{a}_2 \tag{3}$$

and

$$\begin{aligned}\vec{a}_1 &= \cosh\theta'\vec{e}_1'+\sinh\theta'\vec{e}_2'\\ \vec{a}_2 &= \sinh\theta'\vec{e}_1'+\cosh\theta'\vec{e}_2'\end{aligned} \tag{4}$$

$$\overrightarrow{O'B}=\vec{b}'=b_1'\vec{a}_1+b_2'\vec{a}_2 \tag{5}$$

respectively [19]. If we consider equations (2)-(3) and (4)-(5), then the differential equations for the motions L_1/L and L_1/L' are as follows, respectively [19]

$$\begin{aligned}&d\vec{a}_1=d\theta\vec{a}_2, \qquad d\vec{a}_2=d\theta\vec{a}_1\\ &d\vec{b}=(db_1+b_2d\theta)\,\vec{a}_1+(db_2+b_1d\theta)\,\vec{a}_2\end{aligned} \tag{6}$$

and

$$\begin{aligned}&d'\vec{a}_1=d\theta'\vec{a}_2, \qquad d'\vec{a}_2=d\theta'\vec{a}_1\\ &d'\vec{b}'=(db_1'+b_2'd\theta')\,\vec{a}_1+(db_2'+b_1'd\theta')\,\vec{a}_2.\end{aligned} \tag{7}$$

If we use the following abbreviations

$$\begin{aligned}&d\theta=\lambda, && d\theta'=\lambda'\\ &db_1+b_2d\theta=\sigma_1, && db_2+b_1d\theta=\sigma_2\\ &db_1'+b_2'd\theta'=\sigma_1', && db_2'+b_1'd\theta'=\sigma_2'\end{aligned} \tag{8}$$

then the differential equations for L_1/L and L_1/L' become

$$d\vec{a}_1=\lambda\vec{a}_2, \qquad d\vec{a}_2=\lambda\vec{a}_1, \qquad d\vec{b}=\sigma_1\vec{a}_1+\sigma_2\vec{a}_2 \tag{9}$$

and

$$d'\vec{a}_1 = \lambda'\vec{a}_2, \qquad d'\vec{a}_2 = \lambda'\vec{a}_1, \qquad d'\vec{b} = \sigma_1'\vec{a}_1 + \sigma_2'\vec{a}_2 \tag{10}$$

respectively. Here the quantities σ_j, σ_j', λ and λ' are Pfaffian forms of one parameter Lorentzian homothetic motion [19].

For the point X with the coordinates of x_1 and x_2 in plane L_1 we get

$$\begin{aligned} \overrightarrow{BX} &= x_1\vec{a}_1 + x_2\vec{a}_2 \\ \vec{x} &= (hx_1 + b_1)\,\vec{a}_1 + (hx_2 + b_2)\,\vec{a}_2 \\ \vec{x}' &= (hx_1 + b_1')\,\vec{a}_1 + (hx_2 + b_2')\,\vec{a}_2. \end{aligned} \tag{11}$$

Therefore one obtains

$$d\vec{x} = (dhx_1 + hdx_1 + \sigma_1 + hx_2\lambda)\,\vec{a}_1 + (dhx_2 + hdx_2 + \sigma_2 + hx_1\lambda)\,\vec{a}_2 \tag{12}$$

and

$$d'\vec{x} = (dhx_1 + hdx_1 + \sigma_1' + hx_2\lambda')\,\vec{a}_1 + (dhx_2 + hdx_2 + \sigma_2' + hx_1\lambda')\,\vec{a}_2, \tag{13}$$

where $\vec{V}_r = \frac{d\vec{x}}{dt}$ and $\vec{V}_a = \frac{d'\vec{x}}{dt}$ are called relative and absolute velocities of the point X, [19]. If $\vec{V}_r = 0$ (i.e. $d\vec{x} = 0$) and $\vec{V}_a = 0$ (i.e. $d'\vec{x} = 0$), then the point X is fixed in the Lorentzian planes L and L', respectively. Thus, from equations (12) and (13) the condition that the point X are fixed in L and L' are given by following equations

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma_1 - hx_2\lambda \\ hdx_2 &= -dhx_2 - \sigma_2 - hx_1\lambda \end{aligned} \tag{14}$$

and

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma_1' - hx_2\lambda' \\ hdx_2 &= -dhx_2 - \sigma_2' - hx_1\lambda' \end{aligned} \tag{15}$$

respectively. Substituting equation (14) into equation (13), sliding velocities $\vec{V}_f = \frac{d_f\vec{x}}{dt}$ of the point X becomes

$$d_f\vec{x} = \left[(\sigma_1' - \sigma_1) + hx_2\,(\lambda' - \lambda)\right]\vec{a}_1 + \left[(\sigma_2' - \sigma_2) + hx_1\,(\lambda' - \lambda)\right]\vec{a}_2. \tag{16}$$

Thus, for the pole point $P = (p_1, p_2)$ of the motion, we write [19]

$$x_1 = p_1 = -\frac{\sigma_2' - \sigma_2}{h\,(\lambda' - \lambda)}, \qquad x_2 = p_2 = -\frac{\sigma_1' - \sigma_1}{h\,(\lambda' - \lambda)}. \tag{17}$$

§3. Euler-Savary Formula For One Parameter Lorentzian Planar Homothetic Motions

Now, we consider spacelike and timelike pole curves of one parameter lorentzian planar homothetic motions and calculate Euler-Savary formula for both cases individually.

3.1 Canonical Relative System For Spacelike Pole Curves and Euler-Savary Formula

Now, let us choose the moving plane A represented by the coordinate system $\{B; \vec{a}_1, \vec{a}_2\}$ in such way to meet following conditions:

i) The origin of the system B and the instantaneous rotation pole P coincide with each other, i.e. $B = P$;

ii) The axis $\{B; \vec{a}_1\}$ is the pole tangent, that is, it coincides with the common tangent of spacelike pole curves (P) and (P'), (see Figure 1).

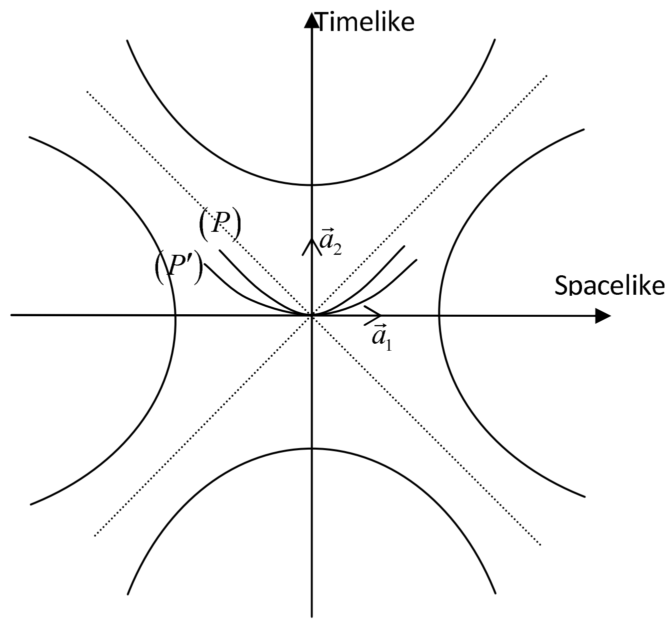

Figure 1. Spacelike Pole Curves (P) and (P')

If we consider the condition (i), then from equation (17) we reach that $\sigma_1 = \sigma_1'$ and $\sigma_2 = \sigma_2'$. Thus, from equation (9) and (10) we get

$$d\vec{b} = d\vec{p} = \sigma_1 \vec{a}_1 + \sigma_2 \vec{a}_2 = d'\vec{p} = d'\vec{b}.$$

Therefore, we have given the tangent of pole and constructed the rolling for the spacelike pole curves (P) and (P'). Considering the condition (ii) yields us that $\sigma_2 = \sigma_2' = 0$. If we choose $\sigma_1 = \sigma_1' = \sigma$ and consider equations (6) and (7), then we get the following equations for the differential equations related to the canonical relative system $\{P; \vec{a}_1, \vec{a}_2\}$ of the plane denoted by L_{1p},

$$d\vec{a}_1 = \lambda \vec{a}_2, \qquad d\vec{a}_2 = \lambda \vec{a}_1, \qquad d\vec{p} = \sigma \vec{a}_1 \tag{18}$$

and

$$d'\vec{a}_1 = \lambda' \vec{a}_2, \qquad d'\vec{a}_2 = \lambda' \vec{a}_1, \qquad d'\vec{p} = \sigma \vec{a}_1 \tag{19}$$

where $\sigma = ds$ is scalar arc element of the spacelike pole curves of (P) and (P') and λ is central cotangent angle, i.e. the angle between two neighboring tangents of (P). Therefore, the curvature of (P) at the point P is λ/σ. Similarly, taking λ' to be central cotangent angle, the curvature (P') at the point P becomes λ'/σ. Therefore, $r = \sigma/\lambda$ and $r' = \sigma/\lambda'$ are the curvature radii of spacelike pole curves (P) and (P'), respectively. Lorentzian plane L with respect to lorentz plane L' rotates about infinitesimal rotation angle $dv = \lambda' - \lambda$ at the time interval dt around the rotation pole P. Thus the rotational motions velocity of L with respect to L' becomes

$$\frac{\lambda' - \lambda}{dt} = \frac{dv}{dt} = \dot{v}\,. \tag{20}$$

Let us suppose that the direction of the unit tangent vector $\vec{a}_1$ is same as the direction of spacelike pole curves (P) and (P') (i.e., $ds/dt > 0$). In this case for the curvature radii (P) and (P'), $r > 0$ and $r' > 0$, respectively.

Now we investigate the velocities of the point X which has the coordinates x_1 and x_2 with respect to canonical relative system. Considering equation (12) and (13) we find

$$d\vec{x} = (dhx_1 + hdx_1 + \sigma + hx_2\lambda)\,\vec{a}_1 + (dhx_2 + hdx_2 + hx_1\lambda)\,\vec{a}_2 \tag{21}$$

$$d'\vec{x} = (dhx_1 + hdx_1 + \sigma + hx_2\lambda')\,\vec{a}_1 + (dhx_2 + hdx_2 + hx_1\lambda')\,\vec{a}_2. \tag{22}$$

Thus, the condition that the point X to be fixed in the Lorentzian planes L and L' becomes

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma - hx_2\lambda \\ hdx_2 &= -dhx_2 - hx_1\lambda \end{aligned} \tag{23}$$

and

$$\begin{aligned} hdx_1 &= -dhx_1 - \sigma - hx_2\lambda' \\ hdx_2 &= -dhx_2 - hx_1\lambda'. \end{aligned} \tag{24}$$

Therefore, the sliding velocity $\vec{V}_f$ is written to be

$$d_f\vec{x} = h\,(x_2\vec{a}_1 + x_1\vec{a}_2)\,(\lambda' - \lambda)\,.$$

Any point X chosen at the moving Lorentzian plane L draws a trajectory at the fixed lorentz plane L' during one parameter Lorentzian planar homothetic motion L/L'. Now we search for the planar curvature center X' of this trajectory at the time t.

The points X and X' have coordinates (x_1, x_2) and (x_1', x_2') with respect to canonical relative system and stay on the trajectory normal of X at every time t with the instantaneous rotation pole P. Generally a curvature center of a planar curve with respect to the point of the plane stays on the normal with respect to the point of the curve. In addition to that, this curvature center can be thought to be the limit of the intersection's normal of two neighboring points on the curve (see Figure 2). Therefore the vectors

$$\begin{aligned} \overrightarrow{PX} &= x_1\vec{a}_1 + x_2\vec{a}_2 \\ \overrightarrow{PX'} &= x_1'\vec{a}_1 + x_2'\vec{a}_2 \end{aligned}$$

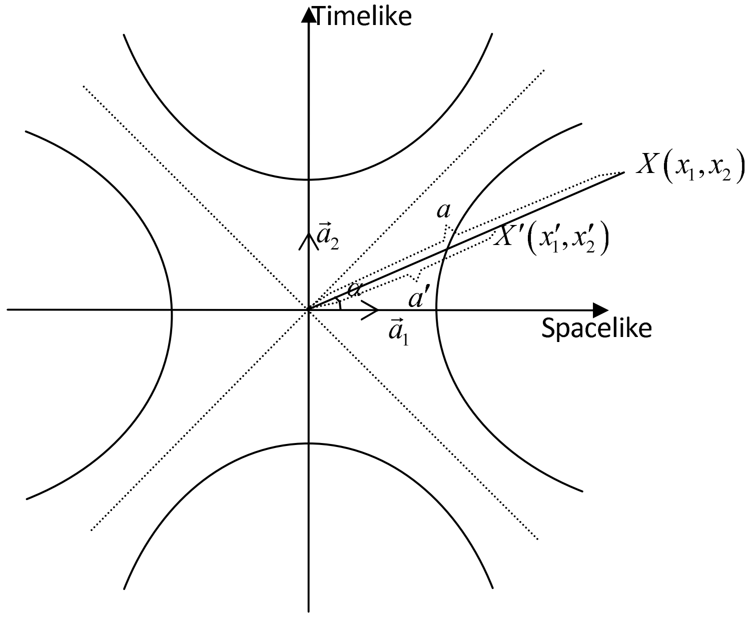

Figure 2. Spacelike vectors $\vec{PX}$ and $\vec{PX'}$

have same direction crossing the point P. Hence, the coordinates of the point X and X' satisfies the following equation:

$$x_1 x_2' - x_2 x_1' = 0. \tag{25}$$

Differentiation the last equation yields

$$dx_1 x_2' + x_1 dx_2' - dx_1' x_2 - x_1' dx_2 = 0. \tag{26}$$

The condition of being fixed of X in the Lorentzian plane L was given in equations (23). Moreover, the condition of being fixed of X' in the Lorentzian plane L' is

$$\begin{aligned} h dx_1' &= -dh x_1' - \sigma - h x_2' \lambda' \\ h dx_2' &= -dh x_2' - h x_1' \lambda'. \end{aligned} \tag{27}$$

Considering equation (26) with equations (23) and (27), we find

$$(x_2' - x_2)\,\sigma + h\,(x_1 x_1' - x_2 x_2')\,(\lambda' - \lambda) = 0. \tag{28}$$

Taking the vectors $\overrightarrow{PX}$ and $\overrightarrow{PX'}$ to be spacelike vectors and switching to the polar coordinates, i.e.,

$$\begin{aligned} x_1 &= a\cosh\alpha, \qquad & x_2 &= a\sinh\alpha \\ x_1' &= a'\cosh\alpha, \qquad & x_2' &= a'\sinh\alpha \end{aligned}$$

we find

$$\sigma\,(a' - a)\sinh\alpha + h a a'\,(\lambda' - \lambda) = 0. \tag{29}$$

From equations (20) and (28) we obtain

$$\left(\frac{1}{a'}-\frac{1}{a}\right)\sinh\alpha = h\left(\frac{1}{r'}-\frac{1}{r}\right) = h\frac{dv}{ds}. \tag{30}$$

This last equation is called Euler-Savary formula for the lorentzian homothetic motion.

Therefore we can give the following theorem.

Theorem 1 *In the one parameter Lorentzian planar homothetic motion of moving Lorentz plane L with respect to fixed Lorentz plane L', any point X at the plane L draws a trajectory with the instantaneous curvature center X' in the plane L'. In reverse motion, any point X' at the plane L' draws a trajectory at the lorentz plane L, being the curvature center at the initial point X. The interrelation between the points X and X' is expressed in equation (30) which is Euler-Savary formula in the sense of Lorentz.*

3.2 Canonical Relative System For Timelike Pole Curves and Euler-Savary Formula

Let us choose the moving plane A represented by the coordinate system $\{B;\vec{a}_1,\vec{a}_2\}$ in such way to meet following conditions:

i) The origin of the system B and the instantaneous rotation pole P coincide with each other, i.e. $B = P$,

ii) The axis $\{B;\vec{a}_2\}$ is the pole tangent, that is, it coincides with the common tangent of timelike pole curves (P) and (P'), (see Figure 3.).

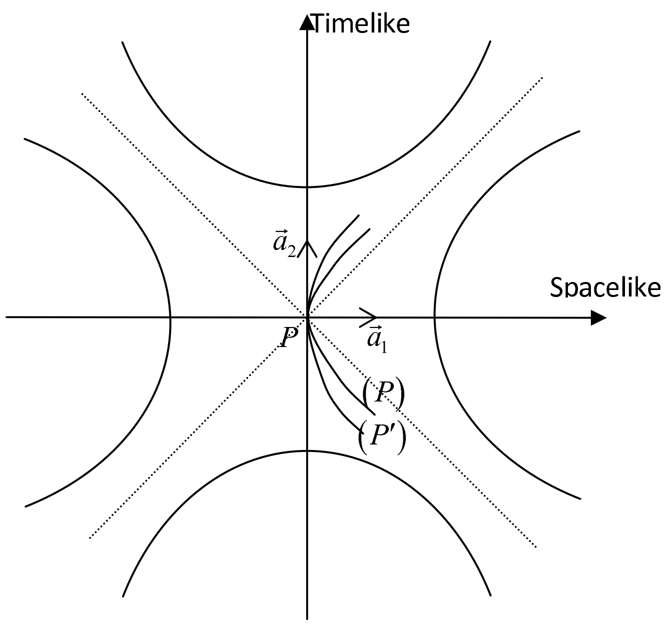

Figure 3. Timelike pole curves (P) and (P')

Thus, if the operations in III.1 section are performed considering the conditions i) and ii), the Euler-Savary formula for one-parameter lorentzian planar homothetic motion remains unchanged, that is, it is the same as in the equation (30), (see Figure 4.).

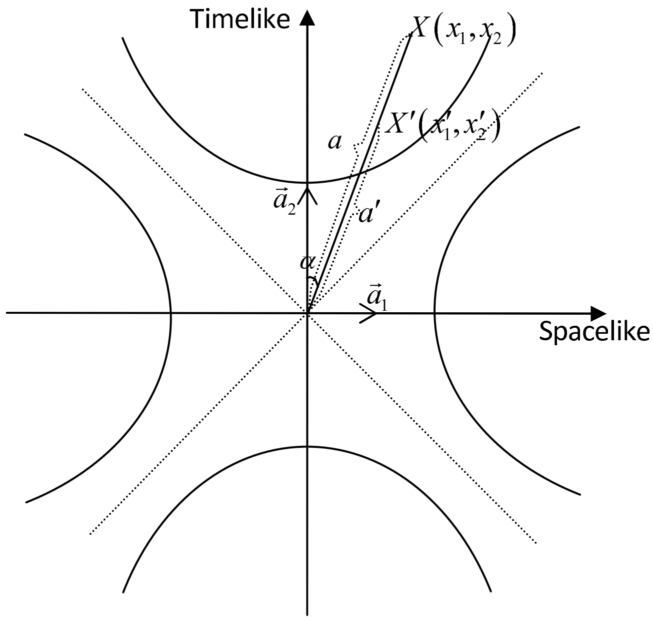

Figure 4. Timelike vectors $\overrightarrow{PX}$ and $\overrightarrow{PX'}$

Following Theorem 1 we reach the following corollaries:

Corollary 1 *In the one parameter Lorentzian homothetic motion L/L', whether the pole curves spacelike or timelike, the interrelation between the points X and X' is given by*

$$\left(\frac{1}{a'} - \frac{1}{a}\right) \sinh \alpha = h \left(\frac{1}{r'} - \frac{1}{r}\right)$$

which is Euler-Savary formula in the sense of Lorentz.

Corollary 2 *If $h \equiv 1$, then we reach the formula*

$$\left(\frac{1}{a'} - \frac{1}{a}\right) \sinh \alpha = \left(\frac{1}{r'} - \frac{1}{r}\right)$$

which is Euler-Savary formula in the Lorentzian plane given in references [1,6,9].

References

[1] I. Aytun, Euler-Savary formula for one-parameter Lorentzian plane motion and its lorentzian geometrical interpretation, M.Sc. Thesis, Celal Bayar University, 2002.

[2] O. Bottema, B. Roth, *Theoretical Kinematics*, North-Holland Series in Applied Mathematics and Mechanics, North-Holland, Amsterdam, 1979.

[3] R. R. Buckley, E. V. Whitfield, The Euler-Savary formula, *The Mathematical Gazette*, 33(306)(1949), 297-299.

[4] Y. J. Chen, B. Ravani, Offsets surface generation and contouring in computer aided design, *ASME J. Mech. Trans. and Autom. Des.* 109(1987), 348-352.

[5] A. A. Ergin, *The kinematic geometry on the lorentz plane*, Ph.D. Thesis, Ankara University, 1989.

[6] M. Ergüt, A. P. Aydın, N. Bildik, The geometry of the canonical system and one-parameter motions in 2-dimensional lorentzian space, *The Journal of Fırat University*, 3(1)(1988), 113-122.

[7] R. T. Farouki, The approximation of non-degenerate offset surface, *Comput. Aided Geom. Des.* 3(1986), 15-43.

[8] H. H. Hacısalihoğlu, On the rolling of one curve or surface upon another, *Proceedings of The Royal Irish Academy*, 71(2)(1971), 13-17.

[9] T. Ikawa, Euler-savary's formula on minkowski geometry, *Balkan Journal of Geometry and Its Applications*, 8(2)(2003), 31-36.

[10] H. Iseri, *Smarandache Manifolds*, Am. Res. Press, 2002.

[11] L. Kuciuk and M. Antholy, An Introduction to Smarandache Geometries, *Mathematics Magazine*, 12(2003), 1-12.

[12] L.F.Mao, *Smarandache Multi-Space Theory*, Hexis, Phoenix,American 2006.

[13] L.F.Mao, *Combinatorial Geometry with applications to Field Theory*, InfoQuest, USA, 2009.

[14] H. R. Müller, Kinematik, Sammlung Göschen, Walter de Gruyter, Berlin, 1963.

[15] H. R. Müller, Zur Bewegunssgeometrie In Raumen Höherer Dimension, *Mh. Math.* 70(1) (1966), 47-57.

[16] K. Nomizu, Kinematics and differential geometry of submanifolds, *Tohoku Math. J.* 30(4) (1978), 623-637.

[17] S. G. Papaioannou, D. Kiritsis, An application of bertrand curves and surface to CAD/CAM., *Comput.-Aided Des.* 17(8)(1985), 348-352.

[18] F.Smarandache, Mixed noneuclidean geometries, eprint: *arXiv: math/0010119, 10/2000.*

[19] A. Tutar, N. Kuruoğlu, M. Düldül, On the moving coordinate system and pole points on the Lorentzian plane, *International Journal of Applied Mathematics*, 7(4)(2001), 439-445.

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By Alfred North Whitehead, a British philosopher and mathematician.

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[12]W.S. Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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