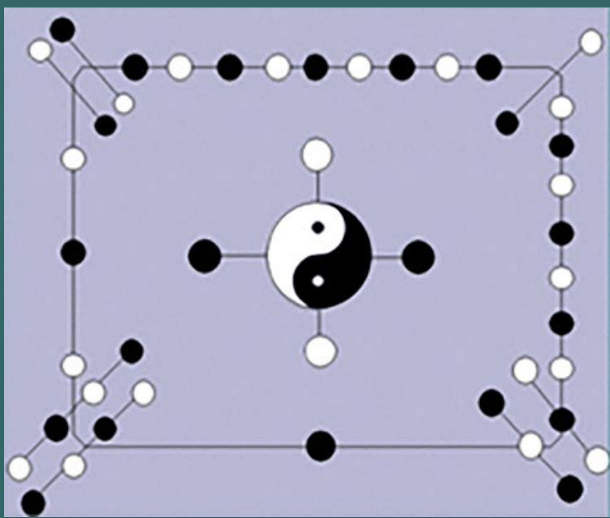
ISSN 1937 - 1055

VOLUME 2, 2026

INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

June, 2026

Vol.2, 2026 **ISSN 1937-1055**

International Journal of

Mathematical Combinatorics

(www.mathcombin.com)

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

June, 2026

Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.L.F. MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055 (Print), 1937-1047 (Online)*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

Smarandache multi-spaces and Smarandache geometries with applications to other sciences;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Mathematical theory on gravitational fields and parallel universes;

Applications of Combinatorics to mathematics and theoretical physics.

Generally, papers on applications of combinatorics to other mathematics and other sciences are welcome by this journal.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews (USA), Zentralblatt Math (Germany), Index EuroPub (UK), Referativnyi Zhurnal (Russia), Mathematika (Russia), EBSCO (USA), Google Scholar, Baidu Scholar, Directory of Open Access (DoAJ), International Scientific Indexing (ISI, impact factor 2.012), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA), CNKI(China).

Subscription A subscription can be ordered by an email directly to

Linfan Mao

The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science Beijing, 100190, P.R.China, and also the President of Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA

Email: maolinfan@163.com

Price: US$48.00

Famous Words:

Mathematics is the door and the key to science, ignoring mathematics will hurt all the knowledge, because the neglect of mathematics can not understand any other science and the world of anything else. More seriously, people who ignore mathematics can not understand his own negligence, and will eventually lead to the inability to seek any remedial measures.

By *Roger Bacon*, a famous British experimental scientist.

International J.Math. Combin. Vol.2(2026), 01-24

Computing the Number of Spanning Trees of Some Families of Graphs Generated by a Triangle and Their Entropies

S. N. Daoud[1,2,*] and Ahmed. N. Elsawy[1,3]

1. Department of Mathematics, College of Science, Taibah University, Al-Madinah 30001, Saudi Arabia
2. Department of Mathematics, Faculty of Science, Menoufia University, Shebin El Kom 32511, Egypt
3. Department of Mathematics, Faculty of Science, Al-Azhar University, Cairo, Egypt

E-mail: salamadaoud@gmail.com, ahnoby@yahoo.com

Abstract: In Intensive matter physics, the 3-dimensional frame of complex artificial atomic structures can be studied quantitatively by combining statistics on graph impractical properties concerning to the topology of atoms. Also, using equivalent transformations, complex circuits that require many mathematical operations to analyze can be reduced to simpler equivalent circuits. These transformations can also be applied to some families of graphs to compute the number of spanning trees. In this work, we compute the explicit formulas for the number of spanning trees of sequences of families of graphs generated by a triangle by equivalent transformations of electric circuits and rules of weighted generating function. Ultimately, we match the entropy of our graphs with other studied graphs with the nearby fee families are in the average degree.

Key Words: Number of spanning trees, electrically equivalent transformations, entropy.

AMS(2010): 05C30, 05C50, 05C63.

§1. Introduction

The number of spanning trees of a connected non-directed graph G is the whole number of different, factor subgraphs of G that are trees. Counting the number of spanning trees in networks (graphs) is a charming and focal issue in statistical physics, because of its inveterate relevance to various sides in concerned fields. For example, the number of spanning trees is a serious measure of reliability of a network [1]. Again, for example, it is precisely the number of repeated configurations of the Abelian sand-pile models [2], which have been studied widely in the past two decades as a sample of the self-arranged criticality [3]. Furthermore, the issue of spanning trees has many connections with other entertaining issues related to networks, such as dimer coverings [4], Potts model [5] random walks [6], origin of fractality for fractal scale-free networks and numerous others. Laplacian matrix of a graph that is a matrix resulting from the difference between the degree matrix of the graph (a diagonal matrix with vertex degrees on the diagonals) and its adjacency matrix (a $(0,1)$-matrix with 1's at places corresponding to

[1]Received November 25, 2025, Accepted April 12, 2026

entries where the vertices are adjacent and 0's otherwise). Kirchhoff [7] gave the known matrix tree theorem, displaying that this number can be counted from the determinant of any cofactor of the Laplacian matrix of the graph. For a given connected graph G with n vertices. Kelmans and Chelnoknov [8] introduced the next formula:

$$\tau(G) = \frac{1}{n}\prod_{i=1}^{n-1}\lambda_i, \tag{1.1}$$

where $n = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ are the eigenvalues of the Laplacian. However, numerical calculation for large size graphs is notoriously intractable since the calculation of eigenvalues is NP-hard with reference to graph size. Enumeration of spanning trees in graphs with certain symmetry and fractals has been excessively studied via ad hoc methods profiting from the particular structures [9C13].

§2. Electrically Equivalent Transformations

Kirchhoff's goal was to study electrical networks: An edge-weighted graph G (with the weight function $\omega : E(G) \to [0, \infty)$) can be considered as an electrical network with the weights being the conductance of the conformable edges. The effect disposal between two specific vertices u, v can be written as the quotient of (weighted) number of spanning trees and the (weighted) number of so-called thickets, the idea of electrically equivalent transformations [14-17] will be applied, which enables us to find the relationship between the numbers of spanning trees in graphs before and after the transformation. Below, we mention the action of some direct transformations on the number of spanning trees. Let G be an edge-weighted graph, G' be the corresponding electrically equivalent graph, $\tau(G)$ indicates the weighted number of spanning trees G.

(1) Double edges: If two analogous edges with disposals u and v in G are combined into one edge with disposals $u + v$ $in\ G'$, then $\tau(G') = \tau(G)$.

(2) Sequent edges: If two sequent edges have disposals u and v in G, are combined into one edge with disposal $\frac{uv}{u+v}$ in G', then $\tau(G') = \frac{1}{u+v}\tau(G)$.

(3) $Wye - Delta$ $(Y - \triangle)$ transformation: means replacing a Y subgraph of a graph with the equivalent $\triangle$ subgraph. The transform preserves the number of edges in a graph, but not the number of vertices or the number of cycles. If a Y graph with disposals u, v and w in G is changed into an electrically equivalent triangle with disposals $x = \frac{vw}{u+v+w}, y = \frac{uw}{u+v+w}$ and $z = \frac{uv}{u+v+w}$ in G', then $\tau(G') = \frac{1}{u+v+w}\tau(G)$.

(4) $Delta - Wye$ $(\triangle - Y)$ transformation: takes a triangle in a graph and replaces it with a new three-valent vertex incident to the three vertices of the triangle. If a triangle with disposals u, v and w in G is changed into an electrically equivalent star graph with disposals $x = \frac{uv+vw+wu}{u}, y = \frac{uv+vw+wu}{v}$ and $z = \frac{uv+vw+wu}{w}$ in G', then $\tau(G') = \frac{(uv+vw+wu)^2}{uvw}\tau(G)$.

In this paper, we derive explicit formulas to the number of spanning trees of three sequences of graphs of the same average degree 5, which we named $A_1^{(n)}, A_2^{(n)}$ and $A_3^{(n)}$ respectively and a sequence of a verage degree 5, named it $A_4^{(n)}$.

§3. **Results and Discussion**

3.1 Number of Spanning Trees in the Sequences of $A_1^{(n)}$ Graph

Consider the sequence of graphs $A_1^{(1)}, A_1^{(2)}, \ldots, A_1^{(n)}$ structured as exhibited in Figure 1. Based on this structure, the number of whole vertices $|V(A_1^{(n)})|$ and edges $|E(A_1^{(n)})|$ are $|V(A_1^{(n)})| = 6n-3$ and $|E(A_1^{(n)})| = 15n-12, n = 1, 2, \cdots$. The average degree of $A_1^{(n)}$ is in the huge n limit which is 5.

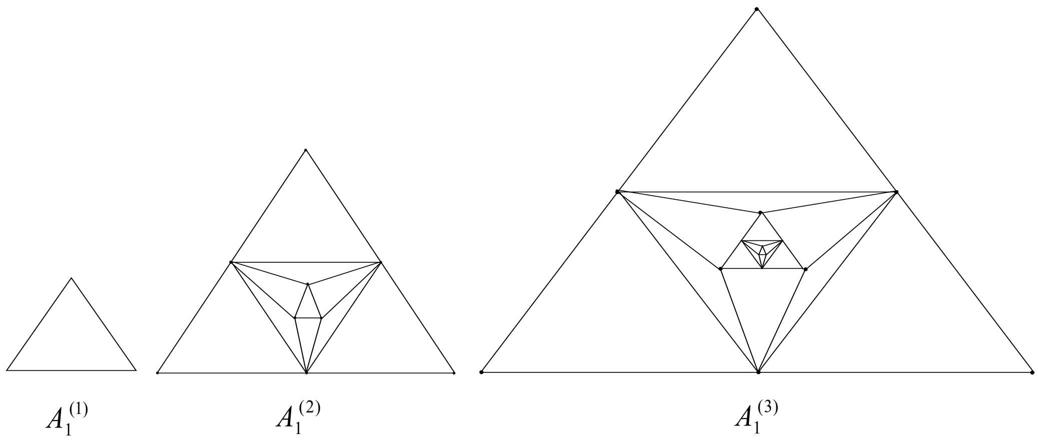


Figure 1. Sample of the family $A_1^{(n)}$

Theorem 3.1 *For $n \geq 1$, the number of spanning trees in the sequence of the graph $A_1^{(n)}$ is specified by*

$$\frac{\left((-714+88\sqrt{42})(8749+1350\sqrt{42})^n+26(13-2\sqrt{42})^n(70329+10852\sqrt{42})\right)^2}{147\left(-13(337+52\sqrt{42})+(29+4\sqrt{42})(337+52\sqrt{42})^n\right)^2}.$$

Proof We employ the electrically equivalent transformation to convert $A_1^{(i)}$ to $A_1^{(i-1)}$. Figures $2-7$ explain the transformation procedure from $A_1^{(2)}$ to $A_1^{(1)}$.

Using the properties given in Section 2, we have the following transformations:

$$\begin{aligned}
\tau(G_1') &= 9s_2\tau(A_1^{(2)}), \quad \tau(G_2') = [\frac{1}{3s_2+2}]3\tau(G_1'), \quad \tau(G_3') = \tau(G_2'), \\
\tau(G_4') &= \left[\frac{3s_2+2}{18s_2}\right]\tau(G_3'), \quad \tau(G_5') = \tau(G_4'), \quad \tau(G_6') = 9\left(\frac{5s_2+3}{3s_2+2}\right)\tau(G_5'), \\
\tau(G_7') &= \left[\frac{3s_2+2}{21s_2+13}\right]^3\tau(G_6'), \quad \tau(G_8') = \tau(G_7'), \quad \tau(G_9') = \left[\frac{21s_2+13}{18(5s_2+3)}\right]\tau(G_8')
\end{aligned}$$

and $\tau(A_1^{(1)}) = \tau(G_9')$.

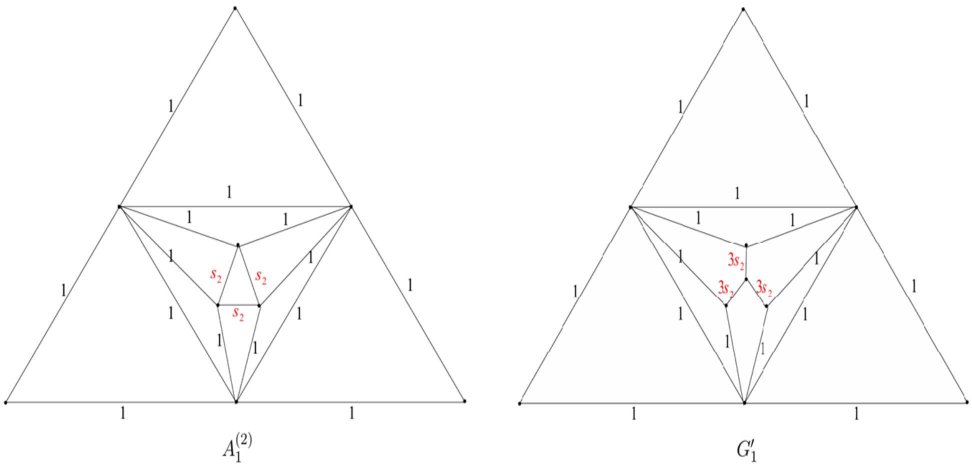


Figure 2.

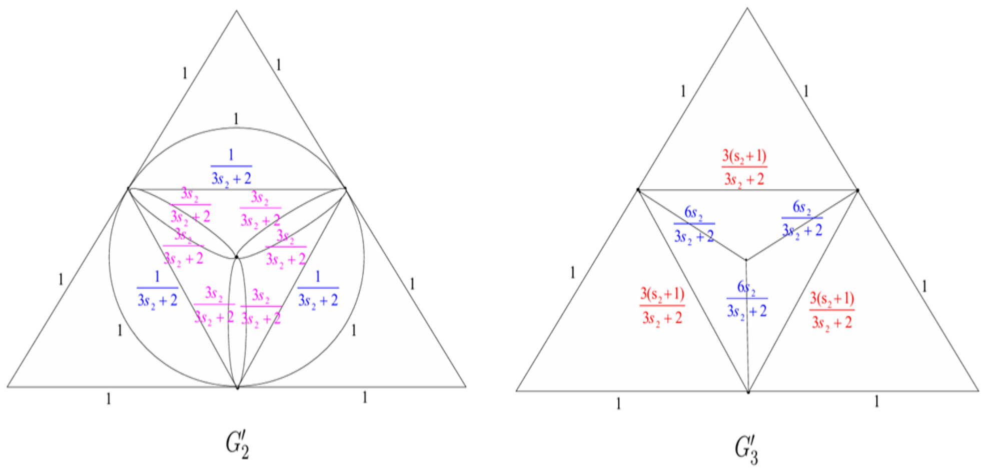


Figure 3.

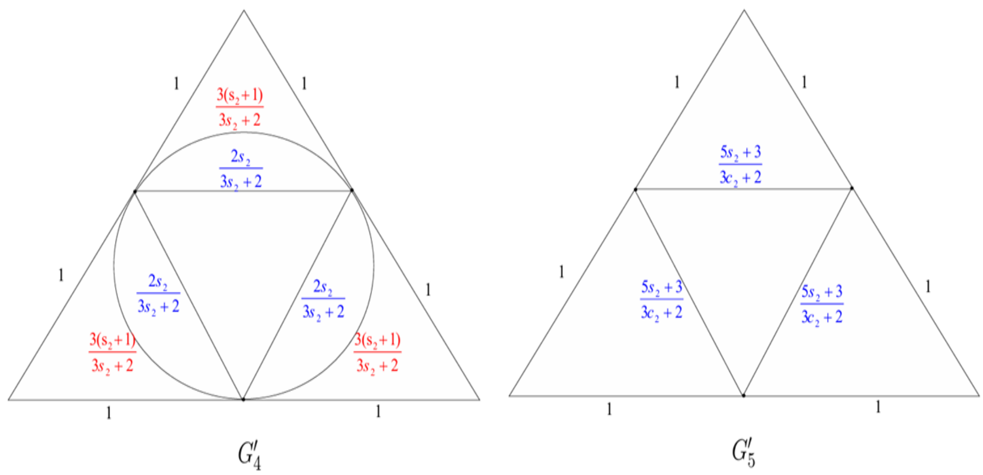


Figure 4.

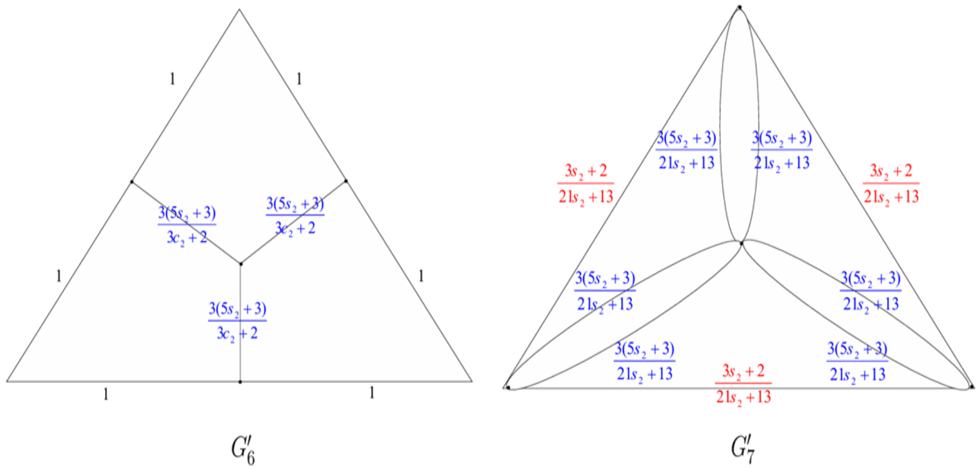


Figure 5.

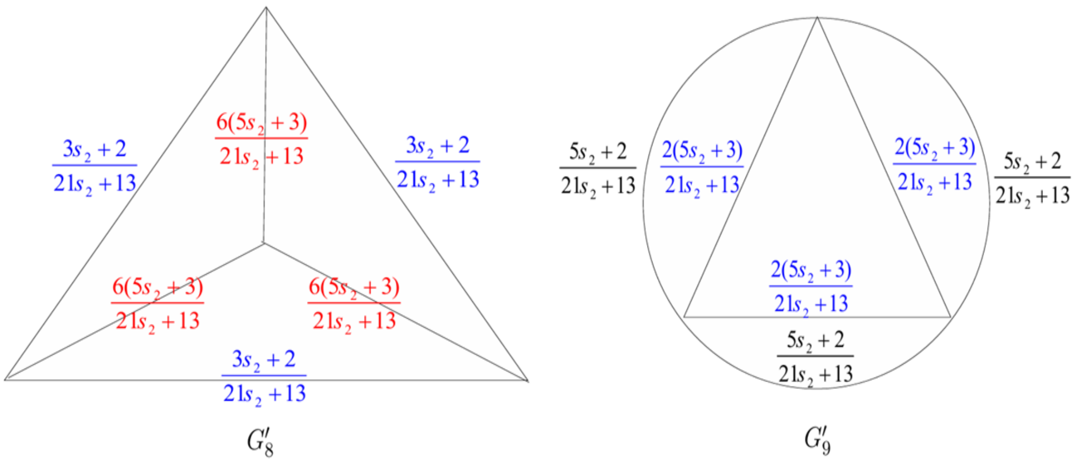


Figure 6.

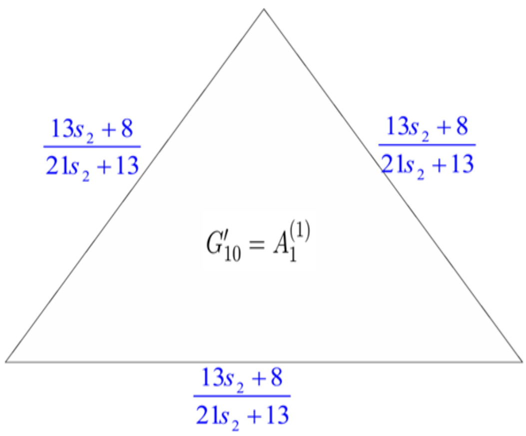


Figure 7.

Combining these ten transformations, we have

$$\tau(A_1^{(2)}) = 4(21s_2+13)^2\tau(A_1^{(1)}). \tag{3.1}$$

Further

$$\tau(A_1^{(n)}) = \prod_{i=2}^{n} 4(21s_i+13)^2\tau\big(A_1^{(1)}\big) = 3\times 4^{n-1}s_1^2\left[\prod_{i=2}^{n}(21s_i+13)\right]^2, \tag{3.2}$$

where, $s_{i-1} = \frac{13s_i+8}{21s_i+13}, i = 2, 3, \cdots, n.$

Its characteristic equation is $21\omega^2 - 8 = 0$ with roots $\omega_1 = \frac{-2\sqrt{42}}{21}$ and $\omega_2 = \frac{2\sqrt{42}}{21}$. Subtracting these two roots into both sides of $s_{i-1} = \frac{13s_i+8}{21s_i+13}$, we get

$$s_{i-1} - \frac{-2\sqrt{42}}{21} = \frac{13s_i+8}{21s_i+13} + \frac{2\sqrt{42}}{21} = \frac{(91+14\sqrt{42})\left(s_i + \frac{2\sqrt{42}}{21}\right)}{7(21s_i+13)}, \tag{3.3}$$

$$s_{i-1} - \frac{2\sqrt{42}}{21} = \frac{13s_i+8}{21s_i+13} - \frac{2\sqrt{42}}{21} = \frac{(91-14\sqrt{42})(s_i - \frac{2\sqrt{42}}{21})}{7(21s_i+13)}. \tag{3.4}$$

Let $y_i = \frac{s_i + \frac{2\sqrt{42}}{21}}{s_i - \frac{2\sqrt{42}}{21}}$. Then, by equations (3.3) and (3.4), we get $y_{i-1} = (337+52\sqrt{42})y_i$ and $y_i = (337+52\sqrt{42})^{n-i}y_n$. Therefore,

$$s_i = \frac{(337+52\sqrt{42})^{n-i}\left(\frac{2\sqrt{42}}{21}\right)y_n + \frac{2\sqrt{42}}{21}}{(337+52\sqrt{42})^{n-i}y_n - 1}.$$

Thus,

$$s_1 = \frac{(337+52\sqrt{42})^{n-1}\left(\frac{2\sqrt{42}}{21}\right)y_n + \frac{2\sqrt{42}}{21}}{(337+52\sqrt{42})^{n-1}y_n - 1}. \tag{3.5}$$

Using the expression $s_{n-1} = \frac{13s_n+8}{21s_n+13}$ and denoting the coefficients of $13s_n+8$ and $21s_n+13$ as h_n and k_n , we have

$$\begin{aligned}
21s_n+13 &= h_0(13s_n+8)+k_0(21s_n+13),\\
21s_{n-1}+13 &= \frac{h_1(13s_n+8)+k_1(21s_n+13)}{h_0(13s_n+8)+k_0(21s_n+13)},\\
21s_{n-2}+13 &= \frac{h_2(13s_n+8)+k_2(21s_n+13)}{h_1(13s_n+8)+k_1(21s_n+13)},\\
\cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
21s_{n-i}+13 &= \frac{h_i(13s_n+8)+k_i(21s_n+13)}{h_{i-1}(13s_n+8)+k_{i-1}(21s_n+13)}, \qquad (3.6)\\
21s_{n-(i+1)}+13 &= \frac{h_{i+1}(13s_n+8)+k_{i+1}(21s_n+13)}{h_i(13s_n+8)+k_i(21s_n+13)}, \qquad (3.7)\\
\cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
21s_2+13 &= \frac{h_{n-2}(13s_n+8)+k_{n-2}(21s_n+13)}{h_{n-3}(13s_n+8)+k_{n-3}(21s_n+13)}.
\end{aligned}$$

Thus, we obtain

$$\tau(A_1^{(n)}) = 3 \times 4^{n-1} s_1^2 [h_{n-2}(13s_n + 8) + k_{n-2}(21s_n + 13)]^2, \tag{3.8}$$

where $h_0 = 0, k_0 = 1$ and $h_1 = 21, k_1 = 13$.

By the expression $s_{n-1} = \frac{13s_n+8}{21s_n+13}$ and using equations (3.6) and (3.7), we have

$$h_{i+1} = 26h_i - h_{i-1}; k_{i+1} = 26k_i - k_{i-1}. \tag{3.9}$$

The characteristic equation of equation (3.9) is $\lambda^2 - 26\lambda + 1 = 0$ with roots $\lambda_1 = 13 + 2\sqrt{42}$ and $\lambda_2 = 13 - 2\sqrt{42}$. The general solution of equation (3.9) is $h_i = a_1\lambda_1^i + a_2\lambda_2^i; k_i = b_1\lambda_1^i + b_2\lambda_2^i$. Using the initial conditions $h_0 = 0, k_0 = 1$ and $h_1 = 21, k_1 = 13$ yields

$$\begin{aligned} h_i &= \frac{\sqrt{42}}{8}(13 + 2\sqrt{42})^i - \frac{\sqrt{42}}{8}(13 - 2\sqrt{42})^i; \\ k_i &= \frac{1}{2}(13 + 2\sqrt{42})^i + \frac{1}{2}(13 - 2\sqrt{42})^i. \end{aligned} \tag{3.10}$$

If $s_n = 1$, it means that $A_1^{(n)}$ is without any electrically equivalent transformation. Plugging equation (3.10) into equation (3.8), we have

$$\begin{aligned} \tau(A_1^{(n)}) &= 3 \times 4^{n-1} s_1^2 \Bigg[\left(\frac{136 + 21\sqrt{42}}{8} \right) (13 + 2\sqrt{42})^{n-2} \\ &\quad + \left(\frac{136 - 21\sqrt{42}}{8} \right) (13 - 2\sqrt{42})^{n-2} \Bigg]^2, \ n \geq 2. \end{aligned} \tag{3.11}$$

When $n = 1, \tau(A_1^{(1)}) = 3$ which satisfies equation (3.11). Therefore, the number of spanning trees in the sequence of the graph $A_1^{(n)}$ is given by

$$\begin{aligned} \tau(A_1^{(n)}) &= 3 \times 4^{n-1} s_1^2 \Bigg[\left(\frac{136 + 21\sqrt{42}}{8} \right) (13 + 2\sqrt{42})^{n-2} \\ &\quad + \left(\frac{136 - 21\sqrt{42}}{8} \right) (13 - 2\sqrt{42})^{n-2} \Bigg]^2, \ n \geq 1, \end{aligned} \tag{3.12}$$

where

$$s_1 = \frac{2(337 + 52\sqrt{42})^{n-1}(168 + 29\sqrt{42}) + 26\sqrt{42}}{21(337 + 52\sqrt{42})^{n-1}(29 + 4\sqrt{42}) - 273}, \quad n \geq 1. \tag{3.13}$$

Inserting equation (3.13) into equation (3.12) we obtain the desired result. □

3.2 Number of Spanning Trees in the Sequences of $A_2^{(n)}$ Graph

Consider the sequence of graphs $A_2^{(1)}, A_2^{(2)}, \cdots, A_2^{(3)}$ structured as exhibited in Figure 8. Based on this structure, the number of whole vertices $|V(A_2^{(n)})|$ and edges $|E(A_2^{(n)})|$ are $|V(A_2^{(n)})| = 6n - 3$ and $|E(A_2^{(n)})| = 15n - 12, n = 1, 2, \cdots$ The average degree of $A_2^{(n)}$ is in the huge n limit which is 5.

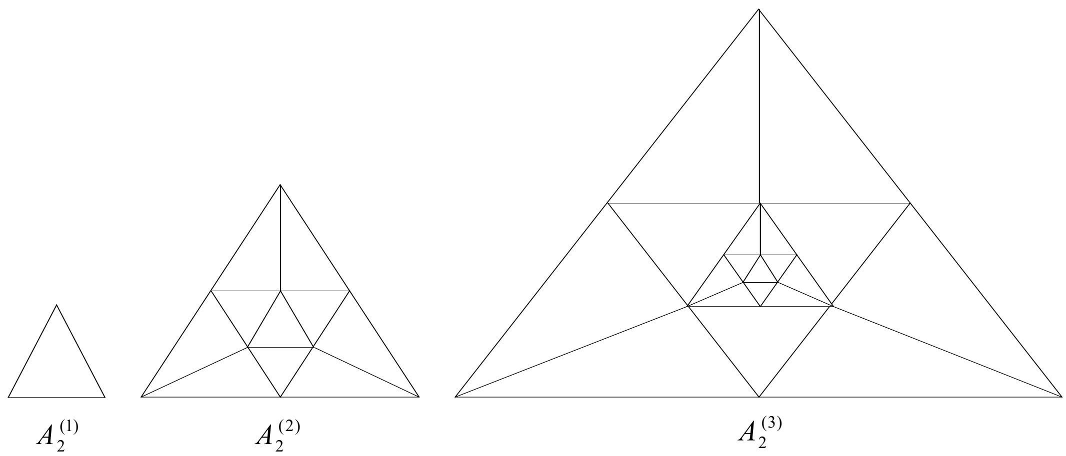


Figure 8. Sample of the family $A_2^{(n)}$

Theorem 3.2 *For $n \geq 1$, the number of spanning trees in the sequence of $A_2^{(n)}$ graph is specified by*

$$\frac{2^{-7+3n}(217+88\sqrt{6})^{2n}\left(-50(11+4\sqrt{6})^{n}(33+13\sqrt{6})+\left(\frac{1}{25}(4499-1836\sqrt{6})\right)^{n}(4962+2018\sqrt{6})\right)^{2}}{1875\left(25^{n}(217+88\sqrt{6})-25(5+2\sqrt{6})(217+88\sqrt{6})^{n}\right)^{2}}.$$

Proof We employ the electrically equivalent transformation to convert $A_2^{(i)}$ to $A_2^{(i-1)}$. Figures $9-13$ explains the transformation procedure from $A_2^{(2)}$ to $A_2^{(1)}$.

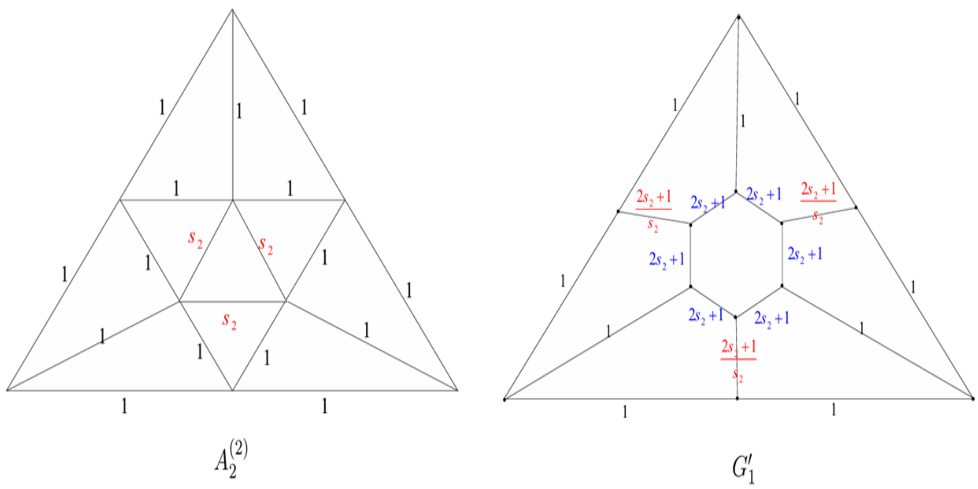


Figure 9.

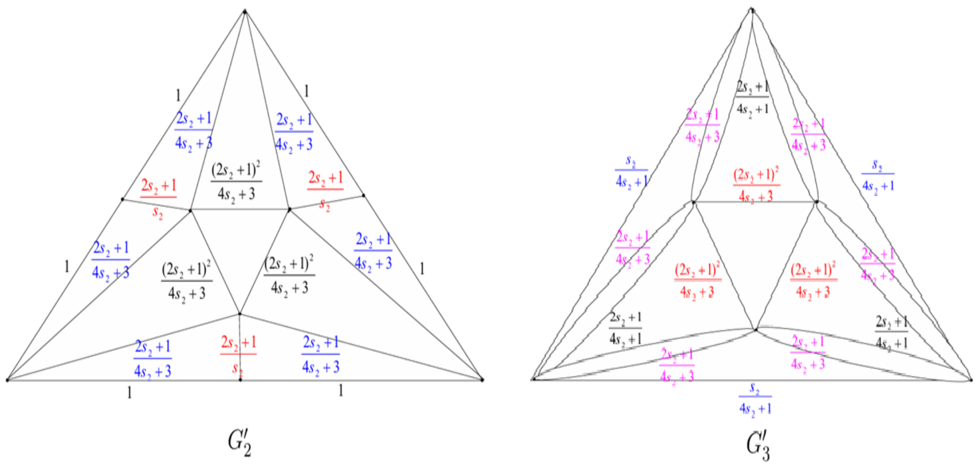


Figure 10.

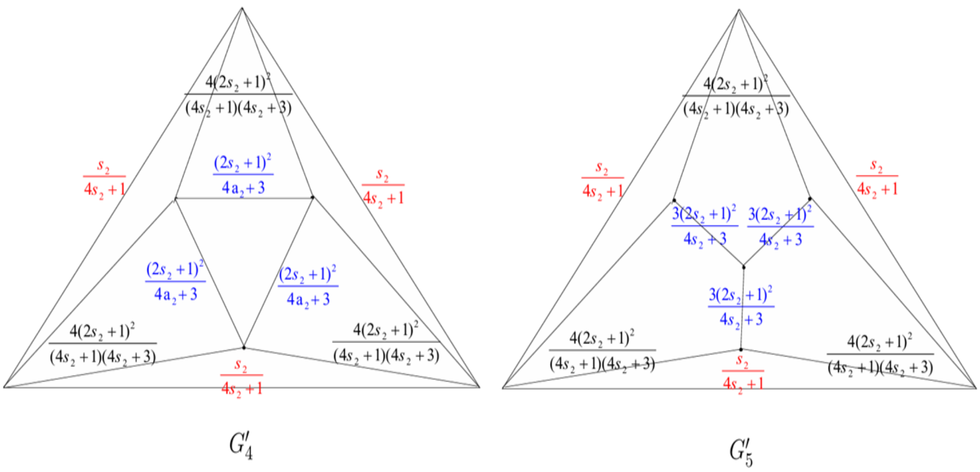


Figure 11.

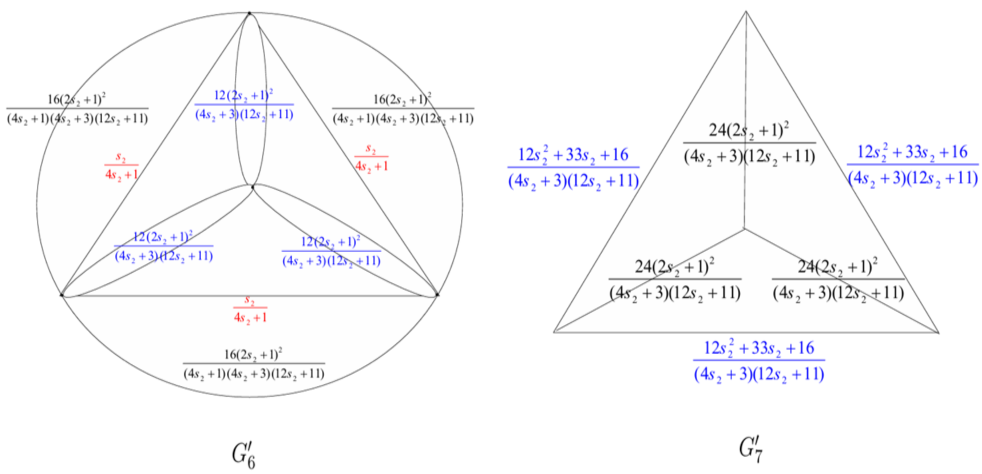


Figure 12.

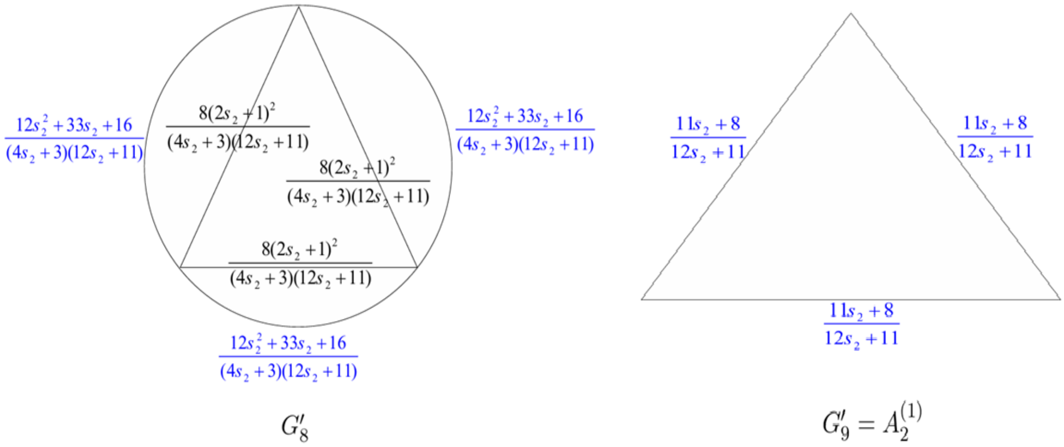


Figure 13.

Using the properties given in Section 2, we have the following transformations:

$$\begin{aligned}
\tau(G'_1) &= \left[\frac{(2s_2+1)^2}{s_2}\right]^3 \tau(A_1^{(2))}, \quad \tau(G'_2) = [14s_2+3]\tau(G'_1), \quad \tau(G'_3) = \left[\frac{s_2}{4s_2+1}\right]^3 \tau(G'_2),\\
\tau(G'_4) &= \tau(G'_3), \quad \tau(G'_5) = 9\left[\frac{(2s_2+1)^2}{4s_2+3}\right]\tau(G'_4), \quad \tau(G'_6) = \left[\frac{(4s_2+1)(4s_2+3)}{(2s_2+1)^2(12s_2+11)}\right]\tau(G'_5),\\
\tau(G'_7) &= \tau(G'_6), \quad \tau(G'_8) = \left[\frac{(4s_2+3)(12s_2+11)72}{(2s_2+1)^2}\right]\tau(G'_7)
\end{aligned}$$

and $\tau(A_2^{(1)}) = \tau(G'_8)$. Combining these ten transformations, we have

$$\tau(A_2^{(2)}) = 8(12s_2+11)^2\tau(A_2^{(1)}). \tag{3.14}$$

Further

$$\tau(A_2^{(n)}) = \prod_{i=2}^{n} 8(12s_2+11)^2\tau(A_2^{(1)}) = 3\times 8^{n-1}s_1^2\left[\prod_{i=2}^{n}(12s_i+11)\right]2, \tag{3.15}$$

where $s_{i-1} = \frac{11s_i+8}{12s_i+11}, i = 2, 3, \cdots, n$.

Its characteristic equation is $3\omega^2 - 2 = 0$ with roots $\omega_1 = \frac{-\sqrt{6}}{3}$ and $\omega_2 = \frac{\sqrt{6}}{3}$. Subtracting these two roots into both sides of $s_{i-1} = \frac{11s_i+8}{12s_i+11}$, we get

$$s_{i-1} - \frac{-\sqrt{6}}{3} = \frac{11s_i+8}{12s_i+11} + \frac{\sqrt{6}}{3} = \frac{(11+4\sqrt{6})(s_i+\frac{\sqrt{6}}{3})}{(12s_i+11)}, \tag{3.16}$$

$$s_{i-1} - \frac{\sqrt{6}}{3} = \frac{11s_i+8}{12s_i+11} - \frac{\sqrt{6}}{3} = \frac{(11-4\sqrt{6})(s_i-\frac{\sqrt{6}}{3})}{(12s_i+11)}. \tag{3.17}$$

Let $y_i = \frac{s_i+\frac{\sqrt{6}}{3}}{s_i-\frac{\sqrt{6}}{3}}$. Then, by equations (3.16) and (3.17), we get $y_{i-1} = (\frac{217+88\sqrt{6}}{25})y_i$ and

$y_i = (\frac{217+88\sqrt{6}}{25})^{n-i} y_n$. Therefore

$$s_i = \frac{\left(\frac{217+88\sqrt{6}}{25}\right)^{n-i}\left(\frac{\sqrt{6}}{3}\right) y_n + \frac{\sqrt{6}}{3}}{\left(\frac{217+88\sqrt{6}}{25}\right)^{n-i} y_n - 1}.$$

Thus,

$$s_1 = \frac{\left(\frac{217+88\sqrt{6}}{25}\right)^{n-1}\left(\frac{\sqrt{6}}{3}\right) y_n + \frac{\sqrt{6}}{3}}{\left(\frac{217+88\sqrt{6}}{25}\right)^{n-1} y_n - 1}. \tag{3.18}$$

Using the expression $s_{n-1} = \frac{11s_n+8}{12s_n+11}$ and denoting the coefficients of $11s_n+8$ and $12s_n+11$ as h_n and k_n , we have

$$\begin{aligned}
12s_n + 11 &= h_0(11s_n+8) + k_0(12s_n+11),\\
12s_{n-1} + 11 &= \frac{h_1(11s_n+8) + k_1(12s_n+11)}{h_0(11s_n+8) + k_0(12s_n+11)},\\
12s_{n-2} + 11 &= \frac{h_2(11s_n+8) + k_2(12s_n+11)}{h_1(11s_n+8) + k_1(12s_n+11)},\\
\cdots \quad & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
12s_{n-i} + 11 &= \frac{h_i(11s_n+8) + k_i(12s_n+11)}{h_{i-1}(11s_n+8) + k_{i-1}(12s_n+11)}, \qquad (3.19)\\
12s_{n-(i+1)} + 11 &= \frac{h_{i+1}(11s_n+8) + k_{i+1}(12s_n+11)}{h_i(11s_n+8) + k_i(12s_n+11)}, \qquad (3.20)\\
\cdots \quad & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
12s_2 + 11 &= \frac{h_{n-2}(11s_n+8) + k_{n-2}(12s_n+11)}{h_{n-3}(11s_n+8) + k_{n-3}(12s_n+11)}.
\end{aligned}$$

Thus we obtain,

$$\tau(A_2^{(n)}) = 3 \times 8^{n-1} s_1^2 [h_{n-2}(11s_n+8) + k_{n-2}(12s_n+11)]^2, \tag{3.21}$$

where $h_0 = 0, k_0 = 1$ and $h_1 = 12, k_1 = 11$.

By the expression $s_{n-1} = \frac{11s_n+8}{12s_n+11}$ and using equations (3.19) and (3.20), we have

$$h_{i+1} = 22h_i - 25h_{i-1}; k_{i+1} = 22k_i - 25k_{i-1}. \tag{3.22}$$

The characteristic equation of Equation (3.22) is $\lambda^2 - 22\lambda + 25 = 0$ with roots $\lambda_1 = 11+4\sqrt{6}$ and $\lambda_2 = 11-4\sqrt{6}$. The general solution of equation (3.22) is $h_i = a_1\lambda_1^i + a_2\lambda_2^i; k_i = b_1\lambda_1^i + b_2\lambda_2^i$. Using the initial conditions $h_0 = 0, k_0 = 1$ and $h_1 = 12, k_1 = 11$, yields

$$\begin{aligned}
h_i &= \frac{\sqrt{6}}{4}(11+4\sqrt{6})^i - \frac{\sqrt{6}}{4}(11-4\sqrt{6})^i;\\
k_i &= \frac{1}{2}(11+4\sqrt{6})^i + \frac{1}{2}(11-4\sqrt{6})^i. \qquad (3.23)
\end{aligned}$$

If $s_n = 1$, it means that $A_2^{(n)}$ is without any electrically equivalent transformation. Plugging equation (3.23) into equation (3.21), we have

$$\tau(A_2^{(n)}) = 3\times 8^{n-1} s_1^2 \left[\left(\frac{46+19\sqrt{6}}{4}\right)(11+4\sqrt{6})^{n-2} + \left(\frac{46-19\sqrt{6}}{4}\right)(11-4\sqrt{6})^{n-2}\right]^2, n \geq 2. \tag{3.24}$$

When $n = 1, \tau(A_2^{(1)}) = 3$ which satisfies equation (3.24). Therefore, the number of spanning trees in the sequence of the graph $A_2^{(n)}$ is given by

$$\tau(A_1^{(n)}) = 3\times 8^{n-1} s_1^2 \left[\left(\frac{46+19\sqrt{6}}{4}\right)(11+4\sqrt{6})^{n-2} + \left(\frac{46-19\sqrt{6}}{4}\right)(11-4\sqrt{6})^{n-2}\right]^2, \ n \geq 1, \tag{3.25}$$

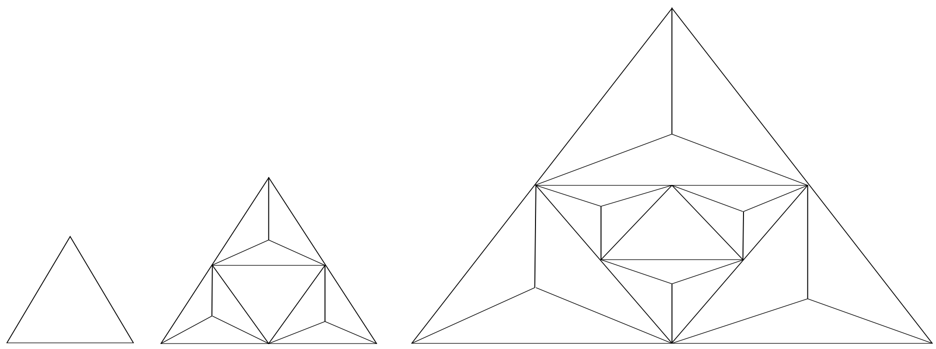
where,

$$s_1 = \frac{\left(\frac{217+88\sqrt{6}}{25}\right)^{n-1}(12+5\sqrt{6})+\sqrt{6}}{3\left(\frac{217+88\sqrt{6}}{25}\right)^{n-1}(5+2\sqrt{6})-3}, \quad n \geq 1. \tag{3.26}$$

Inserting equation (3.26) into equation (3.25) we obtain the desired result. □

3.3 Number of Spanning Trees in the Sequences of $A_3^{(n)}$ Graph

Consider the sequence of graphs $A_3^{(1)}, A_3^{(2)}, \cdots, A_3^{(n)}$ structured as exhibited in Figure . Based on this structure, the number of whole vertices $|V(A_3^{(n)})|$ and edges $|E(A_3^{(n)})|$ are $|V(A_3^{(n)})| = 6n-3$ and $|E(A_3^{(n)})| = 15n-12, n = 1, 2, \cdots$. The average degree of $A_3^{(n)}$ is in the huge n limit, which is 5.



$A_3^{(1)}$ $A_3^{(1)}$ $A_3^{(1)}$

Figure 14. Sample of the family $A_3^{(n)}$

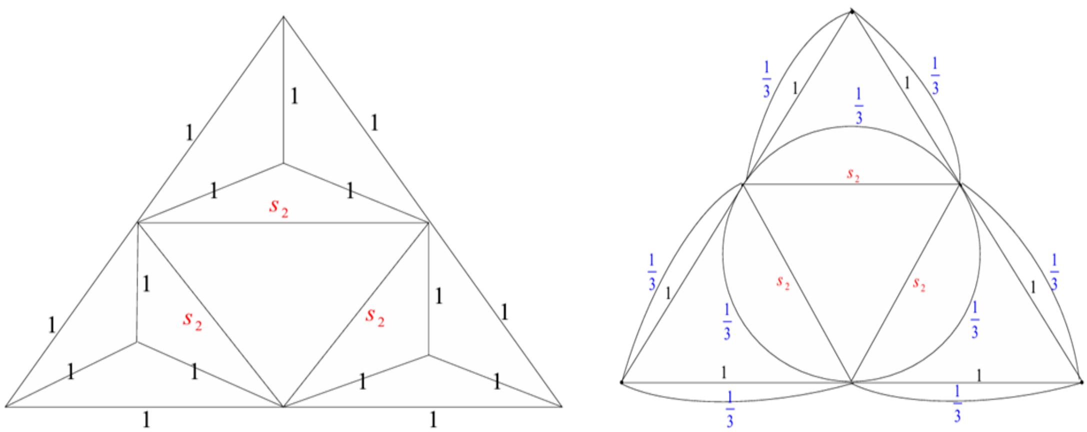
Theorem 3.3 *For $n \geq 1$, the number of spanning trees in the sequence of $A_3^{(n)}$ is specified by*

$$\frac{M_1(n)}{M_2(n)},$$

where

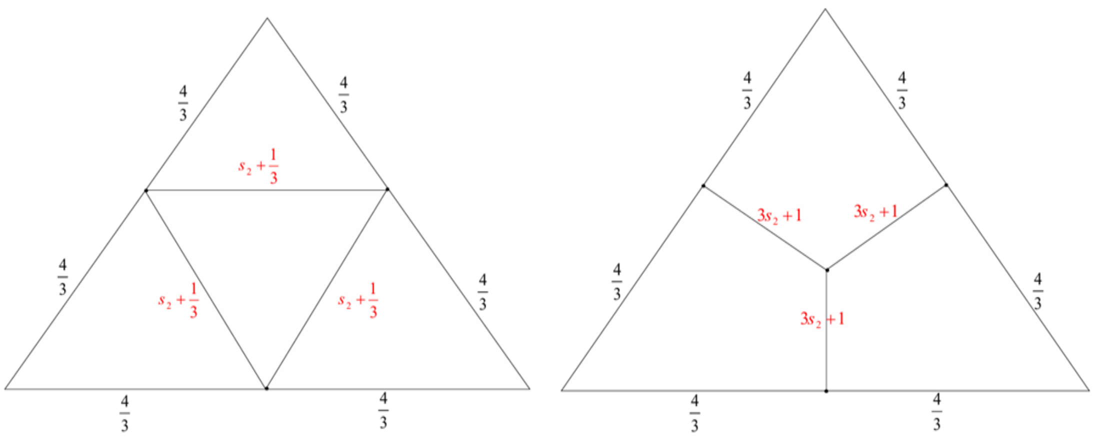
$$
\begin{aligned}
M_1(n) &= 2^{-13-9n}(19-3\sqrt{33})^{2n}\left(33-17\sqrt{33}+\left(\frac{1}{32}(329-57\sqrt{33})\right)^{n}(33+17\sqrt{33})\right)^{2} \\
&\quad \times\left(4(95+17\sqrt{33})(329+57\sqrt{33})^{n}+32^{n}(1105+193\sqrt{33})\right)^{2}, \\
M_2(n) &= 3267\left(-4(41+7\sqrt{33})+32^{n}(329+57\sqrt{33})^{1-n}\right)^{2}.
\end{aligned}
$$

Proof We employ the electrically equivalent transformation to convert $A_3^{(i)}$ to $A_3^{(i-1)}$ Figures $15-18$ explains the transformation procedure from $A_3^{(2)}$ to $A_3^{(1)}$.



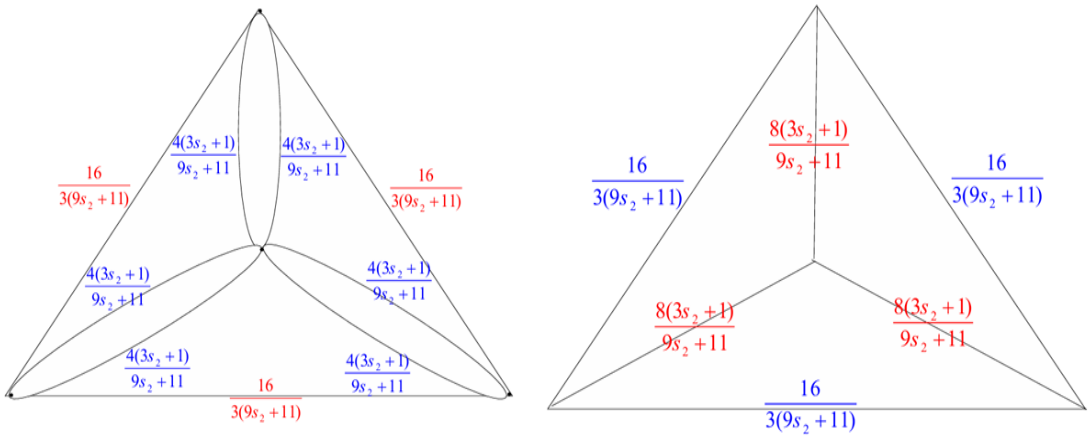
$A_3^{(2)}$ G_1'

Figure 15.



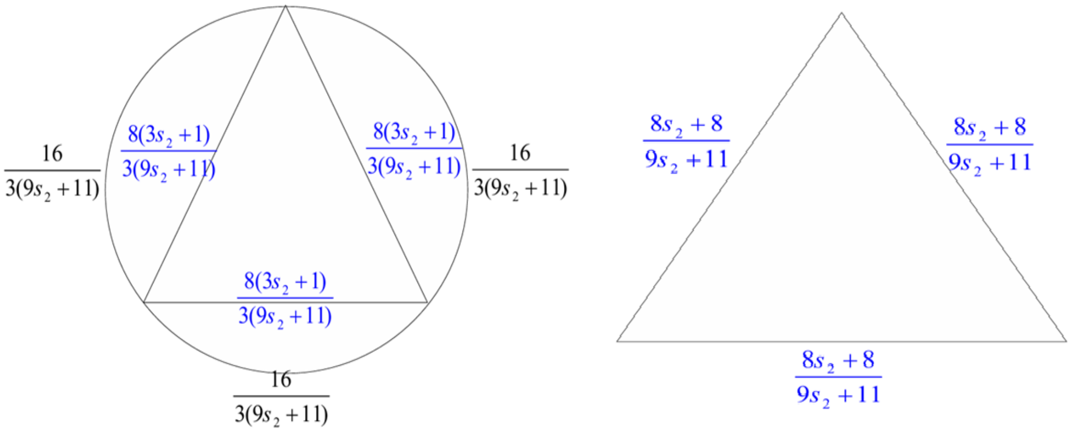
G_2' G_3'

Figure 16.



G'_4 G'_5

Figure 17.



G'_6 $G'_7 = A_3^{(1)}$

Figure 18.

Using the properties given in Section 2, we have the following transformations

$$\begin{aligned}
\tau(G'_1) &= \left[\frac{1}{2}\right]^3 \tau(A_3^{(2)}), \quad \tau(G'_2) = \tau(G'_1), \quad \tau(G'_3) = [9s_2+3]\tau(G'_2), \\
\tau(G'_4) &= [39s_2+11]^3\,\tau(G'_3), \quad \tau(G'_5) = \tau(G'_4), \quad \tau(G'_6) = \left[\frac{(9s_2+1)}{24(3s_2+1)}\right]\tau(G'_5)
\end{aligned}$$

and $\tau(A_3^{(1)}) = \tau(G'_6)$. Combining these transformations, we have

$$\tau(A_3^{(2)}) = 8(9s_2+11)^2\tau(A_3^{(1)}). \tag{3.27}$$

Further,

$$\tau(A_3^{(n)}) = \prod_{i=2}^{n} 8(9s_2+11)^2\tau(A_3^{(1)}) = 3{\times}8^{n-1}s_1^2[i=2n(9si+11)]2, \tag{3.28}$$

where $s_{i-1} = \frac{8s_i+8}{9s_i+11}, i = 2, 3, \cdots, n.$

Its characteristic equation is $9\omega^2 + 3\omega - 8 = 0$ with roots $\omega_1 = \frac{-1-\sqrt{33}}{6}$ and $\omega_2 = \frac{-1+\sqrt{33}}{6}$. Subtracting these two roots into both sides of $s_{i-1} = \frac{8s_i+8}{9s_i+11}$, we get

$$s_{i-1} - \left(\frac{-1-\sqrt{33}}{6}\right) = \frac{8s_i+8}{9s_i+11} + \frac{1+\sqrt{33}}{6} = \frac{(19+3\sqrt{33})\left(s_i + \frac{1+\sqrt{33}}{6}\right)}{2(9s_i+11)}, \tag{3.29}$$

$$s_{i-1} - \left(\frac{-1+\sqrt{33}}{6}\right) = \frac{8s_i+8}{9s_i+11} + \frac{1-\sqrt{33}}{6} = \frac{(19-3\sqrt{33})\left(s_i + \frac{1-\sqrt{33}}{6}\right)}{2(9s_i+11)}. \tag{3.30}$$

Let $y_i = \frac{s_i + \frac{1+\sqrt{33}}{6}}{s_i + \frac{1-\sqrt{33}}{6}}$. Then, by equations (3.29) and (3.30), we get $y_{i-1} = (\frac{329+57\sqrt{33}}{32})y_i$ and $y_i = (\frac{329+57\sqrt{33}}{32})^{n-i}y_n$. Therefore

$$s_i = \frac{\left(\frac{329+57\sqrt{33}}{32}\right)^{n-i}\left(\frac{-1+\sqrt{33}}{6}\right)y_n + \frac{1+\sqrt{33}}{6}}{\left(\frac{329+57\sqrt{33}}{32}\right)^{n-i}y_n - 1}.$$

Thus,

$$s_1 = \frac{\left(\frac{329+57\sqrt{33}}{32}\right)^{n-1}\left(\frac{-1+\sqrt{33}}{6}\right)y_n + \frac{1+\sqrt{33}}{6}}{\left(\frac{329+57\sqrt{33}}{32}\right)^{n-1}y_n - 1}. \tag{3.31}$$

Using the expression $s_{n-1} = \frac{8s_n+8}{9s_n+11}$ and denoting the coefficients of $8s_n+8$ and $9s_n+11$ as h_n and k_n, we have

$$\begin{aligned}
9s_n+11 &= h_0(8s_n+8)+k_0(9s_n+11),\\
9s_{n-1}+11 &= \frac{h_1(8s_n+8)+k_1(9s_n+11)}{h_0(8s_n+8)+k_0(9s_n+11)},\\
9s_{n-2}+11 &= \frac{h_2(8s_n+8)+k_2(9s_n+11)}{h_1(8s_n+8)+k_1(9s_n+11)},\\
\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
9s_{n-i}+11 &= \frac{h_i(8s_n+8)+k_i(9s_n+11)}{h_{i-1}(8s_n+8)+k_{i-1}(9s_n+11)}, && (3.32)\\
9s_{n-(i+1)}+11 &= \frac{h_{i+1}(8s_n+8)+k_{i+1}(9s_n+11)}{h_i(8s_n+8)+k_i(9s_n+11)}, && (3.33)\\
\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
9s_2+11 &= \frac{h_{n-2}(8s_n+8)+k_{n-2}(9s_n+11)}{h_{n-3}(8s_n+8)+k_{n-3}(9s_n+11)}.
\end{aligned}$$

Thus, we obtain

$$\tau(A_3^{(n)}) = 3 \times 8^{n-1} s_1^2 [h_{n-2}(8s_n + 8) + k_{n-2}(9s_n + 11)]^2, \tag{3.34}$$

where $h_0 = 0, k_0 = 1$ and $h_1 = 9, k_1 = 11$. By the expression $s_{n-1} = \frac{8s_n+8}{9s_n+11}$ and using equations (3.32) and (3.33), we have

$$h_{i+1} = 19h_i - 16h_{i-1}; k_{i+1} = 19k_i - 16k_{i-1}. \tag{3.35}$$

The characteristic equation of equation (3.35) is $\lambda^2 - 19\lambda + 16 = 0$ with roots $\lambda_1 = \frac{19+3\sqrt{33}}{2}$ and $\lambda_2 = \frac{19-3\sqrt{33}}{2}$. The general solution of equation (3.35) is $h_i = a_1\lambda_1^i + a_2\lambda_2^i; k_i = b_1\lambda_1^i + b_2\lambda_2^i$. Using the initial conditions $h_0 = 0, k_0 = 1$ and $h_1 = 9, k_1 = 11$, yields

$$\begin{aligned} h_i &= \frac{\sqrt{33}}{11}\left[\left(\frac{19+3\sqrt{33}}{2}\right)^i - \left(\frac{19-3\sqrt{33}}{2}\right)^i\right]; \\ k_i &= \left(\frac{33+\sqrt{33}}{66}\right)\left(\frac{19+3\sqrt{33}}{2}\right)^i + \left(\frac{33-\sqrt{33}}{66}\right)\left(\frac{19-3\sqrt{33}}{2}\right)^i. \end{aligned} \tag{3.36}$$

If $s_n = 1$, it means that $A_3^{(n)}$ is without any electrically equivalent transformation. Plugging equation (3.36) into equation (3.34), we have

$$\begin{aligned} \tau(A_3^{(n)}) &= 3 \times 8^{n-1} s_1^2 \left[\left(\frac{330+58\sqrt{33}}{33}\right)\left(\frac{19+3\sqrt{33}}{2}\right)^{n-2}\right. \\ &\quad \left. + \left(\frac{330-58\sqrt{33}}{33}\right)\left(\frac{19-3\sqrt{33}}{2}\right)^{n-2}\right]^2, \quad n \geq 2. \end{aligned} \tag{3.37}$$

When $n = 1, \tau(A_3^{(1)}) = 3$ which satisfies equation (3.37). Therefore, the number of spanning trees in the sequence of the graph $A_3^{(n)}$ is given by

$$\begin{aligned} \tau(A_1^{(n)}) &= 3 \times 8^{n-1} s_1^2 \left[\left(330+58\sqrt{33}33\right)\left(\frac{19+3\sqrt{33}}{2}\right)^{n-2}\right. \\ &\quad \left. + \left(\frac{330-58\sqrt{33}}{33}\right)\left(\frac{19-3\sqrt{33}}{2}\right)^{n-2}\right]^2, \quad n \geq 1, \end{aligned} \tag{3.38}$$

where,

$$s_1 = \frac{(\frac{329+57\sqrt{33}}{32})^{n-1}(95+17\sqrt{33}) + 4(1+\sqrt{33})}{3(\frac{329+57\sqrt{33}}{32})^{n-1}(41+7\sqrt{33}) - 24}, \quad n \geq 1. \tag{3.39}$$

Inserting equation (3.37) into equation (3.38) we obtain the desired result. □

3.4 Number of Spanning Trees in the Sequences of $A_4^{(n)}$ Graph

Consider the sequence of graphs $A_4^{(1)}, A_4^{(2)}, \cdots, A_4^{(n)}$ structured as exhibited in Figure 19. Based on this structure, the number of whole vertices $|V(A_4^{(n)})|$ and edges $|E(A_4^{(n)})|$ are $|V(A_4^{(n)})| = 6n-3$ and $|E(A_4^{(n)})| = 12n-9, n = 1, 2, \cdots$ The average degree of $A_4^{(n)}$ is in the huge n limit which is 4.

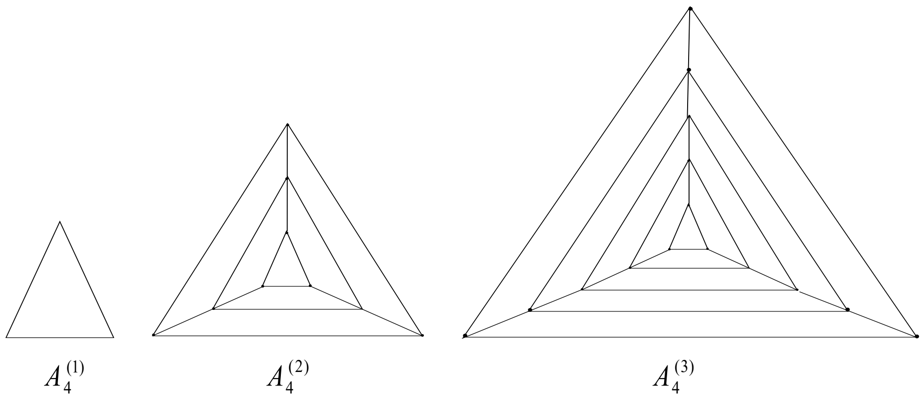


Figure 19 Sample of the family $A_4^{(n)}$

Theorem 3.4 *For $n \geq 1$, the number of spanning trees in the sequence of $A_3^{(n)}$ is specified by*

$$\frac{4^{-n}\left((46+10\sqrt{21})^n(399+87\sqrt{21})-(46-10\sqrt{21})^n(11571+2525\sqrt{21})+2\sqrt{21}(24196+5280\sqrt{21})^n\right)^2}{147\left(2^n(527+115\sqrt{21})+(5+\sqrt{21})(527+115\sqrt{21})^n\right)^2}.$$

Proof We employ the electrically equivalent transformation to convert $A_4^{(i)} to A_4^{(i-1)}$. Figures $20-24$ explains the transformation procedure from $A_4^{(2)}$ to $A_4^{(1)}$.

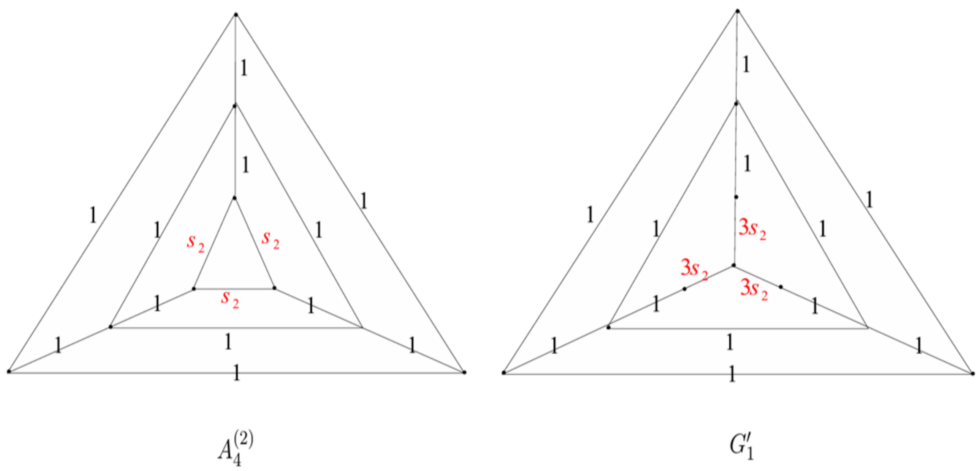


Figure 20.

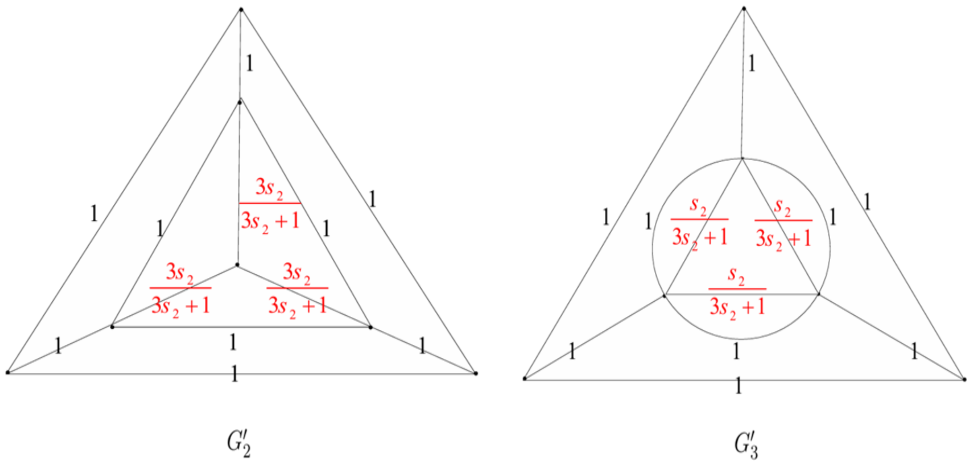


Figure 21.

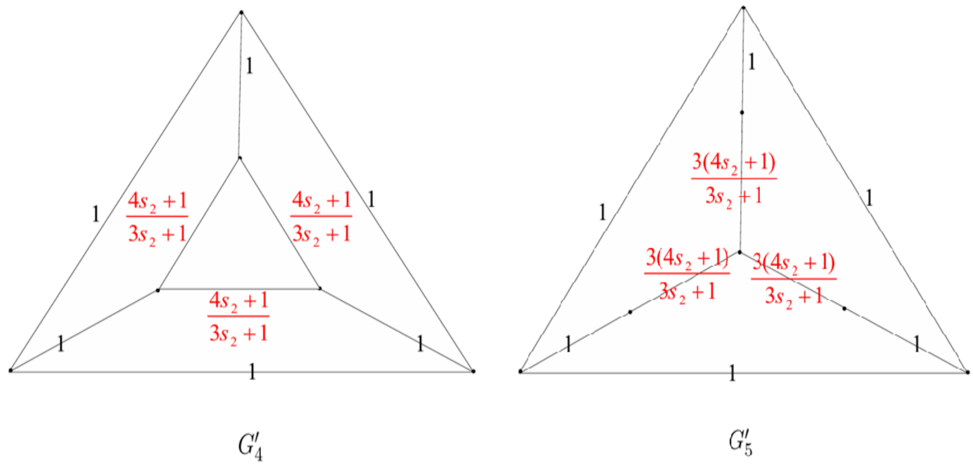


Figure 22.

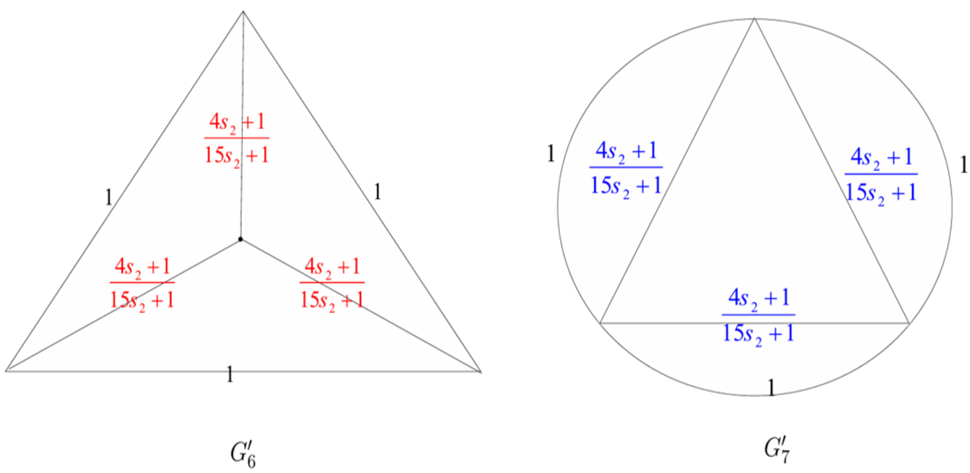


Figure 23.

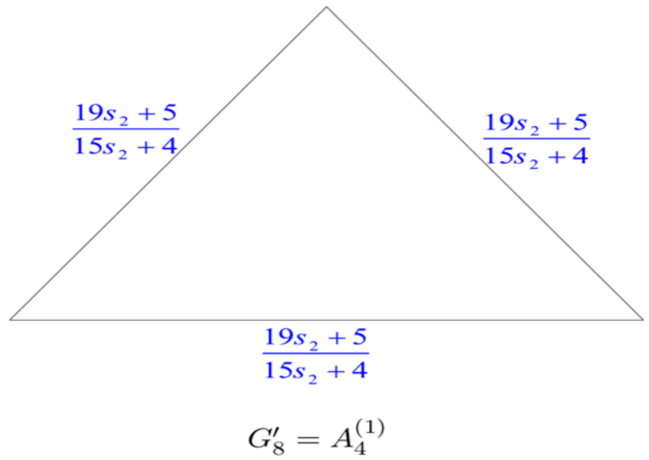


Figure 24.

Using the properties given in Section 2, we have the following transformations

$$\begin{aligned}
\tau(G'_1) &= 9s_2\tau(A_4^{(2)}), \quad \tau(G'_2) = \left[\frac{1}{3s_2+1}\right]^3 \tau(G'_1), \quad \tau(G'_3) = \left[\frac{3s_2+1}{9s_2}\right]\tau(G'_2),\\
\tau(G'_4) &= \tau(G'_3), \quad \tau(G'_5) = 9\left[\frac{4s_2{+}1}{3s_2+1}\right]\tau(G'_4), \quad \tau(G'_6) = \left[\frac{3s_2+1}{15s_2+4}\right]^3\tau(G'_5),\\
\tau(G'_7) &= \left[\frac{15s_2+4}{9(4s_2+1)}\right]\tau(G'_6)
\end{aligned}$$

and $\tau(A_4^{(1)}) = \tau(G'_7)$.

$$\tau(A_4^{(2)}) = (15s_2+4)^2\tau(A_4^{(1)}). \tag{3.40}$$

Further,

$$\tau(A_4^{(n)}) = \prod_{i=2}^{n}(15s_2+4)^2\tau(A_4^{(1)}) = 3\times s_1^2\left[\prod_{i=2}^{n}(15s_i+4)\right]^2, \tag{3.41}$$

where, $s_{i-1} = \frac{19s_i+5}{15s_i+4}, i = 2, 3, \cdots, n$.

Its characteristic equation is $3\omega^2 - 3\omega - 1 = 0$ with roots $\omega_1 = \frac{3-\sqrt{21}}{6}$ and $\omega_2 = \frac{3+\sqrt{21}}{6}$. Subtracting these two roots into both sides of $s_{i-1} = \frac{19s_i+5}{15s_i+4}$, we get

$$s_{i-1} - \frac{3-\sqrt{21}}{6} = \frac{19s_i+5}{15s_i+4} - \frac{3-\sqrt{21}}{6} = \frac{(23+5\sqrt{21})(s_i - \frac{3-\sqrt{21}}{6})}{2(15s_i+4)}, \tag{3.42}$$

$$s_{i-1} - \frac{3+\sqrt{21}}{6} = \frac{19s_i+5}{15s_i+4} - \frac{3+\sqrt{21}}{6} = \frac{(23-5\sqrt{21})(s_i - \frac{3+\sqrt{21}}{6})}{2(15s_i+4)}. \tag{3.43}$$

Let $y_i = \frac{s_i - \frac{3-\sqrt{21}}{6}}{s_i - \frac{3+\sqrt{21}}{6}}$. Then, by equations (3.42) and (3.43), we get $y_{i-1} = (\frac{527+115\sqrt{21}}{2})y_i$ and

$y_i = (\frac{527+115\sqrt{21}}{2})^{n-i} y_n$. Therefore $s_i = \frac{(\frac{527+115\sqrt{21}}{2})^{n-i}(\frac{3+\sqrt{21}}{6})y_n - \frac{3-\sqrt{21}}{6}}{(\frac{527+115\sqrt{21}}{2})^{n-i} y_n - 1}$. Thus,

$$s_1 = \frac{\left(\frac{527+115\sqrt{21}}{2}\right)^{n-1}\left(\frac{3+\sqrt{21}}{6}\right)y_n - \frac{3-\sqrt{21}}{6}}{\left(\frac{527+115\sqrt{21}}{2}\right)^{n-1} y_n - 1}. \tag{3.44}$$

Using the expression $s_{n-1} = \frac{19s_n+5}{15s_n+4}$ and denoting the coefficients of $19s_n + 5$ and $15s_n + 4$ as h_n and k_n, we have

$$\begin{aligned}
15s_n + 4 &= h_0(19s_n + 5) + k_0(15s_n + 4),\\
15s_{n-1} + 4 &= \frac{h_1(19s_n + 5) + k_1(15s_n + 4)}{h_0(19s_n + 5) + k_0(15s_n + 4)},\\
15s_{n-2} + 4 &= \frac{h_2(19s_n + 5) + k_2(15s_n + 4)}{h_1(19s_n + 5) + k_1(15s_n + 4)},\\
\cdots\cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
15s_{n-i} + 4 &= \frac{h_i(19s_n + 5) + k_i(15s_n + 4)}{h_{i-1}(19s_n + 5) + k_{i-1}(15s_n + 4)}, && (3.45)\\
15s_{n-(i+1)} + 4 &= \frac{h_{i+1}(19s_n + 5) + k_{i+1}(15s_n + 4)}{h_i(19s_n + 5) + k_i(15s_n + 4)}, && (3.46)\\
\cdots\cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
15s_2 + 4 &= \frac{h_{n-2}(19s_n + 5) + k_{n-2}(15s_n + 4)}{h_{n-3}(19s_n + 5) + k_{n-3}(15s_n + 4)}.
\end{aligned}$$

Thus, we obtain

$$\tau(A_4^{(n)}) = 3 \times s_1^2[h_{n-2}(19s_n + 5) + k_{n-2}(15s_n + 4)]^2, \tag{3.47}$$

where $h_0 = 0, k_0 = 1$ and $h_1 = 15, k_1 = 4$. By the expression $s_{n-1} = \frac{19s_n+5}{15s_n+4}$ and using equations (3.45) and (3.46), we have

$$h_{i+1} = 23h_i - h_{i-1}; k_{i+1} = 23k_i - k_{i-1}. \tag{3.48}$$

The characteristic equation of equation (3.48) is $\lambda^2 - 23\lambda + 1 = 0 withroots \lambda_1 = \frac{23+5\sqrt{21}}{2}$ and $\lambda_2 = \frac{23-5\sqrt{21}}{2}$. The general solution of equation (3.48) is $h_i = a_1\lambda_1^i + a_2\lambda_2^i; k_i = b_1\lambda_1^i + b_2\lambda_2^i$. Using the initial conditions $h_0 = 0, k_0 = 1$ and $h_1 = 15, k_1 = 4$, yields

$$\begin{aligned}
h_i &= \frac{\sqrt{21}}{7}\left(\frac{23+5\sqrt{21}}{2}\right)^i - \frac{\sqrt{21}}{7}\left(\frac{23-5\sqrt{21}}{2}\right)^i;\\
k_i &= \left(\frac{7-\sqrt{21}}{14}\right)\left(\frac{23+5\sqrt{21}}{2}\right)^i + \left(\frac{7+\sqrt{21}}{14}\right)\left(\frac{23-5\sqrt{21}}{2}\right)^i. && (3.49)
\end{aligned}$$

If $s_n = 1$, it means that $A_4^{(n)}$ is without any electrically equivalent transformation. Plugging equation (3.49) into equation (3.47), we have

$$\tau(A_2^{(n)}) = 3 \times s_1^2 \left[\left(\frac{133 + 29\sqrt{21}}{14} \right) \left(\frac{23 + 5\sqrt{21}}{2} \right)^{n-2} + \left(\frac{133 - 29\sqrt{21}}{14} \right) \left(\frac{23 - 5\sqrt{21}}{2} \right)^{n-2} \right]^2, \quad n \geq 2. \tag{3.50}$$

When $n = 1, \tau(A_4^{(1)}) = 3$ which satisfies equation (3.50). Therefore, the number of spanning trees in the sequence of the graph $A_4^{(n)}$ is given by

$$\tau(A_2^{(n)}) = 3 \times s_1^2 \left[\left(\frac{133 + 29\sqrt{21}}{14} \right) \left(\frac{23 + 5\sqrt{21}}{2} \right)^{n-2} + \left(\frac{133 - 29\sqrt{21}}{14} \right) \left(\frac{23 - 5\sqrt{21}}{2} \right)^{n-2} \right]^2, \quad n \geq 1. \tag{3.51}$$

where,

$$s_1 = \frac{2\left(\frac{527+115\sqrt{21}}{2}\right)^{n-1}(9 + 2\sqrt{21}) + (3 - \sqrt{21})}{3\left(\frac{527+115\sqrt{21}}{2}\right)^{n-1}(5 + \sqrt{21}) + 6}, \quad n \geq 1. \tag{3.52}$$

Inserting equation (3.52) into equation (3.51) we obtain the desired result. □

§4. Numerical Results

Now in Tables 1 and 2 we will list some values for the number of spanning trees from the sequence graphs that were studied.

n	$A_1^{(n)}$	$A_2^{(n)}$
1	3	3
2	5292	8664
3	14257200	29654208
4	38437286592	102551410176
5	103626696377088	354860039811072
6	279376958435404800	1227969544217001984
7	753198621914706751488	4249315364408988008448
8	2030619014650714425311232	m14704504065155326285971456

Table 1 Illustrates values of the number of spanning trees in the graphs $A_1^{(n)}$ and $A_2^{(n)}$.

n	$A_3^{(n)}$	$A_4^{(n)}$
1	3	3
2	6144	1728
3	15925248	910803
4	41789423616	479991603
5	109725622468608	252954664128
6	288112748103991296	133306628004003
7	756514955634621284352	70252340003445603
8	1986426917209987985965056	37022849875187828928

Table 2 Illustrates values of the number of spanning trees in the graphs $A_3^{(n)}$ and $A_4^{(n)}$.

§5. **Spanning Tree Entropy**

After obtaining explicit Formulas for the number of spanning trees of the sequence of the four families of graphs $A_1^{(n)}, A_2^{(n)}, A_3^{(n)}$ and $A_4^{(n)}$, we can compute its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined as in [19]. For a graph G,

$$Z\left(G\right) = lim_{n\to\infty} \frac{ln\tau\left(G\right)}{\left|V\left(G\right)\right|}. \tag{5.1}$$

Then,

$$\begin{aligned} Z(A_1^{(n)}) &= \frac{1}{6}(ln[4] - 2ln[337 + 5242] + 2ln8749 + 135042) = 1.3165970451, \\ Z(A_2^{(n)}) &= \frac{1}{6}(ln[8] + 2ln[11 + 46]) = 1.3581918754, \\ Z(A_3^{(n)}) &= \frac{1}{6}\left(ln\left[2\right] + 2\ ln\left[19 + 3\ \sqrt{33}\right]\ \right) = 1.3121876275, \\ Z(A_4^{(n)}) &= \frac{1}{6}\left(\ln\left[4\right] - 2\ln\left[527 + 115\sqrt{21}\right] + 2ln\left[6049 + 1320\sqrt{21}\right]\right) = 1.044532824. \end{aligned}$$

We compare the value of entropy in our graphs with other graphs. It is clear that the entropy of the $A_2^{(n)}$ graph is greater than the other three graphs and the entropy of the $A_4^{(n)}$ graph is smaller than the other three graphs .In addition the entropy of the graphs ,$A_1^{(n)}, A_2^{(n)}$, $A_3^{(n)}$ and $A_4^{(n)}$ is smaller than the fractal scale free lattice [20] of the average degree 4, which has the entropy.

§6. **Conclusions**

In this work, we calculate the number of spanning trees in the sequences of three sequences of graphs of average degree 4 and 5 utilizing electrically equivalent transformations. The feature of this method lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

References

[1] Tzeng W.J. and Wu F. Y., Spanning trees on hypercubic lattices and nonorientable surfaces, *Applied Mathematics Letters*, Volume 13, Issue 7, October (2000)19-25.

[2] Boccaletti S., Latora V., Moreno Y., Chavez M. and Hwang D. U., A Complex networks: Structure and dynamics, *Physics Reports*, Volume 424, Issues 4C5, February (2006) 175-308.

[3] Boesch F. T., On unreliability polynomials and graph connectivity in reliable network synthesis, *J. Graph Theory*, 10 (1986) 339-352.

[4] Gábor J. Szabó, Mikko Alava and János Kertész, Geometry of minimum spanning trees on scale-free networks, *Physica A Statistical Mechanics and its Applications* Volume 330, Issues 1C2, 1 December (2003) 31-36.

[5] Majumdar S.N. and Deepak Dhar, Equivalence between the Abelian sandpile model and the q0 limit of the Potts model, *Physica A: Statistical Mechanics and its Applications*, Volume 185, Issues 1C4, 15 June (1992)129-145.

[6] Grady L., Random walks for image segmentation, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, Volume: 28, Issue: 11, November (2006) 1768 C 1783.

[7] Kirchhoff G. G., über die Auflösung der Gleichungen auf welche man bei der Untersucher der linearen Verteilung galuanischer Strome gefhrt wird, *Ann. Phg. Chem.*, 72 (1847) 497-508.

[8] Kelmans A. K. and Chelnokov V. M., A certain polynomial of a graph and graphs with an extremal number of trees, *Journal of Combinatorial Theory* B, Vol. 16, (1974) 197C214.

[9] Daoud S. N., The deletion - contraction method for counting the number of spanning trees of graphs, *European Journal of Physical Plus*, Vol.130. No. 10, Oct.(2015) 1-14.

[10] Daoud S. N., Number of spanning trees in different product of complete and complete tripartite graphs, *Ars Combinatoria*, Vol. 139 (2018) 85-103.

[11] Jia-Bao Liu and Daoud S. N., Complexity of some of pyramid graphs created from a gear graph, *Symmetry*, 2018, 10, 689; doi:10.3390/sym10120689.

[12] Daoud S. N., Number of spanning trees of cartesian and composition products of graphs and Chebyshev polynomials, *IEEE Access*, Vol.7 (2019) 71142 C 71157.

[13] S. N. Daoud, Complexity of graphs generated by wheel graph and their asymptotic limits, *Journal of the Egyptian Math. Soc.*, Vol.25, Issue 4, October (2017) 424-433.

[14] Teufl E., Wagner S., Determinant identities for Laplace matrices, *Linear Algebra Appl.*, 432, (2010) 441-457.

[15] S. N. Daoud, Number of spanning trees in the sequence of some nonahedral graphs, *Utilitas Mathematica*, Vol. 115 (2020) (1-18).

[16] S. N. Daoud and Wedad Saleha, Complexity trees of the sequence of some nonahedral graphs generated by triangle, *Heliyon Journal*, 6(9) Sep (2020).

[17] Jia-Bao Liu and S. N. Daoud, Number of spanning trees in the sequence of some graphs, *Complexity*, Volume 2019|Article ID 4271783|, https://doi.org/10.1155/2019/4271783.

[18] Lyons R., Asymptotic enumeration of spanning trees, *Combin. Probab. Comput.*, 14 (2005) 491C522.

[19] Zhang Z., H. Liu., Wu B., Zou T., Spanning trees in a fractal scale Cfree lattice, *Phys. Rev.* E, 83 016116 (2011).

[20] Chang S., Chen L., Yang W., Spanning trees on the Sierpinski gasket, *J. Stat. Phys.*, 126 (2007) 649-667.

International J.Math. Combin. Vol.2(2026), 25-52

Fixed Point Theorems for Nonlinear Rational Contractions in Extended A_b-Metric Spaces

Vikas Ahirwar, Manoj Ughade and Rahul Vishwakarma

(Department of Mathematics, Institute for Excellence in Higher Education, Bhopal, Madhya Pradesh, India)

E-mail: vikyahirwar196@gmail.com, manojhelpyou@gmail.com, rahulvishwakarma1998@gmail.com

Abstract: Fixed point theory plays a fundamental role in nonlinear analysis and has wide applications in integral equations, optimization, computer science, and dynamical systems. The classical Banach contraction principle has inspired numerous generalizations to accommodate mappings that are not strict contractions in the usual metric sense. Among such developments, A-metric and A_b-metric spaces provide a broad framework that unifies and extends classical metric spaces, S-metric spaces, and b-metric spaces. In this article, we introduce the notion of an extended A_b-metric space and investigate fixed point results for several well-known contraction principles within the framework of extended A_b-metric spaces. In particular, we establish Banach-type, Kannan-type, Reich-type, and nonlinear rational contraction results and prove the existence and uniqueness of fixed points under suitable orbit-control conditions. Our approach relies on generalized triangle inequalities associated with A_b-metrics together with the construction of Picard sequences to establish convergence. The obtained results not only extend several classical fixed point theorems to a broader class of generalized metric spaces but also unify various contraction conditions within a single framework. Several illustrative examples are presented to support the main results, and an application to a nonlinear integral equation is provided to demonstrate the effectiveness of the proposed approach. These findings contribute to the growing theory of generalized metric spaces and their applications in nonlinear analysis.

Key Words: Extended A_b-metric space, Banach contraction, Kannan contraction, rational contraction, fixed point, common fixed point.

AMS(2010): 47H10, 54H25, 54E99.

§1. Introduction

Fixed point theory has long been regarded as one of the fundamental branches of nonlinear analysis, with important applications in functional analysis, topology, optimization, differential equations, and applied mathematics. A pioneering contribution to this theory was made by Banach [1] through the celebrated contraction principle, now known as the Banach contraction

[1]Received November 6, 2025, Accepted May 26, 2026

principle, which guarantees the existence and uniqueness of fixed points for contraction mappings on complete metric spaces. Owing to its simplicity and effectiveness, this theorem has become a powerful tool in both pure and applied mathematics and has motivated numerous generalizations in more abstract settings.

A major advancement in this direction was the introduction of partial metric spaces by Matthews [2], where the self-distance of a point need not be zero. This concept provided a suitable framework for applications in domain theory and theoretical computer science. Later, Bukatin et al. [3] developed several foundational properties of partial metric spaces, while Oltra and Valero [11] established various fixed point results in this setting. Subsequently, Shukla [12] introduced partial b-metric spaces, extending the classical metric structure by incorporating a relaxation parameter.

Parallel to these developments, several modifications of the Banach contraction condition were proposed. Kannan [6, 7], Reich [24, 23], and Ćirić [4] introduced alternative contractive conditions that guarantee fixed points without requiring the standard Banach contraction assumption. Later, Khan et al. [5] introduced altering distance functions, while Dass and Gupta [22] investigated rational-type contractions. These generalized contractions significantly broadened the applicability of fixed point theory and unified many classical results.

Further extensions of fixed point theory have been investigated in several generalized spaces. Contributions by Jungck [13], Edelstein [14], Pant [9], and Jaggi [10] enriched the theory of compatible mappings and common ffxed points. More recently, fixed point results have been studied in S-metric spaces [16, 17, 18], A-metric spaces [19], and Ab-metric spaces [20, 21], thereby extending the scope of generalized metric structures.

The concept of extended S_b-metric spaces introduced by Mlaiki [15] further demonstrated how generalized distance functions can provide a unified framework for studying nonlinear contractions. Such generalized spaces preserve essential topological and convergence properties despite weakened metric axioms. Consequently, these developments have attracted considerable attention in nonlinear functional analysis, integral equations, optimization theory, and computer science.

Recently, several authors have investigated generalized contractions in ordered and generalized metric spaces. Bhaskar and Lakshmikantham [27] established important fixed point results in partially ordered metric spaces, while Aghajani et al. [28], Jachymski [29], and Huang et al. [30] obtained various common fixed point and coincidence point results in generalized b-metric settings. Moreover, Taş and Özgür [31] studied generalized fixed point results in S_b-metric spaces.

Motivated by these ongoing developments, in this paper we introduce the notion of an extended A_b-metric space and investigate several fixed point results involving Banach-type [1], Kannan-type [6, 7], Reich-type [24], Ćirić-type [4], and rational-type contractions [22] in this setting. The proposed framework generalizes both A-metric and A_b-metric spaces and provides greater flexibility through the incorporation of a control function. Our results unify and extend several existing theorems in the literature. Moreover, an application to a nonlinear integral equation is presented to illustrate the effectiveness and applicability of the obtained results.

§2. **Preliminaries and Notations**

In this section, we recall some basic definitions and fix the notation that will be used throughout the paper. We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{R}$ the set of real numbers.

The notion of an A-metric space was introduced as a natural generalization of metric and S-metric spaces [19].

Definition 2.1 *Let X be a nonempty set and let $n \geqslant 2$. A function*

$$A : X^n \longrightarrow [0, \infty)$$

is called an A-metric on X if the following conditions hold for all $\xi_1, \xi_2, \cdots, \xi_n, \zeta \in X$:

$(A1)$

$$A(\xi_1, \xi_2, \cdots, \xi_n) = 0 \iff \xi_1 = \xi_2 = \cdots = \xi_n.$$

$(A2)$

$$A(\xi_1, \xi_2, \cdots, \xi_n) = A(\xi_{\pi(1)}, \xi_{\pi(2)}, \cdots, \xi_{\pi(n)})$$

for every permutation π of $[n]$.

$(A3)$ (Generalized triangle inequality)

$$A(\xi_1, \xi_2, \cdots, \xi_n) \leqslant A(\xi_1, \xi_2, \cdots, \xi_{n-1}, \zeta) + A(\underbrace{\zeta, \zeta, \cdots, \zeta}_{n-1}, \xi_n).$$

Then, (X, A) is called an A-metric space.

When $n = 2$, this definition reduces to the classical metric. For $n = 3$, it recovers the S-metric structure. Thus, the A-metric provides a unified framework that generalizes several existing notions of distance.

To allow greater flexibility, the concept of an A_b-metric space was introduced as an extension of A-metric spaces [20, 21].

Definition 2.2 *Let X be a nonempty set, $b \geqslant 1$, and $n \geqslant 2$. A function $A_b : X^n \longrightarrow [0, \infty)$ is called an A_b-metric on X if the following conditions hold for all $\xi_1, \xi_2, \cdots, \xi_n, \zeta \in X$:*

$(Ab1)$

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = 0 \iff \xi_1 = \xi_2 = \cdots = \xi_n.$$

$(Ab2)$

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = A_b(\xi_{\pi(1)}, \xi_{\pi(2)}, \cdots, \xi_{\pi(n)})$$

for every permutation π of $[n]$.

$(Ab3)$ (Generalized triangle inequality)

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) \leq b\Big(A_b\big(\xi_1, \xi_2, \cdots, \xi_{n-1}, \zeta\big) + A_b\big(\underbrace{\zeta, \zeta, \cdots, \zeta}_{n-1}, \xi_n\big)\Big).$$

Then, (X, A_b) is called an A_b-metric space.

Clearly, every A-metric is an A_b-metric with $b = 1$, but the converse does not necessarily hold. The presence of the control parameter b provides additional flexibility in verifying contractive conditions.

Definition 2.3 *A sequence $\{\xi_m\} \subseteq X$ is said to converge to $\xi \in X$ if*

$$\lim_{m\to\infty} A_b(\underbrace{\xi_m, \xi_m, \cdots, \xi_m}_{n-1}, \xi) = 0.$$

In this case, we write

$$\xi_m \to \xi \qquad as \qquad m \to \infty.$$

Definition 2.4 *A sequence $\{\xi_m\} \subseteq X$ is called a Cauchy sequence in (X, A_b) if*

$$\lim_{m,p\to\infty} A_b(\underbrace{\xi_m, \xi_m, \ldots, \xi_m}_{n-1}, \xi_p) = 0.$$

Definition 2.5 *An A_b-metric space (X, A_b) is said to be complete if every Cauchy sequence $\{\xi_m\} \subseteq X$ converges to some $\xi \in X$.*

§3. New Examples of A_b-Metric Spaces

Example 3.1 Let $X = \mathbb{R}$ and fix an integer $n \geqslant 2$. Define

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) := \max_{1\leqslant i<j\leqslant n} |\xi_i - \xi_j|^2, \qquad \xi_1, \xi_2, \cdots, \xi_n \in \mathbb{R}.$$

We verify that A_b defines an A_b-metric on X.

$(Ab1)$ Clearly,

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) \geqslant 0.$$

Moreover,

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = 0$$

if and only if

$$|\xi_i - \xi_j| = 0$$

for all i, j, which implies

$$\xi_1 = \xi_2 = \cdots = \xi_n.$$

$(Ab2)$ Since the maximum of all pairwise squared differences is independent of the ordering of the coordinates, we have

$$A_b(\xi_1, \xi_2, \ldots, \xi_n) = A_b(\xi_{\pi(1)}, \xi_{\pi(2)}, \cdots, \xi_{\pi(n)})$$

for every permutation π of $\{1, 2, \cdots, n\}$.

$(Ab3)$ Let $\zeta \in X$. Using the inequality

$$|a+b|^2 \leqslant 2(|a|^2 + |b|^2),$$

we obtain

$$|\xi_i - \xi_j|^2 = |(\xi_i - \zeta) + (\zeta - \xi_j)|^2 \leqslant 2\big(|\xi_i - \zeta|^2 + |\zeta - \xi_j|^2\big).$$

Taking the maximum over all $1 \leqslant i < j \leqslant n$, it follows that

$$\begin{aligned} A_b(\xi_1, \xi_2, \cdots, \xi_n) \leqslant 2\Big(& A_b(\xi_1, \xi_2, \cdots, \xi_{n-1}, \zeta) \\ &+ A_b(\underbrace{\zeta, \zeta, \cdots, \zeta}_{n-1}, \xi_n)\Big). \end{aligned}$$

Hence, the generalized triangle inequality holds with $b = 2$. Therefore, (X, A_b) is an A_b-metric space.

§4. Extended A_b-Metric Space

In this section, we introduce the notion of an extended A_b-metric space, which generalizes both A-metric and A_b-metric spaces through the incorporation of a variable control function.

Definition 4.1(Extended A_b-Metric Space) *Let X be a nonempty set, $n \geqslant 2$, and let*

$$A_b : X^n \to [0, \infty)$$

be a mapping together with a control function

$$\Theta : X^n \to [1, \infty).$$

Then, A_b is called an extended A_b-metric on X if for all $\xi_1, \xi_2, \cdots, \xi_n, \alpha \in X$, the following conditions hold:

$(EAb1)$

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = 0 \iff \xi_1 = \xi_2 = \cdots = \xi_n.$$

$(EAb2)$

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = A_b(\xi_{\pi(1)}, \xi_{\pi(2)}, \cdots, \xi_{\pi(n)})$$

for every permutation π of $\{1, 2, \cdots, n\}$.

$(EAb3)$ (Generalized triangle inequality)

$$\begin{aligned} A_b(\xi_1, \xi_2, \cdots, \xi_n) \leqslant \Theta(\xi_1, \xi_2, \cdots, \xi_n)\Big(& A_b(\xi_1, \xi_2, \cdots, \xi_{n-1}, \alpha) \\ &+ A_b(\underbrace{\alpha, \alpha, \cdots, \alpha}_{n-1}, \xi_n)\Big). \end{aligned}$$

Then (X, A_b) is called an extended A_b-metric space.

Clearly, every A_b-metric space is an extended A_b-metric space whenever

$$\Theta(\xi_1, \xi_2, \ldots, \xi_n) = b$$

for all $\xi_1, \xi_2, \ldots, \xi_n \in X$. Moreover, when $\Theta \equiv 1$, the structure reduces to an A-metric space.

Definition 4.2 *Let (X, A_b) be an extended A_b-metric space with fixed arity $n \geqslant 2$ and control function Θ. Let $\{\xi_k\} \subset X$ be a sequence and let $\xi \in X$. Then the following notions are defined.*

1. Convergence. *The sequence $\{\xi_k\}$ is said to converge to ξ (written $\xi_k \to \xi$) if for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that*

$$A_b(\underbrace{\xi_k, \xi_k, \cdots, \xi_k}_{n-1}, \xi) < \varepsilon \qquad \textit{for all } k \geqslant N(\varepsilon).$$

2. Cauchy sequence. *The sequence $\{\xi_k\}$ is called an A_b-Cauchy sequence if for every $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that*

$$A_b(\underbrace{\xi_k, \xi_k, \cdots, \xi_k}_{n-1}, \xi_m) < \varepsilon \qquad \textit{for all } k, m \geqslant N(\varepsilon).$$

3. Completeness. *The space (X, A_b) is said to be complete if every A_b-Cauchy sequence in X converges to some point of X.*

4. Diameter. *For a nonempty subset $Y \subset X$, define its diameter by*

$$\operatorname{diam}(Y) := \sup\left\{A_b(\xi_1, \xi_2, \cdots, \xi_n) : \xi_1, \xi_2, \cdots, \xi_n \in Y\right\}.$$

Definition 4.3 *Let (X, A_b) be an extended A_b-metric space. For $\xi \in X$ and $r > 0$, define the open and closed A_b-balls centered at ξ, respectively, by*

$$\mathcal{B}_r^{A_b}(\xi) := \left\{\eta \in X : A_b(\underbrace{\eta, \eta, \cdots, \eta}_{n-1}, \xi) < r\right\},$$

and

$$\overline{\mathcal{B}}_r^{A_b}(\xi) := \left\{\eta \in X : A_b(\underbrace{\eta, \eta, \cdots, \eta}_{n-1}, \xi) \leqslant r\right\}.$$

Furthermore, the following notions hold.

1. Open set. *A subset $U \subset X$ is called open if for every $u \in U$, there exists $\varepsilon > 0$ such that*

$$\mathcal{B}_\varepsilon^{A_b}(u) \subseteq U.$$

2. Closed set. *A subset $V \subset X$ is called closed if whenever a sequence $\{\nu_k\} \subset V$*

converges to some $\nu \in X$, then $\nu \in V$.

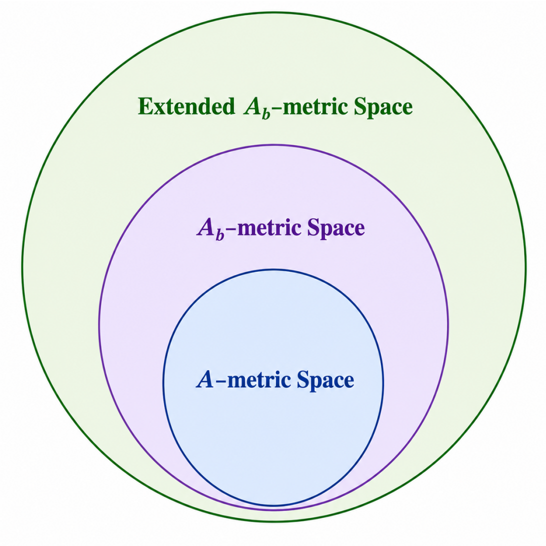


Figure 1. Hierarchical relationship among A-metric, A_b-metric, and extended A_b-metric spaces

§5. Comparative Study of A-Metric, A_b-Metric, and Extended A_b-Metric Spaces

Feature	**A-metric Space**	**A_b-metric Space**	**Extended A_b-metric Space**
Introduced by	Mujahid Abbas et al. (2015)	Manoj Ughade et al. and subsequent extensions	Generalized framework proposed in the present work
Underlying mapping	$A : X^n \to [0, \infty)$	$A_b : X^n \to [0, \infty)$	$A_b : X^n \to [0, \infty)$ with control function $\Theta : X^n \to [1, \infty)$
Parameter involved	None	Constant $b \geqslant 1$	Variable control function Θ
Non-negativity	$A(\xi_1, \ldots, \xi_n) \geqslant 0$	$A_b(\xi_1, \cdots, \xi_n) \geqslant 0$	$A_b(\xi_1, \cdots, \xi_n) \geqslant 0$
Identity property	$A(\xi_1, \cdots, \xi_n) = 0 \iff \xi_1 = \cdots = \xi_n$	Same as the A-metric case	Same condition retained
Symmetry	Invariant under every permutation of coordinates	Invariant under every permutation of coordinates	Invariant under every permutation of coordinates

Feature	**A-metric Space**	**A_b-metric Space**	**Extended A_b-metric Space**
Triangle-type inequality	$A(\xi_1,\cdots,\xi_n)\leqslant A(\xi_1,\cdots,\xi_{n-1},\zeta)+A(\zeta,\cdots,\zeta,\xi_n)$	$A_b(\xi_1,\cdots,\xi_n)\leqslant b\Big(A_b(\xi_1,\cdots,\xi_{n-1},\zeta)+A_b(\zeta,\cdots,\zeta,\xi_n)\Big)$	$A_b(\xi_1,\cdots,\xi_n)\leqslant \Theta(\xi_1,\cdots,\xi_n)\Big(A_b(\xi_1,\cdots,\xi_{n-1},\alpha)+A_b(\alpha,\cdots,\alpha,\xi_n)\Big)$
Flexibility level	Least flexible	More flexible through the constant b	Most flexible through the variable control function Θ
Special cases recovered	Metric space ($n=2$) and S-metric space ($n=3$)	Every A-metric space when $b=1$	Both A-metric and A_b-metric spaces
Nature of control	Fixed structure	Globally controlled by a constant b	Locally controlled by Θ, depending on the points involved
Suitability for fixed point theory	Classical generalized contractions	Wider classes of contractions	Highly adaptable framework for nonlinear contractive mappings
Generalization hierarchy	Base structure	Generalization of A-metric spaces	Generalization of both A-metric and A_b-metric spaces

§6. Examples of Extended A_b-Metric Spaces

In this section, we present several examples of functions

$$A_b : X^n \to [0,\infty)$$

that satisfy the axioms of an extended A_b-metric space introduced in the previous section. In each case, the associated control function Θ is specified and the required properties are verified.

Example 6.1 Let S be a nonempty set and let $\Gamma := S^* \cup S^\omega$ denote the collection of all finite and infinite sequences over S. Fix $n \geqslant 2$. For $\alpha_1,\alpha_2,\cdots,\alpha_n \in \Gamma$, define

$$A_b(\alpha_1,\alpha_2,\cdots,\alpha_n) := 2^{-\ell(\alpha_1,\alpha_2,\cdots,\alpha_n)},$$

where $\ell(\alpha_1,\alpha_2,\cdots,\alpha_n) \in \mathbb{N}_0 \cup \{\infty\}$ denotes the length of the longest common prefix shared by the sequences $\alpha_1,\alpha_2,\cdots,\alpha_n$. By convention,

$$2^{-\infty} = 0$$

and choose the control function $\Theta \equiv 1$.

Proof The nonnegativity is immediate since powers of 2^{-k} are nonnegative.

Moreover,

$$A_b(\alpha_1, \alpha_2, \cdots, \alpha_n) = 0$$

if and only if

$$\ell(\alpha_1, \alpha_2, \cdots, \alpha_n) = \infty,$$

which occurs precisely when all sequences coincide.

Symmetry follows because the common-prefix length is invariant under permutations of the arguments.

For the generalized triangle inequality, let $\zeta \in \Gamma$ and define

$$k_i := \ell(\alpha_i, \zeta), \qquad i = 1, 2, \cdots, n.$$

Then,

$$\ell(\alpha_1, \alpha_2, \cdots, \alpha_n) \geqslant \min_{1 \leqslant i \leqslant n} k_i.$$

Hence,

$$\begin{aligned} A_b(\alpha_1, \alpha_2, \cdots, \alpha_n) &= 2^{-\ell(\alpha_1, \alpha_2, \cdots, \alpha_n)} \\ &\leqslant 2^{-\min_{1 \leqslant i \leqslant n} k_i} \\ &\leqslant \sum_{i=1}^{n} 2^{-k_i} = \sum_{i=1}^{n} A_b(\alpha_i, \cdots, \alpha_i, \zeta). \end{aligned}$$

Therefore, the generalized triangle inequality holds with $\Theta \equiv 1$. Consequently, (Γ, A_b) is an extended A_b-metric space. □

Remark 6.2 The above example provides a natural realization of an extended A_b-metric on a sequence space. The value

$$A_b(\alpha_1, \alpha_2, \cdots, \alpha_n) = 2^{-\ell(\alpha_1, \alpha_2, \cdots, \alpha_n)}$$

depends on the length of the common initial segment shared by the sequences. Consequently, sequences with larger common prefixes are regarded as closer to one another.

An important feature of this construction is that the distance is determined by structural similarity rather than by numerical magnitude. This makes the example suitable for symbolic systems, coding structures, formal languages, and tree-like models where proximity is governed by shared patterns.

Moreover, the choice $\Theta \equiv 1$ shows that the generalized triangle inequality is satisfied without any additional relaxation factor. Hence, the space behaves as a particular case of an extended A_b-metric space with the strongest control condition.

It is also worth mentioning that when $n = 2$, the mapping reduces to the classical prefix-type metric commonly used in sequence analysis and ultrametric constructions. Therefore, this example demonstrates that extended A_b-metric spaces naturally extend several well-known generalized distance frameworks while preserving meaningful geometric behavior.

Example 6.3 Let $\Gamma := [0, \infty)$ and fix constants $c > 0$ and $n \geqslant 2$. Define

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) := c \max\{\xi_1, \xi_2, \cdots, \xi_n\},$$

for all $\xi_i \in \Gamma$ and take $\Theta \equiv 1$.

Proof The nonnegativity and symmetry are immediate.

Furthermore,

$$A_b(\xi_1, \xi_2, \cdots, \xi_n) = 0$$

implies

$$\max\{\xi_1, \xi_2, \cdots, \xi_n\} = 0,$$

and hence

$$\xi_1 = \xi_2 = \cdots = \xi_n = 0.$$

For any $\alpha \in \Gamma$,

$$\begin{aligned}\sum_{i=1}^{n} A_b(\xi_i, \cdots, \xi_i, \alpha) &= c \sum_{i=1}^{n} \max\{\xi_i, \alpha\} \\ &\geqslant c \max_{1 \leqslant i \leqslant n} \xi_i = A_b(\xi_1, \xi_2, \cdots, \xi_n).\end{aligned}$$

Hence, the generalized triangle inequality is satisfied with $\Theta \equiv 1$. Therefore, (Γ, A_b) is an extended A_b-metric space. □

Example 6.4 Let (M, d) be a metric space and set $\Gamma := M$.

For $x_1, x_2, \cdots, x_n \in M$, define

$$A_b(x_1, x_2, \cdots, x_n) := \max_{1 \leqslant i < j \leqslant n} d(x_i, x_j).$$

Choose the constant control function $\Theta \equiv n$.

Proof The nonnegativity and symmetry follow directly from the properties of the metric d.

Moreover,

$$A_b(x_1, x_2, \cdots, x_n) = 0$$

if and only if $d(x_i, x_j) = 0$ for all i, j, which implies $x_1 = x_2 = \cdots = x_n$.

Let $\alpha \in M$. By the triangle inequality,

$$d(x_i, x_j) \leqslant d(x_i, \alpha) + d(\alpha, x_j).$$

Taking the maximum over all $i < j$, we obtain

$$A_b(x_1, \cdots, x_n) \leqslant 2 \max_{1 \leqslant k \leqslant n} d(x_k, \alpha).$$

Since

$$\max_{1 \leqslant k \leqslant n} d(x_k, \alpha) \leqslant \sum_{k=1}^{n} d(x_k, \alpha),$$

it follows that

$$A_b(x_1, \cdots, x_n) \leqslant 2 \sum_{k=1}^{n} d(x_k, \alpha).$$

Now observe that

$$A_b(x_k, \cdots, x_k, \alpha) = d(x_k, \alpha),$$

for every $k = 1, 2, \cdots, n$. Hence,

$$A_b(x_1, \cdots, x_n) \leqslant n \sum_{k=1}^{n} A_b(x_k, \cdots, x_k, \alpha).$$

Thus, the generalized triangle inequality holds with $\Theta \equiv n$. Consequently, (M, A_b) is an extended A_b-metric space. □

Example 6.5 Let (M, d) be a metric space and define

$$A_b(x_1, x_2, \cdots, x_n) := \sum_{1 \leqslant i < j \leqslant n} d(x_i, x_j),$$

for all $x_1, x_2, \cdots, x_n \in M$.

Take the control function $\Theta \equiv n - 1$.

Proof The nonnegativity and symmetry are immediate.

Moreover,

$$A_b(x_1, \cdots, x_n) = 0$$

if and only if all pairwise distances vanish, which implies $x_1 = x_2 = \cdots = x_n$.

For any $\alpha \in M$, using the metric triangle inequality,

$$d(x_i, x_j) \leqslant d(x_i, \alpha) + d(\alpha, x_j).$$

Summing over all pairs $1 \leqslant i < j \leqslant n$, we obtain

$$A_b(x_1, \cdots, x_n) \leqslant (n-1) \sum_{k=1}^{n} d(x_k, \alpha).$$

Since

$$A_b(x_k, \cdots, x_k, \alpha) = d(x_k, \alpha),$$

we conclude that

$$A_b(x_1, \cdots, x_n) \leqslant (n-1) \sum_{k=1}^{n} A_b(x_k, \cdots, x_k, \alpha).$$

Hence, the generalized triangle inequality holds with $\Theta \equiv n - 1$. Therefore, (M, A_b) is an extended A_b-metric space. □

Example 6.6 Let K be a compact set and let $\Gamma := C(K)$, the space of all continuous real-valued functions on K. For $f_1, f_2, \cdots, f_n \in C(K)$, define

$$A_b(f_1, f_2, \cdots, f_n) := \sup_{x\in K} \max_{1\leqslant i<j\leqslant n} |f_i(x) - f_j(x)|$$

and choose $\Theta \equiv n$.

Proof The nonnegativity and symmetry are immediate.

Moreover,

$$A_b(f_1, f_2, \cdots, f_n) = 0$$

if and only if $f_1 = f_2 = \cdots = f_n$.

Let $g \in C(K)$. By the pointwise triangle inequality,

$$|f_i(x) - f_j(x)| \leqslant |f_i(x) - g(x)| + |g(x) - f_j(x)|.$$

Taking maxima and then supremum over K, we obtain

$$A_b(f_1, \cdots, f_n) \leqslant 2 \sup_{x\in K} \max_{1\leqslant k\leqslant n} |f_k(x) - g(x)|.$$

Furthermore,

$$A_b(f_k, \cdots, f_k, g) = \sup_{x\in K} |f_k(x) - g(x)|.$$

Hence,

$$A_b(f_1, \cdots, f_n) \leqslant n \sum_{k=1}^{n} A_b(f_k, \cdots, f_k, g).$$

Therefore, the generalized triangle inequality holds with $\Theta \equiv n$. Consequently, $(C(K), A_b)$ is an extended A_b-metric space. □

Let Γ be a nonempty set and let $A_b : \Gamma^n \to [0, \infty)$ be an extended A_b-metric with control function $\Theta : \Gamma^n \to [1, \infty)$. For $\xi, \eta \in \Gamma$, define

$$A_\eta(\xi) := A_b(\underbrace{\eta, \cdots, \eta}_{n-1}, \xi),$$

whenever no confusion arises.

§7. Main Results

Theorem 7.1 *Let (X, A_b) be a complete extended A_b-metric space of arity $n \geqslant 2$ with control* *function $\Theta : X^n \to [1,\infty)$. Assume that A_b is continuous. Let $T : X \to X$ satisfy*

$$A_b\big(T\xi, T\eta, \underbrace{T\xi,\ldots,T\xi}_{n-2}\big) \leqslant k\, A_b\big(\underbrace{\xi,\ldots,\xi}_{n-1},\eta\big), \qquad \textit{for all } \xi,\eta \in X, \tag{1}$$

for some constant $k \in [0,1)$.

Assume the orbit-control condition: for each $x_0 \in X$, the Picard orbit

$$x_m := T^m x_0$$

satisfies

$$\lim_{m,p\to\infty} \prod_{i=m}^{p} \Theta(x_i,\ldots,x_i,x_p) =: L(x_0) < \frac{1}{k}.$$

Then, T has a unique fixed point $z \in X$, and

$$T^m x_0 \to z$$

for every $x_0 \in X$.

Proof Fix $x_0 \in X$ and define the Picard sequence

$$x_{m+1} := T x_m, \qquad m \geqslant 0.$$

For every $m \geqslant 0$, put

$$d_m := A_b\big(x_{m+1}, x_{m+2}, \underbrace{x_{m+1},\ldots,x_{m+1}}_{n-2}\big).$$

Our goal is to show that $d_m \to 0$ geometrically (up to bounded multiplicative factors), and then deduce that (x_m) is Cauchy.

Applying (1) with

$$\xi = x_m, \qquad \eta = x_{m+1},$$

gives

$$\begin{aligned} d_m &= A_b\big(Tx_m, Tx_{m+1}, \underbrace{Tx_m,\cdots,Tx_m}_{n-2}\big) \\ &\leqslant k\, A_b\big(\underbrace{x_m,\cdots,x_m}_{n-1}, x_{m+1}\big). \end{aligned} \tag{2}$$

Thus, it remains to estimate

$$A_b\big(\underbrace{x_m,\cdots,x_m}_{n-1}, x_{m+1}\big)$$

in terms of d_{m-1}.

Using the generalized triangle inequality repeatedly, we obtain

$$A_b\big(\underbrace{x_m, \cdots, x_m}_{n-1}, x_{m+1}\big) \leqslant C_m d_{m-1},$$

where

$$C_m = \prod_{\ell=1}^{r_m} \Theta(\mathbf{t}_{m,\ell}),$$

for suitable tuples $\mathbf{t}_{m,\ell} \in \{x_0, \cdots, x_{m+1}\}^n$.

Combining this estimate with (2), we obtain

$$d_m \leqslant k C_m d_{m-1}. \tag{3}$$

Iterating (3), we get

$$d_m \leqslant d_0\, k^m \prod_{j=1}^{m} C_j.$$

By the orbit-control hypothesis,

$$k^m \prod_{j=1}^{m} C_j \to 0 \qquad (m \to \infty),$$

and hence

$$d_m \to 0. \tag{4}$$

Next, repeated application of the generalized triangle inequality yields

$$A_b\big(\underbrace{x_q, \cdots, x_q}_{n-1}, x_p\big) \leqslant M_{q,p} \sum_{j=q}^{p-1} d_j, \tag{5}$$

for suitable constants $M_{q,p} \geqslant 1$.

Since $d_j \to 0$, the series $\sum_{j=q}^{\infty} d_j$ converges, and therefore (x_m) is a Cauchy sequence in the A_b-sense.

Notice that the completeness of (X, A_b) implies the existence of $z \in X$ such that $x_m \to z$. Using continuity of A_b and passing to the limit in (1), we obtain

$$A_b\big(Tz, Tz, \underbrace{Tz, \cdots, Tz}_{n-2}\big) \leqslant k A_b\big(\underbrace{z, \cdots, z}_{n}\big).$$

Since $0 \leqslant k < 1$, it follows that

$$A_b\big(\underbrace{Tz, \cdots, Tz}_{n-1}, z\big) = 0,$$

and hence

$$Tz = z.$$

To prove the uniqueness, let $u, v \in X$ be fixed points of T. Then

$$A_b(\underbrace{u, \cdots, u}_{n-1}, v) = A_b(Tu, Tv, \underbrace{Tu, \cdots, Tu}_{n-2})$$
$$\leqslant kA_b(\underbrace{u, \cdots, u}_{n-1}, v).$$

Since $k < 1$, we conclude that

$$A_b(\underbrace{u, \cdots, u}_{n-1}, v) = 0,$$

which implies $u = v$. Therefore, T has a unique fixed point, and every Picard sequence converges to it. □

Theorem 7.2 *Let (X, A_b) be a complete extended A_b-metric space with continuous A_b. Suppose there exists $\alpha \in \left[0, \frac{1}{2}\right)$ such that for all $\xi, \eta \in X$,*

$$A_b(\underbrace{T\xi, \cdots, T\xi}_{n-1}, T\eta) \leqslant \alpha\Big(A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, T\xi) + A_b(\underbrace{\eta, \cdots, \eta}_{n-1}, T\eta)\Big).$$

Assume moreover that, for each $x_0 \in X$, the Picard orbit

$$x_m := T^m x_0$$

satisfies the orbit-control condition

$$\lim_{m,p\to\infty} \prod_{i=m}^{p} \Theta(x_i, \cdots, x_i, x_p) =: L(x_0) < \frac{1}{\alpha}.$$

Then, T has a unique fixed point $z \in X$, and

$$T^m x_0 \to z \qquad \text{for every } x_0 \in X.$$

Proof Let $x_0 \in X$ and define the Picard sequence

$$x_{m+1} := Tx_m, \qquad m \geqslant 0.$$

Set

$$u_m := A_b(x_m, x_{m+1}, \underbrace{x_m, \cdots, x_m}_{n-2}).$$

Applying the Kannan-type inequality with

$$\xi = x_m, \qquad \eta = x_{m+1},$$

we obtain

$$u_{m+1} \leqslant \alpha\Big(A_b\big(\underbrace{x_m,\cdots,x_m}_{n-1},x_{m+1}\big)+A_b\big(\underbrace{x_{m+1},\cdots,x_{m+1}}_{n-1},x_{m+2}\big)\Big).$$

Using the generalized triangle inequality repeatedly, each diagonal term can be estimated by a finite product of Θ-values multiplied by u_m or u_{m+1}. Hence there exist finite constants $C_m, C_{m+1} \geqslant 1$ such that

$$(1-\alpha C_{m+1})u_{m+1} \leqslant \alpha C_m u_m.$$

By the orbit-control hypothesis, the products of Θ-terms remain controlled along the orbit. Consequently, there exists $q \in [0,1)$ such that $u_{m+1} \leqslant q\, u_m$ for all sufficiently large m.

Therefore,

$$u_m \to 0 \qquad (m \to \infty).$$

Proceeding as in Theorem 7.1, one shows that the Picard sequence (x_m) is Cauchy in the A_b-sense. Since (X, A_b) is complete, there exists $z \in X$ such that $x_m \to z$.

Using the continuity of A_b and passing to the limit in the contractive inequality, we obtain

$$Tz = z.$$

To prove uniqueness, let $u, v \in X$ be fixed points of T. Then

$$\begin{aligned} A_b\big(\underbrace{u,\cdots,u}_{n-1},v\big) &= A_b\big(\underbrace{Tu,\cdots,Tu}_{n-1},Tv\big) \\ &\leqslant \alpha\Big(A_b\big(\underbrace{u,\cdots,u}_{n-1},u\big)+A_b\big(\underbrace{v,\cdots,v}_{n-1},v\big)\Big)=0. \end{aligned}$$

Hence $u = v$. Therefore, T has a unique fixed point, and every Picard sequence converges to it. □

Theorem 7.3 *Let (X, A_b) be a complete extended A_b-metric space with continuous A_b. Suppose there exist constants $\alpha, \beta, \gamma \geqslant 0$ satisfying*

$$\alpha+\beta+\gamma<1,$$

such that for all $\xi, \eta \in X$,

$$A_b\big(T\xi,T\eta,\underbrace{T\xi,\cdots,T\xi}_{n-2}\big) \leqslant \alpha A_b\big(\underbrace{\xi,\cdots,\xi}_{n-1},\eta\big)+\beta A_b\big(\underbrace{\xi,\cdots,\xi}_{n-1},T\xi\big)+\gamma A_b\big(\underbrace{\eta,\cdots,\eta}_{n-1},T\eta\big).$$

Assume further that, for each $x_0 \in X$, the Picard orbit $x_m := T^m x_0$ satisfies

$$\lim_{m,p\to\infty}\prod_{i=m}^{p}\Theta(x_i,\cdots,x_i,x_p) =: L(x_0) < \frac{1}{\alpha+\beta+\gamma}.$$

Then, T has a unique fixed point in X.

Proof Let

$$x_{m+1} := Tx_m, \qquad m \geqslant 0,$$

and define

$$v_m := A_b\big(x_m, x_{m+1}, \underbrace{x_m, \cdots, x_m}_{n-2}\big).$$

Applying the Reich-type inequality with

$$\xi = x_m, \qquad \eta = x_{m+1},$$

gives

$$v_{m+1} \leqslant \alpha v_m + \beta A_b\big(\underbrace{x_m, \cdots, x_m}_{n-1}, x_{m+1}\big) + \gamma v_{m+1}.$$

Using the generalized triangle inequality, the diagonal term is bounded by a finite Θ-multiple of v_m. Hence there exists a finite constant $C_m \geqslant 1$ such that

$$(1-\gamma)v_{m+1} \leqslant (\alpha + \beta C_m)v_m.$$

By the orbit-control condition, the products of the corresponding Θ-terms remain bounded along the orbit. Consequently, there exists $q \in [0, 1)$ such that $v_{m+1} \leqslant qv_m$ for all sufficiently large m.

Therefore,

$$v_m \to 0 \qquad (m \to \infty).$$

Proceeding as in Theorem 7.1, one shows that the Picard sequence (x_m) is Cauchy in the A_b-sense. Since (X, A_b) is complete, there exists $z \in X$ such that $x_m \to z$.

Passing to the limit in the contractive condition and using continuity of A_b, we obtain

$$Tz \;=\; z.$$

To prove uniqueness, let $u, v \in X$ be fixed points of T. Then

$$\begin{aligned} A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big) &= A_b\big(Tu, Tv, \underbrace{Tu, \cdots, Tu}_{n-2}\big) \\ &\leqslant \alpha A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big) + \beta A_b\big(\underbrace{u, \cdots, u}_{n-1}, u\big) + \gamma A_b\big(\underbrace{v, \cdots, v}_{n-1}, v\big). \end{aligned}$$

Since

$$A_b\big(\underbrace{u, \cdots, u}_{n-1}, u\big) = A_b\big(\underbrace{v, \cdots, v}_{n-1}, v\big) = 0,$$

we obtain

$$A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big) \leqslant \alpha A_b\big(\underbrace{u, \cdots, u}_{n-1}, v\big).$$

Because $\alpha < 1$, it follows that

$$A_b(\underbrace{u,\cdots,u}_{n-1},v) = 0,$$

and hence $u = v$. Therefore, T has a unique fixed point in X. $\square$

Theorem 7.4 *Let (X, A_b) be a complete extended A_b-metric space with continuous A_b. Suppose there exists $\lambda \in (0,1)$ such that for all $\xi, \eta \in X$,*

$$A_b(T\xi, T\eta, \underbrace{T\xi,\cdots,T\xi}_{n-2}) \leqslant \frac{\lambda \max\Big\{A_b(\underbrace{\xi,\cdots,\xi}_{n-1},\eta),\ A_b(\underbrace{\xi,\cdots,\xi}_{n-1},T\xi),\ A_b(\underbrace{\eta,\cdots,\eta}_{n-1},T\eta)\Big\}}{1 + A_b(\underbrace{\xi,\cdots,\xi}_{n-1},\eta)}.$$

Assume further that the Picard orbit satisfies the same orbit-control condition as in Theorem 7.1*. Then, T has a unique fixed point in X.*

Proof Let

$$x_{m+1} :=\ Tx_m, \qquad m \geqslant 0,$$

and define

$$w_m := A_b(x_{m+1}, x_{m+2}, \underbrace{x_{m+1},\cdots,x_{m+1}}_{n-2}).$$

Applying the rational-type inequality with

$$\xi = x_m, \qquad \eta = x_{m+1},$$

yields

$$w_m \leqslant \frac{\lambda}{1 + A_b(\underbrace{x_m,\cdots,x_m}_{n-1},x_{m+1})} \max\Big\{A_b(\underbrace{x_m,\cdots,x_m}_{n-1},x_{m+1}), A_b(\underbrace{x_m,\cdots,x_m}_{n-1},Tx_m),$$
$$A_b(\underbrace{x_{m+1},\cdots,x_{m+1}}_{n-1},x_{m+2})\Big\}.$$

If $w_m = 0$ for some m, then the Picard sequence becomes stationary and reaches a fixed point. Otherwise, repeated use of the generalized triangle inequality together with the orbit-control condition allows the Θ-factors to be absorbed into a contraction coefficient. Hence, there exists $\lambda' \in (0,1)$ such that

$$w_m \leqslant \lambda' w_{m-1}$$

for all sufficiently large m. Therefore,

$$w_m \to 0 \qquad (m \to \infty).$$

Proceeding as in Theorem 7.1, one shows that the Picard sequence (x_m) is Cauchy in the

A_b-sense. Since (X, A_b) is complete, there exists $z \in X$ such that $x_m \to z$.

Passing to the limit in the contractive inequality and using continuity of A_b, we obtain

$$Tz \;=\; z.$$

To prove uniqueness, let $u, v \in X$ be fixed points of T. Then

$$A_b\big(\underbrace{u,\cdots,u}_{n-1},v\big) = A_b\big(Tu,Tv,\underbrace{Tu,\cdots,Tu}_{n-2}\big) \leqslant \frac{\lambda \max\Big\{A_b\big(\underbrace{u,\cdots,u}_{n-1},v\big),0,0\Big\}}{1+A_b\big(\underbrace{u,\cdots,u}_{n-1},v\big)}.$$

Hence

$$A_b\big(\underbrace{u,\cdots,u}_{n-1},v\big) = 0,$$

which implies $u = v$. Therefore, T has a unique fixed point in X. $\square$

We now present an example illustrating the applicability of Theorem 7.1.

Example 7.5 Let $X = \mathbb{R}$ and define

$$A_b(x_1,x_2,\cdots,x_n) := \max_{1\leqslant i<j\leqslant n} |x_i - x_j|, \qquad x_1,x_2,\cdots,x_n \in \mathbb{R}.$$

Choose the control function $\Theta \equiv n$. Define the mapping

$$T : X \to X, \qquad T(x) = sx,$$

where $0 \leqslant s < 1$. Then, for every $\xi, \eta \in X$, we have

$$A_b\big(T\xi,T\eta,\underbrace{T\xi,\cdots,T\xi}_{n-2}\big) = |s\xi - s\eta| = s|\xi-\eta| = s\,A_b\big(\xi,\eta,\underbrace{\xi,\cdots,\xi}_{n-2}\big).$$

Hence,

$$A_b\big(T\xi,T\eta,\underbrace{T\xi,\cdots,T\xi}_{n-2}\big) \leqslant k\,A_b\big(\xi,\eta,\underbrace{\xi,\cdots,\xi}_{n-2}\big),$$

where $k = s \in [0,1)$. Thus, the contractive condition (1) of Theorem 7.1 is satisfied.

Moreover, for the Picard sequence $x_m = s^m x_0$, the control values $\Theta(x_i,\cdots,x_i,x_p) = n$ remain constant along the orbit. Hence the orbit-control condition is also satisfied.

Therefore, by Theorem 7.1, the mapping T has a unique fixed point, namely 0, and

$$T^m x_0 \to 0 \qquad \text{as } m \to \infty$$

for every $x_0 \in \mathbb{R}$.

Example 7.6 Let $X = \mathbb{R}$ and define

$$A_b(x_1, x_2, \cdots, x_n) := \max_{1 \leqslant i < j \leqslant n} |x_i - x_j|.$$

Define the nonconstant control function

$$\Theta(x_1, x_2, \cdots, x_n) := 1 + \frac{1}{2 + \sum_{i=1}^{n} |x_i|}.$$

Clearly,

$$1 < \Theta(x_1, x_2, \cdots, x_n) \leqslant \frac{3}{2}$$

for all $x_1, x_2, \cdots, x_n \in X$.

Now define the mapping

$$T : X \to X, \qquad T(x) = sx,$$

where $0 \leqslant s < 1$.

For every $\xi, \eta \in X$, we have

$$A_b\big(T\xi, T\eta, \underbrace{T\xi, \cdots, T\xi}_{n-2}\big) = |s\xi - s\eta| = s|\xi - \eta| = s\, A_b\big(\xi, \eta, \underbrace{\xi, \cdots, \xi}_{n-2}\big).$$

Hence, the Banach-type contractive condition holds with $k = s \in [0, 1)$.

Next, consider the Picard sequence $x_m = T^m x_0 = s^m x_0$. Then

$$\begin{aligned}\Theta(x_m, \cdots, x_m, x_p) &= 1 + \frac{1}{2 + (n-1)|x_m| + |x_p|} \\ &= 1 + \frac{1}{2 + (n-1)|s^m x_0| + |s^p x_0|}.\end{aligned}$$

Since

$$s^m x_0 \to 0 \qquad (m \to \infty),$$

we obtain

$$\Theta(x_m, \cdots, x_m, x_p) \to \frac{3}{2}.$$

Also,

$$1 < \Theta(x_m, \cdots, x_m, x_p) \leqslant \frac{3}{2},$$

for all $m, p \in \mathbb{N}$. Therefore, the corresponding finite products of Θ-values remain bounded along the Picard orbit. Consequently, the orbit-control condition required in Theorem 7.1 is satisfied. Hence all assumptions of Theorem 7.1 hold. Therefore, the mapping T has a unique fixed point 0, and

$$T^m x_0 \to 0 \qquad \text{for every } x_0 \in \mathbb{R}.$$

Example 7.7 Let (X, A_b) be an extended A_b-metric space with control function $\Theta : X^n \to [1, \infty)$.

Fix a point $c \in X$ and define the constant mapping

$$T : X \to X, \qquad T(x) = c, \qquad x \in X.$$

Then, for every $\xi, \eta \in X$, we have

$$A_b(\underbrace{T\xi, \cdots, T\xi}_{n-1}, T\eta) = A_b(\underbrace{c, \cdots, c}_{n-1}, c) = 0.$$

Hence,

$$A_b(\underbrace{T\xi, \cdots, T\xi}_{n-1}, T\eta) \leqslant \alpha\Big(A_b(\underbrace{\xi, \cdots, \xi}_{n-1}, T\xi) + A_b(\underbrace{\eta, \cdots, \eta}_{n-1}, T\eta)\Big),$$

for every $\alpha \in \left[0, \frac{1}{2}\right)$. Therefore, the Kannan-type contractive condition of Theorem 7.2 is satisfied.

Moreover, the Picard sequence generated by T is

$$x_m = c, \qquad m \geqslant 0,$$

which is constant. Consequently,

$$\Theta(x_m, \cdots, x_m, x_p) = \Theta(c, \cdots, c),$$

for all $m, p \in \mathbb{N}$. Hence, the orbit-control condition is trivially satisfied.

Therefore, by Theorem 7.2, the mapping T possesses the unique fixed point c, and every Picard sequence converges to c.

Next, we present an example illustrating Theorem 7.3.

Example 7.8 Let $X = \mathbb{R}$ and define

$$A_b(x_1, x_2, \cdots, x_n) := \max_{1 \leqslant i < j \leqslant n} |x_i - x_j|.$$

Define the nonconstant control function

$$\Theta(x_1, x_2, \cdots, x_n) := 1 + \frac{1}{2 + \sum_{i=1}^{n} |x_i|}.$$

Clearly,

$$1 < \Theta(x_1, x_2, \cdots, x_n) \leqslant \frac{3}{2},$$

for all $x_1, x_2, \cdots, x_n \in \mathbb{R}$.

Now define the mapping

$$T : X \to X, \qquad T(x) = sx,$$

where $0 \leqslant s < 1$. Then, for every $\xi, \eta \in X$, we have

$$A_b(T\xi, T\eta, \underbrace{T\xi, \cdots, T\xi}_{n-2}) = |s\xi - s\eta| = s|\xi - \eta| = s\, A_b(\xi, \eta, \underbrace{\xi, \cdots, \xi}_{n-2}).$$

Choose

$$\alpha = s, \qquad \beta = \gamma = 0.$$

Then,

$$\alpha + \beta + \gamma = s < 1,$$

and hence the Reich-type contractive condition of Theorem 7.3 is satisfied.

Now consider the Picard sequence $x_m = T^m x_0 = s^m x_0$.

Then,

$$\begin{aligned}\Theta(x_m, \cdots, x_m, x_p) &= 1 + \frac{1}{2 + (n-1)|x_m| + |x_p|} \\ &= 1 + \frac{1}{2 + (n-1)|s^m x_0| + |s^p x_0|}.\end{aligned}$$

Since

$$s^m x_0 \to 0, \qquad (m \to \infty),$$

we obtain

$$\Theta(x_m, \cdots, x_m, x_p) \to \frac{3}{2}.$$

Also,

$$1 < \Theta(x_m, \cdots, x_m, x_p) \leqslant \frac{3}{2},$$

for all $m, p \in \mathbb{N}$.

Hence, the corresponding products of Θ-values remain bounded along the Picard orbit, and the orbit-control condition is satisfied.

Therefore, all assumptions of Theorem 7.3 hold. Consequently, the mapping T admits the unique fixed point 0, and

$$T^m x_0 \to 0 \qquad \text{for every } x_0 \in \mathbb{R}.$$

Finally, we present an example illustrating Theorem 7.4.

Example 7.9 Let (X, A_b) be an extended A_b-metric space with control function $\Theta : X^n \to [1, \infty)$. Fix a point $p \in X$ and define the constant mapping

$$T : X \to X, \qquad T(x) = p, \qquad x \in X.$$

Then, for every $\xi, \eta \in X$, we have

$$A_b(\underbrace{T\xi, \cdots, T\xi}_{n-1}, T\eta) = A_b(\underbrace{p, \cdots, p}_{n-1}, p) = 0.$$

Therefore,

$$A_b(\underbrace{T\xi,\cdots,T\xi}_{n-1},T\eta)\leqslant\frac{\lambda\max\left\{A_b(\underbrace{\xi,\cdots,\xi}_{n-1},\eta),\ A_b(\underbrace{\xi,\cdots,\xi}_{n-1},T\xi),\ A_b(\underbrace{\eta,\cdots,\eta}_{n-1},T\eta)\right\}}{1+A_b(\underbrace{\xi,\cdots,\xi}_{n-1},\eta)},$$

for every $\lambda\in(0,1)$. Hence, the rational-type contractive condition of Theorem 7.4 is satisfied.

Now consider the Picard sequence

$$x_m:=T^mx_0,\qquad x_0\in X.$$

Since T is constant,

$$x_m=p,\qquad m\geqslant 1.$$

Thus,

$$\Theta(x_m,\cdots,x_m,x_p)=\Theta(p,\cdots,p),$$

for all $m,p\in\mathbb{N}$.

Consequently, the orbit-control condition is trivially satisfied.

Therefore, all assumptions of Theorem 7.4 hold. Hence, the mapping T possesses the unique fixed point p.

§8 Application of Theorem 7.4

Let $X:=C([0,1])$, the Banach space of all real-valued continuous functions on $[0,1]$, equipped with the supremum norm

$$\|f\|_\infty:=\sup_{x\in[0,1]}|f(x)|.$$

Define $A_b:X^n\to[0,\infty)$ by

$$A_b(f_1,f_2,\cdots,f_n):=\max_{1\leqslant i<j\leqslant n}\|f_i-f_j\|_\infty,\qquad f_1,f_2,\cdots,f_n\in X.$$

Choose the control function $\Theta\equiv n$. Then, (X,A_b) forms a complete extended A_b-metric space. Indeed, the convergence and Cauchy properties induced by A_b coincide with the usual uniform convergence on $C([0,1])$, and $C([0,1])$ is complete with respect to the supremum norm.

Let $k:[0,1]\times[0,1]\to\mathbb{R}$ be a continuous kernel, $h\in C([0,1])$, and let $\varphi:\mathbb{R}\to\mathbb{R}$ be a continuous nonlinear function.

Assume that there exist constants

$$K\geqslant 0\qquad\text{and}\qquad L\geqslant 0$$

such that

$$\sup_{x\in[0,1]}\int_0^1|k(x,y)|\,dy\leqslant K,$$

and

$$|\varphi(u) - \varphi(v)| \leqslant L|u - v|, \qquad u, v \in \mathbb{R}.$$

Assume further that

$$\lambda := KL \in (0,1)$$

and define the operator $T : X \to X$ by

$$(Tf)(x) := \int_0^1 k(x,y)\,\varphi(f(y))\,dy + h(x), \qquad x \in [0,1].$$

We show that T satisfies the rational-type contractive condition of Theorem 7.4.

Let $f, g \in X$. Then, for every $x \in [0,1]$,

$$\begin{aligned}
|(Tf)(x) - (Tg)(x)| &= \left|\int_0^1 k(x,y)\big(\varphi(f(y)) - \varphi(g(y))\big)\,dy\right| \\
&\leqslant \int_0^1 |k(x,y)|\,|\varphi(f(y)) - \varphi(g(y))|\,dy \\
&\leqslant L\int_0^1 |k(x,y)|\,|f(y) - g(y)|\,dy \\
&\leqslant L\|f - g\|_\infty \int_0^1 |k(x,y)|\,dy.
\end{aligned}$$

Taking the supremum over $x \in [0,1]$, we obtain

$$\|Tf - Tg\|_\infty \leqslant KL\|f - g\|_\infty = \lambda\|f - g\|_\infty.$$

Hence,

$$A_b\big(Tf, Tg, \underbrace{Tf, \cdots, Tf}_{n-2}\big) = \|Tf - Tg\|_\infty \leqslant \lambda A_b\big(f, g, \underbrace{f, \cdots, f}_{n-2}\big).$$

Since

$$1 + A_b\big(f, g, \underbrace{f, \cdots, f}_{n-2}\big) \geqslant 1,$$

it follows that

$$A_b\big(Tf, Tg, \underbrace{Tf, \cdots, Tf}_{n-2}\big) \leqslant \frac{\lambda \max\Big\{A_b\big(f, g, \underbrace{f, \cdots, f}_{n-2}\big),\ A_b\big(f, Tf, \underbrace{f, \cdots, f}_{n-2}\big),\ A_b\big(g, Tg, \underbrace{g, \cdots, g}_{n-2}\big)\Big\}}{1 + A_b\big(f, g, \underbrace{f, \cdots, f}_{n-2}\big)}.$$

Thus, T satisfies the rational-type contractive condition of Theorem 7.4.

Moreover, the orbit-control condition required in Theorem 7.4 is satisfied since $\Theta \equiv n$ is constant along every Picard orbit. Hence, the corresponding finite products of Θ-values remain bounded. Therefore, all assumptions of Theorem 7.4 are satisfied. Consequently, T possesses a unique fixed point $u \in X$ such that

$$Tu = u.$$

Equivalently, the nonlinear integral equation

$$u(x) = \int_0^1 k(x,y)\,\varphi(u(y))\,dy + h(x), \qquad x \in [0,1],$$

admits a unique continuous solution $u \in C([0,1])$. Furthermore, for every initial function $u_0 \in C([0,1])$, the Picard iterative sequence $u_{m+1} := Tu_m$ converges uniformly to the unique solution u and therefore, the above nonlinear integral equation provides an application of Theorem 7.4 in the framework of extended A_b-metric spaces.

§9. **Conclusion**

In this paper, we investigated fixed point theory in the setting of extended A_b-metric spaces, which generalize several well-known structures including metric spaces, b-metric spaces, extended b-metric spaces, S-metric spaces, S_b-metric spaces, extended S_b-metric spaces, A-metric spaces, and A_b-metric spaces through the introduction of a variable control function. This generalized framework provides greater flexibility for the study of nonlinear contractive mappings and convergence analysis.

Within this setting, we established fixed point results for various classes of contractions, including Banach-type, Kannan-type, Reich-type, and nonlinear rational-type contractions. The obtained theorems were proved under suitable orbit-control conditions that ensure the convergence of Picard iterative sequences. Consequently, our results unify and extend many classical fixed point theorems available in the existing literature on generalized metric spaces.

To demonstrate the applicability of the developed theory, several illustrative examples of extended A_b-metric spaces were presented. In addition, an application to nonlinear integral equations was discussed, showing that the proposed framework is not only theoretically significant but also useful in practical problems arising in nonlinear analysis.

The results obtained in this work open several directions for future research. One may investigate fixed point theory in fuzzy, probabilistic, or stochastic extended A_b-metric spaces, study multivalued and hybrid contractions, or explore applications in optimization theory, differential equations, and computational mathematics. We believe that extended A_b-metric spaces provide a fruitful and flexible framework for further developments in modern fixed point theory and its applications.

Acknowledgement

The authors would like to express their sincere gratitude to the reviewers and editors for their valuable comments, insightful suggestions, and constructive remarks, which significantly improved the quality and presentation of this manuscript. The authors are also thankful to their respective institutions for providing the necessary academic support and research facilities during the preparation of this work.

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundamenta Mathematicae*, Vol. 3, pp. 133–181, 1922. https://doi.org/10.4064/fm-3-1-133-181, DOI: 10.4064/fm-3-1-133-181.

[2] S. G. Matthews, Partial metric topology, *Annals of the New York Academy of Sciences*, Vol. 728, pp. 183–197, 1994. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x. DOI: 10.1111/j.1749-6632.1994.tb44144.x.

[3] M. Bukatin, R. Kopperman, S. Matthews, and H. Pajoohesh, Partial metric spaces, *The American Mathematical Monthly*, Vol. 116, pp. 708–718, 2009. https://doi.org/10.4169/193009709X460831, DOI: 10.4169/193009709X460831.

[4] L. B. Cirić, A generalization of Banach's contraction principle, *Proceedings of the American Mathematical Society*, Vol. 45, pp. 267–273, 1974. https://doi.org/10.1090/S0002-9939-1974-0356011-2, DOI: 10.1090/S0002-9939-1974-0356011-2.

[5] M. S. Khan, M. Swaleh, and S. Sessa, Fixed point theorems by altering distances between the points, *Bulletin of the Australian Mathematical Society*, Vol. 30, pp. 1–9, 1984. https://doi.org/10.1017/S0004972700001659, DOI: 10.1017/S0004972700001659.

[6] R. Kannan, Some results on fixed points, *Bulletin of the Calcutta Mathematical Society*, Vol. 60, pp. 71–76, 1968.

[7] R. Kannan, Some results on fixed points–II, *The American Mathematical Monthly*, Vol. 76, no. 4, pp. 405–408, 1969. https://doi.org/10.1080/00029890.1969.12000228, DOI: 10.1080/00029890.1969.12000228.

[8] F. Yan, Y. Su, and Q. Feng, A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations, *Fixed Point Theory and Applications*, Vol. 2007, Article ID 123456, 2007. https://doi.org/10.1155/2007/123456, DOI: 10.1155/2007/123456.

[9] R. P. Pant, Common fixed point theorems for contractive maps, *Journal of Mathematical Analysis and Applications*, Vol. 226, no. 1, pp. 251–258, 1998. https://doi.org/10.1006/jmaa.1998.6029, DOI: 10.1006/jmaa.1998.6029.

[10] D. S. Jaggi, Some unique fixed point theorems, *Indian Journal of Pure and Applied Mathematics*, Vol. 8, No. 2, pp. 223–230, 1977.

[11] S. Oltra and O. Valero, Banach's fixed point theorem for partial metric spaces, *Rendiconti dell'Istituto di Matematica dell'Università di Trieste*, Vol. 36, pp. 17–26, 2004.

[12] S. Shukla, Partial b-metric spaces and fixed point theorems, *Mediterranean Journal of Mathematics*, Vol. 11, pp. 703–711, 2014. https://doi.org/10.1007/s00009-013-0327-4, DOI: 10.1007/s00009-013-0327-4.

[13] G. Jungck, Compatible mappings and common fixed points, *International Journal of Mathematics and Mathematical Sciences*, Vol. 9, pp. 771–779, 1986. https://doi.org/10.1155/S0161171286000935, DOI: 10.1155/S0161171286000935.

[14] M. Edelstein, On fixed and periodic points under contractive mappings, *Journal of the London Mathematical Society*, Vol. 37, pp. 74–79, 1962.

https://doi.org/10.1112/jlms/s1-37.1.74, DOI: 10.1112/jlms/s1-37.1.74.

[15] N. Mlaiki, Extended S_b-metric spaces, *Journal of Mathematical Analysis*, Vol. 9, no. 1, pp. 124–135, 2018.

[16] S. Sedghi, N. Shobe, and A. Aliouche, A generalization of fixed point theorems in S-metric spaces, *Matematički Vesnik*, Vol. 64, pp. 258–266, 2012.

[17] S. Sedghi and N. V. Dung, Fixed point theorems on S-metric spaces, *Matematički Vesnik*, Vol. 66, pp. 113–124, 2014.

[18] S. Sedghi, I. Altun, N. Shobe, and M. A. Salahshour, Some properties of S-metric spaces and fixed point results, *Kyungpook Mathematical Journal*, Vol. 54, no. 1, pp. 113–122, 2014. https://doi.org/10.5666/KMJ.2014.54.1.113, DOI: 10.5666/KMJ.2014.54.1.113.

[19] M. Abbas, B. Ali, and Y. I. Suleiman, Generalized coupled common fixed point results in partially ordered A-metric spaces, *Fixed Point Theory and Applications*, Vol. 2015, article 64, 2015.
https://doi.org/10.1186/s13663-015-0313-7, DOI: 10.1186/s13663-015-0313-7.

[20] M. Ughade, D. Türkoğlu, S. R. Singh, and R. D. Daheriya, Some fixed point theorems in A_b-metric space, *British Journal of Mathematics & Computer Science*, Vol. 19, no. 6, pp. 1–24, 2016.
https://doi.org/10.9734/BJMCS/2016/29828, DOI: 10.9734/BJMCS/2016/29828.

[21] A. Bakhru, R. Gupta, and M. Ughade, Coincidence points and common fixed points of expansive mappings in A_b-metric spaces, *South East Asian Journal of Mathematics and Mathematical Sciences*, Vol. 17, no. 1, pp. 383–396, 2021.

[22] B. K. Dass and S. Gupta, An extension of Banach contraction principle through rational expressions, *Indian Journal of Pure and Applied Mathematics*, Vol. 6, pp. 1455–1458, 1975.

[23] S. Reich, Kannan's fixed point theorem, *Bollettino della Unione Matematica Italiana*, S-er. 4, Vol. 4, pp. 1–11, 1971.

[24] S. Reich, Some remarks concerning contraction mappings, *Canadian Mathematical Bulletin*, Vol. 14, No. 1, pp. 121–124, 1971.
https://doi.org/10.4153/CMB-1971-024-9, DOI: 10.4153/CMB-1971-024-9.

[25] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, *Journal of Nonlinear and Convex Analysis*, Vol. 7, No. 2, pp. 289–297, 2006.

[26] S. Czerwik, Contraction mappings in b-metric spaces, *Acta Mathematica et Informatica Universitatis Ostraviensis*, Vol. 1, pp. 5–11, 1993.

[27] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis: Theory, Methods & Applications*, Vol. 65, No. 7, pp. 1379–1393, 2006.
https://doi.org/10.1016/j.na.2005.10.017, DOI: 10.1016/j.na.2005.10.017.

[28] A. Aghajani, M. Abbas, and J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces, *Mathematica Slovaca*, Vol. 64, No. 4, pp. 941–960, 2014.
https://doi.org/10.2478/s12175-014-0250-6, DOI: 10.2478/s12175-014-0250-6.

[29] J. Jachymski, Equivalent conditions for generalized contractions on ordered metric spaces, *Nonlinear Analysis: Theory, Methods & Applications*, Vol. 74, No. 3, pp. 768–774, 2011.

https://doi.org/10.1016/j.na.2010.09.025, DOI: 10.1016/j.na.2010.09.025.

[30] H. Huang, S. Radenović, and J. Vujaković, On some recent coincidence results in partially ordered b-metric spaces, *Fixed Point Theory and Applications*, Vol. 2015, Article 63, 2015. https://doi.org/10.1186/s13663-015-0308-3, DOI: 10.1186/s13663-015-0308-3.

[31] N. Taş and N. Y. Özgür, New generalized fixed point results on S_b-metric spaces, *arXiv preprint*, 2017.
https://arxiv.org/abs/1703.01868, arXiv:1703.01868.

[32] S. Cobzaş, b-metric spaces, fixed points and Lipschitz functions, *arXiv preprint*, 2018. https://arxiv.org/abs/1802.02722, arXiv:1802.02722.

[33] M. Jleli and B. Samet, On a new generalization of metric spaces, *Journal of Fixed Point Theory and Applications*, Vol. 20, Article 128, 2018.
https://doi.org/10.1007/s11784-018-0603-9, DOI: 10.1007/s11784-018-0603-9.

[34] M. Cosentino and P. Vetro, Fixed point results for F-contractive mappings of Hardy–Rogers-type, *Filomat*, Vol. 28, No. 4, pp. 715–722, 2014.
https://doi.org/10.2298/FIL1404715C, DOI: 10.2298/FIL1404715C.

[35] E. Karapınar, Revisiting the Kannan type contractions via interpolation, *Advances in the Theory of Nonlinear Analysis and its Applications*, Vol. 2, No. 2, pp. 85–87, 2018.
https://doi.org/10.31197/atnaa.397983, DOI: 10.31197/atnaa.397983.

International J.Math. Combin. Vol.2(2026), 53-65

Automorphic and Holomorphic Characterization of Crazy Loops

Olufemi Olakunle George

(Department of Mathematics, University of Lagos, Akoka, 100213, Nigeria)

E-mail: oogeorge@unilag.edu.ng, femi_george@yahoo.co.uk

Abstract: We study the loop variety defined by the identity $(x \cdot (xy)) \cdot z = (y \cdot x) \ \cdot (x \cdot z)$, which we call *Crazy loops*. We show that Crazy loops are flexible, satisfy left and right inverse properties, and obey the square-symmetric law. For a loop Q with automorphism group $A(Q)$, we characterise when its $A(Q)$-holomorph is itself a Crazy loop: this occurs precisely when the automorphism group satisfies $\alpha^2 \in C_{A(Q)}(\beta)$ for all $\alpha, \beta \in A(Q)$ and when $(x\beta \cdot (x\beta)y)z = yx \cdot xz$ holds in Q for all $x, y, z \in Q$. In particular, it is proved that the $A(Q)$ holomorph $(H, \circ)$ of a loop $(Q, \cdot)$ is a Crazy loop if and only if $(Q, \cdot)$ is a Crazy loop and $(x\beta)^2 \cdot x^\lambda \in N_\rho$. We also describe explicit autotopisms induced by the Crazy identity and show how these restrict to an automorphism of the nucleus. Finally we showed that if $(H, \circ)$ is the $A(Q)$ holomorph of a Crazy loop that is an extra loop and $x\beta \cdot x^\lambda \in N_\lambda$. Then $T(x)^{-1}$ is an automorphism.

Key Words: Loop, Crazy loop, generalized Bol-Moufang type loops, holomorph, automorphism.

AMS(2010): 20N02, 20N05.

§1. Introduction

An identity $\alpha = \beta$ is said to be of *BolCMoufang type* (first Bol-Mog type [8]) if it involves three distinct variables in which two variables occur once on each side while the third occurs twice, and the variables appear in the same relative order on both sides [26, 27]. The broader family of *generalized BolCMoufang* identities arises when the variables do not appear in the same order on both sides of the equal sign. A complete classification of such identities was carried out by Côté, et.al [5], who listed all admissible identities and identified a number of loop varieties defined by them.

Several loop varieties belonging to this scheme have been examined in recent work [6, 18, 20, 21, 23, 25]. Examples include Frute loops, Cheban loops and Mate-1 loops. Among the identities appearing in the classification of Côté et al. is the Crazy identity defined by

$$(x \cdot (xy))z = (yx) \cdot (xz),$$

[1]Received November 25, 2025, Accepted May 28, 2026

The study of holomorphs has provided a powerful framework for analysing loop varieties. Pioneering contributions of Bruck [2] established the role of holomorphs in inverse property loops, while subsequent works by Robinson [28, 29], Huthnance [14], Adeniran [1], Chiboka and Solarin [4], Isere et al.[15] extended holomorphic methods to Bol, extra, weak inverse property loops, generalized Bol loops, conjugacy closed loops and Osborn. More recent developments, particularly those involving WIP and power-associative conjugacy-closed loops [9], have demonstrated that the interaction between loop identities and the automorphism group often encodes the essential algebraic structure of a loop.

Despite this progress, the holomorphic and automorphic structure of Crazy loops has remained completely unexplored. Nothing is known about the behavior of their automorphisms, pseudoautomorphisms, or the internal symmetries that the defining identity imposes. In particular, it is not clear which automorphism-type mappings lift to autotopisms in the holomorph, nor how the Crazy identity shapes the nucleus, inner mappings, or translation structure of the loop.

The aim of this paper is to initiate a systematic study of this variety. We focus on the interaction between the Crazy identity and three central objects: the automorphism group, the pseudoautomorphism group, and the autotopism group of the holomorph. Our goal is to obtain an autotopic characterization of Crazy loops, describe the pseudoautomorphisms they admit, and determine when the holomorph itself inherits the Crazy property.

§2. **Preliminaries**

A loop $(Q, \cdot)$ is a set equipped with a binary operation such that

(1) For every $x \in Q$, the left and right translations

$$L_x(y) = x \cdot y, \qquad R_x(y) = y \cdot x \qquad (y \in Q).$$

are bijections;

(2) There exists an element $e \in Q$ such that

$$ex = xe = x, \ \ \forall x \in Q.$$

Since all translations are bijective, their inverses L_x^{-1} and R_x^{-1} are well defined. We write $x\backslash y = L_x^{-1}(y)$ and $y/x = R_x^{-1}(y)$. For an overview of the theory of loops, readers may check [3, 16, 24, 30]. Generally, a Smarandache loop L is such a loop that a proper subset $A \subset L$ is a subgroup of L respective to the same induced operation. Particularly, if $L = \emptyset$, a Smarandache loop is nothing else but a loop. For the future research, it would be of interest to investigate whether the properties established in this paper can be extended to broader algebraic systems. Specifically, applying these results to Smarandache loops – loops which embed a proper subgroup – could yield interesting algebraic variations.

Definition 2.1 (Nuclei) *The left, middle and right nuclei are*

$$N_\lambda = \{a \in Q : a \cdot (xy) = (ax) \cdot y \ \forall x, y\},$$
$$N_\mu = \{a \in Q : x \cdot (ay) = (xa) \cdot y \ \forall x, y\},$$
$$N_\rho = \{a \in Q : (xy) \cdot a = x \cdot (ya) \ \forall x, y\}$$

and the nucleus $N(Q) = N_\lambda \cap N_\mu \cap N_\rho$.

Remark 2.2 Membership of nuclei admits the usual autotopism description:

$$\begin{aligned} a \in N_\lambda \quad &\iff \quad (L_a, I, L_a) \in \mathrm{Atp}(Q), \quad a \in N_\mu \\ &\iff \quad (R_a^{-1}, L_a, I) \in \mathrm{Atp}(Q), \quad a \in N_\rho \\ &\iff \quad (I, R_a, R_a) \in \mathrm{Atp}(Q). \end{aligned}$$

The commutant of Q is given by

$$C(Q) = \{c : \forall x \in Q, cx = xc\}.$$

A loop Q is said to have the centrum square property if

$$x^2 \in C(Q). \tag{CSP}$$

The center of a loop $Z(Q)$ is defined by $Z(Q) = N(Q) \cap C(Q)$ and a loop has the square symmetric property if it satisfies

$$x \cdot xy = yx \cdot x. \tag{SSP}$$

The left alternative property of a loop is defined as

$$xx \cdot y = x \cdot xy, \tag{LAP}$$

and the right alternative property is given by

$$y \cdot xx = yx \cdot x. \tag{RAP}$$

A loop is an alternative loop if it is left and right alternative. Flexible loops satisfy

$$x \cdot yx = xy \cdot x \tag{FLEX}$$

A loop Q satisfies the left inverse property if

$$x^\lambda \cdot xy = y \tag{LIP}$$

and the right inverse property if

$$xy \cdot y^\rho = x. \tag{RIP}$$

An inverse property loop is a loop that satisfies both the (LIP) and the (RIP) and a loop is a weak inverse property loop if it satisfies any one of the following identities:

$$x(yx)^\rho = y^\rho \quad or \quad (xy)^\lambda x = y^\lambda. \tag{WIP}$$

Definition 2.3(Autotopisms, automorphisms, pseudoautomorphisms) *An ordered triple* (U, V, W) *of bijections of* Q *is an autotopism if*

$$xU \cdot yV = (x \cdot y)W \qquad (\forall x, y \in Q).$$

The set of all autotopisms forms the group $\mathrm{Atp}(Q)$*.*

A bijection $\alpha : Q \to Q$ *is an automorphism when* $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$*, equivalently* $(\alpha, \alpha, \alpha) \in \mathrm{Atp}(Q)$*.*

A map U *is a left pseudoautomorphism with companion* $c \in Q$ *if*

$$(UL_c,\ U,\ UL_c) \in \mathrm{Atp}(Q),$$

where L_c *denotes left multiplication by* c*. Right and middle pseudoautomorphisms are defined analogously.*

A loop $(Q, \cdot)$ is called an extra loop if Q satisfies one (equivalently all) of the following identities

$$(x \cdot yz)y = xy \cdot zy. \tag{RE}$$

$$yz \cdot yx = y(zy \cdot x). \tag{LE}$$

$$(xy \cdot z)x = x(y \cdot zx) \tag{ME}$$

and extra loops of second Bol -Moufang type have been introduced in [12].

Definition 2.45 *Let* $\mathrm{Aut}(Q)$ *be the automorphism group of* Q*. The holomorph* $H = \mathrm{Aut}(Q) \ltimes Q$ *is the semidirect product with multiplication*

$$(\alpha, a) \circ (\beta, b) = (\alpha\beta,\ \alpha(b) \cdot a),$$

where automorphisms act on Q *on the left. The neutral element is* $(\mathrm{id}, 1)$*.*

§3. Main Results

Definition 3.1 *A loop is a Crazy loop if it satisfies*

$$(x(xy))z = (yx)(xz). \tag{1}$$

Proposition 3.2 *In a loop Q, the Crazy identity is equivalent to each of the following:*

$$\begin{aligned} L_{x\cdot xy}L_x^{-1} &= L_{yx}. \\ R_zL_x^2 &= R_{xz}R_x. \\ L_{yx}^{-1}L_{x\cdot xy} &= L_x \end{aligned}$$

Lemma 3.3 *A loop Q is a Crazy loop if and only if $A = (R_xL_x^{-2},\ L_x,\ I)$ is an autotopism of Q.*

Proof Write the crazy identity as $yz = (x \setminus (x \setminus y))x \cdot xz$ and then, the rest is straight forward. □

Lemma 3.4 *In any loop Q, the following are true:*

(1) *Any two of $L_{x^2} = R_x^2$, $L_x^2 = R_x^2$ and $L_{x^2} = L_x^2$ imply the third;*
(2) *Any two of $R_{x^2} = L_x^2$, $L_x^2 = R_x^2$ and $R_{x^2} = R_x^2$ imply the third;*
(3) *Any two of $R_{x^2} = L_x^2$, $L_{x^2} = R_{x^2}$ and $L_{x^2} = L_x^2$ imply the third;*
(4) *Any two of $L_{x^2} = R_{x^2}$, $L_{x^2} = R_x^2$ and $R_{x^2} = R_x^2$ imply the third;*
(5) *$R_{x^2} = R_x^2$, $L_{x^2} = L_x^2$ and $L_x^2 = R_x^2$ imply $R_{x^2} = L_{x^2}$;*
(6) *$R_{x^2} = R_x^2$, $L_{x^2} = L_x^2$ and $R_{x^2} = L_{x^2}$ imply $L_x^2 = R_x^2$;*
(7) *If $x^2 \in C(L)$ and x^2 is in any two of N_λ, N_μ, and N_ρ, then x^2 belongs to all three.*

Proof $(1) - (3)$ and (4) are obvious.

For (5), $L_{x^2} = L_x^2 = R_x^2 = R_{x^2}$, for (6) we have $R_x^2 = R_{x^2} = L_{x^2} = L_x^2$ and for (7), we only consider the case where $x^2 \in C(L)$ and $x^2 \in N_\lambda \bigcap N_\mu$. Then,

$$yz \cdot x^2 = x^2 \cdot yz = x^2y \cdot z = yx^2 \cdot z = y \cdot x^2z = y \cdot zx^2.$$

Thus, $x^2 \in N_\rho$. □

Lemma 3.5 *Let Q be a Crazy loop. Then,*

$(1) L_x^2 = R_x^2;$	$(2) L_{x^2} = L_x^2;$	$(3) R_xL_x^2 = R_{x^2}R_x;$
$(4) x^\lambda = x^\rho;$	$(5) R_xL_x = L_xR_x;$	$(6) L_x^2 = R_{x^2};$
$(7) R_x^2 = R_{x^2};$	$(8) R_{x^2} = L_{x^2};$	$(9) R_xL_x^2 = L_x^2R_x.$

Proof (1) Setting $z = 1$ in the Crazy identity.
(2) Put $y = 1$ in the Crazy identity.
(3) Replace z with x in the Crazy identity.
(4) Put $z = x^\rho$ in the Crazy identity and use Lemma 3.5(1).
(5) With $z = y$ in the Crazy identity and Lemma 3.5(1).
(6) This is from (3) and (5).
(7) Use Lemma 3.5(1). and Lemma 3.5(6).

(8) Use Lemma 3.5(2). and Lemma 3.5(6).

(9) Use Lemma 3.5(6). on the right hand side of Lemma 3.5(3).

This completes the proof. □

Remark 3.6 The identity in Lemma 3.5(9). have been used to study power associative conjugacy closed loops with the Weak inverse power associative loops in [7, 10, 11, 13, 17, 19].

Theorem 3.7 *If Q is a Crazy loop, then, (i) $x^\lambda \cdot xz = z$; (ii) $yx \cdot x^\rho = y$; (iii) $x(yx)^\rho = y^\rho$; (iv) $N_\lambda = N_\rho = N_\mu$.*

Proof (i) The Crazy identity is equivalent to

$$(x\backslash(x\backslash y)) \cdot xz = yz,$$

put $y = 1$ to get

$$(x\backslash(x^\rho))x \ \cdot xz = z. \tag{5}$$

If we set $z = 1$ in (5), we have

$$(x\backslash(x^\rho))x = x^\lambda. \tag{6}$$

From equations (5) and (6), We have

$$x^\lambda \cdot xz = z.$$

(ii) Again if we put $z = x^\rho$ in the Crazy identity, we obtain

$$yx = (x \cdot xy)x^\rho = (yx \cdot x)x^\rho \Leftrightarrow yx \cdot x^\rho = y$$

(iii) Any loop satisfying (i) and (ii) will satisfy (iii).

(iv) is an implication of (i) and (ii). □

Theorem 3.8 *Let Q be an extra loop. Then, the following are equivalent:*

(1) *Q is Crazy loop;*

(2) *$T(x) = T^{-1}(x) = T(x^{-1})$ is an automorphism of Q.*

Proof (1) ⇒ (2): Let Q be a middle extra loop, then $C = (L_x,\ R_x^{-1},\ R_x^{-1}L_x)$ is an autotopism of Q and suppose Q is also a Crazy loop, we have by Lemma 3.3, $A = (R_xL_x^{-2},\ Lx,\ I)$ is also an autotopism of Q. Therefore, $AC = (R_xL_x^{-1},\ L_xR_x^{-1},\ R_x^{-1}L_x)$. The three components are equal since a Crazy loop is flexible, satisfies the square symmetric property and is an inverse property loop.

(2) ⇒ (1): It is similar to the proof of (1) ⇒ (2). □

Theorem 3.9([24]) *Let $A = (U,\ V,\ W)$ be the autotopism of a loop Q. Then,*

(1) *If Q is a left inverse property loop, then $A_\lambda = (IUI,\ W,\ V)$ is also an autotopism of Q;*

(2) *If Q is a right inverse property loop, then $A_\lambda = (W,\ IVI,\ U)$ is also an autotopism of Q.*

Theorem 3.10 *Let $A(Q)$ be the automorphism group of a loop $(Q,\cdot)$. Then, the $A(Q)$ holomorph $(H,\circ)$ of $(Q,\cdot)$ is a Crazy loop if and only if the following two conditions hold:*

(1) *For every $\alpha,\beta\in A(Q)$,*

$$\alpha^2\beta=\beta\alpha^2;$$

(2) *For all $x,y,z\in Q$ and $\alpha^2=id,\ \beta\in A(Q)$,*

$$(x\beta\cdot\ (x\beta)y)z=yx\cdot xz. \tag{7}$$

Proof Let $U=(\alpha,x),\ V=(\beta,y),\ W=(\delta,z)$. Then,

$$\begin{aligned}
U\cdot V &= (\alpha\beta,\ x\beta\cdot y),\\
U\cdot(U\cdot V) &= (\alpha^2\beta,\ \ x(\alpha\beta)\cdot\ x\beta\cdot y),\\
(U\cdot(U\cdot V))\cdot W &= (\alpha^2\beta\delta,\ \ (x\alpha\beta\cdot x\beta\cdot y)\delta\cdot z),\\
V\cdot U &= (\beta\delta,\ y\alpha\cdot x),\\
U\cdot W &= (\alpha\delta,\ x\delta\cdot z),\\
(V\cdot U)\ \cdot(U\cdot W) &= (\beta\alpha^2\delta,\ \ (y\alpha\cdot x)\alpha\delta\cdot\ x\delta\cdot z).
\end{aligned}$$

For $(H,\circ)$ to be a Crazy loop, we must have

$$(U\cdot(U\cdot V))\cdot W=(V\cdot U)\ \cdot(U\cdot W),$$

i.e.,

$$(\alpha^2\beta\delta,\ \ (x\alpha\beta\cdot x\beta\cdot y)\delta\cdot z)=(\beta\alpha^2\delta,\ \ (y\alpha\cdot x)\alpha\delta\cdot\ x\delta\cdot z).$$

Equating the components gives the first component

$$\alpha^2\beta\delta=\beta\alpha^2\delta.$$

Cancel δ on the right to obtain

$$\alpha^2\beta=\beta\alpha^2$$

and second component

$$(x\alpha\beta\cdot(x\beta\cdot y))\delta\cdot z=(y\alpha\cdot x)\alpha\delta\cdot\ x\delta\cdot z,$$

$$(x\beta\ \cdot(x\alpha^{-1}\beta\cdot y\alpha^{-1}))z\delta^{-1}\alpha^{-1}=(y\alpha\cdot x)\cdot(x\alpha^{-1}\cdot z\delta^{-1}\alpha^{-1}).$$

Thus, the equation (7) holds if $\alpha^2=id$. □

Corollary 3.11 *Let Q be a Crazy loop and A an automorphism group of Q. Then, the $A(Q)$ holomorph is Crazy if and only if $B=(R_xL^{-2}_{(x\beta)},\ L_x,\ I)$ is an autotopism of Q for all $x\in Q$*

and $\beta \in A$.

Lemma 3.12([14]) *Let Q be a weak inverse property loop and let A be the automorphisms group of Q. Then, the holomorph $A(Q)$ is a weak inverse property loop.*

Lemma 3.13 *Let $(Q, \cdot)$ be a loop and let $A(Q)$ be the automorphism group of Q.*

(i) If $R_x^2 = R_{x^2}$, then $R^2_{(x\beta)} = R_{(x\beta)^2}$;

(ii) If $L_x^2 = L_{x^2}$, then $L^2_{(x\beta)} = L_{(x\beta)^2}$;

(iii) If $L_x^2 = R_{x^2}$, then $L^2_{(x\beta)} = R_{(x\beta)^2}$.

Proof (*i*) By the hypothesis,
$$(y \cdot x) \cdot x = y \cdot x^2.$$

Applying the automorphism β to the RAP identity

$$((y \cdot x) \cdot x)\beta = (y \cdot x^2)\beta.$$

Since β is homomorphism,

$$(y \cdot x)\beta \cdot x\beta = y\beta \cdot (x^2)\beta \Leftrightarrow (y\beta \cdot x\beta) \cdot x\beta = y\beta \cdot ((x\beta) \cdot (x\beta)).$$

$$(y' \cdot x\beta) \cdot x\beta = y' \cdot ((x\beta) \cdot (x\beta)) \Leftrightarrow y' R^2_{(x\beta)} = y' R_{(x\beta)^2}.$$

(*ii*) and (*iii*) can be prove in a similar way as (*i*). □

Lemma 3.14([22]) *Let Q be a WIPL. If $(U,\ V,\ W) \in Atp(Q)$. Then,*

(1) $(V,\ \lambda W\rho,\ \lambda U\rho) \in Atp(Q)$;

(2) $(\rho W\lambda,\ U,\ \rho V\lambda) \in Atp(Q)$.

Theorem 3.15 *Let $(Q, \cdot)$ be a loop and let $A(Q)$ be the group of automorphisms of $(Q, \cdot)$. Then, the $A(Q)$ holomorph $(H, \circ)$ of $(Q, \cdot)$ is a Crazy loop if and only if $(Q, \cdot)$ is a Crazy loop and $(x\beta)^2 \cdot x^\lambda \in N_\rho$.*

Proof Let $(H, \circ)$ be a Crazy loop, then $(Q, \cdot)$ is a Crazy loop since it is isomorphic to subloop of $(H, \circ)$. Now, by Corollary 3.11, $(R_x L^{-2}_{x\beta},\ L_x,\ I)$ is an autotopism of Q, and by Theorem 3.7, Q has WIP, and since the holomorph of a WIP loop is WIP by Lemma 3.12, we see that
$$(I,\ L^2_{x\beta} R_x^{-1},\ R_x) \tag{8}$$
is an autotopism of Q by Lemma 3.14(2), this is equivalent to

$$y\ \cdot ((x\beta) \cdot (x\beta)(z/x)) = yz \cdot x,$$

with $z = 1$, we have
$$yR((x\beta)^2 \cdot x^\lambda) = yR_x.$$

Therefore, the autotopism in (8) becomes

$$(I,\ R\big((x\beta)^2\cdot x^\lambda\big),\ R\big((x\beta)^2\cdot x^\lambda\big)).$$

Thus, $(x\beta)^2\cdot x^\lambda\in N_\rho$.

Conversely, let Q be a Crazy loop and let each $(x\beta)^2\cdot x^\lambda\in N_\rho$,. i.e.,

$$(I,\ R\big((x\beta)^2\cdot x^\lambda\big),\ R\big((x\beta)^2\cdot x^\lambda\big)). \tag{9}$$

Note that in a Crazy loop,

$$N_\rho=N_\mu=N_\lambda.$$

Let

$$t=(x\beta)^2\cdot x^\lambda\in N_\mu.$$

Then

$$tx=((x\beta)^2\ \cdot x^\lambda)\cdot x=(x\beta)^2.$$

Using the middle nuclear property of t,

$$zt\cdot x=z\cdot tx=z\cdot(x\beta)^2.$$

$$zt\cdot x=z\cdot(x\beta)^2. \tag{10}$$

But by Lemma 3.13,

$$\begin{aligned} z\cdot(x\beta)^2 &= (x\beta)\cdot((x\beta)\cdot z),\\ zt\cdot x &= (x\beta)\cdot((x\beta)\cdot z),\\ zt &= ((x\beta)\cdot((x\beta)\cdot z))/x. \end{aligned}$$

Therefore,

$$zt=R_x^{-1}L^2_{(x\beta)}(z).$$

Again, $zt=R_t(z)$, and on the other hand,

$$L^2_{x\beta}R_x^{-1}(z)=(x\beta)\cdot((x\beta)\cdot(z/x)).$$

Multiply the last expression on the right by x and use the fact that $(x\beta)\in N_\lambda$ if $t\in N_\lambda$,

$$((x\beta)\cdot((x\beta)(z/x)))x=(x\beta)^2\cdot(z/x)x=x\beta\cdot(x\beta)z.$$

Therefore,

$$(x\beta)\cdot((x\beta)(z/x)=((x\beta)\cdot((x\beta)z]/x.$$

That is

$$L^2_{x\beta}R_x^{-1}=R_x^{-1}L^2_{(x\beta)}.$$

Thus, $t = L_{x\beta}^2 R_x^{-1} = R_x^{-1} L_{(x\beta)}^2$ and therefore the autotopism in (9) becomes

$$(I,\ R(t),\ R(t)) = (I,\ L_{x\beta}^2 R_x^{-1},\ L_{x\beta}^2 R_x^{-1}). \tag{11}$$

Now equation (10) says $zt \cdot x = z \cdot (x\beta)^2$, if we put $t = x$ and use Lemma 3.13(iii), we obtain

$$zR_x^2 = zL_{(x\beta)}^2 \Leftrightarrow R_x^2 L_{(x\beta)}^{-2} = id \Leftrightarrow (I,\ I,\ R_x^2 L_{(x\beta)}^{-2})$$

is an autotopism.

Multiplying this autotopism with the autotopism in (11), we obtain

$$(I,\ L_{x\beta}^2 R_x^{-1},\ R_x),$$

and since a Crazy loop has WIP, and by Lemma 3.14(1), the last autotopism becomes

$$(R_x L_{(x\beta)}^{-2},\ L_x,\ I).$$

Thus, $(H, \circ)$ is a Crazy loop. □

Corollary 3.16 *Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop $(Q, \cdot)$. Then, each $\beta \in A(Q)$ is nuclear.*

Theorem 3.17 *Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop $(Q, \cdot)$. Then,*

(1) $(L_x^2 L_{(x\beta)}^{-2},\ I,\ I)$ *is an autotopism of Q;*

(2) *If $N_\lambda(Q) = \{n \in Q : n \cdot (xy) = (nx) \cdot y,\ \forall x, y \in Q\}$. Then*

$$(L_x^2 L_{(x\beta)}^{-2}) N_\lambda(Q) = N_\lambda(Q);$$

(3) *The restriction $(L_x^2 L_{(x\beta)}^{-2}) \mid N_\lambda : N_\lambda(Q) \mapsto N_\lambda(Q)$ is an automorphism of the subloop $N_\lambda(Q)$.*

Proof (1) Compute the autotopism $A^{-1}B$.

(2) Let $n \in N_\lambda(Q)$ and let $u, v, y \in Q$ such that $y = uv$. From (1), we have

$$L_x^2 L_{(x\beta)}^{-2}(n) \cdot y = ny \Leftrightarrow L_x^2 L_{(x\beta)}^{-2}(n) \cdot uv = n \cdot uv. \tag{12}$$

By the left nucleus property of n, we have

$$n \cdot uv = nu \cdot v. \tag{13}$$

Again by Theorem 3.17(1),

$$L_x^2 L_{(x\beta)}^{-2}(n) \cdot u = nu. \tag{14}$$

We therefore have from the three equations above,

$$(L_x^2 L_{(x\beta)}^{-2}(n) \cdot u) \cdot v = n \cdot uv = L_x^2 L_{(x\beta)}^{-2}(n) \cdot uv. \Leftrightarrow L_x^2 L_{(x\beta)}^{-2}(n) \in N_\lambda.$$

Since $L_x^2L_{(x\beta)}^{-2}$ is a bijection and its image is equal to $N_\lambda(Q)$. Thus, $L_x^2L_{(x\beta)}^{-2})N_\lambda(Q) \subseteq N_\lambda(Q)$.

(3) Let $a, b \in N_\lambda(Q)$, then for arbitrary $y \in Q$, compute

$$(L_x^2L_{(x\beta)}^{-2}(a) \cdot L_x^2L_{(x\beta)}^{-2}(b)) \ \cdot y = L_x^2L_{(x\beta)}^{-2})(a) \ \cdot \ (L_x^2L_{(x\beta)}^{-2})(b) \cdot y). \tag{15}$$

Applying the autotopism in Theorem 3.17(1) to the product by

$$L_x^2L_{(x\beta)}^{-2}(b) \cdot y = b \cdot y,$$

equation (15) now becomes

$$(L_x^2L_{(x\beta)}^{-2})(a) \cdot \ (L_x^2L_{(x\beta)}^{-2})(b)) \cdot \ y = L_x^2L_{(x\beta)}^{-2})(a) \ \cdot (b \cdot y). \tag{16}$$

Again apply the autotopism in Theorem 3.17(1) to the product of a and by and use the fact that $a \in N_\lambda$,

$$L_x^2L_{(x\beta)}^{-2}(a) \cdot (b \cdot y) = a \cdot (b \cdot y) = (a \cdot b) \cdot y. \tag{17}$$

Also, apply the autotopism in Theorem 3.17(1) to the product of ab and y,

$$L_x^2L_{(x\beta)}^{-2}(ab) \cdot y = (a \cdot b) \cdot y. \tag{18}$$

Combining equations (15), (16), (17) and (18),

$$(L_x^2L_{(x\beta)}^{-2}(a) \cdot L_x^2L_{(x\beta)}^{-2}(b)) \ \cdot y = L_x^2L_{(x\beta)}^{-2}(ab) \cdot y$$

and by right cancellation,

$$(L_x^2L_{(x\beta)}^{-2}(a) \cdot L_x^2L_{(x\beta)}^{-2}(b)) = L_x^2L_{(x\beta)}^{-2}(ab).$$

This completes the proof. $\square$

Corollary 3.18 *Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop $(Q, \cdot)$. and let $L_x^2L_{(x\beta)}^{-2}(y) \equiv (y)\ (Mod\ N(Q))$ for all $y \in Q$. Then, $L_x^2L_{(x\beta)}^{-2}$ is a nuclear automorphism.*

Proof Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop $(Q, \cdot)$ and let $L_x^2L_{(x\beta)}^{-2}(y) \equiv (y)\ (Mod\ N(Q))$. Since $N_\lambda = N_\rho = N_\mu$ in a Crazy loop, then $L_x^2L_{(x\beta)}^{-2}$ acts trivially on the coset of N and induces the identity map on the quotient $Q/N(Q)$. This combined with (3) of Theorem 3.17, shows that $L_x^2L_{(x\beta)}^{-2}$ restricts to an automorphism of $N(Q)$ and therefore is automorphism on the nucleus. $\square$

Lemma 3.19([29]) *Let $(Q, \cdot)$ be a loop and let $A(Q)$ be a group of automorphism of $(Q, \cdot)$. Then, the $A(Q)$ holomorph $(H, \circ)$ of $(Q, \cdot)$ is an extra loop if and only if $C_\beta = (L_{x\beta},\ R_x^{-1},\ R_x^{-1}L_{x\beta})$ is an autotopism of $(Q, \cdot)$.*

Theorem 3.20 *Let $(H, \circ)$ be the $A(Q)$ holomorph of a Crazy loop that is an extra loop. Then,*

$T(x)^{-1}$ *is a left pseudoautomorphism with companion* $x\beta \cdot x^{\lambda}$.

Proof Let $(H, \circ)$ be the $A(Q)$ holomorph of an extra loop, then

$$C_{\beta} = (L_{x\beta},\ R_x^{-1},\ R_x^{-1}L_{x\beta})$$

and suppose $(H, \circ)$ is also a Crazy loop, we also have

$$B = (R_x L_{(x\beta)}^{-2},\ L_x,\ I).$$

Then

$$BC_{\beta} = (R_x L_{(x\beta)}^{-1},\ L_x R_x^{-1},\ R_x^{-1}L_{x\beta}).$$

Since $L_x R_x^{-1}(1) = 1$, we therefore have $T(x)^{-1}$ is left pseudoautomorphism with companion $R_x^{-1}L_{(x\beta)}(1) = L_{(x\beta)}R_x^{-1}(1) = x\beta \cdot x^{\lambda}$. □

Corollary 3.21 *Let* $(H, \circ)$ *be the* $A(Q)$ *holomorph of a Crazy loop. If* $(H, \circ)$ *is an extra loop and* $x\beta \cdot x^{\lambda} \in N_{\lambda}$*, then* $T(x)^{-1}$ *is an automorphism.*

References

[1] J. O. Adeniran, T. G. Jaiyeola and K. O. Idowu, Holomorph of generalized Bol loops, *Novi Sad Journal of Mathematics*, 44(1) (2014), 37C51.

[2] R. H. Bruck, Contributions to the theory of loops, *Trans. Amer. Math. Soc.*, 55 (1944), 245C354.

[3] R. H. Bruck, *A Survey of Binary Systems*, Springer-Verlag, BerlinCGöttingenCHeidelberg, 1958.

[4] V. O. Chiboka and A. R. T. Solarin, Holomorphs of conjugacy closed loops, *Annals of Al. I. Cuza*, 37(3) (1991), 277C284.

[5] B. Coté, B. Harvill, M. Huhn and A. Kirchman, Classification of loops of generalized Bol-Moufang type, *Quasigroups and Related Systems*, 19 (2011), 193-206.

[6] A. Drapal and J. D. Phillips, The final Moufang variety: Frute loops, *Publicationes Mathematicae*, 95 (2019), 3-4- DOI:10.5486/PMD.2019.8579-P. 477-486.

[7] O. O. George, J. O. Olaleru, J. O. Adéniran and T. G. Jaiyéolá, On a class of power associative LCC-loops, *Extracta Mathematicae*, 37(2) (2022), 185C194.

[8] O. O. George and T. G. Jaiyol, Nuclear identification of some new loop identities of length five, *Bull. Acad. Sci. Moldova, Math.*, 99(2) (2022), 39C58.

[9] O. O. George, On holomorph of WIP PACC loops, *Jordan Journal of Mathematics and Statistics*, 16(3) (2023), 463C482.

[10] O. O. George, Semidirect product of weak inverse property power associative conjugacy closed loops, *Annals of Mathematics and Computer science, Dubai.*, 9(2022), 91-100.

[11] O. O. George, Buchsteiner and conjugacy quasigroups of second Bol-Moufang type, *Romai Journal*, 21(2) (2025), 81C88.

[12] O. O. George, Nuclear identification of extra loop identities of second Bol-Moufang type

with applications to secure information encoding, *Unilag Journal of Mathematics and Applications*, 6(1) (2026), 33C42.

[13] O. O. George and T. G. Jaíyéọlá, Algebraic properties and representations of a class of power associative RCC-loops, *Proyecciones Journal of Mathematics* , 45(1),(2026), 121-135.

[14] E. D. Huthnance, *A Theory of Generalized Moufang Loops*, Ph.D. Thesis, Georgia Institute of Technology, 1968.

[15] A. O. Isere, J. O. Adeniran and T. G. Jaiyeola, Holomorphy of Osborn loops, *Analele Universitatii de Vest Timisoara*, 53(2) (2015), 81C98.

[16] T. G. Jaíyéọlá, *A study of New Concepts in Smarandache Quasigroups and Loops*, ProQuest Information and Learning(ILQ), Ann Arbor, USA, 2009.

[17] T. G. Jaíyéọlá, O. O. George, B. Osoba and E. Ilojide, A class of power associative LCC-loops and some associated total inner mapping group questions, *Algebras, Groups and Geometries*, 41 (2025), 45–65.

[18] T. G. Jaíyéọlá, O. O. George, O. Ajayi and A. Saka, Mate I loops, *Quasigroups and Related Systems*, 34, (2026), 111-130.

[19] T. G. Jaíyéọlá and O. O. George, On loops that satisfy $x\cdot(x\cdot yx)z = (x\cdot xy)\cdot xz$, *Comment. Math. Univ. Carolin.*, 66(1), (2025), 29-36.

[20] Jaíyéọlá, T. G. Adeniregun, A. A. and Asiru, M. A. (2017), Finite FRUTE loops, *J. Algebra Appl.*, 16(2) , 1750040, 10pp.
http://dx.doi.org/10.1142/S0219498817500402.

[21] Jaíyéọlá, T. G. Adeniregun, A. A. Oyebola, O. O. and Adelakun, A. O. (2021), FRUTE loops, *Algebras, Groups and Geometries*, 37(2), 159–179. DOI 10.29083/AGG.37.02.2021.

[22] J. M. Osborn, Loops with the weak inverse property, *Pacific J. Math.*, 10 (1960), 295C304.

[23] B. Osoba, T. G. Jaiyéọlá, and A. O. Abdulkareem, Variations of some inverse properties in Cheban loop, *Quasigroup and Related Systems*, 33(1) (2025), 95-106.
https://doi.org/10.56415/qrs.v33.08.

[24] H. O. Pflugfelder, Quasigroups and loops: Introduction, *Sigma Series in Pure Math.* 7, Heldermann Verlag, Berlin, 1990.

[25] Phillips, J. D. and Shcherbacov, V. A. (2010), Cheban loops, *J. Gen. Lie Theory Appl.* 4, Art. ID G100501, 5 pp.

[26] J. D. Phillips and P. Vojtěchovský, The varieties of quasigroups of Bol-Moufang type, *J. Algebra*, 293 (2005), 17-33.

[27] J. D. Phillips and P. Vojtěchovský, The varieties of loops of Bol-Moufang type, *Algebra Universalis*, 54(3) (2005), 259-271.

[28] D. A. Robinson, *Bol Loops*, Ph.D. Thesis, University of Wisconsin, Madison, 1964.

[29] D. A. Robinson, Holomorphic theory of extra loops, *Publ. Math. Debrecen*,c18 (1971), 59C64.

[30] V. Shcherbacov, *Elements of Quasigroup Theory and Applications,* CRC Press, Boca Raton, 2017.

International J.Math. Combin. Vol.2(2026), 66-84

Non-Newtonian Approach to Some Quaternion Types And Cayley Numbers

Ceremnur Tetik and Abdullah Dertli

(Ondokuz Mayıs University, Samsun, Türkiye)

E-mail: ceremnurtetik@gmail.com, abdullah.dertli@gmail.com

Abstract: In this study, we introduce non-Newtonian quaternions, non-Newtonian Fibonacci quaternions, non-Newtonian Lucas quaternions and non-Newtonian Cayley numbers, and examine some of their properties based on non-Newtonian calculus, which has recently garnered attention in the field of mathematics. Additionally, we obtained the matrix representation of non-Newtonian quaternions using non-Newtonian complex numbers and provided some equations for non-Newtonian Cayley numbers.

Key Words: Non-Newtonian calculus, Non-Newtonian quaternions, Non-Newtonian Fibonacci quaternions, Binet formula, Non-Newtonian Cayley numbers.

AMS(2010): 11U10, 11R52, 11B39.

§1. Introduction

Quaternions were introduced by the Irish mathematician Sir William Rowan Hamilton in the early 19th century and hold a significant place in mathematics, [6]. Quaternions, also known as four-dimensional hypercomplex numbers, are widely utilized in various mathematical fields such as commutative ring theory, number theory, group theory, geometric topology, spectral theory of Riemannian manifolds, algebraic geometry, as well as in physics and engineering. In addition to these studies, research has also been conducted on the Fibonacci and Lucas quaternions [4].

In the mid-19th century, Cayley numbers were introduced by Arthur Cayley, [5]. These numbers play a fundamental role, particularly in the study of group theory and linear algebra. Considered as a special type of matrix, Cayley numbers have been thoroughly investigated in the context of linear transformations, group theory, and matrix theory. As we explore algebraic concepts in higher dimensions, such as real numbers, complex numbers, quaternions, and Cayley numbers, it is observed that at each level of generalization, some characteristics of the algebraic structure diminish. In going from $\mathbb{R}$ to $\mathbb{C}$ the concept of order is lost. In going from $\mathbb{C}$ to $\mathbb{H}$ commutativity in the product is lost and in the transition from $\mathbb{H}$ to $\mathbb{K}$, the associative rule becomes invalid. Consequently, the Cayley numbers represent the culminating point of an intriguing sequence of algebras.

[1]Received November 25, 2025, Accepted May 30, 2026

The non-Newtonian calculus introduced by Grossman and Katz in 1972, provides an alternative perspective to classical Newtonian and Leibnizian calculus, thus establishing a new domain in mathematics, [3]. This emerging field encompasses highly original topics for discussion and exploration. Non-Newtonian calculus includes various types of calculations such as geometric, bigeometric, quadratic and biquadratic computations. Moreover, it holds significant potential for applications in diverse fields, including technology, engineering, physics, finance, dynamic systems and tumor therapy. Particularly in index calculations related to currencies, the preference should be for the bigeometric calculation type within non-Newtonian computations. Recently, Fibonacci and Lucas sequences have been reconsidered with a non-Newtonian approach, [2].

§2. **Preliminaries**

The set of quaternion numbers, denoted by $\mathbb{H}$, can be represented as

$$\mathbb{H} = \{q = q_1 + q_2 i + q_3 j + q_4 k | q_1, q_2, q_3, q_4 \in \mathbb{R}\},$$

where the quaternion bases i, j and k satisfy the following identities known as

$$i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j.$$

The quaternion number $q = q_1 + q_2 i + q_3 j + q_4 k$ consist of two parts, the real number q_1 is called the scalar part and denoted by S_q; the part $V_q = q_2 i + q_3 j + q_4 k$ is called the vector part and denoted the vector $\overrightarrow{V} = (q_2, q_3, q_4) \in \mathbb{R}^3$. So, a quaternion number q can be written as $q = S_q + V_q$. The conjugate of a quaternion number $q = q_1 + q_2 i + q_3 j + q_4 k$ is defined as $\overline{q} = q_1 - q_2 i - q_3 j - q_4 k$. The quaternion multiplication is associative but not commutative.

A complete ordered field is called arithmetic if its realm is a subset of $\mathbb{R}$. A generator is a one-to-one function whose domain $\mathbb{R}$ and whose range is a subset of $\mathbb{R}$. Let α be a generator with range A. We denote by $\mathbb{R}_\alpha$ are called non-Newtonian real numbers.

Let $\alpha_1, \alpha_2, \alpha_3$ and α_4 be arbitrarily chosen generators which image the set $\mathbb{R}$ to $A_i, i = 1, 2, 3, 4$, respectively and $*$-calculus also be the ordered pairs of arithmetics.

For $i = 1, 2, 3, 4$, the following notations will be used

$$\dot{a}_i \oplus_i \dot{b}_i = \alpha_i \left\{ \alpha_i^{-1}\left(\dot{a}_i\right) + \alpha_i^{-1}\left(\dot{b}_i\right) \right\},$$

$$\dot{a}_i \ominus_i \dot{b}_i = \alpha_i \left\{ \alpha_i^{-1}\left(\dot{a}_i\right) - \alpha_i^{-1}\left(\dot{b}_i\right) \right\},$$

$$\dot{a}_i \otimes_i \dot{b}_i = \alpha_i \left\{ \alpha_i^{-1}\left(\dot{a}_i\right) \times \alpha_i^{-1}\left(\dot{b}_i\right) \right\},$$

$$\dot{a}_i \oslash_i \dot{b}_i = \alpha_i \left\{ \alpha_i^{-1}\left(\dot{a}_i\right) / \alpha_i^{-1}\left(\dot{b}_i\right) \right\}, \ \dot{b}_i \neq \dot{0}_{\alpha_i},$$

$$\dot{a}_i \leqslant_i \dot{b}_i \Leftrightarrow \alpha_i^{-1}\left(\dot{a}_i\right) \leqslant \alpha_i^{-1}\left(\dot{b}_i\right).$$

In this study, we will use three unique functions that we defined by taking advantage of the unique ι function used in non-Newtonian calculus. These functions are isomorphisms from α_i-arithmetic to α_{i+1}-arithmetic and have the following three properties:

(i) ι_i is one to one;

(ii) ι_i is on A_i and onto A_{i+1};

(iii) For all $\dot{a}_i, \dot{b}_i \in A_i$

$$\iota_i\left(\dot{a}_i \oplus_i \dot{b}_i\right) = \iota_i\left(\dot{a}_i\right) \oplus_{i+1} \iota_i\left(\dot{b}_i\right),$$

$$\iota_i\left(\dot{a}_i \ominus_i \dot{b}_i\right) = \iota_i\left(\dot{a}_i\right) \ominus_{i+1} \iota_i\left(\dot{b}_i\right),$$

$$\iota_i\left(\dot{a}_i \otimes_i \dot{b}_i\right) = \iota_i\left(\dot{a}_i\right) \otimes_{i+1} \iota_i\left(\dot{b}_i\right),$$

$$\iota_i\left(\dot{a}_i \oslash_i \dot{b}_i\right) = \iota_i\left(\dot{a}_i\right) \oslash_{i+1} \iota_i\left(\dot{b}_i\right), \iota_i\left(\dot{b}_i\right) \neq \dot{0}_{\alpha_{i+1}},$$

$$\dot{a}_i \leqslant_i \dot{b}_i \Leftrightarrow \iota_i\left(\dot{a}_i\right) \leqslant_{i+1} \iota_i\left(\dot{b}_i\right).$$

Also, for every integer n, we set $\iota_i\left(\alpha_i\left(n\right)\right) = \alpha_{i+1}\left(n\right)$.

For $i = 1, 2, 3, 4$, an α_i-positive number is a number $\dot{a}_i$ with $\dot{0}_{\alpha_i} <_i \dot{a}_i$ and an α_i-negative number is a number $\dot{a}_i$ with $\dot{a}_i <_i \dot{0}_i$. α_i-zero and α_i-one numbers are denoted by $\dot{0}_{\alpha_i} = \alpha_i\left(0\right)$ and $\dot{1}_{\alpha_i} = \alpha_i\left(1\right)$, respectively.

§3. Non-Newtonian Quaternions

In this section, we defined the non-Newtonian quaternions and gave some of their properties. We also introduced the matrix form of non-Newtonian quaternions and defined the $\theta : \mathbb{C}\left(N\right) \times \mathbb{C}\left(N\right) \to \mathbb{H}\left(N\right)$ isomorphism, where $\mathbb{C}\left(N\right)$ is the set of non-Newtonian complex numbers and it was introduced in [1]. Let

$$\begin{aligned} &\dot{a}_1 \in \left(A_1, \oplus_1, \ominus_1, \otimes_1, \oslash_1, \leqslant_1\right), \quad \dot{a}_2 \in \left(A_2, \oplus_2, \ominus_2, \otimes_2, \oslash_2, \leqslant_2\right), \\ &\dot{a}_3 \in \left(A_3, \oplus_3, \ominus_3, \otimes_3, \oslash_3, \leqslant_3\right), \quad \dot{a}_4 \in \left(A_4, \oplus_4, \ominus_4, \otimes_4, \oslash_4, \leqslant_4\right) \end{aligned}$$

be arbitrary chosen elements from corresponding arithmetics. Then, the ordered pair $\left(\dot{a}_1, \dot{a}_2, \dot{a}_3, \dot{a}_4\right)$ is called a non-Newtonian point. The set of all non-Newtonian points is called the set of non-Newtonian quaternions and denoted by $\mathbb{H}\left(N\right)$, that is,

$$\mathbb{H}\left(N\right) = \left\{\dot{q} = \left(\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4\right) \mid \dot{q}_s \in A_s \subseteq \mathbb{R}, s = 1, 2, 3, 4\right\}.$$

Define the operations addition $(\oplus)$ and multiplication $(\otimes)$ of non-Newtonian quaternions $\dot{q} = \left(\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4\right)$ and $\dot{p} = \left(\dot{p}_1, \dot{p}_2, \dot{p}_3, \dot{p}_4\right)$ as follows:

$$\begin{aligned} \oplus : \mathbb{H}\left(N\right) \times \mathbb{H}\left(N\right) &\to \mathbb{H}\left(N\right), \\ \left(\dot{q}, \dot{p}\right) &\to \dot{q} \oplus \dot{p} = \left(\dot{q}_1 \oplus_1 \dot{p}_1, \dot{q}_2 \oplus_2 \dot{p}_2, \dot{q}_3 \oplus_3 \dot{p}_3, \dot{q}_4 \oplus_4 \dot{p}_4\right), \end{aligned}$$

$$\otimes \ \mathbb{H}(N) \times \mathbb{H}(N) \to \mathbb{H}(N),$$

$$(\dot{q}, \dot{p}) \to \dot{q} \otimes \dot{p} = \begin{pmatrix} \alpha_1 (q_1p_1 - q_2p_2 - q_3p_3 - q_4p_4), \alpha_2 (q_1p_2 + q_2p_1 - q_3p_4 + q_4p_3), \\ \alpha_3 (q_1p_3 - q_2p_4 + q_3p_1 + q_4p_2), \alpha_4 (q_1p_4 + q_2p_3 - q_3p_2 + q_4p_1) \end{pmatrix}.$$

The scalar product ($\odot$) defined as

$$\begin{aligned} \odot : \mathbb{R}_\alpha \times \mathbb{H}(N) &\to \mathbb{H}(N) \\ \left(\lambda, \dot{q}\right) &\to \lambda \odot \dot{q} = \left(\lambda \odot \dot{q}_1, \lambda \odot \dot{q}_2, \lambda \odot \dot{q}_3, \lambda \odot \dot{q}_4\right). \end{aligned}$$

Example 3.1 Let $\dot{q} = \left(\dot{0}_{\alpha_1}, \dot{1}, \ominus_3\dot{3}, \dot{2}\right), \dot{p} = \left(\ominus_1\dot{1}, \dot{4}, \ominus_3\dot{2}, \dot{3}\right) \in \mathbb{H}(N)$. The non-Newtonian addition and product of $\dot{q}$ and $\dot{p}$ non-Newtonian quaternions are

$$\dot{q} \oplus \dot{p} = \left(\dot{0}_{\alpha_1} \ominus_1 \dot{1}, \dot{1} \oplus_1 \dot{4}, \ominus_3\dot{3} \ominus_3 \dot{2}, \dot{2} \oplus_4 \dot{3}\right) = \left(\ominus_1\dot{1}, \dot{5}, \dot{6}, \dot{5}\right)$$

and

$$\begin{aligned} \dot{q} \otimes \dot{p} &= (\alpha_1 (0 - 4 - 6 - 6), \alpha_2 (0 - 1 + 9 - 4), \alpha_3 (0 - 3 + 3 + 8), \alpha_4 (0 - 2 + 12 - 2)) \\ &= \left(\ominus_1\dot{16}, \dot{4}, \dot{8}, \dot{8}\right), \end{aligned}$$

respectively.

Definition 3.2 *The conjugate of a non-Newtonian quaternion* $\dot{q} = \left(\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4\right)$ *is* $\overline{\dot{q}} = \left(\dot{q}_1, \ominus_2\dot{q}_2, \ominus_3\dot{q}_3, \ominus_4\dot{q}_4\right)$.

Example 3.3 Let $\dot{q} = \left(\ominus_1\dot{2}, \ominus_2\dot{4}, \dot{5}, \dot{1}\right)$. The conjugate of $\dot{q}$ is $\overline{\dot{q}} = \left(\ominus_1\dot{2}, \dot{4}, \ominus_3\dot{5}, \ominus_4\dot{1}\right)$.

Definition 3.4 *For a non-Newtonian quaternion* $\dot{q} = (\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4)$, *the vector* $(\dot{q}_1, \dot{0}_{\alpha_2}, \dot{0}_{\alpha_3}, \dot{0}_{\alpha_4})$ *is called the real or scalar part of* $\dot{q}$ *and the vector* $\left(\dot{0}_{\alpha_1}, \dot{q}_2, \dot{q}_3, \dot{q}_4\right)$ *is called the imaginary part of* $\dot{q}$. *Denoted by* $S^\alpha_{\dot{q}}$ *and* $V^\alpha_{\dot{q}}$, *respectively. Accordingly, a non-Newtonian quaternion* $\dot{q}$ *can be written as* $\dot{q} = S^\alpha_{\dot{q}} \oplus V^\alpha_{\dot{q}}$.

Since $\dot{q} \oplus \dot{p} = \left(\dot{q}_1 \oplus_1 \dot{p}_1, \dot{q}_2 \oplus_2 \dot{p}_2, \dot{q}_3 \oplus_3 \dot{p}_3, \dot{q}_4 \oplus_4 \dot{p}_4\right)$ *for any* $\dot{q}$, $\dot{p}$ *non-Newtonian quaternions can be written as* $S^\alpha_{\dot{q} \oplus \dot{p}} = S^\alpha_{\dot{q}} \oplus S^\alpha_{\dot{p}}$. *Similarly, it can be easily seen that* $V^\alpha_{\dot{q} \oplus \dot{p}} = V^\alpha_{\dot{q}} \oplus V^\alpha_{\dot{p}}$.

Using the above two definitions, the following result can be given.

Proposition 3.5 *The real and imaginary parts of* $\dot{q} \in \mathbb{H}(N)$ *are*

$$S^\alpha_{\dot{q}} = \dot{\left(\frac{1}{2}\right)} \odot \left(\dot{q} \oplus \overline{\dot{q}}\right)$$

and

$$V^\alpha_{\dot{q}} = \dot{\left(\frac{1}{2}\right)} \odot \left(\dot{q} \ominus \overline{\dot{q}}\right),$$

respectively.

Definition 3.6 *The norm of a non-Newtonian quaternion is defined as*

$$N_{\dot{q}} = \dot{q} \otimes \overline{\dot{q}} = \dot{q_1}^{\dot{2}} \oplus \dot{q_2}^{\dot{2}} \oplus \dot{q_3}^{\dot{2}} \oplus \dot{q_4}^{\dot{2}} \in \mathbb{R}_\alpha.$$

Definition 3.7 *The inverse of a non-Newtonian quaternion is defined as*

$$\dot{q}^{-1} = \frac{\dot{1}}{N_{\dot{q}}} \odot \overline{\dot{q}}.$$

Remark 3.8 We can denote a non-Newtonian quaternion

$$\dot{q} = \left(\dot{q_1}, \dot{q_2}, \dot{q_3}, \dot{q_4}\right) = \dot{q_1} \oplus \dot{q_2} \otimes i^\star \oplus \dot{q_3} \otimes j^\star \oplus \dot{q_4} \otimes k^\star,$$

where $i^\star = \left(\dot{0}_{\alpha_1}, \dot{1}_{\alpha_2}, \dot{0}_{\alpha_3}, \dot{0}_{\alpha_4}\right)$, $j^\star = \left(\dot{0}_{\alpha_1}, \dot{0}_{\alpha_2}, \dot{1}_{\alpha_3}, \dot{0}_{\alpha_4}\right)$, $k^\star = \left(\dot{0}_{\alpha_1}, \dot{0}_{\alpha_2}, \dot{0}_{\alpha_3}, \dot{1}_{\alpha_4}\right)$, $(i^\star)^{\dot{2}} = (j^\star)^{\dot{2}} = (k^\star)^{\dot{2}} = \ominus\dot{1}$, $i^\star \otimes j^\star = \ominus j^\star \otimes i^\star = k^\star$, $j^\star \otimes k^\star = \ominus k^\star \otimes j^\star = i^\star$, $k^\star \otimes i^\star = \ominus i^\star \otimes k^\star = j^\star$.

Example 3.9 Let $q = 3 + e^2 i - j + e^3 k = \left(3, e^2, -1, e^3\right) \in \mathbb{H}$. The non-Newtonian quaternion $\dot{q}$ is written as $\dot{q} = \left(\dot{3}, \dot{2}, \ominus_3 \dot{1}, \dot{3}\right) = \dot{3} \oplus \dot{2} \otimes i^\star \ominus j^\star \oplus \dot{3} \otimes k^\star$, where $\alpha_1 = I$, $\alpha_2 = exp$, $\alpha_3 = I$ and $\alpha_4 = exp$.

For $\alpha_1 = exp$, $\alpha_2 = exp$, $\alpha_3 = exp$ and $\alpha_4 = exp$, the non-Newtonian quaternion $\dot{q}$ is written as $\dot{q} = (\alpha_1 (\ln 3), \alpha_2 (2), \alpha_3 (\ln(-1)), \alpha_4 (3)) = \dot{\ln 3} \oplus \dot{2} \otimes i^\star \oplus \ln \dot{(-1)} \otimes j^\star \oplus \dot{3} \otimes k^\star$.

Theorem 3.10 *For* $\forall\ \dot{p}, \dot{q} \in \mathbb{H}(N)$ *and* $\lambda \in \mathbb{R}_\alpha$ *we have,*

- *(i)* $\overline{\dot{p} \otimes \dot{q}} = \overline{\dot{q}} \otimes \overline{\dot{p}}$;
- *(ii)* $\overline{\dot{p} \oplus \dot{q}} = \overline{\dot{p}} \oplus \overline{\dot{q}}$, $\overline{\dot{p} \ominus \dot{q}} = \overline{\dot{p}} \ominus \overline{\dot{q}}$;
- *(iii)* $\overline{\overline{\dot{p}}} = \dot{p}$;
- *(iv)* $\overline{\overline{\dot{p}} \otimes \dot{q}} = \overline{\dot{q}} \otimes \dot{p}$;
- *(v)* $\dot{p} \otimes \overline{\dot{p}} = \overline{\dot{p}} \otimes \dot{p} = N_{\dot{p}}$;
- *(vi)* $\overline{\lambda \odot \dot{p}} = \lambda \odot \overline{\dot{p}}$;
- *(vii)* $\dot{p} \in \mathbb{R}_\alpha \Leftrightarrow \dot{p} = \overline{\dot{p}}$;
- *(viii)* $\dot{p}$ *is a pure non-Newtonian quaternion* $\Leftrightarrow \dot{p} = \ominus\overline{\dot{p}}$;
- *(ix)* $S^\alpha_{\dot{p} \otimes \dot{q}} = S^\alpha_{\dot{q} \otimes \dot{p}}$;
- *(x)* $S^\alpha_{\dot{q} \otimes \dot{p}} = S^\alpha_{\dot{q}} \otimes S^\alpha_{\dot{p}}$;
- *(xi)* $V^\alpha_{\dot{q} \otimes \dot{p}} = V^\alpha_{\dot{q}} \otimes V^\alpha_{\dot{p}}$;
- *(xii)* $N_{\dot{p} \otimes \dot{q}} = N_{\dot{p}} \odot N_{\dot{q}}$;
- *(xiii)* $(\dot{p} \otimes \dot{q})^{-1} = \dot{q}^{-1} \otimes \dot{p}^{-1}$.

Proof (*i*) Let $\dot{p} = \left(\dot{p_1}, \dot{p_2}, \dot{p_3}, \dot{p_4}\right), \dot{q} = \left(\dot{q_1}, \dot{q_2}, \dot{q_3}, \dot{q_4}\right) \in \mathbb{H}(N)$. We have

$$\overline{\dot{p} \otimes \dot{q}} = \begin{pmatrix} \alpha_1 (p_1 q_1 - p_2 q_2 - p_3 q_3 - p_4 q_4), \ominus_2 \alpha_2 (p_1 q_2 + p_2 q_1 + p_3 q_4 - p_4 q_3), \\ \ominus_3 \alpha_3 (p_1 q_3 - p_2 q_4 + p_3 q_1 + p_4 q_2), \ominus_4 \alpha_4 (p_1 q_4 + p_2 q_3 - p_3 q_2 + p_4 q_1) \end{pmatrix}$$

and

$$\overline{\dot{q}} \otimes \overline{\dot{p}} = \begin{pmatrix} \alpha_1 (q_1p_1 - q_2p_2 - q_3p_3 - q_4p_4), \alpha_2 (-q_1p_2 - q_2p_1 + q_3p_4 - q_4p_3), \\ \alpha_3 (-q_1p_3 - q_2p_4 - q_3p_1 + q_4p_2), \alpha_4 (-q_1p_4 + q_2p_3 - q_3p_2 - q_4p_1) \end{pmatrix}.$$

Hence, $\overline{\dot{p} \otimes \dot{q}} = \overline{\dot{q}} \otimes \overline{\dot{p}}$.

(iii) From the definition of $\dot{p}$ and $\overline{\dot{p}}$, we can write $\overline{\overline{\dot{p}}} = \overline{S^{\alpha}_{\overline{\dot{p}}} \ominus V^{\alpha}_{\overline{\dot{p}}}} = S^{\alpha}_{\overline{\dot{p}}} \oplus V^{\alpha}_{\overline{\dot{p}}} = \dot{p}$.

$(xiii)$ From using properties i, xii and definition of $\dot{q}^{-1}$, we can write

$$\begin{aligned} (\dot{p} \otimes \dot{q})^{-1} &= \frac{\dot{1}}{N_{\dot{p}\otimes\dot{q}}} \odot \left(\overline{\dot{p} \otimes \dot{q}}\right) = \frac{\dot{1}}{N_{\dot{p}} \odot N_{\dot{q}}} \odot \left(\overline{\dot{q}} \otimes \overline{\dot{p}}\right) \\ &= \left(\frac{\dot{1}}{N_{\dot{p}}} \odot \overline{\dot{p}}\right) \otimes \left(\frac{\dot{1}}{N_{\dot{q}}} \odot \overline{\dot{q}}\right) = \dot{q}^{-1} \otimes \dot{p}^{-1}. \end{aligned}$$

All other properties can be similarly proven. □

3.1 Matrix Form of Non-Newtonian Quaternions

Definition 3.11 *We define the matrix form of a non-Newtonian quaternion as follows*

$$\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix},$$

where $\dot{\alpha}, \dot{\beta} \in \mathbb{C}(N)$.

Equality between $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$ *and* $\dot{Q} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix}$ *are defined by* $\dot{P} = \dot{Q}$ *if and only if* $\dot{\alpha} = \dot{\gamma}$ *and* $\dot{\beta} = \dot{\delta}$.

Define the operations addition $(\widetilde{\oplus})$ *and multiplication* $(\widetilde{\otimes})$ *of non-Newtonian quaternions* $\dot{P}$ *and* $\dot{Q}$ *as follows:*

$$\dot{P}\widetilde{\oplus}\dot{Q} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \widetilde{\oplus} \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{\alpha}\dot{+}\dot{\gamma} \\ \dot{\beta}\dot{+}\dot{\delta} \end{bmatrix}$$

and

$$\dot{P}\widetilde{\otimes}\dot{Q} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \widetilde{\otimes} \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{\alpha}\dot{\times}\dot{\gamma}\dot{-}\dot{\delta}\dot{\times}\dot{\beta}^{C} \\ \dot{\gamma}\dot{\times}\dot{\beta}\dot{+}\dot{\alpha}^{C}\dot{\times}\dot{\delta} \end{bmatrix},$$

where $\dot{\alpha}^{C}$ *is conjugate of a non-Newtonian complex number. Also, the identity of* $\widetilde{\oplus}$ *is* $\varepsilon = \begin{bmatrix} \dot{0}_{\mathbb{C}(N)} \\ \dot{0}_{\mathbb{C}(N)} \end{bmatrix}$ *and the identity of* $\widetilde{\otimes}$ *is* $E = \begin{bmatrix} \dot{1}_{\mathbb{C}(N)} \\ \dot{0}_{\mathbb{C}(N)} \end{bmatrix}$.

The scalar product $\widetilde{\odot}$ *defined as*

$$\lambda\widetilde{\odot}\dot{P} = \begin{bmatrix} \lambda\dot{\times}\dot{\alpha} \\ \lambda\dot{\times}\dot{\beta} \end{bmatrix},$$

where $\lambda \in \mathbb{R}_{\alpha}$.

Example 3.12 Let $\dot{P}=\begin{bmatrix}(\dot{1},\dot{2})\\(\dot{3},\dot{4})\end{bmatrix}, \dot{Q}=\begin{bmatrix}(\dot{1},\dot{-}\dot{4})\\(\dot{2},\dot{-}\dot{3})\end{bmatrix}$, where $(\dot{1},\dot{2}),\ (\dot{3},\dot{4}),\ (\dot{1},\dot{-}\dot{4}),\ (\dot{2},\dot{-}\dot{3})\in\mathbb{C}(N)$. We calculate

$$\dot{P}\widetilde{\oplus}\dot{Q}=\begin{bmatrix}(\dot{1},\dot{2})\\(\dot{3},\dot{4})\end{bmatrix}\widetilde{\oplus}\begin{bmatrix}(\dot{1},\dot{-}\dot{4})\\(\dot{2},\dot{-}\dot{3})\end{bmatrix}=\begin{bmatrix}(\dot{2},\dot{-}\dot{2})\\(\dot{5},\dot{1})\end{bmatrix}$$

and

$$\begin{aligned}\dot{P}\widetilde{\otimes}\dot{Q} &= \begin{bmatrix}(\dot{1},\dot{2})\\(\dot{3},\dot{4})\end{bmatrix}\widetilde{\otimes}\begin{bmatrix}(\dot{1},\dot{-}\dot{4})\\(\dot{2},\dot{-}\dot{3})\end{bmatrix}=\begin{bmatrix}(\dot{1},\dot{2})\dot{\times}(\dot{1},\dot{-}\dot{4})\dot{-}(\dot{2},\dot{-}\dot{3})\dot{\times}(\dot{3},\dot{-}\dot{4})\\(\dot{1},\dot{-}\dot{4})\dot{\times}(\dot{3},\dot{4})\dot{+}(\dot{1},\dot{-}\dot{2})\dot{\times}(\dot{2},\dot{-}\dot{3})\end{bmatrix}\\ &= \begin{bmatrix}(\dot{9},\dot{-}\dot{2})\dot{-}(\dot{-}\dot{12},\dot{-}\dot{17})\\(\dot{19},\dot{-}\dot{8})\dot{+}(\dot{-}\dot{4},\dot{-}\dot{7})\end{bmatrix}=\begin{bmatrix}(\dot{21},\dot{15})\\(\dot{15},\dot{-}\dot{15})\end{bmatrix}\end{aligned}$$

by using the definition of non-Newtonian complex numbers in [1].

Theorem 3.13 *Let* $\dot{P}=\begin{bmatrix}\dot{\alpha}\\\dot{\beta}\end{bmatrix}$, $\dot{Q}=\begin{bmatrix}\dot{\gamma}\\\dot{\delta}\end{bmatrix}$, $\dot{R}=\begin{bmatrix}\dot{\omega}\\\dot{\eta}\end{bmatrix}$ *be non-Newtonian quaternions. Then* $\dot{P}\widetilde{\otimes}\left(\dot{Q}\widetilde{\otimes}\dot{R}\right)=\left(\dot{P}\widetilde{\otimes}\dot{Q}\right)\widetilde{\otimes}\dot{R}$*.*

Proof Let $\dot{P}=\begin{bmatrix}\dot{\alpha}\\\dot{\beta}\end{bmatrix}, \dot{Q}=\begin{bmatrix}\dot{\gamma}\\\dot{\delta}\end{bmatrix}, \dot{R}=\begin{bmatrix}\dot{\omega}\\\dot{\eta}\end{bmatrix}$ be non-Newtonian quaternions, where $\dot{\alpha},\dot{\beta},\dot{\gamma},\dot{\delta},\dot{\omega},\dot{\eta}\in\mathbb{C}(N)$. Using the properties in $\mathbb{C}(N)$, it can be seen that the multiplication $\widetilde{\otimes}$ is associative as follows:

$$\begin{aligned}\dot{P}\widetilde{\otimes}\left(\dot{Q}\widetilde{\otimes}\dot{R}\right) &= \begin{bmatrix}\dot{\alpha}\\\dot{\beta}\end{bmatrix}\widetilde{\otimes}\begin{bmatrix}\dot{\gamma}\dot{\times}\dot{\omega}\dot{-}\dot{\eta}\dot{\times}\dot{\delta}^{C}\\\dot{\omega}\dot{\times}\dot{\delta}\dot{+}\dot{\gamma}^{C}\dot{\times}\dot{\eta}\end{bmatrix}\\ &= \begin{bmatrix}\dot{\alpha}\dot{\times}\left(\dot{\gamma}\dot{\times}\dot{\omega}\dot{-}\dot{\eta}\dot{\times}\dot{\delta}^{C}\right)\dot{-}\left(\dot{\omega}\dot{\times}\dot{\delta}\dot{+}\dot{\gamma}^{C}\dot{\times}\dot{\eta}\right)\dot{\times}\dot{\beta}^{C}\\\left(\dot{\gamma}\dot{\times}\dot{\omega}\dot{-}\dot{\eta}\dot{\times}\dot{\delta}^{C}\right)\dot{\times}\dot{\beta}\dot{+}\dot{\alpha}^{C}\dot{\times}\left(\dot{\omega}\dot{\times}\dot{\delta}\dot{+}\dot{\gamma}^{C}\dot{\times}\dot{\eta}\right)\end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}\left(\dot{P}\widetilde{\otimes}\dot{Q}\right)\widetilde{\otimes}\dot{R} &= \begin{bmatrix}\dot{\alpha}\dot{\times}\dot{\gamma}\dot{-}\dot{\delta}\dot{\times}\dot{\beta}^{C}\\\dot{\gamma}\dot{\times}\dot{\beta}\dot{+}\dot{\alpha}^{C}\dot{\times}\dot{\delta}\end{bmatrix}\widetilde{\otimes}\begin{bmatrix}\dot{\omega}\\\dot{\eta}\end{bmatrix}\\ &= \begin{bmatrix}\left(\dot{\alpha}\dot{\times}\dot{\gamma}\dot{-}\dot{\delta}\dot{\times}\dot{\beta}^{C}\right)\dot{\times}\dot{\omega}\dot{-}\dot{\eta}\dot{\times}\left(\dot{\gamma}\dot{\times}\dot{\beta}\dot{+}\dot{\alpha}^{C}\dot{\times}\dot{\delta}\right)^{C}\\\dot{\omega}\dot{\times}\left(\dot{\gamma}\dot{\times}\dot{\beta}\dot{+}\dot{\alpha}^{C}\dot{\times}\dot{\delta}\right)\dot{+}\left(\dot{\alpha}\dot{\times}\dot{\gamma}\dot{-}\dot{\delta}\dot{\times}\dot{\beta}^{C}\right)^{C}\dot{\times}\dot{\eta}\end{bmatrix}.\end{aligned}$$

This completes the proof. □

Definition 3.14 *The matrix form of the conjugate of a non-Newtonian quaternion is defined as*

$$\widehat{\dot{P}}=\begin{bmatrix}\dot{\alpha}^{C}\\\dot{-}\dot{\beta}\end{bmatrix}.$$

Definition 3.15 *The scalar and vector parts of the matrix form of a non-Newtonian quaternion are*

$$\widetilde{S}^{\alpha}_{\dot{P}} = \left(\frac{\dot{1}}{2}\right) \widetilde{\odot} \left(\dot{P} \widetilde{\oplus} \widehat{\dot{P}}\right) \quad \textit{and} \quad \widetilde{V}^{\alpha}_{\dot{P}} = \left(\frac{\dot{1}}{2}\right) \widetilde{\odot} \left(\dot{P} \widetilde{\ominus} \widehat{\dot{P}}\right),$$

respectively.

Definition 3.16 *The norm $\widetilde{N}_{\dot{P}}$ of $\dot{P}$ is defined as*

$$\widetilde{N}_{\dot{P}} = \dot{P} \widetilde{\odot} \widehat{\dot{P}} = \begin{bmatrix} \dot{\alpha} \dot{\times} \dot{\alpha}^{C} \dot{+} \dot{\beta} \dot{\times} \dot{\beta}^{C} \\ \dot{0}_{\mathbb{C}(N)} \end{bmatrix}.$$

.

Definition 3.17 *Define an inverse element $\dot{P}^{\widetilde{-1}}$ to every element $\dot{P}$ with non zero norm*

$$\dot{P}^{\widetilde{-1}} = \frac{\dot{1}}{\widetilde{N}_{\dot{P}}} \widetilde{\odot} \widehat{\dot{P}} = \frac{\dot{1}}{\widetilde{N}_{\dot{P}}} \widetilde{\odot} \begin{bmatrix} \dot{\alpha}^{C} \\ \dot{-}\dot{\beta} \end{bmatrix}.$$

Remark 3.18 Non-Newtonian scalar numbers can be written in the following form

$$\lambda \equiv \begin{bmatrix} \dot{s} \\ \dot{0}_{\mathbb{C}(N)} \end{bmatrix},$$

where $\dot{s} \in \mathbb{R}_{\alpha}$. Similarly, the non-Newtonian complex numbers can be written in the following form

$$\dot{z} \equiv \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix},$$

where $\dot{\alpha}, \dot{\beta} \in \mathbb{R}_{\alpha}$. Hence, a non-Newtonian quaternion written as

$$\dot{p} \equiv \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix},$$

where $\dot{\alpha}, \dot{\beta} \in \mathbb{C}(N)$.

Theorem 3.19 *Let $\mathbb{H}_M(N)$ be the set of matrix representations of non-Newtonian quaternions. The map is defined as*

$$\begin{array}{ccc} \Phi & : & \mathbb{H}_M(N) \to \mathbb{H}(N) \\ \Phi\left(\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}\right) & \longmapsto & \dot{x}_0 \dot{+} \dot{x}_1 \dot{\times} i^{\star} \dot{+} \dot{x}_2 \dot{\times} j^{\star} \dot{+} \dot{x}_3 \dot{\times} k^{\star} \end{array}$$

is an isomorphism, where $\dot{x}, \dot{y} \in \mathbb{C}(N)$.

Proof Let $\dot{x}=\dot{x}_1\dot{+}\dot{x}_2\dot{\times}i^*, \dot{y}=\dot{y}_1\dot{+}\dot{y}_2\dot{\times}i^*, \dot{z}=\dot{z}_1\dot{+}\dot{z}_2\dot{\times}i^*, \dot{t}=\dot{t}_1\dot{+}\dot{t}_2\dot{\times}i^*\in\mathbb{C}\left(N\right)$. Since

$$\begin{aligned}
\Phi\left\{\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}\oplus\begin{bmatrix}\dot{z}\\ \dot{t}\end{bmatrix}\right\} &= \Phi\left\{\begin{bmatrix}\dot{x}\dot{+}\dot{z}\\ \dot{y}\dot{+}\dot{t}\end{bmatrix}\right\}\\
&= \Phi\left\{\begin{bmatrix}\dot{x}_1\dot{+}\dot{z}_1\dot{+}\left(\dot{x}_2\dot{+}\dot{z}_2\right)i^*\\ \dot{y}_1\dot{+}\dot{t}_1\dot{+}\left(\dot{y}_2\dot{+}\dot{t}_2\right)i^*\end{bmatrix}\right\}\\
&= \dot{x}_1\dot{+}\dot{z}_1\dot{+}\left(\dot{y}_2\dot{+}\dot{t}_2\right)\dot{\times}i^\star\dot{+}\left(\dot{y}_1\dot{+}\dot{t}_1\right)\dot{\times}j^\star\dot{+}\left(\dot{x}_2\dot{+}\dot{z}_2\right)\dot{\times}k^\star\\
&= \Phi\left\{\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}\right\}\oplus\Phi\left\{\begin{bmatrix}\dot{z}\\ \dot{t}\end{bmatrix}\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\Phi\left\{\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}\otimes\begin{bmatrix}\dot{z}\\ \dot{t}\end{bmatrix}\right\}=\Phi\left\{\begin{bmatrix}\dot{x}\dot{\times}\dot{z}\dot{-}\dot{t}\dot{\times}\dot{y}^C\\ \dot{z}\dot{\times}\dot{y}\dot{+}\dot{x}^C\dot{\times}\dot{t}\end{bmatrix}\right\}\\
&=\Phi\left\{\begin{bmatrix}\dot{x}_1\dot{\times}\dot{z}_1\dot{-}\dot{x}_2\dot{\times}\dot{z}_2\dot{-}\dot{y}_1\dot{\times}\dot{t}_1\dot{-}\dot{y}_2\dot{\times}\dot{t}_2\dot{+}\left(\dot{x}_1\dot{\times}\dot{z}_2\dot{+}\dot{x}_2\dot{\times}\dot{z}_1\dot{-}\dot{y}_1\dot{\times}\dot{t}_2\dot{+}\dot{t}_1\dot{\times}\dot{y}_2\right)\dot{\times}i^*\\ \dot{y}_1\dot{\times}\dot{z}_1\dot{-}\dot{y}_2\dot{\times}\dot{z}_2\dot{+}\dot{x}_1\dot{\times}\dot{t}_1\dot{+}\dot{x}_2\dot{\times}\dot{t}_2\dot{+}\left(\dot{y}_2\dot{\times}\dot{z}_1\dot{+}\dot{y}_1\dot{\times}\dot{z}_2\dot{+}\dot{x}_1\dot{\times}\dot{t}_2\dot{-}\dot{x}_2\dot{\times}\dot{t}_1\right)\dot{\times}i^*\end{bmatrix}\right\}\\
&=\dot{x}_1\dot{\times}\dot{z}_1\dot{-}\dot{x}_2\dot{\times}\dot{z}_2\dot{-}\dot{y}_1\dot{\times}\dot{t}_1\dot{-}\dot{y}_2\dot{\times}\dot{t}_2\dot{+}\left(\dot{y}_2\dot{\times}\dot{z}_1\dot{+}\dot{y}_1\dot{\times}\dot{z}_2\dot{+}\dot{x}_1\dot{\times}\dot{t}_2\dot{-}\dot{x}_2\dot{\times}\dot{t}_1\right)\dot{\times}i^*\dot{+}\left(\dot{y}_1\dot{\times}\dot{z}_1\dot{-}\dot{y}_2\dot{\times}\dot{z}_2\right.\\
&\quad\left.\dot{+}\ \dot{x}_1\dot{\times}\dot{t}_1\dot{+}\dot{x}_2\dot{\times}\dot{t}_2\right)\dot{\times}j^*\dot{+}\left(\dot{x}_1\dot{\times}\dot{z}_2\dot{+}\dot{x}_2\dot{\times}\dot{z}_1\dot{-}\dot{y}_1\dot{\times}\dot{t}_2\dot{+}\dot{t}_1\dot{\times}\dot{y}_2\right)\dot{\times}k^*\\
&=\Phi\left\{\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}\right\}\otimes\Phi\left\{\begin{bmatrix}\dot{z}\\ \dot{t}\end{bmatrix}\right\},
\end{aligned}$$

the map Φ is a homomorphism. The map is clearly one-to-one and onto. So, Φ is an isomorphism. $\square$

§4. Non-Newtonian Fibonacci and Non-Newtonian Lucas Quaternions

In this section, we defined the non-Newtonian Fibonacci and non-Newtonian Lucas quaternions and gave some of their properties.

Definition 4.1 *The n-th non-Newtonian Fibonacci and non-Newtonian Lucas quaternions are defined as*

$$\dot{Q}_n=\left(\dot{F}_n,\dot{F}_{n+1},\dot{F}_{n+2},\dot{F}_{n+3}\right)=\dot{F}_n\oplus\dot{F}_{n+1}\otimes i^\star\oplus\dot{F}_{n+2}\otimes j^\star\oplus\dot{F}_{n+3}\otimes k^\star$$

and

$$\dot{K}_n=\left(\dot{L}_n,\dot{L}_{n+1},\dot{L}_{n+2},\dot{L}_{n+3}\right)=\dot{L}_n\oplus\dot{L}_{n+1}\otimes i^\star\oplus\dot{L}_{n+2}\otimes j^\star\oplus\dot{L}_{n+3}\otimes k^\star,$$

where $\dot{F}_n$ is the nth non-Newtonian Fibonacci number, $\dot{L}_n$ is nth non-Newtonian Lucas number, $i^\star=\left(\dot{0}_{\alpha_1},\dot{1}_{\alpha_2},\dot{0}_{\alpha_3},\dot{0}_{\alpha_4}\right)$, $j^\star=\left(\dot{0}_{\alpha_1},\dot{0}_{\alpha_2},\dot{1}_{\alpha_3},\dot{0}_{\alpha_4}\right)$, $k^\star=\left(\dot{0}_{\alpha_1},\dot{0}_{\alpha_2},\dot{0}_{\alpha_3},\dot{1}_{\alpha_4}\right)$, $\left(i^\star\right)^{\dot{2}}=\left(j^\star\right)^{\dot{2}}=\left(k^\star\right)^{\dot{2}}=\ominus\dot{1}$, $i^\star\otimes j^\star=\ominus j^\star\otimes i^\star=k^\star$, $j^\star\otimes k^\star=\ominus k^\star\otimes j^\star=i^\star$, $k^\star\otimes i^\star=\ominus i^\star\otimes k^\star=j^\star$.

Notice that, for $n \geqslant 0$

$$\dot{Q}_{n+2} = \dot{Q}_{n+1} \oplus \dot{Q}_n \text{ and } \dot{K}_{n+2} = \dot{K}_{n+1} \oplus \dot{K}_n$$

can be written. So, the non-Newtonian Fibonacci and non-Newtonian Lucas quaternions are the second-order linear recurrence sequence.

Theorem 4.2 *Let*

$$T = \{\dot{Q}_n : \dot{Q}_n = \left(\dot{F}_n, \dot{F}_{n+1}, \dot{F}_{n+2}, \dot{F}_{n+3}\right), \dot{F}_n \text{ is nth non-Newtonian Fibonacci number}\}$$

and

$$D = \{Y_n : Y_n = \begin{pmatrix} \dot{w} & \ominus\dot{z} \\ \overline{\dot{z}} & \overline{\dot{w}} \end{pmatrix}, \dot{w}, \dot{z} \in \mathbb{C}(N)\}.$$

Then, there is a isomorphism between T and D such that

$$\dot{Q}_n = \left(\dot{F}_n, \dot{F}_{n+1}, \dot{F}_{n+2}, \dot{F}_{n+3}\right) \longrightarrow Y_n = \begin{pmatrix} \dot{F}_n \oplus \dot{F}_{n+1} \otimes i^\star & \ominus \dot{F}_{n+2} \ominus \dot{F}_{n+3} \otimes i^\star \\ \dot{F}_{n+2} \ominus \dot{F}_{n+3} \otimes i^\star & \dot{F}_n \ominus \dot{F}_{n+1} \otimes i^\star \end{pmatrix}.$$

Corollary 4.3 *We can write*

$$Y_n = \dot{F}_n \otimes E^\star \oplus \dot{F}_{n+1} \otimes I^\star \oplus \dot{F}_{n+2} \otimes J^\star \oplus \dot{F}_{n+3} \otimes K^\star,$$

where $E^\star = \begin{pmatrix} \dot{1} & \dot{0} \\ \dot{0} & \dot{1} \end{pmatrix}$, $I^\star = \begin{pmatrix} i^\star & \dot{0} \\ \dot{0} & \ominus i^\star \end{pmatrix}$, $J^\star = \begin{pmatrix} \dot{0} & \ominus\dot{1} \\ \dot{1} & \dot{0} \end{pmatrix}$ *and* $K^\star = \begin{pmatrix} \dot{0} & \ominus i^\star \\ \ominus i^\star & \dot{0} \end{pmatrix}$.

Since $det(Y_n) \neq \dot{0}$, there is an inverse of the matrix Y_n and it is in D.

Definition 4.4 *The conjugates of non-Newtonian Fibonacci and non-Newtonian Lucas quaternions are*

$$\overline{\dot{Q}_n} = \left(\dot{F}_n, \ominus_2 \dot{F}_{n+1}, \ominus_3 \dot{F}_{n+2}, \ominus_4 \dot{F}_{n+3}\right)$$

and

$$\overline{\dot{K}_n} = \left(\dot{L}_n, \ominus_2 \dot{L}_{n+1}, \ominus_3 \dot{L}_{n+2}, \ominus_4 \dot{L}_{n+3}\right).$$

Theorem 4.5 *Let $\dot{Q}_n$ be the non-Newtonian Fibonacci quaternion and $\overline{\dot{Q}_n}$ be its conjugate. The following equations are satisfied*

(i) $\dot{Q}_n \otimes \overline{\dot{Q}_n} = \dot{\sum}_{i=0}^{3} \dot{F}_{n+i}^{\dot{2}} = \dot{3} \otimes \dot{F}_{2n+3}$;

(ii) $\dot{Q}_n^{\dot{2}} = \dot{2} \otimes \dot{F}_n \otimes \dot{Q}_n \ominus \dot{Q}_n \otimes \overline{\dot{Q}_n}$;

(iii) $\dot{Q}_n \oplus \overline{\dot{Q}_n} = \dot{2} \otimes \dot{F}_n, \dot{Q}_n \neq \dot{0}$.

Proof (i) Notice that

$$\begin{aligned}
\dot{Q}_n \otimes \overline{\dot{Q}_n} &= \left(\dot{F}_n \oplus \dot{F}_{n+1} \otimes i^\star \oplus \dot{F}_{n+2} \otimes j^\star \oplus \dot{F}_{n+3} \otimes k^\star\right) \\
&\otimes \left(\dot{F}_n \ominus \dot{F}_{n+1} \otimes i^\star \ominus \dot{F}_{n+2} \otimes j^\star \ominus \dot{F}_{n+3} \otimes k^\star\right) \\
&= \dot{F}_n^{\dot{2}} \oplus \dot{F}_{n+1}^{\dot{2}} \oplus \dot{F}_{n+2}^{\dot{2}} \oplus \dot{F}_{n+3}^{\dot{2}} \\
&\oplus \left(\ominus \dot{F}_n \otimes \dot{F}_{n+1} \oplus \dot{F}_{n+1} \otimes \dot{F}_n \ominus \dot{F}_{n+2} \otimes \dot{F}_{n+3} \oplus \dot{F}_{n+3} \otimes \dot{F}_{n+2}\right) \otimes i^\star \\
&\oplus \left(\ominus \dot{F}_n \otimes \dot{F}_{n+2} \oplus \dot{F}_{n+1} \otimes \dot{F}_{n+3} \oplus \dot{F}_{n+2} \otimes \dot{F}_n \ominus \dot{F}_{n+3} \otimes \dot{F}_{n+1}\right) \otimes j^\star \\
&\oplus \left(\ominus \dot{F}_n \otimes \dot{F}_{n+3} \ominus \dot{F}_{n+1} \otimes \dot{F}_{n+2} \oplus \dot{F}_{n+2} \otimes \dot{F}_{n+1} \oplus \dot{F}_{n+3} \otimes \dot{F}_n\right) \otimes k^\star \\
&= \dot{\sum}_{i=0}^{3} \dot{F}_{n+i}^{\dot{2}}.
\end{aligned}$$

From $\dot{F}_{2n+1} = \dot{F}_{n+1}^{\dot{2}} \oplus \dot{F}_n^{\dot{2}}$, we can easily see that

$$\dot{Q}_n \otimes \overline{\dot{Q}_n} = \dot{\sum}_{i=0}^{3} \dot{F}_{n+i}^{\dot{2}} = \dot{3} \otimes \dot{F}_{2n+3}.$$

Other equations can be similarly proven. □

Theorem 4.6 *For $n \geqslant 0$, Binet formulas for the non-Newtonian Fibonacci and non-Newtonian Lucas quaternions are respectively as follows:*

$$\dot{Q}_n = \left(\frac{\dot{1}}{\sqrt{5}}\right) \otimes \left(\dot{\tau}_1 \otimes \dot{\alpha}^{\dot{n}} \ominus \dot{\tau}_2 \otimes \dot{\beta}^{\dot{n}}\right)$$

and

$$\dot{K}_n = \dot{\tau}_1 \otimes \dot{\alpha}^{\dot{n}} \oplus \dot{\tau}_2 \otimes \dot{\beta}^{\dot{n}},$$

where $\dot{\tau}_1 = \dot{1} \oplus \dot{\alpha} \otimes i^\star \oplus \dot{\alpha}^{\dot{2}} \otimes j^\star \oplus \dot{\alpha}^{\dot{3}} \otimes k^\star$ *and* $\dot{\tau}_2 = \dot{1} \oplus \dot{\beta} \otimes i^\star \oplus \dot{\beta}^{\dot{2}} \otimes j^\star \oplus \dot{\beta}^{\dot{3}} \otimes k^\star$.

Proof The characteristic equation of recurrence relations of non-Newtonian Fibonacci and non-Newtonian Lucas sequences is

$$\dot{t}^{\dot{2}} \ominus \dot{t} \ominus \dot{1} = \dot{0}.$$

The roots of this equation are $\dot{\alpha} = \frac{\dot{1} \oplus \dot{\sqrt{5}}}{\dot{2}}$ and $\dot{\beta} = \frac{\dot{1} \ominus \dot{\sqrt{5}}}{\dot{2}}$. Using recurrence relation of non-Newtonian Fibonacci sequences and the initial values $\dot{Q}_0 = \left(\dot{0}, \dot{1}, \dot{1}, \dot{2}\right)$ and $\dot{Q}_1 = \left(\dot{1}, \dot{1}, \dot{2}, \dot{3}\right)$ the Binet's formula for $\dot{Q}_n$ is obtained as follows:

$$\dot{Q}_n = \dot{A} \otimes \dot{\alpha}^{\dot{n}} \oplus \dot{B} \otimes \dot{\beta}^{\dot{n}} = \left(\frac{\dot{1}}{\sqrt{5}}\right) \otimes \left(\dot{\tau}_1 \otimes \dot{\alpha}^{\dot{n}} \ominus \dot{\tau}_2 \otimes \dot{\beta}^{\dot{n}}\right).$$

Similarly, we can get

$$\dot{K}_n = \dot{\tau}_1 \otimes \dot{\alpha}^{\dot{n}} \oplus \dot{\tau}_2 \otimes \dot{\beta}^{\dot{n}}.$$

This completes the proof. □

Theorem 4.7 *The generating function for the non-Newtonian Fibonacci quaternion* $\dot{Q}_n$ *is*

$$G\left(\dot{x},\dot{t}\right)=\frac{\dot{t}\oplus i^{\star}\oplus\left(\dot{t}\oplus\dot{1}\right)\otimes j^{\star}\oplus\left(\dot{t}\oplus\dot{2}\right)\otimes k^{\star}}{\dot{1}\ominus\dot{t}\ominus\dot{t}^{\dot{2}}}.$$

Proof From $\dot{Q}_{n+2}=\dot{Q}_{n+1}\oplus\dot{Q}_n$ and $G\left(\dot{x},\dot{t}\right)=\dot{\sum}_{n=0}^{\infty}\dot{Q}_n\left(\dot{x}\right)\otimes\dot{t}^{\dot{n}}$, we can easily see that

$$G\left(\dot{x},\dot{t}\right)=\frac{\dot{t}\oplus i^{\star}\oplus\left(\dot{t}\oplus\dot{1}\right)\otimes j^{\star}\oplus\left(\dot{t}\oplus\dot{2}\right)\otimes k^{\star}}{\dot{1}\ominus\dot{t}\ominus\dot{t}^{\dot{2}}}.\qquad\square$$

Theorem 4.8 *The generating function for the non-Newtonian Lucas quaternion* $\dot{K}_n$ *is*

$$G_L\left(\dot{x},\dot{t}\right)=\frac{\left(\dot{2}\ominus\dot{t}\right)\oplus\left(\dot{1}\oplus\dot{2}\otimes\dot{t}\right)\otimes i^{\star}\oplus\left(\dot{3}\oplus\dot{t}\right)\otimes j^{\star}\oplus\left(\dot{4}\oplus\dot{3}\otimes\dot{t}\right)\otimes k^{\star}}{\dot{1}\ominus\dot{t}\ominus\dot{t}^{\dot{2}}}.$$

Proof The proof is similar to the proof of Theorem 4.7. □

Theorem 4.9 *The following equations are satisfied:*

(*i*) $\dot{Q}_n\ominus\dot{Q}_{n+1}\otimes i^{\star}\ominus\dot{Q}_{n+2}\otimes j^{\star}\ominus\dot{Q}_{n+3}\otimes k^{\star}=\dot{L}_{n+3}$;

(*ii*) $\dot{Q}_n\otimes\dot{L}_n\oplus\dot{K}_n\otimes\dot{F}_n=\dot{2}\otimes\dot{Q}_{2n}$;

(*iii*) $\dot{Q}_n\oplus\dot{K}_n=\dot{2}\otimes\dot{Q}_{n+1}$;

(*iv*) $\dot{K}_n\ominus\dot{Q}_n=\dot{2}\otimes\dot{Q}_{n-1}$.

Proof (*i*) is obvious.

(*ii*) From the Binet formulas,

$$\begin{aligned}\dot{Q}_n\otimes\dot{L}_n\oplus\dot{K}_n\otimes\dot{F}_n &= \left(\frac{\dot{1}}{\sqrt{5}}\right)\otimes\left(\dot{\tau}_1\otimes\dot{\alpha}^{\dot{n}}\ominus\dot{\tau}_2\otimes\dot{\beta}^{\dot{n}}\right)\otimes\left(\dot{\alpha}^{\dot{n}}\oplus\dot{\beta}^{\dot{n}}\right)\\ &\quad\oplus\left(\dot{\tau}_1\otimes\dot{\alpha}^{\dot{n}}\oplus\dot{\tau}_2\otimes\dot{\beta}^{\dot{n}}\right)\otimes\left(\frac{\dot{1}}{\sqrt{5}}\right)\otimes\left(\dot{\alpha}^{\dot{n}}\ominus\dot{\beta}^{\dot{n}}\right)\\ &= \left(\frac{\dot{1}}{\sqrt{5}}\right)\otimes\dot{2}\otimes\left(\dot{\tau}_1\otimes\dot{\alpha}^{\dot{2n}}\ominus\dot{\tau}_2\otimes\dot{\beta}^{\dot{2n}}\right)\\ &= \dot{2}\otimes\dot{Q}_{2n}\end{aligned}$$

can be written. Thus, the proof is completed.

(*iii*) Using $\dot{F}_n\oplus\dot{F}_n=\dot{2}\otimes\dot{F}_{n+1}$, we can easily see that $\dot{Q}_n\oplus\dot{K}_n=\dot{2}\otimes\dot{Q}_{n+1}$.

(*iv*) Using $\dot{F}_n\oplus\dot{F}_n=\dot{2}\otimes\dot{F}_{n+1}$, we can easily see that $\dot{K}_n\ominus\dot{Q}_n=\dot{2}\otimes\dot{Q}_{n-1}$.

This completes the proof. □

§5. Non-Newtonian Cayley Numbers

Definition 5.1 *We define the non-Newtonian Cayley numbers as follows* $\dot{P}=\begin{bmatrix}\dot{\alpha}\\ \dot{\beta}\end{bmatrix}$, *where*

$\dot{\alpha}, \dot{\beta} \in \mathbb{H}(N)$. *Additionally, the set of non-Newtonian Cayley numbers is denoted by* $\mathbb{K}(N)$.

Equality between $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$ *and* $\dot{Q} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix}$ *are defined by* $\dot{P} = \dot{Q}$ *if and only if* $\dot{\alpha} = \dot{\gamma}$ *and* $\dot{\beta} = \dot{\delta}$.

Define the operations addition $(\oplus_M)$ *and multiplication* $(\otimes_M)$ *of non-Newtonian Cayley numbers* $\dot{P}$ *and* $\dot{Q}$ *as follows:*

$$\dot{P} \oplus_M \dot{Q} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \oplus_M \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \oplus \dot{\gamma} \\ \dot{\beta} \oplus \dot{\delta} \end{bmatrix}$$

and

$$\dot{P} \otimes_M \dot{Q} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \otimes_M \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \otimes \dot{\gamma} \ominus \dot{\delta} \otimes \overline{\dot{\beta}} \\ \dot{\gamma} \otimes \dot{\beta} \oplus \overline{\dot{\alpha}} \otimes \dot{\delta} \end{bmatrix},$$

where $\bar{\dot{\alpha}}$ *is conjugate of a non-Newtonian quaternion. Also, the identity of* $\oplus_M$ *is* $\tau = \begin{bmatrix} \dot{0}_{\mathbb{H}(N)} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix}$ *and the identity of* $\otimes_M$ *is* $T = \begin{bmatrix} \dot{1}_{\mathbb{H}(N)} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix}$.

The scalar product $\odot_M$ *is defined as*

$$\lambda \odot_M \dot{P} = \begin{bmatrix} \lambda \otimes \dot{\alpha} \\ \lambda \otimes \dot{\beta} \end{bmatrix},$$

where $\lambda \in \mathbb{R}_\alpha$.

The conjugate of a non-Newtonian Cayley number is defined as

$$\overline{\dot{P}} = \begin{bmatrix} \overline{\dot{\alpha}} \\ \ominus \dot{\beta} \end{bmatrix}.$$

Example 5.2 Let $\dot{P} = \begin{bmatrix} (\dot{1}, \dot{2}, \dot{2}, \dot{3}) \\ (\dot{0}_{\alpha_1}, \dot{1}, \dot{3}, \dot{1}) \end{bmatrix}, \dot{Q} = \begin{bmatrix} (\dot{2}, \dot{0}_{\alpha_2}, \dot{1}, \dot{1}) \\ (\dot{3}, \dot{1}, \dot{2}, \dot{2}) \end{bmatrix} \in \mathbb{K}(N)$. The non-Newtonian addition and multiplication of $\dot{P}$ and $\dot{Q}$ non-Newtonian Cayley numbers are

$$\dot{P} \oplus_M \dot{Q} = \begin{bmatrix} (\dot{1}, \dot{2}, \dot{2}, \dot{3}) \\ (\dot{0}_{\alpha_1}, \dot{1}, \dot{3}, \dot{1}) \end{bmatrix} \oplus_M \begin{bmatrix} (\dot{2}, \dot{0}_{\alpha_2}, \dot{1}, \dot{1}) \\ (\dot{3}, \dot{1}, \dot{2}, \dot{2}) \end{bmatrix} = \begin{bmatrix} (\dot{3}, \dot{2}, \dot{3}, \dot{4}) \\ (\dot{3}, \dot{2}, \dot{5}, \dot{3}) \end{bmatrix}$$

and

$$\dot{P} \otimes_M \dot{Q} = \begin{bmatrix} (\dot{1}, \dot{2}, \dot{2}, \dot{3}) \otimes (\dot{2}, \dot{0}_{\alpha_2}, \dot{1}, \dot{1}) \ominus (\dot{3}, \dot{1}, \dot{2}, \dot{2}) \otimes (\dot{0}_{\alpha_1}, \ominus_2 \dot{1}, \ominus_3 \dot{3}, \ominus_4 \dot{1}) \\ (\dot{2}, \dot{0}_{\alpha_2}, \dot{1}, \dot{1}) \otimes (\dot{0}_{\alpha_1}, \dot{1}, \dot{3}, \dot{1}) \oplus (\dot{1}, \ominus_2 \dot{2}, \ominus_3 \dot{2}, \ominus_4 \dot{3}) \otimes (\dot{3}, \dot{1}, \dot{2}, \dot{2}) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\ominus_1 \dot{12}, \dot{12}, \dot{13}, \dot{13}\right) \\ \left(\dot{11}, \ominus_2 \dot{3}, \dot{4}, \ominus_4 \dot{8}\right) \end{bmatrix},$$

respectively.

Definition 5.3 *The scalar and vector parts of* $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \in \mathbb{K}(N)$ *are*

$$S^{\alpha}_{\dot{P}} = \left(\frac{\dot{1}}{2}\right) \odot_M \left(\dot{P} \oplus_M \bar{\dot{P}}\right) = \begin{bmatrix} S^{\alpha}_{\dot{a}} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix}$$

and

$$V^{\alpha}_{\dot{P}} = \left(\frac{\dot{1}}{2}\right) \odot_M \left(\dot{P} \ominus_M \bar{\dot{P}}\right) = \begin{bmatrix} V^{\alpha}_{\dot{a}} \\ \dot{\beta} \end{bmatrix},$$

respectively.

Definition 5.4 *The norm of a non-Newtonian Cayley numbers* $\dot{P}$ *denoted by* $N_{\dot{P}}$ *and is defined as*

$$N_{\dot{P}} = \dot{P} \otimes_M \overline{\dot{P}} = \begin{bmatrix} \dot{\alpha} \otimes \overline{\dot{\alpha}} \oplus \dot{\beta} \otimes \overline{\dot{\beta}} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix}.$$

Definition 5.5 *Define an inverse element* $\dot{P}^{-1}$ *to every element* $\dot{P}$ *with non zero norm*

$$\dot{P}^{-1} = \frac{\dot{1}}{N_{\dot{P}}} \odot_M \overline{\dot{P}} = \frac{\dot{1}}{N_{\dot{P}}} \odot_M \begin{bmatrix} \overline{\dot{\alpha}} \\ \ominus \dot{\beta} \end{bmatrix}.$$

Theorem 5.6 *Let* $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}, \dot{Q} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix}$ *be non-Newtonian Cayley numbers, where* $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\delta}, \dot{\omega}, \dot{\eta} \in \mathbb{H}(N)$. *There are the following properties:*

(i) $S^{\alpha}_{\dot{P} \otimes_M \dot{Q}} = S^{\alpha}_{\dot{Q} \otimes_M \dot{P}}$;

(ii) $\overline{\left(\dot{P} \otimes_M \dot{Q}\right)} = \overline{\dot{Q}} \otimes_M \overline{\dot{P}}$;

(iii) $N_{\dot{P} \otimes_M \dot{Q}} = N_{\dot{P}} \otimes_M N_{\dot{Q}}$.

Proof Let $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}, \dot{Q} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix}$ be non-Newtonian Cayley numbers, where $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\delta}, \dot{\omega}, \dot{\eta} \in \mathbb{H}(N)$.

(i) It can be easily proved using the property $S^{\alpha}_{\dot{q} \otimes \dot{p}} = S^{\alpha}_{\dot{q}} \otimes S^{\alpha}_{\dot{p}}$.

(ii) $\overline{\left(\dot{P} \otimes_M \dot{Q}\right)} = \begin{bmatrix} \overline{\dot{\gamma}} \otimes \overline{\dot{\alpha}} \ominus \dot{\beta} \otimes \overline{\dot{\delta}} \\ \ominus \dot{\gamma} \otimes \dot{\beta} \ominus \overline{\dot{\alpha}} \otimes \dot{\delta} \end{bmatrix}$

$$\overline{\dot{Q}} \otimes_M \overline{\dot{P}} = \begin{bmatrix} \overline{\dot{\gamma}} \\ \ominus\dot{\delta} \end{bmatrix} \otimes_M \begin{bmatrix} \overline{\dot{\alpha}} \\ \ominus\dot{\beta} \end{bmatrix} = \begin{bmatrix} \overline{\dot{\gamma}} \otimes \overline{\dot{\alpha}} \ominus \dot{\beta} \otimes \overline{\dot{\delta}} \\ \ominus\dot{\gamma} \otimes \dot{\beta} \ominus \overline{\dot{\alpha}} \otimes \dot{\delta} \end{bmatrix} = \overline{\left(\dot{P} \otimes_M \dot{Q}\right)}.$$

(iii)

$$\begin{aligned} N_{\dot{P}} \otimes_M N_{\dot{Q}} &= \begin{bmatrix} \dot{\alpha} \otimes \overline{\dot{\alpha}} \oplus \dot{\beta} \otimes \overline{\dot{\beta}} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix} \otimes_M \begin{bmatrix} \dot{\gamma} \otimes \overline{\dot{\gamma}} \oplus \dot{\delta} \otimes \overline{\dot{\delta}} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix} \\ &= \begin{bmatrix} \left(\dot{\alpha} \otimes \overline{\dot{\alpha}} \oplus \dot{\beta} \otimes \overline{\dot{\beta}}\right) \otimes \left(\dot{\gamma} \otimes \overline{\dot{\gamma}} \oplus \dot{\delta} \otimes \overline{\dot{\delta}}\right) \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix}. \end{aligned}$$

Now, using $\overline{\dot{p} \otimes \dot{q}} = \overline{\dot{q}} \otimes \overline{\dot{p}}$ and $N_{\dot{P}} = \dot{P} \otimes_M \overline{\dot{P}}$, we have

$$N_{\dot{P} \otimes_M \dot{Q}} = \begin{bmatrix} \dot{\alpha} \otimes \dot{\gamma} \ominus \dot{\delta} \otimes \overline{\dot{\beta}} \\ \dot{\gamma} \otimes \dot{\beta} \oplus \overline{\dot{\alpha}} \otimes \dot{\delta} \end{bmatrix} \otimes_M \begin{bmatrix} \overline{\dot{\gamma}} \otimes \overline{\dot{\alpha}} \ominus \dot{\beta}\overline{\dot{\delta}} \\ \ominus\dot{\gamma} \otimes \dot{\beta} \ominus \overline{\dot{\alpha}} \otimes \dot{\delta} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\dot{\alpha} \otimes \dot{\gamma} \ominus \dot{\delta} \otimes \overline{\dot{\beta}}\right) \otimes \left(\overline{\dot{\gamma}} \otimes \overline{\dot{\alpha}} \ominus \dot{\beta} \otimes \overline{\dot{\delta}}\right) \oplus \left(\dot{\gamma} \otimes \dot{\beta} \oplus \overline{\dot{\alpha}} \otimes \dot{\delta}\right) \otimes \left(\overline{\dot{\beta}} \otimes \overline{\dot{\gamma}} \oplus \overline{\dot{\delta}} \otimes \dot{\alpha}\right) \\ \left(\overline{\dot{\gamma}} \otimes \overline{\dot{\alpha}} \ominus \dot{\beta} \otimes \overline{\dot{\delta}}\right) \otimes \left(\dot{\gamma} \otimes \dot{\beta} \oplus \overline{\dot{\alpha}} \otimes \dot{\delta}\right) \ominus \left(\overline{\dot{\gamma}} \otimes \overline{\dot{\alpha}} \ominus \dot{\beta} \otimes \overline{\dot{\delta}}\right) \otimes \left(\dot{\gamma} \otimes \dot{\beta} \oplus \overline{\dot{\alpha}} \otimes \dot{\delta}\right) \end{bmatrix}$$

$$\begin{aligned} &= \begin{bmatrix} \dot{\alpha} \otimes \overline{\dot{\alpha}} \otimes \dot{\gamma} \otimes \overline{\dot{\gamma}} \oplus \dot{\delta} \otimes \overline{\dot{\delta}} \otimes \dot{\beta} \otimes \overline{\dot{\beta}} \oplus \dot{\gamma} \otimes \overline{\dot{\gamma}} \otimes \dot{\beta} \otimes \overline{\dot{\beta}} \oplus \dot{\alpha} \otimes \overline{\dot{\alpha}} \otimes \dot{\delta} \otimes \overline{\dot{\delta}} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix} \\ &= N_{\dot{P}} \otimes_M N_{\dot{Q}}. \end{aligned}$$

This completes the proof. □

Remark 5.7 Let $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$, $\dot{Q} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix}$, $\dot{R} = \begin{bmatrix} \dot{\omega} \\ \dot{\eta} \end{bmatrix}$ be non-Newtonian Cayley numbers, where $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\delta}, \dot{\omega}, \dot{\eta} \in \mathbb{H}(N)$. Using the properties in $\mathbb{H}(N)$, it can be seen that the multiplication $\otimes_M$ is not associative as follows:

$$\dot{P} \otimes_M \left(\dot{Q} \otimes_M \dot{R}\right) = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \otimes_M \begin{bmatrix} \dot{\gamma} \otimes \dot{\omega} \ominus \dot{\eta} \otimes \overline{\dot{\delta}} \\ \dot{\omega} \otimes \dot{\delta} \oplus \overline{\dot{\gamma}} \otimes \dot{\eta} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\alpha} \otimes \left(\dot{\gamma} \otimes \dot{\omega} \ominus \dot{\eta} \otimes \overline{\dot{\delta}}\right) \ominus \left(\dot{\omega} \otimes \dot{\delta} \oplus \overline{\dot{\gamma}} \otimes \dot{\eta}\right) \otimes \overline{\dot{\beta}} \\ \left(\dot{\gamma} \otimes \dot{\omega} \ominus \dot{\eta} \otimes \overline{\dot{\delta}}\right) \otimes \dot{\beta} \oplus \overline{\dot{\alpha}} \otimes \left(\dot{\omega} \otimes \dot{\delta} \oplus \overline{\dot{\gamma}} \otimes \dot{\eta}\right) \end{bmatrix}$$

and

$$\left(\dot{P} \otimes_M \dot{Q}\right) \otimes_M \dot{R} = \begin{bmatrix} \dot{\alpha} \otimes \dot{\gamma} \ominus \dot{\delta} \otimes \overline{\dot{\beta}} \\ \dot{\gamma} \otimes \dot{\beta} \oplus \overline{\dot{\alpha}} \otimes \dot{\delta} \end{bmatrix} \otimes_M \begin{bmatrix} \dot{\omega} \\ \dot{\eta} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\dot{\alpha} \otimes \dot{\gamma} \ominus \dot{\delta} \otimes \overline{\dot{\beta}}\right) \otimes \dot{\omega} \ominus \dot{\eta} \otimes \overline{\left(\dot{\gamma} \otimes \dot{\beta} \oplus \overline{\dot{\alpha}} \otimes \dot{\delta}\right)} \\ \dot{\omega} \otimes \left(\dot{\gamma} \otimes \dot{\beta} \oplus \overline{\dot{\alpha}} \otimes \dot{\delta}\right) \oplus \overline{\left(\dot{\alpha} \otimes \dot{\gamma} \ominus \dot{\delta} \otimes \overline{\dot{\beta}}\right)} \otimes \dot{\eta} \end{bmatrix}.$$

Since non-Newtonian quaternions are not commutative, we have

$$\dot{P} \otimes_M \left(\dot{Q} \otimes_M \dot{R}\right) \neq \left(\dot{P} \otimes_M \dot{Q}\right) \otimes_M \dot{R}.$$

Theorem 5.8 *If* $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}, \dot{Q} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix}$ *be non-Newtonian Cayley numbers, where* $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\delta} \in \mathbb{H}(N)$*, then*

$$\left(\dot{Q} \otimes_M \dot{P}^{-1}\right) \otimes_M \dot{P} = \dot{Q} \quad \textit{and} \quad \dot{P}^{-1} \otimes_M \left(\dot{P} \otimes_M \dot{Q}\right) = \dot{Q}.$$

Proof Let $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}, \dot{Q} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix}$ be non-Newtonian Cayley numbers, where $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\delta} \in \mathbb{H}(N)$. Then, $\dot{P}^{-1} = \frac{\dot{1}}{N_{\dot{P}}} \odot_M \begin{bmatrix} \overline{\dot{\alpha}} \\ \ominus\dot{\beta} \end{bmatrix}$ and

$$\begin{aligned}
\dot{Q} \otimes_M \dot{P}^{-1} &= \frac{\dot{1}}{N_{\dot{P}}} \odot_M \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix} \otimes_M \begin{bmatrix} \overline{\dot{\alpha}} \\ \ominus\dot{\beta} \end{bmatrix} = \frac{\dot{1}}{N_{\dot{P}}} \odot_M \begin{bmatrix} \dot{\gamma} \otimes \overline{\dot{\alpha}} \oplus \dot{\beta} \otimes \overline{\dot{\delta}} \\ \overline{\dot{\alpha}} \otimes \dot{\delta} \ominus \overline{\dot{\gamma}} \otimes \dot{\beta} \end{bmatrix}, \\
\left(\dot{Q} \otimes_M \dot{P}^{-1}\right) \otimes_M \dot{P} &= \frac{\dot{1}}{N_{\dot{P}}} \odot_M \begin{bmatrix} \dot{\gamma} \otimes \overline{\dot{\alpha}} \oplus \dot{\beta} \otimes \overline{\dot{\delta}} \\ \overline{\dot{\alpha}} \otimes \dot{\delta} \ominus \overline{\dot{\gamma}} \otimes \dot{\beta} \end{bmatrix} \otimes_M \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} \\
&= \frac{\dot{1}}{N_{\dot{P}}} \odot_M \begin{bmatrix} \left(\dot{\gamma} \otimes \overline{\dot{\alpha}} \oplus \dot{\beta} \otimes \overline{\dot{\delta}}\right) \otimes \dot{\alpha} \ominus \dot{\beta} \otimes \left(\overline{\dot{\delta}} \otimes \dot{\alpha} \ominus \overline{\dot{\beta}} \otimes \dot{\gamma}\right) \\ \dot{\alpha} \otimes \left(\overline{\dot{\alpha}} \otimes \dot{\delta} \ominus \overline{\dot{\gamma}} \otimes \dot{\beta}\right) \oplus \left(\dot{\alpha} \otimes \overline{\dot{\gamma}} \oplus \dot{\delta} \otimes \overline{\dot{\beta}}\right) \otimes \dot{\beta} \end{bmatrix} \\
&= \frac{\dot{1}}{N_{\dot{P}}} \odot_M \begin{bmatrix} \dot{\gamma} \otimes \left(\overline{\dot{\alpha}} \otimes \dot{\alpha} \oplus \dot{\beta} \otimes \overline{\dot{\beta}}\right) \\ \dot{\delta} \otimes \left(\overline{\dot{\alpha}} \otimes \dot{\alpha} \oplus \overline{\dot{\beta}} \otimes \dot{\beta}\right) \end{bmatrix} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix} = \dot{Q}.
\end{aligned}$$

It can be shown similarly that $\dot{P}^{-1} \otimes_M \left(\dot{P} \otimes_M \dot{Q}\right) = \dot{Q}$.

Generally $\dot{P} \otimes_M \left(\dot{Q} \otimes_M \dot{P}^{-1}\right) \neq \dot{Q}$. This completes the proof. □

Theorem 5.9 *For any non-Newtonian Cayley numbers* $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$, $\dot{Q} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix}$, $\dot{R} = \begin{bmatrix} \dot{\omega} \\ \dot{\eta} \end{bmatrix} \in \mathbb{K}(N)$, *there is*

$$S^{\alpha}_{\left(\dot{R}\otimes_M \dot{P}\right)\otimes_M\left(\dot{Q}\otimes_M \overline{\dot{R}}\right)} = N_{\dot{R}} \odot_M S^{\alpha}_{\dot{P}\otimes_M \dot{Q}}.$$

Proof Let $\dot{P} = \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix}$, $\dot{Q} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix}$, $\dot{R} = \begin{bmatrix} \dot{\omega} \\ \dot{\eta} \end{bmatrix}$ be non-Newtonian Cayley numbers, where $\dot{\alpha}, \dot{\beta}, \dot{\gamma}, \dot{\delta}, \dot{\omega}, \dot{\eta} \in \mathbb{H}(N)$. Then,

$$\dot{R} \otimes_M \dot{P} = \begin{bmatrix} \dot{\omega} \\ \dot{\eta} \end{bmatrix} \otimes_M \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \dot{\omega} \otimes \dot{\alpha} \ominus \dot{\beta} \otimes \overline{\dot{\eta}} \\ \dot{\alpha} \otimes \dot{\eta} \oplus \overline{\dot{\omega}} \otimes \dot{\beta} \end{bmatrix},$$

$$\dot{Q} \otimes_M \overline{\dot{R}} = \begin{bmatrix} \dot{\gamma} \\ \dot{\delta} \end{bmatrix} \otimes_M \begin{bmatrix} \overline{\dot{\omega}} \\ \ominus\dot{\eta} \end{bmatrix} = \begin{bmatrix} \dot{\gamma} \otimes \overline{\dot{\omega}} \oplus \dot{\eta} \otimes \overline{\dot{\delta}} \\ \overline{\dot{\omega}} \otimes \dot{\delta} \ominus \overline{\dot{\gamma}} \otimes \dot{\eta} \end{bmatrix}$$

and

$$\begin{aligned} &\left(\dot{R} \otimes_M \dot{P}\right) \otimes_M \left(\dot{Q} \otimes_M \overline{\dot{R}}\right) \\ &= \begin{bmatrix} \left(\dot{\omega} \otimes \dot{\alpha} \ominus \dot{\beta} \otimes \overline{\dot{\eta}}\right) \otimes \left(\dot{\gamma} \otimes \overline{\dot{\omega}} \oplus \dot{\eta} \otimes \overline{\dot{\delta}}\right) \ominus \left(\overline{\dot{\omega}} \otimes \dot{\delta} \ominus \overline{\dot{\gamma}} \otimes \dot{\eta}\right) \otimes \left(\overline{\dot{\eta}} \otimes \overline{\dot{\alpha}} \oplus \overline{\dot{\beta}} \otimes \dot{\omega}\right) \\ \left(\dot{\gamma} \otimes \overline{\dot{\omega}} \oplus \dot{\eta} \otimes \overline{\dot{\delta}}\right) \otimes \left(\dot{\alpha} \otimes \dot{\eta} \oplus \overline{\dot{\omega}} \otimes \dot{\beta}\right) \oplus \left(\overline{\dot{\alpha}} \otimes \overline{\dot{\omega}} \ominus \dot{\eta} \otimes \overline{\dot{\beta}}\right) \otimes \left(\overline{\dot{\omega}} \otimes \dot{\delta} \ominus \overline{\dot{\gamma}} \otimes \dot{\eta}\right) \end{bmatrix}. \end{aligned}$$

Thus,

$$S^{\alpha}_{\left(\dot{R}\otimes_M \dot{P}\right)\otimes_M\left(\dot{Q}\otimes_M \overline{\dot{R}}\right)} = \left(\dot{\frac{1}{2}}\right) \odot_M \begin{bmatrix} \dot{l} \oplus \dot{m} \oplus \dot{n} \oplus \dot{k} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix},$$

where

$$\begin{aligned} \dot{l} &= \dot{\omega}\left(\dot{\alpha} \otimes \dot{\gamma} \oplus \overline{\dot{\gamma}} \otimes \overline{\dot{\alpha}}\right) \otimes \overline{\dot{\omega}} \ominus \overline{\dot{\omega}}\left(\dot{\delta} \otimes \overline{\dot{\beta}} \oplus \dot{\beta} \otimes \overline{\dot{\delta}}\right) \otimes \dot{\omega}, \\ \dot{m} &= \dot{\eta} \otimes \overline{\dot{\eta}} \otimes \left(\overline{\dot{\gamma}} \otimes \overline{\dot{\alpha}} \oplus \dot{\alpha} \otimes \dot{\gamma} \ominus \dot{\beta} \otimes \overline{\dot{\delta}} \ominus \dot{\delta} \otimes \overline{\dot{\beta}}\right), \\ \dot{n} &= \left(\dot{\delta} \otimes \overline{\dot{\eta}} \otimes \overline{\dot{\alpha}} \otimes \overline{\dot{\omega}} \ominus \overline{\dot{\omega}} \otimes \dot{\delta} \otimes \overline{\dot{\eta}} \otimes \overline{\dot{\alpha}}\right) \oplus \left(\dot{\omega} \otimes \dot{\alpha} \otimes \dot{\eta} \otimes \overline{\dot{\delta}} \ominus \dot{\alpha} \otimes \dot{\eta} \otimes \overline{\dot{\delta}} \otimes \dot{\omega}\right), \\ \dot{k} &= \left(\overline{\dot{\gamma}} \otimes \dot{\eta} \otimes \overline{\dot{\beta}} \otimes \dot{\omega} \ominus \dot{\omega} \otimes \overline{\dot{\gamma}} \otimes \dot{\eta} \otimes \overline{\dot{\beta}}\right) \oplus \left(\overline{\dot{\omega}} \otimes \dot{\beta} \otimes \overline{\dot{\eta}} \otimes \dot{\gamma} \ominus \dot{\beta} \otimes \overline{\dot{\eta}} \otimes \dot{\gamma} \otimes \overline{\dot{\omega}}\right). \\ S^{\alpha}_{\dot{\alpha}\otimes\dot{\gamma}} &= \left(\dot{\frac{1}{2}}\right) \odot \left(\dot{\alpha} \otimes \dot{\gamma} \oplus \overline{\dot{\gamma}} \otimes \overline{\dot{\alpha}}\right) \in \mathbb{R}_{\alpha}, \\ S^{\alpha}_{\dot{\delta}\otimes\overline{\dot{\beta}}} &= \left(\dot{\frac{1}{2}}\right) \odot \left(\dot{\delta} \otimes \overline{\dot{\beta}} \oplus \dot{\beta} \otimes \overline{\dot{\delta}}\right) \in \mathbb{R}_{\alpha} \end{aligned}$$

and using the property $S^{\alpha}_{\dot{p}\otimes\dot{q}} = S^{\alpha}_{\dot{q}\otimes\dot{p}}$ for any $\dot{p},\dot{q} \in \mathbb{H}_{(N)}$, we have

$$\begin{aligned} S^{\alpha}_{\dot{\omega}\otimes\left(\dot{\alpha}\otimes\dot{\eta}\otimes\overline{\dot{\delta}}\right)} &= \dot{\left(\frac{1}{2}\right)} \odot \left(\dot{\omega}\otimes\dot{\alpha}\otimes\dot{\eta}\otimes\overline{\dot{\delta}} \oplus \dot{\delta}\otimes\overline{\dot{\eta}}\otimes\overline{\dot{\alpha}}\otimes\overline{\dot{\omega}}\right) \\ &= S^{\alpha}_{\left(\dot{\alpha}\otimes\dot{\eta}\otimes\overline{\dot{\delta}}\right)\otimes\dot{\omega}} = \dot{\left(\frac{1}{2}\right)} \odot \left(\dot{\alpha}\otimes\dot{\eta}\otimes\overline{\dot{\delta}}\otimes\dot{\omega} \oplus \overline{\dot{\omega}}\otimes\dot{\delta}\otimes\overline{\dot{\eta}}\otimes\overline{\dot{\alpha}}\right) \end{aligned}$$

and

$$\begin{aligned} S^{\alpha}_{\overline{\dot{\omega}}\otimes\left(\dot{\beta}\otimes\overline{\dot{\eta}}\otimes\dot{\gamma}\right)} &= \dot{\left(\frac{1}{2}\right)} \odot \left(\overline{\dot{\omega}}\otimes\dot{\beta}\otimes\overline{\dot{\eta}}\otimes\dot{\gamma} \oplus \overline{\dot{\gamma}}\otimes\dot{\eta}\otimes\overline{\dot{\beta}}\otimes\dot{\omega}\right) \\ &= S^{\alpha}_{\left(\dot{\beta}\otimes\overline{\dot{\eta}}\otimes\dot{\gamma}\right)\otimes\overline{\dot{\omega}}} = \dot{\left(\frac{1}{2}\right)} \odot \left(\dot{\beta}\otimes\overline{\dot{\eta}}\otimes\dot{\gamma}\otimes\overline{\dot{\omega}} \oplus \dot{\omega}\otimes\overline{\dot{\gamma}}\otimes\dot{\eta}\otimes\overline{\dot{\beta}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} S^{\alpha}_{\left(\dot{R}\otimes_M\dot{P}\right)\otimes_M\left(\dot{Q}\otimes_M\overline{\dot{R}}\right)} &= \begin{bmatrix} \left(\dot{\omega}\otimes\overline{\dot{\omega}}\oplus\dot{\eta}\otimes\overline{\dot{\eta}}\right)\odot S^{\alpha}_{\dot{\alpha}\otimes\dot{\gamma}} \ominus \left(\dot{\omega}\otimes\overline{\dot{\omega}}\oplus\dot{\eta}\otimes\overline{\dot{\eta}}\right)\odot S^{\alpha}_{\dot{\delta}\otimes\overline{\dot{\beta}}} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix} \\ &= N_{\dot{R}} \odot_M \begin{bmatrix} S^{\alpha}_{\dot{\alpha}\otimes\dot{\gamma}\ominus\dot{\delta}\otimes\overline{\dot{\beta}}} \\ \dot{0}_{\mathbb{H}(N)} \end{bmatrix} = N_{\dot{R}} \odot_M S^{\alpha}_{\dot{P}\otimes_M\dot{Q}}, \end{aligned}$$

which completes the proof. □

§6. Conclusion

The non-Newtonian quaternions, non-Newtonian Fibonacci quaternions, non-Newtonian Lucas quaternions and non-Newtonian Cayley numbers, created using non-Newtonian calculus, may provide ease of operation in some structures where classical quaternions and Cayley numbers are used and may offer a new approach for some structures. In the future, this approach can be adapted to different types of quaternions and applied to mathematical and physical structures using quaternions. These findings could provide a potential basis for advancing research and applications related to quaternions.

References

[1] Çakmak A. F., Başar, F., Certain spaces of functions over the field of non-Newtonian complex numbers, *Abstract and Applied Analysis*, Vol. 2014, Article ID 23614.

[2] Değirmen N., Some finite sum formulas and inequalities involving non-Newtonian Fibonacci numbers, V, *International Halich Congress on Multidisciplinary Scientific Research*, 864-870, 2023.

[3] Grossman M., Katz, R., Non-Newtonian calculus: A self-contained, Elementary Exposition of the Authors' Investigations, *Non-Newtonian Calculus*, 1972.

[4] Halici S., On Fibonacci quaternions, *Advances in Applied Clifford Algebras*, 22(2), 321-327, 2012.

[5] Ward J. P., Quaternions and Cayley numbers, *Algebra and Applications*, Vol. 403, 2012.

[6] Van Der Waerden B. L., Hamilton's discovery of quaternions, *Mathematics Magazine*, 49(5), 227-234, 1976.

International J.Math. Combin. Vol.2(2026), 85-98

On the Augmented Sombor Index of Polycyclic Molecular Graphs

Bochao Zheng and Zhen Lin*

(School of Mathematics and Statistics,Qinghai Normal University, Xining, 810008, China)

E-mail: lnlinzhen@163.com

Abstract: Let G be a graph with edge set $E(G)$. Recently, Das, Gutman and Ali introduced a new vertex-degree-based topological index, called the augmented Sombor index of G, defined as

$$ASO(G) = \sum_{uv\in E(G)} \sqrt{\frac{d(u)^2 + d(v)^2}{d(u) + d(v) - 2}},$$

where $d(u)$ denotes the degree of vertex u. This paper establishes a linear model for predicting the physicochemical properties of benzenoid hydrocarbons using the augmented Sombor index. Moreover, we characterize all molecular graphs of order n and cyclomatic number ℓ that minimize the augmented Sombor index, under the condition that $n \geq 2(\ell - 1) \geq 4$.

Key Words: Augmented Sombor index, Smarandachely Sombor index, molecular graph, extremal value, cyclomatic number.

AMS(2010): 05C09, 05C07, 05C35, 05C92.

§1. Introduction

A molecular graph is defined in mathematical chemistry as a connected graph whose maximum vertex degree does not exceed four. It serves as an effective abstraction of the skeletal structure of compounds, with carbon atoms represented as vertices and covalent bonds as edges. This graphical framework allows the use of graph-theoretic invariants - termed topological indices - to quantify molecular architecture. Among these, the Sombor index family, which includes the original Sombor index [6], elliptic Sombor index [10], Euler Sombor index [20], diminished Sombor index [15], and augmented Sombor index [5], has attracted considerable research interest in recent years owing to its geometric motivation and strong predictive capability for physicochemical properties of molecular compounds. In Quantitative Structure Property Relationship (QSPR) and Quantitative Structure Activity Relationship (QSAR) studies, such indices are computed over specific molecular graph, for instance, benzenoid hydrocarbons and are correlated with target physicochemical descriptors (e.g., boiling point, π-electron energy,

[1]Supported by the National Science Foundation of China(Nos. 12261074)

[2]*Corresponding author, E-mail addresses: lnlinzhen@163.com

[3]Received December 9, 2025, Accepted June 5, 2026

molecular weight), see for example [3, 13, 19]. This in silico methodology not only offers mechanistic insight into structure-property relationships but also constitutes a cost-efficient and rapid screening strategy before undertaking laborious experimental work.

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is denoted by n. For a vertex $u \in V(G)$, the set of its neighbors in G is denoted as $N(u) = \{v \in V(G) : uv \in E(G)\}$. The degree of u is the number of its neighbors, that is, $d_G(u) = d(u) = |N(u)|$. A vertex of degree one (at least three, respectively) is called a pendant vertex (branching vertex, respectively). An edge incident with a pendent vertex is called a pendent edge. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The cyclomatic number of a graph is the smallest number of edges whose removal makes the graph acyclic.

In 2021, Gutman [8] introduced the Sombor index from a geometric perspective, and defined it as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d^2(u) + d^2(v)}.$$

This index has attracted considerable interest in both mathematics and theoretical chemistry, see, e.g., [6, 12, 14, 17, 18]. On this basis, scholars have proposed various versions related to the Sombor index, such as the first (to Sixth) Sombor-like index [9], the elliptic Sombor index [10], the Euler Sombor index [20], the hyperbolic Sombor index [2] and the diminished Sombor index [15].

Building upon the Sombor index, Das, Gutman, and Ali [5] further extended this metric by incorporating the degrees of line graph vertices, thereby proposing the augmented Sombor (ASO) index, which is defined as follows:

$$ASO(G) = \sum_{uv \in E(G)} \sqrt{\frac{d^2(u) + d^2(v)}{d(u) + d(v) - 2}},$$

and meanwhile, they established lower and upper bounds for the augmented Sombor index within the classes of trees, quasi-trees, and connected graphs, and characterized the corresponding extremal graphs in each case. Notice that the Sombor index $SO(G)$ and augmented Sombor index $ASO(G)$ are global properties on G. Generally, for an induced subgraph $H \prec G$ such as $K_3, K_3 + e, K_{1,3}, K_4 - e, \cdots$, a Smarandachely Sombor index $SSO_H(G)$ is defined by

$$SSO_H(G) = \sum_{uv \in E(G \setminus H)} \sqrt{\frac{d^2(u) + d^2(v)}{d(u) + d(v) - 2}}.$$

Clearly, if $H = \emptyset$ or there are no subgraphs $G' \simeq H$ in G, a Smarandachely Sombor index $SSO_H(G)$ of G is nothing else but the augmented Sombor index $ASO(G)$ of G. However, it characterizes the Sombor index on graph $G \setminus H$.

Alotaibi, Alanazi, Hassan, and Ali [1] determine all graphs that minimize the diminished Sombor index among molecular graphs of order n and with a cyclomatic number of at least three. Inspired by this, we investigate the chemical application of the augmented Sombor index to establish a linear model for predicting the physicochemical properties of benzenoid hydrocarbons (polycyclic molecular graphs). Furthermore, we characterize all molecular graphs of order n and cyclomatic number ℓ that minimize the ASO index, under the condition $n \geq 2(\ell - 1) \geq 4$.

§2. Applications of ASO Index to the Physicochemical Properties of Benzenoid Hydrocarbons

In this section, the chemical applicability of the ASO index is examined in the context of modeling several properties of benzenoid hydrocarbons. Linear regression models are developed for predicting boiling point (BP), π-electron energy (π-ele), molecular weight (MW), polarizability (PO), molar volume (MV), and molar refractivity (MR).

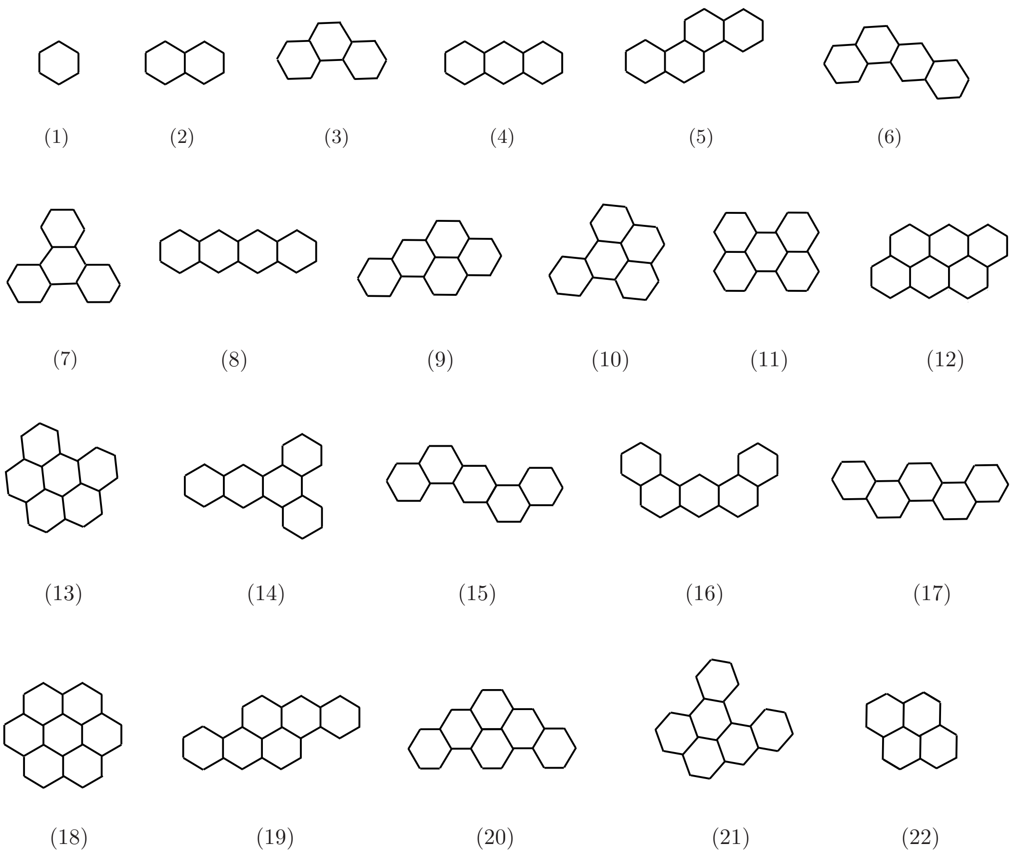


Figure 1. Molecular graphs

((1) Benzene, (2) Naphthalene, (3) Phenanthrene, (4) Anthracene, (5) Chrysene, (6) Benzo[a]anthracene, (7) Triphenylene, (8) Tetracene, (9) Benzo[a]pyrene, (10) Benzo[e]pyrene, (11) Perylene, (12) Anthanthrene, (13) Benzo[ghi]perylene, (14) Dibenz[a,c]anthracene, (15) Dibenz[a,h]anthracene, (16) Dibenz[a,j]anthracene, (17) Picene, (18) Coronene, (19) Dibenzo[a,h]pyrene, (20) Dibenzo[a,i]pyrene, (21) Dibenzo[a,l] pyrene, (22) Pyrene.)

The molecular graph and physicochemical properties of benzenoid hydrocarbons along with the computed ASO index are represented in Figure 1 and Table 1 respectively. For benzenoid hydrocarbons, the correlation coefficients (R) between the ASO index and various properties are as follows:

BP (0.9755), π-ele (0.9991), MW (0.9967), PO (0.9999), MV (0.9456), and MR (0.9999),

as summarized in Table 2. These strong correlations indicate that the ASO index is an effective descriptor for predicting these physicochemical properties.

Table 1. Experimental values of benzenoid hydrocarbons and the corresponding values of augmented Sombor index

Derivatives of benzene	BP	π-ele	MW	PO	MV	MR	ASO
Benzene	78.8	8	78.11	10.4	89.4	26.3	12
Naphthalene	221.5	13.683	128.17	17.5	123.5	44.1	22.4480
Phenanthrene	337.4	19.448	178.23	24.6	157.7	61.9	32.8540
Anthracene	337.4	19.314	178.23	24.6	157.7	61.9	32.8960
Chrysene	448	25.192	228.3	31.6	191.8	79.8	43.2599
Benzo[a]anthracene	436.7	25.101	228.3	31.6	191.8	79.8	43.3019
Triphenylene	425	25.275	228.3	31.6	191.8	79.8	43.2179
Tetracene	436.7	25.188	228.3	31.6	191.8	79.8	43.3440
Benzo[a]pyrene	495	28.222	252.3	35.8	196.1	90.3	49.6659
Benzo[e]pyrene	467.5	28.336	252.3	35.8	196.1	90.3	49.6239
Perylene	467.5	28.245	252.3	35.8	196.1	90.3	49.6239
Anthanthrene	497.1	31.253	276.3	40	200.4	100.8	56.0719
Benzo[ghi]perylene	501	31.425	276.3	40	200.4	100.8	56.0299
Dibenz[a,c]anthracene	518	30.942	278.3	38.7	225.9	97.6	53.6659
Dibenz[a,h]anthracene	524.7	30.881	278.3	38.7	225.9	97.6	53.7079
Dibenz[a,j]anthracene	524.7	30.88	278.3	38.7	225.9	97.6	53.7079
Picene	519	30.943	278.3	38.7	225.9	97.6	53.6659
Coronene	525.6	34.572	300.4	44.1	204.7	111.4	62.4358
Dibenzo[a,h]pyrene	552.3	33.928	302.4	42.9	230.2	108.1	60.0719
Dibenzo[a,i]pyrene	552.3	33.954	302.4	42.9	230.2	108.1	60.0719
Dibenzo[a,l]pyrene	552.3	34.031	302.4	42.9	230.2	108.1	60.0299
Pyrene	404	22.506	202.25	28.7	162	72.5	39.2599

Table 2. The correlation coefficient R from linear regression model between augmented Sombor index and physicochemical properties of benzenoid hydrocarbons

	BP	π-ele	MW	PO	MV	MR
ASO	0.9755	0.9991	0.9967	0.9999	0.9456	0.9999

The linear regression models linking the ASO index with the properties of boiling point (BP), π-electron energy (π-ele), molecular weight (MW), polarizability (PO), molar volume (MV), and molar refractivity (MR) were developed via the least squares fitting procedure, as illustrated in Figure 2. Regression analysis reveals that the properties considered in this study [11, 16] are strongly correlated with the ASO index.

§3. Minimum *ASO* Index of Molecular Graphs with Cyclomatic Number at Least 3

For an edge set $S \subseteq E(G)$, the graph $G - S$ is obtained by removing all edges in S from G. Similarly, $G + S$ is obtained by adding all edges in S to G. In particular, if $S = uv$, we simply write $G - uv$ and $G + uv$ instead of $G - \{uv\}$ and $G + \{uv\}$, respectively.

Let r, k and $a_1, a_2, \cdots, a_k$ be positive integers such that $1 \leqslant a_1 < a_2 < \cdots < a_k \leqslant \lfloor \frac{r}{2} \rfloor$. Such a graph is called circulant graph. The circulant graph, denoted by $C(r; a_1, a_2, \cdots, a_k)$, is the graph with vertex set $\{v_0, v_1, v_2, \cdots, v_{r-1}\}$ and edge set

$$\left\{v_i v_{i+a_j} : 0 \leqslant i \leqslant r-1 \ and \ 1 \leqslant j \leqslant k, \ where \ i + a_j \ is \ taken \ modulo \ r\right\}.$$

If $a_k < \frac{r}{2}$, then $C(r; a_1, a_2, \cdots, a_k)$ is $2k$-regular.

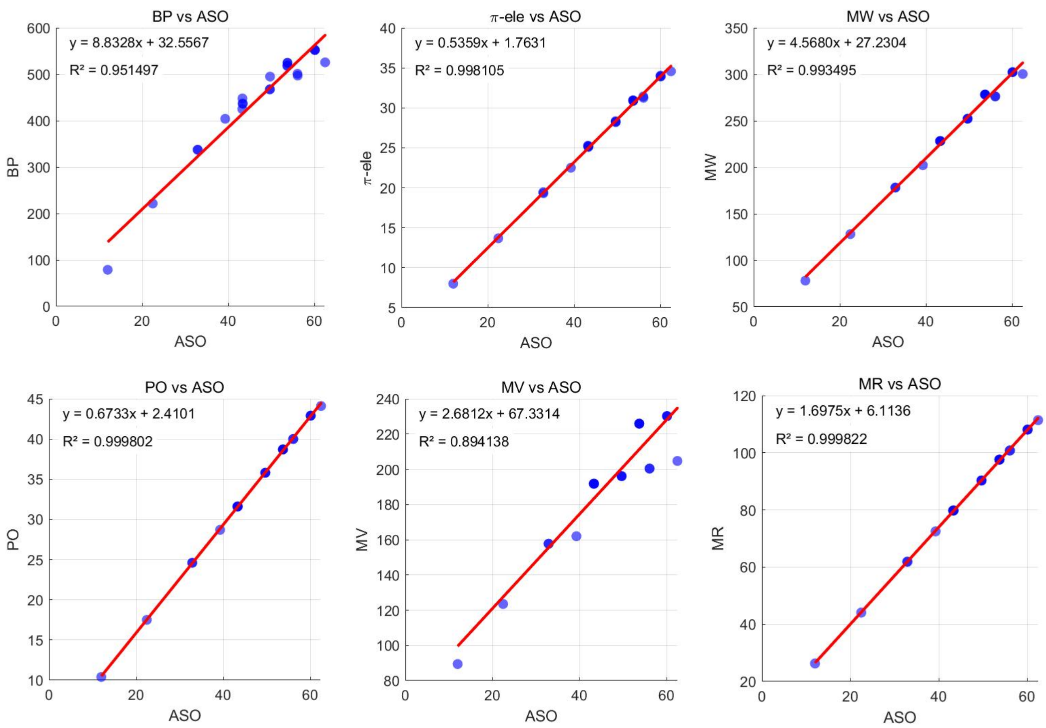


Figure 2. Scatter plots between physicochemical properties of benzenoid hydrocarbons and their *ASO* index

Lemma 3.1([4]) *The circulant graph $C_n(r; a_1, a_2, \cdots, a_k)$ is connected if and only if*

$$\gcd(a_1, a_2, \cdots, a_k, r) = 1.$$

Lemma 3.2 *Let G be a graph minimizing the ASO index among all n-order molecular graphs with cyclomatic number ℓ such that $4 \leqslant 2(\ell - 1) \leqslant n$. Then, G contains no pendant edge incident to a vertex of degree 2.*

Proof This proof is by contradiction. Let G contain at least one pendant edge incident to a vertex of degree 2. Denote such an edge by $e = uv \in E(G)$, where $d_G(u) = 1$ and $d_G(v) = 2$. We consider a branching vertex $w \in V(G)$ such that the distance between w and u is minimized. Then, every vertex on the unique $u - w$ path P, except u and w, has degree 2 in G. Since $\ell \geqslant 3$, w must be adjacent to at least two non-pendent vertices. Let $N_G(w) = \{w_1, w_2, \cdots, w_s\}$, where w_s lies on P and $s \in \{3, 4\}$. Let

$$d_G(w_1) = \max\{d_G(w_i) : 1 \leqslant i \leqslant s-1\}.$$

Case 1. $d_G(w) = 4$.

Since G is a molecular graph, by a direct computation, we have

$$\sqrt{\frac{16}{3}} - \sqrt{5} < \sqrt{5} - \sqrt{\frac{9}{2}} < \sqrt{\frac{17}{3}} - \sqrt{5} < \sqrt{5} - \sqrt{\frac{13}{3}},$$

$$\sqrt{\frac{16}{3}} - \sqrt{5} < \sqrt{\frac{17}{3}} - \sqrt{5} < \sqrt{5} - \sqrt{\frac{13}{3}} < \sqrt{5} - 2.$$

Let $G_1 = G - w_1w + w_1u$. Since $d_G(w_1) \geqslant 2$ and $1 \leqslant d_G(w_i) \leqslant d_G(w_1) \leqslant 4$ for each $i \in \{2, 3\}$, we have

$$\begin{aligned}
&ASO(G) - ASO(G_1)\\
&= \sum_{i=2}^{3}\left[\sqrt{\frac{4^2 + (d_G(w_i))^2}{4 + d_G(w_i) - 2}} - \sqrt{\frac{3^2 + (d_G(w_i))^2}{3 + d_G(w_i) - 2}}\right] + \sqrt{\frac{4^2 + 2^2}{4 + 2 - 2}} - \sqrt{\frac{3^2 + 2^2}{3 + 2 - 2}}\\
&\quad + \sqrt{\frac{1^2 + 2^2}{1 + 2 - 2}} - \sqrt{\frac{2^2 + 2^2}{2 + 2 - 2}} + \sqrt{\frac{4^2 + (d_G(w_1))^2}{4 + d_G(w_1) - 2}} - \sqrt{\frac{2^2 + (d_G(w_1))^2}{2 + d_G(w_1) - 2}}\\
&= \sum_{i=2}^{3}\left[\sqrt{\frac{16 + (d_G(w_i))^2}{2 + d_G(w_i)}} - \sqrt{\frac{9 + (d_G(w_i))^2}{1 + d_G(w_i)}}\right] + \sqrt{5} - \sqrt{\frac{13}{3}} + \sqrt{5} - 2\\
&\quad + \sqrt{\frac{16 + (d_G(w_1))^2}{2 + d_G(w_1)}} - \sqrt{\frac{4 + (d_G(w_1))^2}{d_G(w_1)}}\\
&\geqslant 2\left(\sqrt{\frac{16}{3}} - \sqrt{5}\right) + 2\sqrt{5} - \sqrt{\frac{13}{3}} - 2 + \sqrt{\frac{16}{3}} - \sqrt{5}\\
&> 0.61 > 0,
\end{aligned}$$

that is, $ASO(G) > ASO(G_1)$, a contradiction.

Case 2. $d_G(w) = 3$ and $(d_G(w_1), d_G(w_2)) \neq (4, 4)$.

By a direct computation, we have

$$0 = \sqrt{5} - \sqrt{5} < \sqrt{\frac{9}{2}} - \sqrt{\frac{13}{3}} < \sqrt{\frac{13}{3}} - 2.$$

Let $G_2 = G - w_1w + w_1u$. Since $d_G(w_1) \geqslant 2$, $1 \leqslant d_G(w_2) \leqslant d_G(w_1) \leqslant 4$ and

$$(d_G(w_1), d_G(w_2)) \neq (4, 4),$$

we have

$$\begin{aligned}
&ASO(G) - ASO(G_2)\\
&= \sqrt{\frac{3^2 + (d_G(w_2))^2}{3 + d_G(w_2) - 2}} - \sqrt{\frac{2^2 + (d_G(w_2))^2}{2 + d_G(w_2) - 2}} + \sqrt{\frac{3^2 + 2^2}{3 + 2 - 2}} - \sqrt{\frac{2^2 + 2^2}{2 + 2 - 2}}\\
&\quad + \sqrt{\frac{1^2 + 2^2}{1 + 2 - 2}} - \sqrt{\frac{2^2 + 2^2}{2 + 2 - 2}} + \sqrt{\frac{3^2 + (d_G(w_1))^2}{3 + d_G(w_1) - 2}} - \sqrt{\frac{2^2 + (d_G(w_1))^2}{2 + d_G(w_1) - 2}}\\
&= \sqrt{\frac{9 + (d_G(w_2))^2}{1 + d_G(w_2)}} - \sqrt{\frac{4 + (d_G(w_2))^2}{d_G(w_2)}} + \sqrt{\frac{13}{3}} - 2 + \sqrt{5} - 2\\
&\quad + \sqrt{\frac{9 + (d_G(w_1))^2}{1 + d_G(w_1)}} - \sqrt{\frac{4 + (d_G(w_1))^2}{d_G(w_1)}}\\
&\geqslant \sqrt{5} - \sqrt{5} + \sqrt{\frac{13}{3}} - 2 + \sqrt{5} - 2 + \sqrt{5} - \sqrt{5} > 0.31 > 0,
\end{aligned}$$

that is, $ASO(G) > ASO(G_2)$, a contradiction.

Case 3. $d_G(w) = 3$ and $(d_G(w_1), d_G(w_2)) = (4, 4)$.

Subcase 3.1 Either w_1 or w_2 does not have three neighbors of degree 4.

Without loss of generality, let w_1 does not have three neighbors of degree 4. We choose a vertex $w_1' \in N_G(w_1) \setminus \{w\}$, such that it satisfies $d_G(w_1') = \min\{d_G(w') : w_1' \in N_G(w_1) \setminus \{w\}\}$. Therefore, $d_G(w_1') \leqslant 3$. Let $G_3 = G - w_1'w_1 + w_1'u$. Then, we have

$$\begin{aligned}
ASO(G) - ASO(G_3) = &\sum_{w' \in N_G(w_1)\setminus\{w, w_1'\}} \left[\sqrt{\frac{4^2 + ((d_G(w'))^2}{4 + d_G(w') - 2}} - \sqrt{\frac{3^2 + (d_G(w'))^2}{3 + d_G(w') - 2}}\right]\\
&+ \sqrt{\frac{1^2 + 2^2}{1 + 2 - 2}} - \sqrt{\frac{2^2 + 2^2}{2 + 2 - 2}} + \sqrt{\frac{3^2 + 4^2}{3 + 4 - 2}} - \sqrt{\frac{3^2 + 3^2}{3 + 3 - 2}}\\
&+ \sqrt{\frac{4^2 + (d_G(w_1'))^2}{4 + d_G(w_1') - 2}} - \sqrt{\frac{2^2 + (d_G(w_1'))^2}{2 + d_G(w_1') - 2}}\\
= &\sum_{w' \in N_G(w_1)\setminus\{w, w_1'\}} \left[\sqrt{\frac{16 + ((d_G(w'))^2}{2 + d_G(w')}} - \sqrt{\frac{9 + (d_G(w'))^2}{1 + d_G(w')}}\right] + 2\sqrt{5}\\
&- 2 - \sqrt{\frac{9}{2}} + \sqrt{\frac{16 + (d_G(w_1'))^2}{2 + d_G(w_1')}} - \sqrt{\frac{4 + (d_G(w_1'))^2}{d_G(w_1')}}\\
\geqslant &2\left(\sqrt{\frac{16}{3}} - \sqrt{5}\right) + \sqrt{\frac{17}{3}} - \sqrt{5} + 2\sqrt{5} - 2 - \sqrt{\frac{9}{2}}\\
> &0.64 > 0,
\end{aligned}$$

that is, $ASO(G) > ASO(G_3)$, a contradiction.

Subcase 3.2 Both w_1 and w_2 have three neighbors of degree 4.

Let G^* be the graph obtained from G by removing all vertices on the path P except w and their incident edges. Then, $d_{G^*}(w) = 2$ and $d_{G^*}(x) = d_G(x)$ for each $x \in V(G) \setminus V(P)$, where $V(P)$ is the set of all vertices belonging to the path P. To complete the proof, we first prove a claim.

Claim 1. There exists at least one vertex $y \in V(G^*) \setminus \{w\}$ such that $1 \leqslant d_G(y) \leqslant 3$.

Proof of Claim 1. This proof is by contradiction. Let each vertex $y \in V(G^*) \setminus \{w\}$ satisfy $d_G(y) = 4$, that is, $d_{G^*}(y) = 4$. Since both the graphs G and G^* have the same cyclomatic number ℓ, it follows from the Handshaking Lemma that

$$4(|V(G^*)| - 1) + 2 = 2|E(G^*)| = 2(|V(G^*)| + \ell - 1).$$

Therefore, $|V(G^*)| = \ell$.

If $\ell \leqslant 5$,then we have $|V(G^*)| \leqslant 5$. This contradicts the fact that $d_{G^*}(w) = 2$ because every $y \in V(G^*) \setminus \{w\}$ satisfies $d_{G^*}(y) = 4$, according to our assumption. We now consider the case where $\ell \geqslant 6$, i.e., $|V(G^*)| \geqslant 6$. Next, we first prove that for every ℓ ($\ell \geqslant 6$) ,there exists at least one connected graph of order ℓ and the degree sequence $(\underbrace{4, 4, \cdots, 4}_{\ell-1}, 2)$.

The circulant graph $C(\ell-1; 1, 2)$ is a 4-regular graph for $\ell \geqslant 6$. By Lemma 1, $C(\ell-1; 1, 2)$ is a connected graph. Let $C^*(\ell-1; 1, 2) = C(\ell-1; 1, 2) - \{v_0v_1\} + \{v'v_0, v'v_1\}$, where v' is the newly added vertex. Clearly, the order and the degree sequence of the graph $C^*(\ell-1; 1, 2)$ are ℓ and $(\underbrace{4, 4, \cdots, 4}_{\ell-1}, 2)$,respectively. Therefore, we have

$$m_{1,2}(G) = 1 = m_{2,3}(G), m_{3,4}(G) = 2, m_{4,4}(G) = 2\ell - 3, m_{2,2}(G) = n - \ell - 2,$$

where $m_{i,j}(G)$ denotes the number of edges with end vertices of degrees i and j in G. Then, we have

$$\begin{aligned} ASO(G) =&(n-\ell-2)\sqrt{\frac{2^2+2^2}{2+2-2}} + (2\ell-3)\sqrt{\frac{4^2+4^2}{4+4-2}} + \sqrt{\frac{1^2+2^2}{1+2-2}} \\ &+ \sqrt{\frac{2^2+3^2}{2+3-2}} + 2\sqrt{\frac{3^2+4^2}{3+4-2}} \\ =&2(n-\ell-2) + (2\ell-3)\sqrt{\frac{16}{3}} + 3\sqrt{5} + \sqrt{\frac{13}{3}}. \end{aligned}$$

Let G' be the n-order molecular graph with cyclomatic number ℓ such that $4 \leqslant 2(\ell-1) \leqslant n$ and

$$m_{2,3}(G') = 2, m_{3,3}(G') = 3\ell - 4, m_{2,2}(G') = n - 2\ell + 1.$$

Then, we have

$$\begin{aligned}
ASO(G') =&(n-2\ell+1)\sqrt{\frac{2^2+2^2}{2+2-2}}+(3\ell-4)\sqrt{\frac{3^2+3^2}{3+3-2}}+2\sqrt{\frac{2^2+3^2}{2+3-2}}\\
=&2(n-2\ell+1)+(3\ell-4)\sqrt{\frac{9}{2}}+2\sqrt{\frac{13}{3}}.
\end{aligned}$$

Therefore,

$$ASO(G)-ASO(G')=\left(2+\frac{8\sqrt{3}}{3}-\frac{9\sqrt{2}}{2}\right)\ell-6-4\sqrt{3}+6\sqrt{2}+3\sqrt{5}-\frac{\sqrt{39}}{3}.$$

Since $2+\frac{8\sqrt{3}}{3}-\frac{9\sqrt{2}}{2}>0.25>0$ and $-6-4\sqrt{3}+6\sqrt{2}+3\sqrt{5}-\frac{\sqrt{39}}{3}>0.18>0$, we have $ASO(G)-ASO(G')>0$. Namely, $ASO(G)>ASO(G')$, a contradiction. This completes the proof of Claim 1.

Next, using Claim 1, we proceed with the following three cases based on the minimum vertex degree in $V(G^*)\setminus\{w\}$.

Subcase 3.2.1 There exists a vertex $p\in V(G^*)\setminus\{w\}$ such that $d_G(p)=1$.

Let p_1 be the unique neighbor of p in G and $G_4=G-\{w_1w,w_2w\}+\{w_1w_2,wp\}$. By a direct computation, we have

$$\sqrt{\frac{17}{3}}-\sqrt{5}<\sqrt{5}-\sqrt{\frac{13}{3}}<\sqrt{5}-2.$$

Therefore,

$$\begin{aligned}
ASO(G)-ASO(G_4)=&\sqrt{\frac{1^2+(d_G(p_1))^2}{1+d_G(p_1)-2}}-\sqrt{\frac{2^2+(d_G(p_1))^2}{2+d_G(p_1)-2}}+\sqrt{\frac{2^2+3^2}{2+3-2}}\\
&+2\sqrt{\frac{3^2+4^2}{3+4-2}}-2\sqrt{\frac{2^2+2^2}{2+2-2}}-\sqrt{\frac{4^2+4^2}{4+4-2}}\\
=&\sqrt{\frac{1+(d_G(p_1))^2}{d_G(p_1)-1}}-\sqrt{\frac{4+(d_G(p_1))^2}{d_G(p_1)}}+\sqrt{\frac{13}{3}}+2\sqrt{5}\\
&-4-\sqrt{\frac{16}{3}}\\
\geqslant&\sqrt{\frac{17}{3}}-\sqrt{5}+\sqrt{\frac{13}{3}}+2\sqrt{5}-4-\sqrt{\frac{16}{3}}\\
>&0.38>0,
\end{aligned}$$

that is, $ASO(G)>ASO(G_4)$, a contradiction.

Subcase 3.2.2 There does not exist any vertex $p\in V(G^*)\setminus\{w\}$ such that $d_G(p)=1$, but there is a vertex $q\in V(G^*)\setminus\{w\}$ such that $d_G(q)=2$.

Since G is connected, we choose a vertex q such that at least one of its neighbors has degree at least 3 in G. Let q_1 and q_2 be the neighbors of q such that $d_G(q_1)\geqslant 3$ and $d_G(q_2)\geqslant 2$. Let

$G_5 = G - \{w_1w, w_2w\} + \{w_1w_2, wq\}$. By a direct computation, we have

$$2 - \sqrt{\frac{13}{3}} < \sqrt{\frac{13}{3}} - \sqrt{\frac{9}{2}} < \sqrt{5} - \sqrt{5} = 0$$

Then, we have

$$\begin{aligned}
&ASO(G) - ASO(G_5)\\
&= \sum_{i=1}^{2}\left[\sqrt{\frac{2^2 + (d_G(q_i))^2}{2 + d_G(q_i) - 2}} - \sqrt{\frac{3^2 + (d_G(q_i))^2}{3 + d_G(q_i) - 2}}\right] + 2\sqrt{\frac{3^2 + 4^2}{3 + 4 - 2}} - \sqrt{\frac{2^2 + 2^2}{2 + 2 - 2}}\\
&\quad - \sqrt{\frac{4^2 + 4^2}{4 + 4 - 2}} + \sqrt{\frac{2^2 + 3^2}{2 + 3 - 2}} - \sqrt{\frac{2^2 + 3^2}{2 + 3 - 2}}\\
&= \sum_{i=1}^{2}\left[\sqrt{\frac{4 + (d_G(q_i))^2}{d_G(q_i)}} - \sqrt{\frac{9 + (d_G(q_i))^2}{1 + d_G(q_i)}}\right] + 2\sqrt{5} - 2 - \sqrt{\frac{16}{3}}\\
&\geqslant 2 - \sqrt{\frac{13}{3}} + \sqrt{\frac{13}{3}} - \sqrt{\frac{9}{2}} + 2\sqrt{5} - 2 - \sqrt{\frac{16}{3}}\\
&> 0.04 > 0,
\end{aligned}$$

that is, $ASO(G) > ASO(G_5)$, a contradiction.

Subcase 3.2.3 There does not exist any vertex $p \in V(G^*) \setminus \{w\}$ such that $d_G(p) = 1$ or $d_G(p) = 2$, but there is a vertex $z \in V(G^*) \setminus \{w\}$ such that $d_G(z) = 3$.

Since G is connected, we choose a vertex z such that at least one of its neighbors has degree at least 4 in G. Let z_i for $i \in \{1, 2, 3\}$ be the neighbors of z such that $d_G(z_1) = 4$, $d_G(z_2) \geqslant 3$ and $d_G(z_3) \geqslant 3$.

Subcase 3.2.3.1 $d_G(z_2) = d_G(z_3) = 4$.

Let $G_6 = G - \{w_1w, w_2w, z_2z\} + \{w_1w_2, wz, z_2u\}$. Then, we have

$$\begin{aligned}
ASO(G) - ASO(G_6) =& \sqrt{\frac{1^2 + 2^2}{1 + 2 - 2}} - 2\sqrt{\frac{2^2 + 2^2}{2 + 2 - 2}} + 3\sqrt{\frac{4^2 + 3^2}{4 + 3 - 2}} - \sqrt{\frac{4^2 + 4^2}{4 + 4 - 2}}\\
&- \sqrt{\frac{2^2 + 4^2}{2 + 4 - 2}} + \sqrt{\frac{2^2 + 3^2}{2 + 3 - 2}} - \sqrt{\frac{2^2 + 3^2}{2 + 3 - 2}}\\
=& 3\sqrt{5} - 4 - \sqrt{\frac{16}{3}}\\
>& 0.39 > 0,
\end{aligned}$$

that is, $ASO(G) > ASO(G_6)$, a contradiction.

Subcase 3.2.3.2 $\min\{d_G(z_2), d_G(z_3)\} = 3$.

Without loss of generality, we assume that $d_G(z_2) = 3$. Then, either $d_G(z_3) = 3$ or

$d_G(z_3)=4$.Let $G_7 = G - \{w_1w, w_2w, z_2z\} + \{w_1w_2, wz, z_2u\}$. Then, we have

$$\begin{aligned}ASO(G)-ASO(G_7)=&\sqrt{\frac{1^2+2^2}{1+2-2}}+2\sqrt{\frac{3^2+4^2}{3+4-2}}+\sqrt{\frac{3^2+3^2}{3+3-2}}-2\sqrt{\frac{2^2+2^2}{2+2-2}}\\&-\sqrt{\frac{4^2+4^2}{4+4-2}}-\sqrt{\frac{2^2+3^2}{2+3-2}}+\sqrt{\frac{2^2+3^2}{2+3-2}}-\sqrt{\frac{2^2+3^2}{2+3-2}}\\=&3\sqrt{5}+\sqrt{\frac{9}{2}}-4-\sqrt{\frac{16}{3}}-\sqrt{\frac{13}{3}}\\>&0.43>0,\end{aligned}$$

that is, $ASO(G) > ASO(G_7)$, a contradiction.

Therefore, G contains no pendant edge incident to a vertex of degree 2. □

Theorem 3.3 *Let G be a molecular n-order graph with cyclomatic number ℓ such that $4 \leqslant 2(\ell-1) \leqslant n$.*

(1) *If $n = 2(\ell-1)$, then*

$$ASO(G) \geqslant \frac{9\sqrt{2}}{2}(\ell-1)$$

with equality if and only if G is a 3-regular graph;

(2) *If $n > 2(\ell-1)$, then*

$$ASO(G) \geqslant 2n + \left(\frac{9\sqrt{2}}{2}-4\right)\ell + 2 - 6\sqrt{2} + \frac{2\sqrt{39}}{3}$$

with equality if and only if

$$\delta(G)=2, \quad \Delta(G)=3$$

and

$$m_{3,3}(G)=3\ell-4, \quad m_{2,3}(G)=2, \quad m_{2,2}(G)=n-2\ell+1.$$

Proof Let G^* be a graph minimizing the augmented Sombor index among all molecular n-order graphs with cyclomatic number ℓ such that $4 \leqslant 2(\ell-1) \leqslant n$. Then, we have

$$ASO(G) \ \geqslant \ ASO(G^*).$$

By Lemma 3.2, graph G^* contains no pendant edge incident to a vertex of degree 2. Namely, $m_{1,2}(G^*)=0$. Let $m_{i,j}(G)=m_{i,j}(G^*)$ and

$$A := \{(i,j) : 1 \leqslant i \leqslant j \leqslant 4 \text{ and } i,j \in \mathbb{Z}_+\} \setminus \{(1,1),(1,2)\},$$

where $\mathbb{Z}_+$ denotes the set of positive integers. Since

$$\sum_{(i,j)\in A}\left(\frac{1}{i}+\frac{1}{j}\right)m_{i,j} = |V(G^*)| = n,$$

we have

$$\sum_{(i,j)\in A^*}\left(\frac{1}{i}+\frac{1}{j}\right)m_{i,j}=n-m_{2,2}-\frac{5}{6}m_{2,3}-\frac{2}{3}m_{3,3}, \tag{1}$$

where $A^*:=A\setminus\{(2,2),(2,3),(3,3)\}$.

Moreover, since $\sum_{(i,j)\in A}m_{i,j}=n+\ell-1$, we have

$$\sum_{(i,j)\in A^*}m_{i,j}=n+\ell-1-m_{2,2}-m_{2,3}-m_{3,3}. \tag{2}$$

By solving (1) and (2), we have

$$m_{2,3}=2(n-2\ell-m_{2,2}+2)+\sum_{(i,j)\in A^*}\left(4-6\left(\frac{1}{i}+\frac{1}{j}\right)\right)m_{i,j}$$

and

$$m_{3,3}=5(\ell-1)+m_{2,2}-n+\sum_{(i,j)\in A^*}\left(6\left(\frac{1}{i}+\frac{1}{j}\right)-5\right)m_{i,j}.$$

By substituting $m_{2,3}$ and $m_{3,3}$, we obtain

$$\begin{aligned}
ASO(G^*)=&\sum_{(i,j)\in A}\sqrt{\frac{i^2+j^2}{i+j-2}}m_{i,j}\\
=&\sqrt{\frac{2^2+2^2}{2+2-2}}m_{2,2}+\sqrt{\frac{2^2+3^2}{2+3-2}}m_{2,3}+\sqrt{\frac{3^2+3^2}{3+3-2}}m_{3,3}+\sum_{(i,j)\in A^*}\sqrt{\frac{i^2+j^2}{i+j-2}}m_{i,j}\\
=&2m_{2,2}+\sqrt{\frac{13}{3}}m_{2,3}+\sqrt{\frac{9}{2}}m_{3,3}+\sum_{(i,j)\in A^*}\sqrt{\frac{i^2+j^2}{i+j-2}}m_{i,j}\\
=&\left(\frac{2\sqrt{39}}{3}-\frac{3\sqrt{2}}{2}\right)n+\left(\frac{15\sqrt{2}}{2}-\frac{4\sqrt{39}}{3}\right)\ell+\left(2+\frac{3\sqrt{2}}{2}-\frac{2\sqrt{39}}{3}\right)m_{2,2}\\
&+\frac{4\sqrt{39}}{3}-\frac{15\sqrt{2}}{2}\\
&+\sum_{(i,j)\in A^*}\left[(9\sqrt{2}-2\sqrt{39})\left(\frac{1}{i}+\frac{1}{j}\right)+\sqrt{\frac{i^2+j^2}{i+j-2}}+\frac{4\sqrt{39}}{3}-\frac{15\sqrt{2}}{2}\right]m_{i,j}.
\end{aligned}$$

Let

$$f(i,j)=(9\sqrt{2}-2\sqrt{39})\left(\frac{1}{i}+\frac{1}{j}\right)+\sqrt{\frac{i^2+j^2}{i+j-2}}+\frac{4\sqrt{39}}{3}-\frac{15\sqrt{2}}{2}.$$

By a direct computation, we have

$$\begin{aligned}
&f(1,3)>0.27,\quad f(1,4)>0.39,\quad f(2,4)>0.13,\\
&f(3,4)>0.09,\quad f(4,4)>0.14.
\end{aligned}$$

Hence,

$$\begin{aligned} ASO(G^*) \geqslant & \left(\frac{2\sqrt{39}}{3} - \frac{3\sqrt{2}}{2}\right) n + \left(\frac{15\sqrt{2}}{2} - \frac{4\sqrt{39}}{3}\right) \ell \\ & + \left(2 + \frac{3\sqrt{2}}{2} - \frac{2\sqrt{39}}{3}\right) m_{2,2} + \frac{4\sqrt{39}}{3} - \frac{15\sqrt{2}}{2}. \end{aligned}$$

By the definition of G^*, the inequality above must hold with equality. This implies that $m_{i,j} = 0$ for every $(i, j) \in A^*$, which means G^* contains neither vertices of degree 4 nor pendant edges incident to vertices of degree 3. Therefore, we have either $\delta(G^*) = \Delta(G^*) = 3$ or $(\delta(G^*), \Delta(G^*)) = (2, 3)$.

Case 1. If $n = 2(\ell - 1)$, we can not have $(\delta(G^*), \Delta(G^*)) = (2, 3)$. Otherwise, we would have $n_2(G^*) = n - 2(\ell - 1) = 0$, a contradiction. Therefore, if $n = 2(\ell - 1)$, we have $\delta(G^*) = \Delta(G^*) = 3$, that is, G^* is a 3-regular graph. Consequently, we have

$$ASO(G^*) = \frac{9\sqrt{2}}{2}(\ell - 1).$$

Case 2. If $n > 2(\ell - 1)$, we can not have $\delta(G^*) = \Delta(G^*) = 3$. Otherwise, we would have $3n = 2(n + \ell - 1)$ by the Handshaking Lemma, a contradiction. Therefore, if $n > 2(\ell - 1)$, we have $(\delta(G^*), \Delta(G^*)) = (2, 3)$. Since G^* is a connected graph, it holds that $m_{2,3} > 0$. Moreover, since $2m_{2,2} + m_{2,3} = 2n_2$, it follows that $m_{2,3}$ must be a positive even integer. Hence,

$$2m_{2,2} + 2 \leqslant 2m_{2,2} + m_{2,3} = 2n_2 = 2(n - 2(\ell - 1)),$$

which yields $m_{2,2} \leqslant n - 2\ell + 1$ with equality if and only if $m_{2,3} = 2$. Therefore, from the definition of G^*, we obtain $m_{2,2} = n - 2\ell + 1$, $m_{2,3} = 2$, $m_{3,3} = 3\ell - 4$. Consequently, we have

$$\begin{aligned} ASO(G^*) = & \left(\frac{2\sqrt{39}}{3} - \frac{3\sqrt{2}}{2}\right) n + \left(\frac{15\sqrt{2}}{2} - \frac{4\sqrt{39}}{3}\right) \ell \\ & + \left(2 + \frac{3\sqrt{2}}{2} - \frac{2\sqrt{39}}{3}\right) (n - 2\ell + 1) + \frac{4\sqrt{39}}{3} - \frac{15\sqrt{2}}{2} \\ = & 2n + \left(\frac{9\sqrt{2}}{2} - 4\right) \ell + 2 - 6\sqrt{2} + \frac{2\sqrt{39}}{3}. \end{aligned}$$

This completes the proof. □

References

[1] A.M. Alotaibi, A.M. Alanazi, T.S. Hassan and A. Ali, On the diminished Sombor index of fixed-order molecular graphs with cyclomatic number at least 3, *MATCH Commun. Math. Comput. Chem.*, 95 (2026) 813-828.

[2] J. Barman and S. Das, Geometric approach to degree-based topological index: Hyperbolic

Sombor index, *MATCH Commun. Math. Comput. Chem.*, 95 (2026) 63-94.

[3] J. Barman and S. Das, Chemical applicability of the hyperbolic Sombor index, *Chem. Phys. Lett.*, 878 (2025) 142340.

[4] F. Boesch and R. Tindell, Circulants and their connectivities, *J. Graph Theory*, 8 (1984) 487-499.

[5] K.C. Das, I. Gutman and A. Ali, Augmented Sombor index, *MATCH Commun. Math. Comput. Chem.*, 95 (2026) 523-547.

[6] H. Deng, Z. Tang and R. Wu, Molecular trees with extremal values of Sombor indices, *Int. J. Quantum Chem.*, 121 (2021) e26622.

[7] B. Furtula, I. Gutman and M. Dehmer, On structure-sensitivity of degree-based topological indices, *Appl. Math. Comput.* 219 (2013) 8973-8978.

[8] I. Gutman, Geometric approach to degree-based topological indices: Sombor indices, *MATCH Commun. Math. Comput. Chem.*, 86 (2021) 11-16.

[9] I. Gutman, Sombor indices-back to geometry, *Open J. Discret. Appl. Math.*, 5 (2022) 1-5.

[10] I. Gutman, B. Furtula and M.S. Oz, Geometric approach to vertex-degree-based topological indices-Elliptic Sombor index, theory and application, it Int. J. Quantum Chem., 124 (2024) e27346.

[11] S. Hayat and S. Khan, Quality testing of spectrum-based valency descriptors for polycyclic aromatic hydrocarbons with applications, *J. Mol. Struct.*, 1228 (2021) 129789.

[12] Z. Lin, Sombor-type indices for certain interconnection networks, *MATCH Commun. Math. Comput. Chem.*, 94 (2025) 701-724.

[13] H. Liu, H. Chen, Q. Xiao, X. Fang and Z. Tang, More on Sombor indices of chemical graphs and their applications to the boiling point of benzenoid hydrocarbons, *Int. J. Quantum Chem.*, 121 (2021) e26689.

[14] H. Liu, I. Gutman, L. You and Y. Huang, Sombor index: review of extremal results and bounds, *J. Math. Chem.*, 60 (2022) 771-798.

[15] F. Movahedi, I. Gutman, I. Redžepović and B. Furtula, Diminished Sombor index, *MATCH Commun. Math. Comput. Chem.*, 95 (2026) 141-162.

[16] S. Nikolić, N. Trinajstić and I. Baucčić, Comparison between the vertex- and edge-connectivity indices for benzenoid hydrocarbons, *J. Chem. Inf. Comput. Sci.*, 38 (1998) 42-46.

[17] B. A. Ravindran, Sombor index on trees-A review, *MATCH Commun. Math. Comput. Chem.*, 94 (2025) 291-305.

[18] A. Rauf and S. Ahmad, On sombor indices of tetraphenylethylene, terpyridine rosettes and QSPR analysis on fluorescence properties of several aromatic hetero-cyclic species, *Int. J. Quantum Chem.*, 123 (2023) e27261.

[19] M.C. Shanmukha, A. Usha, V. R. Kulli and K.C. Shilpa, Chemical applicability and curvilinear regression models of vertex-degree-based topological index: Elliptic Sombor index, *Int. J. Quantum Chem.*, 124 (2024) e27376.

[20] Z. Tang, Y. Li and H. Deng, The Euler Sombor index of a graph, *Int. J. Quantum Chem.*, 124 (2024) e27387.

International J.Math. Combin. Vol.2(2026), 99-110

Support Edge Irregularity of FSG

L.Subhalakshmi

Research Scholar, Reg. No. 21212052092001
PG and Research Department of Mathematics, G Venkataswamy Naidu College Kovilpatti
Affiliated to Manonmaniam Sundaranar University, Abishekapatti-627 012
Tirunelveli, Tamil Nadu, India

N.R.Santhi Maheswari

PG and Research Department of Mathematics
G Venkataswamy Naidu College Kovilpatti, Tirunelveli-628502, Tamil Nadu, India

E-mail: lakshmansubha2@gmail.com, nrsmaths@yahoo.com

Abstract: Herein, we define, support of the edges of an FSG and the total support of the edges of an FSG. We also discuss support edge and support edge totally irregular FSG and support edge neighbourly irregular and support edge totally neighbourly irregular FSGs. A relative study of support edge irregularity and support edge neighbourly irregularity of an FSG is made. Some properties of the support edge irregular FSGs are considered and studied.

Key Words: Support edge of FSG, support edge neighbourly irregular FSG, support edge irregular FSG, support edge total irregular FSG.

AMS(2010): 05C12, 03E72, 05C72.

§1. Introduction

The soft set theory as developed by Molodtstov deals with uncertainity and indefinite objects, and its notions was given by him in [6] 1999. Maji.et al [5] merged soft sets with fuzzy which proved much effective and defined basic operations on them with some applications to it. Akram & Nawaz investigated properties of fuzzy soft graphs [7]. Akilandeswari introduced degree of edges in a FSG and edge regular FSG [9] and its properties. Somasundari introduced support of a vertex in FSG [10] and detailed about support neighbourly irregular FSG. We here have defined the support of an edge in an FSG and carried out study of support edge neighbourly irregular FSGs.

§2. Preliminaries

Go through the following to go through the work.

Definition 2.1 *A FGG is nonempty set V with, $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$: $\forall$*

[1]Received November 17, 2025, Accepted June 6, 2026

$u, v \in V$, $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$.

Definition 2.2 *A pair* $(\mathscr{F}, \mathscr{A})$ *is a soft set over the universal set* $\mathscr{U}$, *where* $\mathscr{A} \subseteq \mathscr{E}$ *and* $\mathscr{F} : a \to \mathscr{P}(U)$.

Definition 2.3 *An FSG* $\widetilde{G} = (\mathscr{G}^*, \mathscr{F}, \mathscr{K}, \mathscr{A})$, *a 4-tuple that*

(1) $\mathscr{G}^*$ *is crisp graph;*

(2) $\mathscr{A}$ *is the parameter set;*

(3) $\widetilde{\mathscr{F}}, \mathscr{A}$ *is FSS over vertex set;*

(4) $\widetilde{\mathscr{K}}, \mathscr{A}$ *is FSS over edge set.*

The pair $\left(\widetilde{\mathscr{F}}(a), \widetilde{\mathscr{K}}(a)\right)$ *is FG (subgraph) of* $\mathscr{G}^*, \forall\ a \in \mathscr{A}$ *'*

The membership value of the edge in an FSG is

$$\widetilde{\mathscr{K}}(a)(xy) \leq min\left\{\widetilde{\mathscr{F}}(a)(x), \widetilde{\mathscr{F}}(a)(y)\right\}$$

and a FSG $\left(\widetilde{\mathscr{F}}(a), \widetilde{\mathscr{K}}(a)\right)$ can be denoted as $\widetilde{\mathscr{H}}(a)$.

Definition 2.4 *If* $\widetilde{G}$ *is an FSG, then the vertex degree of* x *is*

$$d_{\widetilde{G}}(x) = \sum_{a_i \in A} (\sum_{x \neq y} \widetilde{\mathscr{K}}(a_i)(xy)).$$

Definition 2.5 *If* $\widetilde{G}$ *is an FSG, then edge degree of* xy *is given as*

$$d_{\widetilde{G}}(xy) = d_{\widetilde{G}}(x) + d_{\widetilde{G}}(y) - 2(\sum_{a_i \in A} \widetilde{\mathscr{K}_{a_i}}(xy)).$$

Definition 2.6 *If* $\widetilde{G}$ *is an FSG, then total degree of* xy *is*

$$td_{\widetilde{G}}(xy) = d_{\widetilde{G}}(xy) + \sum_{a_i \in A} \widetilde{\mathscr{K}_{a_i}}(xy).$$

It can be also expressed as

$$td_{\widetilde{G}}(xy) = d_{\widetilde{G}}(x) + d_{\widetilde{G}}(y) - (\sum_{a_i \in A} \widetilde{\mathscr{K}_{a_i}}(xy)).$$

Definition 2.7 *An FSG* $\widetilde{G}$ *is irregular if* $\widetilde{\mathscr{H}}(e)$ *is irregular for all* $a \in \mathscr{A}$.

Definition 2.8 *The support* $(two - degree)$ *of a vertex* x *in an FSG* $\widetilde{G}$ *is, addition of degrees of its adjacent vertices & given as* $s_{\widetilde{G}}(x)$, *whereas its total support is given as*

$$ts_{\widetilde{G}}(x) = s_{\widetilde{G}}(x) + \sum_{a_i \in A} \widetilde{\mathscr{F}}(a_i)(x).$$

§3. Support and Total Support of an Edge in FSG

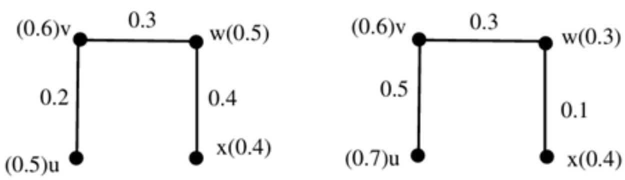
Definition 3.1 *The support* $(2-degree)$ *of a edge in a FSG is the sum of edge degrees of edges which are adjacent to given edge and can be defined as*

$$s_{\widetilde{G}}(uv) = \sum_{e_i \in N(uv), a_i \in A} \widetilde{\mathscr{K}}(a_i)(uv).$$

Definition 3.2 *The total support of an edge in an FSG is found as*

$$ts_{\widetilde{G}}(uv) = s_{\widetilde{G}}(uv) + \sum_{a_i \in A} \mathscr{K}(a_i)(uv).$$

Example 3.3 Consider the following graphs



$\widetilde{H}(a_1)$ $\widetilde{H}(a_2)$

Figure 3.1.

Here the degree of vertex are $d_{\widetilde{G}}(u) = 0.2+0.5 = 0.7$,$d_{\widetilde{G}}(v) = 1.3$,$d_{\widetilde{G}}(w) = 1.1$,$d_{\widetilde{G}}(x) = 0.5$

Table 1. FSS for Figure 3.1

$\widetilde{F}$	**u**	**v**	**w**	**x**	$\widetilde{K}$	**uv**	**vw**	**wx**
(a_1)	0.5	0.6	0.5	0.4	(a_1)	0.2	0.3	0.4
(a_2)	0.7	0.6	0.3	0.4	(a_2)	0.5	0.3	0.1

The degree of the edges are given as $d_{\widetilde{G}}(uv) = 0.6$, $d_{\widetilde{G}}(vw) = 1.2$,$d_{\widetilde{G}}(wx) = 0.6$ and the support of the edges are calculated as $s_{\widetilde{G}}(uv) = d_{\widetilde{G}}(vw) = 1.2$, $s_{\widetilde{G}}(vw) = 0.6 + 0.6 = 1.2$, $s_{\widetilde{G}}(wx) = d_{\widetilde{G}}(vw) = 1.2$.

The total support of the edges are calculated as $ts_{\widetilde{G}}(uv) = s_{\widetilde{G}}(uv) + \sum \widetilde{K}_{a_i}(uv) = 1.2 + 0.7 = 1.9$, $ts_{\widetilde{G}}(vw) = s_{\widetilde{G}}(vw) + \sum \widetilde{K}_{a_i}(vw) = 1.2+0.6 = 1.8$, $ts_{\widetilde{G}}(wx) = s_{\widetilde{G}}(wx) + \sum \widetilde{K}_{a_i}(wx) = 1.2 + 0.5 = 1.7$.

Definition 3.4 *A FSG is support edge irregular (s-EI) if* $\exists$ *atleast an edge in the graph, whose adjacent edges are with different edge supports.*

Definition 3.5 *A FSG is support totally edge irregular (s-TEI) when there exist at least one edge in the graph adjacent to edges with distinct total edge support.*

Example 3.6 Consider the following FSG

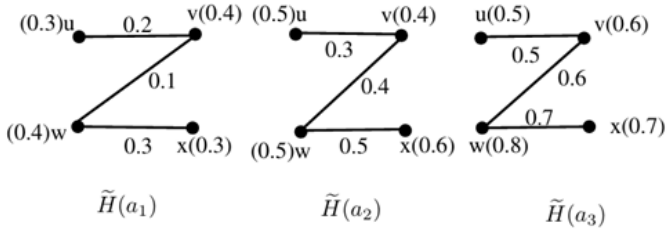


Figure 3.2.

Table 2. FSS for Figure 3.2

$\widetilde{F}$	**u**	**v**	**w**	**x**	$\widetilde{K}$	**uv**	**vw**	**wx**
(a_1)	0.3	0.4	0.4	0.3	(a_1)	0.2	0.1	0.3
(a_2)	0.5	0.4	0.5	0.6	(a_2)	0.3	0.4	0.5
(a_3)	0.5	0.6	0.8	0.7	(a_3)	0.5	0.6	0.7

The degree of the edges are found as $d_{\widetilde{G}}(uv) = 1.1$, $d_{\widetilde{G}}(vw) = 2.5$, $d_{\widetilde{G}}(wx) = 1.1$ and the support of the edges are given as $s_{\widetilde{G}}(uv) = 2.5$, $s_{\widetilde{G}}(vw) = 2.2$, $s_{\widetilde{G}}(wx) = 2.5$. The total support of the edges are given as $ts_{\widetilde{G}}(uv) = 3.5$, $ts_{\widetilde{G}}(vw) = 3.3$, $ts_{\widetilde{G}}(wx) = 1.0$. The 2 adjacent edges in the given example donot have same edge support, so it is s-EI FSG, also considering the total support of the edges, no 2 adjacent edges are alike, hence it is s-TEI FSG.

Remark 3.7 The support edge irregular FSG need not be support totally edge irregular FSGs and vice versa.

Example 3.8 Consider the FSG

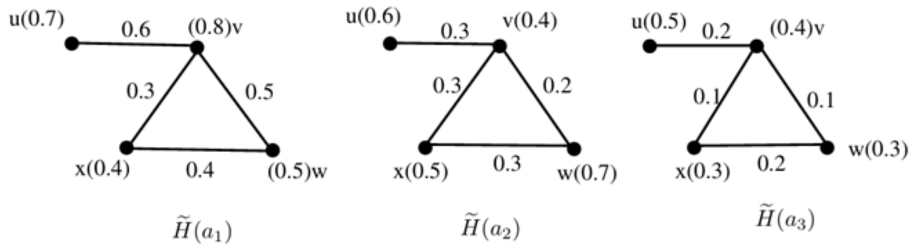


Figure 3.3.

Table 3. FSS for Figure 3.3

$\widetilde{F}$	u	v	w	x	$\widetilde{K}$	uv	vw	wx	vx
(a_1)	0.7	0.8	0.5	0.4	(a_1)	0.6	0.5	0.4	0.3
(a_2)	0.6	0.4	0.7	0.5	(a_2)	0.3	0.2	0.3	0.3
(a_3)	0.5	0.4	0.3	0.3	(a_3)	0.2	0.1	0.2	0.1

Here, the degree of edges are $d_{\widetilde{G}}(uv) = 1.5$, $d_{\widetilde{G}}(vw) = 2.7$, $d_{\widetilde{G}}(wx) = 1.5$, $d_{\widetilde{G}}(vx) = 2.8$ and the support of the edges are $s_{\widetilde{G}}(uv) = 5.5$, $s_{\widetilde{G}}(vw) = 5.8$, $s_{\widetilde{G}}(wx) = 5.5$, $s_{\widetilde{G}}(vx) = 5.7$.

The support of adjacent edges are not same, so it is support edge irregular FSG and the total support of the edges are $ts_{\widetilde{G}}(uv) = 6.6$, $ts_{\widetilde{G}}(vw) = 6.6$, $ts_{\widetilde{G}}(wx) = 6.4$, $ts_{\widetilde{G}}(vx) = 6.4$.

Here, the adjacent edges uv and vw have same total edge support, so it is not support totally edge irregular FSG.

Example 3.9 Consider the Example 3.3, in which the support of edges are same but the total support of edges are distinct. Thus it is support totally edge irregular FSG and not support edge irregular FSG.

Remark 3.10 If $\widetilde{G}$ is a FSG and suppose $\widetilde{\mathscr{K}}$ is constant $\forall a_i \in A$, then the given graph is not support edge irregular and support totally edge irregular FSG.

Remark 3.11 The support and support totally edge irregularity of the FSG is independent of the values of $\widetilde{\mathscr{F}}$ (a_i) $\forall$ $a_i \in A$.

§4. Support Edge and Support Totally Neighbourly Edge Irregular FSG

Definition 4.1 *A FSG $\widetilde{G}$ is support-neighbourly edge irregular (s-NEI), if no 2 adjacent edges have same edge supports.*

Example 4.2 Consider the FSG

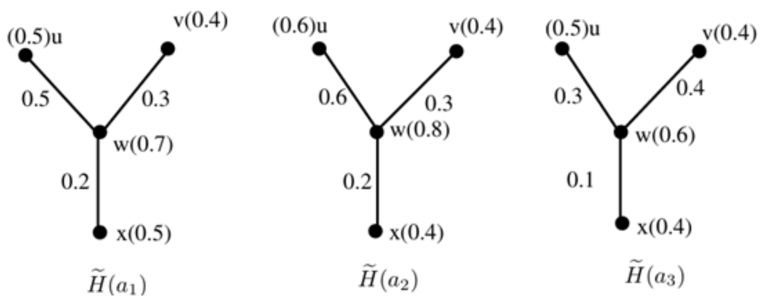


Figure 4.1.

Table 3 FSS for Figure 4.1

$\widetilde{F}$	u	v	w	x	$\widetilde{K}$	uw	vw	wx
(a_1)	0.5	0.4	0.7	0.5	(a_1)	0.5	0.3	0.2
(a_2)	0.6	0.4	0.8	0.4	(a_2)	0.6	0.3	0.2
(a_3)	0.5	0.4	0.6	0.4	(a_3)	0.3	0.4	0.1

Here, the degree of edges are $d_{\widetilde{G}}(uw) = 0.1$, $d_{\widetilde{G}}(vw) = 0.5$, $d_{\widetilde{G}}(wx) = 1.0$ and the support of the edges are $s_{\widetilde{G}}(uw) = 1.5$, $s_{\widetilde{G}}(vw) = 1.1$, $s_{\widetilde{G}}(wx) = 0.6$. Here, no pair of adjacency edges have same support, thus it is s-NEI FSG.

Definition 4.3 *A FSG $\widetilde{G}$ is support-totally NEI (s-TNEI), if no 2 adjacent edges have same total edge supports.*

Example 4.4 In the Example 4.2, the total support of edges are as $ts_{\widetilde{G}}(uw) = 2.9$, $s_{\widetilde{G}}(vw) = 2.1$, $s_{\widetilde{G}}(wx) = 1.1$ and here the total edge support of no two adjacent edges are same, thus it is s-TNEI FSG.

Remark 4.5 A s-NEI FSG need not be s-TNEI and vice versa.

Example 4.6 Consider the Example 3.8, which is s-NEI but not s-TNEI as the edges uv and vw, adjacent to each other have same total edge support. Thus it is only a s-NEI FSG.

Consider the Example 3.3, in which support of all edges are same and the total edge support of no 2 adjacent edges are same, thus it is only s-TNEI FSG.

Example 4.7 Consider the following FSG.

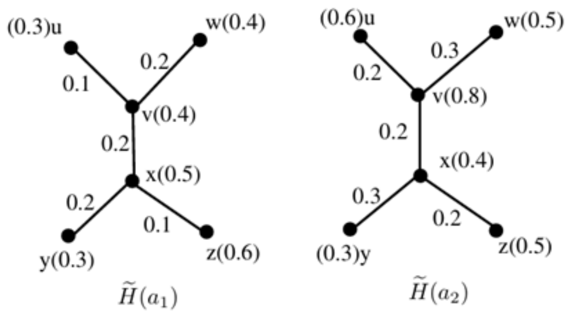


Figure 4.2.

Table 5. FSS for Figure 4.2

$\widetilde{F}$	u	v	w	x	y	z	$\widetilde{K}$	uv	vw	vx	xy	xz
(a_1)	0.3	0.4	0.4	0.5	0.3	0.6	(a_1)	0.1	0.2	0.2	0.2	0.1
(a_2)	0.6	0.8	0.5	0.4	0.3	0.5	(a_2)	0.2	0.3	0.2	0.3	0.2

The degree of edges are given as $d_{\widetilde{G}}(uv) = 0.9$, $d_{\widetilde{G}}(vw) = 0.7$, $d_{\widetilde{G}}(vx) = 1.6$, $d_{\widetilde{G}}(xy) = 0.7$, $d_{\widetilde{G}}(xz) = 0.9$ and the support of the edges are $s_{\widetilde{G}}(uv) = 2.3$, $s_{\widetilde{G}}(vw) = 2.5$, $s_{\widetilde{G}}(vx) = 3.2$,

$s_{\widetilde{G}}(xy) = 2.5$, $s_{\widetilde{G}}(xz) = 2.3$.

The total support of the edges are $ts_{\widetilde{G}}(uv) = 2.6$, $ts_{\widetilde{G}}(vw) = 3.0$, $ts_{\widetilde{G}}(vx) = 3.6$, $ts_{\widetilde{G}}(xy) = 3.0$, $ts_{\widetilde{G}}(xz) = 2.6$. Notice that the above is an example of FSG which is both s-NEI and s-TNEI FSG. Generally, we have

Theorem 4.8 *A s-NEI FSG is always support edge irregular FSG.*

Proof Let $\widetilde{G}$ be a FSG. Assume it is s-NEI, then no 2 adjacent edges have same edge support $\Rightarrow \exists$ atleast one edge, adjacent to edges with different supports. $\Rightarrow \widetilde{G}$ is support edge irregular FSG. $\square$

Theorem 4.9 *A s-TNEI FSG is always support totally edge irregular FSG.*

Proof The proof similar to that of Theorem 4.8. $\square$

Remark 4.10 A FSG in which the support of all edges are distinct, is a s-NEI and it is s-TNEI depending on values of $\sum_{a_i \in A} \widetilde{\mathscr{K}}(a_i)(e_i)$.

Theorem 4.11 *If $\widetilde{G}$ is a FSG, and support of all edges are same, then $\widetilde{G}$ is s-TNEI $\Leftrightarrow$ $\sum \widetilde{\mathscr{K}}(a_i)(e_i)$ are distinct, $\forall\ a_i \in A$ and $e_i \in E$.*

Proof Suppose $\widetilde{G}$ is a FSG, and support of all edges are same is given i.e., $s_G(e_i) = k$, $\forall e_i \in E$. Suppose $\widetilde{G}$ is s-TNEI, then no 2 nearby edges have same total edge support $\Rightarrow \exists$ at least pair of edges with different total edge supports, let e_i and e_j be such edges,

$$
\begin{aligned}
& ts_{\widetilde{G}}(e_m) \neq ts_{\widetilde{G}}(e_n) \\
& \Rightarrow s_{\widetilde{G}}(e_m) + \sum \widetilde{\mathscr{K}}(a_i)(e_m) \neq s_{\widetilde{G}}(e_n) + \sum \widetilde{\mathscr{K}}(a_i)(e_n) \\
& \Rightarrow k + \sum \widetilde{\mathscr{K}}(a_i)(e_m) \neq k + \sum \widetilde{\mathscr{K}}(a_i)(e_n) \\
& \Rightarrow \sum \widetilde{\mathscr{K}}(a_i)(e_m) \neq \sum \widetilde{\mathscr{K}}(a_i)(e_n).
\end{aligned}
$$

Therefore,

$$\sum \widetilde{\mathscr{K}}(a_i)(e_i)$$

are all distinct.

Conversely, suppose we take $\sum \widetilde{\mathscr{K}}(a_i)(e_i)$ are all distinct, and suppose $\widetilde{G}$ is not s-TNEI, then $\exists$ at least one edge with total edge support same as its adjacent edges, let total support of the edges e_m and e_n are same

$$
\begin{aligned}
& \Rightarrow ts_{\widetilde{G}}(e_m) = ts_{\widetilde{G}}(e_n) \\
& \Rightarrow s_{\widetilde{G}}(e_m) + \sum \widetilde{\mathscr{K}}(a_i)(e_m) = s_{\widetilde{G}}(e_n) + \sum \widetilde{\mathscr{K}}(a_i)(e_n) \\
& \Rightarrow k + \sum \widetilde{\mathscr{K}}(a_i)(e_m) = k + \sum \widetilde{\mathscr{K}}(a_i)(e_n) \\
& \Rightarrow \sum \widetilde{\mathscr{K}}(a_i)(e_m) = \sum \widetilde{\mathscr{K}}(a_i)(e_n),
\end{aligned}
$$

which enable us to know the assumption that $\sum \widetilde{\mathscr{K}}(a_i)(e_i)$ are distinct. Thus, $\widetilde{G}$ is s-TNEI

FSG. □

Theorem 4.12 *Let $\widetilde{G}$, is a FSG, and if $\widetilde{\mathscr{H}}(a) = (\widetilde{\mathscr{F}}(a), \widetilde{\mathscr{K}}(a))$ is s-NEI FSG as well as $\widetilde{G}$.*

Proof Suppose $\widetilde{G}$ is not s-NEI FSG, then $\exists$ a pair of edges having same edge support. Let e_m and e_n be such

$$\begin{aligned}
&\Rightarrow s_{\widetilde{G}}(e_m) = s_{\widetilde{G}}(e_n)\\
&\Rightarrow \sum s_{\widetilde{H}_{a_i}}(e_m) = \sum s_{\widetilde{H}_{a_i}}(e_n)\\
&\Rightarrow s_{\widetilde{H}_{a_i}}(e_m) = s_{\widetilde{H}_{a_i}}(e_n)
\end{aligned}$$

for any $a_i \in A$, which enables us to get the assumption. Thus $\widetilde{G}$ is s-NEI FSG. □

Theorem 4.13 *Let $\widetilde{G}$, is a FSG, and if $\widetilde{\mathscr{H}}(a) = (\widetilde{\mathscr{F}}(a), \widetilde{\mathscr{K}}(a))$ is s-TNEI FSG as well as $\widetilde{G}$.*

Proof The proof is same as above theorem. □

Remark 4.14 The converse of both theorems is not necessarily true.

Example 4.15 Consider the Example 3.8 in where the FSG $\widetilde{G}$ is s-NEI and the $H(a_3)$ is

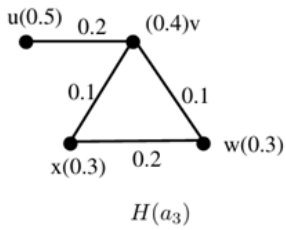


Figure 4.3.

Here, the degree of edges are $d_{\widetilde{G}}(uv) = 0.2$, $d_{\widetilde{G}}(vw) = 0.5$, $d_{\widetilde{G}}(wx) = 0.2$, $d_{\widetilde{G}}(xv) = 0.5$ and the support of the edges are $s_{\widetilde{G}}(uv) = 1.0$, $s_{\widetilde{G}}(vw) = 0.9$, $s_{\widetilde{G}}(wx) = 1.0$, $s_{\widetilde{G}}(xv) = 0.9$. Here, the edges vw and vx which are adjacent have same edge support, thus it is not s-NEI FSG.

Example 4.16 Consider the FSG

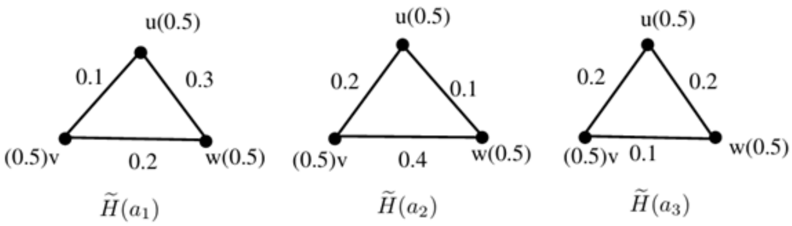


Figure 4.4.

The given FSG is s-TNEI, since the total support of edges are given as $ts_{\widetilde{G}}(uv) = 2.8$, $ts_{\widetilde{G}}(vw) = 3.2$, $ts_{\widetilde{G}}(uw) = 3.0$. But consider $\widetilde{H}(a_3)$ whose edge degree are $d_{\widetilde{G}}(uv) = 0.3$,

$d_{\widetilde{G}}(vw) = 0.4$, $d_{\widetilde{G}}(wu) = 0.3$ and support of edges are $s_{\widetilde{G}}(uv) = 0.7$, $s_{\widetilde{G}}(vw) = 0.6$, $s_{\widetilde{G}}(wu) = 0.7$ and total support of edges are $ts_{\widetilde{G}}(uv) = 0.9$, $ts_{\widetilde{G}}(vw) = 0.7$, $ts_{\widetilde{G}}(uw) = 0.9$, which cleary tells that $\widetilde{H}(a_3)$ is not s-NEI and s-TNEI FSG.

Remark 4.17 The s-NEI FSG is independent of $\widetilde{\mathscr{F}}(a_i)$, where $a_i \in A$

Example 4.18 Considering the Example 4.17, which is s-NEI and s-TNEI we find the value of $\widetilde{\mathscr{F}}(a_i)$, where $a_i \in A$ are same and the support edge irregularity of FSG does not depend on it.

Theorem 4.19 *A fuzzy soft cycle of length n, and $\widetilde{\mathscr{H}}(a_i)$ are distinct for all $a_i \in A$, then it is not s-NEI and s-TNEI FSG.*

Theorem 4.20 *If $\widetilde{G}$ is FS cycle, of length $2m+2$, where $m \geq 1$, with the value of $\widetilde{\mathscr{K}}(a_i)(e_i)$ for alternative edges are same, then, $\widetilde{G}$ is s-NEI and s-TNEI FSG.*

Proof Let $x_1, x_2, \cdots, x_n$ be the vertices of FS cycle and $e_1, e_2, \cdots, e_n$ be its edges. Let

$$\widetilde{\mathscr{K}}(a_i)(e_i) = \begin{cases} m_1, & i \text{ is odd,} \\ m_2, & i \text{ iseven.} \end{cases}$$

The degree of edges are then $d_{\widetilde{G}}(e_i) = m_1 + m_2, \forall e_i \in E$. Let $x_i's$ nearby vertices be x_j and x_k, then $d_{\widetilde{G}}(x_ix_j) = m_1 + m_2 + m_1 + m_2 - 2(m_1) = 2m_2$ and $d_{\widetilde{G}}(x_ix_k) = 2m_1$. Now support of these edges $s_{\widetilde{G}}(x_ix_j) = d_G(x_ix_k) + d_G(x_jx_l) = 2m_2 + 2m_2 = 4m_2$, where x_l is nearby x_j in the cycle. Similarly for the edge x_ix_k, the support is $s_{\widetilde{G}}(x_ix_j) = 4m_1$. This happens for all the edges in cycle with alternate edges having edge supports $4m_1$ and $4m_2$, thus making it s-NEI FSG.

The total support of the edge x_ix_k is $5m_2$, and for x_ix_j it is $5m_1$, and it hold for all edges in an even FS cycle, thus it is s-TNEI FSG. □

§5. Application of FSG in Analysing a Given Situation Using Edge Degree

The solutions of all practical problems can be solved by representing them in the form of graphs. While considering the behavioural study, certain factors influence the ideas and thoughts of others generally. This can be studied with the help of influence diagram in form of graphs, where the vertices and edges are represented based on the group taken for behavioural study. Here we take up certain factors from the society and find how social media influence them. We take the vertices sets as the factors existing the society and the membership values define its social media influence, while the edges connecting each factor show influence of these factors on each other due to social media. We use the FSG model to find out which factor has the most influence by the social media and which factor gets more impacted by others. We consider the parameters like disagreement and comply between the used factors. We consider the factors like politics, entertainment, humanrace, commerce, development, working environment and evaluate their influence on each other in a society through social media. A survey conducted gives the following results.

(1) The politics has an effect on the nature of the human race because of the social media as it provides a medium to communicate and link people and it also influences the development and work nature as these has become a matter of debate openly and sovcial media through various means provide entertainment.

(2) The entertainment industry has now become a part of business and social media helps in gaining a lot of popularity and it attracts a humongous audience and encourages them to be a part of it.

(3) The working environment depends on the population in a given region and social media provides abundant opportunities to look for jobs for the human race.

(4) The development activities take place based on the working environment of a area and depends on the nature of people living in the area. Here again the development opportunities are availed through social medias.

(5) The commerce industry has grown abundantly in recent years with help of social media and affects the population, and development of a region.

(6) The human race is important factor in the society for all the other factors considered here depend on the population in the given region.

Considering the directed graph with vertex set

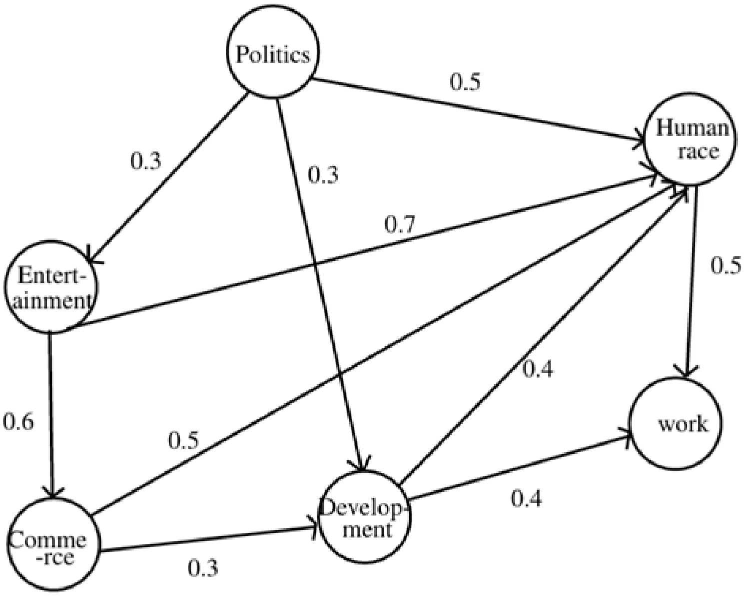
$$V = \{\text{polity}, \text{entertainment}, \text{commerce}, \text{work}, \text{development}, \text{humanrace}\}$$

and it represents the factors existing in the society. The directed edges represent influence of social media on other factors based on our parameters. We set the parameters as $A = \{a_1, a_2\}$, where a_1 = positive influence, a_2 = negative influence. A FS graph is represented below. The membership degree of edges represents the disagreement of a factor in the society. For example membership of edge between politics and human race in $\widetilde{H}(a_1)$ is 0.5, i.e., 50 % positive effect of media over the human population . If there is no link it means no influence between the factors due to media. The degree of membership can be interpresented as % of influence.

Table 6. Fuzzy soft set over the edges

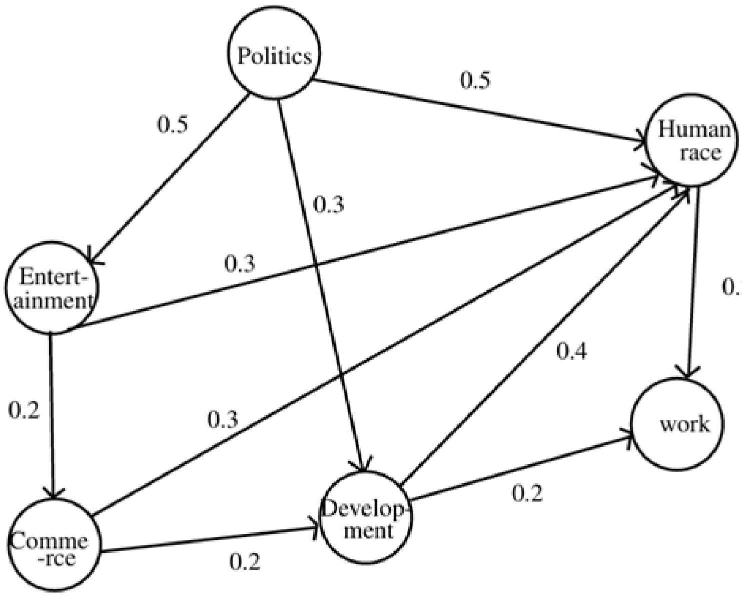
$\widetilde{K}$	$\vec{PD}$	$\vec{PE}$	$\vec{PH}$	$\vec{HW}$	$\vec{DW}$	$\vec{CD}$	$\vec{CH}$	$\vec{EC}$	$\vec{EH}$	$\vec{DH}$
(a_1)	0.3	0.3	0.5	0.5	0.4	0.3	0.5	0.6	0.1	0.4
(a_2)	0.3	0.5	0.5	0.3	0.2	0.2	0.3	0.2	0.4	0.4

The representation of data is given as below.



$H(a_1)$

Figure 5.1.



$H(a_2)$

Figure 5.2.

The degree of vertices are given as $d(P) = 2.4$, $d(E) = 2.1$, $d(H) = 3.9$, $d(W) = 1.4$, $d(D) = 2.5$, $d(C) = 2.1$ which shows that Human race has greater media influence over the other factors. The edge degree of vertices are given as $d(PD) = 3.4$, $d_(PE) = 2.9$, $d(PH) = 4.3$, $d(HW) = 3.7$, $d(DW) = 2.7$, $d(CD) = 3.6$, $d(CH) = 4.2$, $d(EC) = 2.6$, $d(EH) = 5.0$,

$d(DH) = 4.8$.

We infer from above that, entertainment affects the human race more through social media interaction. That is 50% of impact on entertainment on the population. Whereas the impact of entertainment over commerce through media is the least i.e., 26% from the above observation. Thus we can conclude that human race has greater impact of social media as they help to inter connect wide range of people through a common platform.

References

[1] Amei Yu, Mei Lu and Feng Tian, On the Spectral radius of graphs, *Linear Algebra and its Applications*, 387 (2004), 41-49.

[2] Dasong Cao, Bounds on eigen values and chromatic numbers, *Linear algebra and its applications*, 270(1998), 1-13.

[3] Gary Chartrand, Paul Erdos, Ortrud R. Oellermann, How to define an irregular graphs, *College. Math. Journal*, 19(1988).

[4] N.R. Santhi Maheswari and K. Amutha, Support neighbourly edge irregular graphs, *International Journal of Recent Technology and Engineering*, 8(3), September (2019).

[5] P. K. Maji, A. R. Roy and R. Biswas, An application of soft sets in a decision making problem, *Computers and Mathematics with Applications*, 44(89) (2002), 10771083.

[6] D. Molodtsov, Soft set theory-First results, *Computers and Mathematics with Applications*, 37(4-5) (1999) 19-31.

[7] Muhammad Akram and Saira Nawaz, On fuzzy soft graphs, *Italian Journal of Pure and Applied Mathematics-N.* 34(2015), 497-514.

[8] B.Akilandeswari, On edge regular fuzzy soft graph, *International Journal for Research in Applied Science & Engineering Technology*, 2019, 1479-88.

[9] J. Krishnaveni and N.R. Santhi Maheswari, On support neighbourly irregular fuzzy graphs, *International Journal of Advanced Research in Engineering and Technology*, 11(1), 2020, pp. 147-154.

[10] B.Somasundari and S.Mariammal, On support neighbourly irregular fuzzy soft graphs, *National Seminar Proceedings*, (2022), 144-161.

[11] A.Nagoor Gani and S.R.Latha, On irregular fuzzy graphs, *Applied Mathematical Sciences*, Vol.6(2012), pp.517-523.

[12] L.Subha Lakshmi and N.R.Santhi Maheswari, On pseudo highly and pseudo strongly edge irregular graphs, *Proceedings of International Conference- ICRRMCSA'22*, (2022), 61-69.

[13] N.R.SanthiMaheswari and K.Amutha, Stepwise edge irregular graphs, *Advances in Mathematics:Scientific Journal* (Special issue), No.3(2019), pp.766-771.

[14] N.R.Santhi Maheswari and C.Sekar, On strongly edge irregular fuzzy graphs, *Kragujevac journal of Mathematics*, Vol.40, No.1(2016), pp.125-135.

[15] N.R.Santhi Maheswari and C.Sekar, On edge irregular fuzzy graphs, *International Journal of Mathematics and Soft Computing*, Vol.6, No.2(2016), pp.131-143.

[16] L.Subha Lakshmi and N.R.Santhi Maheswari, A novel approach to solving Fuzzy assignment problem, *International Journal of Mathematical Archive* (accepted), 2022,

International J.Math. Combin. Vol.2(2026), 111-124

On Total H-irregularity Strength of Some Families of Graph With an Application of Wireless Sensor Networks

Asma Khalid[1], Faraha Ashraf[2], Muhammad Amir Asif[3],
Humaira Ibrahim[1] and Misbah Hina[1]

1.Department of Mathematics, Air University Islamabad, Multan Campus, Pakistan

2.Department of Computer Science, Green International University Lahore, Pakistan

3.Department of Mathematics, Minhaj University Lahore, Pakistan

E-mail: asmakhalid@aumc.edu.pk, dr.faraha@giu.edu.pk, amirasif460@gmail.com
humairaibrahim75@gmail.com, behakhanxhz@gmail.com

Abstract: This study investigates the total H-irregularity strength of various families of graphs, with a particular focus on their applications in wireless sensor networks (WSNs). For a graph G an H-irregular total k-labeling is a total k-labeling σ with a condition that for every two distinct subgraphs H_α and H_β isomorphic to H, there is, $wt_\sigma(H_\alpha) \neq wt_\sigma(H_\beta)$, where the weight of an element is the sum of all the labels associated to the graph elements. The smallest integer k for which G has a H-irregular total k label is called the total H-irregularity strength of a graph G, denoted by $ths(G,H)$. In this paper the exact value of $ths(G,H)$ is calculated for the book graph, mongolian tent graph, star graph, sunlet graph and chain silicate graph.

Key Words: Vertex (edge) H-irregularity strength, vertex (edge) H-irregular labeling, Smarandachely H-irregular total k-labeling.

AMS(2010): 05C78.

§1. Introduction

Consider a graph $G = (V,E)$ that is connected, undirected, and simple. This is a fact that in a simple graph at least two vertices have the same degree. A natural question arises in mind: is it possible to make the degree of each vertex in a graph different? One way to answer this question is to assign a positive integer to each vertex of a graph. The assignment of each edge on a simple graph G with positive integers from 1 to k is called an irregular assignment if the weight of each vertex is different. The weight of a vertex is the sum of all labels associated with it in that graph. Chartand, Jacobson, Lehel, Oellerman, Ruiz and Saba in [12] were the first to introduce the concept of irregularity strength. The irregularity strength denoted by $s(G)$, is the smallest positive integer k for which each edge in the graph can be replaced by a set of at most k parallel edges so that the degree of each vertex will be different in the new graph. If G contains at most one isolated vertex and no isolated edges then Irregularity strength of a graph

[1]Received September 22, 2025, Accepted June 8, 2026

G is finite. It is not as easy to find this graph invariant as it involves complex computations or algorithms, for instance, see [5, 7, 9, 11, 14]. For a regular graph G of at least order 3, Faudree, Jacobson, Lehel and Schel [13] proved that $s(G) \geqslant 3$. For finite irregularity strength of general graph, Aigner and Triesch [5] showed that if G is connected then $s(G) \leq |V(G)| - 1$, otherwise $s(G) \leq |V(G)| + 1$. Frieze et. al. in [8] provided some important bounds.

Many results on edge irregularity strength and vertex irregularity strength have been proved in several articles [6, 10, 14]. As a varitaion of previous irregularity strength, Ashraf et. al. in [2] introduced a new branch of irregularity strength called H-irregularity strength. The notion of an edge's weight of a graph is extended and generalized by using these new notions. The weight of subgraphs are isomorphic to the given graph H. All vertices and edges with sum of labels belongs to the given H-subgraph is called H-weight or weight of a subgraph H. Total H-irregular labeling is the labeling in which H-weights are distinct in total labeling. To reduce H-weights, The smallest vertex or edge or both labels are used such that the total H-irregular strength stay as minimal as possible. If every edge or vertex of the graph G belongs to at least one of the subgraphs $H_i, i = 1, 2, \cdots, t$, then the graph G is said to admit a $(H_1 H_2, \cdots, H_t)$-(edge or vertex) covering if the family of subgraphs $H_1, H_2, \cdots, H_t$ forms a (edge or vertex) covering of G. The graph G may have an H-covering if the given graph H is isomorphic to its subgraph H_i. A total k-labeling σ of graph G having two distinct subgraphs H_α and H_β isomorphic to H then it is called H-irregular total k-labeling where $wt_\sigma(H_\alpha) \neq wt_\sigma(H_\beta)$. On the other hand, if there are α, β with $wt_\sigma(H_\alpha) = wt_\sigma(H_\beta)$, such a total k-labeling σ is said a Smarandachely H-irregular total k-labeling. An H-irregular total k-labeling is called the total H-irregularity strength of a graph G if there exist a smallest integer k for which total k-labeling is possible, denoted by $ths(G, H)$. The lower bounds of total H-irregularity strength and vertex H-irregularity strength were proved by Ashraf et. al. in [1], which is given in Lemma 1.1.

Lemma 1.1 gives the lower bounds of total H-irregularity strength.

Lemma 1.1 *Let G be a graph admitting an H-covering given by t subgraphs isomorphic to H. Then*

$$ths(G, H) \geq \left\lceil 1 + \frac{t-1}{|V(H)| + |E(H)|} \right\rceil.$$

In 2023, Wahyujati and Susanti [3] discussed the ths of edge comb product of graphs. Faisal et. al. investigated the ths of trees in 2025 with some specific characteristics [4]. Extending the work of Ashraf et. al. [1] we have calculated the exact values of total H-irregularity strength of Mongolian tent graph, Book graph, Sunlet graph, Star graph and chain silicate graph in next section.

§2. Total H-irregularity Strength

In this section, we give our main results about total H-irregularity strength of graphs. Before going to our main results, first recall some necessary definitions.

Definition 2.1 *The Cartesian product of a graph is the graph $G \circledast H$ with vertices $V(G \circledast H) =$*

$V(G)\times V(H)$, and for which $(x,u)(y,v)$ is an edge if $uv \in E(H)$, or $xy \in E(G)$. The Mongolian tent graph is known as the graph obtained for odd n from the Cartesian product of $P_m \circledast P_n$ by inserting an extra vertex above the graph and joining each other vertex of the top row to the additional vertex. It is denoted by $M_{m,n}$ where $m \geq 2, n \geq 3$.

In our first result, we have calculated the exact value of the total H-irregularity strength of Mongolian tent graph.

Theorem 2.1 *Let $M_{m,3}$, for $m \geq 2$ be a Mongolian tent graph having $H = C_4$-covering with $2m-1$ isomorphic copies. Then*

$$ths(M_{m,3}, C_4) = \left\lceil \frac{m+3}{4} \right\rceil$$

Proof A Mongolian tent graph $M_{m,3}, m \geq 2$ contains $3m+1$ vertices and $5m-1$ edges. The vertex set of $M_{m,3}$ is denoted by $V(M_{m,3})$ and edge set is denoted by $E(M_{m,3})$ which is defined as

$$\begin{aligned} V(M_{m,3}) &= \{x, u_j, v_j, w_j : j = 1, 2, 3, \cdots, m\}, \\ E(M_{m,3}) &= \{xu_1, xw_1\} \bigcup \{u_jv_j, v_jw_j : j - 1, 2, 3, \cdots, m\} \\ & \quad \bigcup \{u_ju_{j+1}, w_jw_{j+1}, v_jv_{j+1} : j = 1, 2, 3, \cdots, m-1\}. \end{aligned}$$

Figure 1 shows the Mongolian tent graph $M_{5,3}$. Mongolian tent graph admits exactly $2m-1$ copies of C_4 where, $m \geq 2$. Now put $k = \lceil \frac{m+3}{4} \rceil$, where k is an upper bound for the total C_4 - irregularity strength of $M_{m,3}$. To show this, we define a total C_4-irregular labeling for $m \geq 2$, in this way:

$$\sigma_t : V(M_{m,3}) \cup E(M_{m,3}) \to \{1, 2, 3, \cdots, k\}$$

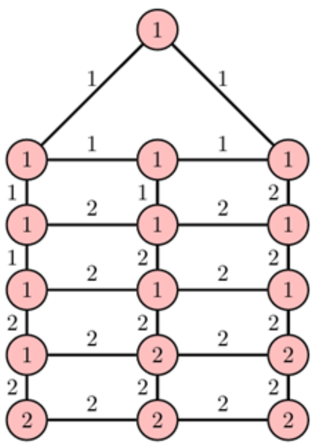


Figure 1. Total C_4-irregular labeling of Mangolian tent graph $M_{5,3}$

$$\begin{aligned}
\sigma_t(x) &= 1, \\
\sigma_t(u_j) &= \left\lceil \frac{j}{4} \right\rceil, && \text{for } j = 1,2,3,\cdots,m\ , \\
\sigma_t(v_j) = \sigma_t(w_j) &= \left\lceil \frac{j+1}{4} \right\rceil, && \text{for } j = 1,2,3,\cdots,m, \\
\sigma_t(xw_1) = \sigma_t(xu_1) &= 1, \\
\sigma_t(v_jv_{j+1}) = \sigma_t(u_jv_j) = \sigma_t(v_jw_j) &= \left\lceil \frac{j+3}{4} \right\rceil, && \text{for } j = 1,2,3,\cdots,m, \\
\sigma_t(u_ju_{j+1}) &= \left\lceil \frac{j+2}{4} \right\rceil, && \text{for } j = 1,2,3,\cdots,m, \\
\sigma_t(w_jw_{j+1}) &= \left\lceil \frac{j+4}{4} \right\rceil, && \text{for } j = 1,2,3,\cdots,m.
\end{aligned}$$

Now, under the total labeling σ_t it can be easily verified that all the vertices labeling and the edges labeling are not more than k, and under the total labeling σ_t, and for the cyclic graph $C_4 - weight$ of C_4^i, where i is from 1 to $2m-1$ we have

$$wt(C_4^i) = \sum_{v \in V(C_4)} \sigma_t(v_i) + \sum_{e \in E(C_4)} \sigma_t(e_i)$$

Lets consider into the difference between $wt(C_4^i)$ and $wt(C_4^{i+1})$ for $\{i = 1,2,3,\cdots,2m-1\}$ following.

$$\begin{aligned}
wt(C_4^{i+1}) &= \sigma_t(u_{i+1}) + \sigma_t(v_{i+1}) + \sigma_t(u_{i+2}) + \sigma_t(v_{i+2}) + \sigma_t(v_{i+1}v_{i+2}) + \sigma_t(u_{i+2}v_{i+2}) \\
&\quad + \sigma_t(u_{i+1}v_{i+1}) + \sigma_t(u_{i+1}u_{i+2}) \\
&= \left\lceil \frac{i+1}{4} \right\rceil + \left\lceil \frac{i+2}{4} \right\rceil + \left\lceil \frac{i+2}{4} \right\rceil + \left\lceil \frac{i+3}{4} \right\rceil + \left\lceil \frac{i+4}{4} \right\rceil \\
&\quad + \left\lceil \frac{i+5}{4} \right\rceil + \left\lceil \frac{i+4}{4} \right\rceil + \left\lceil \frac{i+3}{4} \right\rceil \\
&= \sigma_t(v_i) + \sigma_t(v_{i+1}) + \sigma_t(v_{i+1}) + \sigma_t(u_iu_{i+1}) + \sigma_t(u_{i+1}v_{i+1}) \\
&\quad + \sigma_t(u_i) + 1 + \sigma_t(v_iv_{i+1}) + \sigma_t(u_iv_i)
\end{aligned}$$

and so,

$$wt(C_4^{i+1}) = 1 + wt(C_4^i).$$

Thus, $ths(M_{m,3}, C_4) \leq k$ which concludes the proof. □

Let us recall the definition of star graph before going to find its total H-irregularity strength.

Definition 2.2 *A star graph is the complete bipartite graph $K_{1,n}$, with order $n+1$ and size n. A graph with one internal node of degree n and with n edges are connected with n verties each* *of degree* 1 *is called star graph. It is denoted by* S_n *where* $n \geq 3$. *Sometimes it is also called* n*-star.*

Figure 2 shows the star graph S_8. The next is the result on total H-irregularity strength.

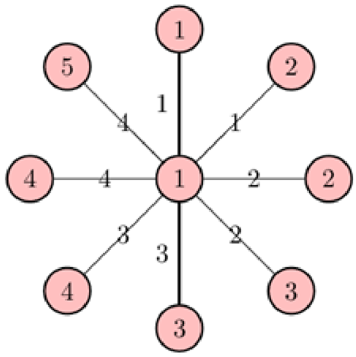


Figure 2. Total P_2-irregular 5−labeling of star graph S_8

Theorem 2.2 *Let* S_n, $n \geq 2$ *be a star on* $n+1$ *vertices having* $H = P_2$*-covering, then*

$$ths(S_n, P_2) = \left\lceil \frac{n+1}{2} \right\rceil.$$

Proof There are $n+1$ vertices in a star graph S_n for $n \geq 2$, say $V(G) = \{w, v_j,\ for\ j = 1, 2, 3, \cdots, n\}$ and n edges say $E(G) = \{wv_j,\ for\ j = 1, 2, 3, \cdots, n\}$. Star graph admits P_2-covering with exactly n copies of path graph. Now put $k = \lceil \frac{n+1}{2} \rceil$, where the total P_2-irregularity strength of S_n has an upper bound of k. To show this, For $n \geq 2$, we obtain a total P_2- irregular labeling of star graph S_n as follows:

$$\sigma_t : V(S_n) \cup E(S_n) \to \{1, 2, 3, \cdots, k\}$$

$$\begin{aligned} \sigma_t(w) &= 1, \\ \sigma_t(v_j) &= \left\lceil \frac{j+1}{2} \right\rceil, \quad \text{for } \ j = 1, 2, 3, \cdots, n, \\ \sigma_t(wv_j) &= \left\lceil \frac{j}{2} \right\rceil, \quad \text{for } \ j = 1, 2, 3, \cdots, n. \end{aligned}$$

Now, all the edge and vertex labels are at most $\lceil \frac{n+1}{2} \rceil$ under σ_t labeling, and for j-weights of P_2 path graph for $\{j = 1, 2, 3, \cdots, n\}$ we have

$$wt(P_2^j) = \sigma_t(v_j) + \sigma_t(wv_j) + \sigma_t(w). \tag{1}$$

For every $\{j = 1, 2, \cdots, n\}$, all the vertex and edge labels form increasing sequences under

σ_t- labeling.

$$\begin{aligned} wt(P_2^{j+1}) &= \sigma_t(v_{j+1}) + \sigma_t(wv_{j+1}) + \sigma_t(w) \\ &= \left\lceil \frac{j+2}{2} \right\rceil + \left\lceil \frac{j+1}{2} \right\rceil + 1 \\ &= \left\lceil \frac{j}{2} \right\rceil + 1 + \left\lceil \frac{j+1}{2} \right\rceil + 1 \\ &= \sigma_t(wv_j) + \sigma_t(w) + \sigma_t(v_j) + 1 = wt(P_2^j) + 1. \end{aligned}$$

So, all the required properties of irregular total labeling σ_t are proved. □

Definition 2.3 *For a graph $G = (V(G), E(G))$, an edge connecting a leaf vertex is called a pendant edge. The n-sunlet graph is a graph that is obtained by adding n pendant edges to the cyclic graph C_n where $n \geq 3$. The sunlet graph is also known as crown graph. Here we use notation Sl_n for sunlet graph with $(n \geq 3)$.*

Figure 3 shows the sunlet graph Sl_6. Here is the result on the total H-irregularity of the sunlet graph.

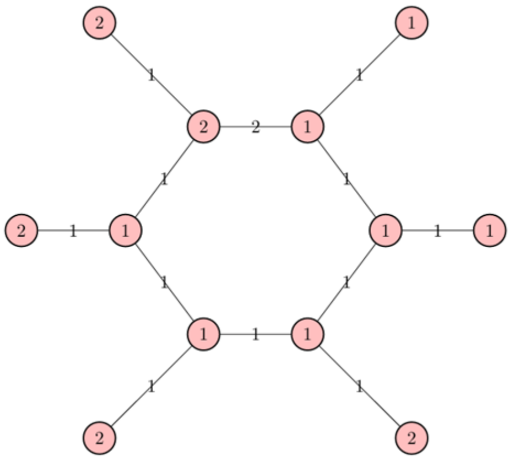


Figure 3. Total $D_{2,1}$-irregular $2-$Labeling of Sunlet Graph Sl_6

Theorem 2.3 *Let Sl_n, for $n \geq 3$, be a Sunlet graph having $H = D_{2,1}$-covering, then*

$$ths(Sl_n, D_{2,1}) = \left\lceil \frac{n+5}{7} \right\rceil$$

Proof By joining pendant edges to a cycle graph C_n, a sunlet Sl_n, for $n \geq 3$, is obtained which has $2n$ vertices and $2n$ edges. The Sunlet graph Sl_n admits the covering of double star graph $D_{2,1}$ with maximum label $k = \lceil \frac{n+5}{7} \rceil$. The vertex set $V(Sl_n)$ and edge set $E(Sl_n)$ of Sl_n-graph are defined as

$$\begin{aligned} V(Sl_n) &= \{x_i, y_i : \text{i is from 1 to n }\}, \\ E(Sl_n) &= \{x_1 x_n\} \bigcup \{x_i y_i : \text{i is from 1 to n }\} \cup \{x_i x_{i+1} : \text{i is from 1 to n-1 }\}. \end{aligned}$$

In order to show the lower bounds of $ths(Sl_n, D_{2,1})$, the existence of a $D_{2,1}$-irregular total k-labeling, which follows, is all that needs to be proved for

$$\sigma_t : V(Sl_n) \cup E(Sl_n) \to \{1, 2, \cdots, k\}$$

we have

$$\begin{aligned}
\sigma_t(x_i) &= \left\lceil \frac{i+4}{9} \right\rceil, \text{ if } 1 \le i \le n-1, \text{ and } n \ge 3,\\
\sigma_t(x_n) &= \left\lceil \frac{n+3}{8} \right\rceil, \text{ if } n \ge 3,\\
\sigma_t(y_i) &= \left\lceil \frac{i+7}{9} \right\rceil, \text{ if } 1 \le i \le n-1, \text{ and } n \ge 3,\\
\sigma_t(y_n) &= \left\lceil \frac{n+7}{8} \right\rceil, \text{ if } n \ge 3,\\
\sigma_t(x_i y_i) &= \left\lceil \frac{i}{9} \right\rceil, \text{ if } 1 \le i \le n-1, \text{ and } n \ge 3,\\
\sigma_t(x_n y_n) &= \left\lceil \frac{n}{7} \right\rceil, \text{ if } n \ge 3,\\
\sigma_t(x_i x_{i+1}) &= \left\lceil \frac{i+2}{9} \right\rceil, \text{ if } 1 \le i \le n-1, \text{ and } n \ge 3,\\
\sigma_t(x_1 x_n) &= \left\lceil \frac{n+3}{7} \right\rceil, \; if\ n \ge 3
\end{aligned}$$

and for every $\{j = 1, 2, \cdots, n-1\}$, under the σ_t-labeling, the j-weight of the double star graph is $D_{2,1}^{j}$ that contains edge and vertex labels and forms increasing sequences.

$$\begin{aligned}
wt(D_{2,1}^{j+1}) &= \sigma_t(x_{j+1}) + \sigma_t(y_{j+1}) + \sigma_t(x_{j+1}y_{j+1}) + \sigma_t(x_{j+1}x_{j+2})\\
&= \left\lceil \frac{j+5}{9} \right\rceil + \left\lceil \frac{j+8}{9} \right\rceil + \left\lceil \frac{j+1}{9} \right\rceil + \left\lceil \frac{j+3}{9} \right\rceil\\
&= \sigma_t(x_j) + \sigma_t(y_j) + \sigma_t(x_j y_j) + \sigma_t(x_j x_{j+1}) + 1 = wt(D_{2,1}^{j}) + 1
\end{aligned}$$

So, all the required properties of irregular total labeling σ_t are proved. □

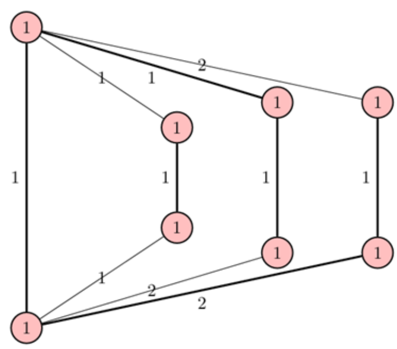


Figure 4. Total C_4-irregular 2−Labeling of Book Graph B_3

Definition 2.4 *The book graph B_n is the cartesian product $B_n = S_n \circledast P_2$, where S_n indicates the star graph with $n \geq 3$ and P_2 represents the path graph on two vertices. This graph is obtained by edge-amalgamation of path and star graph.*

Figure 4 shows the book graph B_3. Our last result on total H-irregularity strength is the next result.

Theorem 2.4 *Let B_n=$S_n \circledast P_2$, for $n \geq 3$, be a book graph revealing a $H = C_4$-covering, then*

$$ths(B_n, C_4) = \left\lceil \frac{n+4}{5} \right\rceil.$$

Proof Let the book graph $B_n = S_n \circledast P_2$ for $n \geq 3$ contains $2n+2$ vertices and $3n+1$ edges. The book graph B_n admits a C_4-covering with exactly n cycles. To show that $k = \lceil \frac{n+4}{5} \rceil$, we define labeling of graph with vertex set $V(B_n)$ and edge set $E(B_n)$ such that

$$V(B_n) = \{x, y, u_i, v_i : \text{i is from 1 to n }\},$$

$$E(B_n) = \{xy\} \cup \{xu_i : \text{i is from 1 to n }\} \cup \{yv_i : \text{i is from 1 to n }\} \cup \{u_iv_i : \text{i is from 1 to n }\}$$

In order to show the lower bounds of $ths(B_n, C_4)$, it is enough to prove the existence of a C_4-irregular total k-labeling which is given as

$$\sigma_t : V(B_n) \cup E(B_n) \to \{1, 2, 3, \cdots, k\},$$

$$\begin{aligned}
\sigma_t(u_i) &= \left\lceil \frac{i+1}{5} \right\rceil \text{ if i is from 1 to n },\\
\sigma_t(v_i) &= \left\lceil \frac{i}{5} \right\rceil \text{ if i is from 1 to n },\\
\sigma_t(x) &= \sigma_t(y) = \sigma_t(xy) = 1,\\
\sigma_t(u_iv_i) &= \left\lceil \frac{i+2}{5} \right\rceil \text{ if i is from 1 to n,}\\
\sigma_t(u_ix) &= \left\lceil \frac{i+3}{5} \right\rceil \text{ if i is from 1 to n,}\\
\sigma_t(v_iy) &= \left\lceil \frac{i+4}{5} \right\rceil \text{ if i is from 1 to n.}
\end{aligned}$$

It is simple to confirm that all of the vertex as well as edge labels are at most k under the total labelling σ_t. We obtain the following for j-weight of the cycle graph C_4^j for $\{j = 1, 2, \cdots, n\}$,

$$\begin{aligned}
wt(C_4^{j+1}) &= \sigma_t(x) + \sigma_t(y) + \sigma_t(xy) + \sigma_t(u_{j+1}) + \sigma_t(v_{j+1}) + \sigma_t(u_{j+1}x)\\
&\quad + \sigma_t(v_{j+1}y) + \sigma_t(u_{j+1}v_{j+1})
\end{aligned}$$

$$
\begin{aligned}
&= 1+1+1+\left\lceil\frac{j+2}{5}\right\rceil+\left\lceil\frac{j+1}{5}\right\rceil+\left\lceil\frac{j+4}{5}\right\rceil+\left\lceil\frac{j+5}{5}\right\rceil+\left\lceil\frac{j+3}{5}\right\rceil \\
&= 1+1+1+\left\lceil\frac{j+1}{5}\right\rceil+\left\lceil\frac{j+2}{5}\right\rceil+\left\lceil\frac{j+3}{5}\right\rceil+\left\lceil\frac{j}{5}\right\rceil+1+\left\lceil\frac{j+4}{5}\right\rceil \\
&= \sigma_t(x)+\sigma_t(y)+\sigma_t(xy)+\sigma_t(u_j)+\sigma_t(u_jv_j)+\sigma_t(u_jx)+\sigma_t(v_j)+\sigma_t(v_jy)+1 \\
&= wt(C_4^j)+1
\end{aligned}
$$

So, all the required properties of irregular total labeling σ_t are proved. □

Definition 2.5 *The structure of chain silicate network is designed by combining the tetrahedrons in a chain form which is attained by arranging m tetrahedron in linear way. An m dimentional chain silicate network is symbolized as CS_m $m \geq 3$, with $3m+1$ and $6m$ the number of vertices and edges respectively.*

In our next results, we have calculated the precise value of the total $H-$irregularity strength of chain silicate graph.

Theorem 2.5 *Let CS_m $m \geq 3$ be the chain silicate graph, where $3m$ is the number of copies of $H = C_3$. Then,*

$$ths(CS_m, C_3) = \left\lceil\frac{m}{2}\right\rceil + 1.$$

Proof Consider a graph CS_m of chain silicate for $m \geq 3$, its vertex and edge sets are

$$
\begin{aligned}
V(CS_m) &= \{v_p, u_p : \text{p is from 1 to m}\}\bigcup\left\{x_p : \text{p is from 1 to } \left\lceil\frac{m+1}{2}\right\rceil\right\} \\
&\quad \bigcup\left\{y_p : \text{p is from 1 to } \left\lceil\frac{m}{2}\right\rceil\right\} \\
E(CS_m) &= \{x_pv_{2p-1}, x_pu_{2p-1}, v_{2p-1}u_{2p-1}, u_{2p}x_{p+1}, u_{2p}v_{2p}, x_{p+1}v_{2p} : \text{p is from 1 to } \left\lceil\frac{m+1}{2}\right\rceil\} \\
&\quad \bigcup\{v_{2p-1}y_p, u_{2p-1}y_p, x_py_p, y_px_{p+1}, y_pu_{2p}, y_pv_{2p} : \text{p is from 1 to } \left\lceil\frac{m}{2}\right\rceil\}.
\end{aligned}
$$

Substitutions $d = \lceil\frac{m}{2}\rceil + 1$ with an C_3- irregularity total d-labeling as

$$\varphi : \mathrm{V}\left(\mathrm{CS_m}\right) \cup \mathrm{E}\left(\mathrm{CS_m}\right) \rightarrow \{1, 2, \cdots, d\}$$

The upper bound of total C_3 - irregularity strength of CS_m- graph is demonstrated by d, the following is how we draw the vertex labeling

$$
\begin{aligned}
&\varphi(v_p) = \left\lceil\frac{p}{2}\right\rceil && \text{where p is from 1 to m,} \\
&\varphi(u_p) = \left\lceil\frac{p}{2}\right\rceil && \text{where p is from 1 to m,} \\
&\varphi(x_p) = p && \text{where p is from 1 to } \left\lceil\frac{m+1}{2}\right\rceil, \\
&\varphi(y_p) = p+1 && \text{where p is from 1 to } \left\lceil\frac{m}{2}\right\rceil
\end{aligned}
$$

and the edge labeling are defined as

$$
\begin{array}{rl}
\varphi\left(x_p v_{2p-1}\right) = p & \text{where p is from 1 to } \left\lceil \dfrac{m+1}{2} \right\rceil, \\
\varphi\left(v_{2p-1} y_p\right) = p+1 & \text{where p is from 1 to } \left\lceil \dfrac{m}{2} \right\rceil, \\
\varphi\left(x_p u_{2p-1}\right) = p & \text{where p is from 1 to } \left\lceil \dfrac{m+1}{2} \right\rceil, \\
\varphi\left(u_{2p-1} y_p\right) = p & \text{where p is from 1 to } \left\lceil \dfrac{m}{2} \right\rceil, \\
\varphi\left(v_{2p-1} u_{2p-1}\right) = p & \text{where p is from 1 to } \left\lceil \dfrac{m+1}{2} \right\rceil, \\
\varphi\left(x_p y_p\right) = p & \text{where p is from 1 to } \left\lceil \dfrac{m}{2} \right\rceil, \\
\varphi\left(y_p x_{p+1}\right) = p+1 & \text{where p is from 1 to } \left\lceil \dfrac{m}{2} \right\rceil, \\
\varphi\left(y_p u_{2p}\right) = p+1 & \text{where p is from 1 to } \left\lceil \dfrac{m}{2} \right\rceil, \\
\varphi\left(u_{2p} x_{p+1}\right) = p+1 & \text{where p is from 1 to } \left\lceil \dfrac{m+1}{2} \right\rceil, \\
\varphi\left(u_{2p} v_{2p}\right) = p+1 & \text{where p is from 1 to } \left\lceil \dfrac{m+1}{2} \right\rceil, \\
\varphi\left(y_p v_{2p}\right) = p & \text{where p is from 1 to } \left\lceil \dfrac{m}{2} \right\rceil, \\
\varphi\left(x_{p+1} v_{2p}\right) = p+1 & \text{where p is from 1 to } \left\lceil \dfrac{m+1}{2} \right\rceil.
\end{array}
$$

Typically, we verify that all edge and node labels have a maximum of $\lceil \frac{m}{2} \rceil + 1$ under labeling. The cycle C_3^p load or weight is represented as

$$\omega t_{\varphi}\left(C_3^p\right) = \sum_{v \varepsilon V(CS_m)} \varphi(v) + \sum_{e \varepsilon E(CS_m)} \varphi(e)$$

The graph CS_m of chain silicate can be covered by C_3 for $m \geq 3$ as shown in Figure 5.

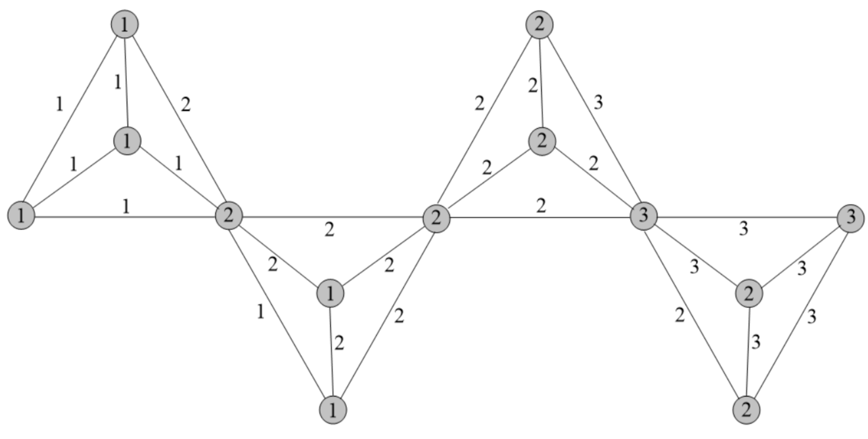


Figure 5. Total C_3-irregular $3-$labeling of chain silicate graph CS_4

It is clear that edge labels forms an increasing sequence showing that $\omega t_{\varphi}\left(C_3^p\right) < \omega t_{\varphi}\left(C_3^{p+1}\right)$, where $p = 1, 2, 3 \ldots, 3m$. Consequently with respect to $\omega t_{\varphi}\left(C_3^{p+1}\right) = 1 + \omega t_{\varphi}\left(C_3^p\right)$, we have the labeling φ that desired C_3-irregular total d-labeling. □

Theorem 2.6 *The ths of chain silicate graph CS_m with m copies of $H = K_4$ for $m \geq 4$ is given as*

$$ths(CS_m) = \left\lceil \frac{m}{11} \right\rceil + 1.$$

Proof Consider a graph CS_m of chain silicate for $m \geq 3$, its vertex and edge sets are

$$\begin{aligned} V(CS_m) &= \{v_p, u_p : \text{p is from 1 to m}\} \bigcup \{x_p : \text{p is from 1 to } \left\lceil \frac{m+1}{2} \right\rceil\} \\ &\bigcup \{y_p : \text{p is from 1 to } \left\lceil \frac{m}{2} \right\rceil\}, \\ E(CS_m) &= \left\{x_p v_{2p-1}, x_p u_{2p-1}, v_{2p-1} u_{2p-1}, u_{2p} x_{p+1}, u_{2p} v_{2p}, x_{p+1} v_{2p} : p = 1, 2, 3, \cdots, \left\lceil \frac{m+1}{2} \right\rceil\right\} \\ &\bigcup \left\{v_{2p-1} y_p, u_{2p-1} y_p, x_p y_p, y_p x_{p+1}, y_p u_{2p}, y_p v_{2p} : p = 1, 2, 3, \cdots, \left\lceil \frac{m}{2} \right\rceil\right\}. \end{aligned}$$

Substitutions $d = \lceil \frac{m}{11} \rceil + 1$ using a K_4- irregularity total d-labeling as

$$\varphi : \mathrm{V}\left(\mathrm{CS_m}\right) \cup \mathrm{E}\left(\mathrm{CS_m}\right) \to \{1, 2, \cdots, d\}.$$

The upper bound of total K_4 - irregularity strength of CS_m- graph is demonstrated by d, the following is how we draw the vertex labeling

$$\begin{aligned} \varphi\left(v_p\right) &= \left\lceil \frac{p+7}{10} \right\rceil && \text{for p from 1 to m,} \\ \varphi\left(u_p\right) &= \left\lceil \frac{p+6}{10} \right\rceil && \text{for p from 1 to m,} \\ \varphi\left(x_p\right) &= \left\lceil \frac{p+4}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m+1}{2} \right\rceil, \\ \varphi\left(y_p\right) &= \left\lceil \frac{p+4}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m}{2} \right\rceil \end{aligned}$$

and the edge labeling are defined as

$$\begin{aligned} \varphi\left(x_p v_{2p-1}\right) &= \left\lceil \frac{p+2}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m+1}{2} \right\rceil, \\ \varphi\left(v_{2p-1} y_p\right) &= \left\lceil \frac{p+1}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m}{2} \right\rceil, \\ \varphi\left(x_p u_{2p-1}\right) &= \left\lceil \frac{p}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m+1}{2} \right\rceil, \\ \varphi\left(u_{2p-1} y_p\right) &= \left\lceil \frac{p}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m}{2} \right\rceil, \\ \varphi\left(v_{2p-1} u_{2p-1}\right) &= \left\lceil \frac{p+1}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m+1}{2} \right\rceil, \end{aligned}$$

$$\begin{aligned}
\varphi\left(x_p y_p\right) &= \left\lceil \frac{p+2}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m}{2} \right\rceil, \\
\varphi\left(y_p x_{p+1}\right) &= \left\lceil \frac{p+3}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m}{2} \right\rceil, \\
\varphi\left(y_p u_{2p}\right) &= \left\lceil \frac{p+1}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m}{2} \right\rceil, \\
\varphi\left(u_{2p} x_{p+1}\right) &= \left\lceil \frac{p}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m+1}{2} \right\rceil, \\
\varphi\left(u_{2p} v_{2p}\right) &= \left\lceil \frac{p+1}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m+1}{2} \right\rceil, \\
\varphi\left(y_p v_{2p}\right) &= \left\lceil \frac{p+2}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m}{2} \right\rceil, \\
\varphi\left(x_{p+1} v_{2p}\right) &= \left\lceil \frac{p+2}{5} \right\rceil && \text{for p from 1 to } \left\lceil \frac{m+1}{2} \right\rceil.
\end{aligned}$$

Typically, we verify that all edge and node labels have a maximum of $\left(\left\lceil \frac{m}{11} \right\rceil + 1\right)$ under labeling. The cycle K_4^p load or weight is represented as

$$\omega t_\varphi\left(K_4^p\right) = \sum_{v \varepsilon V(CS_m)} \varphi(v) + \sum_{e \varepsilon E(CS_m)} \varphi(e)$$

The graph CS_m of chain silicate can be covered by K_4 for $m \geq 3$, as shown in Figure 6.

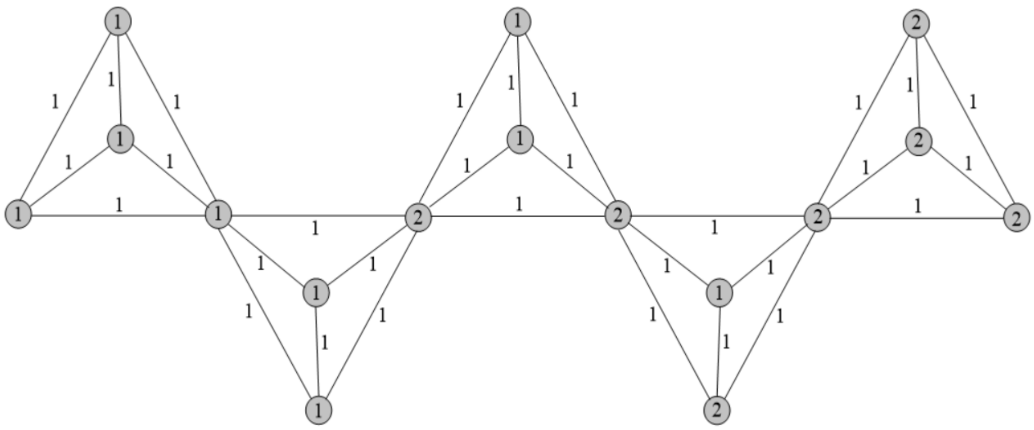


Figure 6. Total K_4-irregular $2-$labeling of chain silicate graph CS_4

It is clear that edge labels forms an increasing sequence showing that

$$\omega t_\varphi\left(K_4^p\right) < \omega t_\varphi\left(K_4^{p+1}\right),$$

where $p = 1, 2, 3 \cdots, m$. Consequently with respect to

$$\omega t_\varphi\left(K_4^{p+1}\right) = 1 + \omega t_\varphi\left(K_4^p\right),$$

we have the labeling φ that desired K_4-irregular total d-labeling. □

§3. Application of Wireless Sensor Networks

In wireless sensor networks, H-irregular total labeling helps to give each sensor (node) and its connections (edges) a unique label to prevent confusion. Each sensor gets a label, and each connection between sensors also gets a label. The total of the sensor's label and the labels of its connections must be different from other sensors. This method helps to make the network more organized, avoid problems, and ensure smooth communication between sensors. These networks can enhance the functionality of a Mongolian tent by monitoring environmental conditions, structural integrity, and energy usage. Sensors embedded in the fabric or frame can track temperature, humidity, air quality, and wind to optimize comfort, while stress and moisture sensors ensure the durability. Additionally, wireless sensor network (WSN) can monitor solar energy use, livestock activity, and provide early warnings. The model with a star-like structure, where all peripheral nodes communicate directly with a central hub, is ideal for applications like environmental monitoring, smart agriculture, home automation, and industrial health monitoring. For instance, in precision farming, sensors measuring soil moisture and temperature send data to a central gateway to optimize irrigation. Similarly, in disaster management, distributed sensors detecting seismic activity or water levels communicate with a central hub to provide early warnings. This topology is energy-efficient, has low latency, and simplifies communication, but it is best suited for small-scale deployments due to its reliance on the central node.

A sunlet graph, consisting of a base cycle with pendant vertices, can be effectively applied to wireless sensor networks (WSNs) to enhance energy efficiency, fault tolerance, and coverage. The base cycle can represent the core cluster of powerful nodes (e.g., cluster heads) that handle communication, while the pendant vertices represent peripheral sensors connecting to the nearest cluster head. This topology reduces energy consumption by limiting communication to short distances and provides redundancy through the cycle, ensuring fault-tolerant routing even if nodes fail. Wireless Sensor Networks (WSNs) can leverage book-like graphs for efficient topology design and communication. In such applications, the "spine" of the book-like graph represents the backbone network, ensuring reliable communication, while the "pages" represent clusters of sensor nodes monitoring specific regions. This model is particularly useful in scenarios like environmental monitoring, where multiple areas must communicate with a central system. Wireless sensor networks (WSNs) can be applied to chain silicate graphs to optimize monitoring and communication in systems modeled after silicate chain structures, such as crystalline materials or geological formations. For instance, in structural health monitoring, sensors deployed on nodes representing SiO_4 tetrahedra can track stress or damage propagation through the interconnected lattice.

§4. Conclusion

The paper successfully determined specific H-irregularity strengths for various graph families, highlighting how these graph properties relate to the design and optimization of wireless sensor networks. In this paper, we have calculated the graph parameter, the total H-irregularity strength $\text{ths}(G, H)$ as the natural extension of the irregularity strength of a graph. We have determined the exact values of the total H-irregularity strength for several families of graphs, namely Mongolian tent graph, Book graph, Sunlet graph, and chain silicate graph graph.

The labeling scheme ensures that no two subnetworks (modeled as subgraphs H) have the same communication burden. This prevents "hot spots" of high energy consumption, extending the overall network's lifespan by distributing the workload more evenly among sensor nodes. The current study likely makes certain assumptions about the WSN (wireless sensor network) environment, such as neglecting interference or the physical effects of signal decay. Future research could explore more complex real-world scenarios. This paper studied specific families of graphs, future work could investigate the total strength of H -irregularity of other graph structures that are relevant to different WSN topologies, such as mobile or hierarchical networks.

References

[1] F. Ashraf, M. Bača, M. Lascsáková, and A. Semaničová-Feňovčíková, On H-irregularity strength of graphs. *Discussiones Mathematicae Graph Theory*, 37, No. 4 (2017), 1067-1078.

[2] F. Ashraf ,M. Bača, Z.Kimáková, A. Semaničová-Feňovčíková, On vertex and edge H-irregularity strengths of graphs. *Discrete Mathematics, Algorithms and Applications.* 2016, Dec 6;8(04):1650070.

[3] Wahyujati, Mohamad Fahruli, and Yeni Susanti. Computing the total H-irregularity strength of edge comb product of graphs, *Analele Stiintifice ale Universitatii Ovidius Constanta*, Seria Matematica 13.2 (2023), 177-189.

[4] Susanto, Faisal, Rinovia Simanjuntak, and Edy Tri Baskoro, Further results on the total vertex irregularity strength of trees, *Electronic Journal of Graph Theory and Applications* (EJGTA), 13.1 (2025), 123-140.

[5] Aigner, Martin, and E. Triesch. Irregular assignments of trees and forests. *SIAM Journal on Discrete Mathematics*, 3.4 (1990): 439-449.

[6] A. Ahmad, M. Bača, and M. K. Siddiqui. On edge irregular total labeling of categorical product of two cycles, *Theory of Computing Systems*, 54, No. 1 (2014), 1-12.

[7] Amar, Denise, and O. Togni, Irregularity strength of trees, *Discrete Mathematics*, 190.1-3 (1998), 15-38.

[8] F.Alan, R. J. Gould, M. Karoski and F. Pfender, On graph irregularity strength, *Journal of Graph Theory*, 41, No. 2 (2002), 120-137.

[9] Anholcer, Marcin and C. Palmer, Irregular labelings of circulant graphs, *Discrete Mathematics*, 312.23 (2012), 3461-3466.

[10] M. Bača, M. Miller and R. Joseph, On irregular total labellings, *Discrete Mathematics*, 307.11-12 (2007), 1378-1388.

[11] Bohman Tom and D. Kravitz, On the irregularity strength of trees, *Journal of Graph Theory*, 45.4 (2004), 241-254.

[12] Chartrand Gary, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, and F. Saba, Irregular networks, *Congr. Numer.*, 64, No. 197-210 (1988), 250th.

[13] Faudree Ralph J., et al., Irregular networks, regular graphs and integer matrices with distinct row and column sums, *Discrete mathematics*, 76.3 (1989), 223-240.

[14] T. Nierhoff, A tight bound on the irregularity strength of graphs, *SIAM Journal on Discrete Mathematics*, 13.3 (2000), 313-323.

International J.Math. Combin. Vol.2(2026), 125-130

On the SuperHyperStructure and Uncertain SuperHyperStructure

Florentin Smarandache

(Mathematics, Physics and Natural Science Division, University of New Mexico, Gallup, NM 87301, USA)

E-mail: fsmarandache@gmail.com

Abstract: In this paper, we introduce the SuperHyperStructure and Neutrosophic SuperHyperStructure on n-th powerset of a set H and discuss how to generalize the classical structures, also the HyperStructures to the more general case, i.e., the SuperHyperStructures and Neutrosophic SuperHyperStructure.

Key Words: HyperStructure, SuperHyperStructure, Neutrosophic SuperHyperStructure, PowerSet.

AMS(2010): 08A05, 08C10.

§1. Introduction

Notice that the *n-th powersets* that are the bases on the *SuperHyperStructure* and *Uncertain SuperHyperStructure* [2, 3, 15], together with their particular cases such as the *SuperHyperAlgebra* and *Uncertain SuperHyperAlgebra* (endowed with *SuperHyperOperations* and *SuperHyperAxioms*) [2016, 2022], *SuperHyperGraph* (including *SuperHyperTree*) and *Uncertain SuperHyperGraph* (including *Uncertain SuperHyperTree*) [2019 - 2022], *SuperHyperSoft Set*, *SuperHyperFunction* and *Uncertain SuperHyperFunction* [2022], *SuperHyperTopology* (all built on the powersets of $P(H)$, or $P^n(H)$, for $n \geq 1$) and *Uncertain SuperHyperTopology* [2022] were all founded by Smarandache [2] and developed between 2016 - 2024.

By *Uncertain* we mean all types of Fuzzy and fuzzy-extensions *Intuitionistic Fuzzy*, *Inconsistent Intuitionistic Fuzzy* (*Picture Fuzzy, Ternary Fuzzy*), *Pythagorean Fuzzy (Atanassov's Intuitionistic Fuzzy* of second type), *q-Rung Orthopair Fuzzy*, *Spherical Fuzzy, n-HyperSpherical Fuzzy,Neutrosophic,Refined Neutrosophic,Plithogenic*, etc.) Sets/Logics/Measures/Probability/ Statistics.

Definition 1.1 *A SuperHyperStructure is a structure built on the n-th PowerSet of a Set H, for $n \geq 1$, as in our real world (because a set (or system) H (that may be a set of items, an organization, country, etc.) is composed by sub-sets that are parts of $P(H)$, which in their turn are organized in sub-sub-sets that are parts of $P(P(H)) = P^2(H)$, then in sub-sub-sub-sets that are parts of $P^3(H)$, and so on, $P^{n+1}(H) = P(P^n(H))$.*

The powerset $P(H)$ means all non-empty and empty subsets of H, including the empty set

[1]Received January 6, 2026, Accepted June 12, 2026

$\emptyset$, which represents the indeterminacy that occurs into the set H (as in our real world where we deal with unclear/indeterminate/unknwon information in any space/set); and similarly for $P^n(H)$.

While $P_*(H)$ means all non-empty subsets of H, or $P_*(H) = P(H) - \emptyset$. And similarly for $P_*^n(H)$.

A structure built on $P_*^n(H)$ is called a SuperHyperStructure (has no indeterminacy), while built on $P^n(H)$ it is called Neutrosophic SuperHyperStructure (does have indeterminacy).

In a SuperHyperStructure we deal with SuperHyperAxioms, SuperHyperLaws, SuperHyperTheorems, SuperHyperProperties, SuperHyperOperators, etc. Similarly for a Neutrosophic SuperHyperStructure.

§2. **"Hyper" and "Super" Prefixes**

The prefix "*Hyper*" (Marty [1], 1934) stand for the codomain of the functions and operations to be $P(H)$, or the powerset of the set H.

While the prefix "*Super*" (Smarandache [2], 2016) stands for using the $P^n(H), n \geq 2$, or the n-th PowerSet of the Set H because the set (or system) H (that may be a set of items, a company, institution, country, region, etc.) is organized in sub-systems that are part of $P(H)$, which in their turn are organized in sub-sub-systems, that are part of $P(P(H)) = P^2(H)$ and so on in the domain and/or codomain of the functions and operations and axioms.

§3. **SuperHyperStructure**

A *SuperHyperStructure* [2, 2016] is a structure built on the n-th powerset $P_*^n(H)$ of a non-empty set H, for integer $n \geq 1$, whose *SuperHyperOperators* are defined as follows:

$$\sharp_{SHS} : (P_*^r(H))^m \to P_*^n(H),$$

where $P_*^r(H)$ is the r-powerset of H, for integer $r \geq 1$, while similarly $P_*^n(H)$ is the n-th powerset of H, and the SuperHyperAxioms act on it. Indeterminacy is not allowed on this structure.

§4. **Neutrosophic SuperHyperStructure**

A *Neutrosophic SuperHyperStructure* is a structure built on the n-th powerset $P^n(H)$ of a non-empty set H, for integer $n \geq 1$, whose *Neutrosophic SuperHyperOperators* are defined as follows:

$$\sharp_{SHS} : (P^r(H))^m \to P^n(H),$$

where $P^r(H)$ is the r-powerset of H, for integer $r \geq 1$, while similarly $P^n(H)$ is the n-th powerset of H, and the SuperHyperAxioms act on it.

Indeterminacy is allowed on this structure.

§5. History: From the classic and HyperStructures to SuperHyperStructures

(*i*) Classical Structure

A classical *Structure* is built on a non-empty set H, whose classical operations $\sharp_S$ are defined as:

$$\sharp_S : H^m \to H, \quad \text{for integer } m \geq 1,$$

and with classical Axioms acting on it.

(*ii*) Classical HyperStructure

A classical *HyperStructure* (Marty [1], 1934) is built on a non-empty set H, whose *HyperOperations* are defined as:

$$\sharp_{HS} : H^m \to P * (H),$$

where $P * (H)$ is the powerset of H, without the empty setand the HyperAxioms acting on it.

(*iii*) Neutrosophic HyperStructure

The *Neutrosophic SuperHyperStructure* is an extension of the Classical HyperStructure, because it allows indeterminacy (uncertainty, unknown), denoted by the empty set $\emptyset$, into the the powerset of H, that it is denoted by $P(H)$, without $*$.

(*iv*) SuperHyperStructure

A *SuperHyperStructure* is built on n-th powerset $P^n(H)$ of a non-empty set H, for integer $n \geq 1$, whose *SuperHyperOperators* are defined as follows:

$$\sharp_{SHS} : (P^r(H))^m \to P^n(H),$$

where $P^r(H)$ is the r-powerset of H, for integer $r \geq 1$, while similarly $P^n(H)$ is the n-th powerset of H, and the SuperHyperAxioms act on it.

Indeterminacy is not allowed in this structure.

(*v*) Neutrosophic SuperHyperStructure

A *Neutrosophic SuperHyperStructure* is built on n-th powerset $P^n(H)$ of a non-empty set H, for integer $n \geq 1$, whose *SuperHyperOperators* are defined as follows:

$$\sharp_{SHS} : (P^r(H))^m \to P^n(H),$$

where $P^r(H)$ is the r-powerset of H, for integer $r \geq 1$, while similarly $P^n(H)$ is the n-th powerset of H, and the SuperHyperAxioms act on it.

Indeterminacy (denoted by the empty set $\emptyset$) is allowed in this structure.

(vi) The n-th powerset of H with indeterminacy (denoted by the empty set $\emptyset$)

The n-th powerset $P^n(H)$ of the set H, that the Neutrosophic SuperHyperStructure is built on, best describe our real world where always the indeterminacy occurs, and a set (or system) H (that may be a set of items, company, institution, country, etc.) is organized in sub-systems, which in their turn are organized each of them in sub-sub-systems, and so on.

The n-th PowerSet $P^n(H)$ is defined recursively:

$$\begin{array}{rcl} P^0(H) & \stackrel{\text{def}}{=} & H, \\ P^1(H) & = & P(H), \\ P^2(H) & = & P(P(H)), \\ P^3(H) & = & P(P^2(H)) = P(P(P(H))), \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot\ , \\ P^n(H) & = & P(P^{n-1}(H)) = \underbrace{P(P(\cdots P(H)\cdots))}_{n}, \end{array}$$

where P is repeated n times into the last formula, and the empty set $\emptyset$ (that represents indeterminacy, uncertainty) is allowed in all sequence terms: $H, P^1(H), P^2(H), P^3(H), \cdots, P^n(H)$.

(vii) The n-th powerset of H without Indeterminacy

The n-th powerset $P_*^n(H)$ of the set H, that the SuperHyperStructure is built on, describe a world that does not contain indeterminacy , where similarly the set (or system) H (that may be a set of items, a company, institution, country, region, etc.) is organized in sub-systems, which in their turn are organized each of them in sub-sub-systems, and so on.

The n-th powerSet $P_*^n(H)$ is also defined recursively:

$$\begin{array}{rcl} P_*^0(H) & \stackrel{\text{def}}{=} & H, \\ P_*^1(H) & = & P_*(H), \\ P_*^2(H) & = & P_*(P_*(H)), \\ P_*^3(H) & = & P_*(P_*^2(H)) = P_*(P_*(P_*(H))), \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot\ , \\ P_*^n(H) & = & P_*(P_*^{n-1}(H)) = \underbrace{P_*(P_*(\cdots P_*(H)\cdots))}_{n}, \end{array}$$

where P_* is repeated n times into the last formula, and the empty set $\emptyset$ (that represents indeterminacy, uncertainty) is not allowed in none of the sequence terms:

$$H, P_*^1(H), P_*^2(H), P_*^3(H), \cdots, P_*^n(H).$$

The n-th powerSet $P^n(H)$ and $P_*^n(H)$ of a non-empty set H, were introduced by Smarandache [2] in 2016.

§6. Particular Cases of SuperHyperStructure, Neutrosophic SuperHyperStructure

Depending on the type of Structure, in any field of knowledge, on has multiple types of *SuperHyperStructures* and respectively their corresponding *Neutrosophic SuperHyperStructures*, such as:

– *SuperHyperAlgebra* (when the structure is an Algebra), *SuperHyperGraph* (when the structure is a Graph), *SuperHyperGeometry* (when the structure is a type of Geometry), *SuperHyperCalculus* etc.

Further on, a *SuperHyperAlgebra* may be as particular cases: *SuperHyper Semigroup*, *SuperHyper Group*, *SuperHyper Field*, etc.

A *SuperHyperGeometry* may be as particular cases: *SuperHyper EuclideanGeometry*, *Superhyper NonEuclideanGeometry*, *SuperHyper ProjectiveGeometry*, etc.

See *https://fs.unm.edu/SHS/* for more details on the SuperHyperStructure and Neutrosophic SuperHyperStructure.

References

[1] F. Marty, Sur une généralisation de la Notion de Groupe, *8th Congress Math. Scandinaves*, Stockholm, Sweden, (1934), 45C49.

[2] F. Smarandache, SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra, Section into the authors book *Nidus Idearum. Scilogs*, II: de rerum consectatione, Second Edition, (2016), pp. 107C 108, https://fs.unm.edu/NidusIdearum2-ed2.pdf

[3] Florentin Smarandache, HyperUncertain, SuperUncertain, and SuperHyperUncertain Sets/Logics/Probabilities/Statistics, *Critical Review*, Vol. XIV, 2017, 10-19.

[4] Abdullah Kargın, Florentin Smarandache, Memet Şahin: New Type Hyper Groups, New Type SuperHyper Groups and Neutro-Type SuperHyper Groups. Chapter One in Florentin Smarandache, Memet Şahin, Derya Bakbak, Vakkas Uluçay & Abdullah Kargın Ed. - *Neutrosophic SuperHyperAlgebra And New Types of Topologies*, Global Knowledge, pp. 10-24, 2023.

[5] Florentin Smarandache, Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra, *Neutrosophic Sets and Systems*, Vol. 33, 2020, pp. 290-296. DOI: 10.5281/zenodo.3783103

[6] S Santhakumar, I R Sumathi and J Mahalakshmi J, A Novel Approach to the Algebraic Structure of Neutrosophic SuperHyper Algebra, *Neutrosophic Sets and Systems*, Vol. 60, 2023, pp. 593-602. DOI: 10.5281/zenodo.10236475

[7] Sirus Jahanpanah and Roohallah Daneshpayeh, On Derived SuperHyper BE-Algebras, *Neutrosophic Sets and Systems*, Vol. 57, 2023, pp. 318-327. DOI: 10.5281/zenodo.8271390

[8] Marzieh Rahmati and Mohammad Hamidi, Extension of G-Algebras to SuperHyper G-Algebras, *Neutrosophic Sets and Systems*, Vol. 55, 2023, pp. 557-567. DOI: 10.5281/zenodo.7879543

[9] Mohammad Hamidi, On Superhyper BCK-Algebras, *Neutrosophic Sets and Systems*, Vol.

53, 2023, pp. 580-588. DOI: 10.5281/zenodo.7536091

[10] Huda E. Khali, Gonca D. GÜNGÖR, Muslim A. Noah Zainal, Neutrosophic SuperHyper Bi-Topological Spaces: Original Notions and New Insights, *Neutrosophic Sets and Systems*, Vol. 51, 2022, pp. 33-45. DOI: 10.5281/zenodo.7135241

[11] Pairote Yiarayong, On 2-SuperHyperLeftAlmostSemihyp regroups, *Neutrosophic Sets and Systems*, Vol. 51, 2022, pp. 516-524. DOI: 10.5281/zenodo.713536

[12] Florentin Smarandache, The SuperHyperFunction and the Neutrosophic SuperHyperFunction, *Neutrosophic Sets and Systems*, Vol. 49, 2022, pp. 594-600. DOI: 10.5281/zenodo.6466524

[13] F. Smarandache, Introduction to the n-SuperHyperGraph - the most general form of graph today, *Neutrosophic Sets and Systems*, Vol. 48, 2022, pp. 483-485, DOI: 10.5281/zenodo.6096894

[14] Florentin Smarandache, Foundation of Revolutionary Topologies: An Overview, Examples, Trend Analysis,Research Issues, Challenges, and Future Directions [including SuperHyperTopology], *Neutrosophic Systems with Applications*, pp. 45-66, Vol. 13, 2024.

[15] F.Smarandache, SuperHyperStructure and Neutrosophic SuperHyperStructure, *Neutrosophic Sets and Systems*, Vol. 63, pp. 367-381, 2024.

[16] M.Hamidi,M.Taghinezhad (2024): Application of SuperHyperGraphs-Based Domination Number in Real World, *Journal of Mahani Mathematical Research*, 13(1), 211-228; DOI:10.22103/jmmr.2023.21203.1415

[17] Huda E. Khalid, Gonca Durmaz Gungor, Muslim A. Noah (2024): Neutrosophic SuperHyper Bi-Topological Spaces: Extra Topics, *Neutrosophic Optimization and Intelligent Systems*, 2, 43-55; https://doi.org/10.61356/j.nois.2024.2217

[18] Mohammad Hamidi, Florentin Smarandache, Mohadeseh Taghinezhad: Decision Making Based on Valued Fuzzy Superhypergraphs, *Computer Modeling in Engineering & Sciences*, Vol. 138, No. 2, 1907-1923, 2023.

Famous Words

In science, where there is no access to any kind of mathematics, where there is no connection with mathematics, are unreliable.

By *Da Vinci*, a famous Italian scholar.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats latex, eps, pdf may be submitted electronically to one member of the Editorial Board for consideration in the **International Journal of Mathematical Combinatorics** (*ISSN 1937-1055 (Print), 1937-1047 (Online)*). An effort is made to publish a paper duly recommended by a referee within a period of $2-4$ months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Theory on the Universe*, Global Knowledge-Publishing House, USA, 2023.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as JPG, EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

June, 2026

Contents

An International Journal on Mathematical Combinatorics

ISSN 1937-1055