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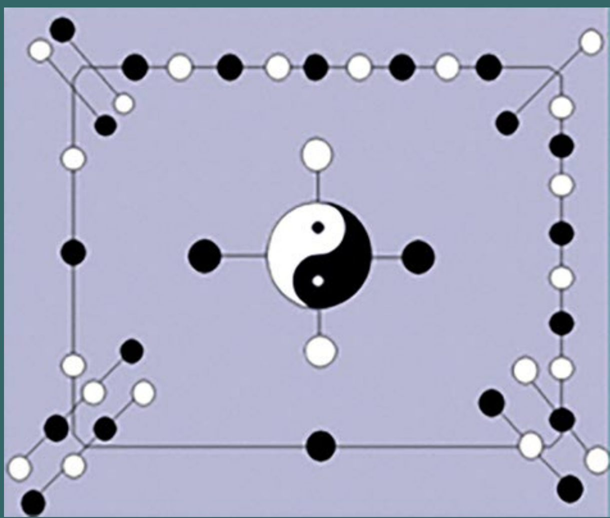

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Famous Words:

In the long process of human history, truth because like gold as heavy, always sink to the bottom and is difficult to be found, instead, fallacy of the cow dung as light down floating everywhere in the above flood.

By *Francis Bacon*, an English philosopher.

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ρ-Yamabe Solitons on 3-Dimensional Hyperbolic Kenmotsu Manifolds

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Abstract: In this paper, we examine the scalar curvature of a 3-dimensional hyperbolic Kenmotsu manifold, admitting a ρ-Yamabe soliton is constant, and the manifold reduces to a ρ-Einstein manifold. We have also examined if a 3-dimensional hyperbolic Kenmotsu manifold admits a ρ-Yamabe soliton, then the manifold reduces to an Einstein under some conditions. Here, we have also studied some different types of curvature properties under certain conditions. Finally, we construct an example of 3-dimensional hyperbolic Kenmotsu manifold admitting ρ-Yamabe soliton.

Key Words: Yamabe soliton, ρ-Yamabe soliton, ρ-Einstein manifold, hyperbolic Kenmotsu manifold.

AMS(2010): 53C21, 53C25, 53C44.

§1. Introduction

The study of self-similar solutions of the Ricci flow, the Yamabe flow and the curvature flow has an important place at the frontier between mathematics and physics. One of the most famous among these which is introduced by Richard Hamilton [10] is the concept of Yamabe solitons [18, 21].

The Yamabe flow is an evolution equation for metrics on a Riemannian manifold (M^n, g) defined as follows:

$$\frac{\partial}{\partial t}(g(t)) = -\kappa g(t), \qquad g(0) = g_0, \tag{1.1}$$

where κ is the scalar curvature of the metric $g(t)$.

The Ricci flow is an evolution equation for metrics on a Riemannian manifold (M^n, g) defined as follows:

$$\frac{\partial}{\partial t}(g(t)) = -2S, \tag{1.2}$$

where S denotes the Ricci tensor. In 2-dimension, the Yamabe flow is equivalent to the Ricci

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flow. However, in dimension stricly greater than two, the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow do not preserve in general.

A self-similar solution of the Yamabe flow is called Yamabe soliton on a (semi-)Riemannian manifold M^n [1, 5, 19] if there exists a smooth vector field V on M^n and a real number λ such that

$$\mathcal{L}_V g = (\kappa - \lambda)g, \tag{1.3}$$

where $\mathcal{L}_V g$ indicates the Lie derivative of g along the vector field V on M^n. Here, V is termed the soliton field of the Yamabe soliton. A Yamabe soliton is denoted by a triplet (g, V, λ). The beauty of Yamabe soliton depends on the soliton scalar λ. A soliton is named as shrinking, steady and expanding according to λ is negative, zero and positive, respectively. Yamabe solitons have been studied by several authors such as [3, 6, 7, 8, 16] and many others.

A Yamabe soliton becomes gradient Yamabe soliton if the soliton field V is gradient of some smooth function γ on M^n. In this case, the equation (1.3) reduces to

$$2Hess\gamma = (\kappa - \lambda)g, \tag{1.4}$$

where $Hess\gamma$ is the Hessian of the smooth function γ on M^n.

A ρ-Yamabe soliton is a generalization of Yamabe soliton [4], defined by the following equation:

$$\mathcal{L}_\zeta g = (\kappa - \lambda)g - \sigma\rho \otimes \rho, \tag{1.5}$$

where λ and σ are constants and ρ is a 1-form defined by $\rho(X) = g(X, \zeta)$, for any smooth vector field X on M^n and $\otimes$ represents the tensor product. If both λ and σ are smooth functions on M^n, then (1.5) is named as almost ρ-Yamabe soliton or a quasi-Yamabe soliton [4]. Moreover if $\sigma = 0$, then equation (1.5) reduces to (1.3) and hence the ρ-Yamabe soliton reduces to a Yamabe soliton whereas an almost ρ-Yamabe soliton reduces to an almost Yamabe soliton if in (1.5), λ is a smooth function on M^n and $\sigma = 0$.

The concept of almost contact hyperbolic (f, g, ρ, ζ)-structure was introduced by Upadhyay and Dube in [20]. Further, it was stadied by several geometers such as R. B. Pal [15], B. B. Sinha and R. N. Singh [17] and many others. A non-zero smooth vector field $U \in T_p(M)$ is called timelike, null, space-like and non-space-like according as $g_p(U, U) < 0$, $g_p(U, U) = 0$, $g_p(U, U) > 0$ and $g_p(U, U) \leq 0$, respectively, where $T_p(M)$ is the tangent space of M at $p \in M$ [14].

Let $\{e_1, e_2, \cdots, e_n = \zeta\}$ be a local orthonormal basis of vector fields in $n(= 2m + 1)$-dimensional semi-Riemannian manifold M^n. Then the Ricci tensor S and the scalar curvature κ of a $n(= 2m + 1)$-dimensional almost hyperbolic contact metric manifold M^n endowed with a semi-Riemannian metric g are defined as follows:

$$S(X, Y) = \sum_{r=1}^{n} g(e_r, e_r)g(R(e_r, X)Y, e_r) \tag{1.6}$$

and

$$\kappa = \sum_{r=1}^{n} g(e_r, e_r) S(e_r, , e_r) \tag{1.7}$$

for all smooth vector fields X, Y on M^n, where ζ is the unit timelike vector field (i.e., $g(\zeta, \zeta) = -1$) and R is the curvature tensor of M^n [14].

In this paper, we start to investigate the ρ-Yamabe solitons on 3-dimensional hyperbolic Kenmotsu manifolds. Here, we establish that the scalar curvature of a 3-dimensional hyperbolic Kenmotsu manifold admitting a ρ-Yamabe soliton is constant, and the manifold reduces to a ρ-Einstein manifold. Next, we have proved that a 3-dimensional hyperbolic Kenmotsu manifold whose metric is ρ-Yamabe soliton reduces to an Einstein manifold under some conditions. Here, we also examine the nature of the manifold in terms of ρ-Yamabe soliton on hyperbolic Kenmotsu manifolds. Next, we have characterized the nature of ρ-Yamabe soliton when the Ricci tensor is parallel along ζ, Ricci operator is parallel along ζ. Next, we have shown that if a 3-dimensional hyperbolic Kenmotsu manifold with a cyclic parallel Ricci tensor admits a ρ-Yamabe soliton, then the manifold reduces to an Einstein manifold. Next, we have studied some different types of curvature properties under certain conditions. Finally, we construct an example of 3-dimensional hyperbolic Kenmotsu manifold admitting ρ-Yamabe soliton.

§2. **Preliminaries**

An odd-dimensional smooth manifold M^{2m+1} is named to be an almost hyperbolic contact metric manifold if it admits a timelike vector field ζ, a 1-form ρ, a fundamental tensor field φ of type $(1, 1)$, and a semi-Riemannian metric g satisfying [20]:

$$\varphi^2(X) = X + \rho(X)\zeta, \tag{2.1}$$

$$\rho(\zeta) = -1 \implies \varphi(\zeta) = 0, \tag{2.2}$$

$$rank(\varphi) = 2m, \tag{2.3}$$

$$\rho \circ \varphi = 0, \tag{2.4}$$

$$g(\varphi X, \varphi Y) = -g(X, Y) - \rho(X)\rho(Y), \tag{2.5}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \tag{2.6}$$

$$g(X, \zeta) = \rho(X) \tag{2.7}$$

for all smooth vector fields X, Y on M^{2m+1}. Then the structure $(\varphi, \zeta, \rho, g)$ on manifold M^{2m+1} named as almost hyperbolic contact metric structure.

If an almost hyperbolic contact metric manifold M^{2m+1} is fulfilled, the following condition [12]:

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\zeta - \rho(Y)\varphi X, \tag{2.8}$$

then, M^{2m+1} is called a hyperbolic Kenmotsu manifold [2], where ∇ denotes the Levi-Civita connection of g. From the antecedent equation, it is clear that

$$\nabla_X \zeta = -X - \rho(X)\zeta, \tag{2.9}$$

and

$$(\nabla_X \rho)Y = g(\varphi X, \varphi Y) = -g(X,Y) - \rho(X)\rho(Y). \tag{2.10}$$

Also in this manifold M^{2m+1} the following relations are satisfied for all smooth vector fields X and X on M^{2m+1} [2]:

$$R(X,Y)\zeta = \rho(Y)X - \rho(X)Y, \tag{2.11}$$

$$R(X,\zeta)\zeta = -X - \rho(X)\zeta, \tag{2.12}$$

$$R(\zeta,X)Y = g(X,Y)\zeta - \rho(Y)X, \tag{2.13}$$

$$S(X,\zeta) = 2m\rho(X), \tag{2.14}$$

$$S(\zeta,\zeta) = -2m, \tag{2.15}$$

$$Q\zeta = 2m\zeta, \tag{2.16}$$

where R, S are the Riemannian curvature tensor, Ricci tensor of the manifold M^{2m+1}, respectively, and Q the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

It is well known that the tensor R on any 3-dimensional Riemannian manifold (M^3, g) always satisfies

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{\kappa}{2}[g(Y,Z)X - g(X,Z)Y] \tag{2.17}$$

for all smooth vector fields X, Y, Z on M^3.

Replacing X and Z by ζ in the above equation (2.17) and using the identities (2.2), (2.7) and (2.11) – (2.16) entails that

$$QX = (\frac{\kappa}{2} - 1)X + (\frac{\kappa}{2} - 3)\rho(X)\zeta. \tag{2.18}$$

Taking inner product of (2.18) with Y and using the relation $g(QX,Y) = S(X,Y)$ yields

$$S(X,Y) = (\frac{\kappa}{2} - 1)g(X,Y) + (\frac{\kappa}{2} - 3)\rho(X)\rho(Y), \tag{2.19}$$

which implies that the three-dimensional hyperbolic Kenmotsu manifolds are ρ Einstein manifolds.

In addition, we know that

$$(\mathcal{L}_\zeta g)(X,Y) = g(\nabla_X \zeta, Y) + g(X, \nabla_Y \zeta). \tag{2.20}$$

Utilizing (2.9) and (2.7) in the above equation (2.20), we get

$$(\mathcal{L}_\zeta g)(X,Y) = -2[g(X,Y) + \rho(X)\rho(Y)] \tag{2.21}$$

for all smooth vector fields X, Y on M^3.

§3. 3-Dimensional Hyperbolic Kenmotsu Manifolds Admitting ρ-Yamabe Solitons

Theorem 3.1 *Let the metric g of a 3-dimensional hyperbolic Kenmotsu manifold M^3 satisfy the ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$, ζ being the timelike vector field of M^3. Then the scalar curvature of the manifold is constant.*

Proof Let M^3 be a 3-dimensional hyperbolic Kenmotsu manifold which admit ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$. Then we have

$$(\mathcal{L}_\zeta g)(X, Y) = (\kappa - \lambda)g(X, Y) - \sigma\rho(X)\rho(Y) \tag{3.1}$$

for all smooth vector fields X, Y on M^3.

Substituting the value of $(\mathcal{L}_\zeta g)(X, Y)$ from (2.21) in the above equation (3.1) yields

$$(\kappa - \lambda + 2)g(X, Y) + (2 - \sigma)\rho(X)\rho(Y) = 0. \tag{3.2}$$

Replacing Y by ζ in equation (3.2) and using the relation (2.7) gives

$$(\kappa - \lambda + \sigma)\rho(X) = 0. \tag{3.3}$$

Since $\rho(X) \neq 0$, we get,

$$\kappa = \lambda - \sigma. \tag{3.4}$$

Since λ and σ both are constants, κ is also constant. □

Theorem 3.2 *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits Yamabe soliton (g, ζ, λ), then the timelike vector field ζ is a Killing vector field.*

Proof Now putting $\sigma = 0$ in equation (3.4) gives $\kappa = \lambda$, so, equation (3.1) reduces to

$$\mathcal{L}_\zeta g = 0. \tag{3.5}$$

This shows that the timelike vector field ζ is a Killing vector field. □

Corollary 3.3 *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$, ζ being the timelike vector field of M^3, then the manifold M^3 reduces to a ρ-Einstein manifold.*

Proof Now, from (2.19) and (3.4), we get

$$S(X, Y) = (\frac{\lambda - \sigma}{2} - 1)g(X, Y) + (\frac{\lambda - \sigma}{2} - 3)\rho(X)\rho(Y). \tag{3.6}$$

This completes the proof. □

§4. ρ-Yamabe Solitons on 3-Dimensional Hyperbolic Kenmotsu Manifold with Ricci Symmetric

Definition 4.1 *A hyperbolic Kenmotsu manifold of dimension three is said to be Ricci symmetric if*

$$(\nabla_Z S)(X,Y) = 0 \tag{4.1}$$

for all smooth vector fields X, Y, Z on M^3 and ∇ is the Riemannian connection.

Theorem 4.1 *If a 3-dimensional Ricci symmetric hyperbolic Kenmotsu manifold M^3 admits ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$, ζ being the timelike vector field of M^3, then $\lambda = \sigma + 6$ and the manifold reduces to an Einstein manifold.*

Proof Now, taking covariant differentiation of (3.6) with respect to Z, we obtain

$$(\nabla_Z S)(X,Y) = (\frac{\lambda-\sigma}{2} - 3)[\rho(X)(\nabla_Z\rho)(Y) + \rho(Y)(\nabla_Z\rho)(X)]. \tag{4.2}$$

In view of (2.10), we have

$$(\nabla_Z S)(X,Y) = (\frac{\lambda-\sigma}{2} - 3)[\rho(X)g(\varphi Z, \varphi Y) + \rho(Y)g(\varphi Z, \varphi X)]. \tag{4.3}$$

Let us assume that the manifold is Ricci symmetric. Then, from (4.3), we get

$$(\frac{\lambda-\sigma}{2} - 3)[\rho(X)g(\varphi Z, \varphi Y) + \rho(Y)g(\varphi Z, \varphi X)] = 0. \tag{4.4}$$

Setting $Y = \zeta$ in the above equation (4.3)and using (2.2) gives

$$(\frac{\lambda-\sigma}{2} - 3)g(\varphi Z, \varphi X) = 0 \tag{4.5}$$

for all smooth vector fields Z, Y on M^3. It follows that

$$\lambda = \sigma + 6.$$

Now, putting $\lambda = \sigma + 6$ in equation (3.6), we have

$$S(X,Y) = 2g(X,Y) \tag{4.6}$$

for all smooth vector fields X, Y on M^3. □

§5. ρ-Recurrent Ricci Tensor of a Hyperbolic Kenmotsu Manifolds of Dimension Three Admitting Rho-Yamabe Solitons

Definition 5.1 *The Ricci tensor S of a hyperbolic Kenmotsu manifold of dimension three is* *called ρ-recurrent if it satisfies*

$$(\nabla_Z S)(X,Y) = \rho(Z)S(X,Y) \tag{5.1}$$

for all smooth vector fields X, Y, Z on M^3.

Theorem 5.1 *There does not exist ρ-recurrent Ricci tensor of a hyperbolic Kenmotsu manifold M^3 of dimension three admitting a ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$, ζ being the timelike vector field of M^3*

Proof Using the relation (4.3) in the equation (5.1), we have

$$\left(\frac{\lambda - \sigma}{2} - 3\right)[\rho(X)g(\varphi Z, \varphi Y) + \rho(Y)g(\varphi Z, \varphi X)] = \rho(Z)S(X,Y). \tag{5.2}$$

Substituting $X = \zeta = Y$ in the above equation (5.2) and using the equations (2.2) and (2.15) yields

$$\rho(Z) = 0,$$

which is a contradiction. Hence, the proof. □

§6. The Ricci Tensor S and Ricci Operator Q Are Parallel Along ζ on 3-Dimensional Hyperbolic Kenmotsu Manifold Admitting ρ-Yamabe Soliton

Definition 6.1 *The Ricci tensor S of a hyperbolic Kenmotsu manifold of dimension three is parallel along the smooth vector field X on M^3 if it satisfies*

$$(\nabla_X S)(Y,Z) = 0 \tag{6.1}$$

for all smooth vector fields X, Y, Z on M^3.

Definition 6.2 *The Ricci operator Q of a hyperbolic Kenmotsu manifold of dimension three is parallel along the smooth vector field X on M^3 if it satisfies*

$$(\nabla_X Q)Y = 0 \tag{6.2}$$

for all smooth vector fields Y on M^3.

Theorem 6.3 *The Ricci tensor S and Ricci operator Q of a 3-dimensional hyperbolic Kenmotsu manifold M^3 admitting an ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$ are parallel along the timelike vector field ζ of M^3.*

Proof From (4.3), we obtain,

$$(\nabla_\zeta S)(X,Y) = \left(\frac{\lambda - \sigma}{2} - 3\right)[\rho(X)g(\varphi\zeta, \varphi Y) + \rho(Y)g(\varphi\zeta, \varphi X)] \tag{6.3}$$

for all smooth vector fields X, Y on M^3.

Using the relation (2.2) in the above equation (6.3), we get,

$$(\nabla_\zeta S)(X,Y) = 0 \tag{6.4}$$

for all smooth vector fields X, Y on M^3,

which implies that S is parallel along the timelike vector field ζ.

Also, from (3.6), we obtain,

$$QX = (\frac{\lambda-\sigma}{2} - 1)X + (\frac{\lambda-\sigma}{2} - 3)\rho(X)\zeta. \tag{6.5}$$

Replace the expression of Q from (6.5) in

$$(\nabla_\zeta Q)X = \nabla_\zeta QX - Q(\nabla_\zeta X), \tag{6.6}$$

we obtain,

$$(\nabla_\zeta Q)X = (\frac{\lambda-\sigma}{2} - 3)((\nabla_\zeta \rho)X)\zeta. \tag{6.7}$$

Now, using the identities (2.10) and (2.2) in the above equation (6.7), we get

$$(\nabla_\zeta Q)X = 0 \tag{6.8}$$

for all smooth vector fields X on M^3.

Hence, Q is parallel to the timelike vector field ζ. □

§7. ρ-Yamabe Solitons on 3-Dimensional Hyperbolic Kenmotsu Manifolds with Cyclic Parallel Ricci Tensor

A. Gray [9] revealed the idea of cyclic parallel Ricci tensor and Ricci tensor of Codazzi type. Codazzi type of Ricci tensor means that the Levi-Civita connection ∇ of such metric is a Yang-Mills connection while keeping the metric of the manifold fixed. A Riemannian manifold (M^n, g) is said to have a cyclic parallel Ricci tensor if its Ricci tensor S of type $(0,2)$ is non-zero and satisfies the following condition:

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \tag{7.1}$$

for all smooth vector fields X, Y, Z on M^3. Ki et al. [13] proved that Carten hypersurfaces are manifolds with non-parallel Ricci tensor that satisfy cyclic parallel Ricci tensor.

Now, from equation (4.3),

$$(\nabla_Z S)(X,Y) = (\frac{\lambda-\sigma}{2} - 3)[\rho(X)g(\varphi Z, \varphi Y) + \rho(Y)g(\varphi Z, \varphi X)]. \tag{7.2}$$

In view of (7.2), it follows that

$$(\nabla_Y S)(Z,X) = (\frac{\lambda-\sigma}{2} - 3)[\rho(Z)g(\varphi Y,\varphi X) + \rho(X)g(\varphi Y,\varphi Z)] \tag{7.3}$$

and

$$(\nabla_X S)(Y,Z) = (\frac{\lambda-\sigma}{2} - 3)[\rho(Y)g(\varphi X,\varphi Z) + \rho(Z)g(\varphi X,\varphi Y)]. \tag{7.4}$$

Let us assume that the Ricci tensor S of type (0,2) is cyclic parallel. Then, using the relations (7.2), (7.3) and $(7,4)$ in equation (7.1), we get

$$(\frac{\lambda-\sigma}{2} - 3)[\rho(X)g(\varphi Z,\varphi Y) + \rho(Y)g(\varphi Z,\varphi X) + \rho(Z)g(\varphi Y,\varphi X)] = 0. \tag{7.5}$$

Putting $X=\zeta$ in the equation (7.5) and using the relation (2.2), we lead,

$$(\frac{\lambda-\sigma}{2} - 3)g(\varphi Z,\varphi Y) = 0 \tag{7.6}$$

for all smooth vector fields Z, Y on M^3,

which gives

$$\lambda = \sigma + 6. \tag{7.7}$$

Now substituting $\lambda = \sigma + 6$ in the equation (3.6), we get

$$S(X,Y) = 2g(X,Y). \tag{7.8}$$

Thus, we can state the following.

Theorem 7.1 *If a* 3*-dimensional hyperbolic Kenmotsu manifold* M^3 *with cyclic parallel Ricci tensor admits a* ρ*-Yamabe soliton* (g,ζ,λ,σ)*,* ζ *being the timelike vector field of* M^3*, then* $\lambda = \sigma + 6$ *and the manifold reduces to an Einstein manifold.*

§8. Curvature Properties on 3-Dimensional Hyperbolic Kenmotsu Manifold Admitting ρ-Yamabe Soliton

Let us assume that 3-dimensional hyperbolic Kenmotsu manifolds with ρ-Yamabe solitons satisfy the condition

$$R(\zeta,X)\cdot S = 0. \tag{8.1}$$

Then we have

$$S(R(\zeta,X)Y,Z) + S(Y,R(\zeta,X)Z) = 0 \tag{8.2}$$

for all smooth vector fields X, Y, Z on M^3.

Then, from (2.13), the equation (8.2) takes the form:

$$S(g(X,Y)\zeta - \rho(Y)X, Z) + S(Y, g(X,Z)\zeta - \rho(Z)X) = 0. \tag{8.3}$$

In view of (3.6), we obtain:

$$(\frac{\lambda-\sigma}{2}-3)[\{-g(X,Y)-\rho(X)\rho(Y)\}\rho(Z)+\{-g(X,Z)-\rho(X)\rho(Z)\}\rho(Y)]=0. \tag{8.4}$$

Using the relation (2.10) in the above equation (8.4), we get

$$(\frac{\lambda-\sigma}{2}-3)[g(\varphi X,\varphi Y)\rho(Z)+g(\varphi X,\varphi Z)\rho(Y)]=0. \tag{8.5}$$

Now putting $Z=\zeta$ in the equation (8.5) and using the relation (2.2) gives

$$(\frac{\lambda-\sigma}{2}-3)g(\varphi X,\varphi Y)=0 \tag{8.6}$$

for all smooth vector fields X, Y on M^3.

It follows that

$$\lambda=\sigma+6. \tag{8.7}$$

Also, we obtain from (3.6),

$$S(X,Y)=2g(X,Y). \tag{8.8}$$

Thus, we are in position to state the following.

Theorem 8.1 *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admitting a ρ-Yamabe soliton (g,ζ,λ,σ), ζ being the timelike vector field of M^3, satisfies the condition $R(\zeta,X)\cdot S=0$, then the soliton scalar $\lambda=\sigma+6$ and the manifold reduces to an Einstein manifold.*

Again, we assume that 3-dimensional hyperbolic Kenmotsu manifolds with ρ-Yamabe solitons satisfy the condition

$$S(\zeta,X)\cdot R=0, \tag{8.9}$$

which implies that

$$\begin{aligned}&S(X,R(Y,Z)W)\zeta-S(\zeta,R(Y,Z)W)X+S(X,Y)R(\zeta,Z)W\\&-S(\zeta,Y)R(X,Z)W+S(X,Z)R(Y,\zeta)W-S(\zeta,Z)R(Y,X)W\\&+S(X,W)R(Y,Z)\zeta-S(\zeta,W)R(Y,Z)X=0\end{aligned} \tag{8.10}$$

for all smooth vector fields X, Y, Z, W on M^3.

Now taking inner product of (8.1) with ζ, we obtain

$$\begin{aligned}&-S(X,R(Y,Z)W)-S(\zeta,R(Y,Z)W)\rho(X)+S(X,Y)\rho(R(\zeta,Z)W)\\&-S(\zeta,Y)\rho(R(X,Z)W)+S(X,Z)\rho(R(Y,\zeta)W)-S(\zeta,Z)\rho(R(Y,X)W)\\&+S(X,W)\rho(R(Y,Z)\zeta)-S(\zeta,W)\rho(R(Y,Z)X)=0.\end{aligned} \tag{8.11}$$

Now using the equations (3.6), (2.2), (2.11) and (2.12) and putting $Z=\zeta, W=\zeta$ in the above equation (8.11) entails that

$$(\frac{\lambda-\sigma}{2}+1)[g(X,Y)+\rho(X)\rho(Y)]=0 \tag{8.12}$$

for all smooth vector fields X, Y on M^3, which implies that

$$\lambda=\sigma-2. \tag{8.13}$$

Thus, we can conclude the following.

Theorem 8.2 *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits an ρ-Yamabe soliton (g,ζ,λ,σ), ζ being the timelike vector field of M^3 satisfies the condition $S(\zeta,X)\cdot R=0$, then the soliton scalar $\lambda=\sigma-2$.*

§9. **Example**

We consider the three-dimensional manifold $M^3=\{(x,y,z)\in\mathbb{R}^3 : z\neq 0\}$, where the standard coordinates are in $\mathbb{R}^3$. Then the vector fields

$$e_1=e^z\frac{\partial}{\partial x}, e_2=e^z\frac{\partial}{\partial y}, e_3=\frac{\partial}{\partial z}=\zeta$$

are linearly independent at each point of M^3, and so they form a basis of the tangent space at each point of M^3. Let g be a semi-Riemannian metric defined by

$$g(e_i,e_j)=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

Let ρ be the 1-form defined by $g(V,e_3)=\rho(V)$ for any vector field V on M^3 and φ is a $(1,1)$ tensor field defined by $\varphi(e_1)=e_2, \varphi(e_2)=e_1, \varphi(e_3)=0$. Then, using the linearity property of φ and g we obtain

$$\rho(e_3)=-1, \varphi^2V=V+\rho(V)e_3, g(\varphi V,\varphi W)=-g(V,W)-\rho(V)\rho(W)$$

for any vector fields V, W on M^3.

Thus, for $e_3=\zeta$, the structure (φ,ζ,ρ,g) defines an almost hyperbolic contact metric structure on M^3. All possible Lie brackets for the example are as follows:

$$[e_1,e_1]=[e_2,e_2]=[e_3,e_3]=[e_1,e_2]=[e_2,e_1]=0, [e_1,e_3]=-e_1,$$

$$[e_3,e_1]=e_1, [e_3,e_2]=e_2, [e_2,e_3]=-e_2$$

.

Let ∇ be a Riemannian connection with respect to the semi-Riemannian metric g. Now using Koszul's formula

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]),$$

we can obtain

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_3} e_1 = \nabla_{e_3} e_2 = \nabla_{e_3} e_3 = 0$$

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_3 = -e_1, \nabla_{e_2} e_2 = e_3, \nabla_{e_2} e_3 = -e_2$$

From the above relations, we get

$$\nabla_X \zeta = -X - \rho(X)\zeta$$

fulfilled for any vector field X on M^3. Hence, the structure $(\varphi, \zeta, \rho, g)$ is a hyperbolic Kenmotsu structure on M^3. Consequently, $M^3(\varphi, \zeta, \rho, g)$ is a 3-dimensional hyperbolic Kenmotsu manifold. The non-zero components of the curvature tensor R as follows:

$R(e_1, e_2)e_1 = -e_2, R(e_2, e_1)e_1 = e_2, R(e_1, e_3)e_1 = -e_3, R(e_3, e_1)e_1 = e_3,$

$R(e_1, e_2)e_2 = -e_1, R(e_2, e_1)e_2 = e_1, R(e_2, e_3)e_2 = e_3, R(e_3, e_2)e_2 = -e_3,$

$R(e_1, e_3)e_3 = -e_1, R(e_1, e_3)e_3 = -e_1, R(e_2, e_3)e_3 = -e_2, R(e_3, e_2)e_3 = e_2.$

The Ricci tensor R is given by

$$S(X,Y) = \sum_{r=1}^{3} g(e_r, e_r) g(R(e_r, X)Y, e_r). \tag{9.1}$$

So, we have

$$S(e_1, e_1) = 2, S(e_2, e_2) = -2, S(e_3, e_3) = -2. \tag{9.2}$$

Again, the scalar curvature of the given hyperbolic Kenmotsu manifold can be calculated as follows:

$$\kappa = \sum_{r=1}^{3} g(e_r, e_r) S(e_r, , e_r) = S(e_1, e_1) - S(e_2, e_2) - S(e_3, e_3) = 6. \tag{9.3}$$

Now, from (3.6) we have

$$S(e_1, e_1) = (\frac{\lambda - \sigma}{2} - 1), S(e_2, e_2) = -(\frac{\lambda - \sigma}{2} - 1), S(e_3, e_3) = -(\frac{\lambda - \sigma}{2} - 1). \tag{9.4}$$

Now, from (9.1) and (9.3) we get

$$\lambda - \sigma = 6. \tag{9.5}$$

So, the values of κ and $\lambda - \sigma$ are the same, and hence the equation (3.4) is satisfied. Therefore, the structure $(g, \zeta, \lambda, \sigma)$ is a ρ-Yamabe soliton on 3-dimensional hyperbolic Kenmotsu manifolds $M^3(\varphi, \zeta, \rho, g)$, ζ being timelike vector field on M^3.

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Coupled Fixed Point Results via New Coupled Implicit Contractive Condition in S-Metric Spaces

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Abstract: In this paper, we prove a coupled and a common coupled fixed point theorems via newly proposed coupled implicit contractive condition in the framework of S-metric spaces. Also, we give some corollaries of the main result. Furthermore, an illustrative example and an application to the Fredholm integral equation are given. Our results extend, generalize and enrich several results from the existing literature.

Key Words: Coupled fixed point, common coupled fixed point, Smarandachely multifixed point, new coupled implicit contractive condition, S-metric space.

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§1. Introduction

In the history of fixed point theorem and applications, it was Stefan Banach ([3]) who introduced the concept of contraction condition in the year 1922 for obtaining fixed as well as unique fixed point. After Banach's remarkable result a number of researchers started working in the area of developing fixed point theory in the lines of Banach. One can refer B. E. Rhoades [Trans. Am. Math. Soc. 266 (1977)] for various types of contraction as well as non-contraction type conditions which facilitates the contraction map to get unique fixed point.

Later on, authors tried to replace the metric space by generalized metric space. In the literature, there are many generalizations of the metric space exists. One of such generalizations is the generalized metric space or S-metric space. In 2012, Sedghi et al. [33] introduced the notion of a S-metric space which is different from other spaces and proved fixed point theorems in such spaces. They also gave some examples of a S-metric space which shows that the S-metric space is different from other spaces. They built up some topological properties in such spaces and proved some fixed point theorems in the setting of S-metric spaces. In 2014, Sedghi and Dung [34] proved new generalized fixed point theorems such as the *Ćirić's* fixed point result on an S-metric space. In 2017, Özgür and Tas obtained the generalizations of the Banach's contraction principle and the Rhoades's condition on an S-metric spaces (see [20, 21] for more details). In 2017, the same authors proved in [22] some fixed point theorems using new contractive conditions of integral type on a complete S-metric spaces and give some illustrative

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examples to verify the obtained results. They also give an application to Fredholm integral equation. Many results which were proved earlier in metric space are valid in the framework of S-metric spaces.

On the other hand, Bhashkar and Lakshmikantham in [24] introduced the concepts of coupled fixed points and mixed monotone property and illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later on these results were further extended and generalized by Ciric and Lakshmikantham [7] to coupled coincidence and coupled common fixed point results for nonlinear contractions in partially ordered metric spaces (see, also [5], [6], [16], [18], [19]).

In the year 2011, Aydi [2] proved some coupled fixed point theorems for various contractive type conditions in the setting of partial metric spaces and give some corollaries of the established results. Recently, Saluja [28] proved some common fixed point theorems for contractive type conditions in the setup of complex valued S-metric spaces. Very recently, Saluja [29] proved some coupled fixed point results for contractive type conditions in the framework of complex partial metric spaces (see, also [31]).

In 2018, Popa and Patricu [24] gave the concept of implicit functions in S-metric space which includes most of the existing literature's well-known contractions besides several new ones (see, also [23, 26, 27, 30]).

Recently, Kim [15] studied existence and uniqueness of coupled fixed point for a family of self-mappings satisfying a new coupled implicit relation in the framework of Hilbert space and also proved well-posedness of a coupled fixed point problem.

Motivated by the results and notions mentioned above, the purpose of this paper is to investigate coupled and common coupled fixed point theorems in the setting of S-metric spaces by using newly proposed coupled implicit contractive condition of three variables. Also we give some corollaries of the main results. Furthermore, we give an application to the Fredholm integral equation. Our results extend, generalize and enrich several results from the existing literature.

§2. Preliminaries

Sedghi et al. [33] gave an interesting generalization of D-metric space to S-metric space by formulating its properties as follows:

Definition 2.1([33]) *Let $\mathcal{X}$ be a nonempty set and let $\mathcal{S}\colon \mathcal{X}^3 \to [0,\infty)$ be a function satisfying the following conditions for all x, y, z, $a \in \mathcal{X}$:*

$(S1)$ $0 < \mathcal{S}(x,y,z)$ for all $x,y,z \in \mathcal{X}$ with $x \neq y \neq z$;

$(S2)$ $\mathcal{S}(x,y,z) = 0$ if and only if $x = y = z$;

$(S3)$ $\mathcal{S}(x,y,z) \leq \mathcal{S}(x,x,a) + \mathcal{S}(y,y,a) + \mathcal{S}(z,z,a)$.

Then, the function $\mathcal{S}$ is called an $\mathcal{S}$-metric on $\mathcal{X}$ and the pair $(\mathcal{X},\mathcal{S})$ is called an $\mathcal{S}$-metric space.

Example 2.2 ([33]) Let $\mathcal{X} = \mathbb{R}^n$ and $\|\cdot\|$ a norm on $\mathcal{X}$. Then,

(1) $\mathcal{S}(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an $\mathcal{S}$-metric on $\mathcal{X}$.

(2) $\mathcal{S}(x, y, z) = \|x - z\| + \|y - z\|$ is an $\mathcal{S}$-metric on $\mathcal{X}$.

Example 2.3 ([34]) Let $\mathcal{X} = \mathbb{R}$ be the real line. Then $\mathcal{S}(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an $\mathcal{S}$-metric on $\mathcal{X}$. This $\mathcal{S}$-metric on $\mathcal{X}$ is called the usual $\mathcal{S}$-metric on $\mathcal{X}$.

Example 2.4 ([13]) Let $\mathcal{X}$ be a non-empty set and d be an ordinary metric on $\mathcal{X}$. Then $\mathcal{S}(x, y, z) = d(x, z) + d(y, z)$ for all $x, y, z \in \mathbb{R}$ is an $\mathcal{S}$-metric on $\mathcal{X}$.

Example 2.5 ([35]) Let $\mathcal{X}$ be a non-empty set and d_1, d_2 be two ordinary metrics on $\mathcal{X}$. Then $\mathcal{S}(x, y, z) = d_1(x, z) + d_2(y, z)$ for all $x, y, z \in \mathcal{X}$ is an $\mathcal{S}$-metric on $\mathcal{X}$.

Sedghi et al. [33] proved that the D-metric space is the S-metric space, but in general the converse is not true.

We give some vivid illustrative examples on S-metric spaces as follows:

Example 2.6 ([33]) Let $\mathcal{X} = \mathbb{R}^2$, d is an ordinary metric on $\mathcal{X}$, therefore $\mathcal{S}(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ for all $x, y, z \in \mathbb{R}$ is a $\mathcal{S}$-metric on $\mathcal{X}$. If we connect the points x, y, z by a line, we have a triangle and if we choose a point a mediating this triangle then the inequality $\mathcal{S}(x, y, z) = \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)$ holds.

Example 2.7 ([33]) Let $\mathcal{X} = \mathbb{R}$, then $\mathcal{S}(x, y, z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$ is an $\mathcal{S}$-metric on $\mathcal{X}$. Define a self-map $\mathcal{F}$ on $\mathcal{X}$ by: $\mathcal{F}(x) = \frac{1}{2}\, sinx$, we have

$$\begin{aligned}
\mathcal{S}(\mathcal{F}x, \mathcal{F}x, \mathcal{F}y) &= \left|\frac{1}{2}(sinx - siny)\right| + \left|\frac{1}{2}(sinx - siny)\right| \\
&\leq \left|\frac{1}{2}(x - y)\right| + \left|\frac{1}{2}(x - y)\right| \\
&\leq \frac{1}{2}\big(|(x - y)| + |(x - y)|\big) \\
&= \frac{1}{2}\mathcal{S}(x, x, y)
\end{aligned}$$

for every $x, y \in \mathcal{X}$.

Definition 2.8 *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. For $r > 0$ and $x \in \mathcal{X}$, we define respectively the open ball $\mathcal{B}_{\mathcal{S}}(x, r)$ and closed ball $\mathcal{B}_{\mathcal{S}}[x, r]$ with center x and radius r as follows:*

$$\mathcal{B}_{\mathcal{S}}(x, r) = \{y \in \mathcal{X} : \mathcal{S}(y, y, x) < r\},$$

$$\mathcal{B}_{\mathcal{S}}[x, r] = \{y \in \mathcal{X} : \mathcal{S}(y, y, x) \leq r\}.$$

Example 2.9 ([34]) Let $\mathcal{X} = \mathbb{R}$. Denote by $\mathcal{S}(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Then

$$\begin{aligned}
\mathcal{B}_{\mathcal{S}}(1, 2) &= \{y \in \mathbb{R} : \mathcal{S}(y, y, 1) < 2\} = \{y \in \mathbb{R} : |y - 1| < 1\} \\
&= \{y \in \mathbb{R} : 0 < y < 2\} = (0, 2),
\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}_{\mathcal{S}}[2,4] &= \{y \in \mathbb{R} : \mathcal{S}(y,y,2) \leq 4\} = \{y \in \mathbb{R} : |y-2| \leq 2\} \\ &= \{y \in \mathbb{R} : 0 \leq y \leq 4\} = [0,4].\end{aligned}$$

Definition 2.10([33], [34]) *Let $(\mathcal{X},\mathcal{S})$ be an $\mathcal{S}$-metric space and $A \subset \mathcal{X}$.*

(a_1) *The subset A is said to be an open subset of $\mathcal{X}$, if for every $x \in A$ there exists $r > 0$ such that $\mathcal{B}_{\mathcal{S}}(x,r) \subset A$.*

(a_2) *A sequence $\{y_n\}$ in $\mathcal{X}$ converges to $y \in \mathcal{X}$ if $\mathcal{S}(y_n,y_n,y) \to 0$ as $n \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\mathcal{S}(y_n,y_n,y) < \varepsilon$. We denote this by $\lim_{n\to\infty} y_n = y$ or $y_n \to y$ as $n \to \infty$.*

(a_3) *A sequence $\{y_n\}$ in X is called a Cauchy sequence if $\mathcal{S}(y_n,y_n,y_m) \to 0$ as $n,m \to \infty$, that is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n,m \geq n_0$ we have $\mathcal{S}(y_n,y_n,y_m) < \varepsilon$.*

(a_4) *The $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ is called complete if every Cauchy sequence in $\mathcal{X}$ is convergent in $\mathcal{X}$.*

(a_5) *Let τ be the set of all $A \subset \mathcal{X}$ with the property that for each $x \in A$ and there exists $r > 0$ such that $\mathcal{B}_{\mathcal{S}}(x,r) \subset A$. Then τ is a topology on $\mathcal{X}$ (induced by the $\mathcal{S}$-metric space).*

(a_6) *A nonempty subset A of $\mathcal{X}$ is $\mathcal{S}$-closed if closure of A coincides with A.*

Definition 2.11([33]) *Let $(\mathcal{X},\mathcal{S})$ be an $\mathcal{S}$-metric space. A mapping $\mathcal{R}\colon \mathcal{X} \to \mathcal{X}$ is said to be a contraction if there exists a constant $0 \leq b < 1$ such that*

$$\mathcal{S}(\mathcal{R}x,\mathcal{R}y,\mathcal{R}z) \leq b\,\mathcal{S}(x,y,z), \tag{2.1}$$

for all $x,y,z \in \mathcal{X}$.

Remark 2.12 If the $\mathcal{S}$-metric space $(\mathcal{X},\mathcal{S})$ is complete then the mapping defined as above has a unique fixed point (see [33], Theorem 3.1).

Definition 2.13([33]) *Let $(\mathcal{X},\mathcal{S})$ and $(\mathcal{Y},\mathcal{S}')$ be two $\mathcal{S}$-metric spaces. A function $W\colon \mathcal{X} \to \mathcal{Y}$ is said to be continuous at a point $x_0 \in \mathcal{X}$ if for every sequence $\{x_n\}$ in $\mathcal{X}$ with $\mathcal{S}(x_n,x_n,x_0) \to 0$, $\mathcal{S}'(W(x_n),W(x_n),W(x_0)) \to 0$ as $n \to \infty$. We say that W is continuous on $\mathcal{X}$ if W is continuous at every point $x_0 \in \mathcal{X}$.*

Definition 2.14 *Let $\mathcal{X}$ be a non-empty set and let $M,N\colon \mathcal{X} \to \mathcal{X}$ be two self-mappings of $\mathcal{X}$. Then a point $\alpha \in \mathcal{X}$ is called a*

(1) *fixed point of operator M if $M(\alpha) = \alpha$;*
(2) *common fixed point of M and N if $M(\alpha) = N(\alpha) = \alpha$*
(3) *Smarandachely multifixed point of M and N if $M(\alpha) = \alpha$ or $N(\alpha) = \alpha$.*

Definition 2.15([1]) *Let M and N be single valued self-mappings on a set $\mathcal{X}$. If $z = M(\alpha) = N(\alpha)$ for some $\alpha \in \mathcal{X}$, then α is called a coincidence point of M and N, and z is called a point of coincidence of M and N. We denote the coincidence point of M and N by $C(M,N)$, that is, $C(M,N) = \{\alpha \in \mathcal{X} : M(\alpha) = N(\alpha)\}$.*

Definition 2.16([11]) *Let M and N be single valued self-mappings on a set $\mathcal{X}$. Mappings M and N are said to be commuting if $MNu = NMu$ for all $u \in \mathcal{X}$.*

Example 2.17 Let $\mathcal{X} = [0, \frac{3}{4}]$ and define $M, N: \mathcal{X} \to \mathcal{X}$ defined by $M(x) = \frac{x^3}{4}$ and $N(x) = x^4$ for all $x, y \in \mathcal{X}$. Then the mappings M and N have two coincidence points 0 and $\frac{1}{4}$. Clearly, they commute at 0 but not at $\frac{1}{4}$.

Definition 2.18([12]) *Let M and N be single valued self-mappings on a set $\mathcal{X}$. Mappings M and N are said to be weakly compatible if they commute at their coincidence points, i.e., if $M\alpha = N\alpha$ for some $\alpha \in \mathcal{X}$ implies $MN\alpha = NM\alpha$.*

Definition 2.19([11]) *An element $(x, y) \in \mathcal{X} \times \mathcal{X}$ is called*

(b_1) A coupled fixed point ([2]) of the mapping $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ if $F(x, y) = x$ and $F(y, x) = y$;

(b_2) A coupled coincidence point ([7]) of the mappings $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $Q: \mathcal{X} \to \mathcal{X}$ if $F(x, y) = Q(x)$ and $F(y, x) = Q(y)$;

(b_3) a common coupled fixed point ([14]) of the mappings $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and $Q: \mathcal{X} \to \mathcal{X}$ if $x = F(x, y) = Q(x)$ and $y = F(y, x) = Q(y)$.

Example 2.20 Let $\mathcal{X} = [0, +\infty)$ and $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ defined by $F(x, y) = \frac{x+y}{3}$ for all $x, y \in \mathcal{X}$. One can easily see that F has a unique coupled fixed point $(0, 0)$.

Example 2.21 Let $\mathcal{X} = [0, +\infty)$ and $F: \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ be defined by $F(x, y) = \frac{x+y}{2}$ for all $x, y \in \mathcal{X}$. Then we see that F has two coupled fixed point $(0, 0)$ and $(1, 1)$, that is, the coupled fixed point is not unique.

Lemma 2.22([33], Lemma 2.5) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. Then, we have $\mathcal{S}(x, x, y) = \mathcal{S}(y, y, x)$ for all $x, y \in \mathcal{X}$.*

Lemma 2.23([33], Lemma 2.12) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $\mathcal{S}(x_n, x_n, y_n) \to \mathcal{S}(x, x, y)$ as $n \to \infty$.*

Lemma 2.24([9], Lemma 8) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space and A is a nonempty subset of $\mathcal{X}$. Then A is said to be $\mathcal{S}$-closed if and only if for any sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$, then $x \in A$.*

Lemma 2.25([33]) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. If $r > 0$ and $x \in \mathcal{X}$, then the ball $\mathcal{B}_{\mathcal{S}}(x, r)$ is an open subset of $\mathcal{X}$.*

Lemma 2.26([34]) *The limit of a sequence $\{x_n\}$ in an $\mathcal{S}$-metric space $(\mathcal{X}, \mathcal{S})$ is unique.*

Lemma 2.27([33]) *Let $(\mathcal{X}, \mathcal{S})$ be an $\mathcal{S}$-metric space. Then any convergent sequence $\{x_n\}$ in $\mathcal{X}$ is Cauchy.*

In the following lemma we see the relationship between a metric and $\mathcal{S}$-metric.

Lemma 2.28([10]) *Let $(\mathcal{X}, d)$ be a metric space. Then the following properties are satisfied:*

(c_1) *$\mathcal{S}_d(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in \mathcal{X}$ is an $\mathcal{S}$-metric on $\mathcal{X}$.*

(c_2) *$x_n \to x$ in $(\mathcal{X}, d)$ if and only if $x_n \to x$ in $(\mathcal{X}, \mathcal{S}_d)$.*

(c_3) *$\{x_n\}$ is Cauchy in $(\mathcal{X}, d)$ if and only if $\{x_n\}$ is Cauchy in $(\mathcal{X}, \mathcal{S}_d)$.*

(c_4) *$(\mathcal{X}, d)$ is complete if and only if $(\mathcal{X}, \mathcal{S}_d)$ is complete.*

We call the function $\mathcal{S}_d$ defined in Lemma 2.28 (c_1) as the $\mathcal{S}$-metric generated by the metric d. It can be found an example of an $\mathcal{S}$-metric which is not generated by any metric in [10, 21].

Example 2.29 Let $\mathcal{X} = \mathbb{R}$ and the function $\mathcal{S}\colon \mathcal{X}^3 \to [0,\infty)$ be defined as

$$\mathcal{S}(x,y,z) = |x-z| + |x+z-2y|,$$

for all $x,y,z \in \mathbb{R}$. Then the function $\mathcal{S}$ is an $\mathcal{S}$-metric on $\mathcal{X}$ and $(\mathcal{X}, \mathcal{S})$ is an $\mathcal{S}$-metric space. Now, we prove that there does not exists any metric d such that $\mathcal{S} = \mathcal{S}_d$. On the contrary, suppose that there exists a metric d such that

$$\mathcal{S}(x,y,z) = d(x,z) + d(y,z),$$

for all $x,y,z \in \mathbb{R}$. Hence, we obtain

$$\mathcal{S}(x,x,z) = 2d(x,z) = 2|x-z|,$$

and

$$d(x,z) = |x-z|.$$

Similarly, we get

$$\mathcal{S}(y,y,z) = 2d(y,z) = 2|y-z|,$$

and

$$d(y,z) = |y-z|,$$

for all $x,y,z \in \mathbb{R}$. Hence, we have

$$|x-z| + |x+z-2y| = |x-z| + |y-z|,$$

which is a contradiction. Therefore, $\mathcal{S} \neq \mathcal{S}_d$ and $(\mathbb{R}, \mathcal{S})$ is a complete $\mathcal{S}$-metric space.

Now, we introduce a new coupled implicit relation.

Definition 2.30 *Let $\mathbb{R}_+$ (where $\mathbb{R}_+ = [0,\infty)$) be the set of all nonnegative real numbers, Ω be the class of all continuous real valued functions $\omega\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ non-decreasing in the third argument satisfying the following conditions: for $x,y,\alpha,\beta > 0$,*

$(CIR1)$ $x \leq \omega\Big(\frac{\alpha+\beta}{2}, \frac{x+\alpha}{2}, \frac{y+\beta}{2}\Big)$ *and* $y \leq \omega\Big(\frac{\alpha+\beta}{2}, \frac{y+\beta}{2}, \frac{x+\alpha}{2}\Big)$ *or*

$(CIR2)$ $x \leq \omega\Big(\frac{\alpha+\beta}{2}, 0, 0\Big)$ *and* $y \leq \omega\Big(\frac{\alpha+\beta}{2}, 0, 0\Big)$ *or*

$(CIR3)$ $x \leq \omega\Big(0, \frac{\alpha}{2}, \frac{\beta}{2}\Big)$ *and* $y \leq \omega\Big(0, \frac{\beta}{2}, \frac{\alpha}{2}\Big)$,

there exists a real number $0 < h < 1$ *such that* $x + y \leq h(\alpha + \beta)$.

§3. Main Results

In this section, we prove some unique coupled fixed point and common coupled fixed point theorems for newly proposed coupled implicit contractive condition in the framework of S-metric spaces.

Theorem 3.1 *Let* $(\mathcal{X}, \mathcal{S})$ *be a complete* $\mathcal{S}$*-metric space. Let* $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ *be a mapping satisfying the following contractive condition: for all* $x, y, u, v, z, w \in \mathcal{X}$*:*

$$\begin{aligned}&\mathcal{S}(F(x,y), F(u,v), F(z,w))\\ &\leq \omega\Big(\frac{\mathcal{S}(x,u,z)+\mathcal{S}(y,v,w)}{2}, \frac{\mathcal{S}(F(x,y),F(x,y),x)+\mathcal{S}(F(z,w),F(z,w),z)}{2},\\ &\quad \frac{\mathcal{S}(F(v,u),F(v,u),v)+\mathcal{S}(F(w,z),F(w,z),w)}{2}\Big), \end{aligned} \tag{3.1}$$

where $\omega \in \Omega$*. If* F *is continuous, then* F *has a unique coupled point in* $\mathcal{X}$*.*

Proof Choose $x_0, y_0 \in X$. Set $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. Repeating this process, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$. Then, from equations (3.1), using $(S2)$ and Lemma 2.22, we have

$$\begin{aligned}\mathcal{S}(x_n, x_n, x_{n+1}) &= \mathcal{S}(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n))\\ &\leq \omega\Big(\frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n)+\mathcal{S}(y_{n-1}, y_{n-1}, y_n)}{2},\\ &\quad \frac{\mathcal{S}(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), x_{n-1})+\mathcal{S}(F(x_n, y_n), F(x_n, y_n), x_n)}{2},\\ &\quad \frac{\mathcal{S}(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), y_{n-1})+\mathcal{S}(F(y_n, x_n), F(y_n, x_n), y_n)}{2}\Big)\\ &= \omega\Big(\frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n)+\mathcal{S}(y_{n-1}, y_{n-1}, y_n)}{2},\\ &\quad \frac{\mathcal{S}(x_n, x_n, x_{n-1})+\mathcal{S}(x_{n+1}, x_{n+1}, x_n)}{2},\\ &\quad \frac{\mathcal{S}(y_n, y_n, y_{n-1})+\mathcal{S}(y_{n+1}, y_{n+1}, y_n)}{2}\Big)\\ &= \omega\Big(\frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n)+\mathcal{S}(y_{n-1}, y_{n-1}, y_n)}{2},\\ &\quad \frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n)+\mathcal{S}(x_n, x_n, x_{n+1})}{2},\\ &\quad \frac{\mathcal{S}(y_{n-1}, y_{n-1}, y_n)+\mathcal{S}(y_n, y_n, y_{n+1})}{2}\Big). \end{aligned} \tag{3.2}$$

Similarly, one can show that

$$\begin{aligned}\mathcal{S}(y_n, y_n, y_{n+1}) &= \mathcal{S}(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq \omega\Big(\frac{\mathcal{S}(y_{n-1}, y_{n-1}, y_n) + \mathcal{S}(x_{n-1}, x_{n-1}, x_n)}{2}, \\ &\qquad \frac{\mathcal{S}(y_{n-1}, y_{n-1}, y_n) + \mathcal{S}(y_n, y_n, y_{n+1})}{2}, \\ &\qquad \frac{\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(x_n, x_n, x_{n+1})}{2}\Big). \end{aligned} \tag{3.3}$$

Let

$$\begin{aligned} x &= \mathcal{S}(x_n, x_n, x_{n+1}), \quad y = \mathcal{S}(y_n, y_n, y_{n+1}), \\ \alpha &= \mathcal{S}(x_{n-1}, x_{n-1}, x_n), \quad \beta = \mathcal{S}(y_{n-1}, y_{n-1}, y_n). \end{aligned}$$

Hence from Definition 2.30 ($CIR1$), there exists $0 < h < 1$ such that

$$\begin{aligned} &\mathcal{S}(x_n, x_n, x_{n+1}) + \mathcal{S}(y_n, y_n, y_{n+1}) \\ &\quad \leq h\,[\mathcal{S}(x_{n-1}, x_{n-1}, x_n) + \mathcal{S}(y_{n-1}, y_{n-1}, y_n)]. \end{aligned} \tag{3.4}$$

Set $\mathcal{D}_n = \mathcal{S}(x_n, x_n, x_{n+1}) + \mathcal{S}(y_n, y_n, y_{n+1})$. Then equation (3.4) implies that

$$\mathcal{D}_n \leq h\,\mathcal{D}_{n-1}. \tag{3.5}$$

Consequently, for each $n \in \mathbb{N}$, we have

$$\mathcal{D}_n \leq h\,\mathcal{D}_{n-1} \leq h^2\,\mathcal{D}_{n-2} \leq \cdots \leq h^n\,\mathcal{D}_0. \tag{3.6}$$

If $\mathcal{D}_0 = 0$, then $\mathcal{S}(x_0, x_0, x_1) + \mathcal{S}(y_0, y_0, y_1) = 0$. Hence, by condition ($S2$), we get $x_0 = x_1 = F(x_0, y_0)$ and $y_0 = y_1 = F(y_0, x_0)$. Thus, (x_0, y_0) is a coupled fixed point of F. Now, we assume that $\mathcal{D}_0 > 0$. For each $m > n$, where $n, m \in \mathbb{N}$, and using ($S3$), we have

$$\begin{aligned} &\mathcal{S}(x_n, x_n, x_m) + \mathcal{S}(y_n, y_n, y_m) \\ &\leq 2\mathcal{S}(x_n, x_n, x_{n+1}) + \mathcal{S}(x_m, x_m, x_{n+1}) \\ &\qquad +2\mathcal{S}(y_n, y_n, y_{n+1}) + \mathcal{S}(y_m, y_m, y_{n+1}) \\ &= 2(\mathcal{S}(x_n, x_n, x_{n+1}) + \mathcal{S}(y_n, y_n, y_{n+1})) \\ &\qquad +\mathcal{S}(x_m, x_m, x_{n+1}) + \mathcal{S}(y_m, y_m, y_{n+1}) \\ &\leq \cdots\cdots\cdots\cdots\cdots\cdots \\ &\leq 2\Big(\mathcal{D}_n + \mathcal{D}_{n+1} + \cdots + \mathcal{D}_{m-1} + \mathcal{D}_m\Big) \\ &\leq 2(h^n + h^{n+1} + \cdots + h^{m-1} + h^m)\mathcal{D}_0 \\ &\leq 2h^n(1 + h + h^2 + \dots)\mathcal{D}_0 \\ &\leq \Big(\frac{2h^n}{1-h}\Big)\mathcal{D}_0 \to 0 \text{ as } n \to \infty, \end{aligned}$$

since $0 < h < 1$. Thus, $\{x_n\}$ and $\{y_n\}$ are $\mathcal{S}$-Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is complete, we get $\{x_n\}$ and $\{y_n\}$ are $\mathcal{S}$-convergent to some $c \in \mathcal{X}$ and $d \in \mathcal{X}$ respectively, that is, $\lim_{n\to\infty} x_n = c$ and $\lim_{n\to\infty} y_n = d$. Since F is continuous, then we have

$$\begin{aligned} c &= \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n) \\ &= F\Big(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\Big) = F(c,d), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} d &= \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n) \\ &= F\Big(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n\Big) = F(d,c). \end{aligned} \tag{3.8}$$

This shows that (c,d) is a coupled fixed point of F.

Now, we show the uniqueness of the coupled fixed point. Assume that (c_1, d_1) is another coupled fixed point of F such that $(c,d) \neq (c_1, d_1)$. Then from equation (3.1), using $(S2)$ and Lemma 2.22, we have

$$\begin{aligned} \mathcal{S}(c,c,c_1) &= \mathcal{S}(F(c,d), F(c,d), F(c_1,d_1)) \\ &\leq \omega\Big(\frac{\mathcal{S}(c,c,c_1) + \mathcal{S}(d,d,d_1)}{2}, \\ &\qquad \frac{\mathcal{S}(F(c,d),F(c,d),c) + \mathcal{S}(F(c_1,d_1),F(c_1,d_1),c_1)}{2}, \\ &\qquad \frac{\mathcal{S}(F(d,c),F(d,c),d) + \mathcal{S}(F(d_1,c_1),F(d_1,c_1),d_1)}{2}\Big) \\ &= \omega\Big(\frac{\mathcal{S}(c,c,c_1) + \mathcal{S}(d,d,d_1)}{2}, \frac{\mathcal{S}(c,c,c) + \mathcal{S}(c_1,c_1,c_1)}{2}, \\ &\qquad \frac{\mathcal{S}(d,d,d) + \mathcal{S}(d_1,d_1,d_1)}{2}\Big) \\ &= \omega\Big(\frac{\mathcal{S}(c,c,c_1) + \mathcal{S}(d,d,d_1)}{2}, 0, 0\Big). \end{aligned} \tag{3.9}$$

Similarly, we have

$$\begin{aligned} \mathcal{S}(d,d,d_1) &= \mathcal{S}(F(d,c), F(d,c), F(d_1,c_1)) \\ &\leq \omega\Big(\frac{\mathcal{S}(d,d,d_1) + \mathcal{S}(c,c,c_1)}{2}, \\ &\qquad \frac{\mathcal{S}(F(d,c),F(d,c),d) + \mathcal{S}(F(d_1,c_1),F(d_1,c_1),d_1)}{2}, \\ &\qquad \frac{\mathcal{S}(F(c,d),F(c,d),c) + \mathcal{S}(F(c_1,d_1),F(c_1,d_1),c_1)}{2}\Big) \\ &= \omega\Big(\frac{\mathcal{S}(d,d,d_1) + \mathcal{S}(c,c,c_1)}{2}, \frac{\mathcal{S}(d,d,d) + \mathcal{S}(d_1,d_1,d_1)}{2}, \\ &\qquad \frac{\mathcal{S}(c,c,c) + \mathcal{S}(c_1,c_1,c_1)}{2}\Big) \\ &= \omega\Big(\frac{\mathcal{S}(d,d,d_1) + \mathcal{S}(c,c,c_1)}{2}, 0, 0\Big) \\ &= \omega\Big(\frac{\mathcal{S}(c,c,c_1) + \mathcal{S}(d,d,d_1)}{2}, 0, 0\Big). \end{aligned} \tag{3.10}$$

Hence from Definition 2.30 ($CIR2$), there exists $0 < h < 1$ such that

$$\mathcal{S}(c, c, c_1) + \mathcal{S}(d, d, d_1) \leq h\,[\mathcal{S}(c, c, c_1) + \mathcal{S}(d, d, d_1)], \tag{3.11}$$

which is a contradiction, since $0 < h < 1$. Hence, we conclude that $\mathcal{S}(c, c, c_1) + \mathcal{S}(d, d, d_1) = 0$, that is, $\mathcal{S}(c, c, c_1) = 0$ and $\mathcal{S}(d, d, d_1) = 0$. Hence by condition ($S2$), we have $c = c_1$ and $d = d_1$. This shows that the coupled fixed point of F is unique. This completes the proof. □

Remark 3.2 If $x = y$ and $\alpha = \beta$ in Definition 2.30, the coupled implicit relation conditions restricted follows implicit relation conditions:

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers, Ω be the class of all continuous real valued functions $\omega\colon \mathbb{R}_+^3 \to \mathbb{R}_+$ non-decreasing in the third argument and satisfying the following conditions: for $x, \alpha > 0$,

($IR1$) $x \leq \omega\Big(\alpha, \frac{x+\alpha}{2}, \frac{x+\alpha}{2}\Big)$, or

($IR2$) $x \leq \omega\big(\alpha, 0, 0\big)$, or

($IR3$) $x \leq \omega\Big(0, \frac{\alpha}{2}, \frac{\alpha}{2}\Big)$,

then, there exists a real number $0 < h < 1$ such that $x \leq h\,\alpha$.

Theorem 3.2 *Let $(\mathcal{X}, \mathcal{S})$ be a complete $\mathcal{S}$-metric space. Suppose that the mappings $F\colon \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ and $A\colon \mathcal{X} \to \mathcal{X}$ satisfy the the following contractive condition: for all $x, y, u, v, z, w \in \mathcal{X}$:*

$$\begin{aligned}\mathcal{S}(F(x, y), F(u, v), F(z, w)) \leq \omega\Big(&\frac{\mathcal{S}(Ax, Au, Az) + \mathcal{S}(Ay, Av, Aw)}{2},\\ &\frac{\mathcal{S}(F(x, y), F(x, y), Ax) + \mathcal{S}(F(z, w), F(z, w), Az)}{2},\\ &\frac{\mathcal{S}(F(v, u), F(v, u), Av) + \mathcal{S}(F(w, z), F(w, z), Aw)}{2}\Big),\end{aligned} \tag{3.12}$$

where $\omega \in \Omega$. Assume that F and A satisfy the following conditions:

(i) $F(\mathcal{X} \times \mathcal{X}) \subseteq A(\mathcal{X})$;

(ii) $A(\mathcal{X})$ *is complete, and*

(iii) A *is continuous and commutes with* F.

Then, F and A have a coupled coincidence point in $\mathcal{X}$. Moreover, if F and A are weakly compatible, then F and A have a unique common coupled fixed point.

Proof Let $x_0, y_0 \in \mathcal{X}$. Since $F(\mathcal{X} \times \mathcal{X}) \subseteq A(\mathcal{X})$, we can choose $x_1, y_1 \in \mathcal{X}$ such that $Ax_1 = F(x_0, y_0)$ and $Ay_1 = F(y_0, x_0)$. Again since $F(\mathcal{X}\times\mathcal{X}) \subseteq A(\mathcal{X})$, we can choose $x_2, y_2 \in \mathcal{X}$ such that $Ax_2 = F(x_1, y_1)$ and $Ay_2 = F(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that $Ax_{n+1} = F(x_n, y_n)$ and $Ay_{n+1} = F(y_n, x_n)$. For

$n \in \mathbb{N}$, by equation (3.12), and using Lemma 2.22, we have

$$\begin{aligned}
&\mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) = \mathcal{S}(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), F(x_n, y_n))\\
&\le \omega\Big(\frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)}{2},\\
&\quad \frac{\mathcal{S}(F(x_{n-1}, y_{n-1}), F(x_{n-1}, y_{n-1}), Ax_{n-1}) + \mathcal{S}(F(x_n, y_n), F(x_n, y_n), Ax_n)}{2},\\
&\quad \frac{\mathcal{S}(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), Ay_{n-1}) + \mathcal{S}(F(y_n, x_n), F(y_n, x_n), Ay_n)}{2}\Big)\\
&= \omega\Big(\frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)}{2},\\
&\quad \frac{\mathcal{S}(Ax_n, Ax_n, Ax_{n-1}) + \mathcal{S}(Ax_{n+1}, Ax_{n+1}, Ax_n)}{2},\\
&\quad \frac{\mathcal{S}(Ay_n, Ay_n, Ay_{n-1}) + \mathcal{S}(Ay_{n+1}, Ay_{n+1}, Ay_n)}{2}\Big)\\
&= \omega\Big(\frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)}{2},\\
&\quad \frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ax_n, Ax_n, Ax_{n+1})}{2},\\
&\quad \frac{\mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})}{2}\Big). \qquad (3.13)
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\mathcal{S}(Ay_n, Ay_n, Ay_{n+1}) = \mathcal{S}(F(y_{n-1}, x_{n-1}), F(y_{n-1}, x_{n-1}), F(y_n, x_n))\\
&\le \omega\Big(\frac{\mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n) + \mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n)}{2},\\
&\quad \frac{\mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})}{2},\\
&\quad \frac{\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ax_n, Ax_n, Ax_{n+1})}{2}\Big). \qquad (3.14)
\end{aligned}$$

Let $x = \mathcal{S}(Ax_n, Ax_n, Ax_{n+1})$, $y = \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})$, $\alpha = \mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n)$ and $\beta = \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)$. Hence from Definition 2.30 ($CIR1$), there exists $0 < h < 1$ such that

$$\begin{aligned}
&\mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})\\
&\le h\,[\mathcal{S}(Ax_{n-1}, Ax_{n-1}, Ax_n) + \mathcal{S}(Ay_{n-1}, Ay_{n-1}, Ay_n)]. \qquad (3.15)
\end{aligned}$$

Set $\mathcal{K}_n = \mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1})$. Then, equation (3.15) implies that

$$\mathcal{K}_n \le h\,\mathcal{K}_{n-1}. \qquad (3.16)$$

Consequently, for each $n \in \mathbb{N}$, we have

$$\mathcal{K}_n \le h\,\mathcal{K}_{n-1} \le h^2\,\mathcal{K}_{n-2} \le \cdots \le h^n\,\mathcal{K}_0. \qquad (3.17)$$

If $\mathcal{K}_0 = 0$, then $\mathcal{S}(Ax_0, Ax_0, Ax_1) + \mathcal{S}(Ay_0, Ay_0, Ay_1) = 0$. Hence, by condition ($S2$), we get $Ax_0 = Ax_1 = F(x_0, y_0)$ and $Ay_0 = Ay_1 = F(y_0, x_0)$. Thus, (Ax_0, Ay_0) is a coupled fixed point of F and A. Now, we assume that $\mathcal{K}_0 > 0$. For each $m > n$, where $n, m \in \mathbb{N}$, and using $(S3)$, we have

$$\begin{aligned}
&\mathcal{S}(Ax_n, Ax_n, Ax_m) + \mathcal{S}(Ay_n, Ay_n, Ay_m)\\
&\leq 2\mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) + \mathcal{S}(Ax_m, Ax_m, Ax_{n+1})\\
&\quad +2\mathcal{S}(Ay_n, Ay_n, Ay_{n+1}) + \mathcal{S}(Ay_m, Ay_m, Ay_{n+1})\\
&= 2(\mathcal{S}(Ax_n, Ax_n, Ax_{n+1}) + \mathcal{S}(Ay_n, Ay_n, Ay_{n+1}))\\
&\quad +\mathcal{S}(Ax_m, Ax_m, Ax_{n+1}) + \mathcal{S}(Ay_m, Ay_m, Ay_{n+1})\\
&\leq \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\leq 2\Big(\mathcal{K}_n + \mathcal{K}_{n+1} + \cdots + \mathcal{K}_{m-1} + \mathcal{K}_m\Big)\\
&\leq 2(h^n + h^{n+1} + \cdots + h^{m-1} + h^m)\mathcal{K}_0\\
&\leq 2h^n(1 + h + h^2 + \dots)\mathcal{K}_0\\
&\leq \Big(\frac{2h^n}{1-h}\Big)\mathcal{K}_0 \to 0 \text{ as } n \to \infty,
\end{aligned}$$

since $0 < h < 1$. Thus, $\{Ax_n\}$ and $\{Ay_n\}$ are $\mathcal{S}$-Cauchy sequence in $A(\mathcal{X})$. Since $A(\mathcal{X})$ is complete, we get $\{Ax_n\}$ and $\{Ay_n\}$ are $\mathcal{S}$-convergent to some $p_1 \in \mathcal{X}$ and $p_2 \in \mathcal{X}$ respectively. Since A is continuous, we have $\{AAx_n\}$ is $\mathcal{S}$-convergent to Ap_1 and $\{AAy_n\}$ is $\mathcal{S}$-convergent to Ap_2. Also, since A and F are commute, we have

$$AAx_{n+1} = A(F(x_n, y_n)) = F(Ax_n, Ay_n),$$

and

$$AAy_{n+1} = A(F(y_n, x_n)) = F(Ay_n, Ax_n).$$

Therefore,

$$\begin{aligned}
&\mathcal{S}(AAx_{n+1}, AAx_{n+1}, F(p_1, p_2)) = \mathcal{S}(F(Ax_n, Ay_n), F(Ax_n, Ay_n), F(p_1, p_2))\\
&\leq \omega\Big(\frac{\mathcal{S}(AAx_n, AAx_n, Ap_1) + \mathcal{S}(AAy_n, AAy_n, Ap_2)}{2},\\
&\qquad \frac{\mathcal{S}(F(Ax_n, Ay_n), F(Ax_n, Ay_n), AAx_n) + \mathcal{S}(F(p_1, p_2), F(p_1, p_2), Ap_1)}{2},\\
&\qquad \frac{\mathcal{S}(F(Ay_n, Ax_n), F(Ay_n, Ax_n), AAy_n) + \mathcal{S}(F(p_2, p_1), F(p_2, p_1), Ap_2)}{2}\Big)\\
&= \omega\Big(\frac{\mathcal{S}(AAx_n, AAx_n, Ap_1) + \mathcal{S}(AAy_n, AAy_n, Ap_2)}{2},\\
&\qquad \frac{\mathcal{S}(AAx_{n+1}, AAx_{n+1}, AAx_n) + \mathcal{S}(F(p_1, p_2), F(p_1, p_2), Ap_1)}{2},\\
&\qquad \frac{\mathcal{S}(AAy_{n+1}, AAy_{n+1}, AAy_n) + \mathcal{S}(F(p_2, p_1), F(p_2, p_1), Ap_2)}{2}\Big). \qquad (3.18)
\end{aligned}$$

Passing to the limit as $n \to \infty$ in equation (3.18), using Lemmas 2.22, 2.23 and the condition $(S2)$, we obtain

$$\begin{aligned}\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2)) &\leq \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ap_2, Ap_2, Ap_2)}{2},\\ &\qquad \frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ap_1, Ap_1, F(p_1,p_2))}{2},\\ &\qquad \frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))}{2}\Big)\\ &= \omega\Big(0, \frac{\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2))}{2}, \frac{\mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))}{2}\Big). \qquad (3.19)\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\mathcal{S}(AAy_{n+1}, AAy_{n+1}, F(p_2,p_1)) = \mathcal{S}(F(Ay_n, Ax_n), F(Ay_n, Ax_n), F(p_2,p_1))\\ &\leq \omega\Big(\frac{\mathcal{S}(AAy_n, AAy_n, Ap_2) + \mathcal{S}(AAx_n, AAx_n, Ap_1)}{2},\\ &\qquad \frac{\mathcal{S}(F(Ay_n, Ax_n), F(Ay_n, Ax_n), AAy_n) + \mathcal{S}(F(p_2,p_1), F(p_2,p_1), Ap_2)}{2},\\ &\qquad \frac{\mathcal{S}(F(Ax_n, Ay_n), F(Ax_n, Ay_n), AAx_n) + \mathcal{S}(F(p_1,p_2), F(p_1,p_2), Ap_1)}{2}\Big)\\ &= \omega\Big(\frac{\mathcal{S}(AAy_n, AAy_n, Ap_2) + \mathcal{S}(AAx_n, AAx_n, Ap_1)}{2},\\ &\qquad \frac{\mathcal{S}(AAy_{n+1}, AAy_{n+1}, AAy_n) + \mathcal{S}(F(p_2,p_1), F(p_2,p_1), Ap_2)}{2},\\ &\qquad \frac{\mathcal{S}(AAx_{n+1}, AAx_{n+1}, AAx_n) + \mathcal{S}(F(p_1,p_2), F(p_1,p_2), Ap_1)}{2}\Big). \qquad (3.20)\end{aligned}$$

Passing to the limit as $n \to \infty$ in equation (3.20), using Lemmas 2.22, 2.23 and the condition $(S2)$, we obtain

$$\begin{aligned}\mathcal{S}(Ap_2, Ap_2, F(p_2,p_1)) &\leq \omega\Big(\frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ap_1, Ap_1, Ap_1)}{2},\\ &\qquad \frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))}{2},\\ &\qquad \frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ap_1, Ap_1, F(p_1,p_2))}{2}\Big)\\ &= \omega\Big(0, \frac{\mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))}{2}, \frac{\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2))}{2}\Big). \qquad (3.21)\end{aligned}$$

Hence from Definition 2.30 $(CIR3)$, there exists $0 < h < 1$ such that

$$\begin{aligned}&\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2)) + \mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))\\ &\qquad \leq h\,[\mathcal{S}(Ap_1, Ap_1, F(p_1,p_2)) + \mathcal{S}(Ap_2, Ap_2, F(p_2,p_1))],\end{aligned}$$

which is a contradiction, since $0 < h < 1$. Hence, we conclude that

$$\mathcal{S}(Ap_1, Ap_1, F(p_1, p_2)) + \mathcal{S}(Ap_2, Ap_2, F(p_2, p_1)) = 0,$$

that is, $\mathcal{S}(Ap_1, Ap_1, F(p_1, p_2)) = 0$ and $\mathcal{S}(Ap_2, Ap_2, F(p_2, p_1)) = 0$. Hence $Ap_1 = F(p_1, p_2)$ and $Ap_2 = F(p_2, p_1)$. Thus (Ap_1, Ap_2) is a coupled coincidence point of the mappings F and A. Since the pair (F, A) is weakly compatible, so by weak compatibility of the mappings F and A, we have

$$A(F(p_1, p_2)) = F(Ap_1, Ap_2) = Ap_1 \text{ and } A(F(p_2, p_1)) = F(Ap_2, Ap_1) = Ap_2. \tag{3.22}$$

Hence (Ap_1, Ap_2) is a common coupled fixed point of F and A.

Now, we show the uniqueness of the common coupled fixed point of F and A. Assume that (Ar_1, Ar_2) is another common coupled fixed point of F and A with $Ap_1 \neq Ar_1$ and $Ap_2 \neq Ar_2$, that is, $(Ap_1, Ap_2) \neq (Ar_1, Ar_2)$. Then by using equation (3.12), using Lemma 2.22 and the condition $(S2)$, we have

$$\begin{aligned}
&S(Ap_1, Ap_1, Ar_1) = S(F(p_1, p_2), F(p_1, p_2), F(r_1, r_2)) \\
&\leq \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)}{2}, \\
&\quad \frac{\mathcal{S}(F(p_1, p_2), F(p_1, p_2), Ap_1) + \mathcal{S}(F(r_1, r_2), F(r_1, r_2), Ar_1)}{2}, \\
&\quad \frac{\mathcal{S}(F(p_2, p_1), F(p_2, p_1), Ap_2) + \mathcal{S}(F(r_2, r_1), F(r_2, r_1), Ar_2)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)}{2}, \\
&\quad \frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ar_1, Ar_1, Ar_1)}{2}, \\
&\quad \frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ar_2, Ar_2, Ar_2)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)}{2}, 0, 0\Big).
\end{aligned} \tag{3.23}$$

Similarly, we obtain

$$\begin{aligned}
&S(Ap_2, Ap_2, Ar_2) = S(F(p_2, p_1), F(p_2, p_1), F(r_2, r_1)) \\
&\leq \omega\Big(\frac{\mathcal{S}(Ap_2, Ap_2, Ar_2) + \mathcal{S}(Ap_1, Ap_1, Ar_1)}{2}, \\
&\quad \frac{\mathcal{S}(F(p_2, p_1), F(p_2, p_1), Ap_2) + \mathcal{S}(F(r_2, r_1), F(r_2, r_1), Ar_2)}{2}, \\
&\quad \frac{\mathcal{S}(F(p_1, p_2), F(p_1, p_2), Ap_1) + \mathcal{S}(F(r_1, r_2), F(r_1, r_2), Ar_1)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ap_2, Ap_2, Ar_2) + \mathcal{S}(Ap_1, Ap_1, Ar_1)}{2}, \\
&\quad \frac{\mathcal{S}(Ap_2, Ap_2, Ap_2) + \mathcal{S}(Ar_2, Ar_2, Ar_2)}{2}, \\
&\quad \frac{\mathcal{S}(Ap_1, Ap_1, Ap_1) + \mathcal{S}(Ar_1, Ar_1, Ar_1)}{2}\Big) \\
&= \omega\Big(\frac{\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)}{2}, 0, 0\Big).
\end{aligned} \tag{3.24}$$

Hence, from Definition 2.30 ($CIR2$), there exists $0 < h < 1$ such that

$$\begin{aligned}&\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)\\&\leq h\,[\mathcal{S}(Ap_1, Ap_1, Ar_1) + \mathcal{S}(Ap_2, Ap_2, Ar_2)],\end{aligned}$$

which is a contradiction, since $0 < h < 1$. Hence, we conclude that $\mathcal{S}(Ap_1, Ap_1, Ar_1)$ $+\mathcal{S}(Ap_2, Ap_2, Ar_2)) = 0$, that is, $\mathcal{S}(Ap_1, Ap_1, Ar_1) = 0$ and $\mathcal{S}(Ap_2, Ap_2, Ar_2) = 0$. Hence $Ap_1 = Ar_1$ and $Ap_2 = Ar_2$. This shows that the common coupled fixed point (Ap_1, Ap_2) of the mappings F and A is unique. This completes the proof of Theorem 3.3. □

We define as $\mathcal{T}x = F(x, x)$. Then, we have the following corollary.

Corollary 3.4 *Let $(\mathcal{X}, \mathcal{S})$ be a complete $\mathcal{S}$-metric space. Suppose that the mapping $\mathcal{T} \colon \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition: for all $x, u, z \in \mathcal{X}$:*

$$\begin{aligned}\mathcal{S}(\mathcal{T}x, \mathcal{T}u, \mathcal{T}z) \quad \leq \quad & \omega\Big(\mathcal{S}(x, u, z), \frac{\mathcal{S}(\mathcal{T}x, \mathcal{T}x, x) + \mathcal{S}(\mathcal{T}z, \mathcal{T}z, z)}{2},\\ & \frac{\mathcal{S}(\mathcal{T}u, \mathcal{T}u, u) + \mathcal{S}(\mathcal{T}z, \mathcal{T}z, z)}{2}\Big),\end{aligned} \tag{3.25}$$

where $\omega \in \Omega$. If $\mathcal{T}$ is continuous, then $\mathcal{T}$ has a unique point in $\mathcal{X}$.

Proof Taking $x = y$, $u = v$ and $z = w$ in Theorem 3.1, then (3.1) coincides with (3.25). Thus, we have the conclusion of the Corollary from Theorem 3.1. □

Next, we give some analogues of coupled fixed point theorems in metric spaces for S-metric spaces by combining Theorem 3.1 with $\omega \in \Omega$ and ω satisfies the conditions ($CIR1$) and ($CIR2$). The following corollary is a generalization and extension of Corollary 2.2 of Aydi [2] from partial metric space to the setting of S-metric space.

Corollary 3.5 *Let $(\mathcal{X}, \mathcal{S})$ be a complete $\mathcal{S}$-metric space. Suppose that the mapping $F \colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ satisfying the following contractive condition: for all $x, y, u, v, z, w \in \mathcal{X}$:*

$$\mathcal{S}(F(x, y), F(u, v), F(z, w)) \leq \frac{k}{2}[\mathcal{S}(x, u, z) + \mathcal{S}(y, v, w)],$$

where $k \in [0, 1)$ is a constant. Then F has a unique coupled fixed point.

Proof The assertion follows using Theorem 3.1 with $\omega(p, q, r) = k\,p$ for some $k \in [0, 1)$ and all $p, q, r \in \mathbb{R}_+$. □

The following corollary is a generalization and extension of Corollary 2.6 of Aydi [2] from partial metric space to the setting of S-metric space.

Corollary 3.6 *Let $(\mathcal{X}, \mathcal{S})$ be a complete $\mathcal{S}$-metric space. Suppose that the mapping $F \colon \mathcal{X} \times \mathcal{X} \to$*

$\mathcal{X}$ satisfying the following contractive condition: for all $x, y, u, v, z, w \in \mathcal{X}$:

$$\mathcal{S}(F(x,y), F(u,v), F(z,w)) \leq \frac{k}{2}[\mathcal{S}(F(x,y), F(x,y), x) + \mathcal{S}(F(z,w), F(z,w), z)],$$

where $k \in [0,1)$ is a constant. Then F has a unique coupled fixed point.

Proof The assertion follows using Theorem 3.1 with $\omega(p,q,r) = k\,q$ for some $k \in [0,1)$ and all $p, q, r \in \mathbb{R}_+$. □

Now, we give an example to validate the result.

Example 3.7 Let $\mathcal{X} = [0,1]$ and the function $\mathcal{S}\colon \mathcal{X}^3 \to [0,\infty)$ be defined as $\mathcal{S}(x,y,z) = |y-z| + |y+z-2x|$ for all $x, y, z \in \mathcal{X}$. Then the function $\mathcal{S}$ is an $\mathcal{S}$-metric on $\mathcal{X}$ and $(\mathcal{X}, \mathcal{S})$ is an $\mathcal{S}$-metric space. Define a map $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by $F(x,y) = \frac{x}{4} + \frac{y}{8}$ for $x, y \in \mathcal{X}$. Then, we have

$$\begin{aligned}
\mathcal{S}(F(x,y), F(u,v), F(z,w)) &= |F(u,v) + F(z,w) - 2F(x,y)| + |F(u,v) - F(z,w)| \\
&= \Big|\frac{u}{4} + \frac{v}{8} + \frac{z}{4} + \frac{w}{8} - \frac{2x}{4} - \frac{2y}{8}\Big| + \Big|\frac{u}{4} + \frac{v}{8} - \frac{z}{4} - \frac{w}{8}\Big| \\
&= \frac{1}{4}|u+z-2x| + \frac{1}{8}|v+w-2y| + \frac{1}{4}|u-z| + \frac{1}{8}|v-w| \\
&= \frac{1}{4}\big(|u+z-2x| + |u-z|\big) + \frac{1}{8}\big(|v+w-2y| + |v-w|\big) \\
&= \frac{1}{4}\mathcal{S}(x,u,z) + \frac{1}{8}\mathcal{S}(y,v,w) \\
&\leq \frac{1}{4}\big[\mathcal{S}(x,u,z) + \mathcal{S}(y,v,w)\big],
\end{aligned}$$

holds for all $x, y, z, u, v, w \in \mathcal{X}$, where $k = \frac{1}{2} < 1$. It is easy to see that F satisfies all the conditions of Corollary 3.5. Thus F has a unique coupled fixed point, namely $F(0,0) = 0$. Similarly, we can verify the result of Corollary 3.6.

As an application of Corollary 3.5, we find an existence and unique result for a type of the following system of Fredholm integral equations:

$$\begin{aligned}
x(t) &= \int_{\mathcal{E}} \mathcal{H}(t, \alpha, x(\alpha), y(\alpha))d(\alpha) + \beta(t), \quad t, \alpha \in \mathcal{E}, \\
y(t) &= \int_{\mathcal{E}} \mathcal{H}(t, \alpha, y(\alpha), x(\alpha))d(\alpha) + \beta(t), \quad t, \alpha \in \mathcal{E},
\end{aligned} \tag{3.26}$$

where $\mathcal{E}$ is measurable, $\mathcal{H}\colon \mathcal{E} \times \mathcal{E} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\beta \in \mathcal{L}^\infty(\mathcal{E})$. Let $\mathcal{X} = \mathcal{L}^\infty(\mathcal{E})$. Now, we define $F\colon \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by

$$F(x,y)(t) = \int_{\mathcal{E}} \mathcal{H}(t, \alpha, x(\alpha), y(\alpha))d(\alpha) + \beta(t), \quad t, \alpha \in \mathcal{E}.$$

Obviously, $(x(t), y(t))$ is a solution of the system of Fredholm integral equations (3.26) if and only if $(x(t), y(t))$ is a coupled fixed point of F. Now, we define the function $S\colon \mathcal{X}^3 \to [0, +\infty)$ by

$$\mathcal{S}(x, y, z) = \sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)| + \sup_{\alpha\in\mathcal{E}} |y(\alpha) + z(\alpha) - 2x(\alpha)|, \tag{3.27}$$

for all $x, y, z \in \mathcal{X}$. Then the function $\mathcal{S}$ is an $\mathcal{S}$-metric. Now, we show that this $\mathcal{S}$-metric can not be generated by any metric d. We assume that this $\mathcal{S}$-metric is generated by any metric d, that is, there exists a metric d such that

$$\mathcal{S}(x, y, z) = d(x, z) + d(y, z), \tag{3.28}$$

for all $x, y, z \in \mathcal{X}$. Then we get

$$\mathcal{S}(x, x, z) = 2d(x, z) = 2 \sup_{\alpha\in\mathcal{E}} |x(\alpha) - z(\alpha)|,$$

and

$$d(x, z) = \sup_{\alpha\in\mathcal{E}} |x(\alpha) - z(\alpha)|. \tag{3.29}$$

Likewise, we obtain

$$\mathcal{S}(y, y, z) = 2d(y, z) = 2 \sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)|,$$

and

$$d(y, z) = \sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)|. \tag{3.30}$$

From equations (3.28), (3.29) and (3.30), we get

$$\sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)| + \sup_{\alpha\in\mathcal{E}} |y(\alpha) + z(\alpha) - 2x(\alpha)| = \sup_{\alpha\in\mathcal{E}} |x(\alpha) - z(\alpha)| + \sup_{\alpha\in\mathcal{E}} |y(\alpha) - z(\alpha)|,$$

which is a contradiction. Hence this $\mathcal{S}$-metric is not generated by any metric d. Thus, $(\mathcal{X}, \mathcal{S})$ is a complete $\mathcal{S}$-metric space.

Now, we state and prove our result as follows.

Theorem 3.8 *Suppose the following:*

1. *There exists a continuous function* $\kappa\colon \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ *such that*

$$\big|\mathcal{H}(t, \alpha, x(\alpha), y(\alpha)) - \mathcal{H}(t, \alpha, u(\alpha), v(\alpha))\big| \le |\kappa(t, \alpha)| \big[|x(\alpha) - u(\alpha)| + |y(\alpha) - v(\alpha)|\big],$$

for all $x, y, u, v \in \mathcal{X}$ *and* $t, \alpha \in \mathcal{E}$.

2. $$\int_{\mathcal{E}} |\kappa(t, \alpha)| d(\alpha) \le \frac{1}{4}.$$

*Then the integral equation (*3.26*) has a unique solution in* $\mathcal{X}$.

Proof Consider

$$\begin{aligned}
&\mathcal{S}(F(x,y),F(x,y),F(u,v)) = 2\,\big|F(x,y)-F(u,v)\big| \\
&= 2\,\Big|\int_{\mathcal{E}} \mathcal{H}(t,\alpha,x(\alpha),y(\alpha))d(\alpha)+\beta(t) \\
&\quad -\Big(\int_{\mathcal{E}} \mathcal{H}(t,\alpha,u(\alpha),v(\alpha))d(\alpha)+\beta(t)\Big)\Big| \\
&= 2\,\Big|\int_{\mathcal{E}} \big[\mathcal{H}(t,\alpha,x(\alpha),y(\alpha))-\mathcal{H}(t,\alpha,u(\alpha),v(\alpha))\big]d(\alpha)\Big| \\
&\le 2\int_{\mathcal{E}} |\kappa(t,\alpha)|\big[|x(\alpha)-u(\alpha)|+|y(\alpha)-v(\alpha)|\big]d(\alpha) \\
&\le \frac{1}{2}\big[|x(\alpha)-u(\alpha)|+|y(\alpha)-v(\alpha)|\big] \\
&= \frac{1}{4}\big[2(|x(\alpha)-u(\alpha)|+|y(\alpha)-v(\alpha)|)\big] \\
&= \lambda\,\big[\mathcal{S}(x,x,u)+\mathcal{S}(y,y,v)\big]
\end{aligned}$$

for all $x,y,u,v\in\mathcal{X}$, where $0\le\lambda=\frac{1}{4}<\frac{1}{2}$. Hence, all the hypothesis of Corollary 3.5 are satisfied and consequently, the integral equation (3.26) has a unique solution. □

§4. Conclusion

In this paper, we prove a unique coupled fixed point and a unique common coupled fixed point theorems under newly proposed coupled implicit relations in the setting of S-metric spaces and give some corollaries of the main results. An illustrative example and an application to the Fredholm integral equation are given. Our results extend and generalize several results from the existing literature.

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Topological Study of Line Graph of Zanamivir and Oseltamivir Used in the Treatment of $H1N1$

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Abstract: Topological indices are the molecular descriptors that characterize the formation of chemical compounds and predict certain physicochemical properties. Reverse degree-based topological indices play an important role in calculating topological descriptors. Line graphs illustrate the molecular graphs of chemical structures in another way and used to predict the physicochemical properties of chemical structures. In this article, we compute some reverse degree-based topological indices of the line graph of Zanamivir and Oseltamivir. This theoretical analysis may help chemists and people working in the pharmaceutical industry to predict the properties of H1N1 drugs without experimenting.

Key Words: Reverse degree, reverse topological indices, Zanamivir, Oseltamivir, line graph, $H1N1$.

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§1. Introduction

The study of causes and treatments of various infectious diseases has been one of the primary areas of study of modern scientific research in medicine. During one such study conducted in 2009, scientists discovered a new influenza A virus subtype, H1N1 (A/H1N1). It gained worldwide attention when it caused a global pandemic. The 2009 H1N1 pandemic resulted in a large number of infections worldwide, but most cases were mild and the overall mortality rate was relatively low. However, certain populations, such as young children, older adults, and individuals with underlying health conditions, were at higher risk of severe complications.

In response to the pandemic, public health organizations and governments implemented various measures to contain the spread of the virus. These measures included widespread vaccination campaigns, antiviral medications, public awareness campaigns, and enhanced surveillance and monitoring. Since the initial pandemic, H1N1 has continued to circulate as a seasonal flu virus. It is now included in the annual influenza vaccine to protect against this strain. Regular vaccination is recommended to prevent seasonal flu and its complications. Until today, various vaccines have been introduced and adopted by different countries worldwide to keep the

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virus under control.

Currently, two classes of antiviral drugs are globally approved to treat influenza in humans, namely, the adamantanes (amantadine and rimantadine) and the neuraminidase inhibitors (Zanamivir, Oseltamivir, and Peramivir). However, the widespread use of these antiviral drugs as monotherapies has resulted in the rise of influenza viruses resistant to both drug classes [10-12,15]. Due to this, only the neuraminidase inhibitors (Zanamivir and Oseltamivir) are presently prescribed for influenza. Combination therapy with Zanamivir and Oseltamivir is reasonable because these agents bind within the neuraminidase active site differently [17]. The likelihood of selecting viruses dually resistant to Zanamivir and Oseltamivir is low. As a result, combining these two drugs in therapy could be a manageable option for combating resistance.

Let $G = (V, E)$ be a graph, where $V(G)$ and $E(G)$ be vertex set and edge set of G, respectively. The *line graph* of G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common.

One of the interesting methods in the studies of mathematical chemistry is to represent molecular graphs by means of parameters calculated for their derived structures. The line graph is a good illustration of derived structures of molecular graphs. Computational aspects of line graph of carbon nanocones are extensively studied in [7]. Zeroth-order general Randić index of line graphs of some chemical structures in drugs are studied in [16]. Topological indices for the line graph of k-subdivided linear [n] tetracene, fullerene networks, tetracenic nanotori, and carbon nanotubes are discussed in [27]. Tomovic [24] and Estrada [3] presented the application of line graphs.

A molecular graph is a set of points denoting the atoms in the molecule and a collection of lines denoting the covalent bonds. The topological indices are helpful in the prediction of physicochemical properties and the biological activity of chemical compounds. Topological indices are obtained from the molecular structure. Topological indices have been used to demonstrate and enhance the statistical features of drugs. In recent times, topological indices of molecular graphs are extensively used for demonstrating correlations connecting the arrangement of a molecular compound and its physicochemical properties or biological activity.

Topological indices are important tools for investigating many physicochemical properties of molecules without performing any testing. They also used to study the Quantitative Structure-Activity Relationship (QSAR) of pharmaceuticals to determine their molecular characteristics by numerical computation.

Various types of topological indices of graphs are classified into distance-based topological indices, degree-based topological indices, and spectrum-based topological indices. Among these, degree-based topological indices play a vital role in theoretical chemistry and pharmacology. Some important degree-based indices are Randić index, Zagreb indices, Harmonic index, sum connectivity index, etc. For example, Randić index is one of the excellent molecular descriptors in QSAR studies and desirable for measuring the extent of branching of the carbon-atom skeleton of saturated hydrocarbons.

Kulli [14] introduced the notion of reverse vertex degree in a graph G. The *reverse vertex degree* in G is defined by $\mathcal{R}_\psi(G) = \Delta(G) - \psi(v) + 1$, where $\psi(v)$ and $\Delta(G)$ are the degree of the vertex v and maximum degree of a vertex in G, respectively. Zhao et al. [28] computed some of the reverse degree-based topological indices, such as, reverse general Randić index, the reverse Balaban index, the reverse atom bond connectivity index, the reverse geometric index, the reverse Zagreb type indices, and the reverse augmented Zagreb index for metal-organic networks TM-TCNB.

Jung et al. [13] computed the first and second reverse Zagreb indices, first and second reverse hyper Zagreb indices, reverse atomic-bond connectivity index and reverse geometric-arithmetic index for TUC4[m,n]. Wei et al. [26] computed some reverse topological indices, namely, the reverse general Randć index, the reverse atom bond connectivity index, the reverse geometric arithmetic index, the reverse forgotten index, the reverse Balaban index, and the reverse Zagreb type indices for Remdisivir compound used in the treatment of Coronavirus (COVID 19). Ravi et al. [22] analyzed the QSPR of the reverse degree-based topological indices. Haoer and Virk [8] computed reverse Zagreb indices, reverse hyper indices, and their polynomials for metal-organic networks. Hashmi et al. [9] computed the reverse ABC index for different generations of dendrimers. Rosary [21] computed some reverse degree-based topological indices for the line graph Remdesivir. Recently, Prasanna Poojari et al. [18] computed some reverse degree-based topological indices for Zanamivir and Oseltamivir.

Motivated by this, we aim to compute the reverse topological indices for the line graph of molecular structures of Zanamivir and Oseltamivir.

Milan Randić [20] introduced the first degree-based index. Wei et al. [26] defined the reverse Randić index as

$$\mathcal{R}R_\alpha(G) = \sum_{mn\in E(G)} \left(R_{\psi(m)} \times R_\psi(n)\right)^\alpha; \quad \alpha = 1, -1, \frac{1}{2}, -\frac{1}{2}. \tag{1.1}$$

If $\alpha = 1$, then it is called the *reverse second Zagreb index* $\mathcal{R}M_2(G)$.

Estrada et al. [2] presented the atom bond connectivity index. Wei et al. [26] defined the reverse atom bond connectivity index as

$$\mathcal{R}ABC(G) = \sum_{mn\in E(G)} \sqrt{\frac{R_{\psi(m)} + R_\psi(n) - 2}{R_{\psi(m)} \times R_\psi(n)}}. \tag{1.2}$$

Vukicevic et al. [25] proposed the geometric arithmetic index. Wei et al. [26] defined the reverse geometric arithmetic index as

$$\mathcal{R}GA(G) = \sum_{mn\in E(G)} 2\frac{\sqrt{R_{\psi(m)} \times R_\psi(n)}}{R_{\psi(m)} + R_\psi(n)}. \tag{1.3}$$

In 1972, Gutman discussed the first and second Zagreb indices [6,4]. Wei et al. [26] defined the reverse first and reverse second Zagreb indices as

$$\mathcal{R}M_1(G) = \sum_{mn\in E(G)} \left(R_{\psi(m)} + R_{\psi(n)}\right), \tag{1.4}$$

$$\mathcal{R}M_2(G) = \sum_{mn \in E(G)} \left(R_{\psi(m)} \times R_{\psi(n)} \right). \tag{1.5}$$

In 2008, Doslic and Gutman [1,5] presented the first and second Zagreb co-indices. Wei et al. [26] defined the reverse first and reverse second Zagreb co-indices as

$$\mathcal{R}\overline{M_1}(G) = 2|E(G)|\left(|(V(G)| - 1\right) - \mathcal{R}M_1(G), \tag{1.6}$$

$$\mathcal{R}\overline{M_2}(G) = 2|E(G)|^2 - \frac{1}{2}\mathcal{R}M_1(G) - \mathcal{R}M_2(G). \tag{1.7}$$

In 2013, Shirdel et al. [23] discussed the concept of hyper Zagreb index. Wei et al. [26] defined the reverse hyper Zagreb index as

$$\mathcal{R}HM(G) = \sum_{mn \in E(G)} \left(R_{\psi(m)} + R_{\psi(n)} \right)^2. \tag{1.8}$$

Furtula and Gutman [6] introduced the Forgotten index. Wei et al. [26] defined the reverse forgotten index as

$$\mathcal{R}F(G) = \sum_{mn \in E(G)} \left((R_{\psi(m)})^2 + (R_{\psi(n)})^2 \right). \tag{1.9}$$

Wei et al. [26] defined the reverse Balaban index for a graph as: Let n and m be order and size of a graph G, respectively. Then,

$$\mathcal{R}J(G) = \frac{m}{m-n+2} \sum_{mn \in E(G)} \frac{1}{\sqrt{R_{\psi(m)} \times R_{\psi(n)}}}. \tag{1.10}$$

Wei et al. [26] defined the reverse first multiple and the reverse second multiple Zagreb indices as

$$\mathcal{R}PM_1(G) = \prod_{mn \in E(G)} \left(R_{\psi(m)} + R_{\psi(n)} \right), \tag{1.11}$$

$$\mathcal{R}PM_2(G) = \prod_{mn \in E(G)} \left(R_{\psi(m)} \times R_{\psi(n)} \right). \tag{1.12}$$

Rajini et al. [19] defined the notion of redefined first, second and third Zagreb indices for a graph. Wei et al. [26] defined the reverse redefined first, second, and third Zagreb indices as

$$\mathcal{R}ReZG_1(G) = \sum_{mn \in E(G)} \frac{R_{\psi(m)} + R_{\psi(n)}}{R_{\psi(m)} \times R_{\psi(n)}}, \tag{1.13}$$

$$\mathcal{R}ReZG_2(G) = \sum_{mn \in E(G)} \frac{R_{\psi(m)} \times R_{\psi(n)}}{R_{\psi(m)} + R_{\psi(n)}}, \tag{1.14}$$

$$\mathcal{R}ReZG_3(G) = \sum_{mn \in E(G)} \left(R_{\psi(m)} + R_{\psi(n)} \right) \left(R_{\psi(m)} \times R_{\psi(n)} \right). \tag{1.15}$$

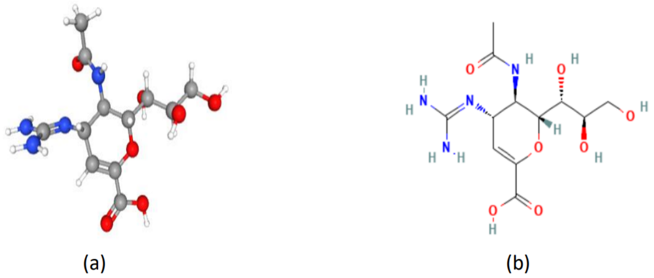
§2. Methods and Techniques

The methods used in this paper contain reverse vertex degree counting, division of vertices based on reverse degree, and partition of edges based on the reverse degree of end vertices. The topological indices as given in formulas (1.1)-(1.15) are calculated with the help of reverse vertex degree counting and partition of edges techniques. The chemical structures (both 2D and 3D) of Zanamivir and Oseltamivir are taken from PubChem, and molecular graphs of chemical structures are drawn using Microsoft Word. Graphical comparison of the reverse topological indices of the molecular structure of Zanamivir and Oseltamivir are plotted by using MATLAB.

§3. Main Results

2.1. Reverse Topological Indices of Line Graph of Zanamivir

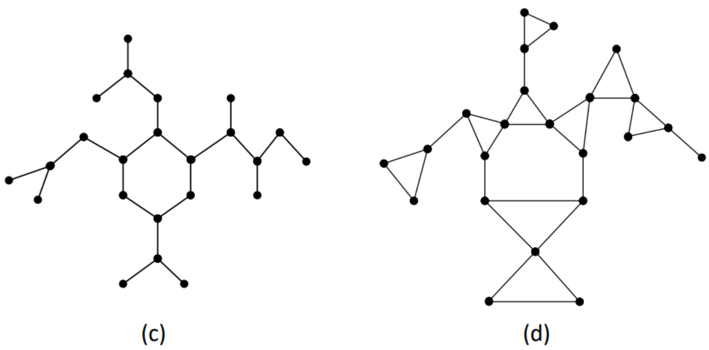
In this subsection, we find the reverse degree-based topological indices for the line graph of Zanamivir. The chemical structure and molecular graph of Zanamivir are shown in Figure 1.



(a) Chemical structure of Zanamivir, (b) Molecular graph of Zanamivir.

Figure 1

We consider hydrogen suppressed molecular graph of compounds since the vertices representing hydrogen atoms do not contribute graph isomorphism. The graph of Zanamivir with vertices and edges; and the line graph of Zanamivir are shown in Figure 2.



(c) Molecular graph of Zanamivir with vertices and edges, (d) Line graph of Zanamivir.

Figure 2

The number of vertices and edges of the line graph of molecular graph of Zanamivir are 23 and 32, respectively. The edge set is partitioned into seven sets based on the reverse degree of end vertices.

The first edge set of the partition includes 3 edges mn, where $R_{\psi(m)} = 1$ and $R_{\psi(n)} = 1$. The second edge set of the partition includes 9 edges mn, where $R_{\psi(m)} = 1$ and $R_{\psi(n)} = 2$. The third edge set of the partition includes 5 edges mn, where $R_{\psi(m)} = 1$ and $R_{\psi(n)} = 3$. The fourth edge set of the partition includes 6 edges mn, where $R_{\psi(m)} = 2$ and $R_{\psi(n)} = 2$. The fifth edge set of the partition includes 5 edges mn, where $R_{\psi(m)} = 2$ and $R_{\psi(n)} = 3$. The sixth edge set of the partition includes an edge mn, where $R_{\psi(m)} = 2$ and $R_{\psi(n)} = 4$. The seventh edge set of the partition includes 3 edges mn, where $R_{\psi(m)} = 3$ and $R_{\psi(n)} = 3$.

Clearly, the number of vertices and edges of the line graph of Zanamivir are 23 and 32, respectively.

The maximum degree of the vertex of the line graph of Zanamivir is 4. By using the definition of reverse vertex degree $R_\psi(L(G)) = \Delta(L(G)) - \psi(v) + 1$, the reverse degree-based edge partition of the line graph of Zanamivir is given in Table 1.

$\left(R_{\psi(m)}, R_{\psi(m)}\right)$	(1,1)	(1,2)	(1,3)	(2,2)	(2,3)	(2,4)	(3,3)
Number of edges	3	9	5	6	5	1	3

Table 1. Edge partition of line graph of Zanamivir

We now find reverse degree-based topological indices for the line graph of Zanamivir.

- Reverse Randić index

$$\mathcal{R}R_\alpha(L(G)) = \sum_{mn\in E(L(G))} \left(R_{\psi(m)} \times R_\psi(n)\right)^\alpha ; \alpha = 1, -1, \frac{1}{2}, -\frac{1}{2}$$

For $\alpha = 1$,

$$\mathcal{R}R_1(L(G)) = 3(1\times 1) + 9(1\times 2) + 5(1\times 3) + 6(2\times 2) + 5(2\times 3) + 1(2\times 4) + 3(3\times 3) = 125.$$

For $\alpha = -1$,

$$\mathcal{R}R_{-1}(L(G)) = \frac{3}{1\times 1} + \frac{9}{1\times 2} + \frac{5}{1\times 3} + \frac{6}{2\times 2} + \frac{5}{2\times 3} + \frac{1}{2\times 4} + \frac{3}{3\times 3} = 11.96.$$

For $\alpha = \frac{1}{2}$,

$$\mathcal{R}R_{\frac{1}{2}}(L(G)) = 3\sqrt{1\times 1} + 9\sqrt{1\times 2} + 5\sqrt{1\times 3} + 6\sqrt{2\times 2} + 5\sqrt{2\times 3} + 1\sqrt{2\times 4} + 3\sqrt{3\times 3} = 60.46.$$

For $\alpha = -\frac{1}{2}$,

$$\mathcal{R}R_{-\frac{1}{2}}(L(G)) = \frac{3}{\sqrt{1\times 1}} + \frac{9}{\sqrt{1\times 2}} + \frac{5}{\sqrt{1\times 3}} + \frac{6}{\sqrt{2\times 2}} + \frac{5}{\sqrt{2\times 3}} + \frac{1}{\sqrt{2\times 4}} + \frac{3}{\sqrt{3\times 3}}$$
$$= 18.64.$$

- Reverse atom bond connectivity index

$$\mathcal{R}ABC(L(G)) = \sum_{mn\in E(L(G))} \sqrt{\frac{R_{\psi(m)} + R_{\psi}(n) - 2}{R_{\psi(m)} \times R_{\psi}(n)}}$$
$$= 3\sqrt{\frac{1+1-2}{1\times 1}} + 9\sqrt{\frac{1+2-2}{1\times 2}} + 5\sqrt{\frac{1+3-2}{1\times 3}} + 6\sqrt{\frac{2+2-2}{2\times 2}} + 5\sqrt{\frac{2+3-2}{2\times 3}}$$
$$+ 1\sqrt{\frac{2+4-2}{2\times 4}} + 3\sqrt{\frac{3+3-2}{3\times 3}}$$
$$= 20.93$$

- Reverse geometric arithmetic index

$$\mathcal{R}GA(L(G)) = \sum_{mn\in E(L(G))} 2\frac{\sqrt{R_{\psi(m)} \times R_{\psi}(n)}}{R_{\psi(m)} + R_{\psi}(n)}$$
$$= 3\left(\frac{2\sqrt{1\times 1}}{1+1}\right) + 9\left(\frac{2\sqrt{1\times 2}}{1+2}\right) + 5\left(\frac{2\sqrt{1\times 3}}{1+3}\right) + 6\left(\frac{2\sqrt{2\times 2}}{2+2}\right)$$
$$+ 5\left(\frac{2\sqrt{2\times 3}}{2+3}\right) + 1\left(\frac{2\sqrt{2\times 4}}{2+4}\right) + 3\left(\frac{2\sqrt{3\times 3}}{3+3}\right)$$
$$= 30.66$$

- Reverse first Zagreb index

$$\mathcal{R}M_1(L(G)) = \sum_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right)$$
$$= 3(1+1) + 9(1+2) + 5(1+3) + 6(2+2) + 5(2+3) + 1(2+4) + 3(3+3)$$
$$= 126$$

- Reverse second Zagreb index

$$\mathcal{R}M_2(L(G)) = \sum_{mn\in E(L(G))} \left(R_{\psi(m)} \times R_{\psi(n)}\right)$$
$$= 3(1\times 1) + 9(1\times 2) + 5(1\times 3) + 6(2\times 2) + 5(2\times 3)$$
$$+ 1(2\times 4) + 3(3\times 3)$$
$$= 125$$

- Reverse first Zagreb co-index

$$
\begin{aligned}
\mathcal{R}\overline{M_1}(L(G)) =& 2|E(L(G))|\,(|(V(L(G))| - 1) - \mathcal{R}M_1(L(G)) \\
&= 2(32)(23-1) - 126 \\
&= 1282
\end{aligned}
$$

- Reverse second Zagreb co-index

$$
\begin{aligned}
\mathcal{R}\overline{M_2}(L(G)) =& 2|E(L(G))|^2 - \frac{1}{2}\mathcal{R}M_1(L(G)) - \mathcal{R}M_2(L(G)) \\
&= 2(32)^2 - \frac{126}{2} - 125 \\
&= 1860
\end{aligned}
$$

- Reverse hyper Zagreb index

$$
\begin{aligned}
\mathcal{R}HM(L(G)) =& \sum_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right)^2 \\
&= 3(1+1)^2 + 9(1+2)^2 + 5(1+3)^2 + 6(2+2)^2 + 5(2+3)^2 \\
&+ 1(2+4)^2 + 3(3+3)^2 \\
&= 538
\end{aligned}
$$

- Reverse forgotten index

$$
\begin{aligned}
\mathcal{R}F(L(G)) =& \sum_{mn\in E(L(G))} (R_{\psi(m)})^2 + (R_{\psi(n)})^2 \\
&= 3(1^2+1^2) + 9(1^2+2^2) + 5(1^2+3^2) + 6(2^2+2^2) + 5(2^2+3^2) \\
&+ 1(2^2+4^2) + 3(3^2+3^2) \\
&= 288
\end{aligned}
$$

- Reverse Balaban index: Let n and m be order and size of $L(G)$, respectively. Then

$$
\begin{aligned}
\mathcal{R}J(L(G)) =& \frac{m}{m-n+2} \sum_{mn\in E(L(G))} \frac{1}{\sqrt{R_{\psi(m)} \times R_{\psi(n)}}} \\
&= \frac{32}{32-23+2}\left(\frac{3}{\sqrt{1\times 1}} + \frac{9}{\sqrt{1\times 2}} + \frac{5}{\sqrt{1\times 3}} \right. \\
&\left. + \frac{6}{\sqrt{2\times 2}} + \frac{5}{\sqrt{2\times 3}} + \frac{1}{\sqrt{2\times 4}}\right) \\
&+ \frac{32}{32-23+2}\left(\frac{3}{\sqrt{3\times 3}}\right) \\
&= 54.24
\end{aligned}
$$

- Reverse first multiple Zagreb index

$$\begin{aligned}\mathcal{R}PM_1(L(G)) &= \prod_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right)\\ &= 3(1+1)\times 9(1+2)\times 5(1+3)\times 6(2+2)\times 5(2+3)\\ &\times 1(2+4)\times 3(3+3) = 209952000\end{aligned}$$

- Reverse second multiple Zagreb index

$$\begin{aligned}\mathcal{R}PM_2(L(G)) &= \prod_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right)\\ &= 3(1\times 1)\times 9(1\times 2)\times 5(1\times 3)\times 6(2\times 2)\\ &\times 5(2\times 3)\times 1(2\times 4)\times 3(3\times 3) = 125971200\end{aligned}$$

- Reverse first refined Zagreb index

$$\begin{aligned}\mathcal{R}ReZG_1(L(G)) &= \sum_{mn\in E(L(G))} \frac{R_{\psi(m)} + R_{\psi(n)}}{R_{\psi(m)} \times R_{\psi(n)}}\\ &= 3\left(\frac{1+1}{1\times 1}\right) + 9\left(\frac{1+2}{1\times 2}\right) + 5\left(\frac{2+3}{2\times 3}\right) + 6\left(\frac{2+2}{2\times 2}\right)\\ &+ 5\left(\frac{2+3}{2\times 3}\right) + 1\left(\frac{2+4}{2\times 4}\right) + 3\left(\frac{3+3}{3\times 3}\right) = 39.08\end{aligned}$$

- Reverse second refined Zagreb index

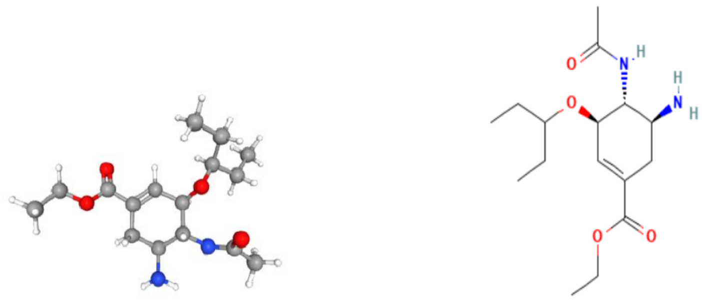
$$\begin{aligned}\mathcal{R}ReZG_2(L(G)) &= \sum_{mn\in E(L(G))} \frac{R_{\psi(m)} \times R_{\psi(n)}}{R_{\psi(m)} + R_{\psi(n)}}\\ &= 3\left(\frac{1\times 1}{1+1}\right) + 9\left(\frac{1\times 2}{1+2}\right) + 5\left(\frac{2\times 3}{2+3}\right) + 6\left(\frac{2\times 2}{2+2}\right)\\ &+ 5\left(\frac{2\times 3}{2+3}\right) + 1\left(\frac{2\times 4}{2+4}\right) + 3\left(\frac{3\times 3}{3+3}\right) = 29.08\end{aligned}$$

- Reverse third refined Zagreb index

$$\begin{aligned}\mathcal{R}ReZG_3(L(G)) &= \sum_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right)\left(R_{\psi(m)} \times R_{\psi(n)}\right)\\ &= 3(2\times 1) + 9(3\times 2) + 5(4\times 3) + 6(4\times 4) + 5(5\times 6)\\ &+ 1(6\times 8) + 3(6\times 9) = 576\end{aligned}$$

2.2. Reverse Topological Indices of Line Graph of Oseltamivir

In this subsection, we find the reverse degree-based topological indices for the line graph of Oseltamivir. Notice that the chemical structure and molecular graph of Oseltamivir are shown in Figure 3.

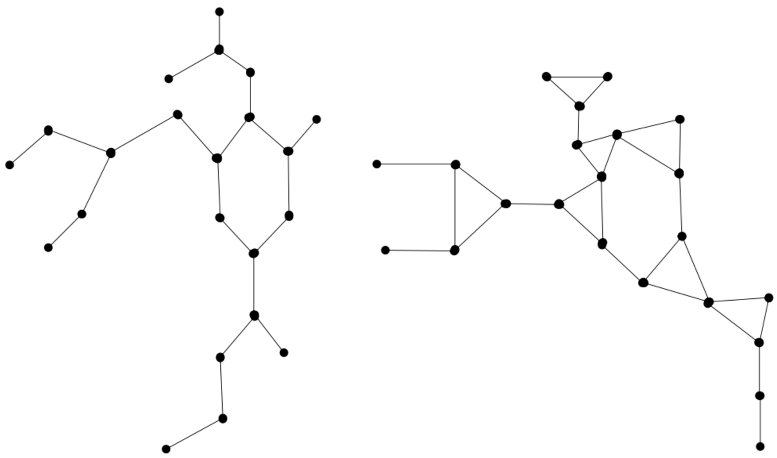


(a) (b)

(a) Chemical structure of Oseltamivir. (b) Molecular graph of Oseltamivir.

Figure 3

The graph of Oseltamivir with vertices and edges; and the line graph of Oseltamivir are shown in Figure 4.



(a) (b)

(a) Molecular graph of Oseltamivir with vertices and edges. (b) Line graph of Oseltamivir.

Figure 4

The number of vertices and edges of the line graph of molecular graph of Oseltamivir are 22 and 29, respectively. The edge set is partitioned into eight sets based on the reverse degree of end vertices.

The first edge set of the partition includes an edge mn, where $R_{\psi(m)} = 1$ and $R_{\psi(n)} = 1$. The second edge set of the partition includes 8 edges mn, where $R_{\psi(m)} = 1$ and $R_{\psi(n)} = 2$. The third edge set of the partition includes 2 edges mn, where $R_{\psi(m)} = 1$ and $R_{\psi(n)} = 3$. The fourth edge set of the partition includes 9 edges mn, where $R_{\psi(m)} = 2$ and $R_{\psi(n)} = 2$. The fifth edge set of the partition includes 5 edges mn, where $R_{\psi(m)} = 2$ and $R_{\psi(n)} = 3$. The sixth edge set of the partition includes 2 edges mn, where $R_{\psi(m)} = 2$ and $R_{\psi(n)} = 4$. The seventh edge set of the partition includes an edge mn, where $R_{\psi(m)} = 3$ and $R_{\psi(n)} = 3$. The eight edge set of the partition includes an edge mn, where $R_{\psi(m)} = 3$ and $R_{\psi(n)} = 4$.

Clearly, the number of vertices and edges of the line graph of Oseltamivir are 22 and 29, respectively.

The maximum degree of the vertex of the line graph of Oseltamivir is 4. The reverse degree-based edge partition of the line graph of Oseltamivir is given in Table 2.

$\big(R_{\psi(m)}, R_{\psi(m)}\big)$	(1,1)	(1,2)	(1,3)	(2,2)	(2,3)	(2,4)	(3,3)	(3,4)
Number of edges	1	8	2	9	5	2	1	1

Table 2. Edge partition of line graph of Oseltamivir

We now find obtain the reverse degree-based topological indices for the line graph of Oseltamivir.

• Reverse Randić index

$$\mathcal{R}R_\alpha(L(G)) = \sum_{mn\in E(L(G))} \big(R_{\psi(m)} \times R_\psi(n)\big)^\alpha; \quad \alpha = 1, -1, \frac{1}{2}, -\frac{1}{2}$$

For $\alpha = 1$,

$$\begin{aligned}\mathcal{R}R_1(L(G)) &= 1(1\times 1) + 8(1\times 2) + 2(1\times 3) + 9(2\times 2) + 5(2\times 3)\\ &\quad + 2(2\times 4) + 1(3\times 3) + 1(3\times 4) = 126.\end{aligned}$$

For $\alpha = -1$,

$$\begin{aligned}\mathcal{R}R_{-1}(L(G)) &= \frac{1}{1\times 1} + \frac{8}{1\times 2} + \frac{2}{1\times 3} + \frac{9}{2\times 2} + \frac{5}{2\times 3} + \frac{2}{2\times 4}\\ &\quad + \frac{1}{3\times 3} + \frac{1}{3\times 4} = 9.19.\end{aligned}$$

For $\alpha = \frac{1}{2}$,

$$\begin{aligned}\mathcal{R}R_{\frac{1}{2}}(L(G)) &= 1\sqrt{1\times 1} + 8\sqrt{1\times 2} + 2\sqrt{1\times 3} + 9\sqrt{2\times 2} + 5\sqrt{2\times 3}\\ &\quad + 2\sqrt{2\times 4} + 1\sqrt{3\times 3} + 1\sqrt{3\times 4}\\ &= 58.15.\end{aligned}$$

For $\alpha = -\frac{1}{2}$,

$$\begin{aligned}\mathcal{R}R_{-\frac{1}{2}}(L(G)) &= \frac{1}{\sqrt{1\times 1}} + \frac{8}{\sqrt{1\times 2}} + \frac{2}{\sqrt{1\times 3}} + \frac{9}{\sqrt{2\times 2}} + \frac{5}{\sqrt{2\times 3}}\\ &\quad + \frac{2}{\sqrt{2\times 4}} + \frac{1}{\sqrt{3\times 3}} + \frac{1}{\sqrt{3\times 4}} = 15.68.\end{aligned}$$

- Reverse atom bond connectivity index

$$\begin{aligned}\mathcal{R}ABC(L(G)) &= \sum_{mn\in E(L(G))} \sqrt{\frac{R_{\psi(m)} + R_{\psi}(n) - 2}{R_{\psi(m)} \times R_{\psi}(n)}} \\ &= 1\sqrt{\frac{1+1-2}{1\times 1}} + 8\sqrt{\frac{1+2-2}{1\times 2}} + 2\sqrt{\frac{1+3-2}{1\times 3}} + 9\sqrt{\frac{2+2-2}{2\times 2}} \\ &\quad + 5\sqrt{\frac{2+3-2}{2\times 3}} + 2\sqrt{\frac{2+4-2}{2\times 4}} + 1\sqrt{\frac{3+3-2}{3\times 3}} + 1\sqrt{\frac{3+4-2}{3\times 4}} \\ &= 19.92\end{aligned}$$

- Reverse geometric arithmetic index

$$\begin{aligned}\mathcal{R}GA(L(G)) &= \sum_{mn\in E(L(G))} 2\frac{\sqrt{R_{\psi(m)} \times R_{\psi}(n)}}{R_{\psi(m)} + R_{\psi}(n)} \\ &= 1\left(\frac{2\sqrt{1\times 1}}{1+1}\right) + 8\left(\frac{2\sqrt{1\times 2}}{1+2}\right) + 2\left(\frac{2\sqrt{1\times 3}}{1+3}\right) + 9\left(\frac{2\sqrt{2\times 2}}{2+2}\right) \\ &\quad + 5\left(\frac{2\sqrt{2\times 3}}{2+3}\right) + 2\left(\frac{2\sqrt{2\times 4}}{2+4}\right) + 1\left(\frac{2\sqrt{3\times 3}}{3+3}\right) + 1\left(\frac{2\sqrt{3\times 4}}{3+4}\right) \\ &= 28.05\end{aligned}$$

- Reverse first Zagreb index

$$\begin{aligned}\mathcal{R}M_1(L(G)) &= \sum_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right) \\ &= 1(1+1) + 8(1+2) + 2(1+3) + 9(2+2) + 5(2+3) + 2(2+4) \\ &\quad + 1(3+3) + 1(3+4) = 120\end{aligned}$$

- Reverse second Zagreb index

$$\begin{aligned}\mathcal{R}M_2(L(G)) &= \sum_{mn\in E(L(G))} \left(R_{\psi(m)} \times R_{\psi(n)}\right) \\ &= 1(1\times 1) + 8(1\times 2) + 2(1\times 3) + 9(2\times 2) + 5(2\times 3) + 2(2\times 4) \\ &\quad + 1(3\times 3) + 1(3\times 4) \\ &= 126\end{aligned}$$

- Reverse first Zagreb co-index

$$\begin{aligned}\mathcal{R}\overline{M_1}(L(G)) =& 2|E(L(G))|\,(|(V(L(G))| - 1) - \mathcal{R}M_1(L(G)) \\ &= 2(29)(22-1) - 120 \\ &= 1098\end{aligned}$$

- Reverse second Zagreb co-index

$$\begin{aligned}\mathcal{R}\overline{M_2}(L(G)) =& 2|E(L(G))|^2 - \frac{1}{2}\mathcal{R}M_1(L(G)) - \mathcal{R}M_2(L(G))\\ &= 2(29)^2 - \frac{120}{2} - 126\\ &= 1496\end{aligned}$$

- Reverse hyper Zagreb index

$$\begin{aligned}\mathcal{R}HM(L(G)) &= \sum_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right)^2\\ &= 1(1+1)^2 + 8(1+2)^2 + 2(1+3)^2 + 9(2+2)^2 + 5(2+3)^2\\ &\quad + 2(2+4)^2 + 1(3+3)^2 + 1(3+4)^2\\ &= 534\end{aligned}$$

- Reverse forgotten index

$$\begin{aligned}\mathcal{R}F(L(G)) &= \sum_{mn\in E(L(G))} (R_{\psi(m)})^2 + (R_{\psi(n)})^2\\ &= 1(1^2+1^2) + 8(1^2+2^2) + 2(1^2+3^2) + 9(2^2+2^2) + 5(2^2+3^2) + 2(2^2+4^2)\\ &+ 1(3^2+3^2) + 1(3^2+4^2)\\ &= 282\end{aligned}$$

- Reverse Balaban index:

$$\begin{aligned}\mathcal{R}J(L(G)) =& \frac{m}{m-n+2} \sum_{mn\in E(L(G))} \frac{1}{\sqrt{R_{\psi(m)} \times R_{\psi(n)}}}\\ &= \left(\frac{29}{29-22+2}\right)\left(\frac{1}{\sqrt{1\times 1}} + \frac{8}{\sqrt{1\times 2}} + \frac{2}{\sqrt{1\times 3}} + \frac{9}{\sqrt{2\times 2}} + \frac{5}{\sqrt{2\times 3}} + \frac{2}{\sqrt{2\times 4}}\right)\\ &+ \left(\frac{29}{29-22+2}\right)\left(\frac{1}{\sqrt{3\times 3}} + \frac{1}{\sqrt{3\times 4}}\right)\\ &= 50.53\end{aligned}$$

- Reverse first multiple Zagreb index

$$\begin{aligned}\mathcal{R}PM_1(L(G)) &= \prod_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right)\\ &= 1(1+1) \times 8(1+2) \times 2(1+3) \times 9(2+2) \times 5(2+3) \times 2(2+4)\\ &\quad \times 1(3+3) \times 1(3+4)\\ &= 174182400\end{aligned}$$

- Reverse second multiple Zagreb index

$$\begin{aligned}\mathcal{R}PM_2(L(G)) &= \prod_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right) \\ &= 1(1\times 1)\times 8(1\times 2)\times 2(1\times 3)\times 9(2\times 2)\times 5(2\times 3)\times 2(2\times 4) \\ &\quad \times 1(3\times 3)\times 1(3\times 4) \\ &= 179159040\end{aligned}$$

- Reverse first refined Zagreb index

$$\begin{aligned}\mathcal{R}ReZG_1(L(G)) &= \sum_{mn\in E(L(G))} \frac{R_{\psi(m)} + R_{\psi(n)}}{R_{\psi(m)} \times R_{\psi(n)}} \\ &= 1\left(\frac{1+1}{1\times 1}\right) + 8\left(\frac{1+2}{1\times 2}\right) + 2\left(\frac{2+3}{2\times 3}\right) + 9\left(\frac{2+2}{2\times 2}\right) + 5\left(\frac{2+3}{2\times 3}\right) \\ &\quad + 2\left(\frac{2+4}{2\times 4}\right) + 1\left(\frac{3+3}{3\times 3}\right) + 1\left(\frac{3+4}{3\times 4}\right) \\ &= 32.58\end{aligned}$$

- Reverse second refined Zagreb index

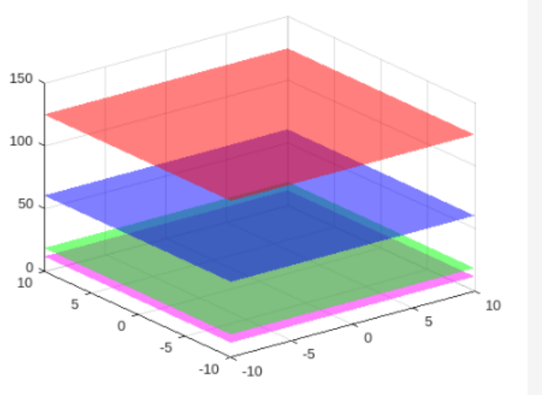
$$\begin{aligned}\mathcal{R}ReZG_2(L(G)) &= \sum_{mn\in E(L(G))} \frac{R_{\psi(m)} \times R_{\psi(n)}}{R_{\psi(m)} + R_{\psi(n)}} \\ &= 1\left(\frac{1\times 1}{1+1}\right) + 8\left(\frac{1\times 2}{1+2}\right) + 2\left(\frac{2\times 3}{2+3}\right) + 9\left(\frac{2\times 2}{2+2}\right) + 5\left(\frac{2\times 3}{2+3}\right) \\ &\quad + 2\left(\frac{2\times 4}{2+4}\right) + 1\left(\frac{3\times 3}{3+3}\right) + 1\left(\frac{3\times 4}{3+4}\right) \\ &= 28.21\end{aligned}$$

- Reverse third refined Zagreb index

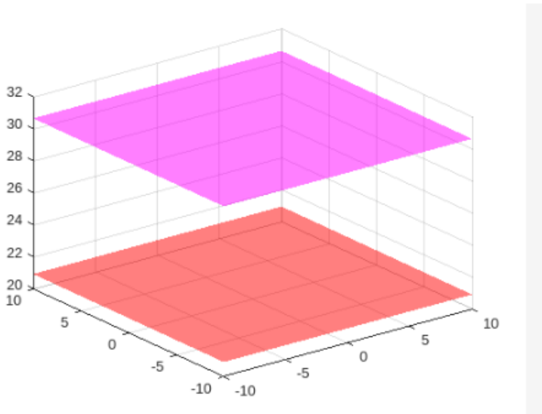
$$\begin{aligned}\mathcal{R}ReZG_3(L(G)) &= \sum_{mn\in E(L(G))} \left(R_{\psi(m)} + R_{\psi(n)}\right)\left(R_{\psi(m)} \times R_{\psi(n)}\right) \\ &= 1(2\times 1) + 8(3\times 2) + 2(4\times 3) + 9(4\times 4) + 5(5\times 6) \\ &\quad + 2(6\times 8) + 1(6\times 9) + 1(7\times 12) \\ &= 602\end{aligned}$$

§4. Graphical Comparison of Line Graph of Zanamivir

In order to understand the similarities between the biological and statistical behavior of the line graph of Zanamivir, we have provided results for topological indices for the structure of line graph of Zanamivir.

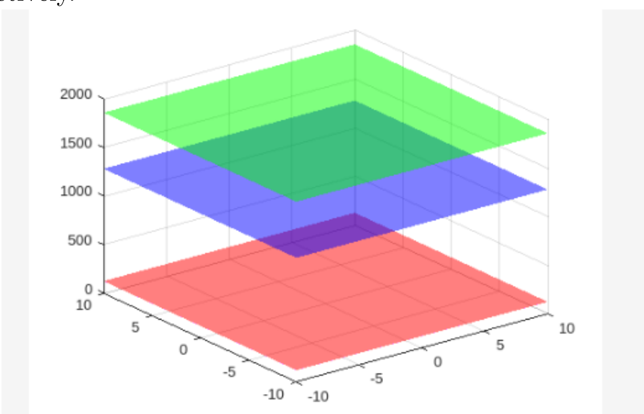


(a)

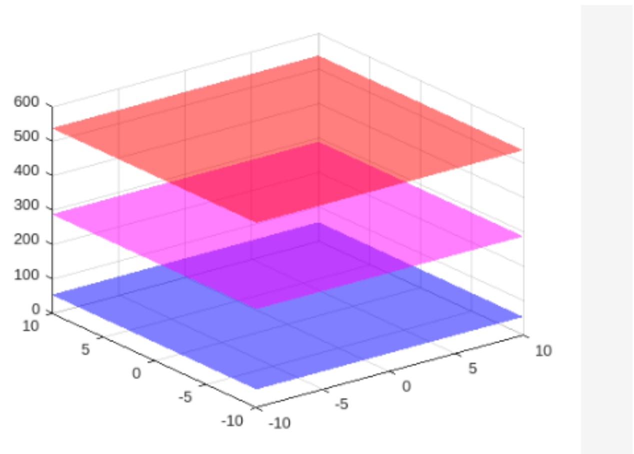


(b)

Figure 5. (a) Comparison of the reverse Randić index of the line graph of Zanamivir for $\alpha = 1, -1, \frac{1}{2}, -\frac{1}{2}$, where red, magenta, blue, and green denotes $\mathcal{R}R_1(L(G)), \mathcal{R}R_{-1}(L(G)), \mathcal{R}R_{\frac{1}{2}}(L(G))$, and $\mathcal{R}R_{-\frac{1}{2}}(L(G))$, respectively; (b) comparison of the reverse atom bond connectivity index and reverse geometric arithmetic index, where red and magenta denotes $\mathcal{R}ABC(L(G))$ and $\mathcal{R}GA(L(G))$, respectively.

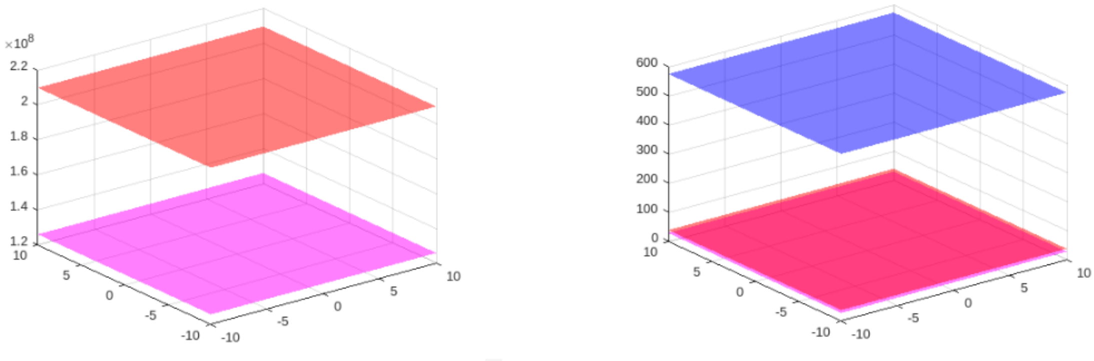


(a)



(b)

Figure 6. (a) Comparison of the reverse first Zagreb index, reverse second Zagreb index, reverse first Zagreb co-index, and reverse second Zagreb co-index of the line graph of Zanamivir, where red, magenta, blue, green denotes $\mathcal{R}M_1(L(G)), \mathcal{R}M_2(L(G)), \mathcal{R}\overline{M_1}(L(G)))$, and $\mathcal{R}\overline{M_2}(L(G))$, respectively. (b) Comparison of the reverse hyper Zagreb index, reverse forgotten index, and reverse Balaban index, where all these of red, magenta and blue denotes $\mathcal{R}HM(L(G)), \mathcal{R}F(L(G))$, and $\mathcal{R}J(L(G))$, respectively.



(a) (b)

Figure 7. (a) Comparison of the reverse first multiple Zagreb index and reverse second multiple Zagreb index of the line graph of Zanamivir, where red and magenta denotes $\mathcal{R}PM_1(L(G))$ and $\mathcal{R}PM_2(L(G))$, respectively; (b) Comparison of the reverse first redefined Zagreb index, reverse second redefined Zagreb index, and reverse third redefined Zagreb index, where red, magenta, and blue denotes $\mathcal{R}ReZG_1(L(G)), \mathcal{R}ReZG_2(L(G))$, and $\mathcal{R}ReZG_3(L(G))$, respectively.

§5. Graphical Comparison of Line Graph of Oseltamivir

The graphical results for topological indices for the structure of line graph of Oseltamivir.

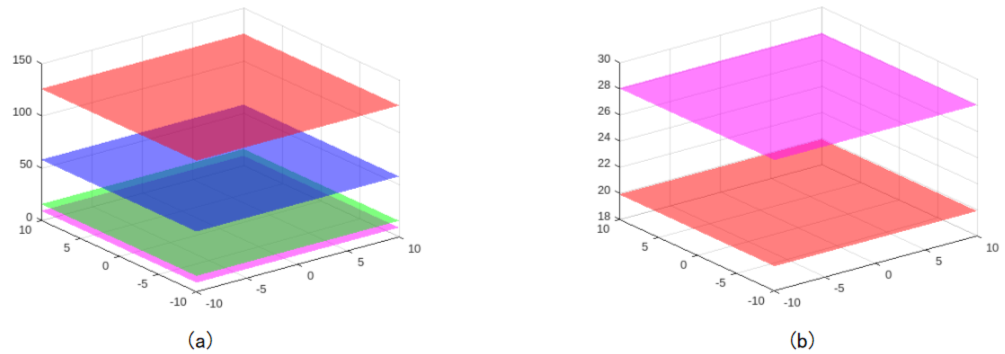


Figure 8. (a) Comparison of the reverse Randić index of the line graph of Oseltamivir for $\alpha = 1, -1, \frac{1}{2}, -\frac{1}{2}$, where red, magenta, blue, green denotes $\mathcal{R}R_1(L(G)), \mathcal{R}R_{-1}(L(G)), \mathcal{R}R_{\frac{1}{2}}(L(G))$ and $\mathcal{R}R_{-\frac{1}{2}}(L(G))$, respectively; (b) comparison of the reverse atom bond connectivity index and reverse geometric arithmetic index, where red, magenta denotes $\mathcal{R}ABC(L(G))$, $\mathcal{R}GA(L(G))$, respectively.

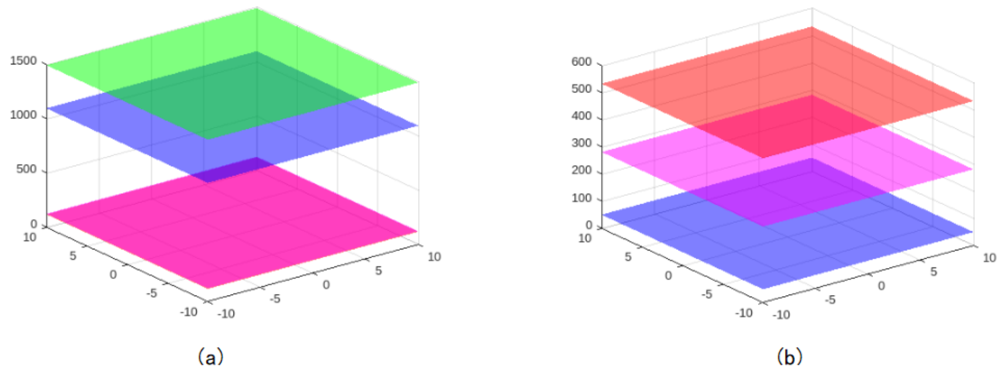


Figure 9. (a) Comparison of the reverse first Zagreb index, reverse second Zagreb index, reverse first Zagreb co-index, and reverse second Zagreb co-index of the line graph of Oseltamivir, where red, magenta, blue, and green denotes $\mathcal{R}M_1(L(G)), \mathcal{R}M_2(L(G)), \mathcal{R}\overline{M_1}(L(G)))$, and $\mathcal{R}\overline{M_2}(L(G))$, respectively; (b) Comparison of the reverse hyper Zagreb index, reverse forgotten index and reverse Balaban index, where these of red, magenta, blue denotes $\mathcal{R}HM(L(G)), \mathcal{R}F(L(G))$, $\mathcal{R}J(L(G))$, respectively.

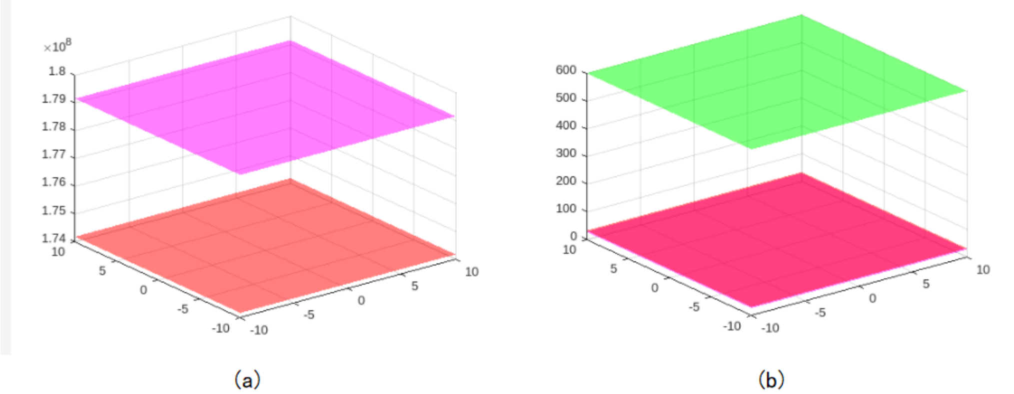


Figure 10. (a) Comparison of the reverse first multiple Zagreb index and reverse second multiple

Zagreb index of the line graph of Oseltamivir, where red and magenta denotes $\mathcal{R}PM_1(L(G))$ and $\mathcal{R}PM_2(L(G))$, respectively; (b) Comparison of the reverse first redefined Zagreb index, reverse second redefined Zagreb index, and reverse third redefined Zagreb index, where red, magenta, blue denotes $\mathcal{R}ReZG_1(L(G)), \mathcal{R}ReZG_2(L(G)), \mathcal{R}ReZG_3(L(G))$, respectively.

§6. Conclusion

In this paper, the reverse degree-based topological indices, namely, reverse Randić index, reverse atomic bond connectivity index, reverse geometric arithmetic index, reverse first and second Zagreb indices, reverse first and second Zagreb co-indices, reverse hyper Zagreb index, reverse forgotten index, reverse Balaban index, and reverse first, second, and third refined Zagreb indices for the line graph of Zanamivir and Oseltamivir are computed. In addition, we have compared these topological indices graphically as shown in Figures 5-10. Using topological indices of the line graph of molecular graphs to predict various qualities and activities, such as entropy, critical pressure, boiling point, enthalpy, and more, can indeed be a valuable approach to drug discovery and design for H1N1 treatment.

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Complexity of Sequence of Some Families of Graphs and Their Asymptotic Behavior

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Abstract: Calculating and analyzing the number of spanning trees of networks (graphs) is an interesting and important research project in wide variety of fields, such as mathematics, theoretical computer science, chemistry, physics and so on. In this paper, we investigated the number of spanning trees in three sequences of families of graphs of the same average degree $\frac{14}{3}$. We used the electrically equivalent transformations and rules of weighted generating function which avoids the laborious computation of the determinant for counting the number of spanning trees. Finally, we determined the entropy of our studied graphs.

Key Words: Number of spanning trees, electrically equivalent transformations, entropy.

AMS(2010): 05C30,05C50,05C63.

§1. Introduction

Complexity (the number of spanning trees) $\tau(G)$ of a finite connected undirected graph G is the total number of distinct spanning subgraphs of G that are trees. As it's known, the problem of evaluating the number of spanning trees of a finite connected undirected graph has been solved by famous Kirchhoff's matrix-tree theorem [1], the product of all nonzero eigenvalues of the Laplacian matrix of the graph. But, for a large-size graph with thousands of vertices and edges, this problem will become more difficult. How to find out the exact solutions of the number of spanning trees of models has been a demanding and exciting mission, in particular on some real-world networks, and always draws many concerns from various science fields, such as mathematics [2], computer science [3], chemistry [4], physics [5] and so on. Luckily, there has been some useful methods, such as the theory of electrical networks, to find the accurate solution for the number of spanning trees of special graph families, for example lattices, grids, Farey graph and Sierpinski gaskets, see [6], [7], [8], [9], [10].

For a summary of further results for calculating umber of spanning trees of graphs, see $[11, 12, 13, 14, 15]$.

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§2. Electrically Equivalent Transformations

To begin with, we briefly review the electrically equivalent transformation technique introduced in [16,17]. An edge-weighted graph G (with the weight function $\omega : E(G) \to [0,\infty)$) can be considered as an electrical network with the weights being the conductances of the corresponding edges. The weighted number of spanning trees in G is defined as: Let G be an edge weighted graph, G' be the corresponding electrically equivalent graph, $\tau(G)$ denotes the weighted number of spanning trees G.

(a) Parallel edges: If two parallel edges with conductances u and v in G are merged into a single edge with conductances $u+v$ in G', then $\tau(G') = \tau(G)$;

(b) Serial edges: If two serial edges with conductances u and v in G are merged into a single edge with conductance $\frac{uv}{u+v}$ in G', then $\tau(G') = \frac{1}{u+v}\tau(G)$;

(c) $\triangle - Y$ transformation: If a triangle with conductances u, v and w in G is changed into an electrically equivalent star graph with conductances $x = \frac{uv+vw+wu}{u}$, $y = \frac{uv+vw+wu}{v}$ and $z = \frac{uv+vw+wu}{w}$ in G', then $\tau(G') = \frac{(uv+vw+wu)^2}{uvw}\tau(G)$;

(d) $Y - \Delta$ transformation: If a star graph with conductances u, v and w in G is changed into an electrically equivalent triangle with conductances $x = \frac{vw}{u+v+w}$, $y = \frac{uw}{u+v+w}$ and $z = \frac{uv}{u+v+w}$ in G', then $\tau(G') = \frac{1}{u+v+w}\tau(G)$.

In this work, we compute the number of spanning trees of three sequences of graphs of the same average degree $\frac{14}{3}$, we named it Θ_n, Π_n and Σ_n respectively.

§3. Number of Spanning Trees in the Sequences of Θ_n Graph

Consider the sequence of graphs $\Theta_1, \Theta_2, \cdots, \Theta_n$ constructed as shown in Figure 1. According to this construction, the number of total vertices $|V(\Theta_n)|$ and edges $|E(\Theta_n)|$ are $|V(\Theta_n)| = 9n-6$ and $|E(\Theta_n)| = 21n-18, n = 1, 2, \cdots$. The average degree of Θ_n is $\dfrac{14}{3}$ in $n \to \infty$.

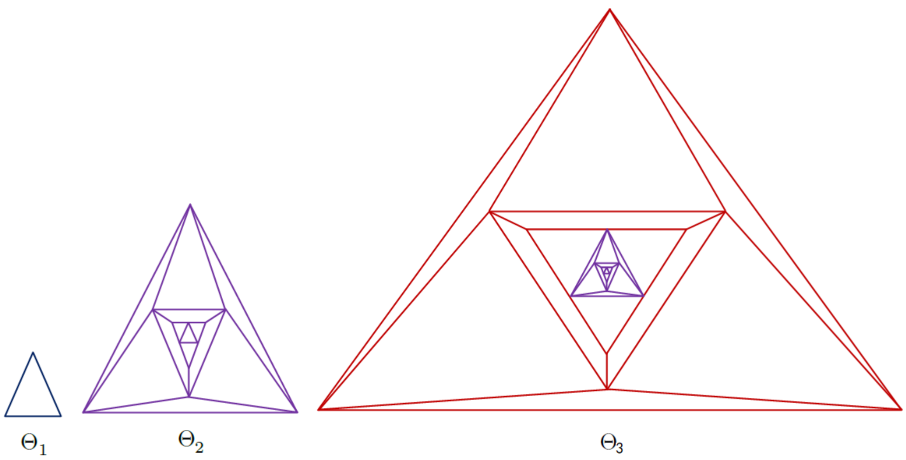
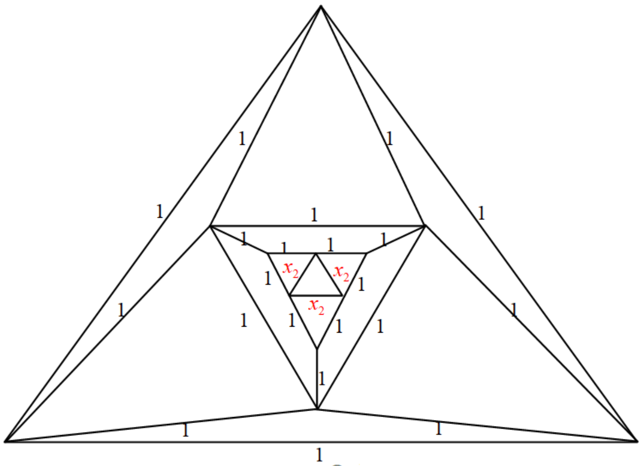


Figure 1. Some sequences of graph Θ_n

Theorem 3.1 *For $n \geq 1$, the number of spanning trees in the sequence of the graph Θ_n is given by $\dfrac{N_1(\Theta_n)}{M_1(\Theta_n)}$, where*

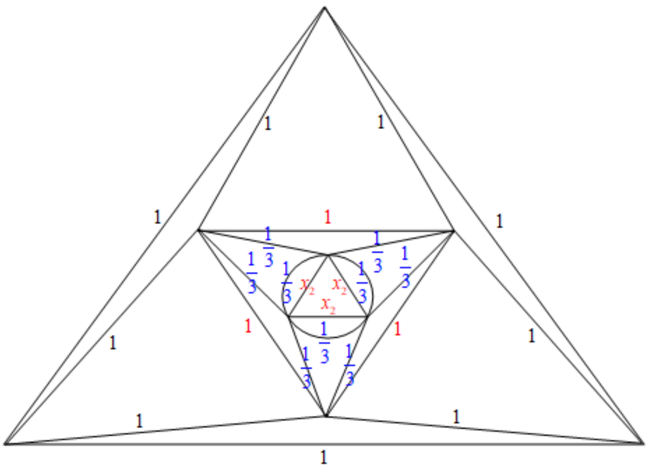
$$\begin{aligned} N_1(\Theta_n) &= 4^{n-2}\left((651-142\sqrt{21})(55+12\sqrt{21})^n+(55-12\sqrt{21})^n(651+142\sqrt{21})\right)^2 \\ &\quad \times\left(9-4\sqrt{21}+(33+8\sqrt{21})(6049+1320\sqrt{21})^{n-1}\right)^2, \\ M_1(\Theta_n) &= 147\left(17+(25+4\sqrt{21})(6049+1320\sqrt{21})^{n-1}\right)^2. \end{aligned}$$

Proof We use the electrically equivalent transformation to transform Θ_i to Θ_{i-1}. Figures 2-6 illustrate the graphs $\Theta_1, \Theta_2, G_1 - G_9$ and the transformation process from Θ_2 to Θ_1.



Θ_2

Figure 2.



G_1

Figure 3.

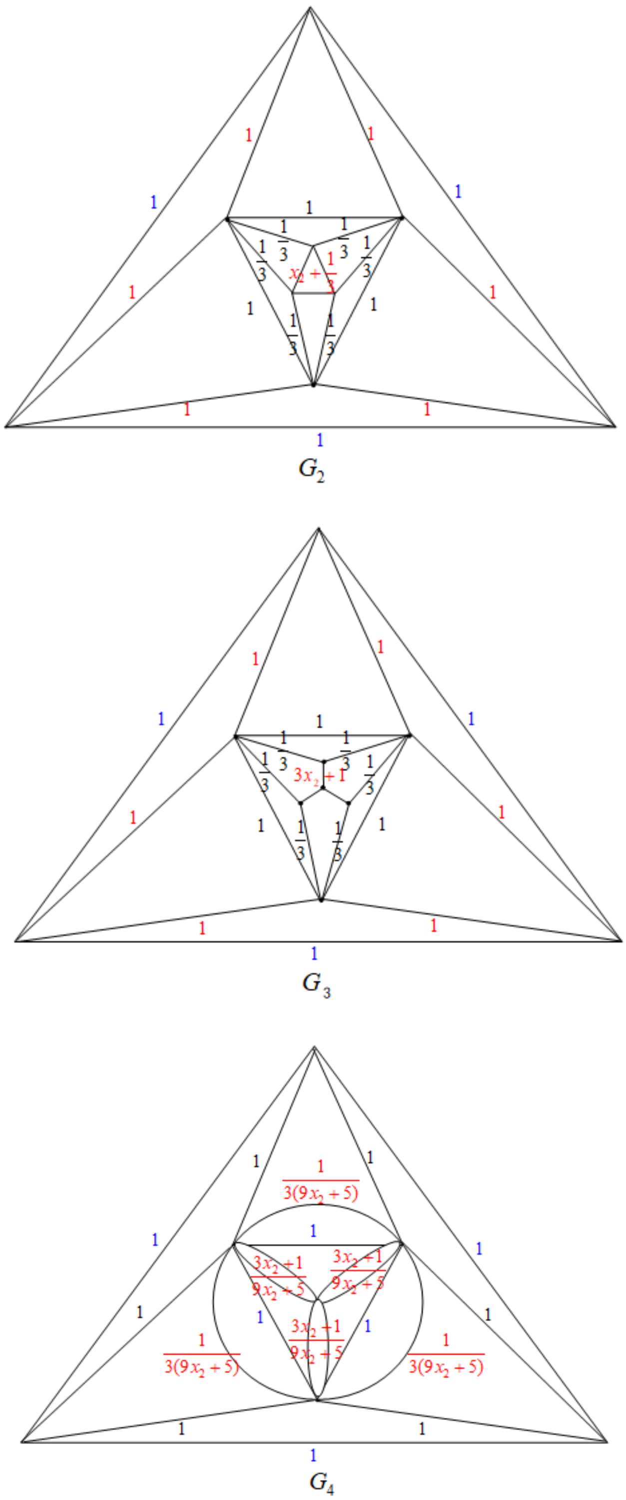


Figure 4.

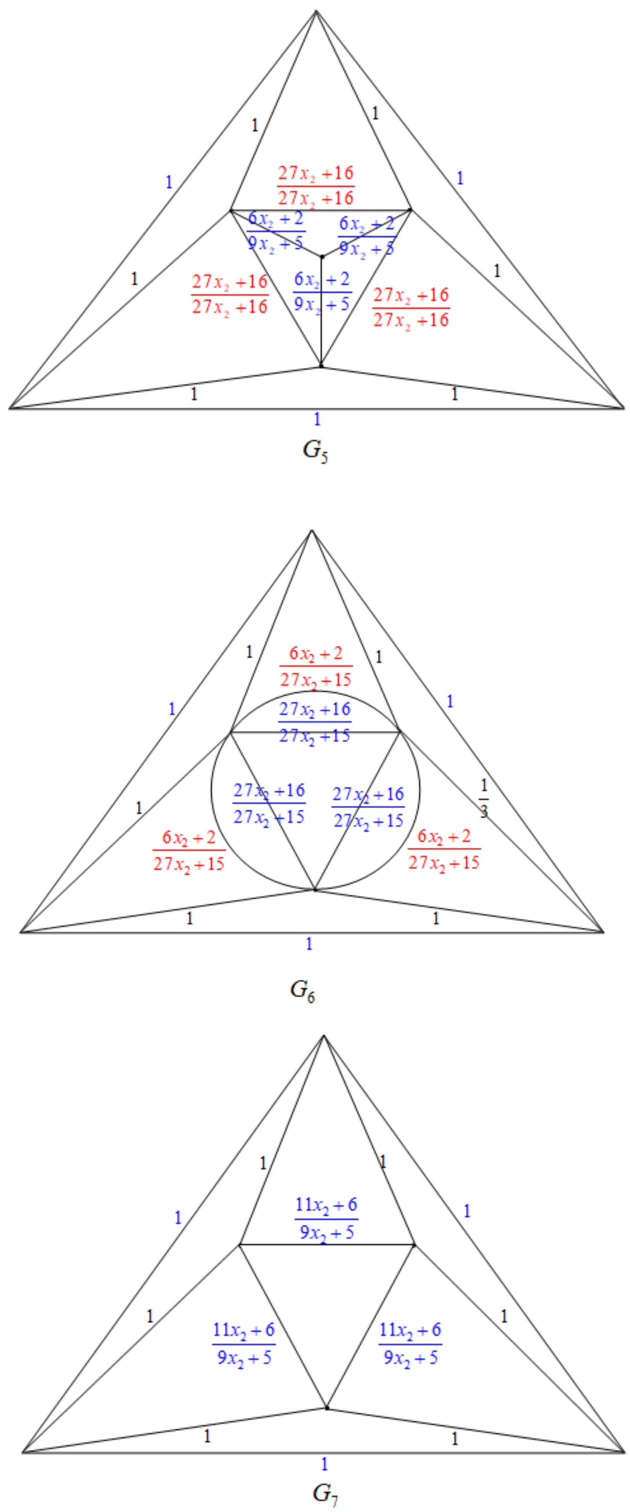

1
1
1
1
1
1
1
1
1
$\frac{27x_2+16}{27x_2+16}$
$\frac{6x_2+2}{9x_2+5}$
G_5
$\frac{6x_2+2}{27x_2+15}$
$\frac{27x_2+16}{27x_2+15}$
$\frac{1}{3}$
G_6
$\frac{11x_2+6}{9x_2+5}$
G_7

Figure 5.

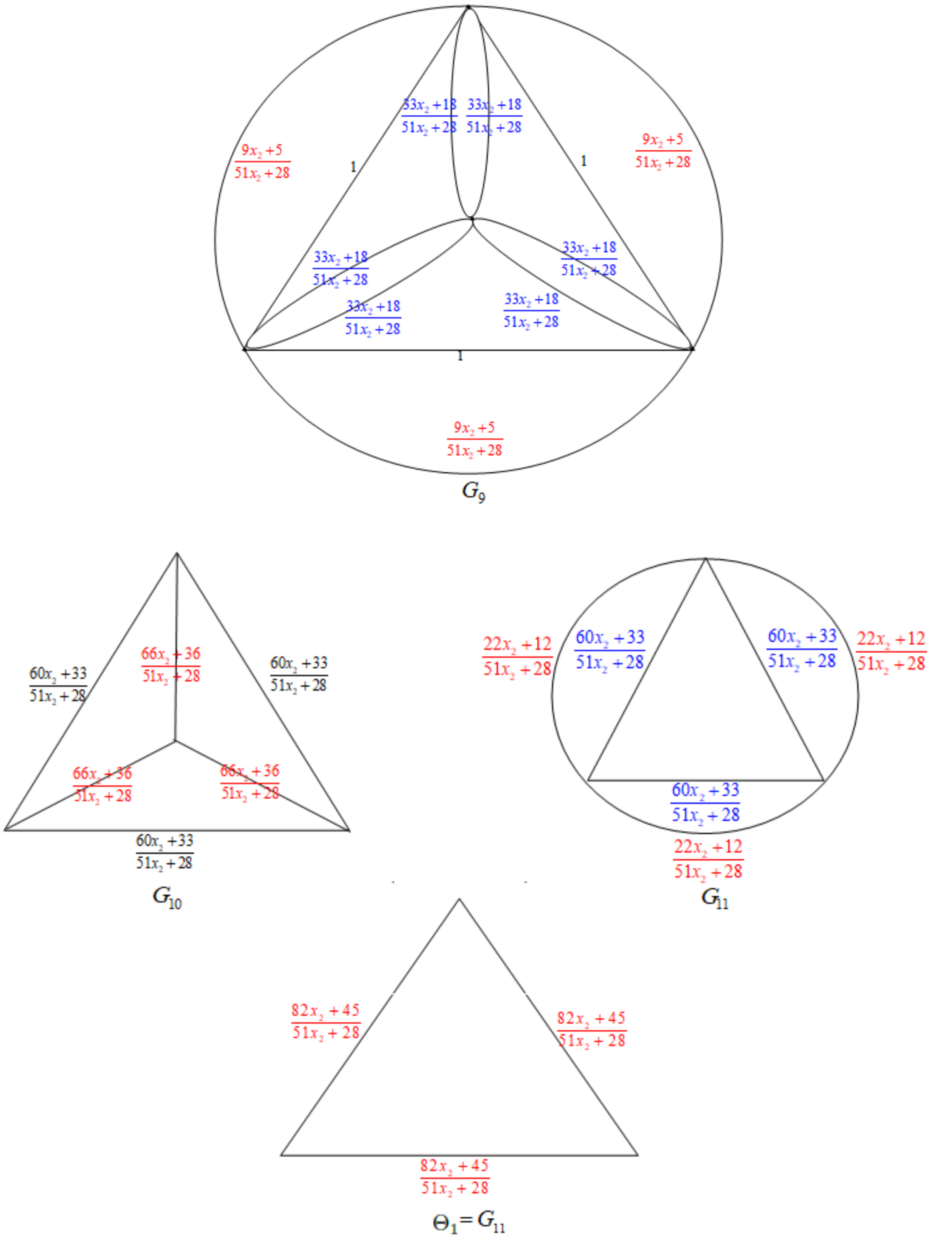


Figure 6. The transformations from θ_2 to θ_1.

Using the properties given in Section 2, we have the following the transformations:

$$\tau(G_1) = \left[\frac{1}{3}\right]^3 \tau(\Theta_2), \tau(G_2) = \tau(G_1), \tau(G_3) = [9x_2 + 3]\, \tau(G_2),$$

$$\tau(G_4) = \left[\frac{3}{9x_2+5}\right]^3 \tau(G_3), \tau(G_5) = \tau(G_4), \tau(G_6) = \left[\frac{9x_2+5}{18x_2+6}\right] \tau(G_5),$$

$$\tau(G_7) = \tau(G_6), \tau(G_8) = 9\left[\frac{11x_2+6}{9x_2+5}\right]\tau(G_7), \tau(G_9) = \left[\frac{9x_2+5}{51x_2+28}\right]^3 \tau(G_8),$$
$$\tau(G_{10}) = \tau(G_9), \tau(G_{11}) = \frac{51x_2+28}{3(66x_2+36)}\tau(G_{10}) \text{ and } \tau(\Theta_1) = \tau(G_{11}).$$

Combining these twelve transformations, we have

$$\tau(\Theta_2) = 4(51x_2+28)^2\tau(\Theta). \tag{3.1}$$

Further

$$\tau(\Theta_n) = \prod_{i=2}^{n} 4(51+28)^2 \tau(\Theta_1) = 3\times(4)^{n-1}x_1^2\left[\prod_{i=2}^{n}(51x_i+28)\right]^2, \tag{3.2}$$

where $x_{i-1} = \frac{82x_i+45}{51x_i+28}, i = 2,3,\cdots,n$. Its characteristic equation is $51\mu^2 - 54\mu - 45 = 0$, which have two roots roots $\mu_1 = \frac{9-4\sqrt{21}}{17}$ and $\mu_2 = \frac{9+4\sqrt{21}}{17}$. Subtracting these two roots into both sides of $x_{i-1} = \frac{82x_i+45}{51x_i+28}$, we get

$$x_{i-1} - \frac{9-4\sqrt{21}}{17} = \frac{82x_i+45}{51x_i+28} - \frac{9-4\sqrt{21}}{17} = (55+12\sqrt{21})\frac{\left(x_i - \frac{9-4\sqrt{21}}{17}\right)}{(51x_i+28)}, \tag{3.3}$$

$$x_{i-1} - \frac{9+4\sqrt{21}}{17} = \frac{82x_i+45}{51x_i+28} - \frac{9+4\sqrt{21}}{17} = (55-12\sqrt{21})\frac{\left(x_i - \frac{9+4\sqrt{21}}{17}\right)}{(51x_i+28)}. \tag{3.4}$$

Let $y_i = \frac{x_i - \frac{9-4\sqrt{21}}{17}}{x_i - \frac{9+4\sqrt{21}}{17}}$. Then by Eqs. (3.3) and (3.4), we get $y_{i-1} = (6049+1320\sqrt{21})y_i$ and $y_i = (6049+1320\sqrt{21})^{n-i}y_n$ Therefore $x_i = \frac{(6049+1320\sqrt{21})^{n-i}\left(\frac{9+4\sqrt{21}}{17}\right)y_n - \frac{9-4\sqrt{21}}{17}}{(6049+1320\sqrt{21})^{n-i}y_n - 1}$. Thus

$$x_1 = \frac{(6049+1320\sqrt{21})^{n-1}\left(\frac{9+4\sqrt{21}}{17}\right)y_n - \frac{9-4\sqrt{21}}{17}}{(6049+1320\sqrt{21})^{n-1}y_n - 1}. \tag{3.5}$$

Using the expression $x_{n-1} = \frac{82x_n+45}{51x_n+28}$ and denoting the coefficients of $82x_n + 45$ and $51x_n + 28$ as σ_n and δ_n we have

$$51x_n + 28 = \sigma_0(82x_n+45) + \delta_0(51x_n+28)$$
$$51x_{n-1} + 28 = \frac{\sigma_1(82x_n+45) + \delta_1(51x_n+28)}{\sigma_0(82x_n+45) + \delta_0(51x_n+28)}$$
$$51x_{n-2} + 28 = \frac{\sigma_2(82x_n+45) + \delta_2(51x_n+28)}{\sigma_1(82x_n+45) + \delta_1(51x_n+28)},$$
$$\vdots$$
$$51x_{n-i} + 28 = \frac{\sigma_i(82x_n+45) + \delta_i(51x_n+28)}{\sigma_{i-1}(82x_n+45) + \delta_{i-1}(51x_n+28)}, \tag{3.6}$$

$$51x_{n-(i+1)}+28=\frac{\sigma_{i+1}\left(82x_n+45\right)+\delta_{i+1}\left(51x_n+28\right)}{\sigma_i\left(82x_n+45\right)+\delta_i\left(51x_n+28\right)}, \tag{3.7}$$

$$\vdots$$

$$51x_2+28=\frac{\sigma_{n-2}\left(82x_n+45\right)+\delta_{n-2}\left(51x_n+28\right)}{\sigma_{n-3}\left(82x_n+45\right)+\delta_{n-3}\left(51x_n+28\right)}$$

Substituting Eq.(4.6) into Eq.(3.2), we obtain

$$\tau\left(\mathrm{E}_n\right)=3\times 4^{n-1}x_1^2\left[\sigma_{n-2}\left(82x_n+45\right)+\sigma_{n-2}\left(51x_n+28\right)\right]^2, \tag{3.8}$$

where $\sigma_0=0,\delta_0=1$ and $\sigma_1=51,\delta_1=28$. By the expression $x_{n-1}=\frac{82x_n+45}{51x_n+28}$ and Eqs. (3.6) and (3.7), we have

$$\sigma_{i+1}=110\sigma_i-\sigma_{i-1};\delta_{i+1}=110\delta_i-\delta_{i-1}. \tag{3.9}$$

The characteristic equation of Eq.(3.9) is $\gamma^2-110\gamma+1=0$ which have two roots $\gamma_1=55+12\sqrt{21}$ and $\gamma_2=55-12\sqrt{21}$. The general solutions of Eq. (3.9) are $\sigma_i=a_1\gamma_1^i+a_2\gamma_2^i;\delta_i=b_1\gamma_1^i+b_2\gamma_2^i$. Using the initial conditions $\sigma_0=0,\delta_0=1$ and $\sigma_1=51,\delta_1=28$, yields

$$\sigma_i=\frac{17\sqrt{21}}{168}(55+12\sqrt{21})^i-\frac{17\sqrt{21}}{168}(55-12\sqrt{21})^i$$

$$\delta_i=\frac{84-9\sqrt{21}}{168}(55+12\sqrt{21})^i+\frac{84+9\sqrt{21}}{168}(55-12\sqrt{21})^i. \tag{3.10}$$

If $x_n=1$, it means that Θ_n without any electrically equivalent transformation. Plugging Eq. (3.10) into Eq.(3.8), we have

$$\begin{aligned}\tau\left(\Theta_n\right) &= 3\times(4)^{n-1}x_1^2\left[\frac{1659+262\sqrt{21}}{42}(55+12\sqrt{15})^{n-2}\right.\\ &\quad\left.+\frac{1659-262\sqrt{21}}{42}(55-12\sqrt{15})^{n-2}\right]^2\end{aligned} \tag{3.11}$$

for integers $n\geq 2$. When $n=1,\tau\left(\Theta_1\right)=3$ which satisfies Eq.(3.11). Therefore, the number of spanning trees in the sequence of the graph Θ_n is given by

$$\begin{aligned}\tau\left(\Theta_n\right) &= 3\times(4)^{n-1}x_1^2\left[\frac{1659+262\sqrt{21}}{42}(55+12\sqrt{15})^{n-2}\right.\\ &\quad\left.+\frac{1659-262\sqrt{21}}{42}(55-12\sqrt{15})^{n-2}\right]^2\end{aligned} \tag{3.12}$$

for integers $n\geq 1$, where

$$x_1=\frac{(6049+1320\sqrt{21})^{n-1}(33+8\sqrt{21})+(9-4\sqrt{21})}{(6049+1320\sqrt{21})^{n-1}(25+4\sqrt{21})+17},n\geq 1. \tag{3.13}$$

Inserting Eq.(3.13) into Eq.(3.12) we obtain the result. □

§4. Number of Spanning Trees in the Sequences of Π_n Graph

Consider the sequence of graphs $\Pi_1, \Pi_2, \cdots, \Pi_n$ constructed as shown in Figure 7. According to this construction, the number of total vertices $|V(\Pi_n)|$ and edges $|E(\Pi_n)|$ are $|V(\Pi_n)| = 9n - 6$ and $|E(\Pi_n)| = 21n - 21, n = 1, 2, \ldots$. The average degree of Π_n is in the large n limit which is $\frac{14}{3}$.

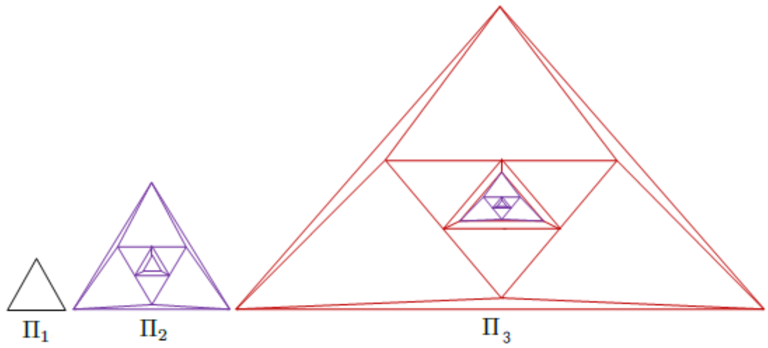
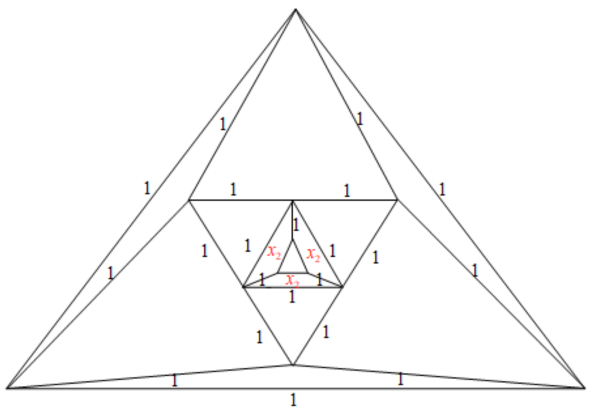


Figure 7. Some sequences of graph Π_n

Theorem 4.1 *For $n \geq 1$, the number of spanning trees in the sequence of Π_n graph is given by* $\dfrac{N_2(\Pi_n)}{M_2(\Pi_n)}$, *where*

$$\begin{aligned} N_2(\Pi_n) &= 3 \times 4^{n-2}\left(-17(-15+\sqrt{455}) + (655+33\sqrt{455})(8191+384\sqrt{455})^{n-1}\right)^2 \\ &\quad \times \left((18200-853\sqrt{455})(64+3\sqrt{455})^n + (64-3\sqrt{455})^n(18200+853\sqrt{455})\right)^2 \\ M_2(\Pi_n) &= 207025\left(391 + (519+16\sqrt{455})(8191+384\sqrt{455})^{n-1}\right)^2. \end{aligned}$$

Proof We use the electrically equivalent transformation to transform Π_i to Π_{i-1}. Figures 8-12 illustrate the graphs $\Pi_1, \Pi_2, G_1 - G_{12}$ and the transformation process from Π_2 to Π_1.



Π_1

Figure 8.

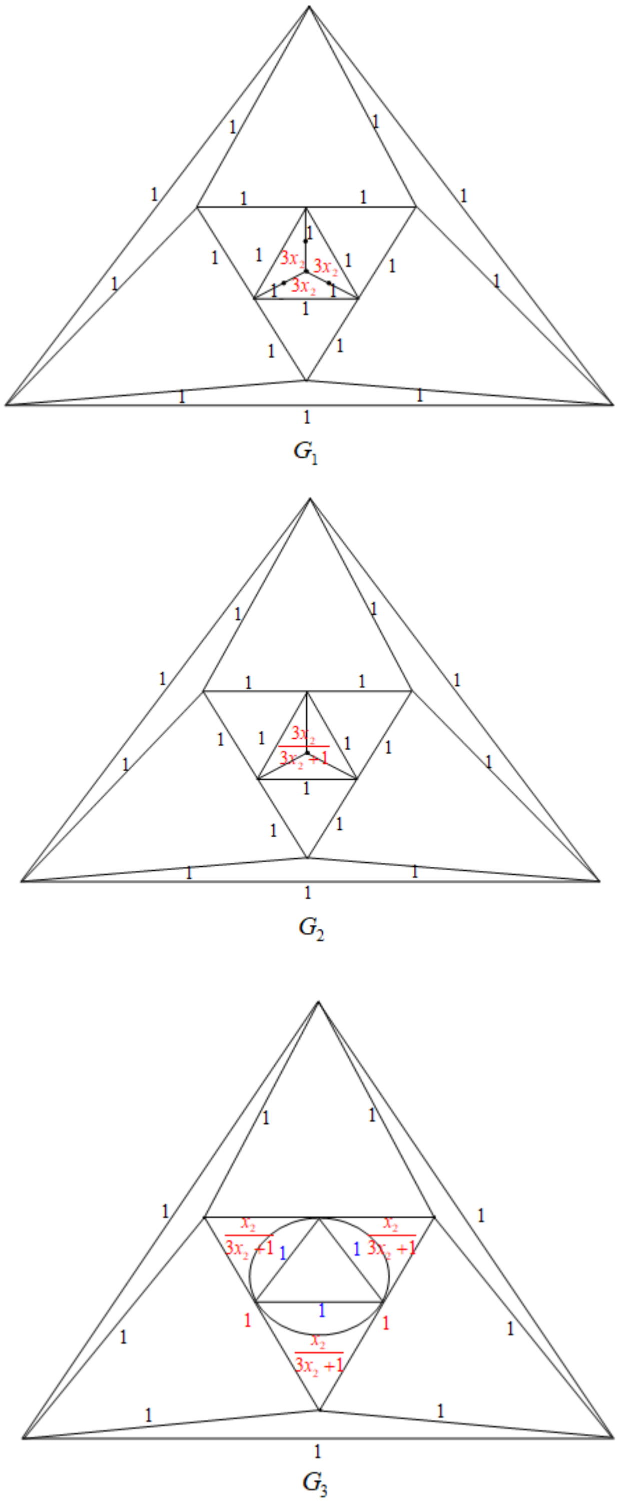


Figure 9.

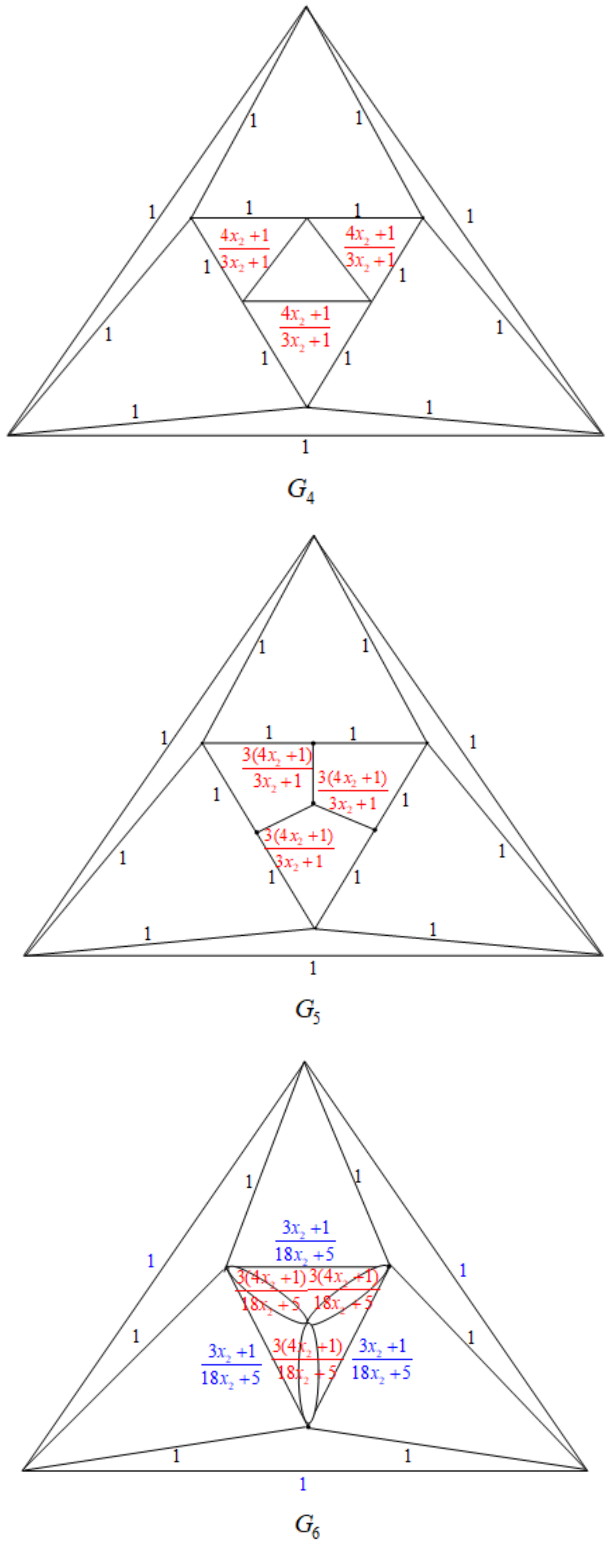

$\frac{4x_2+1}{3x_2+1}$
G_4
$\frac{3(4x_2+1)}{3x_2+1}$
G_5
$\frac{3x_2+1}{18x_2+5}$
$\frac{3(4x_2+1)}{18x_2+5}$
G_6

Figure 10.

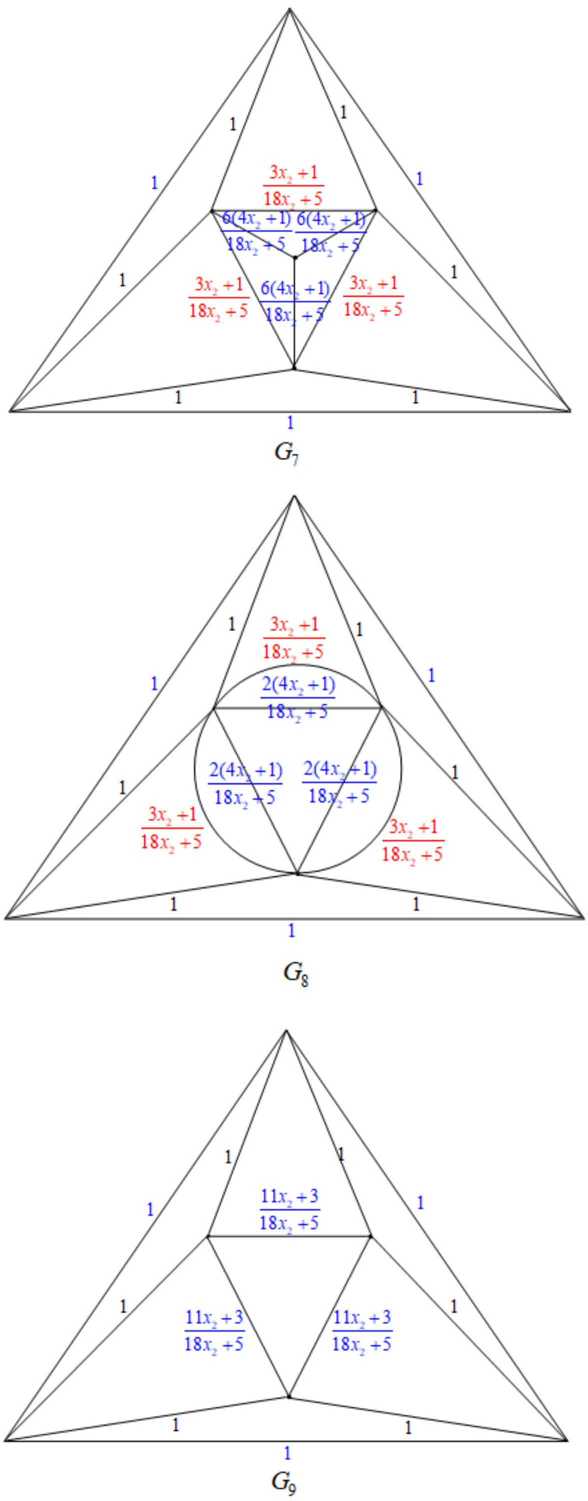


Figure 11.

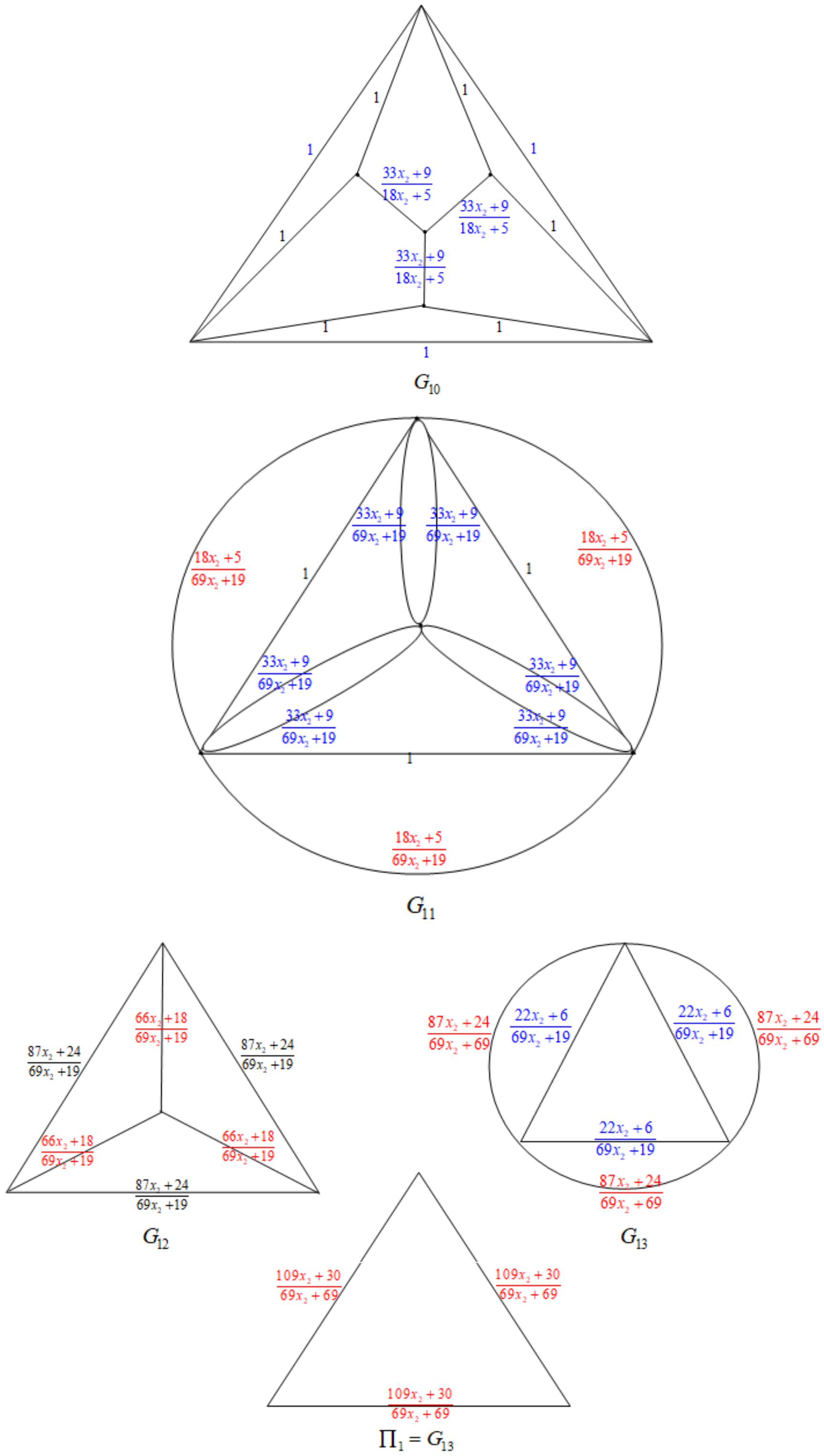


Figure 12. The transformations from Π_2 to Π_1

Using the properties given in Section 2, we have the following the transformations:

$$\begin{aligned}
&\tau(G_1) = 9x_2\tau(\Pi_2), \tau(G_2) = \left[\frac{1}{3x_2+1}\right]^3 \tau(G_1), \tau(G_3) = \frac{3x_2+1}{9x_2}\tau(G_2),\\
&\tau(G_4) = \tau(G_3), \tau(G_5) = 9\left[\frac{4x_2+1}{3x_2+1}\right]\tau(G_4), \tau(G_6) = \left[\frac{3x_2+1}{18x_2+5}\right]^3 \tau(G_5),\\
&\tau(G_7) = \tau(G_6), \tau(G_8) = \left[\frac{18x_2+5}{18(4x_2+1)}\right]\tau(G_7), \tau(G_9) = \tau(G_8),\\
&\tau(G_{10}) = 9\left[\frac{11x_2+3}{18x_2+5}\right]\tau(G_9), \tau(G_{11}) = \left[\frac{18x_2+5}{69x_2+19}\right]^3 \tau(G_{10}),\\
&\tau(G_{12}) = \tau(G_{11}), \tau(G_{13}) = \left[\frac{69x_2+19}{18(11x_2+3)}\right]\tau(G_{12}) \text{ and } \tau(\Pi) = \tau(G_{13}).
\end{aligned}$$

Combining these fourteen transformations, we have

$$\tau(\Pi_2) = 4(69x_2+19)^2\tau(\Pi_1). \tag{4.1}$$

Further

$$\tau(\Pi_n) = \prod_{i=2}^{n} 4(69x_2+19)^2\tau(\Pi_1) = 3\times(4)^{n-1}x_1^2\left[\prod_{i=2}^{n}(69x_i+19)\right]^2, \tag{4.2}$$

where $x_{i-1} = \frac{109x_i+30}{69x_i+19}, i = 2, 3, \ldots, n$. Its characteristic equation is $23\mu^2 - 30\mu - 10 = 0$, which have two roots roots $\mu_1 = \frac{15-\sqrt{455}}{23}$ and $\mu_2 = \frac{15+\sqrt{455}}{23}$. Subtracting these two roots into both sides of $x_{i-1} = \frac{109x_i+30}{69x_i+19}$, we get

$$x_{i-1} - \frac{15-\sqrt{455}}{23} = \frac{109x_i+30}{69x_i+19} - \frac{15-\sqrt{455}}{23} = (64+3\sqrt{455})\frac{x_i - \frac{15-\sqrt{455}}{23}}{69x_i+19}, \tag{4.3}$$

$$x_{i-1} - \frac{15+\sqrt{455}}{23} = \frac{109x_i+30}{69x_i+19} - \frac{15+\sqrt{455}}{23} = (64-3\sqrt{455})\frac{x_i - \frac{15+\sqrt{455}}{23}}{69x_i+19}. \tag{4.4}$$

Let $y_i = \frac{x_i - \frac{15-\sqrt{455}}{23}}{x_i - \frac{15+\sqrt{455}}{23}}$. Then by Eqs. (4.3) and (4.4), we get $y_{i-1} = (8191+384\sqrt{455})y_i$ and $y_i = (8191+384\sqrt{455})^{n-i}y_n$. Therefore $x_i = \frac{(8191+384\sqrt{455})^{n-i}\frac{15+\sqrt{455}}{23}y_n - \frac{15-\sqrt{455}}{23}}{(8191+384\sqrt{455})^{n-i}y_n - 1}$. Thus

$$x_1 = \frac{(8191+384\sqrt{455})^{n-1}\frac{15+\sqrt{455}}{23}y_n - \frac{15-\sqrt{455}}{23}}{(8191+384\sqrt{455})^{n-1}y_n - 1}. \tag{4.5}$$

Using the expression $x_{n-1} = \frac{109x_n+30}{69x_n+19}$ and denoting the coefficients of $109x_n+30$ and $69x_n+19$ as σ_n and δ_n we have

$$\begin{aligned}
69x_n + 19 &= \sigma_0(109x_n+30) + \delta_0(69x_n+19)\\
69x_{n-1} + 19 &= \frac{\sigma_1(109x_n+30) + \delta_1(69x_n+19)}{\sigma_0(109x_n+30) + \delta_0(69x_n+19)}
\end{aligned}$$

$$69x_{n-2}+19=\frac{\sigma_2\left(109x_n+30\right)+\delta_2\left(69x_n+19\right)}{\sigma_1\left(109x_n+30\right)+\delta_1\left(69x_n+19\right)},$$

$$\vdots$$

$$69x_{n-2}+19=\frac{\sigma_2\left(109x_n+30\right)+\delta_2\left(69x_n+19\right)}{\sigma_1\left(109x_n+30\right)+\delta_1\left(69x_n+19\right)}, \tag{4.6}$$

$$69x_{n-(i+1)}+19=\frac{\sigma_{i+1}\left(109x_n+30\right)+\delta_{i+1}\left(69x_n+19\right)}{\sigma_i\left(109x_n+30\right)+\delta_i\left(69x_n+19\right)}, \tag{4.7}$$

$$\vdots$$

$$69x_2+19=\frac{\sigma_{n-2}\left(109x_n+30\right)+\delta_{n-2}\left(69x_n+19\right)}{\sigma_{n-3}\left(109x_n+30\right)+\delta_{n-3}\left(69x_n+19\right)}$$

Substituting Eq.(4.6) into Eq.(4.2), we obtain

$$\tau\left(\mathrm{E}_n\right)=3\times 4^{n-1}x_1^2\left[\sigma_{n-2}\left(109x_n+30\right)+\sigma_{n-2}\left(69x_n+19\right)\right]^2, \tag{4.8}$$

where $\sigma_0=0,\delta_0=1$ and $\sigma_1=69,\delta_1=19$. By the expression $x_{n-1}=\frac{109x_n+30}{69x_n+19}$ and Eqs. (4.6) and (4.7), we have

$$\sigma_{i+1}=128\sigma_i-\sigma_{i-1};\delta_{i+1}=128\delta_i-\delta_{i-1}. \tag{4.9}$$

The characteristic equation of Eq.(4.9) is $\gamma^2-128\gamma+1=0$ which have two roots $\gamma_1=64+3\sqrt{455}$ and $\gamma_2=64-3\sqrt{455}$. The general solutions of Eq. (4.9) are $\sigma_i=a_1\gamma_1^i+a_2\gamma_2^i;\delta_i=b_1\gamma_1^i+b_2\gamma_2^i$. Using the initial conditions $\sigma_0=0,\delta_0=1$ and $\sigma_1=69,\delta_1=19$, yields

$$\sigma_i=\frac{23\sqrt{455}}{910}(64+3\sqrt{455})^i-\frac{23\sqrt{455}}{910}(64+3\sqrt{455})^i$$

$$\delta_i=\frac{455-15\sqrt{455}}{910}(64+3\sqrt{455})^i+\frac{455+15\sqrt{455}}{910}(64-3\sqrt{455})^i. \tag{4.10}$$

If $x_n=1$, it means that Π_n without any electrically equivalent transformation. Plugging Eq. (4.10) into Eq.(4.8), we haven for all $n\geq 2$

$$\begin{aligned}\tau\left(\Pi_n\right) &= 3\times 4^{n-1}x_1^2\left[\frac{40040+1877\sqrt{455}}{910}(64+3\sqrt{455})^{n-2}\right.\\ &\left.+\frac{40040-1877\sqrt{455}}{910}(64-3\sqrt{455})^{n-2}\right]^2.\end{aligned} \tag{4.11}$$

When $n=1,\tau\left(\Pi_1\right)=3$ which satisfies Eq.(4.11). Therefore, the number of spanning trees in the sequence of the graph Π_n where $n\geq 1$, is given by

$$\begin{aligned}\tau\left(\Pi_n\right) &= 3\times 4^{n-1}x_1^2\left[\frac{40040+1877\sqrt{455}}{910}(64+3\sqrt{455})^{n-2}\right.\\ &\left.+\left(\frac{40040-1877\sqrt{455}}{910}\right)(64-3\sqrt{455})^{n-2}\right]^2,\end{aligned} \tag{4.12}$$

where

$$x_1 = \frac{(8191+384\sqrt{455})^{n-1}(655+33\sqrt{455})+17(15-\sqrt{455})}{(8191+384\sqrt{455})^{n-1}(519+16\sqrt{455})+391},\ n \geq 1. \tag{4.13}$$

Inserting Eq.(4.13) into Eq.(4.12) we obtain the result. □

§5. Number of Spanning Trees in the Sequences of Σ_n Graph

Consider the sequence of graphs $\Sigma_1, \Sigma_2, \cdots, \Sigma_n$ constructed as shown in Figure 13. According to this construction, the number of total vertices $|V(\Sigma_n)|$ and edges $|E(\Sigma_n)|$ are $|V(\Sigma_n)| = 9n-6$ and $|E(\Sigma_n)| = 21n-18, n = 1, 2, \cdots$. The average degree of Σ_n is $\frac{14}{3}$ in $n \to \infty$.

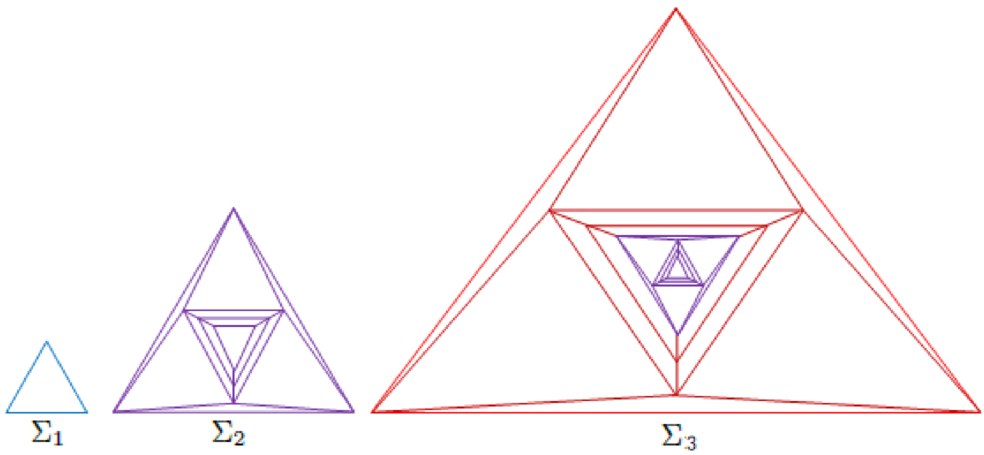


Figure 13. Some sequences of Σ_n

Theorem 5.1 *The number of spanning trees in sequence of Σ_n is given by $\frac{N_3(\Sigma_n)}{M_3(\Sigma_n)}$ for $n \geq 1$, where*

$$\begin{aligned} N_3(\Sigma_n) &= 3 \times 2^{-3-n}(163+\sqrt{26565})^{2n} \\ &\times \left(-\frac{67}{2}(-117+\sqrt{26565}) + \left(\frac{1}{2}(26567-163\sqrt{26565})\right)^{1-n}(9363+62\sqrt{26565})\right)^2 \\ &\times \left(469315-2879\sqrt{26565} + \left(\frac{1}{2}(26567-163\sqrt{26565})\right)^{n}(469315+2879\sqrt{26565})\right)^2, \\ M_3(\Sigma_n) &= 78411025\left(5829+3\times 2^{-n}(4969+19\sqrt{26565})(26567+163\sqrt{26565})^{n-1}\right)^2. \end{aligned}$$

Proof We use the electrically equivalent transformation to transform Σ_i to Σ_{i-1}. Figures 14-17 illustrate the graphs $\Sigma_1, \Sigma_2, G_1 - G_{12}$ and the transformation process from Σ_2 to Σ_1.

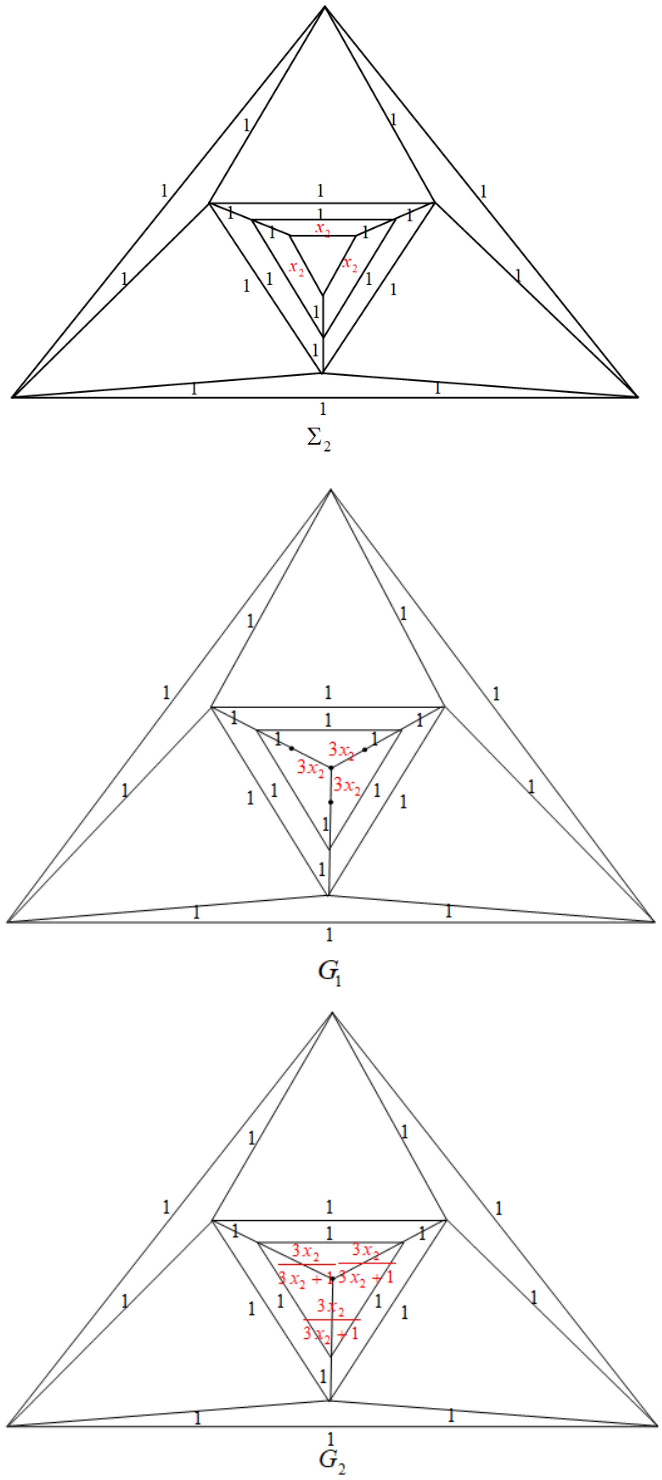


Figure 14.

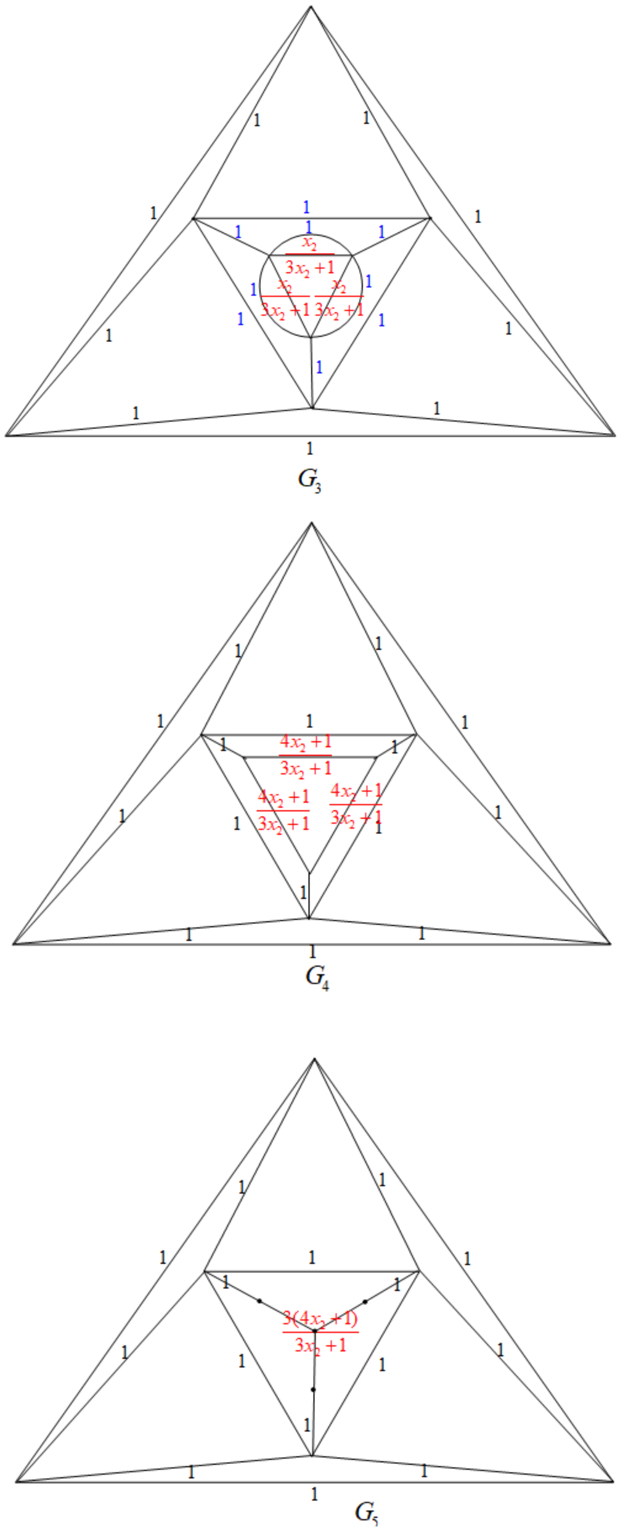
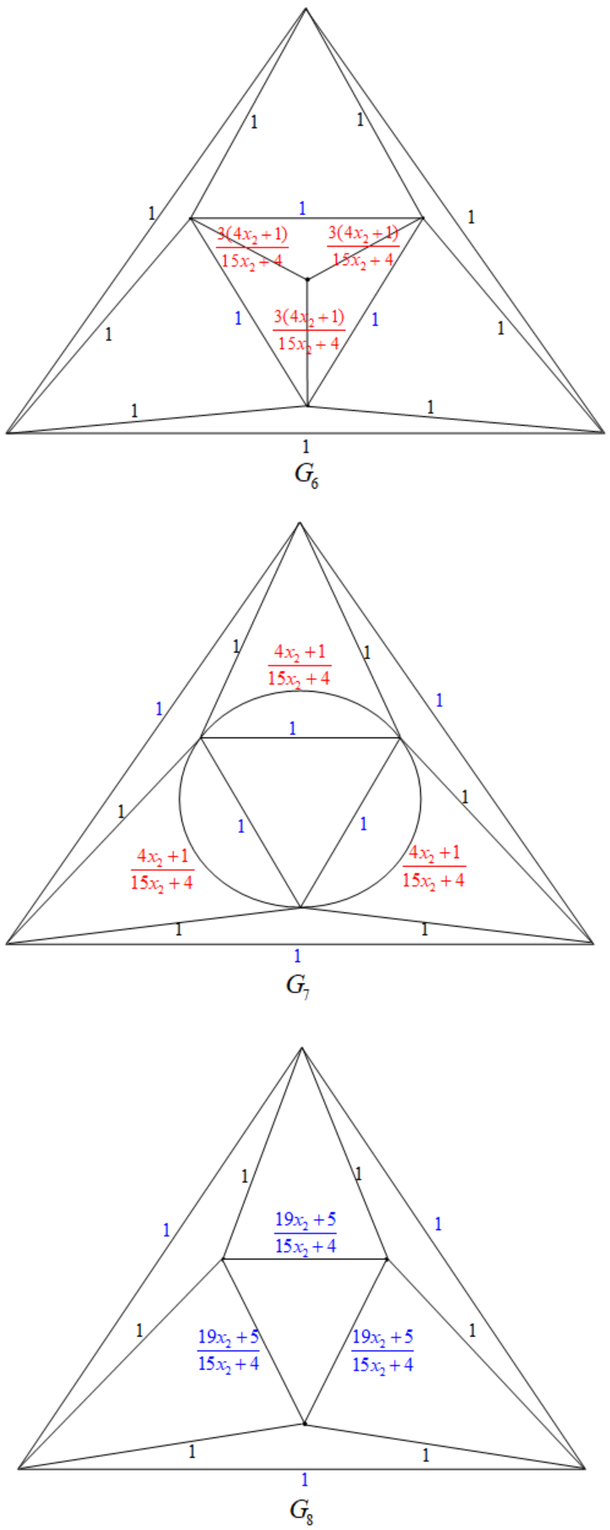


Figure 15.


1
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1
3(4x_2+1)/(15x_2+4)
3(4x_2+1)/(15x_2+4)
3(4x_2+1)/(15x_2+4)
G_6
4x_2+1/(15x_2+4)
4x_2+1/(15x_2+4)
4x_2+1/(15x_2+4)
G_7
19x_2+5/(15x_2+4)
19x_2+5/(15x_2+4)
19x_2+5/(15x_2+4)
G_8

Figure 16.

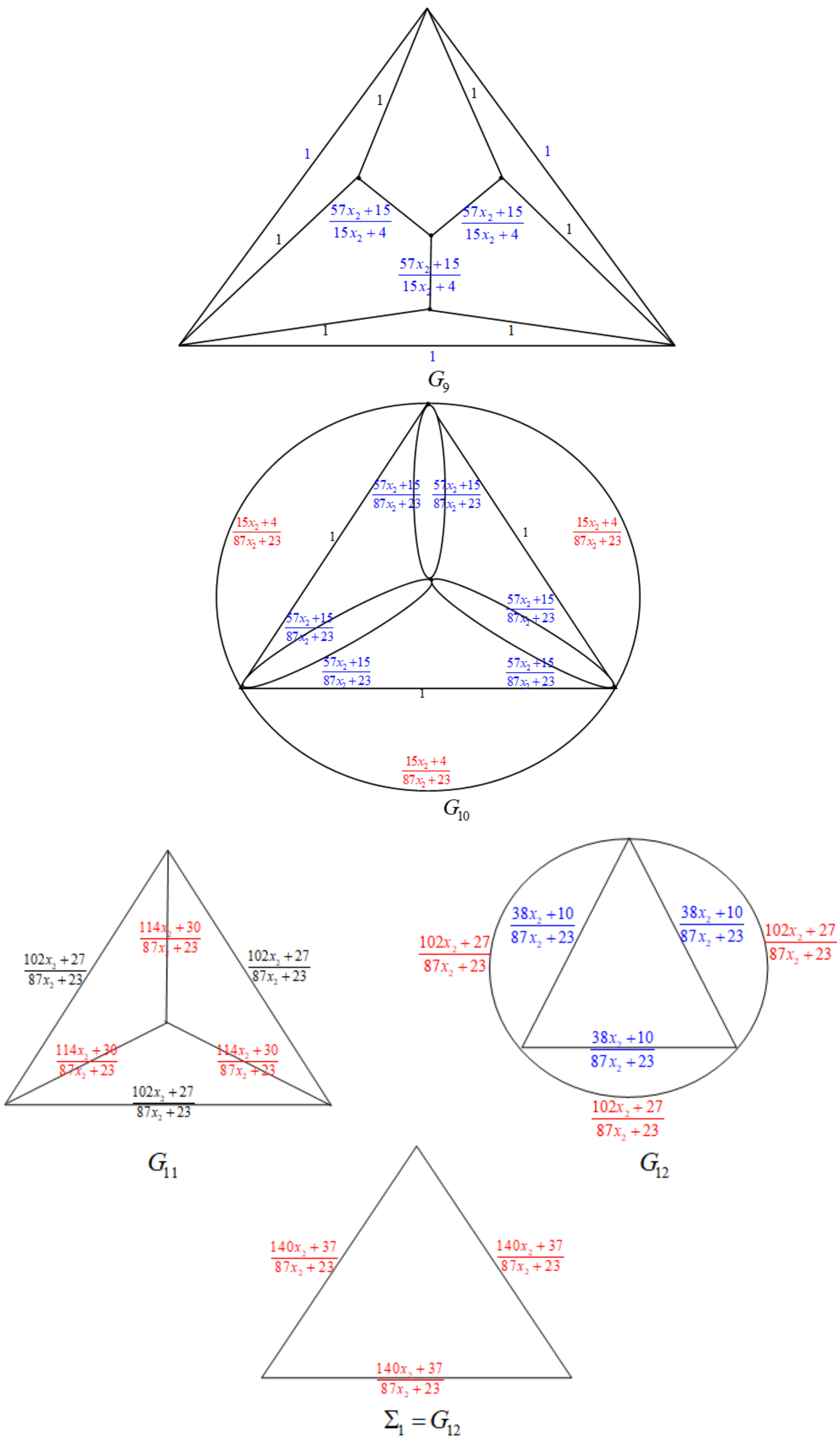

Figure 17. The transformations from Σ_2 to Σ_1

Using the properties given in section 2 , we have the following the transformations:

$\tau\left(G_1\right)=9x_2\tau\left(\Sigma_2\right),\tau\left(G_2\right)=\left[\frac{1}{3x_2+1}\right]^3\tau\left(G_1\right),\tau\left(G_3\right)=\frac{3x_2+1}{9x_2}\tau\left(G_2\right),$
$\tau\left(G_4\right)=\tau\left(G_3\right),\tau\left(G_5\right)=9\left[\frac{4x_2+1}{3x_2+1}\right]\tau\left(G_4\right),\tau\left(G_6\right)=\left[\frac{3x_2+1}{15x_2+4}\right]^3\tau\left(G_5\right),$
$\tau\left(G_7\right)=\frac{15x_2+4}{9\left(4x_2+1\right)}\tau\left(G_6\right),\tau\left(G_8\right)=\tau\left(G_7\right),\tau\left(G_9\right)=9\left[\frac{19x_2+5}{15x_2+4}\right]\tau\left(G_8\right),$
$\tau\left(G_{10}\right)=\left[\frac{15x_2+4}{87x_2+23}\right]^3\tau\left(G_9\right),\tau\left(G_{11}\right)=\tau\left(G_{10}\right),\tau\left(G_{12}\right)=\frac{87x_2+23}{18\left(19x_2+5\right)}\tau\left(G_{11}\right)$ and
$\tau\left(\Sigma_1\right)=\tau\left(G_{12}\right).$

Combining these thirteen transformations, we get

$$\tau\left(\Sigma_2\right)=2\left(87x_2+23\right)^2\tau\left(\Sigma_1\right). \tag{5.1}$$

Further

$$\tau\left(\Sigma_n\right)=\prod_{i=2}^{n}2\left(87x_2+23\right)^2\tau\left(\Sigma_1\right)=3\times(2)^{n-1}x_1^2\left[\prod_{i=2}^{n}\left(87x_i+23\right)\right]^2, \tag{5.2}$$

where $x_{i-1}=\frac{140x_i+37}{87x_i+23}, i=2,3,\ldots,n$. Its characteristic equation is $87\mu^2-117\mu-37=0$, which have two roots roots $\mu_1=\frac{117-\sqrt{26565}}{174}$ and $\mu_2=\frac{117+\sqrt{26565}}{174}$. Subtracting these two roots into both sides of $x_{i-1}=\frac{140x_i+37}{87x_i+23}$, we get

$$x_{i-1}-\frac{117-\sqrt{26565}}{174}=\frac{140x_i+37}{87x_i+23}-\frac{117-\sqrt{26565}}{174}=(163+\sqrt{26565})\frac{x_i-\frac{117-\sqrt{26565}}{174}}{2\left(87x_i+23\right)}, \tag{5.3}$$

$$x_{i-1}-\frac{117+\sqrt{26565}}{174}=\frac{140x_i+37}{87x_i+23}-\frac{117+\sqrt{26565}}{174}=(163-\sqrt{26565})\frac{x_i-\frac{117+\sqrt{26565}}{174}}{2\left(87x_i+23\right)}. \tag{5.4}$$

Let $y_i=\frac{x_i-\frac{117-\sqrt{26565}}{174}}{x_i-\frac{117+\sqrt{26565}}{174}}$. Then by Eqs. (5.3) and (5.4), we get $y_{i-1}=\frac{26567+163\sqrt{26565}}{2}y_i$ and $y_i=\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-i}y_n$. Therefore $x_i=\frac{\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-i}\frac{117+\sqrt{26565}}{174}y_n-\frac{117-\sqrt{26565}}{174}}{\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-i}y_n-1}\cdot$ Thus

$$x_1=\frac{\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-1}\frac{117+\sqrt{26565}}{174}y_n-\frac{117-\sqrt{26565}}{174}}{\left(\frac{26567+163\sqrt{26565}}{2}\right)^{n-1}y_n-1}. \tag{5.5}$$

Using the expression $x_{n-1}=\frac{140x_n+37}{87x_n+23}$ and denoting the coefficients of $140x_n+37$ and

$87x_n + 23$ as σ_n and δ_n we have

$$87x_n + 23 = \sigma_0 (140x_n + 37) + \delta_0 (87x_n + 23)$$
$$87x_{n-1} + 23 = \frac{\sigma_1 (140x_n + 37) + \delta_1 (87x_n + 23)}{\sigma_0 (140x_n + 37) + \delta_0 (87x_n + 23)}$$
$$87x_{n-2} + 23 = \frac{\sigma_2 (140x_n + 37) + \delta_2 (87x_n + 23)}{\sigma_1 (140x_n + 37) + \delta_1 (87x_n + 23)}$$
$$\vdots$$
$$87x_{n-i} + 23 = \frac{\sigma_i (140x_n + 37) + \delta_i (87x_n + 23)}{\sigma_{i-1} (140x_n + 37) + \delta_{i-1} (87x_n + 23)}, \quad (5.6)$$
$$87x_{n-(i+1)} + 23 = \frac{\sigma_{i+1} (140x_n + 37) + \delta_{i+1} (87x_n + 23)}{\sigma_i (140x_n + 37) + \delta_i (87x_n + 23)}, \quad (5.7)$$
$$\vdots$$
$$87x_2 + 23 = \frac{\sigma_{n-2} (140x_n + 37) + \delta_{n-2} (87x_n + 23)}{\sigma_{n-3} (140x_n + 37) + \delta_{n-3} (87x_n + 23)}$$

Substituting Eq.(5.6) into Eq.(5.2), we obtain

$$\tau (\mathrm{E}_n) = 3 \times 2^{n-1} x_1^2 \left[\sigma_{n-2} (140x_n + 37) + \sigma_{n-2} (87x_n + 23)\right]^2, \quad (5.8)$$

where $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 87, \delta_1 = 23$. By the expression $x_{n-1} = \frac{140x_n+37}{87x_n+23}$ and Eqs. (5.6) and (5.7), we have

$$\sigma_{i+1} = 163\sigma_i - \sigma_{i-1}; \delta_{i+1} = 163\delta_i - \delta_{i-1}. \quad (5.9)$$

The characteristic equation of Eq.(5.9) is $\gamma^2 - 163\gamma + 1 = 0$ which have two roots $\gamma_1 = \frac{163-\sqrt{26565}}{2}$ and $\gamma_2 = \frac{163+\sqrt{26565}}{2}$. The general solutions of Eq. (5.9) are $\sigma_i = a_1\gamma_1^i + a_2\gamma_2^i; \delta_i = b_1\gamma_1^i + b_2\gamma_2^i$. Using the initial conditions $\sigma_0 = 0, \delta_0 = 1$ and $\sigma_1 = 87, \delta_1 = 23$, yields

$$\sigma_i = \frac{29\sqrt{26565}}{8855} \left(\frac{163 + \sqrt{26565}}{2}\right)^i - \frac{29\sqrt{26565}}{8855} \left(\frac{163 - \sqrt{26565}}{2}\right)^i,$$

$$\delta_i = \frac{26565 - 117\sqrt{26565}}{53130} \left(\frac{163 + \sqrt{26565}}{2}\right)^i + \frac{26565 + 117\sqrt{26565}}{53130} \left(\frac{163 - \sqrt{26565}}{2}\right)^i. \quad (5.10)$$

If $x_n = 1$, it means that Σ_n without any electrically equivalent transformation. Plugging Eq. (5.10) into Eq.(5.8), we have for all $n \geq 2$

$$\tau (\Sigma_n) = 3 \times 2^{n-1} x_1^2 \left[\frac{487025 + 2988\sqrt{26565}}{8855} \left(\frac{163 + \sqrt{26565}}{2}\right)^{n-2} + \frac{487025 - 2988\sqrt{26565}}{8855} \left(\frac{163 - \sqrt{26565}}{2}\right)^{n-2}\right]^2. \quad (5.11)$$

When $n = 1, \tau(\Sigma_1) = 3$ which satisfies Eq.(5.11). Therefore, the number of spanning trees in the sequence of the graph Σ_n for all $n \geq 1$ is given by

$$\tau(\Sigma_n) = 3 \times 2^{n-1} x_1^2 \left[\frac{487025 + 2988\sqrt{26565}}{8855} \left(\frac{163 + \sqrt{26565}}{2} \right)^{n-2} + \frac{487025 - 2988\sqrt{26565}}{8855} \left(\frac{163 - \sqrt{26565}}{2} \right)^{n-2} \right]^2, \quad (5.12)$$

where

$$x_1 = \frac{\left(\frac{26567+16\sqrt{26565}}{2}\right)^{n-1} (9363 + 62\sqrt{26565}) + \frac{67(117-\sqrt{26565})}{2}}{\frac{3}{2}\left(\frac{2x6677+163\sqrt{26555}}{2}\right)^{n-1} (4969 + 19\sqrt{26565}) + 5829}, n \geq 1. \quad (5.13)$$

This completes the proof. □

§6. Numerical Results

Table 1. some values of the number of spanning trees in the graphs Θ_n, Π_n and Σ_n.

n	$\tau(\Theta_n)$	$\tau(\Pi_n)$	$\tau(\Sigma_n)$
1	3	3	3
2	193548	231852	187974
3	9366382128	15192944688	9987870000
4	453257961670848	695563276393152	530695483947096
5	21934059131316880128	65237270132405699328	28197973804093756848
6	1061432982230559089691648	4274867821307268252675072	1498271137983935741049216

§7. Entropy of Spanning Trees

After having explicit Formulas for the number of spanning trees of the sequence of the three families of graphs Θ_n, Π_n and Σ_n, we can calculate its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined as in [18] as: *For a graph G,*

$$Z(G) = \lim_{n\to\infty} \frac{\ln \tau(G)}{|V(G)|}, \quad (7.1)$$

$$Z(\Theta_n) = \frac{1}{9}(\ln[4] + 2\ln[55 + 12\sqrt{21}]) = 1.198565531,$$
$$Z(\Pi_n) = \frac{1}{9}(\ln[4] + 2\ln[64 + 3\sqrt{455}]) = 1.23224809,$$
$$Z(\Sigma_n) = \frac{1}{9}(\ln[8] - 2\ln[163 - \sqrt{26565}]) = 1.208952478.$$

Now we compare the value of entropy in our graphs with other graphs. The entropy of the graph Π_n is larger than the entropy of the graph Θ_n and Σ_n. In addition the entropy of the families Θ_n, Π_n and Σ_n which have the same average degree 14/3 is larger than the entropy of fractal scale free lattice [6] which has the entropy1.040 and 3-prism graph of average degree 4 which has entropy1.0445 [19] and two dimensional Sierpinski gasket [20] which has the entropy 1.166 of the same average degree 4.

§8. Conclusions

In this work, we enumerate the number of spanning trees in the sequences of three sequences of graphs of average degree 14/3 using electrically equivalent transformations. An advantage of this method lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

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Some Results on Generalized Sasakian Space Forms Admitting Almost Quasi-Yamabe Solitons

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Abstract: In this paper we have studied generalized Sasakian space form admitting almost quasi-Yamabe soliton and gradient almost quasi-Yamabe soliton. It is shown that if a generalized Sasakian space form admits a closed almost quasi-Yamabe soliton, then either soliton vector field is pointwise collinear with ζ or the structure functions are connected by a relation and the manifold becomes Ricci semi-symmetric generalized Sasakian space form. Next, it is proven that, if the metric of a generalized Sasakian space form is a gradient almost quasi-Yamabe soliton, then either the gradient of ψ is pointwise collinear with ζ or the structure functions are connected by a relation.

Key Words: Generalized Sasakian space form, almost quasi-Yamabe soliton, gradient almost quasi-Yamabe soliton

AMS(2010): 53C21, 53C25, 53D10.

§1. Introduction

The notion of Yamabe flow was proposed by R. S. Hamilton [14] as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a Riemannian manifold of dimension greater than or equal to three. The Yamabe soliton as a self-similar solutions to the Yamabe flow

$$\frac{\partial}{\partial}(g(t)) = -\sigma(g(t)),$$

where $g(t)$ is an one parameter family of metrics on a some smooth manifold and σ stands for scalar curvature of the manifold (see [6, 10, 14]). It is well-known that a Riemannian metric g defined on a n-dimensional smooth manifold M is said to be an Yamabe soliton if, for some real constant ω, there exists a smooth vector field $\overline{V}$ on smooth manifold M satisfying the following equation

$$\frac{1}{2}\pounds_{\overline{V}}g = (\sigma - \omega)g, \tag{1.1}$$

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where $\pounds_{\overline{V}}$ indicates the Lie-derivative operator along $\overline{V}$ on M and σ being the scalar curvature of M. The Yamabe soliton is called shrinking, steady and expanding according as ω is positive, zero and negative respectively. Moreover, if $\overline{V}$ is Killing, the Yamabe soliton is said to be trivial, otherwise it is non-trivial. Yamabe solitons have been investigated under different types of manifolds by several geometers (see [8, 9, 15, 21, 23, 26, 28]) and many others.

In 2013, E. Barbosa et al.[3] introduced the concept of almost Yamabe soliton, which is a generalization of the classical Yamabe soliton by setting ω to be a real valued smooth function on M. We denote an almost Yamabe soliton by $(g, \overline{V}, \omega)$. Furthermore, T. Seko et al. [25] completely classified almost Yamabe solitons in the context of hypersurfaces in Euclidean spaces. The Yamabe soliton becomes the so called gradient soliton if $\overline{V} = grad(\psi)$, for some smooth function $\psi : M \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. In this case, the soliton Eq. (1.1) becomes

$$\nabla^2\psi = (\sigma - \omega)g, \tag{1.2}$$

where ∇^2 is the Hessian operator. The function ψ is called potential function of the soliton.

In 2014, G. Huang et al. [16] introduced the concept of quasi-Yamabe gradient soliton, which is a generalization of gradient Yamabe soliton (1.2). The quasi-Yamabe gradient soliton equation is given by

$$\nabla^2\psi = \frac{1}{m} d\psi \otimes d\psi + (\sigma - \omega)g, \tag{1.3}$$

where m is a positive constant and $\omega \in \mathbb{R}$. If $m \to \infty$, the Eq. (1.3) recovers gradient Yamabe soliton. For more details about the quasi-Yamabe gradient solitons, we refer to [16, 19, 27].

In 2017, V. Pirhadi et al. [22] defined gradient almost quasi-Yamabe soliton by setting ω to be a smooth real valued function on M. In their paper, the authors proved that a necessary and sufficient condition under which an arbitrary compact almost Yamabe soliton is necessarily gradient [22].

Very recently in 2020, X. Chen first initiated almost quasi-Yamabe solitons in [7] as follows.

Definition 1.1([7]) *A smooth manifold M of dimension n equipped with a Riemannian metric g is said to be almost quasi-Yamabe soliton if there exist a smooth vector field $\overline{V}$ and a real valued smooth function ω on M satisfying the equation*

$$\frac{1}{2}\pounds_{\overline{V}} g = \frac{1}{m}\overline{V}^b \otimes \overline{V}^b + (\sigma - \omega)g, \tag{1.4}$$

where $\overline{V}^b$ is the 1-form associated to $\overline{V}$ and σ is the scalar curvature of M.

An almost quasi-Yamabe metric is closed if 1-form $\overline{V}^b$ is closed and is trivial if $\overline{V}$ is identically zero. We denote an almost quasi-Yamabe soliton by $(g, \overline{V}, m, \omega)$. From definition, if $m \to \infty$, almost quasi-Yamabe solitons are Yamabe solitons. Furthermore, the almost quasi-Yamabe soliton is said to be gradient if $\overline{V} = grad(\psi) = D\psi$, for some real valued smooth function ψ on M, denoted by (g, ψ, m, ω). After that several authors have studied almost quasi-Yamabe solitons on various geometric contexts like Kenmotsu manifolds [13] and within the frame-work of paracontact geometry [11] and obtained some interesting results.

Motivated by the above studies, the present authors to consider generalized Sasakian space form whose metric as an almost quasi-Yamabe soliton and gradient almost quasi-Yamabe soliton.

The present paper is organized as follows: After the brief introduction, we discuss some fundamental definitions related to generalized Sasakian space forms and curvature formulas, which are contained in Section 2. Section 3 is devoted to study of almost quasi-Yamabe soliton on generalized Sasakian space form. It is proved that if the metric of a generalized Sasakian space form admits a closed almost quasi-Yamabe soliton, then either soliton vector field $\overline{V}$ is pointwise collinear with ζ or the structure functions are connected by relation and the manifold becomes a Ricci semi-symmetric generalized Sasakian space form. Next, it is proven that, if the metric of a generalized Sasakian space form is a gradient almost quasi-Yamabe soliton, then either the gradient of ψ is pointwise collinear with ζ or the structure functions are connected by a relation, which is contained in Section 4.

§2. **Preliminaries**

A smooth manifold M of dimension n equipped with a Riemannian metric g is said to be an almost contact metric manifold [5], if it admits a $(1,1)$-tensor field φ, a characteristic vector field ζ, a 1-form η on M such that

$$\varphi^2(\overline{X}) = -\overline{X} + \eta(\overline{X})\zeta, \qquad \eta(\zeta) = 1, \tag{2.1}$$

$$g(\overline{X},\zeta) = \eta(\overline{X}), \quad \varphi(\zeta) = 0, \quad \eta(\varphi\overline{X}) = 0, \tag{2.2}$$

$$g(\varphi\overline{X},\varphi\overline{Y}) = g(\overline{X},\overline{Y}) - \eta(\overline{X})\eta(\overline{Y}), \tag{2.3}$$

$$g(\overline{X},\varphi\overline{Y}) + g(\varphi\overline{X},\overline{Y}) = 0 \tag{2.4}$$

for any smooth vector fields $\overline{X},\overline{Y}$ on M. An almost contact metric manifold is denoted by (M,φ,ζ,η,g).

A Sasakian space form is a Sasakian manifold with constant φ-sectional curvature c. Similarly, a Kenmotsu manifold with constant φ-sectional curvature c is called Kenmotsu space form. In 2004, P. Alegre et al. [1] initiated the concept of generalized Sasakian space form as a generalization of these spaces. After that several authors ([2, 4, 12, 18]) investigated generalized Sasakian space forms and Sasakian space forms and many beautiful results have been obtained on these spaces. An almost contact metric manifold (M,φ,ζ,η,g) is called a generalized Sasakian space form if its curvature tensor R satisfies

$$\begin{aligned} R(\overline{X},\overline{Y})\overline{Z} = {} & f_1\{g(\overline{Y},\overline{Z})\overline{X} - g(\overline{X},\overline{Z})\overline{Y}\} + f_2\{g(\overline{X},\varphi\overline{Z})\varphi\overline{Y} - g(\overline{Y},\varphi\overline{Z})\varphi\overline{X} \\ & + 2g(\overline{X},\varphi\overline{Y})\varphi\overline{Z}\} + f_3\{g(\overline{X},\overline{Z})\eta(\overline{Y})\zeta - g(\overline{Y},\overline{Z})\eta(\overline{X})\zeta \\ & + \eta(\overline{X})\eta(\overline{Z})\overline{Y} - \eta(\overline{Y})\eta(\overline{Z})\overline{X}\}, \end{aligned} \tag{2.5}$$

where f_1, f_2, f_3 are real valued smooth functions on M. In such a case, we also denote a $(2n+1)$-dimensional generalized Sasakian space form by $M(f_1,f_2,f_3)$. In particular, if $f_1 = \frac{c+3}{4}, f_2 =$

$f_3 = \frac{c-1}{4}$, then generalized Sasakian space form turns into Sasakian space form. Similarly, it turns into Kenmotsu space form if $f_1 = \frac{c+3}{4}, f_2 = f_3 = \frac{c-1}{4}$ (see [17]). Also, if $f_1 = f_2 = f_3 = \frac{c}{4}$, it reduces to cosymplectic space form (see [20]).

In a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$, the following relations are satisfied [1]:

$$\nabla_{\overline{X}}\zeta = (f_3 - f_1)\varphi(\overline{X}), \tag{2.6}$$

$$(\nabla_{\overline{X}}\eta)\overline{Y} = (f_3 - f_1)g(\varphi\overline{X}, \overline{Y}), \tag{2.7}$$

$$(\nabla_{\overline{X}}\varphi)\overline{Y} = (f_3 - f_1)\{\eta(\overline{Y})\overline{X} - g(\overline{X}, \overline{Y})\zeta\}, \tag{2.8}$$

$$R(\overline{X}, \overline{Y})\zeta = (f_1 - f_3)\{\eta(\overline{Y})\overline{X} - \eta(\overline{X})\overline{Y}\}, \tag{2.9}$$

$$R(\zeta, \overline{X})\overline{Y} = (f_3 - f_1)\{\eta(\overline{Y})\overline{X} - g(\overline{X}, \overline{Y})\zeta\}, \tag{2.10}$$

$$S(\overline{X}, \overline{Y}) = (2nf_1 + 3f_2 - f_3)g(\overline{X}, \overline{Y}) - \{3f_2 + (2n-1)f_3\}\eta(\overline{X})\eta(\overline{Y}), \tag{2.11}$$

$$Q\overline{X} = (2nf_1 + 3f_2 - f_3)\overline{X} - \{3f_2 + (2n-1)f_3\}\eta(\overline{X})\zeta, \tag{2.12}$$

$$S(\overline{X}, \zeta) = 2n(f_1 - f_3)\eta(\overline{X}), \tag{2.13}$$

$$Q\zeta = 2n(f_1 - f_3)\zeta \tag{2.14}$$

for any smooth vector fields $\overline{X}, \overline{Y}$ on M, where S and Q denotes the Ricci tensor and Ricci operator respectively.

For a Riemannian manifold (M, g) admitting an almost quasi-Yamabe gradient soliton, we have the following lemma.

Lemma 2.1([7]) *For an almost quasi-Yamabe gradient soliton $(M, g, \sigma, m, \omega)$, the curvature tensor R is given by*

$$R(\overline{X}, \overline{Y})D\psi = \overline{X}(\sigma - \omega)\overline{Y} - \overline{Y}(\sigma - \omega)\overline{X} - \frac{\sigma - \omega}{m}\{\overline{X}(\psi)\overline{Y} - \overline{Y}(\psi)\overline{X}\} \tag{2.15}$$

for any smooth vector fields $\overline{X}, \overline{Y}$ on M.

§3. Almost Quasi-Yamabe Soliton on a Generalized Sasakian Space Form $M(f_1, f_2, f_3)$

In this section, we study on the generalized Sasakian space form admitting almost quasi-Yamabe solitons. First we consider the 1-form $\overline{V}^b$ is closed.

Theorem 3.1 *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a closed almost quasi-Yamabe soliton whose soliton vector field $\overline{V}$, then either $\overline{V}$ is pointwise collinear with ζ or the real valued smooth functions f_2 and f_3 are connected by*

$$3f_2 + (2n-1)f_3 = 0.$$

Proof Since V^b is closed, Eq. (1.4) is identical to

$$\nabla_{\overline{Y}}\overline{V} = (\sigma-\omega)\overline{Y} + \frac{1}{m}g(\overline{V},\overline{Y})\overline{V} \tag{3.1}$$

for all smooth vector field $\overline{Y}$ on M. Covariantly, differentiating the Eq. (3.1) along $\overline{X}$ on M, we obtain

$$\begin{aligned}\nabla_{\overline{X}}\nabla_{\overline{Y}}\overline{V} = (\overline{X}(\sigma-\omega))\overline{Y} + (\sigma-\omega)\nabla_{\overline{X}}\overline{Y} + \tfrac{1}{m}\{g(\nabla_{\overline{X}}\overline{V},\overline{Y})\\ +g(\overline{V},\nabla_{\overline{X}}\overline{Y})\}\overline{V} + \tfrac{1}{m}g(\overline{V},\overline{Y})\nabla_{\overline{X}}\overline{V}.\end{aligned} \tag{3.2}$$

Exchanging $\overline{X}$ and $\overline{Y}$ in Eq. (3.2) gives

$$\begin{aligned}\nabla_{\overline{Y}}\nabla_{\overline{X}}\overline{V} = (\overline{Y}(\sigma-\omega))\overline{X} + (\sigma-\omega)\nabla_{\overline{Y}}\overline{X} + \tfrac{1}{m}\{g(\nabla_{\overline{Y}}\overline{V},\overline{X})\\ +g(\overline{V},\nabla_{\overline{Y}}\overline{X})\}\overline{V} + \tfrac{1}{m}g(\overline{V},\overline{X})\nabla_{\overline{Y}}\overline{V}.\end{aligned} \tag{3.3}$$

Also from Eq. (3.1) it can be written that

$$\nabla_{[\overline{X},\overline{Y}]}\overline{V} = (\sigma-\omega)\{\nabla_{\overline{X}}\overline{Y} - \nabla_{\overline{Y}}\overline{X}\} + \frac{1}{m}\{g(\overline{V},\nabla_{\overline{X}}\overline{Y}) - g(\overline{V},\nabla_{\overline{Y}}\overline{X})\}\overline{V}. \tag{3.4}$$

By virtue of the well-known Riemannian curvature formula

$$R(\overline{X},\overline{Y})\overline{V} = \nabla_{\overline{X}}\nabla_{\overline{Y}}\overline{V} - \nabla_{\overline{Y}}\nabla_{\overline{X}}\overline{V} - \nabla_{[\overline{X},\overline{Y}]}\overline{V},$$

and Eqs. (3.1), (3.3), (3.4), we have

$$R(\overline{X},\overline{Y})\overline{V} = (\overline{X}(\sigma-\omega))\overline{Y} - (\overline{Y}(\sigma-\omega))\overline{X} + \frac{\sigma-\omega}{m}\{g(\overline{V},\overline{Y})\overline{X} - g(\overline{V},\overline{X})\overline{Y}\}. \tag{3.5}$$

Now, taking inner product of Eq. (3.5) with ζ and using (2.2) we obtain

$$\begin{aligned}g(R(\overline{X},\overline{Y})\overline{V},\zeta) = \{(\overline{X}(\sigma-\omega)) - \tfrac{\sigma-\omega}{m}g(\overline{V},\overline{X})\}\eta(\overline{Y})\\ -\{(\overline{Y}(\sigma-\omega)) - \tfrac{\sigma-\omega}{m}g(\overline{V},\overline{Y})\}\eta(\overline{X}).\end{aligned} \tag{3.6}$$

Again, in view of the Eq. (2.9) we obtain

$$g(R(\overline{X},\overline{Y})\overline{V},\zeta) = (f_1-f_3)\{g(\overline{Y},\overline{V})\eta(\overline{X}) - g(\overline{X},\overline{V})\eta(\overline{Y})\}. \tag{3.7}$$

Equating the Eqs. (3.6) and (3.7), we get

$$\begin{aligned}\{(\overline{X}(\sigma-\omega)) - \tfrac{\sigma-\omega}{m}g(\overline{V},\overline{X})\}\eta(\overline{Y}) - \{(\overline{Y}(\sigma-\omega)) - \tfrac{\sigma-\omega}{m}g(\overline{V},\overline{Y})\}\eta(\overline{X})\\ = (f_1-f_3)\{g(\overline{Y},\overline{V})\eta(\overline{X}) - g(\overline{X},\overline{V})\eta(\overline{Y})\}.\end{aligned} \tag{3.8}$$

Replacing $\overline{X}$ by $\varphi\overline{X}$ and $\overline{Y}$ by ζ respectively in Eq. (3.8) and using Eq. (2.2), we have

$$\varphi D(\sigma-\omega) = (\frac{\sigma-\omega}{m} - f_1 + f_3)\varphi V. \tag{3.9}$$

Now consider an orthonormal basis $\{e_1, e_2, e_3, \cdots, e_{2n+1}\}$ of the tangent space at each point of the generalized Sasakian space form $M(f_1, f_2, f_3)$. Then contracting Eq. (3.5) over $\overline{Y}$ and using Eq. (2.11) entails that

$$(2nf_1 + 3f_2 - f_3 - \frac{2n(\sigma - \omega)}{m})\overline{V} - (2nf_3 + 3f_2 - f_3)\eta(\overline{V})\zeta = -2nD(\sigma - \omega). \tag{3.10}$$

Now applying φ on both sides of Eq. (3.10) and making use of Eq. (2.2), we infer that

$$(2nf_1 + 3f_2 - f_3 - \frac{2n(\sigma - \omega)}{m})\varphi\overline{V} = -2n\varphi D(\sigma - \omega). \tag{3.11}$$

In view of Eq. (3.9), the Eq. (3.11) becomes

$$\{3f_2 + (2n - 1)f_3\}\varphi V = 0. \tag{3.12}$$

If $\varphi V = 0$, then it follows from Eq. (2.1) that $\overline{V} = \eta(\overline{V})\zeta$, and hence $\overline{V}$ is pointwise collinear with ζ. Suppose if $\varphi V \neq 0$, then we have from Eq. (3.12) that $3f_2 + (2n - 1)f_3 = 0$, and this proves the theorem. □

From [24], it is known that a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ is Ricci semi-symmetric if and only if $3f_2 + (2n - 1)f_3 = 0$. Thus we can state the following.

Theorem 3.2 *If a $(2n + 1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a closed almost quasi-Yamabe soliton whose soliton vector field $\overline{V}$, then either $\overline{V}$ is pointwise collinear with ζ or the manifold M is Ricci semisymmetric.*

Furthermore, if we put $3f_2 + (2n - 1)f_3 = 0$ in Eq. (2.11), then we get

$$S(\overline{X}, \overline{Y}) = 2n(f_1 - f_3)g(\overline{X}, \overline{Y}) \tag{3.13}$$

for all smooth vector fields $\overline{X}, \overline{Y}$ on M. Thus, we are in a position to state the following.

Corollary 3.3 *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a closed almost quasi-Yamabe soliton whose soliton vector field $\overline{V}$, then either $\overline{V}$ is pointwise collinear with ζ or the manifold M is an Einstein manifold of scalar curvature $\sigma = 2n(2n + 1)(f_1 - f_3)$.*

Again on this generalized Sasakian space form $M(f_1, f_2, f_3)$, considering $f_1 = \frac{c+3}{4}$ and $f_3 = \frac{c-1}{4}$ in Eq. (3.13), we compute that

$$S(\overline{X}, \overline{Y}) = 2ng(\overline{X}, \overline{Y})$$

for all smooth vector fields $\overline{X}, \overline{Y}$ on M. Thus, we are in a position to state the following.

Corollary 3.4 *Let $M(f_1, f_2, f_3)$ be a $(2n + 1)$-dimensional Sasakian space form admitting a closed almost quasi-Yamabe soliton whose soliton vector field $\overline{V}$. Then either $\overline{V}$ is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma =$*

$2n(2n+1)$.

Similarly in a Kenmotsu space form $f_1 = \frac{c-3}{4}$ and $f_3 = \frac{c+1}{4}$. Using these values of f_1 and f_3 in Eq. (3.13) infers that

$$S(\overline{X}, \overline{Y}) = -2ng(\overline{X}, \overline{Y})$$

for all smooth vector fields $\overline{X}, \overline{Y}$ on M.

Thus, we are in a position to state the following.

Corollary 3.5 *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional Kenmotsu space form admitting a closed almost quasi-Yamabe soliton with soliton vector field $\overline{V}$. Then either $\overline{V}$ is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = -2n(2n+1)$.*

Again if $m \to \infty$, then the Eq. (3.9) and Eq. (3.11) reduces to

$$\varphi D(\sigma - \omega) = (f_3 - f_1)\varphi V. \tag{3.14}$$

and

$$(2nf_1 + 3f_2 - f_3)\varphi\overline{V} = -2n\varphi D(\sigma - \omega), \tag{3.15}$$

respectively. Combining Eq. (3.14) and Eq. (3.15) we obtain

$$\{3f_2 + (2n-1)f_3\}\varphi V = 0. \tag{3.16}$$

Thus, we are in a position to state the following.

Corollary 3.6 *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a closed almost Yamabe soliton whose soliton vector field $\overline{V}$, then either $\overline{V}$ is pointwise collinear with ζ or the manifold M is an Einstein manifold of scalar curvature $\sigma = 2n(2n+1)(f_1 - f_3)$.*

Corollary 3.7 *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional Sasakian space form admitting a closed almost Yamabe soliton whose soliton vector field $\overline{V}$. Then either $\overline{V}$ is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = 2n(2n+1)$.*

Corollary 3.8 *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional Kenmotsu space form admitting a closed almost Yamabe soliton whose soliton vector field $\overline{V}$. Then either $\overline{V}$ is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = -2n(2n+1)$.*

§4. Gradient Almost Quasi-Yamabe Soliton on a Generalized Sasakian Space Form $M(f_1, f_2, f_3)$

In this section, we investigate some properties on a generalized Sasakian space form admitting gradient almost quasi-Yamabe solitons whose soliton vector field $\overline{V}$ is gradient of some real valued smooth function ψ on M. In this regard our next result is

Theorem 4.1 *Let a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admit gradient almost quasi-Yamabe soliton, whose potential function ψ. Then either the gradient of ψ is pointwise collinear with ζ or the real valued smooth functions f_2 and f_3 are related by $3f_2 + (2n-1)f_3 = 0$.*

Proof Let us consider a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ and assume that it admits a gradient almost quasi-Yamabe soliton, whose potential function ψ. Taking inner product both sides of Eq. (2.15) in the direction of ζ and then with the help of Eq. (2.2), we get

$$\begin{aligned} g(R(\overline{X},\overline{Y})D\psi,\zeta) = \{&\overline{X}(\sigma-\omega) - \tfrac{\sigma-\omega}{m}\overline{X}(\psi)\}\eta(\overline{Y}) \\ &-\{\overline{Y}(\sigma-\omega) - \tfrac{\sigma-\omega}{m}\overline{Y}(\psi)\}\eta(\overline{X}). \end{aligned} \tag{4.1}$$

On the other hand, in view of Eq. (2.11), we obtain

$$g(R(\overline{X},\overline{Y})D\psi,\zeta) = (f_1 - f_3)\{\overline{Y}(\psi)\eta(\overline{X}) - \overline{X}(\psi)\eta(\overline{Y})\}. \tag{4.2}$$

Comparing the Eqs. (4.1) and (4.2), we get

$$\begin{aligned} (f_1 - f_3)\{\overline{Y}(\psi)\eta(\overline{X}) - \overline{X}(\psi)\eta(\overline{Y})\} = \{&\overline{X}(\sigma-\omega) - \tfrac{\sigma-\omega}{m}\overline{X}(\psi)\}\eta(\overline{Y}) \\ &-\{\overline{Y}(\sigma-\omega) - \tfrac{\sigma-\omega}{m}\overline{Y}(\psi)\}\eta(\overline{X}). \end{aligned} \tag{4.3}$$

Now replacing $\overline{X}$ and $\overline{Y}$ by $\varphi\overline{X}$ and ζ respectively and using Eq. (2.2), we reveal that

$$(f_3 - f_1 + \frac{\sigma-\omega}{m})(\varphi\overline{X})\psi = (\varphi\overline{X})(\sigma-\omega). \tag{4.4}$$

Now, contracting the Eq. (2.15) over $\overline{Y}$, and then recalling the Eq. (2.11), infers that

$$\{2n(\frac{\sigma-\omega}{m} - f_1) - 3f_2 + f_3\}\overline{X}(\psi) + \{3f_2 + (2n-1)f_3\}\eta(\overline{X})\zeta(\psi) = 2n\overline{X}(\sigma-\omega). \tag{4.5}$$

Replacing $\overline{X}$ by $\varphi\overline{X}$ in Eq. (4.5) and using Eq. (2.2), we get

$$\{2n(\frac{\sigma-\omega}{m} - f_1) - 3f_2 + f_3\}(\varphi\overline{X})\psi = 2n(\varphi\overline{X})(\sigma-\omega). \tag{4.6}$$

Then, with the help of Eq. (4.4), the Eq. (4.6) becomes

$$\{3f_2 + (2n-1)f_3\}(\varphi\overline{X})\psi = 0. \tag{4.7}$$

If $(\varphi\overline{X})\psi = 0$, then it follows from Eq. (2.1) that $\overline{X} = \eta(\overline{X})\zeta(\psi)$, from which we have $grad(\psi) = \zeta(\psi)\zeta$ and hence $grad(\psi)$ is pointwise collinear with ζ. Suppose if $(\varphi\overline{X})\psi \neq 0$, then we have from Eq. (4.7) that $3f_2 + (2n-1)f_3 = 0$, and this proves the theorem. □

The following corollaries can be achieved as the similar manner as the Section 3.

Corollary 4.2 *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a gradient almost quasi-Yamabe soliton, whose soliton vector field $\overline{V}$, then either $\overline{V}$ is pointwise collinear with ζ or the manifold M is Ricci semisymmetric.*

Corollary 4.3 *If a $(2n+1)$-dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$ admits a gradient almost quasi-Yamabe soliton, whose potential function ψ, then either the gradient of ψ is pointwise collinear with ζ or the manifold M is an Einstein manifold of scalar curvature $\sigma = 2n(2n+1)(f_1 - f_3)$.*

Corollary 4.4 *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional Sasakian space form admitting a gradient almost quasi-Yamabe soliton, whose potential function ψ. Then either the gradient of ψ is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = 2n(2n+1)$.*

Corollary 4.5 *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional Kenmotsu space form admitting a gradient almost quasi-Yamabe soliton, whose potential function ψ. Then either the gradient of ψ is pointwise collinear with ζ or the manifold M is an Einstein manifold of constant scalar curvature $\sigma = -2n(2n+1)$.*

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An In-depth Exploration on Supra Fuzzy R_0 and R_1 Bitopological Space in Quasi-coincidence Sense

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Abstract: In this paper, we define some notions of R_0-supra fuzzy bitopological space and R_1-supra fuzzy bitopological spaces in the sense of quasi-coincidence. We have explored the relationships among these notions and demonstrated properties such as good extension, hereditary, productive, and projective. We also show that R_0-supra fuzzy bitopological space and R_1-supra fuzzy bitopological spaces are preserved under one-one onto fuzzy pairwise open and continuous mappings.

Key Words: Supra fuzzy bitopological space, quasi-coincidence, R_0-supra fuzzy bitopoloical space, R_1-supra fuzzy bitopoloical space, good extensions, mappings.

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§1. Introduction

The first concept of fuzzy sets was proposed by Zadeh [27] in 1965. By using this concept, Chang [6] defined fuzzy topological spaces in 1968. The concept of bitopological spaces was introduced by J.C. Kelly [8]. A set equipped with two topologies is called a bitopological space. The supra topological spaces were introduced by A.S. Mashhour [13] in 1983. In a topological space, the arbitrary union condition is enough to have a supra topological space. Here, every fuzzy topological space is a supra fuzzy bitopological space but the converse is not always true. Separation axioms [7,14,19] are important parts of fuzzy topological spaces. Also, many researchers have contributed various types of separation axioms [9,16,17] on fuzzy bitopological spaces which were introduced by Kandil and ElShafee [9] in 1991. Fuzzy R_0 and R_1 bitopological spaces were introduced earlier in the literature. There are many articles on fuzzy R_0 and R_1 bitopological spaces created by many authors like P. Wuyts [26] and R. Lowen [11], D.M. Ali [1], M.S. Hossain [7], and others. The purpose of this paper is to further contribute to the development of supra fuzzy Hausdorff bitopological spaces, especially on supra fuzzy bitopological spaces. In this paper, we define Hausdorff supra fuzzy bitopological space and

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show that the definitions satisfy good extension property, hereditary property, order preserving, productive, and projective properties hold on the new concepts, initial and final supra fuzzy bitopologies are discussed also.

§2. Basic Notions and Preliminary Results

Definition 2.1([25]) *A function u from X into the interval I is called a fuzzy set in X. For every $x \in X$, $u(x) \in I$ is called the grade of membership of x in u. Some authors refer to u as a fuzzy subset of X instead of saying that u is a fuzzy set in X. The class of all fuzzy sets from X into the closed unit interval I will be denoted by I^X.*

Definition 2.2([18]) *A fuzzy set u in X is called a fuzzy singleton if and only if $u(x) = r$, $0 < r \leq 1$ for a certain $x \in X$ and $u(y) = 0$ for all other points y of X except x. The fuzzy singleton is denoted by x_r and x is its support. The class of all fuzzy singletons in X will be denoted by $S(X)$. If $u \in I^X$ and $x_r \in S(X)$, then we say that $x_r \in u$ if and only if $r \leq u(x)$.*

Definition 2.3([6]) *A fuzzy singleton x_r is said to be quasi-coincident with u, denoted by $x_r\, qu$ if and only if $u(x) + r > 1$. If x_r is not quasi-coincident with u, we write $x_r\, \bar{q}u$ and define it as $u(x) + r \leq 1$.*

Definition 2.4([6]) *Let f be a mapping from a set X into set Y and u be a fuzzy subset of X. Then f and u induce a fuzzy subset v of Y defined by:*

$$v(y) = \sup\{u(x) : x \in f^{-1}(\{y\}) \neq \emptyset, x \in X\}, \quad \textit{and} \quad = 0 \ \textit{otherwise.}$$

Definition 2.5([6]) *Let f be a mapping from a set X into set Y and v be a fuzzy subset of Y. Then the inverse of v, written as $f^{-1}(v)$, is a fuzzy subset of X defined by $f^{-1}(v)(x) = v(f(x))$, for $x \in X$.*

Definition 2.6([6]) *Let $I = [0, 1]$, X be a non-empty set and I^X be the collection of all mappings from X into I, i.e., the class of all fuzzy sets in X. A fuzzy topology on X is defined as a family t of members of I^X, satisfying the following conditions:*

(*i*) $1, 0 \in t$;
(*ii*) *If $u \in t$ for each $i \in \Lambda$, then $\bigcap_{i\in\Lambda} u_i \in t$, where Λ is an index set;*
(*iii*) *If $u, v \in t$, then $u \cap v \in t$.*

The pair (X, t) is called a fuzzy topological space (in short fts), and members of t are called t-open fuzzy sets. A fuzzy set v is called a t-closed fuzzy set if $1 - v \in t$.

Definition 2.7([19]) *The function $f : (X, t) \to (Y, s)$ is called* fuzzy continuous *if and only if for every $v \in s$, $f^{-1}(v) \in t$. The function f is called* fuzzy homeomorphic *if and only if f is bijective and both f and f^{-1} are fuzzy continuous.*

Definition 2.8([20]) *The function* $f : (X, t) \to (Y, s)$ *is called* fuzzy open *if and only if for every open fuzzy set* u *in* (X, t)*,* $f(u)$ *is an open fuzzy set in* (Y, s)*.*

Definition 2.9([6]) *Let* $\{X_i, i \in \Lambda\}$*, be any class of sets and let* X *denote the Cartesian product of these sets, i.e.,* $X = \prod_{i\in\Lambda} X_i$*. Note that* X *consists of all points* $p = \langle a_i, i \in \Lambda\rangle$*, where* $a_i \in X_i$*. For each* $j_0 \in \Lambda$*, we define the projection* $\pi_{j_0} : X \to X_{j_0}$ *by* $\pi_{j_0}(\langle a_i, i \in \Lambda\rangle) = a_{j_0}$*. These projections are used to define the product topology.*

Definition 2.10([21]) *Let* $\{X_i, i \in \Lambda\}$*, be a family of non-empty sets. Let* $X = \prod_{i\in\Lambda} X_i$ *be the usual product of* X_i*s and let* π_i *be the projection from* X *into* X_i*. Further assume that each* X_i *is a fuzzy topological space with fuzzy topology* t_i*. The fuzzy topology generated by* $\{\pi_i^{-1}(b_i) : b_i \in t_i, i \in \Lambda\}$ *as a subbasis, is called the product fuzzy topology on* X*.*

Definition 2.11([23]) *Let* f *be the real-valued function on a topological space. If* $\{x : f(x) > \alpha\}$ *is open for every real* α*, then* f *is called* lower semi-continuous function.

Definition 2.12([20]) *Let* X *be a non-empty set and* T *be a topology on* X*. Let* $t = \omega(T)$ *be the set of all lower semi-continuous functions from* (X, T) *to* I *(with usual topology). Thus,* $\omega(T) = \{u \in I^X : u^{-1}(\alpha, 1] \in T\}$ *for each* $\alpha \in I_1$*. It can be shown that* $\omega(T)$ *is a fuzzy topology on* X*.*

Definition 2.13([11]) *The initial fuzzy topology on* X *for the family of fuzzy topological spaces* $\{(X_i, t_i)\}_{i\in\Lambda}$ *and the family of the functions* $\{f_i : X \to (X_i, t_i)\}_{i\in\Lambda}$ *is the smallest fuzzy topology on* X *making each* f_i *fuzzy continuous. It is generated by the family* $\{f_i^{-1}(u_i) : u_i \in t_i\}_{i\in\Lambda}$*.*

Definition 2.14([11]) *The final fuzzy topology on* X *for the family of fuzzy topological spaces* $\{(X_i, t_i)\}_{i\in\Lambda}$ *and the family of the functions* $\{f_i : X \to (X_i, t_i)\}_{i\in\Lambda}$ *is the finest fuzzy topology on* X *making each* f_i *fuzzy continuous.*

Definition 2.15([21]) *A bijective mapping from a fuzzy topological space (fts)* (X, t) *to another fts* (Y, s) *preserves the value of a fuzzy singleton (fuzzy points).*

Definition 2.16([6]) *Let* X *be a non-empty set. A subfamily* t^* *of* I^X *is said to be a supra fuzzy topology on* X *if and only if*

(*i*) $1, 0 \in t^*$;
(*ii*) *If* $u_i \in t^*$ *for each* $i \in \Lambda$*, then* $\bigcup_{i\in\Lambda} u_i \in t^*$*.*

Then the pair (X, t^*) *is called a supra fuzzy topological space. The elements of* t^* *are called supra open sets in* (X, t^*) *and the complement of a supra open set is called a supra closed set.*

§3. R_0-Type Separation in Supra Fuzzy Bi-Topological Space

In this section, we define our notions in Supra fuzzy R_0 bi-topological spaces.

Definition 3.1 *A supra fuzzy bi-topological space* (X, s^*, t^*) *is called* (*a*) *$SFPR_0(i)$ if and only if for any pair $x_m, y_n \in S(X)$ for distinct x and y, whenever there exists $\mu \in s^* \cup t^*$ with $x_m q\mu$ and $y_n \bar{q}\mu$, then there exists $\eta \in s^* \cup t^*$ such that $y_n q\eta$ and $x_m \bar{q}\eta$;*

(*b*) *$SFPR_0(ii)$ if and only if for any pair $x_m, y_n \in S(X)$ for distinct x and y, whenever there exists $\mu \in s^* \cup t^*$ with $x_m \in \mu$, $y_n \in \mu$ and $y_n \bar{q}\mu$, then there exists $\eta \in s^* \cup t^*$ such that $y_n \in \eta$ and $x_m \bar{q}\eta$;*

(*c*) *$SFPR_0(iv)$ if and only if for any pair $x, y \in X$ for distinct x and y, whenever there exists $\mu \in s^* \cup t^*$ such that $x_m q\mu$ and $y_n \cap \mu = 0$, then there exists $\eta \in s^* \cup t^*$ such that $y_n q\eta$ and $x_m \cap \eta = 0$.*

Theorem 3.1 *Let (X, S^*, T^*) be a supra fuzzy bi-topological space. Consider the following statements:*

(1) *(X, S^*, T^*) is a R_0 supra bi-topological space;*
(2) *$(X, \omega(S^*), \omega(T^*))$ is an $SFPR_0(i)$ space;*
(3) *$(X, \omega(S^*), \omega(T^*))$ is an $SFPR_0(ii)$ space;*
(4) *$(X, \omega(S^*), \omega(T^*))$ is an $SFPR_0(iii)$ space;*
(5) *$(X, \omega(S^*), \omega(T^*))$ is an $SFPR_0(iv)$ space.*

Then, the following implications are true: (1) $\Leftrightarrow$ (2)*,* (1) $\Leftrightarrow$ (5)*.*

Proof of (1) $\Leftrightarrow$ (2) Let (X, S^*, T^*) be a bi-topological space and (X, S^*, T^*) is $SFPR_0$. We have to prove that $(X, \omega(S^*), \omega(T^*))$ is $SFPR_0(i)$. Let x_m, y_n be fuzzy points in X for distinct x, y and $\mu \in \omega(T^*), \eta \in \omega(S^*)$ with $x_m q\mu$ and $y_n \bar{q}\mu$. Now, $x_m q\mu$ implies $\mu(x) + m > 1$ implying $\mu(x) > 1 - m$ implying $x \in \mu^{-1}(1-m, 1]$ and $y_n \bar{q}\mu$ implies $\mu(y) + m \leq 1$, $\mu(y) \leq 1 - m$ implying $y \notin \mu^{-1}(1-m, 1]$. Since (X, S^*, T^*) is $SFPR_0$ bi-topological space, there exists $\eta \in S^* \cup T^*$ such that $y \in \eta$, $x \notin \eta$. From the definition of lower semi-continuous, we have $1_\eta \in \omega(S^*) \cup \omega(T^*)$ and $1_\eta(y) = 1$, $1_\eta(x) = 0$. Then $1_\eta(y) + m > 1$ implies $y_n q 1_\eta$ and $1_\eta(x) + m \leq 1$ implies $x_m \bar{q} 1_\eta$. It follows that there exists $1_\eta \in \omega(S^*) \cup \omega(T^*)$ such that $y_n q 1_\eta$ and $x_m \bar{q} 1_\eta$. Hence, $(X, \omega(S^*), \omega(T^*))$ is $SFPR_0(i)$.

Conversely, let $(X, \omega(S^*), \omega(T^*))$ is $SFPR_0(i)$. We have to prove that (X, S^*, T^*) is $SFPR_0$. Let x, y be points in X for distinct x, y and $\mu \in S^* \cup T^*$ with $x \in \mu$ and $y \notin \mu$. From the definition of lower semi-continuous, we have $1_\mu \in \omega(T^*)$ and $1_\mu(x) = 1$, $1_\mu(y) = 0$. Then $1_\mu(x) + m > 1$ implies $x_m q 1_\mu$ and $1_\mu(y) + m \leq 1$ implies $y_n \bar{q} 1_\mu$. Since $(X, \omega(S^*), \omega(T^*))$ is $SFPR_0(i)$ bi-topological space, there exists $\eta \in \omega(S^*) \cup \omega(T^*)$ such that $y_n q\eta$ and $x_m \bar{q}\eta$. Now, $y_n q\eta$ implies $\eta(y) + m > 1$ implying $\eta(y) > 1 - m$ implying $y \in \eta^{-1}(1-m, 1]$ and $x_m \bar{q}\eta$ implies $\eta(x) + m \leq 1$ implying $\eta(x) \leq 1 - m$ implying $x \notin \eta^{-1}(1-m, 1]$. It follows that there exists $\mu^{-1}(1-m, 1]$ in $S^* \cup T^*$ such that $y \in \eta^{-1}(1-m, 1]$, $x \notin \eta^{-1}(1-m, 1]$. Hence, (X, S^*, T^*) is $SFPR_0$. Thus, (1) $\Leftrightarrow$ (2) holds.

Proof of (1) $\Leftrightarrow$ (5) Let (X, S^*, T^*) be a bi-topological space and assume that (X, S^*, T^*) is $SFPR_0$. We must prove that $(X, \omega(S^*), \omega(T^*))$ is $SFPR_0(iv)$. Let x_m, y_n be fuzzy points in X for distinct x, y, and let $\mu \in \omega(T^*)$, $\eta \in \omega(S^*)$ with $x_m q\mu$ and $y_n \cap \mu = 0$. Now, $x_m q\mu$ implies $\mu(x) \geq m$ implying $x \in \mu^{-1}(1-m, 1]$ and $y_n \cap \mu = 0$ implies $\mu(y) = 0$, $\mu(y) + m \leq 1$ implies $\mu(y) \leq 1 - m$ implying $y \notin \mu^{-1}(1-m, 1]$. Since (X, S^*, T^*) is $SFPR_0$ bi-topological space, there exists $\eta \in \omega(S^*) \cup \omega(T^*)$ such that $y \in \eta$, $x \notin \eta$. From the definition of lower semi-continuous, $1_\eta \in \omega(S^*) \cup \omega(T^*)$ and $1_\eta(y) = 1$, $1_\eta(x) = 0$. Then $1_\eta(y) + m > 1$ implies $y_n q 1_\eta$ and $1_\eta(x) + m \leq 1$ implies $x_m \bar{q} 1_\eta$. It follows that there exists $1_\eta \in \omega(S^*) \cup \omega(T^*)$ such that $y_n q 1_\eta$ and $x_m \bar{q} 1_\eta$. Hence, $(X, \omega(S^*), \omega(T^*))$ is $SFPR_0(iv)$.

Conversely, assume that $(X, \omega(S^*), \omega(T^*))$ is $SFPR_0(iv)$. We must prove that (X, S^*, T^*) is $SFPR_0$. Let x, y be points in X for distinct x, y and let $\mu \in S^* \cup T^*$ with $x \in \mu$ and $y \notin \mu$. From the definition of lower semi-continuous, $1_\mu \in \omega(T^*)$ and $1_\mu(x) = 1$, $1_\mu(y) = 0$. Then $1_\mu(x) + m > 1$ implies $x_m q 1_\mu$ and $1_\mu(y) + m \leq 1$ implies $y_n \cap 1_\mu = 0$. Since $(X, \omega(S^*), \omega(T^*))$ is $SFPR_0(iv)$ bi-topological space, there exists $\eta \in \omega(S^*) \cup \omega(T^*)$ such that $y_n q \eta$ and $x_m \cap \eta = 0$. Now, $y_n q \eta$ implies $\eta(y) + m > 1$ implying $\eta(y) > 1 - m$ implying $y \in \eta^{-1}(1 - m, 1]$ and $x_m \cap \eta$ implies $\eta(x) = 0$ implying $\eta(x) + m \leq 1$ implying $x \notin \eta^{-1}(1 - m, 1]$. It follows that there exists $\mu^{-1}(1 - m, 1]$ in $S^* \cup T^*$ such that $y \in \eta^{-1}(1 - m, 1]$, $x \notin \eta^{-1}(1 - m, 1]$. Hence, (X, S^*, T^*) is $SFPR_0$. Thus, (1) ⇔ (5) holds. □

3.1.Subspaces in Supra Fuzzy R_0 Bi-Topological Spaces. In this section, we shall show that our notions satisfy the hereditary property.

Theorem 3.2 *Let (X, s^*, t^*) be a supra fuzzy bi-topological space, $A \subseteq X$, $t^*_A = \{\mu/A : \mu \in s^*_A \cup t^*_A\}$, $s^*_A = \{\eta/A : \eta \in s^*_A \cup t^*_A\}$. Then (X, s^*, t^*) is $SFPR_0(j) \Rightarrow (A, s^*_A, t^*_A)$ is $SFPR_0(j)$, where $j = i, ii, iii, iv$.*

Proof Assume (X, s^*, t^*) is a supra fuzzy bi-topological space and (X, s^*, t^*) is $SFPR_0(i)$. We aim to prove that (A, s^*_A, t^*_A) is $SFPR_0(i)$. Let x_m, y_n be fuzzy points in A for distinct x and y, and let $\mu \in s^*_A \cup t^*_A$ such that $x_m\ q\ \mu$ and $y_n \bar{q} \mu$. Since $A \subseteq X$, these fuzzy points are also fuzzy points in X. Moreover, since (X, s^*, t^*) is $SFPR_0(i)$ space, there exists $\mu \in s^* \cup t^*$ such that $x_m \bar{q} \mu, y_n\ q\ \mu$. From the definition of t^*_A, $\mu/A \in s^*_A \cup t^*_A$. Now, $y_n\ q\ \mu \Rightarrow \mu(x) + n > 1, y \in X \Rightarrow (\mu/A)(y) + n > 1, y \in A \subseteq X \Rightarrow y_n\ q\ (\mu/A)$ and $x_m \bar{q} \mu \Rightarrow \mu(x) + m \leq 1, x \in X \Rightarrow (\mu/A)(x) + m \leq 1, x \in A \subseteq X \Rightarrow x_m \bar{q}(\mu/A)$. It follows that there exists $\mu/A \in t^*_A$ such that $y_n\ q\ (\mu/A)$ and $x_m \bar{q}(\mu/A)$. Hence, (A, s^*_A, t^*_A) is $SFPR_0(i)$. The proof for $j = ii, iii, iv$ follows in a similar manner. □

3.2. Productivity and Projectivity in Supra Fuzzy R_0 Bi-Topological Spaces. In this section, we shall show that our notions satisfy productive and projective properties.

Theorem 3.3 *Let (X_i, s^*_i, t^*_i), $i \in \Lambda$ be a family of supra fuzzy bi-topological spaces, and let $X = \prod_{i \in \Lambda} X_i$ be the product space with s^* and t^* forming the product bi-topology on X. Then, for all $i \in \Lambda$, (X_i, s^*_i, t^*_i) is $SFPR_0(j)$ if and only if (X, s^*, t^*) is $SFPR_0(j)$, where $j = i, ii, iii, iv$.*

Proof Assume for all $i \in \Lambda$, (X_i, s^*_i, t^*_i) is $SFPR_0(ii)$. We aim to show that (X, s^*, t^*) is $SFPR_0(ii)$. Let x_m, y_n be fuzzy points in X for distinct x and y, and let $\mu \in s^* \cup t^*$ such that $x_m \in \mu$, $y_n \bar{q} \mu$. Then $(x_i)_m, (y_i)_n$ are fuzzy points with $x_i \neq y_i$ for some $i \in \Lambda$, and we can find $\mu_i \in s^*_i \cup t^*_i$ such that $(x_i)_m \in \mu_i$, $(y_i)_n \bar{q} \mu_i$. Since (X_i, s^*_i, t^*_i) is $SFPR_0(ii)$, there exists $\eta_i \in s^*_i \cup t^*_i$ such that $(y_i)_n \in \eta_i$, and $(x_i)_m \bar{q} \eta_i$. Now, $(y_i)_n \in \eta_i$ implies $\eta_i(y_i) \geq n$, $y \in X$ implies $\eta_i(\pi_i(y)) \geq n$ implies $(\eta_i \circ \pi_i)(y) \geq n$ implies $y_n \in (\eta_i \circ \pi_i)$ and $(x_i)_m \bar{q} \eta_i$ implies $\eta_i(x_i) + m \leq 1$,

$x \in X$ implies $\eta_i(\pi_i(x)) + m \leq 1$ implies $(\eta_i \circ \pi_i)(x) + m \leq 1$ implies $x_m \bar{q}(\eta_i \circ \pi_i)$. It follows that there exists $(\eta_i \circ \pi_i) \in s_i^* \cup t_i^*$ such that $y_n \in (\eta_i \circ \pi_i)$, $x_m \bar{q}(\eta_i \circ \pi_i)$. Thus, (X, s^*, t^*) is $SFPR_0(ii)$.

Conversely, assume (X, s^*, t^*) is a supra fuzzy bi-topological space and (X, s^*, t^*) is $SFPR_0(ii)$. To demonstrate that each (X_i, s_i^*, t_i^*), $i \in \Lambda$ is $SFPR_0(ii)$, let a_i be a fixed element in X_i. Define $A_i = \{x \in X = \prod_{i\in\Lambda} X_i : x_j = a_j \text{ for some } i \neq j\}$. Then A_i is a subset of X, and hence $(A_i, s^*_{A_i}, t^*_{A_i})$ is a subspace of (X, s^*, t^*). Since (X, s^*, t^*) is $SFPR_0(ii)$, so is $(A_i, s^*_{A_i}, t^*_{A_i})$. Now, A_i is homeomorphic to X_i. Hence, for all $i \in \Lambda$, (X_i, s_i^*, t_i^*) is $SFPR_0(ii)$ space. Other results can similarly be proved. □

3.3.Mapping in Supra Fuzzy R_0 Bi-Topological Spaces. In this section, we shall show that our notions satisfy the order preserving property.

Theorem 3.4 *Let (X, s_1^*, t_1^*) and (Y, s_2^*, t_2^*) be two supra fuzzy bi-topological spaces, and let $f : X \to Y$ be a one-one, onto, and supra fuzzy open map. Then (X, s_1^*, t_1^*) is $SFPR_0(j) \Rightarrow (Y, s_2^*, t_2^*)$ is $SFPR_0(j)$, where $j = i, ii, iii, iv$.*

Proof Assume (X, s_1^*, t_1^*) is a supra fuzzy bi-topological space and (X, s_1^*, t_1^*) is $SFPR_0(iii)$. To prove that (Y, s_2^*, t_2^*) is $SFPR_0(iii)$, consider fuzzy points x_m, y_n in Y for distinct x, y, and let $\mu \in s_2^* \cup t_2^*$ such that $x_m \in \mu$ and $y_n \cap \mu = 0$. Since f is onto, there exist x'_m, y'_n in X with $f(x') = x$, $f(y') = y$ for distinct x', y' since f is one-one. Given that f is continuous and $\mu \in s_2^* \cup t_2^*$, $f^{-1}(\mu) \in s_1^* \cup t_1^*$. Now, $x_m \in \mu \Rightarrow \mu(x) \geq m \Rightarrow \mu(f(x')) \geq m \Rightarrow (f^{-1}(\mu))(x') \geq m \Rightarrow x'_m \in f^{-1}(\mu)$ and $y_n \cap \mu = 0 \Rightarrow \mu(y) = 0 \Rightarrow \mu(f(y')) = 0 \Rightarrow (f^{-1}(\mu))(y') = 0 \Rightarrow y'_n \cap f^{-1}(\mu) = 0$. Since (X, s_1^*, t_1^*) is $SFPR_0(iii)$ space, there exists $\eta \in s_1^* \cup t_1^*$ such that $y'_n \in \eta$, $x'_m \cap \eta = 0$. Then $y'_n \in \eta \Rightarrow \eta(y') \geq n \Rightarrow \sup \eta(y') \geq n \Rightarrow (f(\eta))(y) \geq n$, where $f(\eta)(y) = \sup\{\eta(y') : f(y') = y\} \Rightarrow y_n \in f(\eta)$ and $x'_m \cap \eta = 0 \Rightarrow \eta(x') = 0 \Rightarrow (f(\eta))(x) = 0 \Rightarrow x_m \cap f(\eta) = 0$. Since f is fuzzy open, $f(\eta) \in s_2^* \cup t_2^*$. It follows that there exists $f(\eta) \in s_2^* \cup t_2^*$ such that $y_n \in f(\eta)$, $x_m \cap f(\eta) = 0$. Hence, (Y, s_2^*, t_2^*) is $SFPR_0(iii)$ space. □

Theorem 3.5 *Let (X, s_1^*, t_1^*) and (Y, s_2^*, t_2^*) be two supra fuzzy bi-topological spaces and $f : X \to Y$ be a one-one and supra fuzzy continuous map. Then (Y, s_2^*, t_2^*) is $SFPR_0(j) \Rightarrow (X, s_1^*, t_1^*)$ is $SFPR_0(j)$, where $j = i, ii, iii, iv$.*

Proof Let (Y, s_2^*, t_2^*) be a supra fuzzy bi-topological space and assume that (Y, s_2^*, t_2^*) is $SFPR_0(iv)$. We have to prove that (X, s_1^*, t_1^*) is $SFPR_0(iv)$. Let x_m, y_n be fuzzy points in X for distinct x, y and let $\mu \in s_1^* \cup t_1^*$ such that $x_m q \mu, y_n \cap \mu = 0$. Then there exists fuzzy points x'_m, y'_n in Y with $f(x) = x', f(y) = y'$ for distinct x', y' as f is one-one. Again, since f is open and $\mu \in s_1^* \cup t_1^*$, $f(\mu) \in s_2^* \cup t_2^*$. Now, $x_m q \mu \Rightarrow \mu(x) + m > 1 \Rightarrow (f(\mu))(x') + m > 1$, where $f(\mu)(x') = \sup\{\mu(x) : f(x) = x'\} = \mu(x)$, $x'_m q f(\mu)$ and, $y_n \cap \mu = 0 \Rightarrow \mu(y) = 0 \Rightarrow (f(\mu))(y') = 0$, where $f(\mu)(y') = \sup\{\mu(y) : f(y) = y'\} = \mu(y) \Rightarrow y'_n \cap f(\mu) = 0$. Since (Y, s_2^*, t_2^*) is $SFPR_0(iv)$ space, there exists $\eta \in s_2^* \cup t_2^*$ such that $y'_n q \eta, x'_m \cap \eta = 0$.

Now, $y'_n q \eta \Rightarrow \eta(y') + n > 1 \Rightarrow \eta(f(y)) + n > 1 \Rightarrow (f^{-1}(\eta))(y) + n > 1$, since f is continuous $f^{-1}(\eta) \in s_1^* \cup t_1^* \Rightarrow y_n q f^{-1}(\eta)$ and $x'_m \cap \eta = 0 \Rightarrow \eta(x') = 0 \Rightarrow \eta(f(x)) = 0 \Rightarrow f^{-1}(\eta)(x) = 0 \Rightarrow x_m \cap f^{-1}(\eta) = 0$.

It follows that there exists $f^{-1}(\eta) \in s_1^* \cup t_1^*$ such that $y_n q f^{-1}(\eta), x_m \cap f^{-1}(\eta) = 0$. Hence (X, s_1^*, t_1^*) is $SFPR_0(iv)$ space. □

§4. R_1-Type Separation in Supra Fuzzy Bi-Topological Space

In this section, we define our notions in Supra fuzzy R_1 bi-topological spaces.

Definition 4.1 *A supra fuzzy bi-topological space (X, s^*, t^*) is called*

(a) $SFPR_1$ (i) space if and only if for any pair of fuzzy points x_m, y_n in X with $x \neq y$, whenever there exists $\omega \in s^* \cup t^*$ with $\omega(x) \neq \omega(y)$, then there exists $\mu, \eta \in s^* \cup t^*$ such that $x_m \; q \; \mu, y_n \; q \; \eta$ and $\mu \cap \eta = \emptyset$;

(b) $SFPR_1$ (ii) space if and only if any pair of fuzzy points x_m, y_n in X for distinct x and y, whenever there exists $\omega \in s^* \cup t^*$ with $\omega(x) \neq \omega(y)$, then there exists $\mu, \eta \in s^* \cup t^*$ such that $x_m \in \mu, y_n \in \eta$ and $\mu \bar{q} \eta$;

(c) $SFPR_1$ (iii) space if and only if for any pair of fuzzy points x_m, y_n in X for distinct x and y, whenever there exists $\omega \in s^* \cup t^*$ with $\omega(x) \neq \omega(y)$, then there exists $\mu, \eta \in s^* \cup t^*$ such that $x_m \; q \; \mu, y_n \; q \; \eta$ and $\mu \bar{q} \eta$.

4.1.Subspaces in Supra Fuzzy R_1 Bi-Topological Spaces. In this section, we shall show that our notions satisfy the hereditary property.

Theorem 4.1 *Let (X, s^*, t^*) be a supra fuzzy bi-topological space, $A \subseteq X$, $t_A^* = \{\mu/A : \mu \in s^* \cup t^*\}$, $s_A^* = \{\eta/A : \eta \in s^* \cup t^*\}$, then*

(a) If (X, s^*, t^*) is $SFPR_1(i)$, then (A, s_A^*, t_A^*) is $SFPR_1(i)$;
(b) If (X, s^*, t^*) is $SFPR_1(ii)$, then (A, s_A^*, t_A^*) is $SFPR_1(ii)$;
(c) If (X, s^*, t^*) is $SFPR_1(iii)$, then (A, s_A^*, t_A^*) is $SFPR_1(iii)$.

Proof Let (X, s^*, t^*) be a supra fuzzy bi-topological space and (X, s^*, t^*) is $SFPR_1(i)$. We have to prove that (A, s_A^*, t_A^*) is $SFPR_1(i)$. Let x_m and y_n be fuzzy points in A for distinct x and y. Then x_m and y_n are also fuzzy points in X for distinct x and y. Consider $p \in s_A^* \cup t_A^*$ with $p(x) \neq p(y)$. Here p can be written as μ/A, where $\mu \in s^* \cup t^*$ and hence $\mu(x) \neq \mu(y)$. Since (X, s^*, t^*) is $SFPR_1(i)$ supra fuzzy bi-topological space, we have, there exists $\eta, \omega \in s^* \cup t^*$ such that $x_m q \eta$, $y_n q \omega$ and $\eta \cap \omega = 0$. From the definition of t_A^*, we get η/A, $\omega/A \in s_A^* \cup t_A^*$. Now, $x_m q \eta \Rightarrow \eta(x) + m > 1$, $x \in X \Rightarrow (\eta/A)(x) + m > 1$, $x \in A \subseteq X \Rightarrow x_m q(\eta/A)$. Also, $y_n q \omega \Rightarrow \omega(y) + n > 1$, $y \in X \Rightarrow (\omega/A)(y) + n > 1$, $y \in A \subseteq X \Rightarrow y_n q(\omega/A)$ and $\eta \cap \omega = 0 \Rightarrow (\eta \cap \omega)(x) = 0$, $x \in X \Rightarrow \min(\eta(x), \omega(x)) = 0 \Rightarrow \min((\eta/A)(x), (\omega/A)(x)) = 0$, $x \in A \subseteq X \Rightarrow ((\eta/A) \cap (\omega/A))(x) = 0 \Rightarrow (\eta/A) \cap (\omega/A) = 0$.

It follows that there exists η/A, $\omega/A \in s_A^* \cup t_A^*$ such that $x_m q(\eta/A)$, $y_n q(\omega/A)$ and $(\eta/A) \cap (\omega/A) = 0$. Hence, (A, s_A^*, t_A^*) is $SFPR_1(i)$. Thus, (a) holds. Others are of similar manner. □

4.2.Productivity and Projectivity in Supra Fuzzy R_1 Bi-Topological Spaces. In this section, we shall show that our notions satisfy productive and projective properties.

Theorem 4.2 *Let (X_i, s_i^*, t_i^*), $i \in \Lambda$ be a supra fuzzy bi-topological space and $X = \prod_{i\in\Lambda} X_i$ and s^*, $X = \prod_{i\in\Lambda} X_i$ and t^* be the product topologies on X, then for all $i \in \Lambda$, (X_i, s_i^*, t_i^*) is $SFPR_1(j)$ if and only if (X, s^*, t^*) is $SFPR_1(j)$, where $j = i, ii, iii$.*

Proof Let for all $i \in \Lambda$, (X_i, s_i^*, t_i^*) be $SFPR_1(i)$ space. We have to prove that (X, s^*, t^*) is $SFPR_1(i)$. Let x_m and y_n be fuzzy points in X for distinct x, y and $\omega \in s^* \cup t^*$ with $\omega(x) \neq \omega(y)$. But we have $\omega(x) = \min\{\omega_i(x_i) : i \in \Lambda\}$, $\omega(y) = \min\{\omega_i(y_i) : i \in \Lambda\}$. Then there is at least one $\omega_i \in s_i^* \cup t_i^*$ and $(x_i)_m$, $(y_i)_n$ are fuzzy points with $x_i \neq y_i$ for some $i \in \Lambda$ with $\omega_i(x_i) \neq \omega_i(y_i)$. Since (X_i, s_i^*, t_i^*) is $SFPR_1(i)$, there exists μ_i, $\eta_i \in s_i^* \cup t_i^*$ such that $(x_i)_m q\mu_i$, $(y_i)_n q\eta_i$ and $\mu_i \cap \eta_i = 0$. Again, we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$. Now, $(x_i)_m q\mu_i \Rightarrow \mu_i(x_i) + m > 1 \Rightarrow \mu_i(\pi_i(x)) + m > 1 \Rightarrow (\mu_i \circ \pi_i)(x) + m > 1 \Rightarrow x_m q(\mu_i \circ \pi_i)$. Again, $(y_i)_n q\eta_i \Rightarrow \eta_i(y_i) + n > 1 \Rightarrow \eta_i(\pi_i(y)) + n > 1 \Rightarrow (\eta_i \circ \pi_i)(y) + n > 1 \Rightarrow y_n q(\eta_i \circ \pi_i)$ and $\mu_i \cap \eta_i = 0 \Rightarrow \mu_i \cap \eta_i(x_i) = 0 \Rightarrow \min(\mu_i(x_i), \eta_i(x_i)) = 0 \Rightarrow \min((\mu_i \circ \pi_i)(x), (\eta_i \circ \pi_i)(x)) = 0 \Rightarrow ((\mu_i \circ \pi_i) \cap (\eta_i \circ \pi_i))(x) = 0$.

It follows that there exists $\mu_i \circ \pi_i$, $\eta_i \circ \pi_i \in s^* \cup t^*$ such that $x_m q(\mu_i \circ \pi_i)$, $y_n q(\eta_i \circ \pi_i)$ and $(\mu_i \circ \pi_i) \cap (\eta_i \circ \pi_i) = 0$. Hence, (X, s^*, t^*) is $SFPR_1(i)$ space.

The rest of the results can be proved similarly. □

4.3.Mapping in Supra Fuzzy R_1 Bi-Topological Spaces. In this section, we shall show that our notions satisfy the order-preserving property.

Theorem 4.3 *Let (X, s_1^*, t_1^*) and (Y, s_2^*, t_2^*) be two supra fuzzy bi-topological spaces, and let $f : X \to Y$ be a one-one, onto, and supra fuzzy open map. Then, if (X, s_1^*, t_1^*) is $SFPR_1(j)$, it implies that (Y, s_2^*, t_2^*) is $SFPR_1(j)$, where $j = i, ii, iii$.*

Proof Let (X, s_1^*, t_1^*) be a supra fuzzy bi-topological space and (X, s_1^*, t_1^*) is $SFPR_1(ii)$. We have to prove that (Y, s_2^*, t_2^*) is $SFPR_1(ii)$. Let x'_m and y'_n be fuzzy points in Y for distinct x', y' and let $w \in s_2^* \cup t_2^*$ with $w(x') \neq w(y')$. Since f is onto, then there exists fuzzy points x_r, y_s in X with $f(x) = x'$, $f(y) = y'$ and $x \neq y$ as f is one-one. Since f is continuous, then $f^{-1}(w) \in s_1^* \cup t_1^*$ with $f^{-1}(w)(x) \neq f^{-1}(w)(y)$. Again, since (X, s_1^*, t_1^*) is $SFPR_1(ii)$, then there exists μ, $\eta \in s_1^* \cup t_1^*$ such that $x_m \in \mu$, $y_n \in \eta$ and $\mu\bar{q}\eta$. Now, $x_m \in \mu \Rightarrow \mu(x) \geq m \Rightarrow \mu(f^{-1}(x')) \geq m \Rightarrow f(\mu)(x') \geq m$, as f is open then $f(\mu) \in s_2^* \cup t_2^* \Rightarrow x'_m \in f(\mu)$. Again, $y_n \in \eta \Rightarrow \eta(y) \geq n \Rightarrow \eta(f^{-1}(y')) \geq n \Rightarrow f(\eta)(y') \geq n$, as f is open then $f(\eta) \in s_2^* \cup t_2^*$ $\Rightarrow y'_n \in f(\eta)$ and $f(\mu)(x') + f(\eta)(x') = \mu(f^{-1}(x')) + \eta(f^{-1}(x')) = \mu(x) + \eta(x) \leq 1$, as $\mu\bar{q}\eta$. Therefore, $f(\mu)(x') + f(\eta)(x') \leq 1 \Rightarrow f(\mu)\bar{q}f(\eta)$. It follows that there exists $f(\mu), f(\eta) \in s_2^* \cup t_2^*$ such that $x'_m \in f(\mu)$, $y'_n \in f(\eta)$ and $f(\mu)\bar{q}f(\eta)$. Hence (Y, s_2^*, t_2^*) is $SFPR_1(ii)$ space. □

Theorem 4.4 *Let (X, s_1^*, t_1^*) and (Y, s_2^*, t_2^*) be two supra fuzzy bi-topological spaces and let $f : X \to Y$ be a one-one and supra fuzzy continuous map. Then, if (Y, s_2^*, t_2^*) is $SFPR_1(j)$, it implies that (X, s_1^*, t_1^*) is $SFPR_1(j)$, where $j = i, ii, iii$.*

Proof Let (Y, s_2^*, t_2^*) be a supra fuzzy bi-topological space and (Y, s_2^*, t_2^*) is $SFPR_1(iii)$. We have to prove that (X, s_1^*, t_1^*) is $SFPR_1(iii)$. Let x_m and y_n be fuzzy points in X for distinct x, y and let $\omega \in s_2^* \cup t_2^*$ with $\omega(x) \neq \omega(y)$. Since f is one-one, let $f(x) = x'$, $f(y) = y'$

with $x' \neq y'$ and then x'_m and y'_n are fuzzy points in Y. As f is open, $f(\omega) \in s_2^* \cup t_2^*$ with $f(\omega)(x') \neq f(\omega)(y')$. Again, since (Y, s_2^*, t_2^*) is $SFPR_1(iii)$, then there exists $\mu, \eta \in s_2^* \cup t_2^*$ such that $x'_m q\mu$, $y'_n q\eta$ and $\mu q \eta = 0$. Now, $x'_m q\mu \Rightarrow \mu(x') + m > 1 \Rightarrow \mu(f(x)) + m > 1 \Rightarrow f^{-1}(\mu)(x) + m > 1$, as f is continuous then $f^{-1}(\mu) \in s_1^* \cup t_1^* \Rightarrow x_m q f^{-1}(\mu)$. Again, $y'_n q\eta \Rightarrow \eta(y') + n > 1, \eta(f(y)) + n > 1 \Rightarrow f^{-1}(\eta)(y') + n > 1$, as f is continuous then $f^{-1}(\eta) \in s_1^* \cup t_1^*$ $\Rightarrow y_n q f^{-1}(\eta)$ and $f^{-1}(\mu)(x) + f^{-1}(\eta)(x) = (\mu(f)(x) + \eta(f)(x)) = \mu(x') + \eta(x') \leq 1$, as $\mu q \eta$. Therefore $(f^{-1}(\mu) + f^{-1}(\eta))(x) \leq 1 \Rightarrow f^{-1}(\mu) q \overline{f^{-1}(\eta)}$.

It follows that there exists $f^{-1}(\mu)$, $f^{-1}(\eta) \in s_1^* \cup t_1^*$ such that $x_m q f^{-1}(\mu)$, $y_n q f^{-1}(\eta)$ and $f^{-1}(\mu) q \overline{f^{-1}(\eta)}$. Hence (X, s_1^*, t_1^*) is $SFPR_1(iii)$ space. □

§5. Conclusion

The main result of this paper is introducing some new concepts of R_0- type and R_1- type separations on supra fuzzy bitopological spaces in sense of quasi-coincidence.We discuss some features of these concepts and present their good extension, hereditary, projective and productive properties.

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4-Total Mean Cordial Graphs Derived From Star, Jellyfish and Fan

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Abstract: Let G be a graph. Let $f : V(G) \to \{0, 1, 2, \cdots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called k-total mean cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$, for all $i, j \in \{0, 1, \ldots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \ldots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph.

Key Words: Star, bistar, jellyfish, fan, total mean cordial labeling, Smarandachely total mean cordial labeling.

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§1. Introduction

All graphs in this paper are finite, simple and undirected. Cordial labeling was introduced by Cahit [1]. Subsequently cordial related labeling was studied by several authors [9, 10, 11, 12, 13]. The notation of k-total mean cordial labeling have been introduced in [4]. Also 4-total mean cordial labeling of certain graphs like path, cycle, star, bistar, comb, crown, square of path, double comb, double crown, double fan, subdivision of star, subdivision of comb, subdivision of ladder, helm, flower graph, gear graph and web graph have been investigated [4, 5, 6, 7, 8]. In this paper, we investigate the 4-total mean cordial behavior of $K_{1,n} \odot K_1$, $K_{1,n} \odot \overline{K_2}$, $K_{1,n} \odot \overline{K_3}$, $B_{n,n} \odot \overline{K_2}$, $B_{n,n} \odot \overline{K_3}$, $J_{n,n} \odot K_1$, $J_{n,n} \odot \overline{K_2}$, $J_{n,n} \odot \overline{K_3}$, $F_n \odot K_1$, $F_n \odot \overline{K_3}$. Let x be any real number. Then $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms not defined here follow from Harary [3] and Gallian [2].

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§2. k-Total Mean Cordial Graph

Definition 2.1 *Let G be a graph. Let $f : V(G) \to \{0, 1, 2, \ldots, k-1\}$ be a function where $k \in \mathbb{N}$ and $k > 1$. For each edge uv, assign the label $f(uv) = \left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called k-total mean*

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cordial labeling of G if $|t_{mf}(i) - t_{mf}(j)| \leq 1$, for all $i, j \in \{0, 1, \cdots, k-1\}$, where $t_{mf}(x)$ denotes the total number of vertices and edges labelled with x, $x \in \{0, 1, 2, \cdots, k-1\}$. A graph with admit a k-total mean cordial labeling is called k-total mean cordial graph.

Conversely, such a labeling f is called a Smarandachely k-total mean cordial labeling of G if there are integers $i, j \in \{0, 1, 2, \cdots, k-1\}$ hold with $|t_{mf}(i) - t_{mf}(j)| \geq 2$ and G is called a Smarandachely k-total mean cordial graph.

§3. Preliminaries

Definition 3.1 *A complete bipartite graph $K_{1,n}$ is called a star.*

Definition 3.2 *A bistar $B_{m,n}$ is the graph obtained by joining the two central vertices of $K_{1,m}$ and $K_{1,n}$.*

Definition 3.3 *A jellyfish graph $J(m, n)$ is obtained from a cycle $C_4 : uxvwu$ by joining x and w with an edge and appending m pendent edges to u and n pendent edges to v.*

Definition 3.4 *A graph $F_n = P_n + K_1$ is called a fan graph, where P_n is a path.*

Definition 3.5 *Let G_1, G_2 respectively be (p_1, q_1), (p_2, q_2) graphs. A corona of G_1 with G_2 is the graph $G_1 \odot G_2$ obtained by taking one copy of G_1 , p_1 copies of G_2 and joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2 where $1 \leq i \leq p_1$.*

Definition 3.6 *The complement $\overline{G}$ of a graph G also has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.*

§4. Main Results

Theorem 4.1 *A graph $K_{1,n} \odot K_1$ is 4-total mean cordial for all n.*

Proof Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$, $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Let v_1, v_2, $\cdots$, v_n be the pendent vertices adjacent to u_1, u_2, $\cdots$, u_n and v be the pendent vertex adjacent to u.

Clearly,

$$|V(K_{1,n} \odot K_1)| + |E(K_{1,n} \odot K_1)| = 4n + 3.$$

Assign the labels 1, 3 to the vertices u, v respectively. Consider the vertices u_1, u_2, $\cdots$, u_n. Now we assign the label 0 to the n vertices u_1, u_2, $\ldots$, u_n. We now consider the vertices v_1, v_2, $\cdots$, v_n. Finally we assign the label 3 to the n vertices v_1, v_2, $\ldots$, v_n.

Obviously, $t_{mf}(0) = n$, $t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = n + 1$. □

Theorem 4.2 *A graph $K_{1,n} \odot \overline{K_2}$ is 4-total mean cordial for all n.*

Proof Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$, $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Let v, w be the pendent vertices adjacent to u and v_1, v_2, $\cdots$, v_n, w_1, w_2, $\cdots$, w_n be the pendent vertices adjacent to $u_1, u_2, \cdots, u_n$. Obviously,

$$\left|V\left(K_{1,n}\odot\overline{K_2}\right)\right|+\left|E\left(K_{1,n}\odot\overline{K_2}\right)\right|=6n+5.$$

Case 1. $n\equiv 0 \pmod 4$.

Let $n=4r$, where $r\in N$. Assign the labels 0, 2, 3 to the vertices u, v, w respectively.

Now, we assign the label 0 to the $3r$ vertices $u_1, u_2, \cdots, u_{3r}$. We now assign the label 1 to the r vertices $u_{3r+1}, u_{3r+2}, \cdots, u_{4r}$. Next we assign the label 1 to the $2r$ vertices $v_1, v_2, \cdots, v_{2r}$. We now assign the label 3 to the $2r$ vertices $v_{2r+1}, v_{2r+2}, \cdots, v_{4r}$. Finally we assign the label 3 to the $4r$ vertices $w_1, w_2, \cdots, w_{4r}$.

Case 2. $n\equiv 1 \pmod 4$.

Let $n=4r+1$, where $r\in N$. Assign the labels 0, 2, 3 to the vertices u, v, w respectively.

We now assign the label 0 to the $3r+1$ vertices $u_1, u_2, \cdots, u_{3r+1}$. Now we asisgn the label 1 to the r vertices $u_{3r+2}, u_{3r+3}, \cdots, u_{4r+1}$. Next we assign the label 1 to the $2r+1$ vertices $v_1, v_2, \cdots, v_{2r+1}$. Now we assign the label 3 to the $2r$ vertices $v_{2r+2}, v_{2r+3}, \cdots, v_{4r+1}$. Finally we assign the label 3 to the $4r+1$ vertices $w_1, w_2, \cdots, w_{4r+1}$.

Case 3. $n\equiv 2 \pmod 4$.

Let $n=4r+2$, where $r\geq 0$. Assign the labels 0, 3, 3 to the vertices u, v, w respectively.

We now assign the label 0 to the $3r+2$ vertices $u_1, u_2, \cdots, u_{3r+2}$. Now we asisgn the label 1 to the r vertices $u_{3r+3}, u_{3r+4}, \cdots, u_{4r+2}$. Next we assign the label 1 to the $2r+2$ vertices $v_1, v_2, \cdots, v_{2r+2}$. Now we assign the label 3 to the $2r$ vertices $v_{2r+3}, v_{2r+4}, \cdots, v_{4r+2}$. Finally we assign the label 3 to the $4r+2$ vertices $w_1, w_2, \ldots, w_{4r+2}$.

Case 4. $n\equiv 3 \pmod 4$.

Let $n=4r+3$, where $r\geq 0$. Assign the labels 0, 3, 3 to the vertices u, v, w respectively.

Now we assign the label 0 to the $3r+2$ vertices $u_1, u_2, \cdots, u_{3r+2}$. We now asisgn the label 1 to the $r+1$ vertices $u_{3r+3}, u_{3r+4}, \cdots, u_{4r+3}$. Next we assign the label 1 to the $2r+2$ vertices $v_1, v_2, \cdots, v_{2r+2}$. We now assign the label 3 to the $2r+1$ vertices $v_{2r+3}, v_{2r+4}, \cdots, v_{4r+3}$. Finally we assign the label 3 to the $4r+3$ vertices $w_1, w_2, \cdots, w_{4r+3}$.

This vertex labeling f is a 4-total mean cordial labeling follows from the Table 1.

Nature of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n=4r$	$6r+1$	$6r+1$	$6r+2$	$6r+1$
$n=4r+1$	$6r+3$	$6r+3$	$6r+3$	$6r+2$
$n=4r+2$	$6r+5$	$6r+4$	$6r+4$	$6r+4$
$n=4r+3$	$6r+5$	$6r+6$	$6r+6$	$6r+6$

Table 1.

This completes the proof. □

Theorem 4.3 *A graph $K_{1,n}\odot\overline{K_3}$ is 4-total mean cordial for all n.*

Proof Let $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$, $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$. Let x, y, z be the pendent vertices adjacent to u and $x_1, x_2, \cdots, x_n, y_1, y_2, \ldots, y_n, z_1, z_2, \cdots, z_n$ be the pendent vertices adjacent to $u_1, u_2, \cdots, u_n$ respectively. Note that

$$\left|V\left(K_{1,n} \odot \overline{K_3}\right)\right| + \left|E\left(K_{1,n} \odot \overline{K_3}\right)\right| = 8n + 7.$$

Assign the labels 0, 1, 3, 3 to the vertices u, x, y, z respectively. Now, we assign the label 0 to the n vertices $u_1, u_2, \cdots, u_n$. We now assign the label 1 to the n vertices $x_1, x_2, \cdots, x_n$. Next we assign the label 3 to the n vertices $y_1, y_2, \cdots, y_n$. Finally we assign the label 3 to the n vertices $z_1, z_2, \cdots, z_n$.

Clearly,

$$t_{mf}(0) = 2n + 1,\ t_{mf}(1) = 2n + 2,\ \ t_{mf}(2) = 2n + 2 \text{ and } t_{mf}(3) = 2n + 2. \qquad \square$$

Example 4.1 A 4 - total mean cordial labeling of $K_{1,3} \odot \overline{K_3}$ is given in Figure 1.

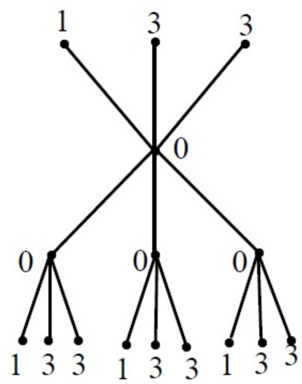

Figure 1. $K_{1,3} \odot \overline{K_3}$

Theorem 4.4 *A graph* $B_{n,n} \odot \overline{K_2}$ *is* 4*-total mean cordial for all* n.

Proof Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \leq i \leq n\}$. Let x, y be the pendent vertices adjacent to u and $x_1, x_2, \ldots, x_n, y_1, y_2, \cdots, y_n$ be the pendent vertices adjacent to $u_1, u_2, \cdots, u_n$. Let p, q be the pendent vertices adjacent to v and $p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_n$ be the pendent vertices adjacent to $v_1, v_2, \cdots, v_n$.

Clearly,

$$\left|V\left(B_{n,n} \odot \overline{K_2}\right)\right| + \left|E\left(B_{n,n} \odot \overline{K_2}\right)\right| = 12n + 11.$$

Assign the labels 1, 3, 0, 0, 2, 2 to the vertices u, v, x, y, p, q respectively. Now, we assign the label 0 to the n vertices $u_1, u_2, \cdots, u_n$. We now assign the label 2 to the n vertices $v_1, v_2, \cdots, v_n$. Next we assign the label 0 to the n vertices $x_1, x_2, \cdots, x_n$. We now assign the label 1 to the n vertices $y_1, y_2, \cdots, y_n$. Now we assign the label 2 to the n vertices $p_1, p_2, \cdots, p_n$. Finally, we assign tne label 3 to the n vertices $q_1, q_2, \cdots, q_n$.

Thus,

$$t_{mf}(0) = 3n + 2,\ \ t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = 3n + 3. \qquad \square$$

Example 4.2 A 4-total mean cordial labeling of $B_{2,2} \odot \overline{K_2}$ is shown in Figure 2.

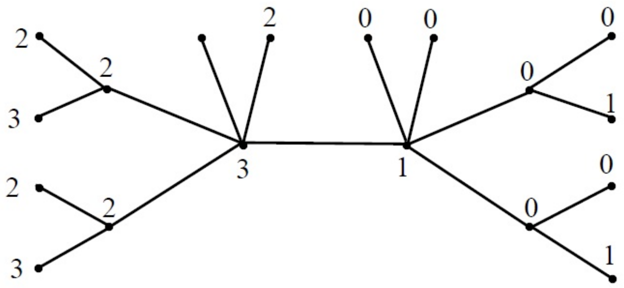

Figure 2. $B_{2,2} \odot \overline{K_2}$

Theorem 4.5 *A graph $B_{n,n} \odot \overline{K_3}$ is 4-total mean cordial for all n.*

Proof Let $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \le i \le n\}$ and $E(B_{n,n}) = \{uv, uu_i, vv_i : 1 \le i \le n\}$. Let x, y, z be the pendent vertices adjacent to u and x_1, x_2, $\cdots$, x_n, y_1, y_2, $\cdots$, y_n, z_1, z_2, $\cdots$, z_n be the pendent vertices adjacent to u_1, u_2, $\cdots$, u_n. Let p, q, r be the pendent vertices adjacent to v and p_1, p_2, $\cdots$, p_n, q_1, q_2, $\cdots$, q_n, r_1, r_2, $\cdots$, r_n be the pendent vertices adjacent to v_1, v_2, $\cdots$, v_n. Note that

$$\left|V\left(B_{n,n} \odot \overline{K_3}\right)\right| + \left|E\left(B_{n,n} \odot \overline{K_3}\right)\right| = 16n + 15.$$

Assign the labels 0, 1, 3, 3, 0, 1, 3, 3 to the vertices u, x, y, z, v, p, q, r respectively. Now we assign the label 0 to the $2n$ vertices u_1, u_2, $\ldots$, u_n, v_1, v_2, $\cdots$, v_n. We now assign the label 1 to the $2n$ vertices x_1, x_2, $\cdots$, x_n, p_1, p_2, $\ldots$, p_n. Finally, we assign the label 3 to the $4n$ vertices y_1, y_2, $\ldots$, y_n, z_1, z_2, $\cdots$, z_n, q_1, q_2, $\cdots$, q_n, r_1, r_2, $\cdots$, r_n.

Clearly,

$$t_{mf}(0) = 4n + 3, \; t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = 4n + 4. \qquad \square$$

Example 4.3 A 4 - total mean cordial labeling of $B_{3,3} \odot \overline{K_3}$ is given in Figure 3.

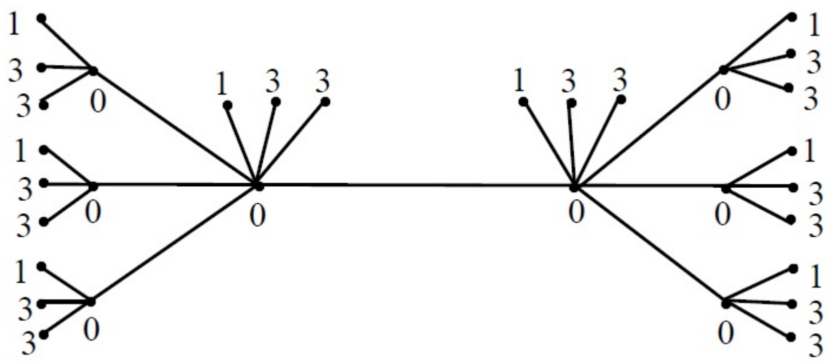

Figure 3. $B_{3,3} \odot \overline{K_3}$

Theorem 4.6 *A graph $J_{n,n} \odot K_1$ is 4-total mean cordial for all n.*

Proof Let $V(J_{n,n} \odot K_1) = \{u, v, x, y, p, q, r, s\} \cup \{u_i, v_i, x_i, y_i : 1 \le i \le n\}$, $E(J_{n,n} \odot K_1) = \{ux, xv, vy, yu, xy, pu, qx, rv, sy\} \cup \{uu_i, u_ix_i, vv_i, v_iy_i : 1 \le i \le n\}$. Clearly,

$$|V(J_{n,n} \odot K_1)| + |E(J_{n,n} \odot K_1)| = 8n + 17.$$

Assign the labels 0, 2, 1, 3, 0, 0, 0, 3 to the vertices u, v, x, y, p, q, r, s respectively. We now assign the label 0 to the n vertices $u_1, u_2, \cdots, u_n$. Now we assign the label 1 to the n vertices $x_1, x_2, \cdots, x_n$. Next we assign the label 2 to the n vertices $v_1, v_2, \cdots, v_n$. Finally we assign the label 3 to the n vertices $y_1, y_2, \cdots, y_n$.

Clearly,

$$t_{mf}(0) = 2n + 5, \quad t_{mf}(1) = t_{mf}(2) = t_{mf}(3) = 2n + 4. \qquad \square$$

Example 4.4 A 4 - total mean cordial labeling of $J_{3,3} \odot K_1$ is shown in Figure 4.

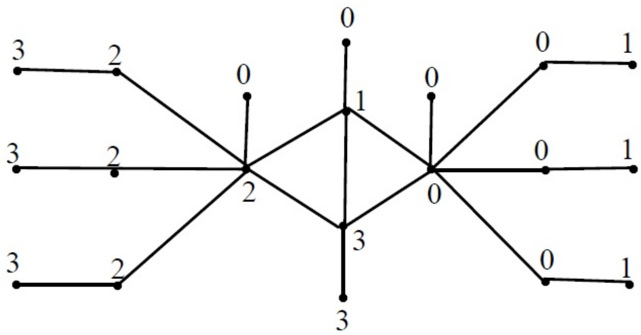

Figure 4. $J_{3,3} \odot K_1$

Theorem 4.7 *A graph $J_{n,n} \odot \overline{K_2}$ is 4-total mean cordial for all n.*

Proof Let $V\left(J_{n,n} \odot \overline{K_2}\right) = \{u, v, x, y, a, b, c, d, e, f, g, h\} \cup \{u_i, v_i, x_i, y_i, p_i, q_i : 1 \le i \le n\}$, $E\left(J_{n,n} \odot \overline{K_2}\right) = \{ux, xv, vy, yu, xy, au, bu, cx, dx, ev, fv, gy, hy\} \cup \{uu_i, u_ix_i, u_iy_i, vv_i, v_ip_i, v_iq_i : 1 \le i \le n\}$.

Obviously,

$$\left|V\left(J_{n,n} \odot \overline{K_2}\right)\right| + \left|E\left(J_{n,n} \odot \overline{K_2}\right)\right| = 12n + 25.$$

Assign the labels 1, 3, 1, 0, 0, 2, 3, 3, 3, 3, 0, 0 to the vertices u, v, x, y, a, b, c, d, e, f, g, h respectively. Now we assign the label 0 to the $2n$ vertices $u_1, u_2, \cdots, u_n, x_1, x_2, \cdots, x_n$. We now assign the label 1 to the n vertices $y_1, y_2, \cdots, y_n$. Next we assign the label 2 to the $2n$ vertices $v_1, v_2, \cdots, v_n, p_1, p_2, \cdots, p_n$. Finally we assign the label 3 to the n vertices $q_1, q_2, \cdots, q_n$.

Clearly,

$$t_{mf}(0) = t_{mf}(1) = t_{mf}(2) = 3n + 6, \; t_{mf}(3) = 3n + 7. \qquad \square$$

Example 4.5 A 4 - total mean cordial labeling of $J_{2,2} \odot \overline{K_2}$ is given in Figure 5.

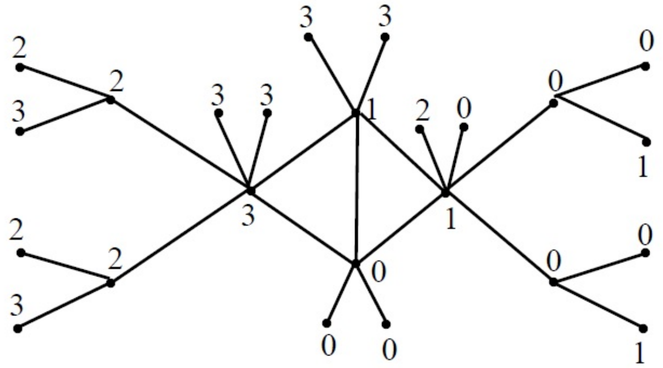

Figure 5. $J_{2,2} \odot \overline{K_2}$

Theorem 4.8 *A graph $J_{n,n} \odot \overline{K_3}$ is 4-total mean cordial.*

Proof Let $V\left(J_{n,n} \odot \overline{K_3}\right) = \{u, v, x, y\} \cup \{u_i, v_i, x_i, y_i : 1 \leq i \leq 3\} \cup \{z_j, p_j, q_j, r_j, w_j, a_j, b_j, c_j : 1 \leq i \leq n\}$, $E\left(J_{n,n} \odot \overline{K_3}\right) = \{ux, xv, vy, yu, xy\} \cup \{uu_i, xx_i, vv_i, yy_i : 1 \leq i \leq 3\} \cup \{uz_j, z_jp_j, z_jq_j, z_jr_j, vw_j, w_ja_j, w_jb_j, w_jc_j : 1 \leq j \leq n\}$.

Clearly,

$$\left|V\left(J_{n,n} \odot \overline{K_3}\right)\right| + \left|E\left(J_{n,n} \odot \overline{K_3}\right)\right| = 16n + 33.$$

Assign the label 0, 0, 3, 2 to the vertices u, v, x, y respectively. We now assign the label 0 to the vertices u_1, u_2, u_3. Now we assign the label 1 to the vertices v_1, v_2, v_3. Next we assign the label 3 to the vertices x_1, x_2, x_3. Now we assign the label 2 to the vertices y_1, y_2, y_3. We now assign the label 0 to the $2n$ vertices $z_1, z_2, \cdots, z_n, w_1, w_2, \cdots, w_n$. Next we assign the label 1 to the $2n$ vertices $p_1, p_2, \cdots, p_n, a_1, a_2, \cdots, a_n$. Finally we assign the label 3 to the $4n$ vertices $q_1, q_2, \ldots, q_n, r_1, r_2, \cdots, r_n, b_1, b_2, \cdots, b_n, c_1, c_2, \cdots, c_n$.

Clearly,

$$t_{mf}(0) = t_{mf}(1) = t_{mf}(3) = 4n + 8, \; t_{mf}(2) = 4n + 9.$$ □

Example 4.6 A 4 - total mean cordial labeling of $J_{2,2} \odot \overline{K_3}$ is shown in Figure 6.

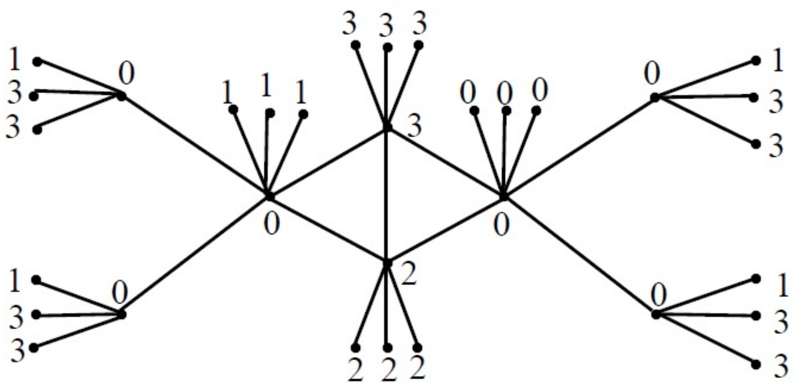

Figure 6. $J_{2,2} \odot \overline{K_3}$

Theorem 4.9 *A graph $F_n \odot K_1$ is 4-total mean cordial for all $n \geq 2$.*

Proof Let P_n be the path $u_1 u_2 \cdots u_n$. Let $V(F_n \odot K_1) = \{u, x\} \cup V(P_n) \cup \{x_i : 1 \leq i \leq n\}$ and

$$E(F_n \odot K_1) = \{uu_i : 1 \leq i \leq n\} \cup E(P_n) \cup \{u_i x_i : 1 \leq i \leq n\}.$$

Clearly

$$|V(F_n \odot K_1)| + |E(F_n) \odot K_1| = 5n + 2.$$

Assign the labels 2, 0 to the vertices u, x respectively.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, where $r \in N$. Consider the path vertices $u_1, u_2, \cdots, u_n$. Assign the labels 0, 0, 2, 3 respectively to the vertices u_1, u_2, u_3, u_4. Now we assign the labels 0, 0, 2, 3 to the vertices u_5, u_6, u_7, u_8 respectively. We now assign the labels 0, 0, 2, 3 respectively to the vertices u_9, u_{10}, u_{11}, u_{12}. Proceeding like this until reach the vertex u_{4r}. Obviously the vertices u_{4r-3}, u_{4r-2}, u_{4r-1}, u_{4r} receive the labels 0, 0, 2, 3. Now we assign the labels 0, 1, 2, 3 respectively to the vertices x_1, x_2, x_3, x_4. We now assign the labels 0, 1, 2, 3 to the vertices x_5, x_6, x_7, x_8 respectively. Next we assign the labels 0, 1, 2, 3 respectively to the vertices x_9, x_{10}, x_{11}, x_{12}. Continuing like this until reach the vertices x_{4r}. Clearly the vertices x_{4r-3}, x_{4r-2}, x_{4r-1}, x_{4r} receive the labels 0, 1, 2, 3.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r + 1$, where $r \in N$.

As in Case 1 assign the label to the vertices u_i, x_i $(1 \leq i \leq 4r)$. Finally, we assign the labels 0, 3 to the vertices u_{4r+1}, x_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r + 2$, where $r \in N$. Label the vertices u_i, x_i $(1 \leq i \leq 4r)$ as in Case 1. Now we assign the labels 3, 0, 0, 2 to the vertices u_{4r+1}, u_{4r+2}, x_{4r+1},x_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4r+3$, where $r \in N$. In this case assign the label for the vertices u_i, x_i $(1 \leq i \leq 4r)$ as in Case 1. We now assign the labels 3, 0, 0, 0, 2, 3 to the vertices u_{4r+1}, u_{4r+2}, u_{4r+3}, x_{4r+1},x_{4r+2}, x_{4r+3}.

This vertex labeling f is a 4-total mean cordial labeling of $F_n \odot K_1$ follows from the Table 2.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$5r + 1$	$5r + 1$	$5r$	$5r + 1$
$n = 4r + 1$	$5r + 2$	$5r + 2$	$5r + 2$	$5r + 1$
$n = 4r + 2$	$5r + 3$	$5r + 3$	$5r + 3$	$5r + 3$
$n = 4r + 3$	$5r + 5$	$5r + 4$	$5r + 4$	$5r + 4$

Table 2.

Case 5. $n = 2, 3$.

A 4-total mean cordial labeling is given in Table 3.

n	u	v	u_1	u_2	u_3	v_1	v_2	v_3
2	1	3	0	0		3	3	
3	1	3	0	0	0	3	3	3

Table 3.

This completes the proof. □

Theorem 4.10 *A graph $F_n \odot \overline{K_3}$ is 4-total mean cordial for all values of $n \geq 2$.*

Proof Let P_n be the path $u_1u_2 \ldots u_n$. Let $V\left(F_n \odot \overline{K_3}\right) = \{u, x, y, z\} \cup V(P_n) \cup \{x_i, y_i, z_i : 1 \leq i \leq n\}$ and $E\left(F_n \odot \overline{K_3}\right) = \{uu_i : 1 \leq i \leq n\} \cup E(P_n) \cup \{u_ix_i, u_iy_i, u_iz_i : 1 \leq i \leq n\}$.

Clearly,

$$\left|V\left(F_n \odot \overline{K_3}\right)\right| + \left|E\left(F_n\right) \odot \overline{K_3}\right| = 9n + 6.$$

Assign the labels 2, 0, 2, 3 to the vertices u, x, y, z respectively.

Case 1. $n \equiv 0 \pmod 4$.

Let $n = 4r$, where $r \in N$. Label the vertices u_i, x_i $(1 \leq i \leq n)$ as in Case 1 of Theorem 4.2. Assign the labels 0, 1, 2, 3 respectively to the vertices y_1, y_2, y_3, y_4. Now we assign the labels 0, 1, 2, 3 to the vertices y_5, y_6, y_7, y_8 respectively. We now assign the labels 0, 1, 2, 3 respectively to the vertices y_9, y_{10}, y_{11}, y_{12}. Proceeding like this until reach the vertex y_{4r}. Obviously the vertices y_{4r-3}, y_{4r-2}, y_{4r-1}, y_{4r} receive the labels 0, 1, 2, 3. Next we assign the labels 0, 1, 2, 3 respectively to the vertices z_1, z_2, z_3, z_4. We now assign the labels 0, 1, 2, 3 to the vertices z_5, z_6, z_7, z_8 respectively. Now we assign the labels 0, 1, 2, 3 respectively to the vertices z_9, z_{10}, z_{11}, x_{12}. Continuing like this until reach the vertices z_{4r}. Clearly, the vertices z_{4r-3}, z_{4r-2}, z_{4r-1}, z_{4r} receive the labels 0, 1, 2, 3.

Case 2. $n \equiv 1 \pmod 4$.

Let $n = 4r + 1$, where $r \in N$. As in Case 1 assign the label to the vertices u_i, x_i, y_i, z_i $(1 \leq i \leq 4r)$. Finally we assign the labels 2, 0, 0, 0 to the vertices u_{4r+1}, x_{4r+1}, y_{4r+1}, z_{4r+1}.

Case 3. $n \equiv 2 \pmod 4$.

Let $n = 4r + 2$, where $r \in N$. Label the vertices u_i, x_i, y_i, z_i $(1 \leq i \leq 4r)$ as in Case 1. Now we assign the labels 3, 0, 2, 0, 1, 0, 1, 1 to the vertices u_{4r+1}, u_{4r+2}, x_{4r+1},x_{4r+2}, y_{4r+1}, y_{4r+2}, z_{4r+1},z_{4r+2}.

Case 4. $n \equiv 3 \pmod 4$.

Let $n = 4r + 3$, where $r \in N$. In this case assign the label for the vertices u_i, x_i $(1 \leq i \leq 4r + 2)$ as in Case 3. We now assign the labels 2, 0, 0, 3 to the vertices u_{4r+3}, x_{4r+3}, y_{4r+3}, z_{4r+3}.

This labeling f is a 4-total mean cordial labeling of $F_n \odot K_1$ follows from the Table 4.

Order of n	$t_{mf}(0)$	$t_{mf}(1)$	$t_{mf}(2)$	$t_{mf}(3)$
$n = 4r$	$5r+1$	$5r+1$	$5r$	$5r+1$
$n = 4r+1$	$5r+2$	$5r+2$	$5r+2$	$5r+1$
$n = 4r+2$	$5r+3$	$5r+3$	$5r+3$	$5r+3$
$n = 4r+3$	$5r+5$	$5r+4$	$5r+4$	$5r+4$

Table 4.

Case 5. $n = 2, 3$.

A 4-total mean cordial labeling is given in Table 5.

n	u	x	y	z	u_1	u_2	u_3	x_1	x_2	x_3	y_1	y_2	y_3	z_1	z_2	z_3
2	2	0	2	3	3	0		2	0		1	0		1	1	
3	2	0	2	3	3	0	2	3	0	1	2	0	1	2	1	3

Table 5.

This completes the proof. □

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June, 2024

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