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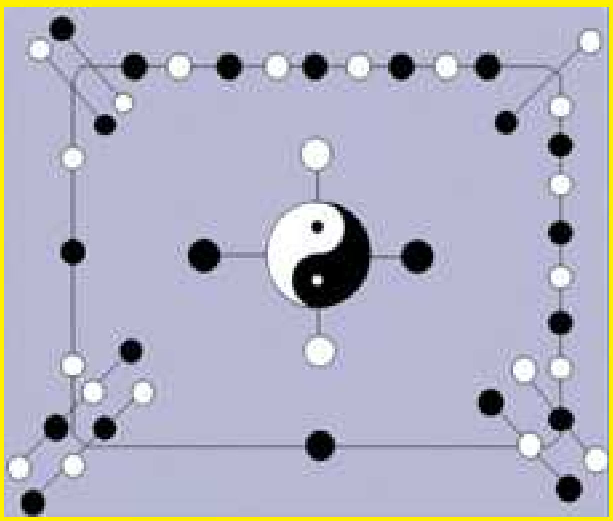

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Famous Words:

The world can be changed by man's endeavor, and that this endeavor can lead to something new and better. No man can sever the bonds that unite him to his society simply by averting his eyes. He must ever be receptive and sensitive to the new; and have sufficient courage and skill to face novel facts and to deal with them.

Franklin Roosevelt, an American president.

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Neutrosophic Groups and Subgroups

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Abstract: This paper is devoted to the study of neutrosophic groups and neutrosophic subgroups. Some properties of neutrosophic groups and neutrosophic subgroups are presented. It is shown that the product of a neutrosophic subgroup and a pseudo neutrosophic subgroup of a commutative neutrosophic group is a neutrosophic subgroup and their union is also a neutrosophic subgroup even if neither is contained in the other. It is also shown that all neutrosophic groups generated by the neutrosophic element I and any group isomorphic to Klein 4-group are Lagrange neutrosophic groups. The partitioning of neutrosophic groups is also presented.

Key Words: Neutrosophy, neutrosophic, neutrosophic logic, fuzzy logic, neutrosophic group, neutrosophic subgroup, pseudo neutrosophic subgroup, Lagrange neutrosophic group, Lagrange neutrosophic subgroup, pseudo Lagrange neutrosophic subgroup, weak Lagrange neutrosophic group, free Lagrange neutrosophic group, weak pseudo Lagrange neutrosophic group, free pseudo Lagrange neutrosophic group, smooth left coset, rough left coset, smooth index.

AMS(2010): 03B60, 20A05, 97H40

§1. Introduction

In 1980, Florentin Smarandache introduced the notion of neutrosophy as a new branch of philosophy. Neutrosophy is the base of neutrosophic logic which is an extension of the fuzzy logic in which indeterminancy is included. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminancy in a subset I, and the percentage of falsity in a subset F. Since the world is full of indeterminancy, several real world problems involving indeterminancy arising from law, medicine, sociology, psychology, politics, engineering, industry, economics, management and decision making, finance, stocks and share, meteorology, artificial intelligence, IT, communication etc can be solved by neutrosophic logic.

Using Neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache introduced the concept of neutrosophic algebraic structures in [1,2]. Some of the neutrosophic algebraic

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structures introduced and studied include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and so on. In [5], Agboola et al studied the structure of neutrosophic polynomial. It was shown that Division Algorithm is generally not true for neutrosophic polynomial rings and it was also shown that a neutrosophic polynomial ring $\langle R \cup I \rangle [x]$ cannot be an Integral Domain even if R is an Integral Domain. Also in [5], it was shown that $\langle R \cup I \rangle [x]$ cannot be a Unique Factorization Domain even if R is a unique factorization domain and it was also shown that every non-zero neutrosophic principal ideal in a neutrosophic polynomial ring is not a neutrosophic prime ideal. In [6], Agboola et al studied ideals of neutrosophic rings. Neutrosophic quotient rings were also studied. In the present paper, we study neutrosophic group and neutrosophic subgroup. It is shown that the product of a neutrosophic subgroup and a pseudo neutrosophic subgroup of a commutative neutrosophic group is a neutrosophic subgroup and their union is also a neutrosophic subgroup even if neither is contained in the other. It is also shown that all neutrosophic groups generated by I and any group isomorphic to Klein 4-group are Lagrange neutrosophic groups. The partitioning of neutrosophic groups is also studied. It is shown that the set of distinct smooth left cosets of a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group (resp. finite Lagrange neutrosophic group) is a partition of the neutrosophic group (resp. Lagrange neutrosophic group).

§2. Main Results

Definition 2.1 *Let $(G, *)$ be any group and let $\langle G \cup I \rangle = \{a + bI : a, b \in G\}$. $N(G) = (\langle G \cup I \rangle, *)$ is called a neutrosophic group generated by G and I under the binary operation $*$. I is called the neutrosophic element with the property $I^2 = I$. For an integer n, n+I, and nI are neutrosophic elements and $0.I = 0$. I^{-1}, the inverse of I is not defined and hence does not exist.*

N(G) is said to be commutative if $ab = ba$ for all $a, b \in N(G)$.

Theorem 2.2 *Let $N(G)$ be a neutrosophic group.*

(i) $N(G)$ in general is not a group;
(ii) $N(G)$ always contain a group.

Proof (*i*) Suppose that $N(G)$ is in general a group. Let $x \in N(G)$ be arbitrary. If x is a neutrosophic element then $x^{-1} \notin N(G)$ and consequently N(G) is not a group, a contradiction.

(*ii*) Since a group G and an indeterminate I generate $N(G)$, it follows that $G \subset N(G)$ and N(G) always contain a group. □

Definition 2.3 *Let $N(G)$ be a neutrosophic group.*

(i) A proper subset $N(H)$ of $N(G)$ is said to be a neutrosophic subgroup of $N(G)$ if $N(H)$ is a neutrosophic group such that is $N(H)$ contains a proper subset which is a group;

(ii) $N(H)$ is said to be a pseudo neutrosophic subgroup if it does not contain a proper subset which is a group.

Example 2.4 (*i*) $(N(\mathcal{Z}),+)$, $(N(\mathcal{Q}),+)$ $(N(\mathcal{R}),+)$ and $(N(\mathcal{C}),+)$ are neutrosophic groups of integer, rational, real and complex numbers respectively.
(ii) $(\langle\{\mathcal{Q}-\{0\}\}\cup I\rangle,.)$, $(\langle\{\mathcal{R}-\{0\}\}\cup I\rangle,.)$ and $(\langle\{\mathcal{C}-\{0\}\}\cup I\rangle,.)$ are neutrosophic groups of rational, real and complex numbers respectively.

Example 2.5 Let $N(G)=\{e,a,b,c,I,aI,bI,cI\}$ be a set where $a^2=b^2=c^2=e$, $bc=cb=a, ac=ca=b, ab=ba=c$, then N(G) is a commutative neutrosophic group under multiplication since $\{e,a,b,c\}$ is a Klein 4-group. $N(H)=\{e,a,I,aI\}$, $N(K)=\{e,b,I,bI\}$ and $N(P)=\{e,c,I,cI\}$ are neutrosophic subgroups of N(G).

Theorem 2.6 *Let N(H)be a nonempty proper subset of a neutrosophic group $(N(G),\star)$. N(H)is a neutrosophic subgroup of N(G) if and only if the following conditions hold:*

(i) $a,b\in N(H)$ implies that $a\star b\in N(H)\ \forall\ a,b\in N(H)$;

(ii) there exists a proper subset A of N(H) such that $(A,\star)$ is a group.

Proof Suppose that N(H) is a neutrosophic subgroup of $((N(G),\star)$. Then $(N(G),\star)$ is a neutrosophic group and consequently, conditions (*i*) and (*ii*) hold.

Conversely, suppose that conditions (i) and (ii) hold. Then $N(H)=\langle A\cup I\rangle$ is a neutrosophic group under $\star$. The required result follows. □

Theorem 2.7 *Let N(H) be a nonempty proper subset of a neutrosophic group (N(G),*). N(H) is a pseudo neutrosophic subgroup of N(G) if and only if the following conditions hold:*

(i) $a,b\in N(H)$ implies that $a*b\in N(H)\ \forall\ a,b\in N(H)$;

(ii) N(H) does not contain a proper subset A such that (A,) is a group.*

Definition 2.8 *Let N(H) and N(K) be any two neutrosophic subgroups (resp. pseudo neutrosophic subgroups) of a neutrosophic group N(G). The product of N(H) and N(K) denoted by N(H).N(K) is the set $N(H).N(K)=\{hk:h\in N(H),k\in N(K)\}$.*

Theorem 2.9 *Let N(H) and N(K) be any two neutrosophic subgroups of a commutative neutrosophic group N(G). Then:*

(i) $N(H)\cap N(K)$ is a neutrosophic subgroup of N(G);

(ii) $N(H).N(K)$ is a neutrosophic subgroup of N(G);

(iii) $N(H)\cup N(K)$ is a neutrosophic subgroup of N(G) if and only if $N(H)\subset N(K)$ or $N(K)\subset N(H)$.

Proof The proof is the same as the classical case. □

Theorem 2.10 *Let N(H) be a neutrosophic subgroup and let N(K) be a pseudo neutrosophic subgroup of a commutative neutrosophic group N(G). Then:*

(i) *$N(H).N(K)$ is a neutrosophic subgroup of N(G);*

(ii) *$N(H) \cap N(K)$ is a pseudo neutrosophic subgroup of N(G);*

(iii) *$N(H) \cup N(K)$ is a neutrosophic subgroup of N(G) even if $N(H) \not\subseteq N(K)$ or $N(K) \not\subseteq N(H)$.*

Proof (i) Suppose that N(H) and N(K) are neutrosophic subgroup and pseudo neutrosophic subgroup of N(G) respectively. Let $x, y \in N(H).N(K)$. Then $xy \in N(H).N(K)$. Since $N(H) \subset N(H).N(K)$ and $N(K) \subset N(H).N(K)$, it follows that N(H).N(K) contains a proper subset which is a group. Hence N(H).N(K) is a neutrosophic of N(G).

(ii) Let $x, y \in N(H) \cap N(K)$. Since N(H) and N(K) are neutrosophic subgroup and pseudo neutrosophic of N(G) respectively, it follows that $xy \in N(H) \cap N(K)$ and also since $N(H) \cap N(K) \subset N(H)$ and $N(H) \cap N(K) \subset N(K)$, it follows that $N(H) \cap N(K)$ cannot contain a proper subset which is a group. Therefore, $N(H) \cap N(K)$ is a pseudo neutrosophic subgroup of N(G).

(iii) Suppose that N(H) and N(K) are neutrosophic subgroup and pseudo neutrosophic subgroup of N(G) respectively such that $N(H) \not\subseteq N(K)$ or $N(K) \not\subseteq N(H)$. Let $x, y \in N(H) \cup N(K)$. Then $xy \in N(H) \cup N(K)$. But then $N(H) \subset N(H) \cup N(K)$ and $N(K) \subset N(H) \cup N(K)$ so that $N(H) \cup N(K)$ contains a proper subset which is a group. Thus $N(H) \cup N(K)$ is a neutrosophic subgroup of N(G). This is different from what is obtainable in classical group theory. □

Example 2.11 $N(G) = \langle \mathcal{Z}_{10} \cup I \rangle = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, I, 2I, 3I, 4I, 5I, 6I, 7I, 8I, 9I, 1 + I, 2 + I, 3 + I, 4 + I, 5 + I, 6 + I, 7 + I, 8 + I, 9 + I, \cdots, 9 + 9I\}$ is a neutrosophic group under multiplication modulo 10. $N(H) = \{1, 3, 7, 9, I, 3I, 7I, 9I\}$ and $N(K) = \{1, 9, I, 9I\}$ are neutrosophic subgroups of N(G) and $N(P) = \{1, I, 3I, 7I, 9I\}$ is a pseudo neutrosophic subgroup of N(G). It is easy to see that $N(H) \cap N(K)$, $N(H) \cup N(K)$, $N(H).N(K)$, $N(P) \cup N(H)$, $N(P) \cup N(K)$, $N(P).N(H)$ and $N(P).N(K)$ are neutrosophic subgroups of N(G) while $N(P) \cap N(H)$ and $N(P) \cup N(K)$ are pseudo neutrosophic subgroups of N(G).

Definition 2.12 *Let N(G) be a neutrosophic group. The center of N(G) denoted by Z(N(G)) is the set $Z(N(G)) = \{g \in N(G) : gx = xg \ \forall \ x \in N(G)\}$.*

Definition 2.13 *Let g be a fixed element of a neutrosophic group N(G). The normalizer of g in N(G) denoted by N(g) is the set $N(g) = \{x \in N(G) : gx = xg\}$.*

Theorem 2.14 *Let N(G) be a neutrosophic group. Then*

(i) *$Z(N(G))$ is a neutrosophic subgroup of N(G);*

(ii) *$N(g)$ is a neutrosophic subgroup of N(G);*

Proof (i) Suppose that $Z(N(G))$ is the neutrosophic center of N(G). If $x, y \in Z(N(G))$, then $xy \in Z(N(G))$. Since $Z(G)$, the center of the group G is a proper subset of $Z(N(G))$, it follows that $Z(N(G))$ contains a proper subset which is a group. Hence $Z(N(G))$ is a neutrosophic subgroup of N(G).

(ii) The proof is the same as (i). □

Theorem 2.15 *Let N(G) be a neutrosophic group and let $Z(N(G))$ be the center of N(G) and $N(x)$ the normalizer of x in N(G). Then*

(*i*) *N(G) is commutative if and only if $Z(N(G)) = N(G)$;*
(*ii*) *$x \in Z(N(G))$ if and only if $N(x) = N(G)$.*

Definition 2.16 *Let $N(G)$ be a neutrosophic group. Its order denoted by $o(N(G))$ or $\mid N(G) \mid$ is the number of distinct elements in $N(G)$. $N(G)$ is called a finite neutrosophic group if $o(N(G))$ is finite and infinite neutrosophic group if otherwise.*

Theorem 2.17 *Let N(H)and N(K)be two neutrosophic subgroups (resp. pseudo neutrosophic subgroups) of a finite neutrosophic group N(G). Then $o(N(H).N(K)) = \frac{o(N(H)).o(N(K))}{o(N(H) \cap N(K))}$.*

Definition 2.18 *Let N(G)and N(H)be any two neutrosophic groups. The direct product of N(G) and N(H) denoted by $N(G) \times N(H)$ is defined by $N(G) \times N(H) = \{(g, h) : g \in N(G), h \in N(H)\}$.*

Theorem 2.19 *If $(N(G), *_1)$ and $(N(H), *_2)$ are neutrosophic groups, then $(N(G) \times N(H), *)$ is a neutrosophic group if $(g_1, h_1) * (g_2, h_2) = (g_1 *_1 g_2, h_1 *_2 h_2)$ $\forall$ $(g_1, h_1), (g_2, h_2) \in N(G) \times N(H)$.*

Theorem 2.20 *Let N(G)be a neutrosophic group and let H be a classical group. Then $N(G) \times H$ is a neutrosophic group.*

Definition 2.21 *Let $N(G)$ be a finite neutrosophic group and let $N(H)$ be a neutrosophic subgroup of $N(G)$.*

(*i*) *$N(H)$ is called a Lagrange neutrosophic subgroup of $N(G)$ if $o(N(H)) \mid o(N(G))$;*

(*ii*) *$N(G)$ is called a Lagrange neutrosophic group if all neutrosophic subgroups of $N(G)$ are Lagrange neutrosophic subgroups;*

(*iii*) *$N(G)$ is called a weak Lagrange neutrosophic group if $N(G)$ has at least one Lagrange neutrosophic subgroup;*

(*iv*) *$N(G)$ is called a free Lagrange neutrosophic group if it has no Lagrange neutrosophic subgroup.*

Definition 2.22 *Let $N(G)$ be a finite neutrosophic group and let $N(H)$ be a pseudo neutrosophic subgroup of $N(G)$.*

(*i*) *$N(H)$ is called a pseudo Lagrange neutrosophic subgroup of $N(G)$ if $o(N(H)) \mid o(N(G))$;*

(*ii*) *$N(G)$ is called a pseudo Lagrange neutrosophic group if all pseudo neutrosophic subgroups of $N(G)$ are pseudo Lagrange neutrosophic subgroups;*

(*iii*) *$N(G)$ is called a weak pseudo Lagrange neutrosophic group if $N(G)$ has at least one pseudo Lagrange neutrosophic subgroup;*

(*iv*) *$N(G)$ is called a free pseudo Lagrange neutrosophic group if it has no pseudo Lagrange neutrosophic subgroup.*

Example 2.23 (*i*) Let N(G) be the neutrosophic group of Example 2.5. The only neutrosophic subgroups of N(G)are $N(H)=\{e,a,I,aI\}$, $N(K)=\{e,b,I,bI\}$ and $N(P)=\{e,c,I,cI\}$. Since $o(N(G))=8$ and $o(N(H))=o(N(K))=o(N(P))=4$ and $4\mid 8$, it follows that N(H), N(K)and N(P)are Lagrange neutrosophic subgroups and N(G) is a Lagrange neutrosophic group.

(*ii*) Let $N(G)=\{1,3,5,7,I,3I,5I,7I\}$ be a neutrosophic group under multiplication modulo 8. The neutrosophic subgroups $N(H)=\{1,3,I,3I\}$, $N(K)=\{1,5,I,5I\}$ and $N(P)=\{1,7,I,7I\}$ are all Lagrange neutrosophic subgroups. Hence N(G) is a Lagrange neutrosophic group.

(*iii*) $N(G)=N(\mathcal{Z}_2)\times N(\mathcal{Z}_2)=\{(0,0),(0,1),(1,0),(1,1),(0,1+I),(1,I),\cdots,(1+I,1+I)\}$ is a neutrosophic group under addition modulo 2. N(G) is a Lagrange neutrosophic group since all its neutrosophic subgroups are Lagrange neutrosophic subgroups.

(*iv*) Let $N(G)=\{e,g,g^2,g^3,I,gI,g^2I,g^3I\}$ be a neutrosophic group under multiplication where $g^4=e$. $N(H)=\{e,g^2,I,g^2I\}$ and $N(K)=\{e,I,g^2I\}$ are neutrosophic subgroups of N(G). Since $o(N(H))\mid o(N(G))$ but $o(N(K))$ does not divide $o(N(G))$ it shows that N(G) is a weak Lagrange neutrosophic group.

(*v*) Let $N(G)=\{e,g,g^2,I,gI,g^2I\}$ be a neutrosophic group under multiplication where $g^3=e$. N(G) is a free Lagrange neutrosophic group.

Theorem 2.24 *All neutrosophic groups generated by I and any group isomorphic to Klein 4-group are Lagrange neutrosophic groups.*

Definition 2.25 *Let $N(H)$ be a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) of a neutrosophic group $N(G)$. For a $g\in N(G)$, the set $gN(H)=\{gh:h\in N(H)\}$ is called a left coset (resp. pseudo left coset) of N(H) in N(G). Similarly, for a $g\in N(G)$, the set $N(H)g=\{hg:h\in N(H)\}$ is called a right coset (resp. pseudo right coset) of N(H) in N(G). If N(G) is commutative, a left coset (resp. pseudo left coset) and a right coset (resp. pseudo right coset) coincide.*

Definition 2.26 *Let N(H)be a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group N(G). A left coset xN(H)of N(H) in N(G) determined by x is called a smooth left coset if $\mid xN(H)\mid=\mid N(H)\mid$. Otherwise, xN(H) is called a rough left coset of N(H) in N(G).*

Definition 2.27 *Let N(H)be a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) of a finite neutrosophic group N(G). The number of distinct left cosets of N(H) in N(G)denoted by [N(G):N(H)] is called the index of N(H) in N(G).*

Definition 2.28 *Let N(H)be a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group N(G). The number of distinct smooth left cosets of N(H) in N(G) denoted by [N(H):N(G)] is called the smooth index of N(H) in N(G).*

Theorem 2.29 *Let X be the set of distinct smooth left cosets of a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group (resp. finite Lagrange neutrosophic group) N(G). Then X is a partition of N(G).*

Proof Suppose that $X=\{X_i\}_{i=1}^{n}$ is the set of distinct smooth left cosets of a Lagrange neutrosophic subgroup (resp. pseudo Lagrange neutrosophic subgroup) of a finite neutrosophic group (resp. finite Lagrange neutrosophic group) N(G). Since $o(N(H)) \mid o(N(G))$ and $\mid xN(H) \mid = \mid N(H) \mid \forall\, x \in N(G)$, it follows that X is not empty and every member of N(G) belongs to one and only one member of X. Hence $\cap_{i=1}^{n} X_i = \emptyset$ and $\cup_{i=1}^{n} X_i = N(G)$. Consequently, X is a partition of N(G). □

Corollary 2.30 *Let $[N(H) : N(G)]$ be the smooth index of a Lagrange neutrosophic subgroup in a finite neutrosophic group (resp. finite Lagrange neutrosophic group) N(G). Then $\mid N(G) \mid = \mid N(H) \mid [N(H) : N(G)]$.*

Proof The proof follows directly from Theorem 2.29. □

Theorem 2.31 *Let X be the set of distinct left cosets of a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) of a finite neutrosophic group N(G). Then X is not a partition of N(G).*

Proof Suppose that $X = \{X_i\}_{i=1}^{n}$ is the set of distinct left cosets of a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) of a finite neutrosophic group N(G). Since N(H) is a non-Lagrange pseudo neutrosophic subgroup, it follows that $o(N(H))$ is not a divisor of $o(N(G))$ and $\mid xN(H) \mid \neq \mid N(H) \mid \forall\, x \in N(G)$. Clearly, X is not empty and every member of N(G) can not belongs to one and only one member of X. Consequently, $\cap_{i=1}^{n} X_i \neq \emptyset$ and $\cup_{i=1}^{n} X_i \neq N(G)$ and thus X is not a partition of N(G). □

Corollary 2.32 *Let $[N(G) : N(H)]$ be the index of a neutrosophic subgroup (resp. pseudo neutrosophic subgroup) in a finite neutrosophic group N(G). Then $\mid N(G) \mid \neq \mid N(H) \mid [N(G) : N(H)]$.*

Proof The proof follows directly from Theorem 2.31. □

Example 2.33 Let N(G)be a neutrosophic group of Example 2.23(iv).

(a) Distinct left cosets of the Lagrange neutrosophic subgroup $N(H) = \{e, g^2, I, g^2I\}$ are: $X_1 = \{e, g^2, I, g^2I\}$, $X_2 = \{g, g^3, gI, g^3I\}$, $X_3 = \{I, g^2I\}$, $X_4 = \{gI, g^3I\}$. X_1, X_2 are smooth cosets while X_3, X_4 are rough cosets and therefore $[N(G) : N(H)] = 4$, $[N(H) : N(G)] = 2$.
$\mid N(H) \mid [N(G) : N(H)] = 4 \times 4 \neq \mid N(G) \mid$ and $\mid N(H) \mid [N(H) : N(G)] = 4 \times 2 = \mid N(G) \mid$.
$X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = N(G)$ and hence the set $X = \{X_1, X_2\}$ is a partition of N(G).

(b) Distinct left cosets of the pseudo non-Lagrange neutrosophic subgroup $N(H) = \{e, I, g^2I\}$ are: $X_1 = \{e, I, g^2I\}$, $X_2 = \{g, gI, g^3I\}$, $X_3 = \{g^2, I, g^2I\}$, $X_4 = \{g^3, gI, g^3I\}$, $X_5 = \{I, g^2I\}$, $X_6 = \{gI, g^3I\}$. X_1, X_2, X_3, X_4 are smooth cosets while X_5, X_6 are rough cosets. $[N(G) : N(H)] = 6$, $[N(H) : N(G)] = 4$, $\mid N(H) \mid [N(G) : N(H)] = 3 \times 6 \neq \mid N(G) \mid$ and $\mid N(H) \mid [N(H) : N(G)] = 3 \times 4 \neq \mid N(G) \mid$. Members of the set $X = \{X_1, X_2, X_3, X_4\}$ are not mutually disjoint and hence do not form a partition of N(G).

Example 2.34 Let $N(G) = \{1, 2, 3, 4, I, 2I, 3I, 4I\}$ be a neutrosophic group under multiplication modulo 5. Distinct left cosets of the non-Lagrange neutrosophic subgroup $N(H) = \{1, 4, I, 2I, 3I, 4I\}$ are $X_1 = \{1, 4, I, 2I, 3I, 4I\}$, $X_2 = \{2, 3, I, 2I, 3I, 4I\}$, $X_3 = \{I, 2I, 3I, 4I\}$. X_1, X_2 are smooth cosets while X_3 is a rough coset and therefore $[N(G) : N(H)] = 3$,

$[N(H) : N(G)] = 2$, $| N(H) | [N(G) : N(H)] = 6 \times 3 \neq | N(G) |$ and $| N(H) | [N(H) : N(G)] = 6 \times 2 \neq | N(G) |$. Members of the set $X = \{X_1, X_2\}$ are not mutually disjoint and hence do not form a partition of N(G).

Example 2.35 Let N(G)be the Lagrange neutrosophic group of Example 2.5. Distinct left cosets of the Lagrange neutrosophic subgroup $N(H) = \{e, a, I, aI\}$ are: $X_1 = \{e, a, I, aI\}$, $X_2 = \{b, c, bI, cI\}$, $X_3 = \{I, aI\}$, $X_4 = \{bI, cI\}$. X_1, X_2 are smooth cosets while X_3, X_4 are rough cosets and thus $[N(G) : N(H)] = 4$, $[N(H) : N(G)] = 2$, $| N(H) | [N(G) : N(H)] = 4 \times 4 = 16 \neq | N(G) |$ and $| N(H) | [N(H) : N(G)] = 4 \times 2 = 8 = | N(G) |$. Members of the set $X = \{X_1, X_2\}$ are mutually disjoint and $N(G) = X_1 \cup X_2$. Hence X is a partition of N(G).

Example 2.36 Let N(G)be the Lagrange neutrosophic group of Example 2.23(iii).

(a) Distinct left cosets of the Lagrange neutrosophic subgroup $N(H) = \{(0,0), (0,1), (0,I), (0,1+I)\}$ are respectively $X_1 = \{(0,0), (0,1), (0,I), (0,1+I)\}$, $X_2 = \{(1,0), (1,1), (1,I), (1,1+I)\}$, $X_3 = \{(I,0), (I,1), (I,I), (I,1+I)\}$, $X_4 = \{(I+I,0), (1+I,1), (1+I,I), (1+I,1+I)\}$, $X_5 = \{(1+I,0), (1+I,1), (1+I,1+I)\}$. X_1, X_2, X_3, X_4 are smooth cosets while X_5 is a rough coset. Thus, $[N(G) : N(H)] = 5$, $[N(H) : N(G)] = 4$, $| N(H) | [N(G) : N(H)] = 4 \times 5 = 20 \neq | N(G) | = 16$ and $| N(H) | [N(H) : N(G)] = 4 \times 4 = 16 = | N(G) |$. Members of the set $X = \{X_1, X_2, X_3, X_4\}$ are mutually disjoint and $N(G) = X_1 \cup X_2 \cup X_3 \cup X_4$ so that X is a partition of N(G).

(b) Distinct left cosets of the pseudo Lagrange neutrosophic subgroup $N(H) = \{(0,0), (0,I), (I,0), (I,I)\}$ are respectively $X_1 = \{(0,0), (0,I), (I,0), (I,1)\}$, $X_2 = \{(0,1), (0,1+I), (I,1), (I,1+I)\}$, $X_3 = \{(1,0), (1,I), (1+I,0), (1+I,I)\}$, $X_4 = \{(1,1), (1,1+I), (1+I,1), (1+I,1+I)\}$. X_1, X_2, X_3, X_4 are smooth cosets and $[N(G) : N(H)] = [N(H) : N(G)] = 4$. Consequently, $| N(H) | [N(G) : N(H)] = | N(H) | [N(H) : N(G)] = 4 \times 4 = 16 = | N(G) |$. Members of the set $X = \{X_1, X_2, X_3, X_4\}$ are mutually disjoint, $N(G) = X_1 \cup X_2 \cup X_3 \cup X_4$ and hence X is a partition of N(G).

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On Bitopological Supra B-Open Sets

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Abstract: In this paper, we introduce and investigate a new class of sets and maps between bitopological spaces called supra(1,2) b-open, and supra (1,2) b-continuous maps, respectively. Furthermore, we introduce the concepts of supra(1,2) locally-closed, supra(1,2) locally b-closed sets. We also introduce supra(1,2) extremely disconnected. Finally, additional properties of these sets are investigated.

Key Words: Supra(1,2) b-open set, supra(1,2) locally closed, supra(1,2) b-closed, supra(1,2) extremely disconnected.

AMS(2010): 54C55

§1. Introduction

In 1983 A.S.Mashhour et al [5] introduced supra topological spaces and studied s-continuous maps and s^*-continuous maps. Andrijevic [1] introduced a class of generalized open sets in a topological space,the called b-open sets in 1996. In 1963, J.C.Kelly [3] introduced the concept of bitopological spaces. The purpose of this present paper is to define some properties by using supra(1,,2) b-open sets, supra(1,2) locally-closed, supra(1,2) locally b-closed in supra bitopological spaces and investigate the relationship between them.

§2. Preliminaries

Throughout this paper by (X, τ_1, τ_2), (Y, σ_1, σ_2) and (Z, η_1, η_2). (or simply X, Y and Z) represent bitopological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of X, A^c denote the complement of A. A subcollection μ is called a supra topology [5] on X if $X \in \mu$, where μ is closed under arbitrary union. (X, μ) is called a supra topological space. The elements of μ are said to be supra open in (X, μ) and the complement of a supra open set is called a supra closed set. The supra topology μ is associated with the topol-

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ogy τ if $\tau \subset \mu$. A subset A of X is $\tau_1\tau_2$-open [4] if A$\in \tau_1 \cup \tau_2$ and $\tau_1\tau_2$-closed if its complement is $\tau_1\tau_2$-open in X. The $\tau_1\tau_2$-closure of A is denoted by $\tau_1\tau_2 cl(A)$ and $\tau_1\tau_2 cl(A) = \cap\{\ F : A \subset F$ and F^c is $\tau_1\tau_2$-open$\}$. Let (X,μ_1, μ_2) be a supra bitopological space. A set A is $\mu_1\mu_2$-open if $A \in \mu_1 \cup \mu_2$ and $\mu_1\mu_2$-closed if its complement is $\mu_1\mu_2$-open in (X, μ_1, μ_2). The $\mu_1\mu_2$-closure of A is denoted by $\mu_1\mu_2 cl(A)$ and $\mu_1\mu_2 cl(A) = \cap\{\ F : A \subset F$ and F^c is $\mu_1\mu_2 - \text{open}\}$.

Definition 2.1 *Let (X, μ) be a supra topological space. A set A is called*

(1) *supra α-open set [2] if $A \subseteq int^\mu(cl^\mu(int^\mu(A)))$;*
(2) *supra semi-open set [2] if $A \subseteq cl^\mu(int^\mu(A))$;*
(3) *supra b-open set [6] if $A \subseteq cl^\mu(int^\mu(A)) \cup int^\mu(cl^\mu(A))$.*

Definition 2.2([4]) *Let (X, τ_1, τ_2) be a bitopological space. A subset A of (X, τ_1, τ_2) is called*

(1) *(1,2)semi-open set if $A \subseteq \tau_1\tau_2 cl(\tau_1 int(A))$;*
(2) *(1,2)pre-open set if $A \subseteq \tau_1 int(\tau_1\tau_2 cl(A))$;*
(3) *(1,2)α-open-set if $A \subseteq \tau_1 int(\tau_1\tau_2 cl(\tau_1 int(A)))$;*
(4) *(1,2)b-open-set $A \subseteq \tau_1\tau_2 cl(\tau_1 int(A)) \cup \tau_1 int(\tau_1\tau_2 cl(A))$.*

§3. Comparison

In this section we introduce a new class of generalized open sets called supra(1,2) b-open sets and investigate the relationship between some other sets.

Definition 3.1 *Let (X, τ_1, τ_2) be a supra bitopological space. A set A is called a supra(1,2) b-open set if $A \subseteq \mu_1\mu_2 cl(\mu_1 int(A)) \cup \mu_1 int(\mu_1\mu_2 cl(A))$.The compliment of a supra(1,2) b-open is called a supra(1,2) b-closed set.*

Definition 3.2 *Let X be a supra bitopological space. A set A is called*

(1) *supra (1,2) semi-open set if $A \subseteq \mu_1\mu_2 cl(\mu_1 int(A))$;*
(2) *supra (1,2) pre-open set if $A \subseteq \mu_1 int(\mu_1\mu_2 cl(A))$;*
(3) *supra (1,2) α-open-set if $A \subseteq \mu_1 int(\mu_1\mu_2 cl(\mu_1 int(A)))$.*

Theorem 3.3 *In a supra bitopological space (X, μ_1, μ_2), any supra open set in (X, μ_1) is supra(1,2) b-open set and any supra open set in (X, μ_2) is supra (2,1) b-open set.*

Proof Let A be any supra open in (X, μ_1). Then $A = \mu_1 int(A)$. Now $A \subseteq \mu_1\mu_2 cl(A) = \mu_1\mu_2 cl(\mu_1 int(A)) \subseteq \mu_1\mu_2 cl(\mu_1 int(A) \cup \mu_1 int(\mu_1\mu_2 cl(A))$. Hence A is supra(1,2) b-open set. Similarly, any supra open in (X, μ_2) is supra(2,1) b-open set. □

Remark 3.4 The converse of the above theorem need not be true as shown by the following example.

Example 3.5 Let $X = \{a, b, c, d\}$, $\mu_1 = \{\phi, X, \{a, b\}, \{a, c, d\}\}$, $\mu_2 = \{\phi, \{a\}, \{a, b\}, \{b, c, d\}, X\}$; $\mu_1\mu_2$-open $= \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, \{b, c, d\}, X\}$, $\mu_1\mu_2$-closed $= \{\phi, \{a\}, \{b\}, \{c, d\}, \{b, c, d\}, X\}$, supra(1,2) $bO(X) = \{\phi, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, X\}$. It is obvious that {a,d}$\in$ supra(1,2) b-open but $\{a,d\} \notin \mu_1$-open. Also, supra(2,1) $bO(X) = (\phi, \{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,c,d\}, \{a,b,d\}, \{b,c,d\}, X\}$. Here $\{a,c\} \in$ supra(2,1) b-open set but $\{a,c\} \notin \mu_2$-open.

Theorem 3.6 *In a supra bitopological space (X, μ_1, μ_2), any supra open set in (X, μ_1) is supra(1,2) α-open set and any supra open set in (X, μ_2) is supra (2,1) α-open set.*

Proof Let A be any supra open in (X, μ_1). Then $A = \mu_1 int(A)$. Now $A \subseteq \mu_1\mu_2 cl(A)$. Then $\mu_1 int(A) \subseteq \mu_1 int(\mu_1\mu_2 cl(A))$. Since $A = \mu_1 int(A)$, $A \subseteq \mu_1 int(\mu_1\mu_2 cl(\mu_1 int(A)))$. Hence A is supra(1,2) α-open set. Similarly, any supra open in (X, μ_2) is supra(2,1) α-open set. $\square$

Remark 3.7 The converse of the above theorem need not be true as shown in the following example.

Example 3.8 Let $X = \{a,b,c,d\}$, $\mu_1 = \{\phi, \{a,c\}, \{a,b,c\}, \{a,b,d\}, X\}$, $\mu_2 = \{\phi, \{c,d\}, \{a,b,d\}, \{b,c,d\}, X\}$, $\mu_1\mu_2$-open $= \{\phi, \{a,c\}, \{c,d\}, \{a,b,c\}, \{a,b,d, \{b,c,d\}, X\}$, $\mu_1\mu_2$-closed $= \{\phi, \{a\}, \{c\}, \{d\}, \{a,b\}, \{b,d\}, X\}$. supra(1,2) $\alpha O(X) = \{\phi, \{a,c\}, \{a,b,c\}, \{a,c,d\}, \{a,b,d\}\}, X\}$. Here $\{a,c,d\} \in supra(1,2)$ α-open but $\{a,c,d\} \notin \mu_1$-open. Also, supra(2,1) $\alpha O(X) = (\phi, \{c,d\}, \{a,c,d\}, \{a,b,d\}, \{b,c,d\}, X\}$. Here $\{a,c,d\} \in supra(2,1)\alpha$-open but $\{a,c,d\} \notin \mu_2$-open.

Theorem 3.9 *Every supra(1,2) α-open is supra(1,2) semi-open.*

Proof Let A be a supra (1,2) α-open set in X. Then $A \subseteq \mu_1 int(\mu_1\mu_2 cl(\mu_1 int(A))) \subseteq \mu_1\mu_2 cl(\mu_1 int(A))$. Therefore, $A \subseteq \mu_1\mu_2 cl(\mu_1 int(A))$. Hence A is supra(1,2) semi-open set. $\square$

Remark 3.10 The converse of the above theorem need not be true as shown below.

Example 3.11 Let $X = \{a,b,c,d\}$, $\mu_1 = \{\phi, \{b\}, \{a,d\}, \{a,b,c\}, X\}$, $\mu_2 = \{\phi, \{b,c\}, \{a,b,d\}, X\}$, $\mu_1\mu_2$-open $=\{\phi, \{b\}, \{a,d\}, \{b,c\}, \{a,b,c, \{a,b,d\}, X\}$, $\mu_1\mu_2$-closed $= \{\phi, \{c\}, \{d\}, \{a,d\}, \{b,c\}, \{a,c,d\}, X\}$. supra(1,2) $\alpha O(X) = \{\phi, \{b\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}\}, X\}$, supra(1,2) $SO(X) = \{\phi, \{b\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}\}, X\}$. Here $\{b,c\}$ is a supra(1,2) α-open but not supra(1,2) semi-open.

Theorem 3.12 *Every supra(1,2) semi-open set is supra(1,2) b-open.*

Proof Let A be a supra(1,2) semi-open set X. Then $A \subseteq \mu_1\mu_2 cl(\mu_1 int(A))$. Hence $A \subseteq \mu_1\mu_2 cl(\mu_1 int(A)) \cup \mu_1 int(\mu_1\mu_2 cl(A))$. Thus A is supra(1,2) b-open set. $\square$

Remark 3.13 The converse of the above theorem need not be true as shown in the following example.

Example 3.14 Let $X = \{a,b,c,d\}$, $\mu_1 = \{\phi, \{a\}, \{a,b\}, \{b,c,d\}, X\}$, $\mu_2 = \{\phi, \{b\}, \{a,d\}, \{a,b,d\}, \{b,c,d\}, X\}$, $\mu_1\mu_2$-open $= \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,d\}, \{a,b,d\}, \{b,c,d\}, X\}$, $\mu_1\mu_2$-closed $= \{\phi, \{a\}, \{c\}, \{b,c\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}, X\}$, supra(1,2) $bO(X) = \{\phi, \{a\}, \{a,b\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}, X\}$, supra(1,2) $SO(X) = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\}, X\}$. Here $\{b,d\} \in supra(1,2)b$-open set but $\{b,d\} \notin$ supra(1,2) semi-open.

Theorem 3.15 *Every supra(1,2) α-open is supra(1,2) b-open.*

Proof Let A be an supra(1,2) α-open in X. Then $A \subseteq \mu_1 int(\mu_1\mu_2 cl(\mu_1 int(A)))$. It is obvious that $\mu_1 int(\mu_1\mu_2 cl(\mu_1 int(A))) \subseteq \mu_1\mu_2 cl(\mu_1 int(A)) \subseteq \mu_1\mu_2 cl(\mu_1 int(A)) \cup \mu_1 int(\mu_1\mu_2 cl(A))$. Hence $A \subseteq \mu_1\mu_2 cl(\mu_1 int(A)) \cup \mu_1 int(\mu_1\mu_2 cl(A))$. Thus A is supra(1,2) b-open set. □

Remark 3.16 The reverse claim in Theorem 3.15 is not usually true.

Example 3.17 Let $X = \{a,b,c,d\}$,$\mu_1 = \{\phi,\{a,c\},\{a,b,c\},\{a,b,d\},X\}$,$\mu_2 = \{\phi,\{c,d\},\{b,c,d\},\{a,b,d\}X\}$,$\mu_1\mu_2$-open= $\{\phi,\{a,c\},\{c,d\},\{a,b,c\},\{a,b,d\{b,c,d\},X\}$ $\mu_1\mu_2$-closed $= \{\phi,\{a\},\{c\},\{d\},\{a,b\},\{b,d\},X\}$, supra(1,2) $\alpha O(X) = \{\phi,\{a,c\},\{a,b,c\},\{a,c,d\},\{a,b,d\}\},X\}$, supra(1,2) $bO(X) = \{\phi,\{a,c\},\{a,d\},\{b,c\},\{c,d\}\{a,b,c\},\{a,c,d\},\{a,b,d\},\{b,c,d\},X\}$. Here $\{a,d\} \in$ supra(1,2) b-open but $\{a,d\} \notin$ supra(1,2) α-open.

Theorem 3.18 *In a supra bitopological space (X,μ_1,μ_2), any supra open set in (X,μ_1) is supra(1,2) semi-open set and any supra open set in (X,μ_2) is supra (2,1) semi-open set.*

Proof This follows immediately from Theorems 3.6 and 3.9. □

Remark 3.19 The converse of the above theorem need not be true as shown in the Example 3.8, $\{a,c,d\}$ is both supra(1,2) semi-open and supra(2,1) semi-open but it is not supra μ_1-open and also is not μ_2-open.

Remark 3.20 From the above discussions we have the following diagram. $A \rightarrow$B represents A implies B, $A \nrightarrow B$ represents A does not implies B.

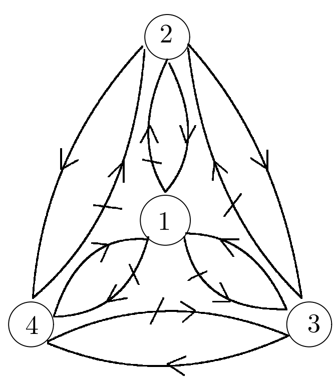

Fig. 1 1=supra (1,2) b-open, 2=μ_1-open,
3=supra (1,2) α-open, 4=supra (1,2) semi-open

§4. Properties of Supra(1,2) b-Open Sets

Theorem 4.1 *A finite union of supra(1,2) b-open sets is always supra(1,2) b-open.*

Proof Let A and B be two supra(1,2) b-open sets. Then $A \subseteq \mu_1\mu_2 cl(\mu_1 int(A)) \cup$

$\mu_1 int(\mu_1\mu_2 cl(A))$ and $B \subseteq \mu_1\mu_2 cl(\mu_1 int(B)) \cup \mu_1 int(\mu_1\mu_2 cl(B))$. Now, $A \cup B \subseteq \mu_1\mu_2 cl(\mu_1 int(A \cup B)) \cup \mu_1 int(\mu_1\mu_2 cl(A \cup B))$. Hence $A \cup B$ is supra(1,2) b-open set. □

Remark 4.2 Finite intersection of supra(1,2) b-open sets may fail to be supra(1,2) b-open since, in Example 3.14, both $\{a, b\}$ and $\{b, d\}$ are supra(1,2) b-open sets, but their intersection $\{c\}$ is not supra(1,2) b-open.

Definition 4.3 *The supra(1,2) b-closure of a set A is denoted by supra$(1,2)bcl(A)$ and defined as supra$(1,2)bcl(A) = \cap\{B :$ B is a supra(1,2) b-closed set and $A \subset B\}$. The supra(1,2) interior of a set A is denoted by supra(1,2)bint(A), and defined as supra(1,2)b int(A)$= \cup\{B :$ B is a supra(1,2) b-open set and $A \supseteq B\}$.*

Remark 4.4 It is clear that $supra(1,2)bint(A)$ is a supra(1,2) b-open and $supra(1,2)bcl(A)$ is supra(1,2) b-closed set.

Definition 4.5 *A subset A of supra bitopological space X is called*

(1) *supra(1,2)locally-closed if $A = U \cap V$, where $U \in \mu_1$ and V is supra$\mu_1\mu_2$ closed;*
(2) *supra(1,2) locally b-closed if $A = U \cap V$, where $U \in \mu_1$ and V is supra(1,2) b-closed;*
(3) *supra (1,2)D(c,b) set if $\mu_1 int(A) = supra(1,2)bint(A)$.*

Theorem 4.6 *The intersection of a supra open in (X, μ_1) and a supra(1,2) b-open set is a supra(1,2) b-open set.*

Proof Let A be supra open in (X, μ_1). Then A is supra(1,2) b-open and $A = \mu_1 int(A) \subseteq supra(1,2)bint(A)$. Let B be supra(1,2) b-open then $B = supra(1,2)bint(B)$. Now $A \cap B \subseteq supra(1,2)bint(A) \cap supra(1,2)bint(B) = supra(1,2)bint(A \cap B)$. Hence the intersection of supra open set in (X, μ_1) and a supra(1,2) b-open set is a supra(1,2) b-open set. □

Theorem 4.7 *For a subset A of X, the following are equivalent:*

(1) *A is supra-open in (X, μ_1);*
(2) *A is supra (1,2) b-open and supra(1,2) D(c,b)-set.*

Proof (1)⇒(2) If A is supra-open in (X, μ_1), then A is supra (1,2) b-open and $A = \mu_1 int(A)$, $A = supra(1,2)bint(A)$. Hence $\mu_1 int(A) = supra(1,2)bint(A)$. Therefore, A is $supra(1,2)D(c,b)$-set.

(2)⇒(1) Let A be supra (1,2) b-open and supra $(1,2)D(c,b)$-set. Then $A = supra(1,2)bint(A)$ and $\mu_1 int(A) = supra(1,2)bint(A)$. Hence $A = \mu_1 int(A)$. This implies that A is supra-open in (X, μ_1). □

Definition 4.8 *A space X is called an supra(1,2) extremely disconnected space (briefly supra(1,2) E.D) if supra$\mu_1\mu_2$ closure of each supra-open in (X, μ_1) is supra open set in (X, μ_1). Similarly supra$\mu_1\mu_2$ closure of each supra-open in (X, μ_2) is supra open set in (X, μ_2).*

Example 4.9 Let $X = \{a, b, c\}$, $\mu_1 = \{\phi, \{b\}, \{a, b\}, \{a, c\}, X\}$, $\mu_2 = \{\phi, \{a\}, \{b, c\}, \{a, c\}\}$, $\mu_1\mu_2$open $= \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$, $\mu_1\mu_2$ closed $= \{\phi, \{a\}, \{b\}, \{c\}, \{a, c\}\{b, c\}, X\}$.

Hence every $\mu_1\mu_2$ closure of supra-open is (X, μ_1) and also every supra$\mu_1\mu_2$ closure of supra-open in (X, μ_2).

Theorem 4.10 *Let A be a subset of supra bitopological space (X, μ_1, μ_2) if A is supra(1,2) locally b-closed, then*

(1) *$supra(1,2)bcl(A) - A$ is supra(1,2) b-closed set:*
(2) *$[A \cup (X - supra(1,2)bcl(A))]$ is supra(1,2) b-open;*
(3) *$A \subseteq supra(1,2)bint(A \cup (X - supra(1,2)bcl(A))$.*

Proof (1) If A is an supra(1,2) locally b-closed, there exist an U is supra-open in (X, μ_1) such that $A = U \cap supra(1,2)bcl(A)$. Now, $supra(1,2)bcl(A) - A = supra(1,2)bcl(A) - [U \cap supra(1,2)bcl(A)] = supra(1,2)bcl(A) \cap [X - (U \cap supra(1,2)bcl(A))] = supra(1,2)bcl(A) \cap [(X - U) \cup (X - supra(1,2)bcl(A))] = supra(1,2)bcl(A) \cap (X - U)$, which is supra(1,2) b-closed by Theorem 4.5.

(2) Since $supra(1,2)bcl(A) - A$ is supra(1,2) b closed, then $[X - (supra(1,2)bcl(A) - A)]$ is supra(1,2) b-open and $[X - (supra(1,2)bcl(A) - A)] = (X - supra(1,2)bcl(A)) \cup (X \cap A) = A \cup [X - supra(1,2)bcl(A)]$. Hence $[A \cup (X - supra(1,2)bcl(A))]$ is supra(1,2) b-open.

(3) It is clear that

$$A \subseteq [A \cup (X - supra(1,2)bcl(A)] = supra(1,2)bint[A \cup (X - supra(1,2)bcl(A))]. \qquad \square$$

§5. Supra (1,2) b-Continuous Functions

In this section, We introduce a new class of continuous maps called a supra (1,2) b-continuous maps and obtain some of their properties.

Definition 5.1 *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and μ_1, μ_2 be an associated supra bitopology with τ_1, τ_2. A map $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called a supra (1,2) b-continuous map [resp. supra (1,2) α-continuous, supra (1,2) semi-continuous] if the inverse image of each $\sigma_1\sigma_2$-open set in Y is supra (1,2) b-open set [resp. supra (1,2) α-open, supra (1,2) semi-open] in X.*

Definition 5.2 *Let (X, τ_1, τ_2) and (Y, σ_1, σ_2) be two bitopological spaces and μ_1, μ_2 be an associated supra bitopology with τ_1, τ_2. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called supra (1,2) continuous if $f^{-1}(V)$ is μ_1-open in X for each $\sigma_1\sigma_2$-open set V of Y.*

Theorem 5.3 *Every (1,2) continuous is supra (1,2) b-continuous.*

Proof Let $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be an (1,2)-continuous map and let A be an $\sigma_1\sigma_2$-open set in (Y, σ_1, σ_2). Then $f^{-1}(A)$ is an τ_1-open set in (X, τ_1, τ_2). Since μ_1 and μ_2 are associated with τ_1 and τ_2, then $\tau_1 \subseteq \mu_1$. This implies that $f^{-1}(A)$ is μ_1-open in X and it is supra (1,2) b-open in X. Hence f is supra (1,2)b-continuous. $\square$

Theorem 5.4 *Every supra (1,2)-continuous is supra (1,2) b-continuous function.*

Proof Let $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ be an supra (1,2)-continuous and let A be an $\sigma_1\sigma_2$ open set in Y. Since f is supra (1,2)-continuous and μ_1,μ_2 associated with τ_1,τ_2, $f^{-1}(A)$ is μ_1-open in X and it is supra (1,2) b-open in X. Hence f is supra (1,2) b-continuous. □

Remark 5.5 The converse of Theorems 5.3 and 5.4 need not be true. We can shown this by the following example.

Example 5.6 Let $X=\{a,b,c,d\}$, $Y=\{p,q,r,s\}$, $\tau_1=\{\phi,\{a\},\{a,b\},\{a,d\},X\}$ and $\tau_2=\{\phi,\{a\},\{a,b\},X\}$ are topologies on (X,τ_1,τ_2), $\sigma_1=\{\phi,\{p\},\{r\},\{p,r\},Y\}$, $\sigma_2=\{\phi,\{p,r\},Y\}$, $\sigma_1\sigma_2$-open $=\{\phi,\{p\},\{r\},\{p,r\},Y\}$. The supra topologies μ_1,μ_2 are defined as follows:

$\mu_1=\{\phi,\{a\},\{a,b\},\{a,d\},\{b,c\},X\}$, $\mu_2=\{\phi,\{a\},\{a,b\},\{b,c\},X\}$, $\mu_1\mu_2$ open $=\{\phi,\{a\},\{a,b\},\{a,d\},\{b,c\},X\}$, $\mu_1\mu_2$ closed $=\{\phi,\{a,d\},\{b,c\},\{b,d\},\{b,c,d\},X\}$, supra (1,2) b-open $=\{\phi,\{a\},\{c\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{a,b,c\},\{a,c,d\},\{a,b,d\},X\}$. Define a map $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ by $f(a)=p,\ f(b)=q,\ f(c)=r,\ f(d)=s$. Clearly f is supra (1,2) b-continuous. But $f^{-1}(\{p,r\})=\{a,c\}$ is not μ_1-open set in X where $\{p,r\}$ is $\sigma_1\sigma_2$-open in Y. So f is not supra (1,2) continuous. And also f is not (1,2)-continuous functions because $f^{-1}(\{p,r\})=\{a,c\}$ is not τ_1-open in X where where $\{p,r\}$ is $\sigma_1\sigma_2$-open in Y.

Theorem 5.7 *Every supra (1,2) α-continuous map is supra (1,2)b-continuous.*

Proof It is obvious that every supra (1,2) α-open is (1,2) b-open. □

Remark 5.8 The converse of the above theorem need not be true as shown in the following example.

Example 5.9 Let $X=\{a,b,c,d\}$, $Y=\{p,q,r,s\}$, $\tau_1=\{\phi,\{a,b,c\},X\}$ and $\tau_2=\{\phi,\{a,b,d\},X\}$ are topologies on (X,τ_1,τ_2), $\sigma_1=\{\phi,\{p,r\},Y\}$, $\sigma_2=\{\phi,\{p,q\},\{p,q,s\},Y\}$, $\sigma_1\sigma_2$-open $=\{\phi,\{p,q\},\{p,r\},\{p,q,s\},Y\}$. The supra topologies μ_1,μ_2 are defined as follows:

$\mu_1=\{\phi,\{a,c\},\{a,b,c\},\{a,b,d\},X\}$, $\mu_2=\{\phi,\{c,d\},\{b,c,d\},\{a,b,d\},X\}$. Define a function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ by $f(a)=q,\ f(b)=r,\ f(c)=p,\ f(d)=s$. Then f is supra (1,2) b-continuous but not (1,2) α-continuous because $f^{-1}(\{p,r\})=\{b,c\}$ is not supra (1,2) α-open where $\{p,r\}$ is $\sigma_1\sigma_2$-open in Y.

Theorem 5.10 *Let (X,τ_1,τ_2), (Y,σ_1,σ_2) and (Z,η_1,η_2) be three bitopological spaces. If a map $f:(X,\tau_1,\tau_2\to(Y,\sigma_1,\sigma_2)$ is supra(1,2) b-continuous and $g:(Y,\sigma_1,\sigma_2)\to(Z,\eta_1,\eta_2)$ is a (1,2)-continuous map, then $g\circ f:(X,\tau_1,\tau_2)\to(Z,\eta_1,\eta_2)$ is a supra(1,2) b-continuous.*

Proof Let A be a $\eta_1\eta_2$-open set in Z. Since g is (1,2)-continuous, then $g^{-1}(A)$ is σ_1-open in Y. Every σ_1-open is $\sigma_1\sigma_2$-open. Thus $g^{-1}(A)$ is $\sigma_1\sigma_2$-open in Y. Since f is supra (1,2) b-continuous, then $f^{-1}(g^{-1}(A))=(g\circ f)^{-1}(A)$ is supra (1,2) b-open set in X. Therefore $g\circ f$ is supra(1,2) b-continuous. □

Theorem 5.11 *Let (X,τ_1,τ_2) and (Y,σ_1,σ_2) be bitopological spaces. Let μ_1,μ_2 and v_1,v_2 be the associated supra bitopologies with τ_1,τ_2 and σ_1,σ_2, respectively. Then $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is a supra(1,2) b-continuous map if one of the following holds:*

(1) $f^{-1}(supra(1,2)bint(A)) \subseteq \tau_1 int(f^{-1}(A))$ *for every set* A *in* Y*;*

(2) $\tau_1\tau_2 cl(f^{-1}(A)) \subseteq f^{-1}(supra(1,2)bcl(A))$ *for every set* A *in* Y*;*

(3) $f(\tau_1\tau_2 cl(B)) \subseteq supra(1,2)bcl(f(B))$ *for every set* B *in* X*.*

Proof Let A be any $\sigma_1\sigma_2$-open set of Y. If condition (1) is satisfied, then

$$f^{-1}(supra(1,2)bint(A)) \subseteq \tau_1 int(f^{-1}(A)).$$

We get, $f^{-1}(A) \subseteq \tau_1 int(f^{-1}(A))$. Therefore $f^{-1}(A)$ is supra open set in (X, μ_1). Every supra open set in (X, μ_1) is supra(1,2) b-open set. Hence f is supra (1,2) b-continuous function.

If condition (2) is satisfied, then we can easily prove that f is supra(1,2) b-continuous function.

Now if the condition (3) is satisfied and A be any $\sigma_1\sigma_2$-open set of Y. Then $f^{-1}(A)$ is a set in X and $f(\tau_1\tau_2 cl(f^{-1}(A))) \subseteq supra(1,2)bcl(f(f^{-1}(A)))$. This implies $f(\tau_1\tau_2 cl(f^{-1}(A))) \subseteq supra(1,2)bcl(A)$. It is nothing but just the condition (2). Hence f is a supra(1,2) b-continuous map. □

§6. Applications

Now we introduce a new class of space called a supra(1,2)-extremely disconnected space.

Definition 6.1 *A space X is called an supra(1,2)-extremely disconnected space (briefly supra (1,2)-E.D) if* $\mu_1\mu_2$ *closure of each supra-open in* (X, μ_1) *is supra-open set in* (X, μ_1)*. Similarly* $\mu_1\mu_2$*-closure of each supra-open in* (X, μ_2) *is supra-open set in* (X, μ_2)*.*

Theorem 6.2 *For a subset* A *of a supra(1,2) extremely disconnected space* X*, the following are equivalent:*

(1) *A is supra-open in (X, μ_1);*

(2) *A is supra(1,2) b-open and supra(1,2) locally closed.*

Proof (1)⇒(2) It is obvious.

(2)⇒(1) Let A be supra(1,2) b-open and supra(1,2) locally closed. Then

$$A \subseteq \mu_1\mu_2 cl(\mu_1 int(A)) \cup \mu_1 int(\mu_1\mu_2 cl(A)) \text{ and } A = U \cap \mu_1\mu_2 cl(A),$$

where U is supra-open in (X, μ_1). So $A \subseteq U \cap (\mu_1 int(\mu_1\mu_2 cl(A)) \cup \mu_1\mu_2 cl(\mu_1 int(A)) \subseteq [\mu_1 int(U \cap \mu_1\mu_2 cl(A))] \cup [U \cap \mu_1\mu_2 cl(\mu_1 int(A))] \subseteq [\mu_1 int(U \cap \mu_1\mu_2 cl(A))] \cup [U \cap \mu_1 int(\mu_1\mu_2 cl(A))]$ (since X is supra(1,2) E.D)$\subseteq [\mu_1 int(U \cap \mu_1\mu_2 cl(A))] \cup [\mu_1 int(U \cap \mu_1\mu_2 cl(A))] = \mu_1 int(A) \cup \mu_1 int(A) = \mu_1 int(A)$. Hence $A \subseteq \mu_1 int(A)$. Therefore A is supra-open in (X, μ_1). □

Theorem 6.3 *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be two bitopological spaces and* μ_1, μ_2 *be associated supra topologies with* τ_1, τ_2*. Let* $f: X \to Y$ *be a map. Then the following are equivalent.*

(1) *f is supra (1,2) b-continuous map;*

(2) *The inverse image of a $\sigma_1\sigma_2$-closed set in Y is a supra (1,2) b-closed set in X;*

(3) *Supra* $(1,2)bcl(f^{-1}(A) \subseteq f^{-1}(\sigma_1\sigma_2 cl(A))$ *for every set* A *in* Y*;*

(4) $f(supra(1,2)bcl(A)) \subseteq \sigma_1\sigma_2 cl(f(A))$ *for every set* $A \in X$*;*

(5) $f^{-1}(\sigma_1(B)) \subseteq supra(1,2)int(f^{-1}(B))$ *for every* B *in* Y*.*

Proof (1)⇒(2) Let A be a $\sigma_1\sigma_2$ closed set in Y, then $Y - A$ is $\sigma_1\sigma_2$ open set in Y. Since f is supra (1,2) b-continuous, $f^{-1}(Y - A) = X - f^{-1}(A)$ is a supra (1,2) b-open set in X. This implies that $f^{-1}(A)$ is a supra (1,2) b-closed subset of X.

(2)⇒(3) Let A be any subset of Y. Since $\sigma_1\sigma_2 cl(A)$ is $\sigma_1\sigma_2$ closed in Y, then $f^{-1}(\sigma_1\sigma_2 cl(A))$ is supra (1,2) b-closed in X. Hence supra $(1,2)bcl(f^{-1}(A) \subseteq supra(1,2)bcl(f^{-1}(\sigma_1\sigma_2 cl(A))) = f^{-1}(\sigma_1\sigma_2 cl(A))$.

(3)⇒(4) Let A be any subset of X. By (3), we obtain

$$f^{-1}(\sigma_1\sigma_2 cl(f(A))) \supseteq supra(1,2)bclf^{-1}(f(A)) \supseteq supra(1,2)bcl(A).$$

Hence $f(supra(1,2)cl(A)) \subseteq \sigma_1\sigma_2 cl(f(A))$.

(4)⇒(5) Let B be any subset of Y. By (5), $f(supra(1,2)bcl(X - f^{-1}(B))) \subset \sigma_1\sigma_2 cl(f(X - f^{-1}(B)))$ and $f(X - supra(1,2)bint(f^{-1}(B))) \subseteq \sigma_1\sigma_2 cl(Y - B) = Y - \sigma_1 int(B)$. Therefore we have

$$X - supra(1,2)bint(f^{-1}(B)) \subset f^{-1}(Y - \sigma_1 int(B))$$

and

$$f^{-1}(\sigma_1 int(B)) \subset supra(1,2)bint(f^{-1}(B)).$$

(5)⇒(1) Let B be a σ_1-open set in Y. Then by (4), $f^{-1}(\sigma_1 int(B)) \subseteq supra(1,2)int(f^{-1}(B))$. Therefore $f^{-1}(B) \subseteq supra(1,2)int(f^{-1}(B))$. But $supra(1,2)bint(f^{-1}(B)) \subseteq f^{-1}(B)$. Hence $f^{-1}(B) = supra(1,2)bint(f^{-1}(B))$. Therefore $f^{-1}(B)$ is supra (1,2) b-open in X. Thus f is supra (1,2) b-continuous map. □

We introduce the following definition.

Definition 6.4 *Let* (X, τ_1, τ_2) *and* (Y, σ_1, σ_2) *be two bitopological spaces and* μ_1, μ_2 *be associated supra bitopologies with* τ_1, τ_2*. A map* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is called a supra (1,2) locally closed continuous [resp. supra (1,2) D(c,b) continuous, supra (1,2) locally b-closed continuous] if* $f^{-1}(B)$ *is supra (1,2) locally closed [resp. supra (1,2) D(c,b) set, supra (1,2) locally b-closed] in* X *for each* $\sigma_1\sigma_2$ *open set* V *of* Y*.*

Theorem 6.5 *Let* X *be supra (1,2) extremely disconnected space, the function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is supra (1,2)-continuous iff* f *is supra (1,2) b-continuous and supra (1,2) locally closed continuous.*

Proof Let V be a $\sigma_1\sigma_2$-open set in Y. Since f is supra (1,2)-continuous, $f^{-1}(V)$ is μ_1-open in X. Then by Theorem 3.3, $f^{-1}(V)$ is supra (1,2) b-open and supra (1,2) locally closed in X. Hence f is supra (1,2) b-continuous and supra (1,2) locally closed continuous. Conversely, let U be a $\sigma_1\sigma_2$-open set in Y. Since f is supra (1,2) b-continuous and supra (1,2) locally closed continuous, $f^{-1}(U)$ is supra (1,2) b-open and supra (1,2) locally-closed in X. Since X is supra (1,2) extremely disconnected, by Theorem 6.1, $f^{-1}(U)$ is μ_1-open in X. Hence f is supra (1,2)-continuous. □

Theorem 6.6 *The function $f:\ (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is supra (1,2) continuous iff f is supra (1,2) b-continuous and supra (1,2) D(c,b)-continuous.*

Proof Let V be a $\sigma_1\sigma_2$ open set in Y. Since f is supra (1,2) continuous, $f^{-1}(V)$ is μ_1-open in X. By Theorem 4.7, $f^{-1}(V)$ is supra (1,2) b-open and supra (1,2) D(c,b)set. Then f is supra (1,2) b-continuous and supra (1,2) D(c,b)continuous. Conversely, let U be a $\sigma_1\sigma_2$ open in Y. Since f is supra (1,2) b-continuous and supra (1,2) D(c,b)-continuous, $f^{-1}(U)$ is supra (1,2) b-open and supra (1,2) D(c,b)set. By Theorem 4.7, $f^{-1}(U)$ is supra open in (X,μ_1). Hence f is supra (1,2) continuous. □

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On Finsler Space with Randers Conformal Change — Main Scalar, Geodesic and Scalar Curvature

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Abstract: Let M^n be an n-dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x,y), (y^i = \dot{x}^i)$ of M^n. In the present paper we define Randers conformal change as

$$L(x,y) \to L^*(x,y) = e^{\sigma(x)} L(x,y) + \beta(x,y)$$

where $\sigma(x)$ is a function of x and $\beta(x,y) = b_i(x)y^i$ is a 1- form on M^n.
This transformation is more general as it includes conformal, Randers and homothetic transformation as particular cases. In the present paper we have found out the expressions for scalar curvature and main scalar of two-dimensional Finsler space obtained by Randers conformal change of F^n. We have also obtained equation of geodesic for this transformed space.

Key Words: two-dimensional Finsler space, β-change, homothetic change, conformal change, one form metric, main scalar, scalar curvature, geodesic.

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§1. Introduction

Let M^n be an n-dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x,y), (y^i = \dot{x}^i)$ of M^n. If a differential 1-form $\beta(x,y) = b_i(x)y^i$ is given on M^n, then M. Matsumoto [1] introduced another Finsler space whose fundamental function is given by

$$\bar{L}(x,y) = L(x,y) + \beta(x,y)$$

This change of Finsler metric has been called β-change [2,3].

The conformal theory of Finsler spaces has been initiated by M.S. Knebelman [4] in 1929 and has been investigated in detail by many authors [5-8] etc. The conformal change is defined as

$$L(x,y) \to e^{\sigma(x)} L(x,y),$$

where $\sigma(x)$ is a function of position only and known as conformal factor.

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In the present paper, we construct a theory which generalizes all the above mentioned changes. In fact, we consider a change of the form

$$L(x,y) \to L^*(x,y) = e^{\sigma(x)}L(x,y) + \beta(x,y), \tag{1}$$

where $\sigma(x)$ is a function of x and $\beta(x,y) = b_i(x)y^i$ is a 1- form on M^n, which we call a Randers conformal change. This change generalizes various types of changes. When $\beta = 0$, it reduces to a conformal change. When $\sigma = 0$, it reduces to a Randers change. When $\beta = 0$ and σ is a non-zero constant then it reduces to homothetic change.

In the present paper we have obtained the relations between

(1) the main scalars of F^2 and F^{*2};

(2) the scalar curvatures of F^2 and F^{*2}.

Further, we have derived the equation of geodesic for F^{*n}.

§2. Randers Conformal Change

Definition 2.1 *Let (M^n, L) be a Finsler space F^n, where M^n is an n-dimensional differentiable manifold equipped with a fundamental function L. A change in fundamental metric L, defined by equation (1), is called Randers conformal change, where $\sigma(x)$ is conformal factor and function of position only and $\beta(x,y) = b_i(x)y^i$ is a 1- form on M^n. A space equipped with fundamental metric $L^*(x,y)$ is called Randers conformally changed space F^{*n}.*

This change generalizes various changes studied by Randers [11], Matsumuto [12], Shibata [13], Pandey [10] etc. Differentiating equation (1) with respect to y^i, the normalized supporting element $l_i^* = \dot{\partial}_i L^*$ is given by

$$l_i^*(x,y) = e^{\sigma(x)}l_i(x,y) + b_i(x), \tag{2}$$

where $l_i = \dot{\partial}_i L$ is the normalized supporting element in the Finsler space F^n. Differentiating (2) with respect to y^j, the angular metric tensor $h_{ij}^* = L^*\dot{\partial}_i\dot{\partial}_j L^*$ is given by

$$h_{ij}^* = e^{\sigma(x)}\frac{L^*}{L}h_{ij} \tag{3}$$

where $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ is the angular metric tensor in the Finsler space F^n .

Again the fundamental tensor $g_{ij}^* = \dot{\partial}_i\dot{\partial}_j\frac{L^{*2}}{2} = h_{ij}^* + l_i^* l_j^*$ is given by

$$g_{ij}^* = \tau g_{ij} + b_i b_j + e^{\sigma(x)}L^{-1}(b_i y_j + b_j y_i) - \beta e^{\sigma(x)}L^{-3}y_i y_j \tag{4}$$

where we put $y_i = g_{ij}(x,y)y^j$, $\tau = e^{\sigma(x)}\frac{L^*}{L}$ and g_{ij} is the fundamental tensor of the Finsler space F^n. It is easy to see that the $\det(g_{ij}^*)$ does not vanish, and the reciprocal tensor with components g^{*ij} is given by

$$g^{*ij} = \tau^{-1}g^{ij} + \phi y^i y^j - L^{-1}\tau^{-2}(y^i b^j + y^j b^i) \tag{5}$$

where $\phi = e^{-2\sigma(x)}(Le^{\sigma(x)}b^2 + \beta)L^{*-3}$, $b^2 = b_i b^i$, $b^i = g^{ij}b_j$ and g^{ij} is the reciprocal tensor of g_{ij}. Here it will be more convenient to use the tensors

$$h_{ij} = g_{ij} - L^{-2}y_i y_j, \qquad a_i = \beta L^{-2}y_i - b_i \tag{6}$$

both of which have the following interesting property:

$$h_{ij}y^j = 0, \qquad a_i y^i = 0 \tag{7}$$

Now differentiating equation (4) with respect to y^k and using relation (6), the Cartan covariant tensor C^* with the components $C^*_{ijk} = \dot{\partial}_k(\frac{g^*_{ij}}{2})$ is given as:

$$C^*_{ijk} = \tau[C_{ijk} - \frac{1}{2L^*}(h_{ij}a_k + h_{jk}a_i + h_{ki}a_j)] \tag{8}$$

where C_{ijk} is (h)hv-torsion tensor of Cartan's connection $C\Gamma$ of Finsler space F^n.

In order to obtain the tensor with the components C^*_{ijk}, paying attention to (7), we obtain from (5) and (8),

$$\begin{aligned} C^{*j}_{ik} &= C^j_{ik} - \frac{1}{2L^*}(h^j_i a_k + h^j_k a_i + h_{ik}a^j) \\ &\quad -(\tau L)^{-1}C_{ikr}y^j b^r - \frac{\tau^{-1}}{2LL^*}(2a_i a_j + a^2 h_{ij})y^j \end{aligned} \tag{9}$$

where $a_i a^i = a^2$.

Proposition 2.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the normalized supporting element l^*_i, angular metric tensor h^*_{ij}, fundamental metric tensor g^*_{ij} and (h)hv-torsion tensor C^*_{ijk} of F^{*n} are given by (2), (3), (4) and (8) respectively.*

§3. Main Scalar of Randers Conformally Changed Two-Dimensional Finsler Space

The (h)hv-torsion tensor for a two-dimensional Finsler space F^2 is given by [9]:

$$C_{ijk} = Im_i m_j m_k \tag{10}$$

where $I = C_{222}$ is the main scalar of F^2.

Similarly, the (h)hv-torsion tensor for a two-dimensional Finsler space F^{*2} is given by

$$C^*_{ijk} = I^* m^*_i m^*_j m^*_k \tag{11}$$

where I^* is the main scalar of F^{*2}, and m^*_i is unit vector orthogonal to l^*_i in two-dimensional Finsler space.

Putting $j = k$ in equation (9), we get

$$C^*_i = C_i - \frac{(n+1)}{2L^*}a_i \tag{12}$$

The normalized torsion vectors are $m^i = \frac{C^i}{C}$ in F^2 and $m^{*i} = \frac{C^{*i}}{C^*}$ in F^{*2}, where C and C^* are the lengths of C^i and C^{*i} in F^2 and F^{*2} respectively. The equation (12) can also be written as

$$m_i^* = \lambda m_i + \mu a_i \tag{13}$$

where $\lambda = \frac{C}{C^*}$ and $\mu = -\frac{(n+1)}{2C^*}L^{*-1}$.

Now

$$C^{*2} = g^{*ij}C_i^*C_j^* = \tau^{-1}[C^2 + \frac{(n+1)}{L^*}A_\gamma], \tag{14}$$

where $A_\gamma = C_\gamma + \frac{(n+1)}{4L^*}a^2$ and $C_\gamma = C_i b^i$ are scalars.

The contravariant components of l_i^* and m_i^* are given below:

$$l^{*i} = g^{*ij}l_j^* = Al^i + Bb^i \tag{15}$$

where $A = e^{\sigma(x)}\tau^{-1} - \tau{-}2\beta e^{\sigma(x)} + \beta\phi L - b^2\tau^{-2} + e^{\sigma(x)}L^2$ and $B = (-e^{\sigma(x)}\tau^{-2} - \tau^{-1} - \beta L^{-1}\tau^{-2})$ are scalars, $l_i l^i = 1$ and $b_i l^i = b^i l_i = L\beta$. Also

$$m^{*i} = Dm^i + El^i + Fa^i \tag{16}$$

where $D = \tau^{-1}\lambda$, $E = (-\tau^{-2}\lambda H - \tau^{-2}\mu(\beta^2 L^{-1} - b^2))$, $F = \mu\tau^{-1}$ and $H = m_i b^i$ are scalars. Hence, we have

Proposition 3.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then contravariant components of the Berwald frame (l, m) in two-dimensional Finsler space are given by (15) and (16), whereas covariant components are given by (2) and (13) respectively.*

Proposition 3.2 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the relationship between the lengths of the components C_i and C_i^* is given by (14).*

Since the (h)hv-torsion tensor given by (8) can be rewritten in two-dimensional form as follows:

$$I^* m_i^* m_j^* m_k^* = \tau[I m_i m_j m_k - \frac{3}{2L^*}a_2 m_i m_j m_k] \tag{17}$$

where $h_{ij} = m_i m_j$ and $a_i = a_1 l_i + a_2 m_i$, then $a_i y^i = 0 \Longrightarrow a_1 = 0$. So, $a_i = a_2 m_i$, a_1 and a_2 are certain scalars.

From equations (13) and (17), we have

$$I^*(\lambda + \mu a_2)^3 m_i m_j m_k = \tau[I m_i m_j m_k - \frac{3}{2L^*}a_2 m_i m_j m_k] \tag{18}$$

Contracting (18) by $m_i m_j m_k$, we have

$$I^* = \frac{\tau}{(\lambda + \mu a_2)^3}[I - \frac{3}{2L^*}a_2] \tag{19}$$

Theorem 3.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the relationship between the Main scalars I^* and I of the Finsler space F^{*2} and F^2 is given by (19).*

Corollary 3.1 *For $\sigma(x) = 0$, i.e. for Randers change, the relationship between the Main scalars I^* and I of the Finsler space F^{*2} and F^2 is given by [10]:*

$$I^* = \frac{(L+\beta)L^{-1}}{(\lambda+\mu a_2)^3} I - \frac{3L^{-1}}{2(\lambda+\mu a_2)^3} a_2.$$

Corollary 3.2 *For $\beta = 0$, i.e. for conformal change, the relationship between the Main scalars I^* and I of the Finsler space F^{*2} and F^2 is given by*

$$I^* = \frac{e^{\sigma(x)}}{\lambda^3} I.$$

Corollary 3.3 *For $\beta = 0$ and $\sigma =$ a non-zero constant i.e. for homothetic change, the relationship between the Main scalars I^* and I of the Finsler space F^{*2} and F^2 is given by*

$$I^* = \frac{e^{\sigma}}{\lambda^3} I.$$

§4. Geodesic of Randers Conformally Changed Space

Let s be the arc-length, then the equation of a geodesic [14] of $F^n = (M^n, L)$ is written in the well-known form:

$$\frac{d^2x^i}{ds^2} + 2G^i(x, \frac{dx}{ds}) = 0, \tag{20}$$

where functions $G^i(x, y)$ are given by

$$2G^i = g^{ir}(y^j \dot{\partial}_r \partial_j F - \partial_r F), \qquad F = \frac{L^2}{2}.$$

Now suppose s^* is the arc-length in the Finsler space $F^{*n} = (M^n, L^*)$, then the equation of geodesic in F^{*n} can be written as

$$\frac{d^2x^i}{ds^{*2}} + 2G^{*i}(x, \frac{dx}{ds^*}) = 0, \tag{21}$$

where functions $G^{*i}(x, y)$ are given by

$$2G^{*i} = g^{*ir}(y^j \dot{\partial}_r \partial_j F^* - \partial_r F^*), \qquad F^* = \frac{L^{*2}}{2}.$$

Since $ds^* = L^*(x, dx)$, this is also written as

$$ds^* = e^{\sigma(x)} L(x, dx) + b_i(x) dx^i = e^{\sigma(x)} ds + b_i(x) dx^i$$

Since $ds = L(x, dx)$, we have

$$\frac{dx^i}{ds} = \frac{dx^i}{ds^*}[e^{\sigma(x)} + b_i \frac{dx^i}{ds}] \tag{22}$$

Differentiating (22) with respect to s, we have

$$\frac{d^2x^i}{ds^2}=\frac{d^2x^i}{ds^{*2}}[e^{\sigma(x)}+b_i\frac{dx^i}{ds}]^2+\frac{dx^i}{ds^*}(\frac{de^{\sigma(x)}}{ds}+\frac{db_i}{ds}\frac{dx^i}{ds}+b_i\frac{d^2x^i}{ds^2}).$$

Substituting the value of $\frac{dx^i}{ds^*}$ from (22), the above equation becomes

$$\begin{aligned}\frac{d^2x^i}{ds^2} &= \frac{d^2x^i}{ds^{*2}}[e^{\sigma(x)}+b_i\frac{dx^i}{ds}]^2\\ &\quad+\frac{\frac{dx^i}{ds}}{[e^{\sigma(x)}+b_i\frac{dx^i}{ds}]}(\frac{de^{\sigma(x)}}{ds}+\frac{db_i}{ds}\frac{dx^i}{ds}+b_i\frac{d^2x^i}{ds^2})\end{aligned} \tag{23}$$

Since $2G^{*i}=g^{*ir}(y^j\dot{\partial}_r\partial_j\frac{L^{*2}}{2}-\partial_r\frac{L^{*2}}{2})$, we have

$$\begin{aligned}2G^*_i &= e^{2\sigma(x)}G_i+y^j[e^{\sigma(x)}L\dot{\partial}_i(\partial_je^{\sigma(x)})L+e^{\sigma(x)}L\dot{\partial}_i\partial_j\beta+\\ &\quad\beta\dot{\partial}_i(\partial_je^{\sigma(x)})L+\beta e^{\sigma(x)}\dot{\partial}_i\partial_jL+\beta\dot{\partial}_i\partial_j\beta+(e^{\sigma(x)}l_i+b_i)((\partial_je^{\sigma(x)})L\\ &\quad+\partial_j\beta)+e^{\sigma(x)}b_r\partial_jL]-[e^{\sigma(x)}L(\partial_ie^{\sigma(x)})L+e^{\sigma(x)}L\partial_i\beta+\beta\partial_i(e^{\sigma(x)})L\\ &\quad+\beta e^{\sigma(x)}\partial_iL+\beta\partial_i\beta]\end{aligned} \tag{24}$$

Now we have

$$2G^{*i}=g^{*ir}G^*_r=JG^i+M^i \tag{25}$$

where $J=e^{2\sigma(x)}\tau^{-1}$ and

$$\begin{aligned}M^i &= e^{2\sigma(x)}G_r[\phi y^iy^r-L^{-1}\tau^{-2}(y^ib^r+y^rb^i)]+[\tau^{-1}g^{ir}+\phi y^iy^r\\ &\quad-L^{-1}\tau^{-2}(y^ib^r+y^rb^i)][y^j[e^{\sigma(x)}L\dot{\partial}_r(\partial_je^{\sigma(x)})L+e^{\sigma(x)}L\dot{\partial}_r\partial_j\beta\\ &\quad+\beta\dot{\partial}_r(\partial_je^{\sigma(x)})L+\beta e^{\sigma(x)}\dot{\partial}_r\partial_jL+\beta\dot{\partial}_r\partial_j\beta+(e^{\sigma(x)}l_r+b_r)((\partial_je^{\sigma(x)})L\\ &\quad+\partial_j\beta)+e^{\sigma(x)}b_r\partial_jL]-[e^{\sigma(x)}L(\partial_re^{\sigma(x)})L+e^{\sigma(x)}L\partial_r\beta+\beta\partial_r(e^{\sigma(x)})L\\ &\quad+\beta e^{\sigma(x)}\partial_rL+\beta\partial_r\beta]\end{aligned} \tag{26}$$

Proposition 4.1 *Let $F^{*n}=(M^n,L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n=(M^n,L)$, then the relationship between the Berwald connection function G^{*i} and G^i is given by (25).*

Theorem 4.1 *Let $F^{*n}=(M^n,L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n=(M^n,L)$, then the equation of geodesic of F^{*n} is given by (21), where $\frac{d^2x^i}{ds^{*2}}$ and G^{*i} are given by (23) and (25) respectively.*

Corollary 4.1 *For $\sigma(x)=0$, i.e. for Randers change, the equation of geodesic of F^{*n} is given by (21), where $\frac{d^2x^i}{ds^{*2}}$ and G^{*i} are given below [10]:*

$$\frac{d^2x^i}{ds^2}=\frac{d^2x^i}{ds^{*2}}[1+b_i\frac{dx^i}{ds}]^2+\frac{dx^i}{ds^*}(\frac{db_i}{ds}\frac{dx^i}{ds}+b_i\frac{d^2x^i}{ds^2})$$

and

$$\begin{aligned}2G^{*i} \;=\;& L(L+\beta)^{-1}G^i + G_r[-L(L+\beta)^{-2}((y^ib^r+y^rb^i)+(Lb^2+\beta)(L+\beta)^{-3}y^iy^r]\\ &+[L(L+\beta)^{-1}g^{ir} - L(L+\beta)^{-2}((y^ib^r+y^rb^i)+(Lb^2+\beta)(L+\beta)^{-3}y^iy^r][y^j(2L\partial_jb_r\\ &+\beta\dot{\partial}_j\partial_rL+2\beta\partial_jb_r+b_r\partial_jL+)-(\beta l_r+(L+\beta)\partial_jb_ry^j)].\end{aligned}$$

Corollary 4.2 *For $\beta = 0$, i.e. for conformal change, the equation of geodesic of F^{*n} is given by (21), where $\frac{d^2x^i}{ds^{*2}}$ and G^{*i} are given below:*

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{ds^{*2}}e^{2\sigma(x)} + \frac{dx^i}{ds^*}\frac{de^{\sigma(x)}}{ds}$$

and

$$2G^{*i} = G^i + e^{-2\sigma(x)}g^{ir}[y^j[e^{\sigma(x)}L\dot{\partial}_r(\partial_je^{\sigma(x)})L + e^{\sigma(x)}l_r(\partial_je^{\sigma(x)})L] - \quad e^{\sigma(x)}L(\partial_re^{\sigma(x)})L].$$

Corollary 4.3 *For $\beta = 0$ and $\sigma =$ a non-zero constant i.e. for homothetic change, the equation of geodesic of F^{*n} is given by (21), where $\dfrac{d^2x^i}{ds^{*2}}$ and G^{*i} are given below*

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{ds^{*2}}e^{2\sigma}$$

and $2G^{*i} = G^i$.

§5. Scalar Curvature of Randers Conformally Changed Two-Dimensional Finsler Space

The (v)h-torsion tensor R^i_{jk} in two-dimensional Finsler space may be written as [9]

$$R^i_{jk} = LRm^i(l_jm_k - l_km_j), \tag{27}$$

where R is the h-scalar curvature in F^2.

Similarly the (v)h-torsion tensor R^{*i}_{jk} in Finsler space F^{*2} is given by

$$R^{*i}_{jk} = L^*R^*m^{*i}(l^*_jm^*_k - l^*_km^*_j), \tag{28}$$

where R^* is the h-scalar curvature in F^{*2}. If we are concerned with Berwald connection $B\Gamma$, the non-vanishing (v)h-torsion tensor R^i_{jk} [9] is given as

$$R^i_{jk} = \delta_kG^i_j - \delta_jG^i_k = \partial_kG^i_j - \partial_jG^i_k + G^r_jG^i_{rk} - G^r_kG^i_{rj}, \tag{29}$$

where $\delta_i = \partial_i - G^r_i\dot{\partial}_r$, $G^i_j = \dot{\partial}_jG^i$ and $G^i_{jk} = \dot{\partial}_kG^i_j$.

Similarly the (v)h-torsion tensor R^{*i}_{jk} for Berwald connection $B\Gamma$ in F^{*n} is

$$R^{*i}_{jk} = \delta_kG^{*i}_j - \delta_jG^{*i}_k = \partial_kG^{*i}_j - \partial_jG^{*i}_k + G^{*r}_jG^{*i}_{rk} - G^{*r}_kG^{*i}_{rj}, \tag{30}$$

where $\delta_i = \partial_i - G_i^{*r}\dot{\partial}_r$, $G_j^{*i} = \dot{\partial}_j G^{*i}$ and $G_k^{*i} = \dot{\partial}_k G_j^{*i}$.

Using relation (25) we have

$$G_j^{*i} = \dot{\partial}_j G^{*i} = \frac{1}{2}[JG_j^i + M_j^i], \tag{31}$$

where $\dot{\partial}_j M^i = M_j^i$, and

$$G_{jk}^{*i} = \dot{\partial}_k G_j^{*i} = \frac{1}{2}[JG_{jk}^i + M_{jk}^i], \tag{32}$$

where $\dot{\partial}_k M_j^i = M_{jk}^i$.

Using equation (30) and (31) in (29), we have

$$\begin{aligned} R_{jk}^{*i} \quad = \quad & \frac{J}{2}[\partial_k G_j^i - \partial_j G_k^i] + \frac{1}{2}[\partial_k M_j^i - \partial_j M_k^i] + \frac{J^2}{4}[G_j^r G_{kr}^i - G_k^r G_{jr}^i] \\ & + \frac{J}{2}[G_j^r M_{kr}^i + M_j^r G_{kr}^i - G_k^r M_{jr}^i - M_k^r G_{jr}^i] + [M_j^r M_{kr}^i - M_k^r M_{jr}^i] \end{aligned} \tag{33}$$

From equation (27) we have

$$\frac{R_{jk}^{*i}}{R^*} = L^* m^{*i}(l_j^* m_k^* - l_k^* m_j^*).$$

In view of (1), (2), (13) and (16), we have

$$\begin{aligned} \frac{R_{jk}^{*i}}{L^* R^*} \quad = \quad & D\lambda e^{\sigma(x)} m^i (l_j m_k - l_k m_j) + D\lambda m^i (b_j m_k - b_k m_j) \\ & + (El^i + \mu e^{\sigma(x)} m^i (l_j b_k - l_k b_j) \\ & + Fa^i)[\lambda e^{\sigma(x)} (l_j m_k - l_k m_j) \mu e^{\sigma(x)} (l_j b_k - l_k b_j) + \lambda (b_j m_k - b_k m_j)] \end{aligned} \tag{34}$$

Using (26), (28), (32) and (33), we have

$$\begin{aligned} & \frac{1}{R^*}(\frac{J}{2}[\partial_k G_j^i - \partial_j G_k^i] + \frac{1}{2}[\partial_k M_j^i - \partial_j M_k^i] + \frac{J^2}{4}[G_j^r G_{kr}^i - G_k^r G_{jr}^i] \\ & + \frac{J}{2}[G_j^r M_{kr}^i + M_j^r G_{kr}^i - G_k^r M_{jr}^i - M_k^r G_{jr}^i] + [M_j^r M_{kr}^i - M_k^r M_{jr}^i]) \\ & = \frac{D\lambda\tau}{R}(\partial_k G_j^i - \partial_j G_k^i + G_j^r G_{rk}^i - G_k^r G_{rj}^i) + (e^{\sigma(x)} L + \beta)(\mu e^{\sigma(x)} m^i (l_j b_k \\ & \quad - l_k b_j) + D\lambda m^i (b_j m_k - b_k m_j) + (El^i + Fa^i)[\lambda e^{\sigma(x)} (l_j m_k - l_k m_j) \\ & \quad + \mu e^{\sigma(x)} (l_j b_k - l_k b_j) + \lambda (b_j m_k - b_k m_j)]) \end{aligned} \tag{35}$$

Theorem 5.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the relationship between scalar curvatures of the Finsler space F^{*2} and F^2 is given by (34).*

Corollary 5.1 *For $\sigma(x) = 0$, i.e. for Randers change, the relationship between scalar curvatures* *of the Finsler space F^{*2} and F^2 is given as [10]:*

$$\frac{1}{R^*}(\frac{J_1}{2}[\partial_k G^i_j - \partial_j G^i_k] + \frac{1}{2}[\partial_k M^i_{1j} - \partial_j M^i_{1k}] + \frac{J_1^2}{4}[G^r_j G^i_{kr} - G^r_k G^i_{jr}]$$
$$+\frac{J_1}{2}[G^r_j M^i_{1kr} + M^r_{1j} G^i_{kr} - G^r_k M^i_{1jr} - M^r_{1k} G^i_{jr}] + [M^r_{1j} M^i_{1kr} - M^r_{1k} M^i_{1jr}])$$
$$= \frac{D_1 \lambda \tau_1}{R}(\partial_k G^i_j - \partial_j G^i_k + G^r_j G^i_{rk} - G^r_k G^i_{rj}) + (L+\beta)(\mu_1 m^i (l_j b_k - l_k b_j)$$
$$+D_1 \lambda m^i (b_j m_k - b_k m_j) + (E_1 l^i + F_1 a^i)[\lambda(l_j m_k - l_k m_j) + \mu_1 (l_j b_k - l_k b_j)$$
$$+\lambda(b_j m_k - b_k m_j)]]),$$

where

$$J_1 = \frac{L(L+\beta)^{-1}}{2}, \quad \tau_1 = \frac{L+\beta}{L}, \quad \mu_1 = -\frac{(n+1)}{2C^*}(L+\beta)^{-1},$$
$$D_1 = \frac{L}{L+\beta}\frac{C}{C^*}, \quad E_1 = -(\frac{L+\beta}{L})^{-2}(\lambda H + \mu_1(\beta^2 L^{-1} - b^2)), \quad F_1 = \mu_1 \frac{L}{L+\beta}$$

and

$$M^i_1 = \frac{1}{2}[G_r[-L(L+\beta)^{-2}((y^i b^r + y^r b^i) + (Lb^2+\beta)(L+\beta)^{-3} y^i y^r]$$
$$+[L(L+\beta)^{-1} g^{ir} - L(L+\beta)^{-2}((y^i b^r + y^r b^i) + (Lb^2+\beta)(L+\beta)^{-3} y^i y^r]$$
$$\times[y^j(2L\partial_j b_r + \beta\dot{\partial}_j \partial_r L + 2\beta \partial_j b_r + b_r \partial_j L) - (\beta l_r + (L+\beta)\partial_j b_r y^j)]],$$

$$M^i_{1j} = \dot{\partial}_j M^i_1, \quad M^i_{1jk} = \dot{\partial}_k M^i_{1j}.$$

Corollary 5.2 *For $\beta = 0$, i.e. for conformal change, the relationship between scalar curvatures of the Finsler space F^{*2} and F^2 is given as:*

$$\frac{1}{R^*}(\frac{1}{2}[\partial_k G^i_j - \partial_j G^i_k] + \frac{1}{2}[\partial_k M^i_{2j} - \partial_j M^i_{2k}] + \frac{1}{4}[G^r_j G^i_{kr} - G^r_k G^i_{jr}]$$
$$+\frac{1}{2}[G^r_j M^i_{2kr} + M^r_{2j} G^i_{kr} - G^r_k M^i_{2jr} - M^r_{2k} G^i_{jr}] + [M^r_{2j} M^i_{2kr} - M^r_{2k} M^i_{2jr}])$$
$$= \frac{D_2 \lambda \tau_2}{R}(\partial_k G^i_j - \partial_j G^i_k + G^r_j G^i_{rk} - G^r_k G^i_{rj}),$$

where

$$\tau_2 = e^{\sigma(x)}, \quad D_2 = e^{-\sigma(x)}\frac{C}{C^*}$$

and

$$M^i_2 = e^{-2\sigma(x)} g^{ir}[y^j[e^{\sigma(x)} L \dot{\partial}_r(\partial_j e^{\sigma(x)})L + e^{\sigma(x)} l_r(\partial_j e^{\sigma(x)})L] - e^{\sigma(x)} L(\partial_r e^{\sigma(x)})L],$$
$$M^i_{2j} = \dot{\partial}_j M^i_2, \quad M^i_{2jk} = \dot{\partial}_k M^i_{2j}.$$

Corollary 5.3 *For $\beta = 0$ and $\sigma =$ a non-zero constant i.e. for homothetic change, the relationship between scalar curvatures of the Finsler space F^{*2} and F^2 is given as:*

$$\frac{1}{R^*}(\frac{1}{2}[\partial_k G^i_j - \partial_j G^i_k] + \frac{1}{4}[G^r_j G^i_{kr} - G^r_k G^i_{jr}]) = \frac{D_3 \lambda \tau_3}{R}(\partial_k G^i_j - \partial_j G^i_k + G^r_j G^i_{rk} - G^r_k G^i_{rj}),$$

where

$$\tau_3 = e^{\sigma}, D_3 = e^{-\sigma}\frac{C}{C^*}.$$

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On the Forcing Hull and Forcing Monophonic Hull Numbers of Graphs

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Abstract: For a connected graph $G = (V, E)$, let a set M be a minimum monophonic hull set of G. A subset $T \subseteq M$ is called a forcing subset for M if M is the unique minimum monophonic hull set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing monophonic hull number of M , denoted by $f_{mh}(M)$, is the cardinality of a minimum forcing subset of M. The forcing monophonic hull number of G, denoted by $f_{mh}(G)$, is $f_{mh}(G) = min\{f_{mh}(M)\}$, where the minimum is taken over all minimum monophonic hull sets in G. Some general properties satisfied by this concept are studied. Every monophonic set of G is also a monophonic hull set of G and so $mh(G) \leq h(G)$, where $h(G)$ and $mh(G)$ are hull number and monophonic hull number of a connected graph G. However, there is no relationship between $f_h(G)$ and $f_{mh}(G)$, where $f_h(G)$ is the forcing hull number of a connected graph G. We give a series of realization results for various possibilities of these four parameters.

Key Words: hull number, monophonic hull number, forcing hull number, forcing monophonic hull number, Smarandachely geodetic k-set, Smarandachely hull k-set.

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§1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Harary [1,9]. A convexity on a finite set V is a family C of subsets of V, convex sets which is closed under intersection and which contains both V and the empty set. The pair(V, E) is called a convexity space. A finite graph convexity space is a pair (V, E), formed by a finite connected graph $G = (V, E)$ and a convexity C on V such that (V, E) is a convexity space satisfying that every member of C induces a connected subgraph of G. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of R^n is convex if

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every segment joining two points of X is entirely contained in it. Similarly a vertex set W of a finite connected graph is said to be convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W[2,8]. The distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. An $u-v$ path of length $d(u,v)$ is called an $u-v$ geodesic. A vertex x is said to lie on a $u-v$ geodesic P if x is a vertex of P including the vertices u and v. For two vertices u and v, let $I[u,v]$ denotes the set of all vertices which lie on $u-v$ geodesic. For a set S of vertices, let $I[S]=\bigcup_{(u,v)\in S} I[u,v]$. The set S is convex if $I[S]=S$. Clearly if $S=\{v\}$or $S=V$, then S is convex. The convexity number, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of V. The smallest convex set containing S is denoted by $I_h(S)$ and called the convex hull of S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S\subseteq I[S]\subseteq I_h[S]\subseteq V$. For an integer $k\geq 0$, a subset $S\subseteq V$ is called a *Smarandachely geodetic k-set* if $I[S\bigcup S^+]=V$ and a *Smarandachely hull k-set* if $I_h(S\bigcup S^+)=V$ for a subset $S^+\subset V$ with $|S^+|\leq k$. Particularly, if $k=0$, such Smarandachely geodetic 0-set and Smarandachely hull 0-set are called the *geodetic set* and *hull set*, respectively. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a minimum geodetic set or simply a g- set of G. Similarly, the hull number $h(G)$ of G is the minimum order of its hull sets and any hull set of order $h(G)$ is a minimum hull set or simply a h- set of G. The geodetic number of a graph is studied in [1,4,10] and the hull number of a graph is studied in [1,6].A subset $T\subseteq S$ is called a forcing subset for S if S is the unique minimum hull set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of M. The forcing hull number of S, denoted by $f_h(S)$, is the cardinality of a minimum forcing subset of S. The forcing hull number of G, denoted by $f_h(G)$,is $f_h(G)=min\{f_h(S)\}$, where the minimum is taken over all minimum hull sets S in G. The forcing hull number of a graph is studied in[3,14]. A chord of a path $u_o,u_1,u_2,...,u_n$ is an edge u_iu_j with $j\geq i+2(0\leq i,j\leq n)$. A $u-v$ path P is called a monophonic path if it is a chordless path. A vertex x is said to lie on a $u-v$ monophonic path P if x is a vertex of P including the vertices u and v. For two vertices u and v, let $J[u,v]$ denotes the set of all vertices which lie on $u-v$ monophonic path. For a set M of vertices, let $J[M]=\cup_{u,v\in M}J[u,v]$. The set M is monophonic convex or m-convex if $J[M]=M$. Clearly if $M=\{v\}$ or $M=V$, then M is m-convex. The m-convexity number, denoted by $C_m(G)$, is the cardinality of a maximum proper m-convex subset of V. The smallest m-convex set containing M is denoted by $J_h(M)$ and called the monophonic convex hull or m-convex hull of M. Since the intersection of two m-convex set is m-convex, the m-convex hull is well defined. Note that $M\subseteq J[M]\subseteq J_h(M)\subseteq V$. A subset $M\subseteq V$ is called a monophonic set if $J[M]=V$ and a m-hull set if $J_h(M)=V$. The monophonic number $m(G)$ of G is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a m- set of G. Similarly, the monophonic hull number $mh(G)$ of G is the minimum order of its m-hull sets and any m-hull set of order $mh(G)$ is a minimum monophonic set or simply a mh- set of G. The monophonic number of a graph is studied in [5,7,11,15] and the monophonic hull number of a graph is studied in [12]. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is complete.Let G be a connected graph and M a minimum monophonic hull set of G. A subset $T\subseteq M$ is called a forcing subset for M

if M is the unique minimum monophonic hull set containing T. A forcing subset for M of minimum cardinality is a minimum forcing subset of M. The forcing monophonic hull number of M, denoted by $f_{mh}(M)$, is the cardinality of a minimum forcing subset of M. The forcing monophonic hull number of G, denoted by $f_{mh}(G)$, is $f_{mh}(G) = min\{f_{mh}(M)\}$, where the minimum is taken over all minimum monophonic hull sets M in G.For the graph G given in Figure 1.1, $M = \{v_1, v_8\}$ is the unique minimum monophonic hull set of G so that $mh(G) = 2$ and $f_{mh}(G) = 0$. Also $S_1 = \{v_1, v_5, v_8\}$ and $S_2 = \{v_1, v_6, v_8\}$ are the only two h-sets of G such that $f_h(S_1) = 1, f_h(S_2) = 1$ so that $f_h(G) = 1$. For the graph G given in Figure 1.2, $M_1 = \{v_1, v_4\}, M_2 = \{v_1, v_6\}, M_3 = \{v_1, v_7\}$ and $M_4 = \{v_1, v_8\}$ are the only four mh-sets of G such that $f_{mh}(M_1) = 1, f_{mh}(M_2) = 1, f_{mh}(M_3) = 1$ and $f_{mh}(M_4) = 1$ so that $f_{mh}(G) = 1$. Also, $S = \{v_1, v_7\}$ is the unique minimum hull set of G so that $h(G) = 2$ and $f_h(G) = 0$. Throughout the following G denotes a connected graph with at least two vertices.

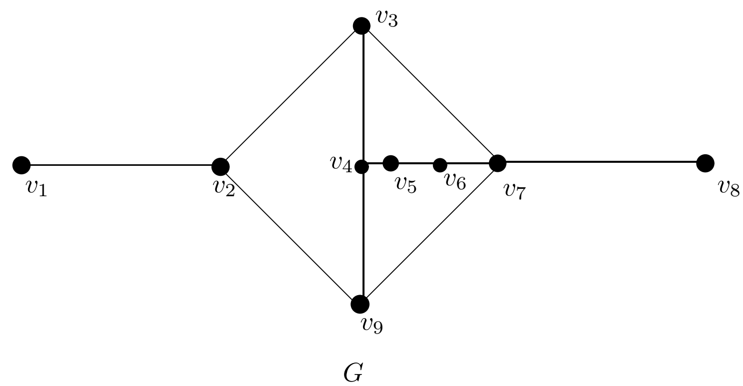

Figure 1.1

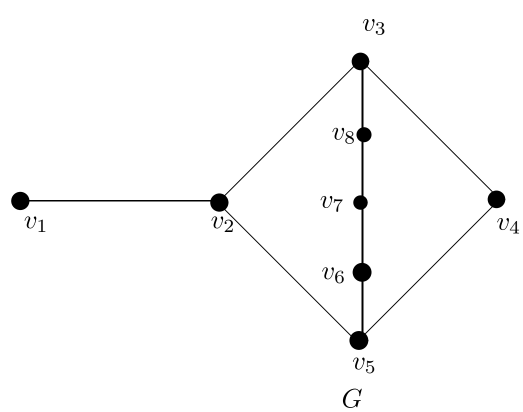

Figure 1.2

The following theorems are used in the sequel

Theorem 1.1 ([6]) *Let G be a connected graph. Then*

a) Each extreme vertex of G belongs to every hull set of G;

(b) $h(G) = p$ *if and only if* $G = K_p$.

Theorem 1.2 ([3]) *Let* G *be a connected graph. Then*

(a) $f_h(G) = 0$ *if and only if* G *has a unique minimum hull set;*

(b) $f_h(G) \leq h(G) - |W|$, *where* W *is the set of all hull vertices of* G.

Theorem 1.3 ([13]) *Let* G *be a connected graph. Then*

(a) Each extreme vertex of G *belongs to every monophonic hull set of* G*;*

(b) $mh(G) = p$ *if and only if* $G = K_p$.

Theorem 1.4 ([12]) *Let* G *be a connected graph. Then*

(a) $f_{mh}(G) = 0$ *if and only if* G *has a unique mh-set;*

(b) $f_{mh}(G) \leq mh(G) - |S|$, *where* S *is the set of all monophonic hull vertices of* G.

Theorem 1.5 ([12]) *For any complete Graph* $G = K_p (p \geq 2), f_{mh}(G) = 0$.

§2. Special Graphs

In this section, we present some graphs from which various graphs arising in theorem are generated using identification.

Let $U_i : \alpha_i, \beta_i, \gamma_i, \delta_i, \alpha_i (1 \leq i \leq a)$ be a copy of cycle C_4. Let V_i be the graph obtained from U_i by adding three new vertices η_i, f_i, g_i and the edges $\beta_i\eta_i, \eta_i f_i, f_i g_i, g_i\delta_i, \eta_i\gamma_i, f_i\gamma_i, g_i\gamma_i (1 \leq i \leq a)$. The graph T_a given in Figure 2.1 is obtained from V_i's by identifying γ_{i-1} of V_{i-1} and α_i of $V_i (2 \leq i \leq a)$.

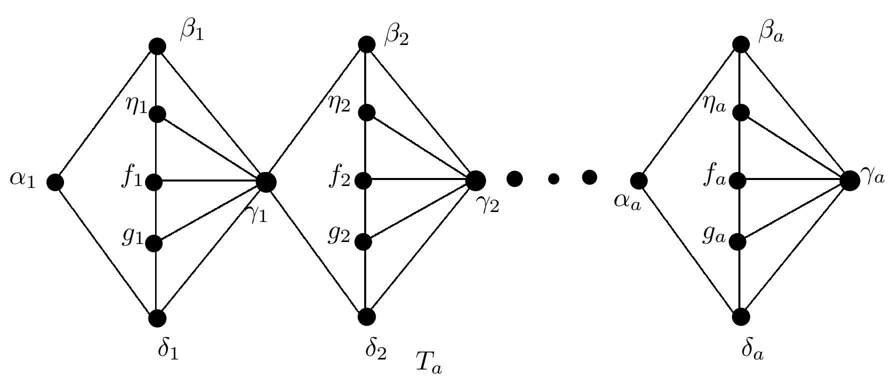

Figure 2.1

Let $P_i : k_i, l_i, m_i, n_i, k_i (1 \leq i \leq b)$ be a copy of cycle C_4. Let Q_i be the graph obtained from P_i by adding three new vertices h_i, p_i and q_i and the edges $l_i h_i, h_i p_i, p_i q_i$, and $q_i m_i (1 \leq i \leq b)$. The graph W_b given in Figure 2.2 is obtained from Q_i's by identifying m_{i-1} of Q_{i-1} and k_i of $Q_i (2 \leq i \leq b)$.

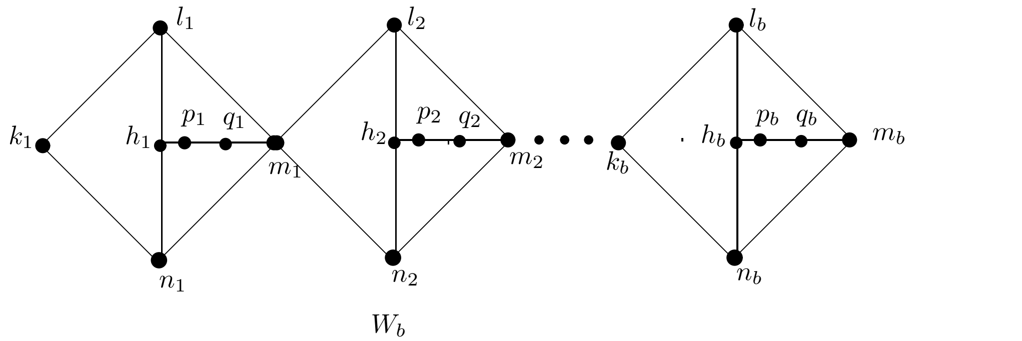

Figure 2.2

The graph Z_b given in Figure 2.3 is obtained from W_b by joining the edge $l_in_i(1 \leq i \leq b)$.

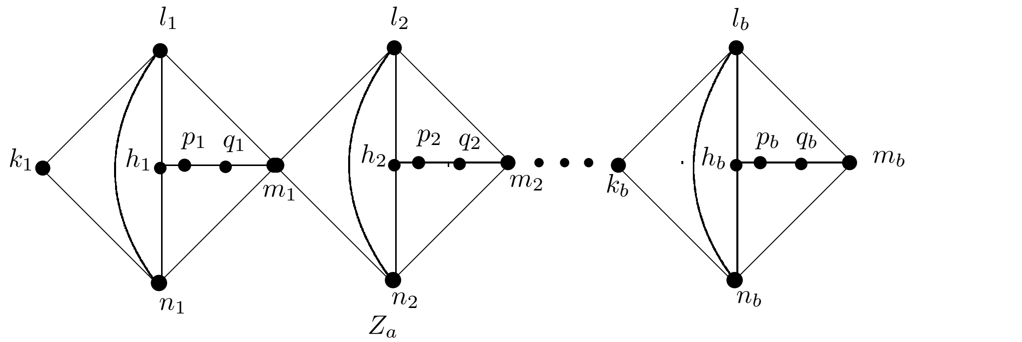

Figure 2.3

Let $F_i : s_i, t_i, x_i, w_i, s_i(1 \leq i \leq c)$ be a copy of cycle C_4. Let R_i be the graph obtained from F_i by adding two new vertices u_i, v_i and joining the edges $t_iu_i, u_iw_i, t_iw_i, u_iv_i$ and $v_ix_i(1 \leq i \leq c)$. The graph H_c given in Figure 2.4 is obtained from R_i's by identifying the vertices x_{i-1} of R_{i-1} and s_i of $R_i(1 \leq i \leq c)$.

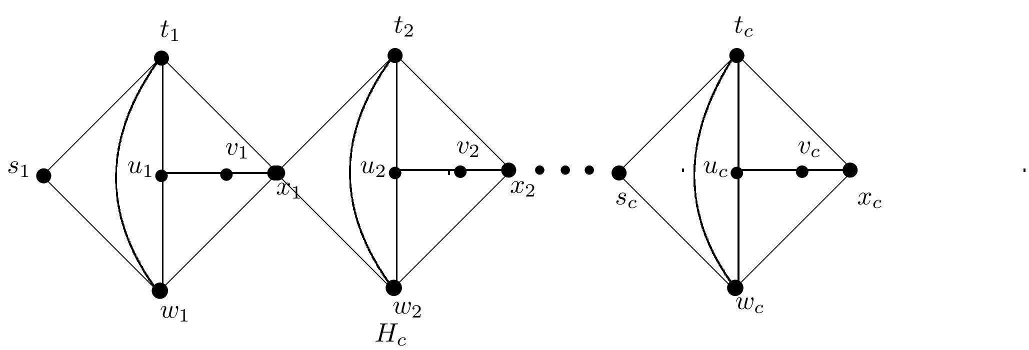

Figure 2.4

Every monophonic set of G is also a monophonic hull set of G and so $mh(G) \leq h(G)$, where $h(G)$ and $mh(G)$ are hull number and monophonic hull number of a connected graph G. However, there is no relationship between $f_h(G)$ and $f_{mh}(G)$, where $f_h(G)$ is the forcing hull number of a connected graph G. We give a series of realization results for various possibilities of these four parameters.

§3. Some Realization Results

Theorem 3.1 *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $f_{mh}(G) = f_h(G) = 0$, $mh(G) = a$ and $h(G) = b$.*

Proof If $a = b$, let $G = K_a$. Then by Theorems1.3(b) and 1.1(b), $mh(G) = h(G) = a$ and by Theorems 1.5 and 1.2(a), $f_{mh}(G) = f_h(G) = 0$. For $a < b$, let G be the graph obtained from T_{b-a} by adding new vertices $x, z_1, z_2, \cdots, z_{a-1}$ and joining the edges $x\alpha_1, \gamma_{b-a}z_1, \gamma_{b-a}z_2, \cdots,$ $\gamma_{b-a}z_{a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{a-1}\}$ be the set of end-vertices of G. Then it is clear that Z is a monophonic hull set of G and so by Theorem 1.3(a), Z is the unique mh-set of G so that $mh(G) = a$ and hence by Theorem 1.4(a), $f_{mh}(G) = 0$. Since $I_h(Z) \neq V, Z$ is not a hull set of G. Now it is easily seen that $W = Z \cup \{f_1, f_2, \cdots, f_{b-a}\}$ is the unique h-set of G and hence by Theorem 1.1(a) and Theorem 1.2(a), $h(G) = b$ and $f_h(G) = 0$. □

Theorem 3.2 *For every integers a, b and c with $0 \leq a < b < c$ and $c > a + b$, there exists a connected graph G such that $f_{mh}(G) = 0, f_h(G) = a, mh(G) = b$ and $h(G) = c$.*

Proof We consider two cases.

Case 1. $a = 0$. Then the graph T_b constructed in Theorem 3.1 satisfies the requirements of the theorem.

Case 2. $a \geq 1$. Let G be the graph obtained from W_a and $T_{c-(a+b)}$ by identifying the vertex m_a of W_a and α_1 of $T_{c-(a+b)}$ and then adding new vertices $x, z_1, z_2, \cdots, z_{b-1}$ and joining the edges $xk_1, \gamma_{c-b-a}z_1, \gamma_{c-b-a}z_2, \cdots, \gamma_{c-b-a}z_{b-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{b-1}\}$. Since $J_h(Z) = V$, Z is a monophonic hull set G and so by Theorem 1.3(a), Z is the unique mh- set of G so that $mh(G) = b$ and hence by Theorem 1.4(a), $f_{mh}(G) = 0$. Next we show that $h(G) = c$. Let S be any hull set of G. Then by Theorem 1.1(a), $Z \subseteq S$. It is clear that Z is not a hull set of G. For $1 \leq i \leq a$, let $H_i = \{p_i, q_i\}$. We observe that every h-set of G must contain at least one vertex from each $H_i (1 \leq i \leq a)$ and each $f_i (1 \leq i \leq c-b-a)$ so that $h(G) \geq b+a+c-a-b = c$. Now, $M = Z \cup \{q_1, q_2, \cdots, q_a\} \cup \{f_1, f_2, \cdots, f_{c-b-a}\}$ is a hull set of G so that $h(G) \leq b+a+c-b-a = c$. Thus $h(G) = c$. Since every h-set contains $S_1 = Z \cup \{f_1, f_2, \cdots, f_{c-b-a}\}$, it follows from Theorem 1.2(b) that $f_h(G) = h(G) - |S_1| = c - (c - a) = a$. Now, since $h(G) = c$ and every h-set of G contains S_1, it is easily seen that every h-set S is of the form $S_1 \cup \{d_1, d_2, \cdots, d_a\}$,where $d_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \phi$, which shows that $f_h(G) = a$. □

Theorem 3.3 *For every integers a, b and c with $0 \leq a < b \leq c$ and $b > a + 1$, there exists a connected graph G such that $f_h(G) = 0, f_{mh}(G) = a, mh(G) = b$ and $h(G) = c$.*

Proof We consider two cases.

Case 1. $a = 0$. Then the graph G constructed in Theorem 3.1 satisfies the requirements of the theorem.

Case 2. $a \geq 1$.

Subcase 2a. $b = c$. Let G be the graph obtained from Z_a by adding new vertices $x, z_1, z_2, \cdots, z_{b-a-1}$ and joining the edges $xk_1, m_az_1, m_az_2, \cdots, m_az_{b-a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{b-a-1}\}$ be the set of end-vertices of G. Let S be any hull set of G. Then by Theorem 1.1(a), $Z \subseteq S$. It is clear that Z is not a hull set of G. For $1 \leq i \leq a$, let $H_i = \{h_i, p_i, q_i\}$. We observe that every h-set of G must contain only the vertex p_i from each H_i so that $h(G) \leq b - a + a = b$. Now $S = Z \cup \{p_1, p_2, p_3, \cdots, p_a\}$ is a hull set of G so that $h(G) \geq b - a + a = b$. Thus $h(G) = b$. Also it is easily seen that S is the unique h-set of G and so by Theorem 1.2(a), $f_h(G) = 0$.Next we show that $mh(G) = b$. Since $J_h(Z) \neq V, Z$ is not a monophonic hull set of G. We observe that every mh-set of G must contain at least one vertex from each H_i so that $mh(G) \geq b - a + a = b$. Now $M_1 = Z \cup \{q_1, q_2, q_3, \cdots, q_a\}$ is a monophonic hull set of G so that $mh(G) \leq b - a + a = b$. Thus $mh(G) = b$. Next we show that $f_{mh}(G) = a$. Since every mh-set contains Z, it follows from Theorem 1.4(b) that $f_{mh}(G) \leq mh(G) - |Z| = b - (b - a) = a$. Now, since $mh(G) = b$ and every mh-set of G contains Z, it is easily seen that every mh-set M is of the form $Z \cup \{d_1, d_2, d_3, \cdots, d_a\}$, where $d_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of M with $|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \phi$, which shows that $f_{mh}(G) = a$.

Subcase 2b. $b < c$. Let G be the graph obtained from Z_a and T_{c-b} by identifying the vertex m_a of Z_a and α_1 of T_{c-b} and then adding the new vertices $x, z_1, z_2, \cdots, z_{b-a-1}$ and joining the edges $x\alpha_1, \gamma_{c-b}z_1, \gamma_{c-b}z_2, \cdots, \gamma_{c-b}z_{b-a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{b-a-1}\}$ be the set of end vertices of G. Let S be any hull set of G. Then by Theorem 1.1(a), $Z \subseteq S$. Since $I_h(Z) \neq V, Z$ is not a hull set of G. For $1 \leq i \leq a$, let $H_i = \{h_i, p_i, q_i\}$. We observe that every h-set of G must contain only the vertex p_i from each H_i and each $f_i (1 \leq i \leq c - b)$ so that $h(G) \geq b - a + a + c - b = c$. Now $S = Z \cup \{p_1, p_2, p_3, \cdots, p_a\} \cup \{f_1, f_2, f_3, \cdots, f_{c-b}\}$is a hull set of G so that $h(G) \leq b - a + a + c - b = c$. Thus $h(G) = c$. Also it is easily seen that S is the unique h-set of G and so by Theorem 1.2(a), $f_h(G) = 0$. Since $J_h(Z) \neq V$, Z is not a monophonic hull set of G. We observe that every mh-set of G must contain at least one vertex from each $H_i (1 \leq i \leq a)$ so that $mh(G) \geq b - a + a = b$. Now, $M_1 = Z \cup \{h_1, h_2, h_3, \cdots, h_a\}$ is a monophonic hull set of G so that $mh(G) \leq b - a + a = b$. Thus $mh(G) = b$. Next we show that $f_{mh}(G) = a$. Since every mh-set contains Z, it follows from Theorem 1.4(b) that $f_{mh}(G) \leq mh(G) - |Z| = b - (b - a) = a$. Now, since $mh(G) = b$ and every mh-set of G contains Z, it is easily seen that every mh-set S is of the form $Z \cup \{d_1, d_2, d_3, \cdots, d_a\}$, where $d_i \in H_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap H_j = \phi$, which shows that $f_{mh}(G) = a$. □

Theorem 3.4 *For every integers a, b and c with $0 \leq a < b \leq c$ and $b > a + 1$, there exists a connected graph G such that $f_{mh}(G) = f_h(G) = a$, $mh(G) = b$ and $h(G) = c$.*

Proof We consider two cases.

Case 1. $a = 0$, then the graph G constructed in Theorem 3.1 satisfies the requirements of the theorem.

Case 2. $a \geq 1$.

Subcase 2a. $b = c$. Let G be the graph obtained from H_a by adding new vertices $x, z_1, z_2, \cdots, z_{b-a-1}$ and joining the edges $xs_1, x_a z_1, x_a z_2, \cdots, x_a z_{b-a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{b-a-1}\}$ be the set of end-vertices of G. Let M be any monophonic hull set of G. Then by Theorem 1.3(a), $Z \subseteq M$. First we show that $mh(G) = b$. Since $J_h(Z) \neq V, Z$ is not a monophonic hull set of G. Let $F_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$. We observe that every mh-set of G must contain at least one vertex from each $F_i (1 \leq i \leq a)$. Thus $mh(G) \geq b - a + a = b$. On the other hand since the set $M = Z \cup \{v_1, v_2, v_3, \cdots, v_a\}$ is a monophonic hull set of G, it follows that $mh(G) \leq |M| = b$. Hence $mh(G) = b$. Next we show that $f_{mh}(G) = a$. By Theorem 1.3(a), every monophonic hull set of G contains Z and so it follows from Theorem 1.4(b) that $f_{mh}(G) \leq mh(G) - |Z| = a$. Now, since $mh(G) = b$ and every mh-set of G contains Z, it is easily seen that every mh-set M is of the form $Z \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap F_j = \phi$, which shows that $f_{mh}(G) = a$. By similar way we can prove $h(G) = b$ and $f_h(G) = a$.

Subcase 2b. $b < c$. Let G be the graph obtained from H_a and T_{c-b} by identifying the vertex x_a of H_a and the vertex α_1 of T_{c-b} and then adding the new vertices $x, z_1, z_2, \cdots, z_{b-a-1}$ and joining the edges $xs_1, \gamma_{c-b} z_1, \gamma_{c-b} z_2, \cdots, \gamma_{c-b} z_{b-a-1}$. First we show that $mh(G) = b$. Since $J_h(Z) \neq V, Z$ is not a monophonic hull set of G. Let $F_i = \{u_i, v_i\}$ $(1 \leq i \leq a)$ .We observe that every mh-set of G must contain at least one vertex from each $F_i (1 \leq i \leq a)$. Thus $mh(G) \geq b - a + a = b$. On the other hand since the set $M = Z \cup \{v_1, v_2, v_3, \cdots, v_a\}$ is a monophonic hull set of G, it follows that $mh(G) \geq |M| = b$. Hence $mh(G) = b$. Next, we show that $f_{mh}(G) = a$. By Theorem 1.3(a), every monophonic hull set of G contains Z and so it follows from Theorem 1.4(b) that $f_{mh}(G) \leq mh(G) - |Z| = a$. Now, since $mh(G) = b$ and every mh-set of G contains Z, it is easily seen that every mh-set is of the form $M = Z \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i (1 \leq i \leq a)$. Let T be any proper subset of M with $|T| < a$. Then it is clear that there exists some j such that $T \cup F_j = \phi$, which shows that $f_{mh}(G) = a$. Next we show that $h(G) = c$. Since $I_h(Z) \neq V, Z$ is not a hull set of G. We observe that every h-set of G must contain at least one vertex from each $F_i (1 \leq i \leq a)$ and each $f_i (1 \leq i \leq c-b)$ so that $h(G) \geq b - a + a + c - b = c$. On the other hand, since the set $S_1 = Z \cup \{u_1, u_2, u_3, \cdots, u_a\} \cup \{f_1, f_2, f_3, \cdots, f_{c-b}\}$ is a hull set of G, so that $h(G) \leq |S_1| = c$. Hence $h(G) = c$. Next we show that $f_h(G) = a$. By Theorem 1.1(a), every hull set of G contains $S_2 = Z \cup \{f_1, f_2, f_3, \cdots, f_{c-b}\}$ and so it follows from Theorem 1.2(b) that $f_h(G) \leq h(G) - |S_2| = a$. Now, since $h(G) = c$ and every h-set of G contains S_2, it is easily seen that every h-set S is of the form $S = S_2 \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i (1 \leq i \leq a)$. Let T be any proper subset of S with $|T| < a$. Then it is clear that there exists some j such that $T \cap F_j = \phi$, which shows that $f_h(G) = a$. □

Theorem 3.5 *For every integers a, b, c and d with $0 \leq a \leq b < c < d, c > a + 1, d > c - a + b$, there exists a connected graph G such that $f_{mh}(G) = a, f_h(G) = b, mh(G) = c$ and $h(G) = d$.*

Proof We consider four cases.

Case 1. $a = b = 0$. Then the graph G constructed in Theorem 3.1 satisfies the requirements of this theorem.

Case 2. $a = 0, b \geq 1$. Then the graph G constructed in Theorem 3.2 satisfies the requirements of this theorem.

Case 3. $1 \leq a = b$. Then the graph G constructed in Theorem 3.4 satisfies the requirements of this theorem.

Case 4. $1 \leq a < b$. Let G_1 be the graph obtained from H_a and W_{b-a} by identifying the vertex x_a of H_a and the vertex k_1 of W_{b-a}. Now let G be the graph obtained from G_1 and $T_{d-(c-a+b)}$ by identifying the vertex m_{b-a} of G_1 and the vertex α_1 of $T_{d-(c-a+b)}$ and adding new vertices $x, z_1, z_2, \cdots, z_{c-a-1}$ and joining the edges $xs_1, \gamma_{d-(c-a+b)}z_1, \gamma_{d-(c-a+b)}z_2, \cdots, \gamma_{d-(c-a+b)}z_{c-a-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{c-a-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}\,(1 \leq i \leq a)$. It is clear that any mh-set S is of the form $S = Z \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i(1 \leq i \leq a)$. Then as in earlier theorems it can be seen that $f_{mh}(G) = a$ and $mh(G) = c$. Let $Q_i = \{p_i, q_i\}$. It is clear that any h-set W is of the form $W = Z \cup \left\{f_1, f_2, f_3, \cdots, f_{d-(c-a+b)}\right\} \cup \{c_1, c_2, c_3, \cdots, c_a\} \cup \{d_1, d_2, d_3, \cdots, d_{b-a}\}$, where $c_i \in F_i(1 \leq i \leq a)$ and $d_j \in Q_j(1 \leq j \leq b-a)$. Then as in earlier theorems it can be seen that $f_h(G) = b$ and $h(G) = d$. $\square$

Theorem 3.6 *For every integers a, b, c and d with $a \leq b < c \leq d$ and $c > b + 1$ there exists a connected graph G such that $f_h(G) = a, f_{mh}(G) = b, mh(G) = c$ and $h(G) = d$.*

Proof We consider four cases.

Case 1. $a = b = 0$. Then the graph G constructed in Theorem 3.1 satisfies the requirements of this theorem.

Case 2. $a = 0, b \geq 1$. Then the graph G constructed in Theorem 3.2 satisfies the requirements of this theorem.

Case 3. $1 \leq a = b$. Then the graph G constructed in Theorem 3.4 satisfies the requirements of this theorem.

Case 4. $1 \leq a < b$.

Subcase 4a. $c = d$. Let G be the graph obtained from H_a and Z_{b-a} by identifying the vertex x_a of H_a and the vertex k_1 of Z_{b-a} and then adding the new vertices $x, z_1, z_2, ..., z_{c-b-1}$ and joining the edges $xs_1, m_{b-a}z_1, m_{b-a}z_2, ..., m_{b-a}z_{c-b-1}$. First we show that $mh(G) = c$. Let $Z = \{x, z_1, z_2, ..., z_{c-b-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}\,(1 \leq i \leq a)$ and $H_i = \{h_i, p_i, q_i\}\,(1 \leq i \leq b-a)$. It is clear that any mh-set of G is of the form $S = Z \cup \{c_1, c_2, c_3, ..., c_a\} \cup \{d_1, d_2, d_3, ..., d_{b-a}\}$, where $c_i \in F_i(1 \leq i \leq a)$ and $d_j \in H_j(1 \leq j \leq b-a)$. Then as in earlier theorems it can be seen that $f_{mh}(G) = b$ and $mh(G) = c$. It is clear that any h-set W is of the form $W = Z \cup \{p_1, p_2, p_3, ..., p_{b-a}\} \cup \{c_1, c_2, c_3, ..., c_a\}$, where $c_i \in F_i(1 \leq i \leq a)$. Then as in earlier theorems it can be seen that $f_h(G) = a$ and $h(G) = c$.

Subcase 4b. $c < d$. Let G_1 be the graph obtained from H_a and Z_{b-a} by identifying the vertex x_a of H_a and the vertex k_1 of Z_{b-a}. Now let G be the graph obtained from G_1 and T_{d-c} by identifying the vertex m_{b-a} of G_1 and the vertex α_1 of T_{d-c} and then adding new vertices $x, z_1, z_2, \cdots, z_{c-b-1}$ and joining the edges $xs_1, \gamma_{d-c}z_1, \gamma_{d-c}z_2, \cdots, \gamma_{d-c}z_{c-b-1}$. Let $Z = \{x, z_1, z_2, \cdots, z_{c-b-1}\}$ be the set of end vertices of G. Let $F_i = \{u_i, v_i\}\,(1 \leq i \leq a)$ and $H_i = \{h_i, p_i, q_i\}\,(1 \leq i \leq b-a)$. It is clear that any mh-set of G is of the form $S = Z \cup$

$\{c_1, c_2, c_3, \cdots, c_a\} \cup \{d_1, d_2, d_3, \cdots, d_{b-a}\}$, where $c_i \in F_i (1 \leq i \leq a)$ and $d_j \in H_j (1 \leq j \leq b-a)$. Then as in earlier theorems it can be seen that $f_{mh}(G) = b$ and $mh(G) = c$. It is clear that any h-set W is of the form $W = Z \cup \{p_1, p_2, p_3, \cdots, p_{b-a}\} \cup \{f_1, f_2, f_3, \cdots, f_{d-c}\} \cup \{c_1, c_2, c_3, \cdots, c_a\}$, where $c_i \in F_i (1 \leq i \leq a)$. Then as in earlier theorems it can be seen that $f_h(G) = a$ and $h(G) = d$. □

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Simplicial Branched Coverings of the 3-Sphere

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Abstract: In this article we give, for each $d > 1$, a simplicial branched covering map $\lambda_d : S^3_{3(d+1)} \to S^3_6$ of degree d. And by using the simplicial map λ_2 we demonstrate a well known topological fact that the space obtained by identifying diagonally opposite points of the 3-sphere is homeomorphic to the 3-sphere.

Key Words: Branched Covering, simplicial map, triangulation of map.

AMS(2010): 57M12, 57N12, 55M25, 57M20

§1. Introduction

In articles [5] and [6] we have given *simplicial branched coverings* of the Real Projective Plane and the 2-Sphere respectively. The present article is in continuation of these articles. Here we give, for each $d > 1$, a simplicial branched covering map, $\lambda_d : S^3_{3(d+1)} \to S^3_6$, from a 3(d+1) vertex triangulation of the 3-sphere onto a 6 vertex triangulation of the 3-sphere. For $d = 2$, we show that the simplicial branched covering map $\lambda_2 : S^3_9 \to S^3_6$ is a minimal triangulation of the well known two fold branched covering map $S^3 \to S^3/(x, y) \sim (y, x)$. Moreover the simplicial map λ_2 verifies a familiar topological fact that after identifying diagonally symmetric points of the 3-sphere we get a homeomorphic copy of the 3-sphere. Branched coverings of the low dimensional manifolds have been discussed extensively (e.g. see [1], [3] and [4]) but the explicit constructions, which we are giving here are missing. The purpose here is to give some concrete examples, which are not at all trivial but explain some important topological facts.

§2. Preliminary Notes

Definition 2.1 *An abstract simplicial complex K on a finite set V is a collection of subsets of V, which is closed under inclusion i.e. if $s \in K$ and $s' \subset s$ then $s' \in K$. The elements of K are called simplices and in particular a set $\gamma \in K$ of cardinality n+1 is called an n-simplex; 0-simplices are called vertices, 1-simplices are called edges and so on.*

A geometric n-simplex is the convex hull of $n + 1$ affinely independent points of $\mathbb{R}^N$(see [2]). A geometric simplicial complex is a collection of geometric simplices such that all faces of

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these simplices are also in the collection and intersection of any two of these simplices is either empty or a common face of both of these simplices. It is easy to see that corresponding to each geometric simplicial complex there is an abstract simplicial complex. Converse is also true i. e. corresponding to any abstract simplicial complex K there is a topological space $|K| \subset R^N$, made up of geometric simplices, called its geometric realization (see [2], [7], [8]). If K is an abstract simplicial complex and M is a subspace of $\mathbb{R}^N$ such that there is a homeomorphism $h : |K| \to M$ then we say $(|K|, h)$ is a triangulation of M or K triangulates the topological space M.

Definition 2.2 *A map $f : K \to L$, between two abstract simplicial complexes K and L, is called a simplicial map if image, $f(\sigma) = \{f(v_0), f(v_1), ..., f(v_k)\}$, of any simplex $\sigma = \{v_0, v_1, ..., v_k\}$ of K is a simplex of L. Further if $|K|$ and $|L|$ are geometric realizations of K and L respectively then there is a piecewise-linear continuous map $|f| : |K| \to |L|$ defined as follows. As each point x of $|K|$ is an interior point of exactly one simplex (say $\sigma = \{v_0, .., v_k\}$) of $|K|$, so for each $x \in \sigma$ we have $x = \sum_{i=1}^{k} \lambda_i v_i$ where $\lambda_i \geq 0$, $\sum \lambda_i = 1$. Therefore we may define $|f|(x) = \sum_{i=1}^{k} \lambda_i f(v_i)$.*

Definition 2.3 *A simplicial branched covering map between two triangulated n-manifolds K and L is defined by a dimension preserving piecewise linear map $p : |K| \to |L|$, which is an ordinary covering over the complement of some specific co-dimension 2 sub-complex L' of L (for more detailed definition see [1], [3], [4], [5]). The sub-complex L' is called branch set of the branched covering map and a point $x \in p^{-1}(L')$ is called a singular point if p fails to be a local homeomorphism at x.*

§3. Main Results

3.1 Simplicial branched covering map $\lambda_d : S^3_{3(d+1)} \to S^3_6$

We first define a simplicial branched covering map $\lambda_2 : S^3_9 \to S^3_6$ of degree 2 and then show that the same method gives, for each $d > 2$, a simplicial branched covering map $\lambda_d : S^3_{3(d+1)} \to S^3_6$ of degree d.

Since the join of two 1-spheres is a 3-sphere, so in order to get the desired 9 vertex 3-sphere S^3_9, we take join of a three vertex 1-sphere $S^1_3 = \{A_0, E_0, F_0, A_0E_0, E_0F_0, F_0A_0\}$ with the six vertex 1-sphere $S^1_6 = \{B_0, C_0, D_0, B_1, C_1, D_1, C_0B_0, B_0D_0, D_0C_1, C_1B_1, B_1D_1, D_1C_0\}$. The 3-sipmlices of $S^3_9 = S^1_3 * S^1_6$ are shown in Figure 1.

We define a map on the vertex set of S^3_9, as $A_0 \to A, E_0 \to E, F_0 \to F, X_i \to X$ for each $X \in \{B, C, D\}, i \in \{0, 1\}$ and extend it linearly on the 3-simplices of S^3_9. The image of this map is a simplicial complex whose 3-simplices are ABCE, ACDE, ABDE, EDBF, EBCF, EDCF, CDAF, DBAF and CBAF. This simplicial complex triangulates the 3-sphere because its geometric realization is homeomorphic to the 3-sphere as it is a disjoint union of two 3-balls having a common boundary S^2 (see figure 2 below). We denote this simplicial complex by S^3_6 and the map just defined is the simplicial map $\lambda_2 : S^3_9 \to S^3_6$. Notice that the map $\lambda_2 : S^3_9 \to S^3_6$ is a 2-fold simplicial branched covering map because pre-image of each 3-simplex of S^3_6 consists of exactly two 3-simplices of S^3_9; each is being mapped, under the map λ_2, with the same orientation. Branching set and the singular set of the map are $AE + EF + FA$ and $A_0E_0 + E_0F_0 + F_0A_0$ respectively.

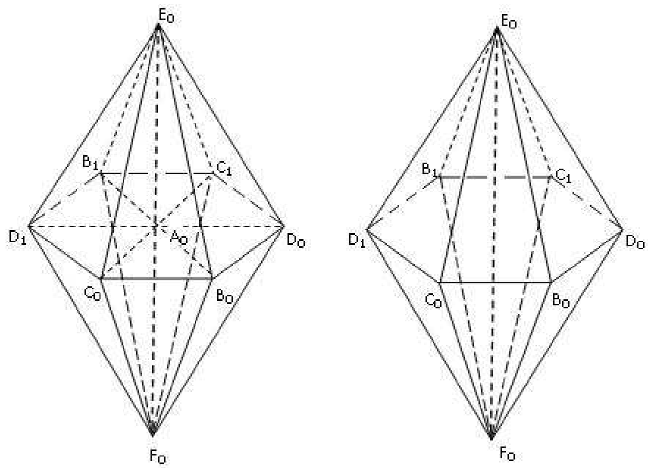

Figure 1

In order to get a simplicial branched covering map, $\lambda_d : S^3_{3(d+1)} \to S^3_6$, of degree d (for each $d > 2$) we consider the join of a 3 vertex 1-sphere with the 3d vertex 1-sphere. i.e. $S^3_{3(d+1)} = S^1_3 * S^1_{3d} = \{A_0, E_0, F_0, A_0E_0, E_0F_0, F_0A_0\} * \{B_i, C_i, D_i, C_iB_i, B_iD_i, D_iC_{i+1} : i \in Z_d\}$.

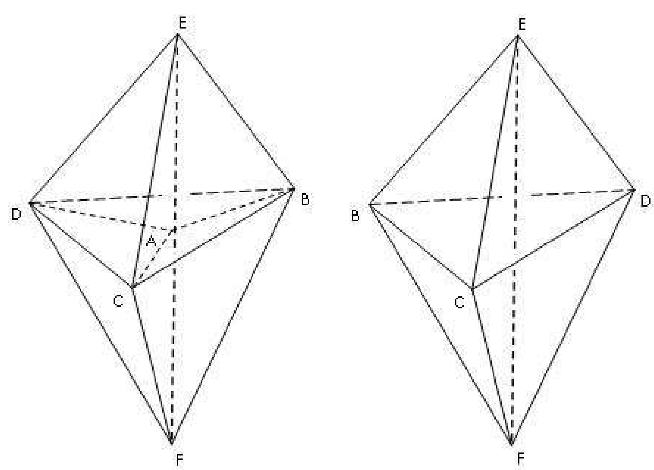

Figure 2

The 3-simplices of $S^3_{3(d+1)}$ are $\{A_0E_0C_iB_i, A_0E_0B_iD_i, A_0E_0D_iC_{i+1}, E_0F_0C_iB_i, E_0F_0B_iD_i,$ $E_0F_0D_iC_{i+1}, F_0A_0C_iB_i, F_0A_0B_iD_i, F_0A_0D_iC_{i+1} : i \in Z_d$, addition in the subscripts is mod d$\}$. Notice that a map, defined on the vertices of the simplicial complex $S^3_{3(d+1)}$, as $A_0 \to A, E_0 \to$

$E, F_0 \to F, X_i \to X$ for each $X \in \{B, C, D\}$ and for $i \in \{0, 1, , d-1\}$ is a simplicial branched covering map $\lambda_d : S^3_{3(d+1)} \to S^3_6$ of degree d.

Remark 3.1.1 We shall now show that the simplicial map $\lambda_2 : S^3_9 \to S^3_6$ triangulates the 2-fold branched covering map $S^3 \to S^3/(x,y) \sim (y,x)$ but before that we prove the following theorem.

Theorem 3.1.1 *The simplicial map $\lambda_2 : S^3_9 \to S^3_6$ is a minimal triangulation of the 2-fold branched covering map q: $S^3 \to S^3/(x,y) \sim (y,x)$.*

Proof Notice that branching of the map q occurs along the diagonal circle of the quotient space and pre-image of the branching circle is the diagonal circle of the domain of the map q. Let $\lambda_2 : S^3_{\alpha_0} \to S^3_{\beta_0}$ be a minimal triangulation of the map q, so the branching circle and the singular circle are at least triangles. Since the polygonal link of any singular 1-simplex of $S^3_{\alpha_0}$, is to be mapped with degree 2 by the map λ_2 so the link will have at least 6-vertices. The image of this link will be a circle with at least 3 vertices, which are different from the vertices of the branching circle. This implies that the domain 3-sphere of the map λ_2 will have at least 9 vertices and its image will have at least 6 vertices i.e. $\alpha_0 \geq 9$ and $\beta_0 \geq 6$. $\square$

Note 3.1.1 Following description of the simplicial complex S^3_9 enables us to show that the simplicial map $\lambda_2 : S^3_9 \to S^3_6$ triangulates the 2-fold branched covering map q: $S^3 \to S^3/(x,y) \sim (y,x)$. It also leads to a combinatorial proof of the fact that after identification of diagonally symmetric points of the 3-sphere we get the 3-sphere again.

3.2 Diagonally Symmetric Triangulation of the 3-Sphere

The diagonal of the standard 3-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ is the subspace $\Delta = \{(z_1, z_2) \in S^3 : z_1 = z_2\}$. A triangulation of S^3 will be called diagonally symmetric if whenever there is a vertex at a point (z_1, z_2) there is a vertex at the point (z_2, z_1) and whenever there is a 3-simplex on the vertices $(z_{i_1}, z_{i_2}), (z_{i_3}, z_{i_4}), (z_{i_5}, z_{i_6}), (z_{i_7}, z_{i_8})$, there is a 3-simplex on the vertices $(z_{i_2}, z_{i_1}), (z_{i_4}, z_{i_3}), z_{i_6}, z_{i_5}), (z_{i_8}, z_{i_7})$. We show that the simplicial complex S^3_9 obtained above is a diagonally symmetric triangulation of the 3-sphere and the simplicial branched covering map $\lambda_2 : S^3_9 \to S^3_6$ is equivalent to the map q: $S^3 \to S^3/(x,y) \sim (y,x)$. In order to show this we consider the following description of the 3-sphere:

$$S^3 = T_1 \bigcup T_2,$$

where $T_1 = \{(z_1, z_2) \in S^3 : |z_1| \leq |z_2|\}$, $T_2 = \{(z_1, z_2) \in S^3 : |z_1| \geq |z_2|\}$ and

$$T = T_1 \bigcap T_2 = \{(z_1, z_2) \in S^3 : |z_1| = |z_2| = 1/\sqrt{2}\} \cong S^1 \times S^1.$$

A map $\theta : S^3 \to S^3$ defined as $(z_1, z_2) \to (z_2, z_1)$ swaps the interiors of the solid tori T_1 and T_2 homeomorphically. We triangulate T_1 and T_2 in such a way that the homeomorphism θ induces a simplicial isomorphism between the triangulations of T_1 and T_2. The triangulations of T, T_1 and T_2 are described as follows.

In Figure 3 below we give a triangulated 2-torus T, which is the common boundary of both of the solid tori T_1 and T_2. The vertices X_0, X_1 for each $X \in \{B, C, D\}$ are symmetric about the diagonal Δ and the vertices A_0, E_0, F_0 triangulate the diagonal.

Since there are precisely two ways to fold a square to get a torus, viz (i) first identify vertical boundaries and then identify horizontal boundaries of the square (ii) first identify horizontal boundaries and then vertical boundaries of the square, so we use this fact to obtain the solid tori T_1 and T_2.

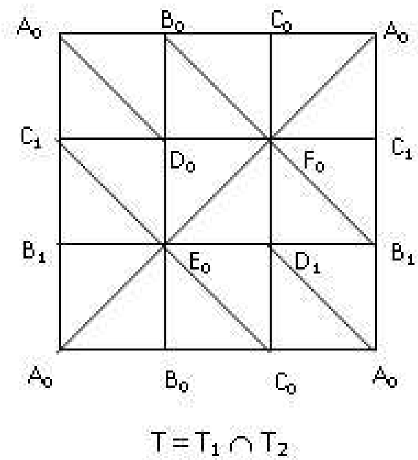

Figure 3

The solid torus T_1 has been obtained by first identifying the vertical edges, of the square of Figure 3, and then top and bottom edges (see Figure 4 below). Its three, of the total nine, 3-simplices are $A_0B_0C_0E_0, A_0C_0E_0D_1$ and $A_0D_1E_0B_1$ and remaining six 3-simplices can be obtained from an automorphism defined by $A_0 \to E_0 \to F_0 \to A_0, B_0 \to D_1 \to C_1 \to B_0$ and $C_0 \to B_1 \to D_0 \to C_0$.

The solid torus T_2 has been obtained by first identifying the horizontal edges, of the square of Figure 3, and then the other two sides as shown in Figure 4. The nine 3-simplices of T_2 are $\{A_0D_0E_0B_0, E_0A_0C_1D_0, A_0B_1C_1E_0, E_0C_0F_0D_1, F_0E_0B_0C_0, E_0D_0B_0F_0, F_0B_1A_0C_1, A_0F_0D_1B_1, F_0C_0D_1A_0\}$. These simplices can also be obtained from the 3-simplices of T_1 by using the permutation $\rho = (B_0B_1)(C_0C_1)(D_0D_1)$, which is equivalent to the Z_2-action defined by the map $\theta : (x, y) \to (y, x)$ on S^3.

The nine 3-simplices of T_1 together with the nine 3-simplices of T_2 constitute a diagonally symmetric triangulation of the 3-sphere with 9 vertices. And since the list of 3-simplices of $S^1_3 * S^1_6$ is same as that of the 3-simplices of the simplicial complex obtained now, so the two simplicial complexes are isomorphic.

Notice that the identification of diagonally symmetric vertices / simplices of the 3-sphere (obtained now) is equivalent to the identifications provided by the simplicial map λ_2. This equivalence implies that the identification of diagonally symmetric points of the 3-sphere gives a 3-sphere.

Remark 3.2.1 In Figure 3 (triangulation of T) if we replace the edges A_0D_0 and A_0D_1 by the edges B_0C_1 and B_1C_0 respectively then we get another triangulation of T, which is also symmetric about the diagonal. But this triangulation under the diagonal action does not give a simplicial branched covering map.

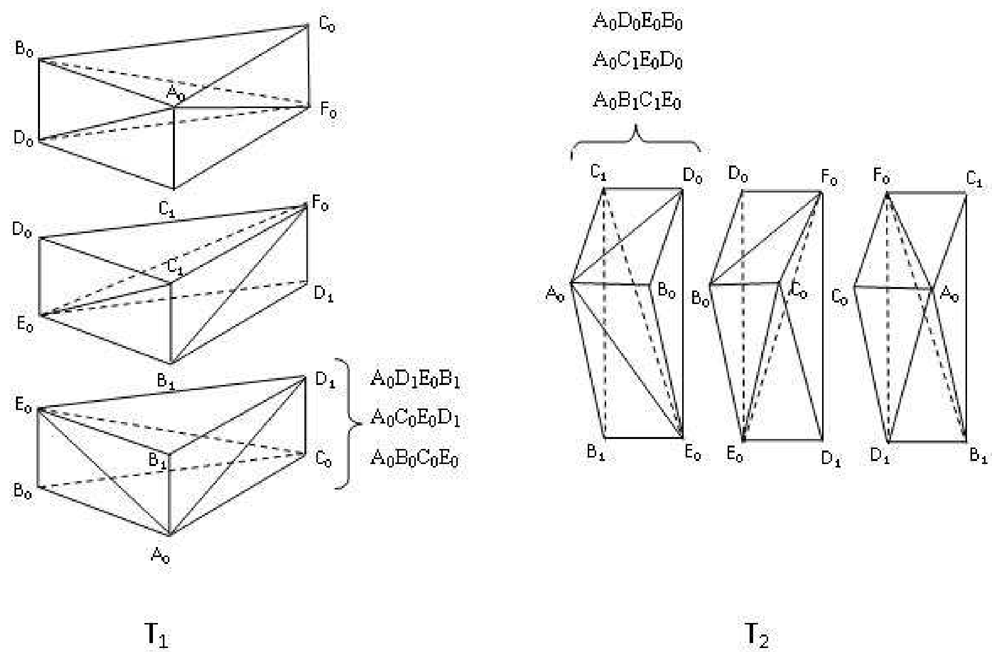

Figure 4

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On the Osculating Spheres of a Real Quaternionic Curve In the Euclidean Space E^4

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Abstract: In the Euclidean space E^4, there is a unique quaternionic sphere for a real quaternionic curve $\alpha : I \subset \mathbb{R} \to Q_H$ such that it touches α at the fourth order at $\alpha(0)$. In this paper, we studied some characterizations of the osculating sphere of the real quaternionic curves in the four dimensional Euclidean space.

Key Words: Euclidean space, quaternion algebra, osculating spheres.

AMS(2010): 14Q05, 53A17, 53C50

§1. Introduction

The quaternions introduced by Hamilton in 1843 are the number system in four dimensional vector space and an extension of the complex number. There are different types of quaternions, namely: real, complex dual quaternions. A real quaternion is defined as $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ is composed of four units $\{1, e_1, e_2, e_3\}$ where e_1, e_2, e_3 are orthogonal unit spatial vectors, q_i $(i = 0, 1, 2, 3)$ are real numbers and this quaternion can be written as a linear combination of a real part (scalar) and vectorial part (a spatial vector) [1,5,8].

The space of quaternions Q are isomorphic to E^4, four dimensional vector space over the real numbers. Then, Clifford generalized the quaternions to bi-quaternions in 1873 [11]. Hence they play an important role in the representation of physical quantities up to four dimensional space. Also they are used in both theoretical and applied mathematics. They are important number systems which use in Newtonian mechanics, quantum physics, robot kinematics, orbital mechanics and three dimensional rotations such as in the three dimensional computer graphics and vision. Real quaternions provide us with a simple and elegant representation for describing finite rotation in space. On the other hand, dual quaternions offer us a better way to express both rotational and translational transformations in a robot kinematic [5].

In 1985, the Serret-Frenet formulas for a quaternionic curve in Euclidean spaces E^3 and E^4 are given by Bharathi and Nagaraj [9]. By using of these formulas Karadağ and Sivridağ gave some characterizations for quaternionic inclined curves in the terms of the harmonic curvatures in Euclidean spaces E^3 and E^4 [10]. Gök et al. defined the real spatial quaternionic b-slant

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helix and the quaternionic B_2-slant helix in Euclidean spaces E^3 and E^4 respectively and they gave new characterization for them in the terms of the harmonic curvatures [7].

In the Euclidean space E^3, there is a unique sphere for a curve $\alpha : I \subset \mathbb{R} \to E^3$ such that the sphere contacts α at the third order at $\alpha(0)$. The intersection of the sphere with the osculating plane is a circle which contacts α at the second order at $\alpha(0)$ [2,3,6]. In [4], the osculating sphere and the osculating circle of the curve are studied for each of timelike, spacelike and null curves in semi- Euclidean spaces; E_1^3, E_1^4 and E_2^4.

In this paper, we define osculating sphere for a real quaternionic curve $\alpha : I \subset \mathbb{R} \to E^4$ such that it contacts α at the fourth order at $\alpha(0)$. Also some characterizations of the osculating sphere are given in Euclidean space E^4.

§2. Preliminaries

We give basic concepts about the real quaternions. Let Q_H denote a four dimensional vector space over a field H whose characteristic grater than 2. Let e_i $(1 \leq i \leq 4)$ denote a basis for the vector space. Let the rule of multiplication on Q_H be defined on e_i $(1 \leq i \leq 4)$ and extended to the whole of the vector space by distributivity as follows:

A real quaternion is defined by $q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + de_4$ where a, b, c, d are ordinary numbers such that

$$\begin{aligned}
&\overrightarrow{e_1} \times \overrightarrow{e_2} = \overrightarrow{e_3} = -\overrightarrow{e_2} \times \overrightarrow{e_1},\\
&\overrightarrow{e_2} \times \overrightarrow{e_3} = \overrightarrow{e_1} = -\overrightarrow{e_3} \times \overrightarrow{e_2},\\
&\overrightarrow{e_3} \times \overrightarrow{e_1} = \overrightarrow{e_2} = -\overrightarrow{e_1} \times \overrightarrow{e_3},\\
&\overrightarrow{e_1}^2 = \overrightarrow{e_2}^2 = \overrightarrow{e_3}^2 = -1, \quad e_4^2 = 1.
\end{aligned}$$

We can write a real quaternion as a linear combination of scalar part $S_q = d$ and vectorial part $V_q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$. Using these basic products we can now expand the product of two quaternions as

$$p \times q = S_p S_q - \left\langle \overrightarrow{V_p}, \overrightarrow{V_q} \right\rangle + S_p \overrightarrow{V_q} + S_q \overrightarrow{V_p} + \overrightarrow{V_p} \wedge \overrightarrow{V_q} \quad \text{for every} \quad p, q \in Q_H,$$

where $\langle , \rangle$ and $\wedge$ are inner product and cross product on E^3, respectively. There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol γ and defined as follows:

$$\gamma q = -a\overrightarrow{e_1} - b\overrightarrow{e_2} - c\overrightarrow{e_3} + d$$

for every $q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + de_4 \in Q_H$ which is called the *Hamiltonian conjugation.* This defines the symmetric, real valued, non-degenerate, bilinear form h as follows:

$$h(p, q) = \frac{1}{2}(p \times \gamma q + q \times \gamma p) \quad \text{for every } p, q \in Q_H.$$

Now we can give the definition of the norm for every quaternion. the norm of any q real quaternion is denoted by

$$\|q\|^2 = h(q, q) = q \times \gamma q = a^2 + b^2 + c^2 + d^2$$

in [5,8].

The four-dimensional Euclidean space E^4 is identified with the space of unit quaternions. A real quaternionic sphere with origin m and radius $R > 0$ in E^4 is

$$S^3(m, R) = \{p \in Q_H : h(p - m, p - m) = R^2\}.$$

The Serret-Frenet formulas for real quaternionic curves in E^4 are as follows:

Theorem 2.1([10]) *The four-dimensional Euclidean space E^4 is identified with the space of unit quaternions. Let $I = [0, 1]$ denotes the unit interval in the real line $\mathbb{R}$ and $\overrightarrow{e_4} = 1$. Let*

$$\begin{aligned} &\alpha : I \subset \mathbb{R} \to Q_H \\ &s \to \alpha(s) = \sum_{i=1}^{4} \alpha_i(s)\overrightarrow{e_i}, \end{aligned}$$

be a smooth curve in E^4 with nonzero curvatures $\{K, k, r - K\}$ and the Frenet frame of the curve α is $\{T, N, B_1, B_2\}$. Then Frenet formulas are given by

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r-K) \\ 0 & 0 & -(r-K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2.1}$$

where K is the principal curvature, k is torsion and $(r - K)$ is bitorsion of α.

§3. Osculating Sphere of a Real Quaternionic Curve in E^4

We assume that the real quaternionic curve $\alpha : I \subset \mathbb{R} \to Q_H$ is arc-length parametrized, i.e, $\|\alpha'(s)\| = 1$. Then the tangent vector $T(s) = \alpha'(s) = \sum_{i=1}^{4} \alpha_i'(s)\overrightarrow{e_i}$ has unit length. Let (y_1, y_2, y_3, y_4) be a rectangular coordinate system of $\mathbb{R}^4$. We take a real quaternionic sphere $h(y - d, y - d) = R^2$ with origin d and radius R, where $y = (y_1, y_2, y_3, y_4)$. Let $f(s) = h(\alpha(s) - d, \alpha(s) - d) - R^2$. If we have the following equations

$$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 0, \quad f'''(0) = 0, \quad f^{(4)}(0) = 0$$

then we say that the sphere contacts at fourth order to the curve α at $\alpha(0)$. The sphere is called osculating sphere.

Theorem 3.1 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve with nonzero curvatures $K(0), k(0)$ and $(r - K)(0)$ at $\alpha(0)$. Then there exists a sphere which contacts at the fourth order to the curve α at $\alpha(0)$ and the equation of the osculating sphere according to the Frenet frame $\{T_0, N_0, B_{1_0}, B_{2_0}\}$ is*

$$x_1^2+(x_2-\rho_0)^2+(x_3-\rho_0'\sigma_0)^2+(x_4-\omega_0((\rho_0'\sigma_0)'+\frac{\rho_0}{\sigma_0}))^2 = \rho_0^2+(\rho_0'\sigma_0)^2+\omega_0^2((\rho_0'\sigma_0)'+\frac{\rho_0}{\sigma_0})^2, \tag{3.1}$$

where

$$\rho_0 = \frac{1}{K(0)},\ \sigma_0 = \frac{1}{k(0)},\ \omega_0 = \frac{1}{r(0) - K(0)}.$$

Proof If $f(0) = 0$ then $h(\alpha(0) - d, \alpha(0) - d) = R^2$. Since we have

$$f' = 2h(\alpha', \alpha - d) \ \text{ and } \ f'(0) = 0$$

then

$$h(T_0, \alpha(0) - d) = 0. \tag{3.2}$$

Similarly we have

$$f'' = 2[h(\alpha'', \alpha - d) + h(\alpha', \alpha')] \ \text{ and } \ f''(0) = 0$$

implies $h(K(0)N_0, \alpha(0) - d) + h(T_0, T_0) = 0$. Since $h(T_0, T_0) = 1$, then

$$h(N_0, \alpha(0) - d) = -\frac{1}{K(0)} = -\rho_0. \tag{3.3}$$

Considering

$$f''' = 2[h(\alpha''', \alpha - d) + 3h(\alpha'', \alpha')] \ \text{ and } \ f'''(0) = 0$$

we get

$$h(-K^2(0)T_0 + K'(0)N_0 + K(0)k(0)B_{1_0}, \alpha(0) - d) = 0.$$

From the equations (3.2) and (3.3) we obtain

$$h(B_{1_0}, \alpha(0) - d) = \frac{K'(0)}{K^2(0)k(0)} = -\rho_0'\sigma_0. \tag{3.4}$$

Since

$$f^{(4)} = 2[h(\alpha^{(4)}, \alpha - d) + 4h(\alpha''', \alpha') + 3h(\alpha'', \alpha'')] \ \text{ and } \ f^{(4)}(0) = 0,$$

from the equations (2.1), (3.1)-(3.4), we obtain

$$h(B_{2_0}, \alpha(0) - d) = -\frac{1}{r(0) - K(0)}\left[(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0})\right] = -\omega_0\left[(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0})\right]. \tag{3.5}$$

Now we investigate the numbers u_1, u_2 , u_3 and u_4 such that

$$\alpha(0) - d = u_1T_0 + u_2N_0 + u_3B_{1_0} + u_4B_{2_0}.$$

From $h(T_0, \alpha(0) - d) = u_1$ and the equation (3.2), then we find $u_1 = 0$. From $h(N_0, \alpha(0) - d) = u_2$ and the equation (3.3), then we find $u_2 = -\rho_0$. From $h(B_{1_0}, \alpha(0) - d) = u_3$ and the equation (3.4), then we obtain $u_3 = -\rho_0'\sigma_0$. From the equation (3.5), we obtain $u_4 = -\omega_0\left[(\rho_0'\sigma_0)' + \frac{\rho_0}{\sigma_0})\right]$. Also the origin of the sphere that contacts at the fourth order to the curve at the point $\alpha(0)$ is

$$d = \alpha(0) - u_1T_0 - u_2N_0 - u_3B_{1_0} - u_4B_{2_0} \tag{3.6}$$

Given a real quaternionic variable P on the osculating sphere, suppose

$$P = \alpha(0) + x_1T_0 + x_2N_0 + x_3B_{1_0} + x_4B_{2_0}$$

and from the equation (3.6)

$$P - d = x_1 T_0 + (x_2 - \rho_0) N_0 + (x_3 - \rho_0' \sigma_0) B_{1_0} + (x_4 - \omega_0 \left[(\rho_0' \sigma_0)' + \frac{\rho_0}{\sigma_0}) \right]) B_{2_0}.$$

Also

$$h(P - d, P - d) = x_1^2 + (x_2 - \rho_0)^2 + (x_3 - \rho_0' \sigma_0)^2 + (x_4 - \omega_0((\rho_0' \sigma_0)' + \frac{\rho_0}{\sigma_0}))^2$$

and using (3.6), we obtain

$$R^2 = h(\alpha(0) - d, \alpha(0) - d) = \rho_0^2 + (\rho_0' \sigma_0)^2 + \omega_0^2((\rho_0' \sigma_0)' + \frac{\rho_0}{\sigma_0})^2. \qquad \square$$

Definition 3.2 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve with nonzero curvatures K, k and $r - K$. The functions $m_i : I \to \mathbb{R}$, $1 \leq i \leq 4$ such that*

$$\begin{cases} m_1 = 0, \\ m_2 = \dfrac{1}{K}, \\ m_3 = \dfrac{m_2'}{k}, \\ m_4 = \dfrac{m_3' + k m_2}{r - K} \end{cases} \tag{3.7}$$

is called m_i curvature function.

Corollary 3.3 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve with nonzero curvatures K, k, $r - K$ and the Frenet frame $\{T, N, B_1, B_2\}$. If $d(s)$ is the center of the osculating sphere at $\alpha(s)$, then*

$$d = \alpha(s) + m_2(s) N(s) + m_3(s) B_1(s) + m_4(s) B_2(s). \tag{3.8}$$

Moreover the radius of the osculating sphere at $\alpha(s)$ is

$$R = \sqrt{m_2^2(s) + m_3^2(s) + m_4^2(s)}. \tag{3.9}$$

Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve. If $\alpha(I) \subset S^3(m, R)$, then α is called spherical curve. We obtain new characterization for spherical curve α.

Theorem 3.4 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve and $\alpha(I) \subset S^3(0, R)$. Then*

$$h(\alpha(s), V_j(s)) = -m_j(s), \quad 1 \leq j \leq 4,$$

where $V_1 = T$, $V_2 = N$, $V_3 = B_1$ and $V_4 = B_2$.

Proof Since $\alpha(s) \in S^3(0, R)$ for all $s \in I$, then $h(\alpha(s), \alpha(s)) = R^2$. Derivating of this equation with respect to s four times and from the equation (3.7), we get

$$h(V_1(s), \alpha(s)) = h(T(s), \alpha(s)) = 0,$$

$$h(V_2(s), \alpha(s)) = h(N(s), \alpha(s)) = -\frac{1}{K(s)} = -m_2(s),$$

$$h(V_3(s),\alpha(s)) = h(B_1(s),\alpha(s)) = -\left(\frac{1}{K(s)}\right)'\frac{1}{k(s)} = -\frac{m_2'(s)}{k(s)} = -m_3(s)$$

and

$$\begin{aligned} h(V_4(s),\alpha(s)) &= h(B_2(s),\alpha(s)) \\ &= -\left[\left(\left(\frac{1}{K(s)}\right)'\frac{1}{k(s)}\right)' + \frac{k(s)}{K(s)}\right]\frac{1}{r(s)-K(s)} \\ &= -\frac{m_3'(s)+k(s)m_2(s)}{r(s)-K(s)} \\ &= -m_4(s). \end{aligned}$$

□

Theorem 3.5 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve. If $\alpha(I) \subset S^3(0,R)$, then the osculating sphere at $\alpha(s)$ for each $s \in I$ is $S^3(0,R)$.*

Proof We assume $\alpha(I) \subset S^3(0,R)$. From the equation (3.8), the center of the osculating sphere at $\alpha(s)$ is

$$\begin{aligned} d &= \alpha(s) + m_2(s)N(s) + m_3(s)B_1(s) + m_4(s)B_2(s) \\ &= \alpha(s) + m_2(s)V_2(s) + m_3(s)V_3(s) + m_4(s)V_4(s). \end{aligned}$$

According to Theorem 3.4

$$d = \alpha(s) - \sum_{j=2}^{4} h(\alpha(s),V_j(s))V_j(s). \tag{3.10}$$

On the other hand

$$\alpha(s) = \sum_{j=1}^{4} h(\alpha(s),V_j(s))V_j(s)$$

and since $h(\alpha(s),V_1(s)) = 0$, we have

$$\alpha(s) = \sum_{j=2}^{4} h(\alpha(s),V_j(s))V_j(s). \tag{3.11}$$

From the equations (3.10) and (3.11), we get $d = 0$. In addition we have

$$h(\alpha(s),d) = R.$$

□

In general, above theorem is valid for the sphere $S^3(b,R)$ with the center b. As well as $S^3(0,R)$ isometric to $S^3(b,R)$, the truth can be avowable. Now, we give relationship between center and radius of the osculating sphere following.

Theorem 3.6 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve with nonzero curvatures K, k, $r-K$ and m_4. The radii of the osculating spheres at $\alpha(s)$ for all $s \in I$ is constant iff the centers of the osculating spheres at $\alpha(s)$ are fixed.*

Proof We assume that the radius of the osculating sphere at $\alpha(s)$ for all $s \in I$ is constant. From the equation (3.9)

$$R(s)^2 = m_2^2(s) + m_3^2(s) + m_4^2(s).$$

Derivating of the equation with respect to s, we obtain

$$m_2(s)m_2'(s) + m_3(s)m_3'(s) + m_4(s)m_4'(s) = 0.$$

Since $m_3(s) = \dfrac{m_2'(s)}{k(s)}$ and $m_4(s) = \dfrac{m_3'(s) + k(s)m_2(s)}{r(s) - K(s)}$, then

$$(r(s) - K(s))m_3(s) + m_4'(s) = 0. \tag{3.12}$$

On the other hand derivating of the equation (3.8) with respect to s and from the equations (3.7), (3.12), we get

$$d'(s) = 0.$$

Thus the center $d(s)$ of the osculating sphere at $\alpha(s)$ is fixed.

Conversely, let the center $d(s)$ of the osculating sphere at $\alpha(s)$ for all $s \in I$ be fixed. Since

$$h(d(s) - \alpha(s), d(s) - \alpha(s)) = R^2(s),$$

derivating of the equation with respect to s, we obtain

$$h(T(s), \alpha(s) - d(s)) = R'(s)R(s).$$

Left hand side this equation is zero. Hence $R'(s) = 0$ and than the radius of the osculating sphere at $\alpha(s)$ for all $s \in I$ is constant. □

Theorem 3.7 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve. The curve is spherical iff the centers of the osculating spheres at $\alpha(s)$ are fixed.*

Proof We assume $\alpha(I) \subset S^3(b, R)$. According to Theorem 3.6 the proof is clearly. Conversely, according to Theorem 3.5 if the centers $d(s)$ of the osculating spheres at $\alpha(s)$ for all $s \in I$ are fixed point b, then the radii of the osculating spheres is constant R. Thus $h(\alpha(s), b) = R$ and than α is spherical. □

Now we give a characterization for spherical curve α in terms of its curvatures K, k and $r - K$ in following theorem.

Theorem 3.8 *Let $\alpha : I \subset \mathbb{R} \to Q_H$ be a real quaternionic curve with nonzero curvatures K, k, $r - K$ and m_4. The curve α is spherical iff*

$$\frac{r-K}{k}\left(\frac{1}{K}\right)' + \left\{\left[\left(\left(\frac{1}{K}\right)'\frac{1}{k}\right)' + \frac{k}{K}\right]\frac{1}{r-K}\right\}' = 0. \tag{3.13}$$

Proof Let the curve α be spherical. According to Theorem 3.7 the centers $d(s)$ of the osculating spheres at $\alpha(s)$ for all $s \in I$ are fixed. From the equations (3.7) and (3.12) we obtain (3.13).

Conversely we assume

$$\frac{r-K}{k}\left(\frac{1}{K}\right)' + \left\{\left[\left(\left(\frac{1}{K}\right)'\frac{1}{k}\right)' + \frac{k}{K}\right]\frac{1}{r-K}\right\}' = 0$$

From the equation (3.7), we get

$$(r-K)m_3 + m_4' = 0.$$

Derivating equation (3.8) with respect to s and from the last equation and (3.7), we obtain $d'(s) = 0$. Hence $d(s)$ is fixed point. According to Theorem 3.7 the curve α is spherical. □

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Cover Pebbling Number for Square of a Path

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Abstract: Given a graph G and a configuration C of pebbles on the vertices of G, a pebbling step (move) removes two pebbles from one vertex and places one pebble on an adjacent vertex. The cover pebbling number $\gamma(G)$ is the minimum number so that every configuration of $\gamma(G)$ pebbles has the property that after some sequence of pebbling steps(moves), every vertex has a pebble on it. In this paper we determine the cover pebbling number for square of a path.

Key Words: Cover pebbling, square of a path, Smarandachely cover H-pebbling.

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§1. Introduction

The game of pebbling was first suggested by Lagarias and Saks as a tool for solving a number-theoretical conjecture of Erdos. Chung successfully used this tool to prove the conjecture and established other results concerning pebbling numbers. In doing so she introduced pebbling to the literature [1].

Begin with a graph G and a certain number of pebbles placed on its vertices. A pebbling step consists of removing two pebbles from one vertex and placing one on an adjacent vertex. In pebbling, the target is selected, and the goal is to move a pebble to the target vertex. The minimum number of pebbles such that regardless of their initial placement and regardless of the target vertex, we can pebble that target is called the pebbling number of G. In cover pebbling, the goal is to cover all the vertices with pebbles, that is, to move a pebble to every vertex simultaneously. Generally, for a connected subgraph $H < G$, a *Smarandachely cover H-pebbling* is to move a pebble to every vertex in H but not in $G \setminus H$ simultaneously. The minimum number of pebbles required such that, regardless of their initial placement on G, there is a sequence of pebbling steps at the end of which every vertex has at least one pebble on it, is called the cover pebbling number of G. In [2], Crull et al. determine the cover pebbling number of several families of graphs, including trees and complete graphs. Hulbert and Munyan [4] have also announced a proof for the cover pebbling of the n-dimensional cube. In [5], Maggy Tomova and Cindy Wyles determine the cover pebbling number for cycles and certain graph products. In the next section, we determine the cover pebbling number for square of a path.

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§2. The Cover Pebbling Number for square of a Path

Definition([6]) *Let $G = (V(G), E(G))$ be a connected graph. The nth power of G, denoted by G^p, is the graph obtained from G by adding the edge uv to G whenever $2 \leq d(u,v) \leq p$ in G, that is, $G^p = (V(G), E(G) \cup \{uv : 2 \leq d(u,v) \leq p \text{ in } G\})$. If p=1, we define G^1=G. We know that if p is large enough, that is, $p \geq n-1$, then $G^p = K_n$.*

Notation 2.2 The Labeling of P_n^2 is $P_n^2 : \ v_1v_2\cdots v_{n-1}v_n$. Let $p(v_i)$ denote the number of pebbles on the vertex v_i and $p(P_A)$ denote the number of pebbles on the path P_A.

It is easy to see that $\gamma(P_3^2) = 5$ since $P_3^2 \cong K_3$ [2].

Theorem 2.3 *The cover pebbling number of P_4^2 is $\gamma(P_4^2) = 9$.*

Proof Consider the distribution of eight pebbles on v_1. Clearly, we cannot cover at least one of the vertices of P_4^2. Thus,$\gamma(P_4^2) \geq 9$.

Now, consider the distribution of nine pebbles on the vertices of P_4^2. If v_4 has zero pebbles on it, then using at most four pebbles from $P_3^2 : v_1v_2v_3$ we can move a pebble to v_4. After moving a pebble to v_4, P_3^2 contains at least five pebbles and we are done. Next assume that v_4 has at least one pebble. If $p(v_4) \leq$4, then $p(P_3^2) \geq$5 and we are done. If $p(v_4) = 5$ or 6 or 7, clearly we are done. If $p(v_4) \geq$8, then move as many as possible to the vertices of P_3^2 using at most four moves while retaining one or two pebbles on v_4, we cover all the vertices of P_4^2 in these distributions also. Thus, $\gamma(P_4^2) \leq 9$. Therefore, $\gamma(P_4^2) = 9$. □

Theorem 2.4 *The cover pebbling number of P_5^2 is $\gamma(P_5^2) = 13$.*

Proof Consider the distribution of twelve pebbles on v_1. Clearly, we cannot cover at least one of the vertices of P_5^2. Thus,$\gamma(P_5^2) \geq 13$.

Now, consider the distribution of thirteen pebbles on the vertices of P_5^2. If v_5 has zero pebbles on it, then using at most four pebbles from $P_4^2 : v_1v_2v_3v_4$ we can move a pebble to v_5. After moving a pebble to v_5, P_4^2 contains at least nine pebbles and we are done. Next assume that v_5 has at least one pebble. If $p(v_5) \leq$4, then $p(P_4^2) \geq$9 and we are done. If $p(v_5) = 5$ or 6 or 7, then clearly we are done. If $p(v_5) \geq 8$, then move as many as possible to the vertices of P_4^2 using at most four moves while retaining one or two pebbles on v_5, we cover all the vertices of P_5^2 in these distributions also. Thus, $\gamma(P_5^2) \leq 13$. Therefore, $\gamma(P_5^2) = 13$. □

Theorem 2.5 *The cover pebbling number of P_n^2 is*

$$\gamma(P_n^2) = \begin{cases} 2^{k+2} - 3 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k - 1) & if\ n = 2k\ (k \geq 2). \end{cases}$$

Proof Consider the following distribution

$$p(v_1) = \begin{cases} 2^{k+2} - 4 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k) - 4 & if\ n = 2k\ (k \geq 2). \end{cases}$$

and $p(v_i) = 0, i \neq 1$. Notice that we cannot cover at least one of the vertices of P_n^2. Thus,

$$\gamma(P_n^2) \geq \begin{cases} 2^{k+2} - 3 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k - 1) & if\ n = 2k\ (k \geq 2). \end{cases}$$

. Next, we are going to prove the upper bound by induction on n. Obviously, the result is true for n = 4 and 5, by Theorem 2.3 and Theorem 2.4. So, assume the result is true for m $\leq$ n-1. If v_n has zero pebbles on it, then using at most 2k pebbles from the vertices of $P_{n-1}^2 : v_1 v_2 \cdots v_{n-2} v_{n-1}$ we can cover the vertex v_n. Then P_{n-1}^2contains at least

$$\begin{cases} 3(2^k - 1), & where\ k = \dfrac{n-1}{2}; \\ 2^{k+1} - 3, & where\ k = \dfrac{n}{2} \end{cases}$$

pebbles and we are done by induction. Next, assume that v_n has a pebble on it. If $p(v_n) \leq 2(2^k - 1)$, then

$$p(P_{n-1}^2) \geq \begin{cases} 2^{k+1} - 3 & if\ n\ is\ odd; \\ 2^k - 1 & if\ n\ is\ even. \end{cases}$$

In these both cases, either P_{n-1}^2 has enough pebbles or we can make it by retaining one or two pebbles on v_n and moving as many pebbles as possible from v_n to v_{n-1} or v_{n-2}. So, we are done easily if $p(v_n) \leq 2(2^k - 1)$. Suppose $p(v_n) \geq 2(2^k - 1) + 1$, then by moving as many pebbles as possible to the vertices of P_{n-1}^2, using at most

$$\begin{cases} 2^{k+1} - 2 & if\ n\ is\ odd; \\ 3(2^{k-1}) - 2 & if\ n\ is\ even \end{cases}$$

pebbling steps while retaining one or two pebbles on v_n, and hence we are done. Thus,

$$\gamma(P_n^2) \leq \begin{cases} 2^{k+2} - 3 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k - 1) & if\ n = 2k\ (k \geq 2). \end{cases}$$

Therefore,

$$\gamma(P_n^2) = \begin{cases} 2^{k+2} - 3 & if\ n = 2k+1\ (k \geq 1); \\ 3(2^k - 1) & if\ n = 2k\ (k \geq 2). \end{cases} \qquad \square$$

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Switching Equivalence in Symmetric n-Sigraphs-V

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Abstract: An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \to H_n$ $(\mu : V \to H_n)$ is a function. In this paper, we introduced a new notion $\mathcal{S}$-antipodal symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also we give the relation between antipodal symmetric n-sigraphs and $\mathcal{S}$-antipodal symmetric n-sigraphs. Further, we discuss structural characterization of $\mathcal{S}$-antipodal symmetric n-sigraphs.

Key Words: Symmetric n-sigraphs, Smarandachely symmetric n-marked graph, symmetric n-marked graphs, balance, switching, antipodal symmetric n-sigraphs, $\mathcal{S}$-antipodal symmetric n-sigraphs, complementation.

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§1. Introduction

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [1]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \cdots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \cdots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \lceil \frac{n}{2} \rceil$.

A *Smarandachely k-marked graph* (*Smarandachely k-signed graph*) is an ordered pair $S = (G, \mu)$ $(S = (G, \sigma))$ where $G = (V, E)$ is a graph called *underlying graph* of S and $\mu : V \to \{\overline{e}_1, \overline{e}_2, ..., \overline{e}_k\}$ $(\sigma : E \to \{\overline{e}_1, \overline{e}_2, ..., \overline{e}_k\})$ is a function, where $\overline{e}_i \in \{+, -\}$. An *n-tuple* $(a_1, a_2, ..., a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, ..., a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. A *Smarandachely symmetric n-marked graph* (*Smarandachely symmetric n-signed graph*) is an ordered pair $S_n = (G, \mu)$ $(S_n = (G, \sigma))$ where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\mu : V \to H_n$ $(\sigma : E \to H_n)$ is a function. Particularly, a Smarandachely 1-marked graph (Smarandachely 1-signed graph) is called a *marked graph* (*signed graph*).

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In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple/symmetric n-sigraph/symmetric n-marked graph.

An n-tuple $(a_1, a_2, \cdots, a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*. Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [7], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P.S.K.Reddy [4]):

Definition 1.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then,*

(i) S_n is identity balanced *(or* i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and*

(ii) S_n is balanced*, if every cycle in S_n contains an even number of non-identity edges.*

Note 1.1 An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [7].

Proposition 1.1 (E. Sampathkumar et al. [7]) *An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of S_n is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma^c)$, where for any edge $e = uv \in \overline{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1.1 ([10]).

In [7], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows (See also [2,5,6,10]):

Let $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$, be two n-sigraphs. Then S_n and S'_n are said to be *isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \cdots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S'_n with label $(a_1, a_2, \cdots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $\mathcal{S}_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*. Further, an n-sigraph S_n *switches* to n-sigraph S'_n (or that they are *switching equivalent* to each other), written as $S_n \sim S'_n$, whenever there exists an n-marking of S_n such that $\mathcal{S}_\mu(S_n) \cong S'_n$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S'_n = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \to G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\sigma(\phi(C))$ in S'_n. We make use of the following known result (see [7]).

Proposition 1.2 (E. Sampathkumar et al. [7]) *Given a graph G, any two n-sigraphs with G* *as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the n-tuples on the edges incident at v. *Complement* of S is an n-sigraph $\overline{S_n} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ as defined here is an i-balanced n-sigraph due to Proposition 1.1.

§2. $\mathcal{S}$-Antipodal n-Sigraphs

Radhakrishnan Nair and Vijayakumar [3] has introduced the concept of $\mathcal{S}$-antipodal graph of a graph G as the graph $A^*(G)$ has the vertices in G with maximum eccentricity and two vertices of $A^*(G)$ are adjacent if they are at a distance of $diam(G)$ in G.

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of $\mathcal{S}$-antipodal graphs to n-sigraphs as follows:

The *$\mathcal{S}$-antipodal n-sigraph* $A^*(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $A^*(G)$ and the n-tuple of any edge uv is $A^*(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called $\mathcal{S}$-antipodal n-sigraph, if $S_n \cong A^*(S'_n)$ for some n-sigraph S'_n. The following result indicates the limitations of the notion $A^*(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be $\mathcal{S}$-antipodal n-sigraphs.

Proposition 2.1 *For any n-sigraph $S_n = (G, \sigma)$, its $\mathcal{S}$-antipodal n-sigraph $A^*(S_n)$ is i-balanced.*

Proof Since the n-tuple of any edge uv in $A^*(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Proposition 1.1, $A^*(S_n)$ is i-balanced. $\square$

For any positive integer k, the k^{th} iterated $\mathcal{S}$-antipodal n-sigraph $A^*(S_n)$ of S_n is defined as follows:

$$(A^*)^0(S_n) = S_n,\ (A^*)^k(S_n) = A^*((A^*)^{k-1}(S_n))$$

Corollary 2.2 *For any n-sigraph $S_n = (G, \sigma)$ and any positive integer k, $(A^*)^k(S_n)$ is i-balanced.*

In [3], the authors characterized those graphs that are isomorphic to their $\mathcal{S}$-antipodal graphs.

Proposition 2.3(Radhakrishnan Nair and Vijayakumar [3]) *For a graph $G = (V, E)$, $G \cong A^*(G)$ if, and only if, G is a regular self-complementary graph.*

We now characterize the n-sigraphs that are switching equivalent to their $\mathcal{S}$-antipodal n-sigraphs.

Proposition 2.4 *For any n-sigraph $S_n = (G, \sigma)$, $S_n \sim A^*(S_n)$ if, and only if, G is regular* *self-complementary graph and S_n is i-balanced n-sigraph.*

Proof Suppose $S_n \sim A^*(S_n)$. This implies, $G \cong A^*(G)$ and hence G is is a regular self-complementary graph. Now, if S_n is any n-sigraph with underlying graph as regular self-complementary graph, Proposition 2.1 implies that $A^*(S_n)$ is i-balanced and hence if S is i-unbalanced and its $A^*(S_n)$ being i-balanced can not be switching equivalent to S_n in accordance with Proposition 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is an i-balanced n-sigraph and G is regular self-complementary. Then, since $A^*(S_n)$ is i-balanced as per Proposition 2.1 and since $G \cong A^*(G)$, the result follows from Proposition 1.2 again. □

Proposition 2.5 *For any two vs S_n and S'_n with the same underlying graph, their $\mathcal{S}$-antipodal n-sigraphs are switching equivalent.*

Remark 2.6 If G is regular self-complementary graph, then $G \cong \overline{G}$. The above result is holds good for $\overline{S_n} \sim A^*(S_n)$.

In [16], P.S.K.Reddy et al. introduced antipodal n-sigraph of an n-sigraph as follows:

The *antipodal n-sigraph* $A(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $A(G)$ and the n-tuple of any edge uv in $A(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called antipodal n-sigraph, if $S_n \cong A(S'_n)$ for some n-sigraph S'_n.

Proposition 2.7(P.S.K.Reddy et al. [16]) *For any n-sigraph $S_n = (G, \sigma)$, its antipodal n-sigraph $A(S_n)$ is i-balanced.*

We now characterize n-sigraphs whose $\mathcal{S}$-antipodal n-sigraphs and antipodal n-sigraphs are switching equivalent. In case of graphs the following result is due to Radhakrishnan Nair and Vijayakumar [3].

Proposition 2.8 *For a graph $G = (V, E)$, $A^*(G) \cong A(G)$ if, and only if, G is self-centred.*

Proposition 2.9 *For any n-sigraph $S_n = (G, \sigma)$, $A^*(S_n) \sim A(S_n)$ if, and only if, G is self-centred.*

Proof Suppose $A^*(S_n) \sim A(S_n)$. This implies, $A^*(G) \cong A(G)$ and hence by Proposition 2.8, we see that the graph G must be self-centred.

Conversely, suppose that G is self centred. Then $A^*(G) \cong A(G)$ by Proposition 2.8. Now, if S_n is an n-sigraph with underlying graph as self centred, by Propositions 2.1 and 2.7, $A^*(S_n)$ and $A(S_n)$ are i-balanced and hence, the result follows from Proposition 1.2.

In [3], the authors shown that $A^*(G) \cong A^*(\overline{G})$ if G is either complete or totally disconnected. We now characterize n-sigraphs whose $A^*(S_n)$ and $A^*(\overline{S_n})$ are switching equivalent.

Proposition 2.10 *For any signed graph $S = (G, \sigma)$, $A^*(S_n) \sim A^*(\overline{S_n})$ if, and only if, G is either complete or totally disconnected.*

The following result characterize n-sigraphs which are $\mathcal{S}$-antipodal n-sigraphs.

Proposition 2.11 *An n-sigraph $S_n = (G, \sigma)$ is a $\mathcal{S}$-antipodal n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a $\mathcal{S}$-antipodal graph.*

Proof Suppose that S_n is i-balanced and G is a $A(G)$. Then there exists a graph H such that $A^*(H) \cong G$. Since S_n is i-balanced, by Proposition 1.1, there exists an n-marking μ of G such that each edge uv in S_n satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the n-sigraph $S'_n = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $A^*(S'_n) \cong S_n$. Hence S_n is a $\mathcal{S}$-antipodal n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a $\mathcal{S}$-antipodal n-sigraph. Then there exists an n-sigraph $S'_n = (H, \sigma')$ such that $A^*(S'_n) \cong S_n$. Hence G is the $A^*(G)$ of H and by Proposition 2.1, S_n is i-balanced. □

§3. Complementation

In this section, we investigate the notion of complementation of a graph whose edges have signs (a *sigraph*) in the more general context of graphs with multiple signs on their edges. We look at two kinds of complementation: complementing some or all of the signs, and reversing the order of the signs on each edge.

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2, \cdots., a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$. For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written (S_n^m), is the same graph but with each edge label $a = (a_1, a_2, ..., a_n)$ replaced by a^m. For an n-sigraph $S_n = (G, \sigma)$, the $A^*(S_n)$ is i-balanced (Proposition 2.1). We now examine, the condition under which m-complement of $A(S_n)$ is i-balanced, where for any $m \in H_n$.

Proposition 3.1 *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $A^*(G)$ is bipartite then $(A^*(S_n))^m$ is i-balanced.*

Proof Since, by Proposition 2.1, $A^*(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $A^*(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $A^*(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $A^*(S_n)$ whose k^{th} co-ordinate are $+$ is also even. This implies that the same thing is true in any m-complement, where for any $m, \in H_n$. Hence $(A^*(S_n))^t$ is i-balanced. □

Problem 3.2 *Characterize these n-sigraphs for which*

(1) $(S_n)^m \sim A^*(S_n)$;
(2) $(\overline{S_n})^m \sim A(S_n)$;
(3) $(A^*(S_n))^m \sim A(S_n)$;
(4) $A^*(S_n) \sim (A(S_n))^m$;
(5) $(A^*(S))^m \sim A^*(\overline{S_n})$;
(6) $A^*(S_n) \sim (A^*(\overline{S_n}))^m$.

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Further Results on Product Cordial Labeling

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Abstract: We prove that closed helm CH_n, web graph Wb_n, flower graph Fl_n, double triangular snake DT_n and gear graph G_n admit product cordial labeling.

Key Words: Graph labeling, cordial labeling, Smarandachely p-product cordial labeling, product cordial labeling.

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§1. Introduction

We begin with finite, connected and undirected graph $G = (V(G), E(G))$ without loops and multiple edges. For any undefined notations and terminology we rely upon Clark and Holton [3]. In order to maintain compactness we provide a brief summery of definitions and existing results.

Definition 1.1 *A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling).*

According to Beineke and Hegde [1] labeling of discrete structure serves as a frontier between graph theory and theory of numbers. A dynamic survey of graph labeling is carried out and frequently updated by Gallian [4].

Definition 1.2 *A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f.*

The induced edge labeling $f^* : E(G) \to \{0, 1\}$ is given by $f^*(e = uv) = |f(u) - f(v)|$. Let us denote $v_f(0)$, $v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0)$, $e_f(1)$ be the number of edges of G having labels 0 and 1 respectively under f^*.

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Definition 1.3 *A binary vertex labeling of a graph G is called a cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called cordial if it admits cordial labeling.*

The concept of cordial labeling was introduced by Cahit [2] in which he investigated several results on this newly defined concept. After this some labelings like prime cordial labeling, A - cordial labeling, H-cordial labeling and product cordial labeling are also introduced as variants of cordial labeling.

This paper is aimed to report some new families of product cordial graphs.

Definition 1.4 *For an integer $p > 1$. A mapping $f : V(G) \to \{0, 1, 2, \cdots, p\}$ is called a Smarandachely p-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2, \cdots, p-1\}$, where $v_f(i)$ denotes the number of vertices labeled with i, $e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i(mod\ p)$. Particularly, if $p = 2$, i.e., a binary vertex labeling of graph G with an induced edge labeling $f^* : E(G) \to \{0, 1\}$ defined by $f^*(e = uv) = f(u)f(v)$, such a Smarandachely 2-product cordial labeling is called product cordial labeling. A graph with product cordial labeling is called a product cordial graph.*

The product cordial labeling was introduced by Sundaram et al. [5] and they investigated several results on this newly defined concept. They have established a necessary condition showing that a graph with p vertices and q edges with $p \geq 4$ is product cordial then $q < (p^2 - 1)/4 + 1$.

The graphs obtained by joining apex vertices of k copies of stars, shells and wheels to a new vertex are proved to be product cordial by Vaidya and Dani [6] while some results on product cordial labeling for cycle related graphs are reported in Vaidya and Kanani [7].

Vaidya and Barasara [8] have proved that the cycle with one chord, the cycle with twin chords, the friendship graph and the middle graph of path admit product cordial labeling. The same authors in [9] have proved that the graphs obtained by duplication of one edge, mutual vertex duplication and mutual edge duplication in cycle are product cordial graphs. Vaidya and Vyas [10] have discussed product cordial labeling in the context of tensor product of some graphs while Vaidya and Barasara [11] have investigated some results on product cordial labeling in the context of some graph operations.

Definition 1.5 *The wheel graph W_n is defined to be the join $K_1 + C_n$. The vertex corresponding to K_1 is known as apex vertex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges. We continue to recognize apex of respective graphs obtained from wheel in Definitions 1.6 to 1.9.*

Definition 1.6 *The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.*

Definition 1.7 *The closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle.*

Definition 1.8 *The web graph Wb_n is the graph obtained by joining the pendant vertices of a helm H_n to form a cycle and then adding a pendant edge to each vertex of outer cycle.*

Definition 1.9 *The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.*

Definition 1.10 *The double triangular snake DT_n is obtained from a path P_n with vertices $v_1, v_2, \cdots, v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, \cdots, n-1$ and to a new vertex u_i for $i = 1, 2, \cdots, n-1$.*

Definition 1.11 *Let $e = uv$ be an edge of graph G and w is not a vertex of G. The edge e is subdivided when it is replaced by edges $e' = uw$ and $e'' = wv$.*

Definition 1.12 *The gear graph G_n is obtained from the wheel by subdividing each of its rim edge.*

§2. Main Results

Theorem 2.1 *Closed helm CH_n is a product cordial graph.*

Proof Let v be the apex vertex, $v_1, v_2, \ldots, v_n$ be the vertices of inner cycle and $u_1, u_2, \ldots, u_n$ be the vertices of outer cycle of CH_n. Then $|V(CH_n)| = 2n + 1$ and $|E(CH_n)| = 4n$.

We define $f : V(CH_n) \to \{0, 1\}$ to be $f(v) = 1$, $f(v_i) = 1$ and $f(u_i) = 0$ for all i. In view of the above labeling patten we have $v_f(0) = v_f(1) - 1 = n$, $e_f(0) = e_f(1) = 2n$. Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence CH_n is a product cordial graph. $\square$

Illustration 2.2 The Fig.1 shows the closed helm CH_5 and its product cordial labeling.

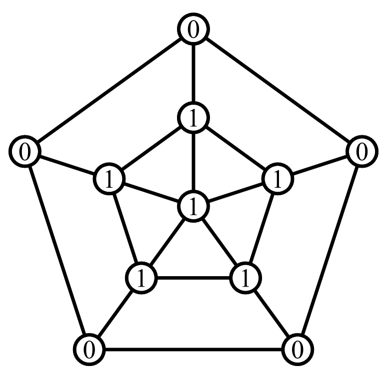

Fig.1

Theorem 2.3 *Web graph Wb_n admits product cordial labeling.*

Proof Let v be the apex vertex, $v_1, v_2, \cdots, v_n$ be the vertices of inner cycle, $v_{n+1}, v_{n+2}, \cdots, v_{2n}$ be the vertices of outer cycle and $v_{2n+1}, v_{2n+2}, \cdots, v_{3n}$ be the pendant vertices in Wb_n. Then $|V(Wb_n)| = 3n + 1$ and $|E(Wb_n)| = 5n$.

To define $f : V(Wb_n) \to \{0, 1\}$ we consider following two cases.

Case 1. n is odd

Define $f(v) = 1$, $f(v_i) = 1$ for $1 \leq i \leq n$, $f(v_{2i}) = 1$ for $\left\lceil \frac{n}{2} \right\rceil \leq i \leq n-1$ and $f(v_i) = 0$ otherwise. In view of the above labeling patten we have $v_f(0) = v_f(1) = \frac{3n+1}{2}$, $e_f(0) - 1 = e_f(1) = \frac{5n-1}{2}$.

Case 2. n is even

Define $f(v) = 1$, $f(v_i) = 1$ for $1 \leq i \leq n$, $f(v_{2i+1}) = 1$ for $\frac{n}{2} \leq i \leq n-1$ and $f(v_i) = 0$ otherwise. In view of the above labeling patten we have $v_f(0) = v_f(1) - 1 = \frac{3n}{2}$, $e_f(0) = e_f(1) = \frac{5n}{2}$. Thus in each case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence Wb_n admits product cordial labeling. □

Illustration 2.4 The Fig.2 shows the web graph Wb_5 and its product cordial labeling.

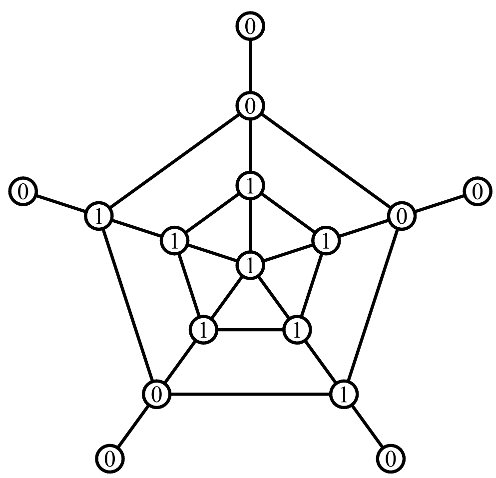

Fig.2

Theorem 2.5 *Flower graph Fl_n admits product cordial labeling.*

Proof Let H_n be a helm with v as the apex vertex, $v_1, v_2, \cdots, v_n$ be the vertices of cycle and $v_{n+1}, v_{n+2}, \cdots, v_{2n}$ be the pendant vertices. Let Fl_n be the flower graph obtained from helm H_n. Then $|V(Fl_n)| = 2n + 1$ and $|E(Fl_n)| = 4n$.

We define $f : V(Fl_n) \to \{0, 1\}$ to be $f(v) = 1$, $f(v_i) = 1$ for $1 \leq i \leq n$ and $f(v_i) = 0$ for $n + 1 \leq i \leq 2n$. In view of the above labeling patten we have $v_f(0) = v_f(1) - 1 = n$, $e_f(0) = e_f(1) = 2n$. Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Hence Fl_n admits product cordial labeling. □

Illustration 2.6 The Fig.3 shows flower graph Fl_5 and its product cordial labeling.

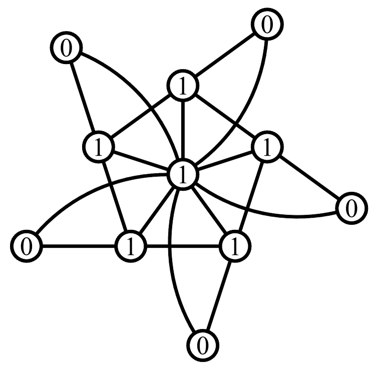

Fig.3

Theorem 2.7 *Double triangular snake DT_n is a product cordial graph for odd n and not a product cordial graph for even n.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of path P_n and $v_{n+1}, v_{n+2}, \cdots, v_{3n-2}$ be the newly added vertices in order to obtain DT_n. Then $|V(DT_n)| = 3n-2$ and $|E(DT_n)| = 5n-5$.

To define $f : V(DT_n) \to \{0,1\}$ we consider following two cases.

Case 1. n is odd

$f(v_i) = 0$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $f(v_i) = 0$ for $n+1 \leq i \leq n + \left\lfloor \frac{n}{2} \right\rfloor$ and $f(v_i) = 1$ otherwise. In view of the above labeling patten we have $v_f(0) + 1 = v_f(1) = \left\lceil \frac{3n-2}{2} \right\rceil$, $e_f(0) - 1 = e_f(1) = \frac{5n-5}{2}$. Thus we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 2. n is even

Subcase 1. $n = 2$.

The graph DT_2 has $p = 4$ vertices and $q = 5$ edges since

$$\frac{p^2-1}{4} + 1 = \frac{19}{4} < q.$$

Thus the necessary condition for product cordial graph is violated. Hence DT_2 is not a product cordial graph.

Subcase 2. $n \neq 2$

In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to $\frac{3n-2}{2}$ vertices out of $3n-2$ vertices. The vertices with label 0 will give rise at least $\frac{5n}{2} - 1$ edges with label 0 and at most $\frac{5n}{2} - 4$ edge with label 1 out of total $5n-5$ edges. Therefore $|e_f(0) - e_f(1)| = 3$. Thus the edge condition for product cordial graph is violated. Therefore DT_n is not a product cordial graph for even n.

Hence Double triangular snake DT_n is a product cordial graph for odd n and not a product cordial graph for even n. $\square$

Illustration 2.8 The Fig.4 shows the double triangular snake DT_7 and its product cordial labeling.

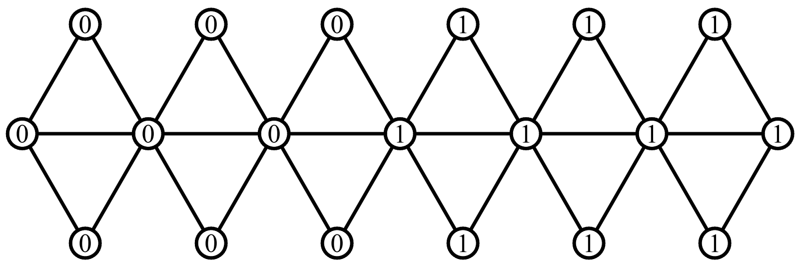

Fig.4

Theorem 2.9 *Gear graph G_n is a product cordial graph for odd n and not product cordial graph for even n.*

Proof Let W_n be the wheel with apex vertex v and rim vertices $v_1, v_2, \cdots, v_n$. To obtain the gear graph G_n subdivide each rim edge of wheel by the vertices $u_1, u_2, \cdots, u_n$. Where each u_i subdivides the edge $v_i v_{i+1}$ for $i = 1, 2, \cdots, n-1$ and u_n subdivides the edge $v_1 v_n$. Then $|V(G_n)| = 2n + 1$ and $|E(G_n)| = 3n$.

To define $f : V(G_n) \to \{0, 1\}$ we consider following two cases.

Case 1. n is odd

$$f(v) = 1;\ f(v_i) = 1 \text{ for } 1 \le i \le \left\lceil \frac{n}{2} \right\rceil;\ f(v_i) = 0, \text{ otherwise;}$$
$$f(u_i) = 1 \text{ for } 1 \le i \le n + \left\lfloor \frac{n}{2} \right\rfloor;\ f(u_i) = 0, \text{ otherwise.}$$

In view of the above labeling patten we have $v_f(0) = v_f(1) - 1 = n$, $e_f(0) = e_f(1) + 1 = \frac{3n+1}{2}$. Thus we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$.

Case 2. n is even

In order to satisfy the vertex condition for product cordial graph it is essential to assign label 0 to n vertices out of $2n + 1$ vertices. The vertices with label 0 will give rise at least $\frac{3n}{2} + 1$ edges with label 0 and at most $\frac{3n}{2} - 1$ edge with label 1 out of total $3n$ edges. Therefore $|e_f(0) - e_f(1)| = 2$. Thus the edge condition for product cordial graph is violated. So G_n is not a product cordial graph for even n.

Hence gear graph is a product cordial graph for odd n and not product cordial graph for even n. $\square$

Illustration 2.10 The Fig.5 shows the gear graph G_7 and its product cordial labeling.

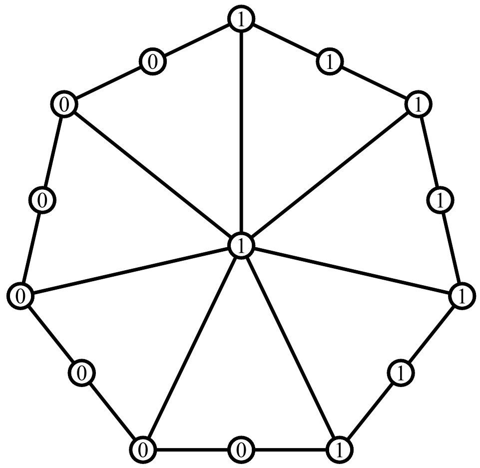

Fig.5

§3. Concluding Remarks

Some new families of product cordial graphs are investigated. To investigate some characterization(s) or sufficient condition(s) for the graph to be product cordial is an open area of research.

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Around The Berge Problem And Hadwiger Conjecture

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Abstract: We say that a graph B is *berge*, if every graph $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5 [$\bar{B}$ is the complementary graph of B]. A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$, where $\chi(G')$ is the *chromatic number* of G' and $\omega(G')$ is the *clique number* of G'. The Berge conjecture states that a graph H is *perfect* if and only if H is *berge*. Indeed, the difficult part of the Berge conjecture consists to show that $\chi(B) = \omega(B)$ for every *berge* graph B. The Hadwiger conjecture states that every graph G satisfies $\chi(G) \leq \eta(G)$ [where $\eta(G)$ is the *hadwiger number* of G (i.e., the maximum of p such that G is *contractible* to the complete graph K_p)]. The Berge conjecture (see [1] or [2] or [3] or [5] or [6] or [7] or [9] or [10] or [11]) was proved by Chudnovsky, Robertson, Seymour and Thomas in a paper of at least 140 pages (see [1]), and an elementary proof of the Berge conjecture was given by Ikorong Nemron in a detailled paper of 37 pages long (see [9]). The Hadwiger conjecture (see [4] or [5] or [7] or [8] or [10] or [11] or [12] or [13] or [15] or [16]) was proved by Ikorong Nemron in a detailled paper of 28 pages long (see [13]), by using arithmetic calculus, arithmetic congruences, elementary complex analysis, induction and reasoning by reduction to absurd. That being so, in this paper, via two simple Theorems, we rigorously show that the difficult part of the Berge conjecture (solved) and the Hadwiger conjecture (also solved), are exactly the same conjecture. The previous immediately implies that, the Hadwiger conjecture is only a non obvious special case of the Berge conjecture.

Key Words: True pal, parent, berge, the berge problem, the berge index, representative, the hadwiger index, son.

AMS(2010): 05CXX

§0. Preliminary and Some Denotations

We recall that in a graph $G = [V(G), E(G), \chi(G), \omega(G), \bar{G}]$, $V(G)$ is the set of vertices, $E(G)$ is the set of edges, $\chi(G)$ is the chromatic number, $\omega(G)$ is the clique number and $\bar{G}$ is the complementary graph of G. We say that a graph B is *berge* if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5. A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. The Berge conjecture states that a graph H is *perfect* if

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and only if H is berge. Indeed the difficult part of the Berge conjecture consists to show that $\chi(B) = \omega(B)$ for every berge graph B. Briefly, the difficult part of the Berge conjecture will be called the Berge problem. In this topic, we rigorously show that the Berge problem and the Hadwiger conjecture are exactly the same problem [the Hadwiger conjecture states that every graph G is $\eta(G)$ colorable (i.e. we can color all vertices of G with $\eta(G)$ colors such that two adjacent vertices do not receive the same color). $\eta(G)$ is the hadwiger number of G and is the maximum of p such that G is contractible to the complete graph K_p]. That being so, this paper is divided into six simple Sections. In Section 1, we present briefly some standard definitions known in Graph Theory. In Section 2 , we introduce definitions that are not standard, and some elementary properties. In Section 3 we define a graph parameter denoted by β (β is called the berge index) and we give some obvious properties of this parameter. In Section 4 we introduce another graph parameter denoted by τ (τ is called the *hadwiger index*) and we present elementary properties of this parameter. In Section 5 , using the couple (β, τ), we show two simple Theorems which are equivalent to the Hadwiger conjecture and the Berge problem. In Section 6, using the two simple Theorems stated and proved in Section 5, we immediately deduce that the Berge problem and the Hadwiger conjecture are exactly the same problem, and therefore, the Hadwiger conjecture is only a non obvious special case of the Berge conjecture. In this paper, all results are simple, and every graph is finite, is simple and is undirected. We start.

§1. Standard Definitions Known in Graph Theory

Recall (see [2] or [14]) that in a graph $G = [V(G), E(G)]$, $V(G)$ is the set of vertices and $E(G)$ is the set of edges. $\bar{G}$ is the complementary graph of G (recall $\bar{G}$ is the *complementary* graph of G, if $V(G) = V(\bar{G})$ and two vertices are adjacent in G if and only if they are not adjacent in $\bar{G}$). A graph F is a *subgraph* of G, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We say that a graph F is an *induced subgraph* of G by Z, if F is a subgraph of G such that $V(F) = Z$, $Z \subseteq V(G)$, and two vertices of F are adjacent in F, if and only if they are adjacent in G. For $X \subseteq V(G)$, $G \setminus X$ denotes the *subgraph* of G induced by $V(G) \setminus X$. A *clique* of G is a subgraph of G that is complete; such a subgraph is necessarily an induced subgraph (recall that a graph K is complete if every pair of vertices of K is an edge of K); $\omega(G)$ is the size of a largest clique of G, and $\omega(G)$ is called the *clique number* of G. A **stable set** of a graph G is a set of vertices of G that induces a subgraph with no edges; $\alpha(G)$ is the size of a largest stable set, and $\alpha(G)$ is called the *stability number* of G. The *chromatic number* of G (denoted by $\chi(G)$) is the smallest number of colors needed to color all vertices of G such that two adjacent vertices do not receive the same color. It is easy to see:

Assertion 1.0 *Let G be a graph. Then $\omega(G) \leq \chi(G)$*

The *hadwiger number* of a graph G (denoted by $\eta(G)$), is the maximum of p such that G is contractible to the complete graph K_p. Recall that, if e is an edge of G incident to x and y, we can obtain a new graph from G by removing the edge e and identifying x and y so that the resulting vertex is incident to all those edges (other than e) originally incident to x or to y. This is called *contracting* the edge e. If a graph F can be obtained from G by a succession of such edge-contractions, then, G is *contractible* to F. The maximum of p such that G is contractible to the complete graph K_p is the hadwiger number of G, and is denoted by $\eta(G)$. The Hadwiger conjecture states that $\chi(G) \leq \eta(G)$ for every graph G. Clearly we have:

Assertion 1.1 *Let G be a graph, and let F be a subgraph of G. Then $\eta(F) \leq \eta(G)$.*

§2. Non-Standard Definitions and Some Elementary Properties

In this section, we introduce definitions that are not standard. These definitions are crucial for the two theorems which we will use in Section 6 to show that the Berge problem and the Hadwiger conjecture are exactly the same problem. We say that a graph B is *berge*, if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5. A graph G is *perfect*, if every induced subgraph G' of G is $\omega(G')$-colorable. The Berge conjecture states that a graph G is perfect if and only if G is berge. Indeed, the Berge problem (i.e. the difficult part of the Berge conjecture, see Preliminary of this paper) consists to show that $\chi(B) = \omega(B)$, for every berge graph B. We will see in Section6 that the Berge problem and the Hadwiger conjecture are exactly the same problem.

We say that a graph G is a *true pal* of a graph F, if F is a subgraph of G and $\chi(F) = \chi(G)$; $trpl(F)$ denotes the set of all true pals of F (so, $G \in trpl(F)$ means G is a *true pal* of F).

Recall that a set X is a *stable* subset of a graph G, if $X \subseteq V(G)$ and if the subgraph of G induced by X has no edges. A graph G is a *complete $\omega(G)$-partite* graph (or a *complete multipartite* graph), if there exists a partition $\Xi(G) = \{Y_1, \cdots, Y_{\omega(G)}\}$ of $V(G)$ into $\omega(G)$ stable sets such that $x \in Y_j \in \Xi(G)$, $y \in Y_k \in \Xi(G)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in G. *It is immediate that $\chi(G) = \omega(G)$, for every complete $\omega(G)$-partite graph.* Ω denotes the set of graphs G which are complete $\omega(G)$-partite. So, $G \in \Omega$ means G is a complete $\omega(G)$-partite graph. Using the definition of Ω, then the following Assertion becomes immediate.

Assertion 2.0 *Let $H \in \Omega$ and let F be a graph. Then we have the following two properties.*

(2.0.0) $\chi(H) = \omega(H)$;

(2.0.1) There exists a graph $P \in \Omega$ such that P is a true pal of F.

Proof Property (2.0.0) is immediate (use definition of Ω and note $H \in \Omega$). Property (2.0.1) is also immediate. Indeed, let F be graph and let $\Xi(F) = \{Y_1, \cdots, Y_{\chi(F)}\}$ be a partition of $V(F)$ into $\chi(F)$ stable sets (it is immediate that such a partition $\Xi(F)$ exists). Now let Q be a graph defined as follows: (i) $V(Q) = V(F)$; (ii) $\Xi(Q) = \{Y_1, ..., Y_{\chi(F)}\}$ is a partition of $V(Q)$ into $\chi(F)$ stable sets such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q. Clearly $Q \in \Omega$, $\chi(Q) = \omega(Q) = \chi(F)$, and F is visibly a subgraph of Q; in particular Q is a true pal of F such that $Q \in \Omega$ (because F is a subgraph of Q and $\chi(Q) = \chi(F)$ and $Q \in \Omega$). Now putting $Q = P$, the property (2.0.1) follows. $\square$

So, we say that a graph P is a *parent* of a graph F, if $P \in \Omega \bigcap trpl(F)$. In other words, P is a *parent* of F, if P is a complete $\omega(P)$-partite graph and P is also a true pal of F (observe that such a P exists, via property (2.0.1) of Assertion 2.0). $parent(F)$ denotes the set of all parents of a graph F (so, $P \in parent(F)$ means P is a parent of F). Using the definition of a parent, then the following Assertion is immediate.

Assertion 2.1 *Let F be a graph and let $P \in parent(F)$. We have the folowing two properties.*

(2.1.0) Suppose that $F \in \Omega$. Then $\chi(F) = \omega(F) = \omega(P) = \chi(P)$;
(2.1.1) Suppose that $F \notin \Omega$. Then $\chi(F) = \omega(P) = \chi(P)$.

§3. The Berge Index of a Graph

In this section, we define a graph parameter called the berge index and we define a representative of a graph; we also give some elementary properties concerning the berge index. We recall that a graph B is berge, if every $B' \in \{B, \bar{B}\}$ does not contain an induced cycle of odd length ≥ 5. A graph G is perfect, if every induced subgraph G' of G is $\omega(G')$-colorable. The Berge conjecture states that a graph G is perfect if and only if G is berge. Indeed the Berge problem, consists to show that $\chi(B) = \omega(B)$ for every berge graph B. Using the definition of a berge graph and the definition of Ω the following assertion becomes immediate.

Assertion 3.0 *Let $G \in \Omega$. Then, G is berge.*

Assertion 3.0 says that the set Ω is an obvious example of berge graphs. Now, we define the berge index of a graph G. Let G be a graph. Then the berge index of G (denoted by $\beta(G)$) is defined in the following two cases (namely case where $G \in \Omega$ and case where $G \notin \Omega$).

First, we define the berge index of G in the case where $G \in \Omega$.

Case *i* Suppose that $G \in \Omega$, and put $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \textit{ is berge}\}$; clearly $\mathcal{B}(G)$ is the set of graphs F such that G is a parent of F and F is berge. Then, $\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$. In other words, $\beta(G) = \omega(F'')$, where $F'' \in \mathcal{B}(G)$, and $\omega(F'')$ is *minimum* for this property.

We prove that such a β clearly exists via the following remark.

Remark *i* Suppose that $G \in \Omega$. Then, the berge index $\beta(G)$ exists. Indeed put $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \textit{ is berge}\}$. Recall $G \in \Omega$, so G is berge (use Assertion 3.0); clearly $G \in \mathcal{B}(G)$, so $\min_{F \in \mathcal{B}(G)} \omega(F)$ exists, and the previous clearly says that $\beta(G)$ exists.

Now, we define the berge index of G, in the case where $G \notin \Omega$.

Case *ii* Suppose that $G \notin \Omega$ and let $parent(G)$ be the set of all parents of G. Then, $\beta(G) = \min_{P \in parent(G)} \beta(P)$. In other words, $\beta(G) = \beta(P'')$, where $P'' \in parent(G)$, and $\beta(P'')$ is *minimum* for this property.

We prove that such a β clearly exists, via the following remark.

Remark *ii* Suppose that $G \notin \Omega$. Then, the berge index $\beta(G)$ exists. Indeed, let $P \in \Omega$ such that P is a true pal of G [such a P exists (use property (2.0.1) of Assertion 2.0)], clearly $P \in parent(G)$; note $P \in \Omega$, and Remark.*(i)* implies that $\beta(P)$ exists. So $\min_{P \in parent(G)} \beta(P)$ exists, and clearly $\beta(G)$ also exists.

Remark *iii* Let G be a graph. Then the berge index $\beta(G)$ exists. In fact, applying Remark *i* if $G \in \Omega$, and Remark *ii* if $G \notin \Omega$, we get the conclusion.

To conclude, note that the berge index of a graph G is $\beta(G)$, where $\beta(G)$ is defined as follows.

$\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$ if $G \in \Omega$; and $\beta(G) = \min_{P \in parent(G)} \beta(P)$ if $G \notin \Omega$. Recall $\mathcal{B}(G) = \{F; G \in parent(F)$ and F *is berge*$\}$, and $parent(G)$ is the set of all parents of G.

We recall (see Section 1) that $\eta(G)$ is the hadwiger number of G, and we clearly have.

Proposition 3.1 *Let K be a complete graph and let $G \in \Omega$. Then, we have the following three properties.*

*(*3.1.0*) If $\omega(G) \leq 1$, then $\beta(G) = \omega(G) = \chi(G) = \eta(G)$;*
*(*3.1.1*) $\beta(K) = \omega(K) = \chi(K) = \eta(K)$;*
*(*3.1.2*) $\omega(G) \geq \beta(G)$.*

Proof Property (3.1.0) is immediate. We prove property (3.1.1). Indeed let $\mathcal{B}(K) = \{F; K \in parent(F)$ and F *is berge*$\}$, recall K is complete, and clearly $\mathcal{B}(K) = \{K\}$; observe $K \in \Omega$, so $\beta(K) = \min_{F \in \mathcal{B}(K)} \omega(F)$ (use definition of parameter β and note $K \in \Omega$), and we easily deduce that $\beta(K) = \omega(K) = \chi(K)$. Note $\eta(K) = \chi(K)$ (since K is complete), and using the previous, we clearly have $\beta(K) = \omega(K) = \chi(K) = \eta(K)$. Property (3.1.1) follows.

Now we prove property (3.1.2). Indeed, let $\mathcal{B}(G) = \{F; G \in parent(F)$ and F *is berge*$\}$, recall $G \in \Omega$, and so $\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$ (use definition of parameter β and note $G \in \Omega$); observe G is berge (use Assertion 3.0), so $G \in \mathcal{B}(G)$, and the previous equality implies that $\omega(G) \geq \beta(G)$. □

Using the definition of the berge index, then we clearly have:

Proposition 3.2 *Let B be berge, and let $P \in parent(B)$. Then, $\beta(P) \leq \omega(B)$.*

Proof Let $\mathcal{B}(P) = \{F; P \in parent(F)$ and F *is berge*$\}$, clearly $B \in \mathcal{B}(P)$; observe $P \in \Omega$, so $\beta(P) = \min_{F \in \mathcal{B}(P)} \omega(F)$, and we immediately deduce that $\beta(P) \leq \omega(B)$. □

Now, we define a representative of a graph. Let G be a graph and let $\beta(G)$ be the berge index of G [observe $\beta(G)$ exists, by using Remark *iii*]; we say that a graph S is a *representative* of G if S is defined in the following two cases (namely case where $G \in \Omega$ and case where $G \notin \Omega$.

First, we define a *representative* of G in the case where $G \in \Omega$.

Case i' Suppose that $G \in \Omega$. Put $\mathcal{B}(G) = \{F; G \in parent(F)$ and F *is berge*$\}$. Then S is a *representative* of G, if $S \in \mathcal{B}(G)$ and $\omega(S) = \beta(G)$. In other words, S is a representative of G, if S is berge and $G \in parent(S)$ and $\omega(S) = \beta(G)$. In other terms again, S is a representative of G if S is berge, $G \in parent(S)$, and $\omega(S)$ is minimum for this property. Via Remarks i' and $i'.0$, we prove that such a S exists, and we have $\chi(S) = \chi(G) = \omega(G)$.

Remark i' Suppose that $G \in \Omega$. Then, there exists a graph S such that S is a representative of

G. Indeed, let $\beta(G)$ be the berge index of G, recalling that $G \in \Omega$, clearly $\beta(G) = \min_{F \in \mathcal{B}(G)} \omega(F)$, where $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \ is\ berge\}$ (use definition of parameter β and note $G \in \Omega$) ; now let $B \in \mathcal{B}(G)$ such that $\omega(B) = \beta(G)$ (such a B exists, since $\beta(G)$ exists (use Remark *iii*), clearly B is a representative of G. Now put $B = S$; then Remark i' clearly follows.

Remark $i'.0$ Suppose that $G \in \Omega$. Now let S be a representative of G (such a S exists, by using Remark i'). Then, $\chi(S) = \chi(G) = \omega(G)$. Indeed, let $\mathcal{B}(G) = \{F; G \in parent(F) \text{ and } F \ is\ berge\}$, and let S be a representative of G. Recall $G \in \Omega$, and clearly $S \in \mathcal{B}(G)$ (use definition of a representative and note $G \in \Omega$) ; so $G \in parent(S)$, and clearly $\chi(S) = \chi(G)$. Note $\chi(G) = \omega(G)$ (since $G \in \Omega$), and the last two equalities immediately imply that $\chi(S) = \chi(G) = \omega(G)$. Remark $i'.0$ follows.

Now, we define a representative of G, in the case where $G \notin \Omega$.

Case ii' Suppose that $G \notin \Omega$. Now let $parent(G)$ be the set of all parents of G, and let $P' \in parent(G)$ such that $\beta(P') = \beta(G)$ (observe that such a P' exists, since $G \notin \Omega$, and by using the definition of $\beta(G)$); put $\mathcal{B}(P') = \{F'; P' \in parent(F') \text{ and } F' \ is\ berge\}$. Then S is a *representative* of G if $S \in \mathcal{B}(P')$ and $\omega(S) = \beta(P') = \beta(G)$. In other words, S is a representative of G (recall $G \notin \Omega$), if S is berge and $P' \in parent(S)$ and $\omega(S) = \beta(P') = \beta(G)$ [where $P' \in parent(G)$ and $\beta(P') = \beta(G)$]. Via Remarks ii' and Remark $ii'.0$, we prove that such a S exists, and we have $\chi(S) = \chi(G)$.

Remark ii' Suppose that $G \notin \Omega$. Then, there exists a graph S such that S is a representative of G. Indeed, let $\beta(G)$ be the berge index of G, recalling that $G \notin \Omega$, clearly $\beta(G) = \min_{P \in parent(G)} \beta(P)$. Now, let $P' \in parent(G)$ such that $\beta(P') = \beta(G)$ [observe that such a P' exists, since $G \notin \Omega$, and by using the definition of $\beta(G)$]; note $P' \in \Omega$, and clearly $\beta(P') = \min_{F' \in \mathcal{B}(P')} \omega(F')$ (note $\mathcal{B}(P') = \{F'; P' \in parent(F') \text{ and } F' \ is\ berge\}$). Now, let $B' \in \mathcal{B}(P')$ such that $\omega(B') = \beta(P')$. Clearly B' is berge and $\omega(B') = \beta(P') = \beta(G)$. It is easy to see that B' is a representative of G. Now put $S = B'$, then Remark ii' follows.

Remark $ii'.0$ Suppose that $G \notin \Omega$. Now let S be a representative of G (such a S exists by using Remark ii'). Then $\chi(S) = \chi(G)$. Indeed, let S be a representative of G, and consider $P' \in parent(G)$ such that P' is a parent of S and $\beta(P') = \beta(G)$ (such a P' clearly exists, by observing that S be a representative of G, $G \notin \Omega$ and by using the definition of a representative of G), clearly $\chi(S) = \omega(P') = \chi(P') = \chi(G)$ (since P' is a parent of G and S). So $\chi(S) = \chi(G)$, and Remark.*(ii'.0)* follows.

Remark iii' Let G be a graph. Then, there exists a graph S such that S is a representative of G. Applying Remark i' if $G \in \Omega$ and applying Remark ii' if $G \notin \Omega$, we get the conclusion.

Remark iv Let G be a graph and let S be a representative of G (such a S exists, by using Remark iii'). Then, $\chi(G) = \chi(S)$. Applying Remark $i'.0$ if $G \in \Omega$, and Remark $ii'.0$ if $G \notin \Omega$, the conclusion follows.

It is clear that a representative of a graph G is not necessarily unique, and in all the cases, we have $\chi(G) = \chi(S)$ for every representative S of G [use Remark *iv*].

To conclude, note that a graph S is a representative of a graph G if S is defined in the following two cases.

Case 1. Suppose that $G \in \Omega$. Then S is a representative of G, if and only if S is berge and $G \in parent(S)$ and $\omega(S) = \beta(G)$.

Case 2. Suppose that $G \notin \Omega$. Now let $P \in parent(G)$ such that $\beta(P) = \beta(G)$. Then S is a representative of G if and only if S is berge and $P \in parent(S)$ and $\omega(S) = \beta(P) = \beta(G)$; in other words, S is a representative of G if and only if S is a representative of P, where $P \in parent(G)$ and $\beta(P) = \beta(G)$.

We will see in Section 5 that the berge index and a representative help to obtain an original reformulation of the Berge problem, and this original reformulation of the Berge problem is crucial for the result of Section 6 which clearly implies that the Hadwiger conjecture is only a non obvious special case of the Berge conjecture.

§4. The Hadwiger Index of a Graph

Here, we define the hadwiger index of a graph and a son of a graph, and we also give some elementary properties related to the hadwiger index. Using the definition of a true pal, the following assertion is immediate.

Assertion 4.0 *Let G be a graph. Then, there exists a graph S such that G is a true pal of S and $\eta(S)$ is minimum for this property.*

Now we define the hadwiger index and a son. Let G be a graph and put $\mathcal{A}(G) = \{H; G \in trpl(H)\}$; clearly $\mathcal{A}(G)$ is the set of all graphs H, such that G is a true pal of H. The *hadwiger index* of G is denoted by $\tau(G)$, where $\tau(G) = \min_{F \in \mathcal{A}(G)} \eta(F)$. In other words, $\tau(G) = \eta(F'')$, where $F'' \in \mathcal{A}(G)$, and $\eta(F'')$ is minimum for this property . We say that a graph S is a *son* of G if $G \in trpl(S)$ and $\eta(S) = \tau(G)$. In other words, a graph S is a son of G, if $S \in \mathcal{A}(G)$ and $\eta(S) = \tau(G)$. In other terms again, a graph S is a son of G, if G is a true pal of S and $\eta(S)$ is *minimum* for this property. Observe that such a son exists, via Assertion 4.0. It is immediate that, if S is a son of a graph G, then $\chi(S) = \chi(G)$ and $\eta(S) \leq \eta(G)$.

We recall that $\beta(G)$ is the berge index of G, and we clearly have.

Proposition 4.1 *Let K be a complete graph and let $G \in \Omega$. We have the following three properties.*

(4.1.0) If $\omega(G) \leq 1$, then $\beta(G) = \omega(G) = \chi(G) = \eta(G) = \tau(G)$;
(4.1.1) $\beta(K) = \omega(K) = \chi(K) = \eta(K) = \tau(K)$;
(4.1.2) $\omega(G) \geq \tau(G)$.

Proof Properties (4.1.0) and (4.1.1) are immediate. Now we show property (4.1.2). Indeed, recall $G \in \Omega$, and clearly $\chi(G) = \omega(G)$. Now, put $\mathcal{A}(G) = \{H; G \in trpl(H)\}$ and let K' be a complete graph such that $\omega(K') = \omega(G)$ and $V(K') \subseteq V(G)$; clearly K' is a subgraph of G and

$$\chi(G) = \omega(G) = \chi(K') = \omega(K') = \eta(K') = \tau(K') \qquad (4.1.2.0).$$

In particular K' is a subgraph of G with $\chi(G) = \chi(K')$, and therefore, G is a true pal of K'. So $K' \in \mathcal{A}(G)$ and clearly

$$\tau(G) \leq \eta(K') \qquad (4.1.2.1).$$

Note $\omega(G) = \eta(K')$ (use *(4.1.2.0)*), and inequality *(4.1.2.1)* immediately becomes $\tau(G) \leq \omega(G)$. □

Observe Proposition 4.1 resembles to Proposition 3.1. Using the definition of τ, the following proposition becomes immediate.

Proposition 4.2 *Let F be a graph and let $G \in trpl(F)$. Then $\tau(G) \leq \tau(F)$.*

Proof Put $\mathcal{A}(G) = \{H; G \in trpl(H)\}$, and let S be a son of F, recalling that $G \in trpl(F)$, clearly $G \in trpl(S)$; so $S \in \mathcal{A}(G)$ and clearly $\tau(G) \leq \eta(S)$. Now, observe $\eta(S) = \tau(F)$ (because S is a son of F), and the previous inequality immediately becomes $\tau(G) \leq \tau(F)$. □

Corollary 4.3 *Let F be a graph and let $P \in parent(F)$. Then $\tau(P) \leq \tau(F)$.*

Proof Observe that $P \in trpl(F)$ and apply Proposition 4.2. □

We will see in Section 5 that the hadwiger index and a son help to obtain an original reformulation of the Hadwiger conjecture, and this original reformulation of the Hadwiger conjecture is also crucial for the result of Section 6 which clearly implies that the Hadwiger conjecture is only a non obvious special case of the Berge conjecture.

§5. An Original Reformulation of the Berge Problem and the Hadwiger Conjecture

In this section, we prove two simple Theorems which are equivalent to the Berge problem and the Hadwiger conjecture. These original reformulations will help in Section 6 to show that the Berge problem and the Hadwiger conjecture are exactly the same problem. That being so, using the berge index β, then the following first simple Theorem is an original reformulation of the Berge problem.

Theorem 5.1 *The following are equivalent.*

(1) The Berge problem is true (i.e. $\chi(B) = \omega(B)$ for every berge graph B).
(2) $\chi(F) = \beta(F)$, for every graph F.
(3) $\omega(G) = \beta(G)$, for every $G \in \Omega$.

Proof (2) $\Rightarrow$ (3) Let $G \in \Omega$, in particular G is a graph, and so $\chi(G) = \beta(G)$; observe $\chi(G) = \omega(G)$ (since $G \in \Omega$), and the last two equalities imply that $\omega(G) = \beta(G)$. So (2) $\Rightarrow$ (3)].

(3) $\Rightarrow$ (1) Let B be berge and let $P \in parent(B)$; Proposition 3.2 implies that $\beta(P) \leq \omega(B)$. Note $\beta(P) = \omega(P)$ (because $P \in \Omega$), and the previous inequality becomes $\omega(P) \leq \omega(B)$. It is immediate that $\chi(B) = \chi(P) = \omega(P)$ [since $P \in parent(B)$], and the last inequality becomes $\chi(B) \leq \omega(B)$; observe $\chi(B) \geq \omega(B)$, and the previous two inequalities imply that

$\chi(B) = \omega(B)$. So (3) $\Rightarrow$ (1)].

(1) $\Rightarrow$ (2) Let F be a graph and let S be a representative of F, in particular S is berge (because S is a representative of F) and clearly $\chi(S) = \omega(S)$, now, observing that $\omega(S) = \beta(F)$ (because S is a representative of F), then the previous two equalities imply that $\chi(S) = \beta(F)$; note $\chi(S) = \chi(F)$ (by observing that S is a representative of F and by using Remark *iv* of Section 3), and the last two equalities immediately become $\chi(F) = \beta(F)$. So (1) $\Rightarrow$ (2)], and Theorem 5.1 follows. □

We recall that the Hadwiger conjecture states that $\chi(G) \leq \eta(G)$ for every graph G. Using the hadwiger index τ, then the following is a corresponding original reformulation of the Hadwiger conjecture.

Theorem 5.2 *The following are equivalent.*

(1) The Hadwiger conjecture is true, i.e., $\chi(H) \leq \eta(H)$ for every graph H;
(2) $\chi(F) \leq \tau(F)$, for every graph F;
(3) $\omega(G) = \tau(G)$, for every $G \in \Omega$.

Proof (2) $\Rightarrow$ (3) Let $G \in \Omega$, clearly G is a graph and so $\chi(G) \leq \tau(G)$. Note $\chi(G) = \omega(G)$ (since $G \in \Omega$), and the previous inequality becomes $\omega(G) \leq \tau(G)$; now, using property (4.1.2) of Proposition 4.1, we have $\omega(G) \geq \tau(G)$, and the last two inequalities imply that $\omega(G) = \tau(G)$.

(3) $\Rightarrow$ (1) Let H be a graph and let $P \in parent(H)$, then $\tau(P) \leq \tau(H)$ (use Corollary 4.3); observe $P \in \Omega$ (since $P \in parent(H)$), clearly $\omega(P) = \tau(P)$ (since $P \in \Omega$), and $\chi(H) = \chi(P) = \omega(P)$ (since $P \in parent(H)$). Clearly $\tau(P) = \chi(H)$ and the previous inequality becomes $\chi(H) \leq \tau(H)$. Recall $\tau(H) \leq \eta(H)$, and the last two inequalities become $\chi(H) \leq \tau(H) \leq \eta(H)$. So $\chi(H) \leq \eta(H)$, and clearly (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) Indeed, let F be a graph and let S be a son of F, clearly $\chi(S) \leq \eta(S)$; now observing that $\chi(S) = \chi(F)$ (since $F \in trpl(S)$) and $\eta(S) = \tau(F)$ (because S is a son of F), then the previous inequality immediately becomes $\chi(F) \leq \tau(F)$. So (1) $\Rightarrow$ (2)] and Theorem 5.2 follows. □

Theorems 5.1 and 5.2 immediately imply that the Berge problem and the Hadwiger conjecture are exactly the same problem, and therefore, the Hadwiger conjecture is only a special non-obvious case of the Berge conjecture.

§6. Conclusion

Indeed, the following two theorems follow immediately from Theorems 5.1 and 5.2.

Theorem 6.1 *The following are equivalent.*

(i) The Berge problem is true;
(ii) $\omega(G) = \beta(G)$, for every $G \in \Omega$.

Proof Indeed, it is an immediate consequence of Theorem 5.1. □

Theorem 6.2 *The following are equivalent.*

(i) The Hadwiger conjecture is true;
(ii) $\omega(G) = \tau(G)$ *for every* $G \in \Omega$.

Proof Indeed, it is an immediate consequence of Theorem 5.2. □

Using Theorems 6.1 and 6.2, the following Theorem becomes immediate.

Theorem 6.3 *The Berge problem and the Hadwiger conjecture are exactly the same problem.*

Proof Indeed observing that the Berge conjecture is true (see [1] or see [9]), then in particular the Berge problem is true.Now using Theorem 6.1 and the previous, then it becomes immediate to deduce that

$$\omega(G) = \beta(G), \ for\ every\ G \in \Omega \qquad (6.3.1).$$

That being so, noticing that the Hadwiger conjecture is true (see [13]) and using Theorem 6.2, then it becomes immediate to deduce that

$$\omega(G) = \tau(G), \ for\ every\ G \in \Omega \qquad (6.3.2).$$

(6.3.1) and *(6.3.2)* clearly say that the Berge problem and the Hadwiger conjecture are exactly the same problem. □

From Theorem 6.3, then it comes:

Theorem 6.4(Tribute to Claude Berge) *The Hadwiger conjecture is a special case of the Berge conjecture.*

Proof It is immediate to see that

$$the\ Berge\ conjecture\ implies\ the\ Berge\ problem \qquad (6.4.1).$$

Now by Theorem 6.3

$$the\ Berge\ problem\ and\ the\ Hadwiger\ conjecture\ are\ exactly\ the\ same\ problem \qquad (6.4.2).$$

That being so, using (6.4.1) and (6.4.2) , then it becomes immediate to deduce that the Hadwiger conjecture is a special case of the Berge conjecture. □

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Graphs and Cellular Foldings of 2-Manifolds

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Abstract: In this paper we considered the set of regular CW-complexes or simply complexes. We obtained the necessary and sufficient condition for the composition of cellular maps to be a cellular folding. Also the necessary and sufficient condition for the composition of a cellular folding with a cellular map to be a cellular folding is declared. Then we proved that the Cartesian product of two cellular maps is a cellular folding iff each map is a cellular folding. By using these results we proved some other results. Once again we generalized the first three results and in each case we obtained the folding graph of the new map in terms of the original ones.

Key Words: Graph, cellular folding, 2-manifold, Cartesian product.

AMS(2010): 57M10, 57M20

§1. Introduction

A *cellular folding* is a folding defined on regular CW-complexes first defined by E. El-Kholy and H. Al-Khursani [1], and various properties of this type of folding are also studied by them. By a cellular folding of regular CW-complexes, it is meant a cellular map $f : K \to L$ which maps i-cells of K to i- cells of L and such that $f|_{e^i}$ for each i-cells e is a homeomorphism onto its image.

The set of regular CW-complexes together with cellular foldings form a category denoted by $C(K, L)$. If $f \in C(K, L)$, then $x \in K$ is said to be a *singularity* of f iff f is not a local homeomorphism at x. The set of all singularities of f is denoted by $\sum f$. This set corresponds to the folds of map. It is noticed that for a cellular f, the set $\sum f$ of singularities of f is a proper subset of the union of cells of dimension$\leq n-1$. Thus, when we consider any $f \in C(K, L)$, where K and L are connected regular CW-complexes of dimension 2, the set $\sum f$ will consists of 0- cells, 1-cells, and each 0-cell (vertex) has an even valency [2]. Of course, $\sum f$ need not be connected. Thus in this case $\sum f$ has the structure of a locally finite graph Γ_f embedded in K, for which every vertex has an even valency. Note that if K is compact, then Γ_f is finite, also any

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compact connected 2-manifold without boundary (surface) K with a finite cell decomposition is a regular CW-complex, then the 0-and 1-cells of the decomposition K from a finite graph Γ_f without loops and f folds K along the edges or 1-cells of Γ_f. Let K and L be complexes of the same dimension n. A neat cellular folding $f : K \to L$ is a cellular folding such that $L^n - L^{n-1}$ consists of a single n-cell, Int$L$ that is f satisfies the following:

(i) f maps i-cells to i-cells;

(ii) for each $\overline{e}$ which contains n vertices, $\overline{f(e)}$ is mapped on the single n- cell, $\overline{\mathrm{Int}L}$, [3].

The set of regular CW-complexes together with neat cellular foldings form a category which is denoted by $NC(K, L)$. This category is a subcategory of cellular foldings $C(K, L)$. From now we mean by a complex a regular CW-complex in this paper.

§2. Main Results

Theorem 2.1 *Let M, N and L be complexes of the same dimension 2 such that $L \subset N \subset M$. Let $f : M \to N$, $g : N \to L$ be cellular maps such that $f(M) = N$, $g(N) = L$. Then $g \circ f$ is a cellular folding iff f and g are cellular foldings. In this case, $\Gamma_{g\circ f} = \Gamma_f \bigcup f^{-1}(\Gamma_g)$.*

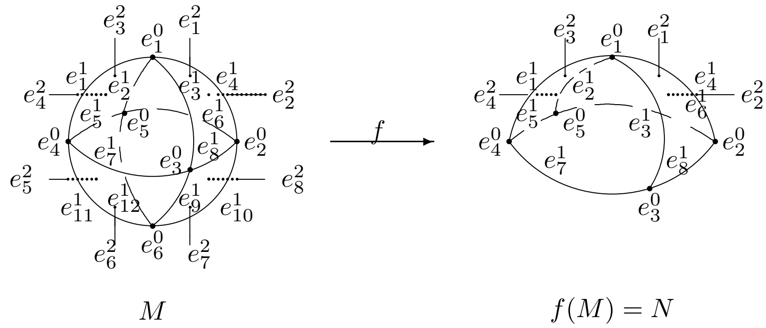
Proof Let M, N and L be complexes of the same dimension 2, let $f : M \to N$ be a cellular folding such that $\sum f \neq \emptyset$, i.e., $f(M) = N \neq M$. Then $\sum f$ form a graph Γ_f embedded in M. Let $g : N \to N$ be a cellular folding such that $g(N) = L \neq N$, $\sum g = \Gamma_g$ is embedded in N. Now, let $\sigma \in M^{(i)}$, $i = 0, 1, 2$ be an arbitrary i-cell in M such that $\overline{\sigma}$ has S distinct vertices then $(g \circ f)(\sigma) = g(f(\sigma)) = g(\sigma')$, where $\sigma' \in N^{(i)}$ such that $\overline{\sigma}$ has S distinct vertices since f is a cellular folding. Also $g(\sigma') \in L^{(i)}$ such that $\overline{g(\sigma')}$ has S distinct vertices since g is a cellular folding. Thus $g\circ f$ is a cellular folding. In this case $\sum g \circ f$ is $\sum f \bigcup f^{-1}(\sum g)$. In other words, $\Gamma_{f\circ g} = \Gamma_f \bigcup f^{-1}(\Gamma_g)$.

Conversely, suppose $f : M \to N$ and $g : N \to L$ are cellular maps such that $g \circ f : M \to L$ is a cellular folding. Now, let $\sigma \in M^{(i)}$ be an i-cell in M. Suppose $f(\sigma) = \sigma'$ is a j-cell in N, such that $j \neq i$. Then since f is a cellular map, then $j \leq i$. But $j \neq i$, thus $j < i$. Since $f(\sigma) = \sigma'$, then $(g \circ f)(\sigma) = g(f(\sigma)) = g(\sigma')$. But $g \circ f$ is a cellular folding, thus $(g \circ f)(\sigma)$ is an i- cell in L and so is $g(\sigma')$. Since σ' is a j- cell in N and g is a cellular map, then i must be less than j and this contradicts the assumption that $j < i$. Hence the only possibly is that $i = j$. Note that the above theorem is true if we consider f and g are neat cellular foldings instead of cellular folding. □

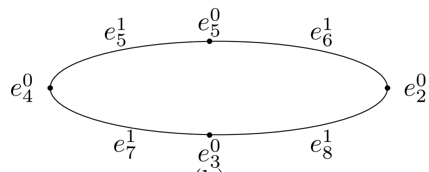
Example 2.2 Consider a complex on $M = S^2$ with cellular subdivision consists of six-vertices, twelve 1-cells and eight 2-cells. Let $f : M \to N$ be a cellular folding given by:

$$f(e_1^0, e_2^0, e_3^0, e_4^0, e_5^0, e_6^0) = (e_1^0, e_2^0, e_3^0, e_4^0, e_5^0, e_1^0),$$
$$f(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1, e_9^1, e_{10}^1, e_{11}^1, e_{12}^1) = (e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1, e_3^1, e_4^1, e_1^1, e_2^1),$$
$$f(e_1^2, e_2^2, e_3^2, e_4^2, e_5^2, e_6^2, e_7^2, e_8^2) = (e_1^2, e_2^2, e_3^2, e_4^2, e_1^2, e_2^2).$$

In this case $f(M) = N$ is a complex with five vertices, eight 1-cells and four 2-cells, see Fig.1(a). The folding graph Γ_f is shown Fig.1(b).



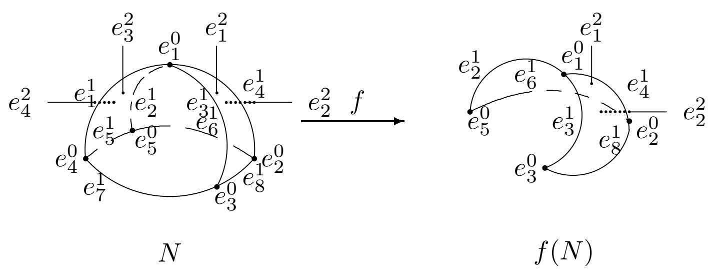
(a)



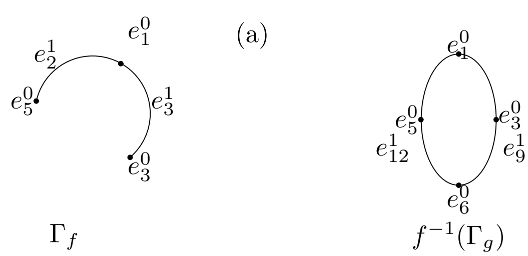
(b)

Fig.1

Now, let $g : N \to N$ be given by : $g(e^0_1, e^0_2, e^0_3, e^0_4, e^0_5) = (e^0_1, e^0_2, e^0_3, e^0_4, e^0_5)$, $g(e^1_1, e^1_2, e^1_3, e^1_4, e^1_5, e^1_6, e^1_7, e^1_8) = (e^1_4, e^1_2, e^1_3, e^1_4, e^1_6, e^1_6, e^1_8, e^1_8)$, $g(e^2_1, e^2_2, e^2_3, e^2_4) = (e^2_1, e^2_2, e^2_1, e^2_2)$. See Fig.2(a). Again g is a cellular folding and the folding graphs Γ_g and $f^{-1}(\Gamma_g)$ are shown in Fig.2(b).



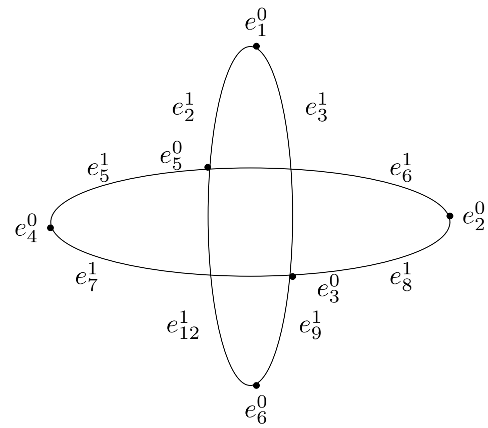
(a)



(b)

Fig.2

Then $g \circ f : M \to L$ is a cellular folding with folding graph $\Gamma_{g \circ f}$ shown in Fig.3.



$$\Gamma_{g\circ f} = \Gamma_f \bigcup f^{-1}(\Gamma_g)$$

Fig.3

Theorem 2.1 can be generalized for a series of cellular foldings as follows:

Theorem 2.3 *Let $M, M_1, M_2, \cdots, M_n$ be complexes of the same dimension 2 such that $M_n \subset M_{n-1} \subset M_1 \subset M$, and consider the cellular maps $M \overset{f_1}{\to} M_1 \overset{f_2}{\to} M_2 \cdots \overset{f_n}{\to} M_n$. Then the composition of these cellular maps $\phi :: M \to M_n$ is a cellular folding iff each f_r, $r = 1, 2, \cdots, n$ is a cellular folding. In this case the folding graphs satisfy the condition*

$$\begin{aligned}\Gamma_\phi &= \Gamma_{f_1} \bigcup f_1^{-1}(\Gamma_{f_2}) \bigcup (f_1 \circ f_1)^{-1}(\Gamma_{f_3} \bigcup (f_3 \circ f_2 \circ f_1)^{-1}(\Gamma_{f_4}))\\ &\quad \bigcup \cdots \bigcup (f_{n-1} \circ f_{n-2} \circ \cdots \circ f_1)^{-1}(\Gamma_{f_n}).\end{aligned}$$

Theorem 2.4 *Let M, N and L be complexes of the same dimension 2 such that $L \subset N \subset M$. Let $f : M \to N$ be a cellular folding such that $f(M) = N$. Then a cellular map $g : N \to L$ is a cellular folding iff $g \circ f : M \to L$ is a cellular folding. In this case $\Gamma_g = f[(\Gamma_{g\circ f} \setminus E(\Gamma_f)) \setminus \{V\}]$, where $E(\Gamma_f)$ is the set of edges of Γ_f and $\{V\}$ is the set of the isolated vertices remains in $\Gamma_{g\circ f}$.*

Proof Suppose $g \circ f$ is a cellular folding, $f \in C(M, N)$, $\sum f \neq \emptyset$. Let $\sigma \in M^{(i)}, i = 0, 1, 2$ be an arbitrary i-cell in M such that σ has S vertices. Since $g \circ f$ is a cellular folding, then $g \circ f(\sigma) = \sigma'$ is an i-cell in L such that σ' has S distinct vertices. But $g \circ f(\sigma) = g(f(\sigma))$ and $f(\sigma)$ is an i-cell in N such that $\overline{f(\sigma)}$ has S distinct vertices, then g maps i-cells to i-cells and satisfies the second condition of cellular folding, consequently, g is a cellular folding. In this case, $\Gamma_g = f[(\Gamma_{g\circ f} \setminus E(\Gamma_f)) \setminus \{V\}]$,where $E(\Gamma_f)$ is the set of edges of Γ_f and $\{V\}$ is the set of the isolated vertices remains in $\Gamma_{g\circ f}$.

Conversely, suppose $g : N \to L$ is a cellular folding. Since $f : M \to N$ is a cellular folding, by Theorem 2.1, $g \circ f$ is a cellular folding. Notice that this conclusion is also true if we consider g and $g \circ f$ neat cellular foldings instead of cellular foldings. $\square$

Example 2.5 Consider a complex on $|M| = S^2$ with cellular subdivision consisting of six vertices, twelve 1-cells and eight 2-cells. Let $f : M \to M, f(M) = N$ be a cellular folding given as shown in Fig.1(a) with folding graph Γ_f shown in Fig.1(b).

Now, let L be a 2-cell with boundary consists of three 0-cells and three 1-cells , see Fig.4(a) and let $h : M \to L$ be a cellular folding defined by:

$$h(e_1^0, e_2^0, e_3^0, e_4^0, e_5^0, e_6^0) = (e_1^0, e_2^0, e_3^0, e_2^0, e_3^0, e_1^0),$$
$$h(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1, e_9^1, e_{10}^1, e_{11}^1, e_{12}^1) = (e_4^1, e_4^1, e_3^1, e_4^1, e_8^1, e_8^1, e_8^1, e_8^1, e_3^1, e_4^1, e_4^1, e_4^1),$$
$$h(e_1^2, e_2^2, e_3^2, e_4^2, e_5^2, e_6^2, e_7^2, e_8^2) = (e_1^2).$$

The folding graph Γ_h is shown in Fig.4(b).

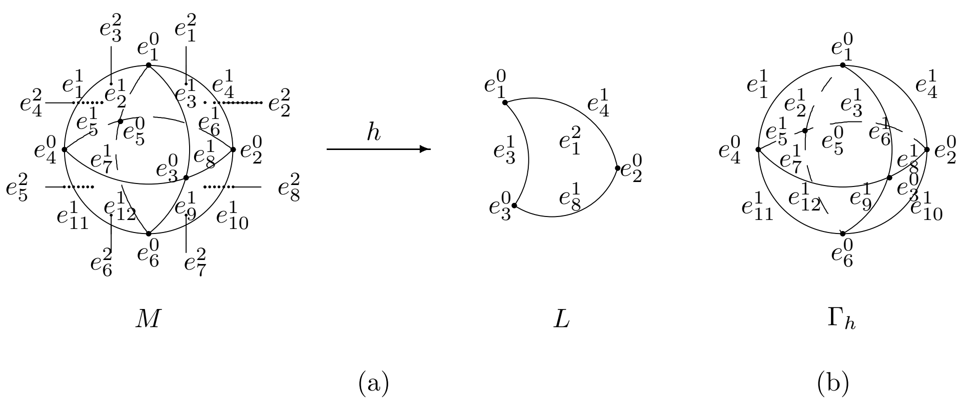
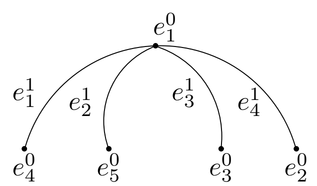


Fig.4

The cellular folding h is the composition of f with a cellular folding $g : N \to L$ which folds N onto L. The graph Γ_g is given is given in Fig.5.



$\Gamma_g = f[(\Gamma_{g\circ f} \setminus E(\Gamma_f)) \setminus \{V\}]$

Fig.5

where $E(\Gamma_f)$ is the edges of Γ_f and $\{V\}$ is the set of the isolated vertices remains in $\Gamma_{g\circ f} = \Gamma_h$.

Theorem 2.4 can be generalized for a finite series of cellular foldings as follows:

Theorem 2.6 *Let $M, M_1, M_2, \cdots, M_n$ be complexes of the same dimension* 2 *such that $M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M$, and consider the cellular maps $M \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \cdots \xrightarrow{f_{n-1}} M_{n-1}$. Then a cellular map $f_n : M_{n-1} \to M_n$ is a cellular folding iff the composition $f_n \circ f_{n-1} \circ \cdots \circ f_1$:*

$M \to M_n$ is a cellular folding. In this case the folding graph of f_n is given by:

$$\Gamma_{f_n} = (f_{n-1} \circ \cdots \circ f_1)[(\Gamma_{f_{n-1}\circ\cdots\circ f_1} \setminus E(\Gamma_{f_{n-1}\circ\cdots\circ f_1}) \setminus \{V\}],$$

where $E(\Gamma_{f_{n-1}\circ\cdots\circ f_1})$ is the set of edges of $\Gamma_{f_{n-1}\circ\cdots\circ f_1}$ and $\{V\}$ is the set of the isolated vertices remains in $\Gamma_{f_n\circ f_{n-1}\circ\cdots\circ f_1}$.

Theorem 2.7 *Suppose K, L, X and Y are complexes of the same dimension 2. Let $f : K \to X$ and $g : L \to Y$ be cellular maps. Then $f \times g : K \times L \to X \times Y$ is a cellular folding iff f and g are cellular foldings. In this case, $\Gamma_{f\times g} = (\Gamma_f \times L)\bigcup(\Gamma_g \times K)$.*

Proof Suppose f and g are cellular foldings. We claim that $f \times g$ is a cellular folding. Let e^i be an arbitrary i-cell in K, $e'j$ be an arbitrary j-cell in L. Then $(e^i, e^{'j})$ is an $(i + j)$-cell in $K \times L$. Since $(f \times g)[(e^i, e^{'j})] = (f(e^i), g(e'j))$, thus $(f \times g)(e^i, e^{'j})$ is an $(i + j)$-cell in $X \times Y$ (since $f(e^i)$ is an i-cell in X, $g(e^{'j})$ is a j - cell in Y, f and g are cellular foldings). Then $f \times g$ sends cells to cells of the same dimension. Also, if $\sigma = (e^i, e^{'j})$, $\overline{\sigma}$ and $\overline{(f \times g)(\sigma)}$ contains the same number of vertices because each of f and g is a cellular folding.

Suppose now $f \times g$ is a cellular folding, then $f \times g$ maps p-cells to p-cells, i.e., if (e, e') is a p-cell in $K \times L$, then $(f \times g)(e, e') = (f(e), g(e'))$ is a p-cell in $X \times Y$. Let e be an i-cell in K and e' be a $(p - i)$-cell in L. The all cellular maps must map i-cells to j-cells such that $j \le i$. If $i = j$, there are nothing needed to prove. So let $i > j$. In this case g will map $(p - i)$-cells to $(p - j)$-cells and hence it is not a cellular map. This is a contradiction and hence $i = j$ is the only possibility. The second condition of cellular folding certainly satisfied in this case. □

It should be noted that this conclusion is also true for neat cellular foldings, but it is not true for simplecial complexes since the product of two positive-dimensional simplexes is not a simplex any more.

Example 2.8 Let K be complex such that $|K| = S^1$ with four vertices and four 1-cells, and let $f : K \to K$ be a cellular folding defined by $f(v^1, v^2, v^3, v^4) = (v^1, v^2, v^1, v^4)$ and L a complex such that $|L| = I$ with three vertices and two 1-cells and let $g : L \to L$ be a neat cellular folding $g(u^1, u^2, u^3) = (u^1, u62, u^1)$, see Fig.6.

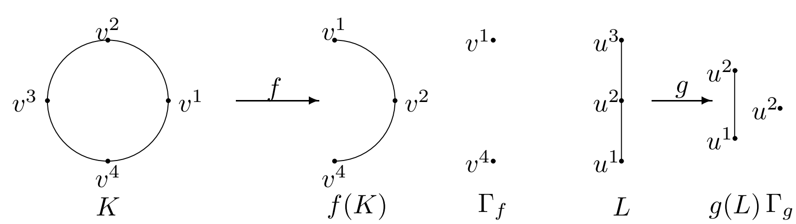
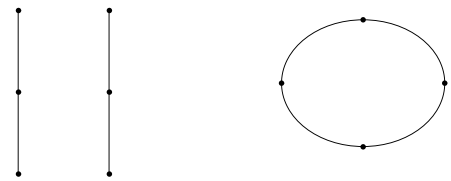


Fig.6

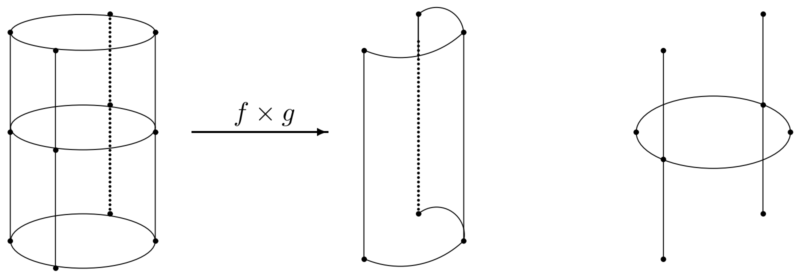
Then the folding graphs $\Gamma_f \times L$ and $\Gamma_g \times K$ have the form shown in Fig.7.



$\Gamma_f \times L$ $\Gamma_g \times K$

Fig.7

Now $f \times g : K \times L \to K \times L$ is a cellular folding but not neat. The cell decomposition of $K \times L$ and $(f \times g)(K \times L)$ are shown in Fig.8(a). In this case, $\Gamma_{f\times g}$ has the form shown in Fig.8(b).



$K \times L$ $(f \times g)(K \times L)$ $\Gamma_{f\times g} = (\Gamma_f \times L)\bigcup(\Gamma_g \times K)$

Fig.8

Theorem 2.7 can be generalized for the product of finite numbers of complexes as follows:

Theorem 2.9 *Suppose $K_1, K_2, \cdots, K_n$ and $X_1, X_2, \cdots, X_n$ are complexes of the same dimension 2 and $f_i : K_i \to X_i$ for $i = 1, 2, \cdots, n$ are cellular maps. Then the product map $f_1 \times f_2 \times \cdots f_n : K_1 \times K_2 \times \cdots \times K_n \to X_1 \times X_2 \times \cdots \times X_n$ is a cellular folding iff each of f_i is a cellular folding for $i = 1, 2, \cdots, n$. In this case,*

$$\begin{aligned}\Gamma_{f_1\times f_2\times\cdots\times f_n} &= \Gamma_{f_1} \times (K_2 \times K_3 \times \cdots \times K_n)\bigcup \Gamma_{f_2} \times (K_1 \times K_3 \times \cdots \times K_n)\\ &\bigcup \cdots \bigcup \Gamma_{f_n} \times (K_1 \times K_2 \times \cdots \times K_{n-1}).\end{aligned}$$

Theorem 2.10 *Let A, B, A_1, A_2, B_1, B_2 be complexes and let $f : A \to A_1$, $g : B \to B_1$, $h : A_1 \to A_2$, $k : B_1 \to B_2$ be cellular foldings. Then $(h \times k) \circ (f \times g) = (h \circ f) \times (k \circ g)$ is a cellular folding with folding graph*

$$\Gamma_{(h\times k)\circ(f\times g)} = \Gamma_{f\times g}\bigcup(f \times g)^{-1}(\Gamma_{h\times k}) = \Gamma_{(h\circ f)\times(k\circ g)} = (\Gamma_{h\circ f} \times B)\bigcup(\Gamma_{k\circ g} \times A).$$

Proof Since $h : A_1 \to A_2$, $k : B_1 \to B_2$ are cellular foldings, then $h\times k : A_1\times B_1 \to A_2\times B_2$ is a cellular folding. Also, since $f : A \to A_1$, $g : B \to B_1$ are cellular foldings, then so is

$f \times g : A \times B \to A_1 \times B_1$. Thus $(h \times k) \circ (f \times g) : A \times B \to A_2 \times B_2$ is a cellular folding with folding graph $\Gamma_{(h\times k)\circ(f\times g)} = \Gamma_{f\times g} \bigcup (f \times g)^{-1}(\Gamma_{h\times k})$.

On the other hand, because both of $(h \circ f)$ and $(k \circ g)$ are cellular foldings, then $(h \circ f) \times (k \circ g)$ is a cellular folding with folding graph

$$\Gamma_{(h\circ f)\times(k\circ g)} = (\Gamma_{h\circ f} \times B) \bigcup (\Gamma_{k\circ g} \times A). \qquad \square$$

The above theorem can be generalized for a finite number of cellular foldings.

Example 2.11 Suppose A, B, A_1, A_2, B_1, B_2 are complexes such that $A = S^1$, $B = |A_1| = |A_2| = |B_1| = |B_2| = I$ with cell decompositions shown in Fig.9.

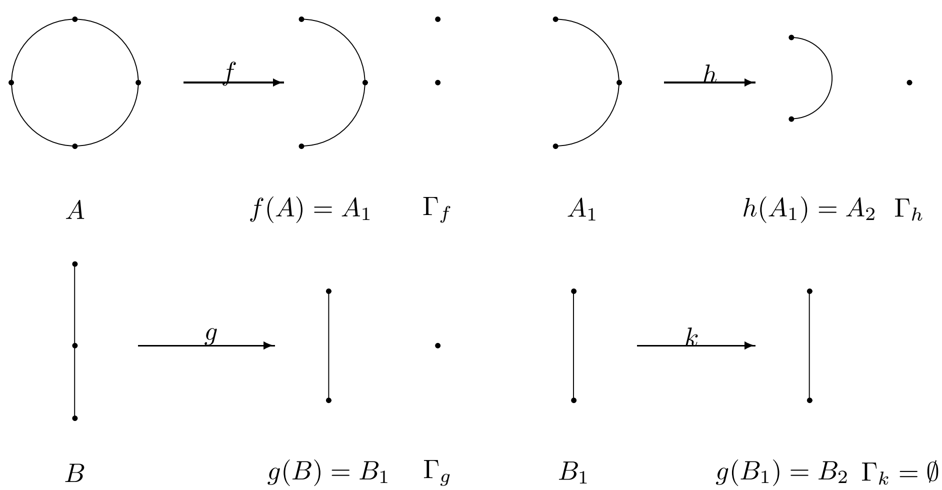
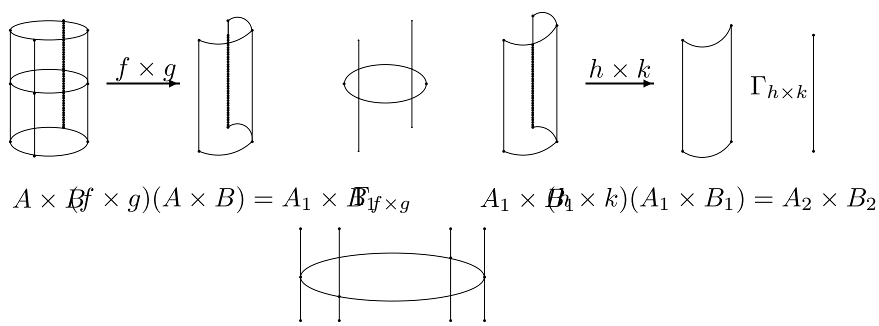


Fig.9

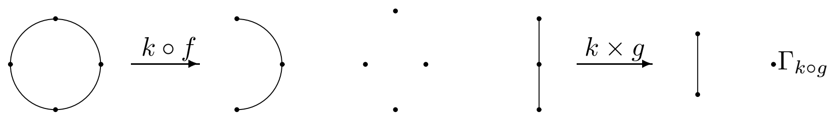
Suppose $f : A \to A_1$, $g : B \to B_1$, $h : A_1 \to A_2$ and $k : B_1 \to B_2$ are cellular foldings. The cellular foldings $f \times g$, $h \times k$ and the folding graphs $\Gamma_{f\times g}$, $\Gamma_{h\times k}$, $\Gamma_{(h\times k)\circ(f\times g)}$ are shown in Fig.10.



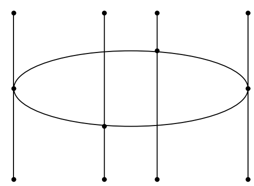
$\Gamma_{(h\times k)\circ(f\times g)} = \Gamma_{f\times g} \bigcup (f \times g)^{-1}(\Gamma_{h\times k})$

Fig.10

Also the cellular folding $h \circ f$, $k \circ g$ and the folding graphs $\Gamma_{h \circ f}$, $\Gamma_{k \circ g}$, $\Gamma_{(h \circ f)\times(k \circ g)}$ are shown in Fig.11.



A $\quad (h \circ f)(A) = A_2 \quad \Gamma_{h \circ f} \quad B \quad (k \circ g)(B) = B_2$



$$\Gamma_{(h \circ f)\times(k \circ g)} = (\Gamma_{h \circ f} \times B) \bigcup (\Gamma_{k \circ g} \times A)$$

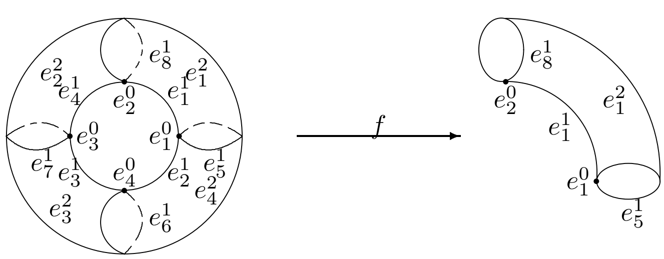
Fig.11

Proposition 2.11 *Let X be a complex and $f : X \to X$ any neat cellular folding. Then f restricted to any subcomplex A of X is again a neat cellular folding over the image $f(X) = Y$.*

This is due to the fact that f_{e^i} with e^i an i-cell of X, is a homeomorphism onto its image and in the case of neat cellular folding of surfaces the image, Y must has only one 2-cell, Int$Y$, and thus the restriction of f to any subcomplex of X will maps each 2-cells of A onto the 2-cell of Y and it does so for the 0 and 1-cells of A since f in fact is cellular. Consequently $f|_A$ is a neat cellular folding of A to Y.

Example 2.12 Consider a complex X such that $|X|$ is a torus with a cellular subdivision shown in Fig.12 and let $f : X \to X$ be given by

$$f(e_1^0, e_2^0, e_3^0, e_4^0) = (e_1^0, e_2^0, e_1^0, e_1^0),$$
$$f(e_1^1, e_2^1, e_3^1, e_4^1, e_5^1, e_6^1, e_7^1, e_8^1) = (e_1^1, e_1^1, e_1^1, e_1^1, e_5^1, e_8^1, e_5^1, e_8^1),$$
$$f(e_n^2) = e_1^2 \quad \text{for } n = 1, 2, 3, 4.$$



X $\qquad f(X) = Y$

Fig.12

The map f is a neat cellular folding with image $f(X) = Y$ which is a subcomplex of X consists of two 0-cells, three 1-cells and a single 2-cell. Now let $A \subset X$ shown in Fig.13. Then $f|_A : A \to Y$ given by

$$f|_A(e_1^0, e_2^0, e_3^0) = (e_1^0, e_2^0, e_1^0),$$
$$f|_A(e_1^1, e_4^1, e_5^1, e_7^1, e_8^1) = (e_1^1, e_1^1, e_5^1, e_1^1, e_8^1),$$
$$f|_A(e_n^2) = e_1^2 \text{ for} n = 1, 2, 3, 4$$

is a neat cellular folding.

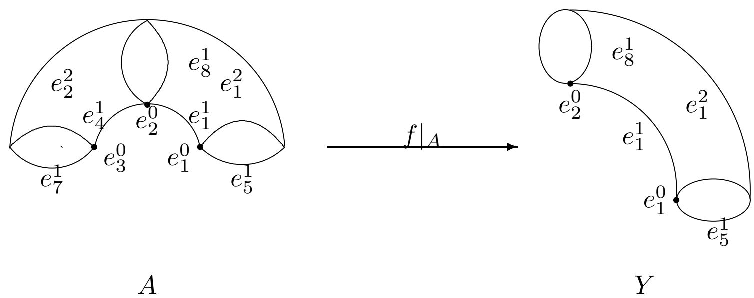


Fig.13

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Triple Connected Domination Number of a Graph

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Abstract: The concept of triple connected graphs with real life application was introduced in [7] by considering the existence of a path containing any three vertices of a graph G. In this paper, we introduce a new domination parameter, called Smarandachely triple connected domination number of a graph. A subset S of V of a nontrivial graph G is said to be Smarandachely triple connected dominating set, if S is a dominating set and the induced sub graph $\langle S\rangle$ is triple connected. The minimum cardinality taken over all Smarandachely triple connected dominating sets is called the Smarandachely triple connected domination number and is denoted by γ_{tc}. We determine this number for some standard graphs and obtain bounds for general graphs. Its relationship with other graph theoretical parameters are also investigated.

Key Words: Domination number, triple connected graph, Smarandachely triple connected domination number.

AMS(2010): 05C69

§1. Introduction

By a graph we mean a finite, simple, connected and undirected graph $G(V, E)$, where V denotes its vertex set and E its edge set. Unless otherwise stated, the graph G has p vertices and q edges. Degree of a vertex v is denoted by $d(v)$, the maximum degree of a graph G is denoted by $\Delta(G)$. We denote a cycle on p vertices by C_p, a path on p vertices by P_p, and a complete graph on p vertices by K_p. A graph G is connected if any two vertices of G are connected by a path. A maximal connected subgraph of a graph G is called a component of G. The number of components of G is denoted by $\omega(G)$. The complement $\overline{G}$ of G is the graph with vertex set V in which two vertices are adjacent if and only if they are not adjacent in G. A tree is a connected acyclic graph. A bipartite graph (or bigraph) is a graph whose vertex set can be divided into two disjoint sets V_1 and V_2 such that every edge has one end in V_1 and another end in V_2. A complete bipartite graph is a bipartite graph where every vertex of V_1 is adjacent to every

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vertex in V_2. The complete bipartite graph with partitions of order $|V_1| = m$ and $|V_2| = n$, is denoted by $K_{m,n}$. A star, denoted by $K_{1,p-1}$ is a tree with one root vertex and $p-1$ pendant vertices. A bistar, denoted by $B(m,n)$ is the graph obtained by joining the root vertices of the stars $K_{1,m}$ and $K_{1,n}$. A wheel graph, denoted by W_p is a graph with p vertices, formed by joining a single vertex to all vertices of C_{p-1}. A helm graph, denoted by H_n is a graph obtained from the wheel W_n by attaching a pendant vertex to each vertex in the outer cycle of W_n. Corona of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$ is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 in which i^{th} vertex of G_1 is joined to every vertex in the i^{th} copy of G_2. If S is a subset of V, then $\langle S\rangle$ denotes the vertex induced subgraph of G induced by S. The open neighbourhood of a set S of vertices of a graph G, denoted by $N(S)$ is the set of all vertices adjacent to some vertex in S and $N(S) \cup S$ is called the closed neighbourhood of S, denoted by $N[S]$. The diameter of a connected graph is the maximum distance between two vertices in G and is denoted by $diam(G)$. A cut-vertex (cut edge) of a graph G is a vertex (edge) whose removal increases the number of components. A vertex cut, or separating set of a connected graph G is a set of vertices whose removal results in a disconnected graph. The connectivity or vertex connectivity of a graph G, denoted by $\kappa(G)$ (where G is not complete) is the size of a smallest vertex cut. A connected subgraph H of a connected graph G is called a H-cut if $\omega(G-H) \geq 2$. The chromatic number of a graph G, denoted by $\chi(G)$ is the smallest number of colors needed to colour all the vertices of a graph G in which adjacent vertices receive different colours. For any real number x, $\lfloor x\rfloor$ denotes the largest integer less than or equal to x. A Nordhaus-Gaddum-type result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. Terms not defined here are used in the sense of [2].

A subset S of V is called a dominating set of G if every vertex in $V-S$ is adjacent to at least one vertex in S. The domination number $\gamma(G)$ of G is the minimum cardinality taken over all dominating sets in G. A dominating set S of a connected graph G is said to be a connected dominating set of G if the induced sub graph $\langle S\rangle$ is connected. The minimum cardinality taken over all connected dominating sets is the connected domination number and is denoted by γ_c.

Many authors have introduced different types of domination parameters by imposing conditions on the dominating set [11-12]. Recently, the concept of triple connected graphs has been introduced by Paulraj Joseph et. al. [7] by considering the existence of a path containing any three vertices of G. They have studied the properties of triple connected graphs and established many results on them. A graph G is said to be triple connected if any three vertices lie on a path in G. All paths, cycles, complete graphs and wheels are some standard examples of triple connected graphs. In this paper, we use this idea to develop the concept of Smarandachely triple connected dominating set and Smarandachely triple connected domination number of a graph.

Theorem 1.1([7]) *A tree T is triple connected if and only if $T \cong P_p; p \geq 3$.*

Theorem 1.2([7]) *A connected graph G is not triple connected if and only if there exists a H-cut with $\omega(G-H) \geq 3$ such that $|V(H) \cap N(C_i)| = 1$ for at least three components C_1, C_2 and C_3 of $G-H$.*

Notation 1.3 Let G be a connected graph with m vertices $v_1, v_2, \ldots, v_m$. The graph obtained from G by attaching n_1 times a pendant vertex of P_{l_1} on the vertex v_1, n_2 times a pendant vertex of P_{l_2} on the vertex v_2 and so on, is denoted by $G(n_1P_{l_1}, n_2P_{l_2}, n_3P_{l_3}, \ldots, n_mP_{l_m})$ where $n_i, l_i \geq 0$ and $1 \leq i \leq m$.

Example 1.4 Let v_1, v_2, v_3, v_4, be the vertices of K_4. The graph $K_4(2P_2, P_3, P_4, P_3)$ is obtained from K_4 by attaching 2 times a pendant vertex of P_2 on v_1, 1 time a pendant vertex of P_3 on v_2, 1 time a pendant vertex of P_4 on v_3 and 1 time a pendant vertex of P_3 on v_4 and is shown in Figure 1.1.

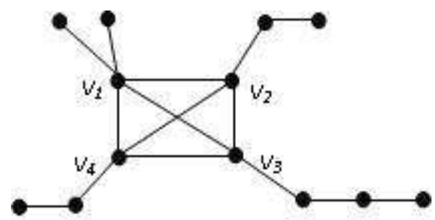

Figure 1.1 $K_4(2P_2, P_3, P_4, P_3)$

§2. Triple Connected Domination Number

Definition 2.1 *A subset S of V of a nontrivial connected graph G is said to be a Smarandachely triple connected dominating set, if S is a dominating set and the induced subgraph $\langle S \rangle$ is triple connected. The minimum cardinality taken over all Smarandachely triple connected dominating sets is called the Smarandachely triple connected domination number of G and is denoted by $\gamma_{tc}(G)$. Any Smarandachely triple connected dominating set with γ_{tc} vertices is called a γ_{tc}-set of G.*

Example 2.2 For the graph G_1 in Figure 2.1, $S = \{v_1, v_2, v_5\}$ forms a γ_{tc}-set of G_1. Hence $\gamma_{tc}(G_1) = 3$.

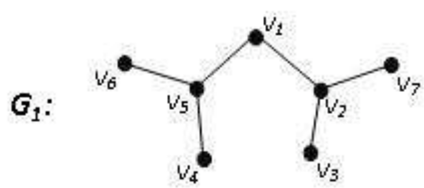

Figure 2.1 Graph with $\gamma_{tc} = 3$

Observation 2.3 Triple connected dominating set (tcd-set) does not exist for all graphs and if exists, then $\gamma_{tc}(G) \geq 3$.

Example 2.4 For the graph G_2 in Figure 2.2, any minimum dominating set must contain all the supports and any connected subgraph containing these supports is not triple connected and hence γ_{tc} does not exist.

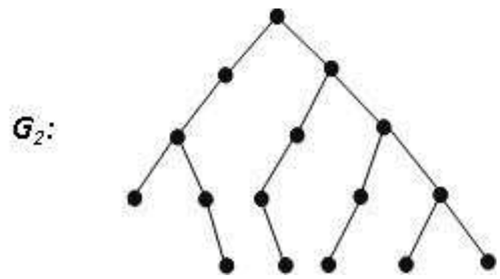

Figure 2.2 Graph with no tcd-set

Throughout this paper we consider only connected graphs for which triple connected dominating set exists.

Observation 2.5 The complement of the triple connected dominating set need not be a triple connected dominating set.

Example 2.6 For the graph G_3 in Figure 2.3, $S = \{v_1, v_2, v_3\}$ forms a triple connected dominating set of G_3. But the complement $V - S = \{v_4, v_5, v_6, v_7, v_8, v_9\}$ is not a triple connected dominating set.

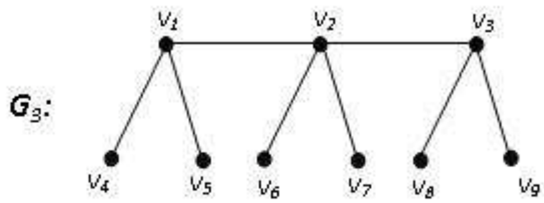

Figure 2.3 Graph in which $V - S$ is not a tcd-set

Observation 2.7 Every triple connected dominating set is a dominating set but not conversely.

Observation 2.8 For any connected graph $G, \gamma(G) \leq \gamma_c(G) \leq \gamma_{tc}(G)$ and the bounds are sharp.

Example 2.9 For the graph G_4 in Figure 2.4, $\gamma(G_4) = 4, \gamma_c(G_4) = 6$ and $\gamma_{tc}(G_4) = 8$. For the graph G_5 in Figure 2.4, $\gamma(G_5) = \gamma_c(G_5) = \gamma_{tc}(G_5) = 3$.

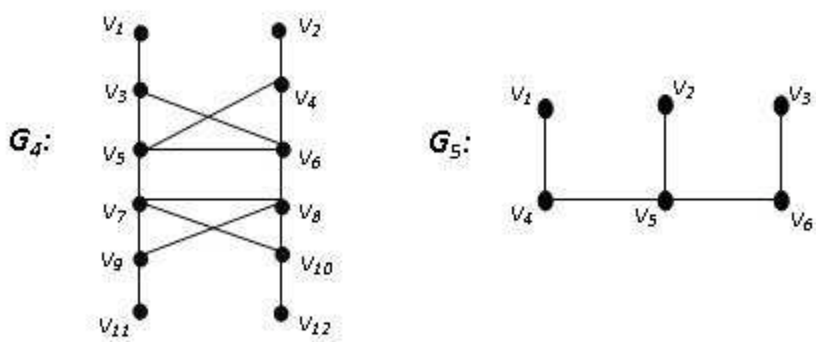

Figure 2.4

Theorem 2.10 *If the induced subgraph of each connected dominating set of G has more than two pendant vertices, then G does not contain a triple connected dominating set.*

Proof The proof follows from Theorem 1.2. □

Some exact value for some standard graphs are listed in the following:

1. Let P be the petersen graph. Then $\gamma_{tc}(P) = 5$.
2. For any triple connected graph G with p vertices, $\gamma_{tc}(G \circ K_1) = p$.
3. For any path of order $p \geq 3, \gamma_{tc}(P_p) = \begin{cases} 3 & \text{if } p < 5 \\ p-2 & \text{if } p \geq 5. \end{cases}$
4. For any cycle of order $p \geq 3, \gamma_{tc}(C_p) = \begin{cases} 3 & \text{if } p < 5 \\ p-2 & \text{if } p \geq 5. \end{cases}$
5. For any complete bipartite graph of order $p \geq 4, \gamma_{tc}(K_{m,n}) = 3$. (where $m, n \geq 2$ and $m + n = p$).
6. For any star of order $p \geq 3, \gamma_{tc}(K_{1,p-1}) = 3$.
7. For any complete graph of order $p \geq 3, \gamma_{tc}(K_p) = 3$.
8. For any wheel of order $p \geq 4, \gamma_{tc}(W_p) = 3$.
9. For any helm graph of order $p \geq 7, \gamma_{tc}(H_n) = \frac{p-1}{2}$ (where $2n - 1 = p$).
10. For any bistar of order $p \geq 4, \gamma_{tc}(B(m,n)) = 3$ (where $m, n \geq 1$ and $m + n + 2 = p$).

Example 2.11 For the graph G_6 in Figure 2.5, $S = \{v_6, v_2, v_3, v_4\}$ is a unique minimum connected dominating set so that $\gamma_c(G_6) = 4$. Here we notice that the induced subgraph of S has three pendant vertices and hence G does not contain a triple connected dominating set.

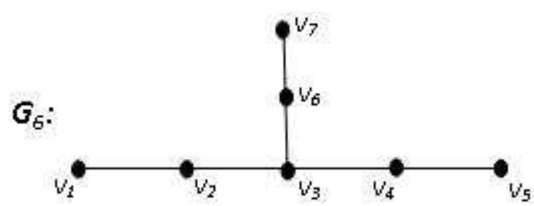

Figure 2.5 Graph having cd set and not having tcd-set

Observation 2.12 If a spanning sub graph H of a graph G has a triple connected dominating set, then G also has a triple connected dominating set.

Observation 2.13 Let G be a connected graph and H be a spanning sub graph of G. If H has a triple connected dominating set, then $\gamma_{tc}(G) \leq \gamma_{tc}(H)$ and the bound is sharp.

Example 2.14 Consider the graph G_7 and its spanning subgraphs G_8 and G_9 shown in Figure 2.6.

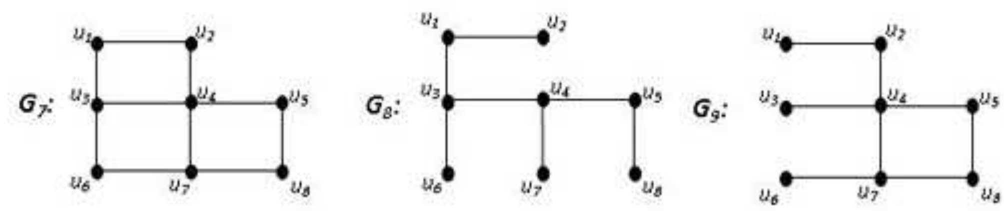

Figure 2.6

For the graph $G_7, S = \{u_2, u_4, u_7\}$ is a minimum triple connected dominating set and so $\gamma_{tc}(G_7) = 3$. For the spanning subgraph G_8 of $G_7, S = \{u_1, u_3, u_4, u_5\}$ is a minimum triple connected dominating set so that $\gamma_{tc}(G_8) = 4$. Hence $\gamma_{tc}(G_7) < \gamma_{tc}(G_8)$. For the spanning subgraph G_9 of $G_7, S = \{u_2, u_4, u_7\}$ is a minimum triple connected dominating set so that $\gamma_{tc}(G_9) = 3$. Hence $\gamma_{tc}(G_7) = \gamma_{tc}(G_9)$.

Observation 2.15 For any connected graph G with p vertices, $\gamma_{tc}(G) = p$ if and only if $G \cong P_3$ or C_3.

Theorem 2.16 *For any connected graph G with p vertices, $\gamma_{tc}(G) = p - 1$ if and only if $G \cong P_4, C_4, K_4, K_{1,3}, K_4 - \{e\}, C_3(P_2)$.*

Proof Suppose $G \cong P_4, C_4, K_4 - \{e\}, K_4, K_{1,3}, C_3(P_2)$, then $\gamma_{tc}(G) = 3 = p-1$. Conversely, let G be a connected graph with p vertices such that $\gamma_{tc}(G) = p-1$. Let $S = \{u_1, u_2, \ldots, u_{p-1}\}$ be a γ_{tc}-set of G. Let x be in $V - S$. Since S is a dominating set, there exists a vertex v_i in S such that v_i is adjacent to x. If $p \geq 5$, by taking the vertex v_i, we can construct a triple connected dominating set S with fewer elements than $p - 1$, which is a contradiction. Hence $p \leq 4$. Since $\gamma_{tc}(G) = p - 1$, by Observation 2.5, we have $p = 4$. Let $S = \{v_1, v_2, v_3\}$ and $V - S = \{v_4\}$. Since S is a γ_{tc}-set of $G, \langle S\rangle = P_3$ or C_3.

Case *i* $\langle S\rangle = P_3 = v_1v_2v_3$

Since G is connected, v_4 is adjacent to v_1 (or v_3) or v_4 is adjacent to v_2. Hence $G \cong P_4$ or $K_{1,3}$.

Case *ii* $\langle S\rangle = C_3 = v_1v_2v_3v_1$

Since G is connected, v_4 is adjacent to v_1 (or v_2 or v_3). Hence $G \cong C_3(P_2)$. Now by adding edges to $P_4, K_{1,3}$ or $C_3(P_2)$ without affecting γ_{tc}, we have $G \cong C_4, K_4 - \{e\}, K_4$. □

Theorem 2.17 *For any connected graph G with $p \geq 5$, we have $3 \leq \gamma_{tc}(G) \leq p - 2$ and the bounds are sharp.*

Proof The lower bound follows from Definition 2.1 and the upper bound follows from Observation 2.15 and Theorem 2.16. Consider the dodecahedron graph G_{10} in Figure 2.7, the path P_5 and the cycle C_9.

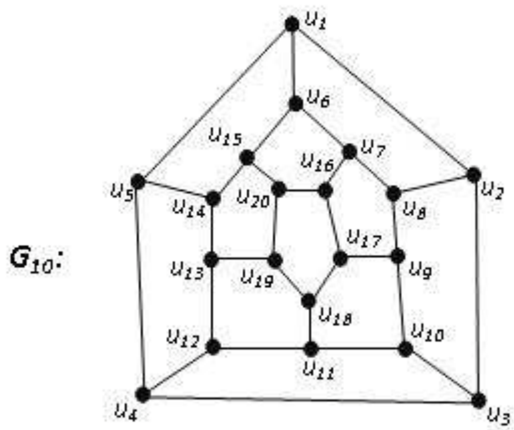

Figure 2.7

One can easily check that $S = \{u_6, u_7, u_8, u_9, u_{10}, u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}$ is a minimum triple connected dominating set of G_{10} and $\gamma_{tc}(G_{10}) = 10 > 3$. In addition, $\gamma_{tc}(G_{10}) = 10 < p - 2$. For P_5, the lower bound is attained and for C_9 the upper bound is attained. □

Theorem 2.18 *For a connected graph G with 5 vertices, $\gamma_{tc}(G) = p - 2$ if and only if G is isomorphic to $P_5, C_5, W_5, K_5, K_{1,4}, K_{2,3}, K_1 \circ 2K_2, K_5 - \{e\}, K_4(P_2), C_4(P_2), C_3(P_3), C_3(2P_2), C_3(P_2, P_2, 0)$, $P_4(0, P_2, 0, 0)$ or any one of the graphs shown in Figure* 2.8.

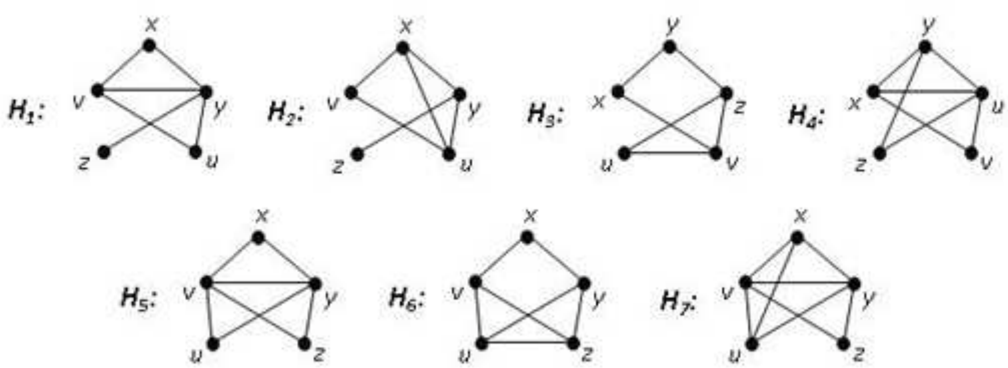

Figure 2.8 Graphs with $\gamma_{tc} = p - 2$

Proof Suppose G is isomorphic to $P_5, C_5, W_5, K_5, K_{1,4}, K_{2,3}, K_1 \circ 2K_2, K_5 - \{e\}$, $K_4(P_2), C_4(P_2), C_3(P_3), C_3(2P_2), C_3(P_2, P_2, 0), P_4(0, P_2, 0, 0)$ or any one of the graphs H_1 to H_7 given in Figure 2.8., then clearly $\gamma_{tc}(G) = p - 2$. Conversely, let G be a connected graph with 5 vertices and $\gamma_{tc}(G) = 3$. Let $S = \{x, y, z\}$ be a γ_{tc}-set. Then clearly $\langle S \rangle = P_3$ or C_3. Let $V - S = V(G) - V(S) = \{u, v\}$. Then $\langle V - S \rangle = K_2$ or $\overline{K}_2$.

Case *i* $\langle S \rangle = P_3 = xyz$

Subcase *i* $\langle V-S\rangle = K_2 = uv$

Since G is connected, there exists a vertex say x (or z) in P_3 which is adjacent to u (or v) in K_2. Then $S=\{x,y,u\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(y)=2, d(z)=1$, then $G\simeq P_5$. Since G is connected, there exists a vertex say y in P_3 is adjacent to u (or v) in K_2. Then $S=\{y,u,v\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(z)=1, d(y)=3$, then $G\cong P_4(0,P_2,0,0)$. Now by increasing the degrees of the vertices, by the above arguments, we have $G\cong C_5, W_5, K_5, K_{2,3}, K_5-\{e\}, K_4(P_2), C_4(P_2), C_3(P_3), C_3(2P_2), C_3(P_2,P_2,0)$ and H_1 to H_7 in Figure 2.8. In all the other cases, no new graph exists.

Subcase *ii* $\langle V-S\rangle = 2$

Since G is connected, there exists a vertex say x (or z) in P_3 is adjacent to u and v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=3, d(y)=2, d(z)=1$, then $G\cong P_4(0,P_2,0,0)$. In all the other cases, no new graph exists. Since G is connected, there exists a vertex say y in P_3 which is adjacent to u and v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(z)=1, d(y)=4$, then $G\cong K_{1,4}$. In all the other cases, no new graph exists. Since G is connected, there exists a vertex say x in P_3 which is adjacent to u in $\overline{K}_2$ and y in P_3 is adjacent to v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=2, d(y)=3, d(z)=1$, then $G\cong P_4(0,P_2,0,0)$. In all the other cases, no new graph exists. Since G is connected, there exists a vertex say x in P_3 which is adjacent to u in $\overline{K}_2$ and z in P_3 which is adjacent to v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(y)=d(z)=2$, then $G\cong P_5$. In all the other cases, no new graph exists.

Case *ii* $\langle S\rangle = C_3 = xyzx$

Subcase *i* $\langle V-S\rangle = K_2 = uv$

Since G is connected, there exists a vertex say x (or y,z) in C_3 is adjacent to u (or v) in K_2. Then $S=\{x,y,u\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=3, d(y)=d(z)=2$, then $G\cong C_3(P_3)$. If $d(x)=4, d(y)=d(z)=2$, then $G\cong K_1\circ 2K_2$. In all the other cases, no new graph exists.

Subcase *ii* $\langle V-S\rangle = \overline{K}_2$

Since G is connected, there exists a vertex say x (or y,z) in C_3 is adjacent to u and v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=4, d(y)=d(z)=2$, then $G\cong C_3(2P_2)$. In all the other cases, no new graph exists. Since G is connected, there exists a vertex say x(or y,z) in C_3 is adjacent to u in $\overline{K}_2$ and y (or z) in C_3 is adjacent to v in $\overline{K}_2$. Then $S=\{x,y,z\}$ is a minimum triple connected dominating set of G so that $\gamma_{tc}(G)=p-2$. If $d(x)=d(y)=3, d(z)=2$, then $G\cong C_3(P_2,P_2,0)$. In all other cases, no new graph exists. □

Theorem 2.19 *For a connected graph G with $p>5$ vertices, $\gamma_{tc}(G)=p-2$ if and only if G*

is isomorphic to P_p or C_p.

Proof Suppose G is isomorphic to P_p or C_p, then clearly $\gamma_{tc}(G) = p-2$. Conversely, let G be a connected graph with $p > 5$ vertices and $\gamma_{tc}(G) = p-2$. Let $S = \{v_1, v_2, \ldots, v_{p-2}\}$ be a γ_{tc}-set and let $V - S = V(G) - V(S) = \{v_{p-1}, v_p\}$. Then $\langle V - S\rangle = K_2, \overline{K}_2$.

Claim. *$\langle S\rangle$ is a tree.*

Suppose $\langle S\rangle$ is not a tree. Then $\langle S\rangle$ contains a cycle. Without loss of generality, let $C = v_1v_2\cdots v_qv_1, q \le p-2$ be a cycle of shortest length in $\langle S\rangle$. Now let $\langle V-S\rangle = K_2 = v_{p-1}v_p$. Since G is connected and S is a γ_{tc}-set of G, v_{p-1}(or v_p) is adjacent to a vertex v_k in $\langle S\rangle$. If v_k is in C, then $S = \{v_{p-1}, v_i, v_{i+1}, \ldots, v_{i-3}\} \cup \{x \in V(G) : x \notin C\}$ forms a γ_{tc}-set of G so that $\gamma_{tc}(G) < p-2$, which is a contradiction. Suppose v_{p-1} (or v_p) is adjacent to a vertex v_i in $\langle S\rangle - C$, then we can construct a γ_{tc}-set which contains v_{p-1}, v_i with fewer elements than $p-2$, which is a contradiction. Similarly if $\langle V - S\rangle = \overline{K}_2$, we can prove that no graph exists. Hence $\langle S\rangle$ is a tree. But S is a triple connected dominating set. Therefore by Theorem 1.1, we have $\langle S\rangle \cong P_{p-2}$.

Case *i* $\langle V - S\rangle = K_2 = v_{p-1}v_p$

Since G is connected and S is a γ_{tc}-set of G, there exists a vertex, say, v_i in P_{p-2} which is adjacent to a vertex, say, v_{p-1} in K_2. If $v_i = v_1$ (or) v_{p-2}, then $G \cong P_p$. If $v_i = v_1$ is adjacent to v_{p+1} and v_{p-2} is adjacent to v_p, then $G \cong C_p$. If $v_i = v_j$ for $j = 2, 3, \ldots, p-3$, then $S_1 = S - \{v_1, v_{p-2}\} \cup \{v_{p-1}\}$ is a triple connected dominating set of cardinality $p-3$ and hence $\gamma_{tc} \le p-3$, which is a contradiction.

Case *ii* $\langle V - S\rangle = \overline{K}_2$

Since G is connected and S is a γ_{tc}-set of G, there exists a vertex say v_i in P_{p-2} which is adjacent to both the vertices v_{p-1} and v_p in $\overline{K}_2$. If $v_i = v_1$ (or v_{p-2}), then by taking the vertex v_1 (or v_{p-2}), we can construct a triple connected dominating set which contains fewer elements than $p-2$, which is a contradiction. Hence no graph exists. If $v_i = v_j$ for $j = 2, 3, \ldots, n-3$, then by taking the vertex v_j, we can construct a triple connected dominating set which contains fewer elements than $p-2$, which is a contradiction. Hence no graph exists. Suppose there exists a vertex say v_i in P_{p-2} which is adjacent to v_{p-1} in $\overline{K}_2$ and a vertex $v_j (i \ne j)$ in P_{p-2} which is adjacent to v_p in $\overline{K}_2$. If $v_i = v_1$ and $v_j = v_{p-2}$, then $S = \{v_1, v_2, \ldots, v_{p-2}\}$ is a γ_{tc}-set of G and hence $G \cong P_p$. If $v_i = v_1$ and $v_j = v_k$ for $k = 2, 3, \ldots, n-3$, then by taking the vertex v_1 and v_k, we can construct a triple connected dominating set which contains fewer elements than $p-2$, which is a contradiction. Hence no graph exists. If $v_i = v_k$ and $v_j = v_l$ for $k, l = 2, 3, \ldots, n-3$, then by taking the vertex v_k and v_l, we can construct a triple connected dominating set which contains fewer elements than $p-2$, which is a contradiction. □

Corollary 2.20 *Let G be a connected graph with $p > 5$ vertices. If $\gamma_{tc}(G) = p-2$, then $\kappa(G) = 1$ or 2, $\Delta(G) = 2, \chi(G) = 2$ or 3, and diam$(G) = p-1$ or $\lfloor \frac{p}{2} \rfloor$.*

Proof Let G be a connected graph with $p > 5$ vertices and $\gamma_{tc}(G) = p-2$. Since $\gamma_{tc}(G) = p-2$, by Theorem 2.19, G is isomorphic to P_p or C_p. We know that for $P_p, \kappa(G) = 1, \Delta(G) =$

$2, \chi(G) = 2$ and $diam(G) = p-1$. For $C_p, \kappa(G) = 2, \Delta(G) = 2$, $diam(G) = \lfloor \frac{p}{2} \rfloor$ and

$$\chi(G) = \begin{cases} 2 & \text{if p is even,} \\ 3 & \text{if p is odd.} \end{cases} \qquad \square$$

Observation 2.21 Let G be a connected graph with $p \geq 3$ vertices and $\Delta(G) = p-1$. Then $\gamma_{tc}(G) = 3$.

For, let v be a full vertex in G. Then $S = \{v, v_i, v_j\}$ is a minimum triple connected dominating set of G, where v_i and v_j are in $N(v)$. Hence $\gamma_{tc}(G) = 3$.

Theorem 2.22 *For any connected graph G with $p \geq 3$ vertices and $\Delta(G) = p-2, \gamma_{tc}(G) = 3$.*

Proof Let G be a connected graph with $p \geq 3$ vertices and $\Delta(G) = p-2$. Let v be a vertex of maximum degree $\Delta(G) = p-2$. Let $v_1, v_2, \ldots$ and v_{p-2} be the vertices which are adjacent to v, and let v_{p-1} be the vertex which is not adjacent to v. Since G is connected, v_{p-1} is adjacent to a vertex v_i for some i. Then $S = \{v, v_i, v_j | i \neq j\}$ is a minimum triple connected dominating set of G. Hence $\gamma_{tc}(G) = 3$. $\square$

Theorem 2.23 *For any connected graph G with $p \geq 3$ vertices and $\Delta(G) = p-3, \gamma_{tc}(G) = 3$.*

Proof Let G be a connected graph with $p \geq 3$ vertices and $\Delta(G) = p-3$ and let v be the vertex of G with degree $p-3$. Suppose $N(v) = \{v_1, v_2, \ldots, v_{p-3}\}$ and $V - N(v) = \{v_{p-2}, v_{p-1}\}$. If v_{p-1} and v_{p-2} are not adjacent in G, then since G is connected, there are vertices v_i and v_j for some $i, j, 1 \leq i, j \leq p-3$, which are adjacent to v_{p-2} and v_{p-1} respectively. Here note that i can be equal to j. If $i = j$, then $\{v, v_i, v_{p-1}\}$ is a required triple connected dominating set of G. If $i \neq j$, then $\{v_i, v, v_j\}$ is a required triple connected dominating set of G. If v_{p-2} and v_{p-1} are adjacent in G, then there is a vertex v_j, for some $j, 1 \leq j \leq p-3$, which is adjacent to v_{p-1} or to v_{p-1} or to both. In this case, $\{v, v_i, v_{p-1}\}$ or $\{v, v_i, v_{p-2}\}$ is a triple connected dominating set of G. Hence in all the cases, $\gamma_{tc}(G) = 3$. $\square$

The Nordhaus - Gaddum type result is given below:

Theorem 2.24 *Let G be a graph such that G and $\overline{G}$ are connected graphs of order $p \geq 5$. Then $\gamma_{tc}(G) + \gamma_{tc}(\overline{G}) \leq 2(p-2)$ and the bound is sharp.*

Proof The bound directly follows from Theorem 2.17. For the cycle $C_5, \gamma_{tc}(G) + \gamma_{tc}(\overline{G}) = 2(p-2)$. $\square$

§3. Relation with Other Graph Parameters

Theorem 3.1 *For any connected graph G with $p \geq 5$ vertices, $\gamma_{tc}(G) + \kappa(G) \leq 2p-3$ and the bound is sharp if and only if $G \cong K_5$.*

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\kappa(G) \leq p-1$ and by Theorem 2.17, $\gamma_{tc}(G) \leq p-2$. Hence $\gamma_{tc}(G) + \kappa(G) \leq 2p-3$. Suppose G is isomorphic to K_5. Then clearly $\gamma_{tc}(G) + \kappa(G) = 2p - 3$. Conversely, let $\gamma_{tc}(G) + \kappa(G) = 2p - 3$. This is possible only if $\gamma_{tc}(G) = p - 2$ and $\kappa(G) = p - 1$. But $\kappa(G) = p - 1$, and so $G \cong K_p$ for which $\gamma_{tc}(G) = 3 = p - 2$ so that $p = 5$. Hence $G \cong K_5$. $\square$

Theorem 3.2 *For any connected graph G with $p \geq 5$ vertices, $\gamma_{tc}(G) + \chi(G) \leq 2p - 2$ and the bound is sharp if and only if $G \cong K_5$.*

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\chi(G) \leq p$ and by Theorem 2.17, $\gamma_{tc}(G) \leq p - 2$. Hence $\gamma_{tc}(G) + \chi(G) \leq 2p - 2$. Suppose G is isomorphic to K_5. Then clearly $\gamma_{tc}(G) + \chi(G) = 2p - 2$. Conversely, let $\gamma_{tc}(G) + \chi(G) = 2p - 2$. This is possible only if $\gamma_{tc}(G) = p - 2$ and $\chi(G) = p$. Since $\chi(G) = p, G$ is isomorphic to K_p for which $\gamma_{tc}(G) = 3 = p - 2$ so that $p = 5$. Hence $G \cong K_5$. $\square$

Theorem 3.3 *For any connected graph G with $p \geq 5$ vertices, $\gamma_{tc}(G) + \Delta(G) \leq 2p - 3$ and the bound is sharp if and only if G is isomorphic to $W_5, K_5, K_{1,4}, K_1 \circ 2K_2, K_5 - \{e\}, K_4(P_2), C_3(2P_2)$ or any one of the graphs shown in Figure 3.1.*

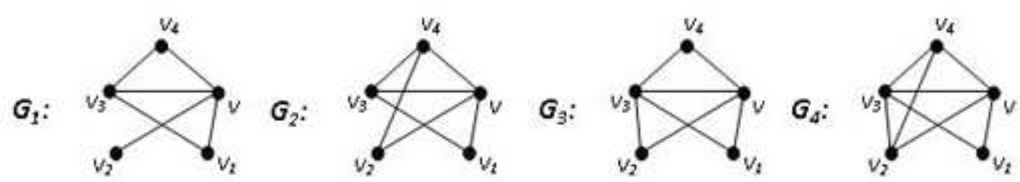

Figure 3.1 Graphs with $\gamma_{tc} + \Delta = 2p - 3$

Proof Let G be a connected graph with $p \geq 5$ vertices. We know that $\Delta(G) \leq p - 1$ and by Theorem 2.17, $\gamma_{tc}(G) \leq p - 2$. Hence $\gamma_{tc}(G) + \Delta(G) \leq 2p - 3$. Let G be isomorphic to $W_5, K_5, K_{1,4}, K_1 \circ 2K_2, K_5 - \{e\}, K_4(P_2), C_3(2P_2)$ or any one of the graphs G_1 to G_4 given in Figure 3.1. Then clearly $\gamma_{tc}(G) + \Delta(G) = 2p - 3$. Conversely, let $\gamma_{tc}(G) + \Delta(G) = 2p - 3$. This is possible only if $\gamma_{tc}(G) = p - 2$ and $\Delta(G) = p - 1$. Since $\Delta(G) = p - 1$, by Observation 2.21, we have $\gamma_{tc}(G) = 3$ so that $p = 5$. Let v be the vertex having a maximum degree and let v_1, v_2, v_3, v_4 be the vertices which are adjacent to the vertex v. If $d(v) = 4, d(v_1) = d(v_2) = d(v_3) = d(v_4) = 1$, then $G \cong K_{1,4}$. Now by adding edges to $K_{1,4}$ without affecting the value of γ_{tc}, we have $G \cong W_5, K_5, K_1 \circ 2K_2, K_5 - \{e\}, K_4(P_2), C_3(2P_2)$ and the graphs G_1 to G_4 given in Figure 3.1. $\square$

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Odd Harmonious Labeling of Some Graphs

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Abstract: The labeling of discrete structures is a potential area of research due to its wide range of applications. The present work is focused on one such labeling called odd harmonious labeling. A graph G is said to be odd harmonious if there exist an injection $f : V(G) \to \{0, 1, 2, \ldots, 2q-1\}$ such that the induced function $f^* : E(G) \to \{1, 3, \ldots, 2q-1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. Here we investigate odd harmonious labeling of some graphs. We prove that the shadow graph and the splitting graph of bistar $B_{n,n}$ are odd harmonious graphs. Moreover we show that the arbitrary supersubdivision of path P_n admits odd harmonious labeling. We also prove that the joint sum of two copies of cycle C_n for $n \equiv 0(mod\ 4)$ and the graph $H_{n,n}$ are odd harmonious graphs.

Key Words: Harmonious labeling, Smarandachely p-harmonious labeling, odd harmonious labeling, shadow graph, splitting graph, arbitrary supersubdivision.

AMS(2010): 05C78

§1. Introduction

We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with $|V(G)| = p$ and $|E(G)| = q$. For standard terminology and notation we follow Gross and Yellen [5]. We will provide brief summary of definitions and other information which are necessary and useful for the present investigations.

Definition 1.1 *If the vertices are assigned values subject to certain condition(s) then it is known as graph labeling.*

Any graph labeling will have the following three common characteristics:

(1) a set of numbers from which the vertex labels are chosen;
(2) a rule that assigns a value to each edge;
(3) a condition that these values must satisfy.

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Graph labelings is an active area of research in graph theory which is mainly evolved through its rigorous applications in coding theory, communication networks, optimal circuits layouts and graph decomposition problems. According to Beineke and Hegde [1] graph labeling serves as a frontier between number theory and structure of graphs. For a dynamic survey of various graph labeling problems along with an extensive bibliography we refer to Gallian [2].

Definition 1.2 *A function f is called graceful labeling of a graph G if $f : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^* : E(G) \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e = uv) = |f(u) - f(v)|$ is bijective.*

A graph which admits graceful labeling is called a *graceful graph*. Rosa [8] called such a labeling a $\beta - valuation$ and Golomb [3] subsequently called it *graceful labeling*. Several infinite families of graceful as well as non-graceful graphs have been reported. The famous Ringel-Kotzig tree conjecture [7] and many illustrious works on graceful graphs brought a tide of different ways of labeling the elements of graph such as odd graceful labeling, harmonious labeling etc. Graham and Sloane [4] introduced harmonious labeling during their study of modular versions of additive bases problems stemming from error correcting codes.

Definition 1.3 *A graph G is said to be harmonious if there exist an injection $f : V(G) \rightarrow Z_q$ such that the induced function $f^* : E(G) \rightarrow Z_q$ defined by $f^*(uv) = (f(u) + f(v))\,(mod\ q)$ is a bijection and f is said to be harmonious labelling of G.*

If G is a tree or it has a component that is a tree, then exactly one label may be used on two vertices and the labeling function is not an injection. After this many researchers have studied harmonious labeling. A labeling is also introduced with minor variation in harmonious theme, which is defined as follows.

Definition 1.4 *Let k, p be integers with $p \geq 1$ and $k \leq p$. A graph G is said to be Smarandachely p-harmonious labeling if there exist an injection $f : V(G) \rightarrow \{0, 1, 2, \ldots, kq - 1\}$ such that the induced function $f^* : E(G) \rightarrow \{1, p + 1, \ldots, pq - 1\}$ defined by $f^*(uv) = f(u) + f(v)$ is a bijection. Particularly, if $p = k = 2$, such a Smarandachely 2-harmonious labeling is called an odd harmonious labeling of G, f and f^* are called vertex function and edge function respectively.*

Liang and Bai [6] have obtained a necessary conditions for the existence of odd harmonious labelling of graph. It has been also shown that many graphs admit odd harmonious labeling and odd harmoniousness of graph is useful for the solution of undetermined equations. In the same paper many ways to construct an odd harmonious graph were reported. Vaidya and Shah [9] have also proved that the shadow and the splitting graphs of path P_n and star $K_{1,n}$ are odd harmonious graphs.

Generally there are three types of problems that can be considered in this area.

(1) How odd harmonious labeling is affected under various graph operations;

(2) To construct new families of odd harmonious graph by investigating suitable function which generates labeling;

(3) Given a graph theoretic property P, characterize the class/classes of graphs with prop-

erty P that are odd harmonious.

The problems of second type are largely discussed while the problems of first and third types are not so often but they are of great importance. The present work is aimed to discuss the problems of first kind.

Definition 1.5 *The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G say G' and G''. Join each vertex u' in G' to the neighbours of the corresponding vertex v' in G''.*

Definition 1.6 *For a graph G the splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.*

Definition 1.7 *The arbitrary supersubdivision of a graph G produces a new graph by replacing each edge of G by complete bipartite graph K_{2,m_i}(where m_i is any positive integer) in such a way that the ends of each e_i are merged with two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from the graph G.*

Definition 1.8 *Consider two copies of a graph G and define a new graph known as joint sum is the graph obtained by connecting a vertex of first copy with a vertex of second copy.*

Definition 1.9 *$H_{n,n}$ is the graph with vertex set $V(H_{n,n}) = \{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and the edge set $E(H_{n,n}) = \{v_i u_j : 1 \leqslant i \leqslant n, n-i+1 \leqslant j \leqslant n\}$.*

§2. Main Results

Theorem 2.1 *$D_2(B_{n,n})$ is an odd harmonious graph.*

Proof Consider two copies of $B_{n,n}$. Let $\{u, v, u_i, v_i, 1 \leq i \leq n\}$ and $\{u', v', u'_i, v'_i, 1 \leq i \leq n\}$ be the corresponding vertex sets of each copy of $B_{n,n}$. Denote $D_2(B_{n,n})$ as G. Then $|V(G)| = 4(n+1)$ and $|E(G)| = 4(2n+1)$. To define $f : V(G) \to \{0, 1, 2, 3, \ldots, 16n+7\}$, we consider following two cases.

Case 1. n is even

$$\begin{aligned}
&f(u) = 2,\ f(v) = 1,\ f(u') = 0,\ f(v') = 5,\\
&f(u_i) = 9 + 4(i-1),\ 1 \leqslant i \leqslant n,\ f(u'_i) = f(u_n) + 4i,\ 1 \leqslant i \leqslant n,\\
&f(v_1) = f(u'_n) + 3,\ f(v_{2i+1}) = f(v_1) + 8i,\ 1 \leqslant i \leqslant \tfrac{n}{2} - 1,\\
&f(v_2) = f(u'_n) + 5,\ f(v_{2i}) = f(v_2) + 8(i-1),\ 2 \leqslant i \leqslant \tfrac{n}{2},\\
&f(v'_1) = f(v_n) + 6,\ f(v'_{2i+1}) = f(v'_1) + 8i,\ 1 \leqslant i \leqslant \tfrac{n}{2} - 1,\\
&f(v'_2) = f(v_n) + 8,\ f(v'_{2i}) = f(v'_2) + 8(i-1),\ 2 \leqslant i \leqslant \tfrac{n}{2}
\end{aligned}$$

Case 2: n is odd

$$f(u) = 2,\ f(v) = 1,\ f(u') = 0,\ f(v') = 5,$$
$$f(u_i) = 9 + 4(i-1),\ 1 \leqslant i \leqslant n,\ f(u'_i) = f(u_n) + 4i,\ 1 \leqslant i \leqslant n,$$
$$f(v_1) = f(u'_n) + 3,\ f(v_{2i+1}) = f(v_1) + 8i,\ 1 \leqslant i \leqslant \tfrac{n-1}{2},$$
$$f(v_2) = f(u'_n) + 5,\ f(v_{2i}) = f(v_2) + 8(i-1),\ 2 \leqslant i \leqslant \tfrac{n-1}{2},$$
$$f(v'_1) = f(v_n) + 2,\ f(v'_{2i+1}) = f(v'_1) + 8i,\ 1 \leqslant i \leqslant \tfrac{n-1}{2},$$
$$f(v'_2) = f(v_n) + 8,\ f(v'_{2i}) = f(v'_2) + 8(i-1),\ 2 \leqslant i \leqslant \tfrac{n-1}{2}$$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \to \{1, 3, \ldots, 16n + 7\}$. Thus f is an odd harmonious labeling for $G = D_2(B_{n,n})$. Hence G is an odd harmonious graph. $\square$

Illustration 2.2 Odd harmonious labeling of the graph $D_2(B_{5,5})$ is shown in Fig. 1.

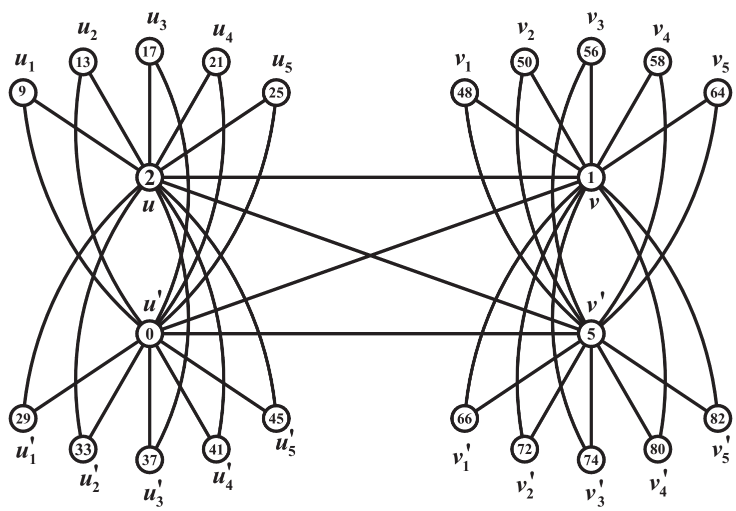

Fig. 1

Theorem 2.3 *$S'(B_{n,n})$ is an odd harmonious graph.*

Proof Consider $B_{n,n}$ with vertex set $\{u, v, u_i, v_i, 1 \leq i \leq n\}$, where u_i, v_i are pendant vertices. In order to obtain $S'(B_{n,n})$, add u', v', u'_i, v'_i vertices corresponding to u, v, u_i, v_i where, $1 \leq i \leq n$. If $G = S'(B_{n,n})$ then $|V(G)| = 4(n + 1)$ and $|E(G)| = 6n + 3$. We define vertex labeling $f : V(G) \to \{0, 1, 2, 3, \ldots, 12n + 5\}$ as follows.

$$f(u) = 0,\ f(v) = 3,\ f(u') = 2,\ f(v') = 1,$$
$$f(u_i) = 7 + 4(i-1),\ 1 \leqslant i \leqslant n,\ f(v_1) = f(u_n) + 3,$$
$$f(v_{i+1}) = f(v_1) + 4i,\ 1 \leqslant i \leqslant n-1,$$
$$f(u'_1) = f(v_n) + 5,\ f(u'_{i+1}) = f(u'_1) + 2i,\ 1 \leqslant i \leqslant n-1,$$
$$f(v'_1) = f(u'_n) - 1,\ f(v'_{i+1}) = f(v'_1) + 2i,\ 1 \leqslant i \leqslant n-1.$$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \rightarrow \{1, 3, \cdots, 12n+5\}$. Thus f is an odd harmonious labeling of $G = S'(B_{n,n})$ and G is an odd harmonious graph. $\square$

Illustration 2.4 Odd harmonious labeling of the graph $S'(B_{5,5})$ is shown in Fig. 2.

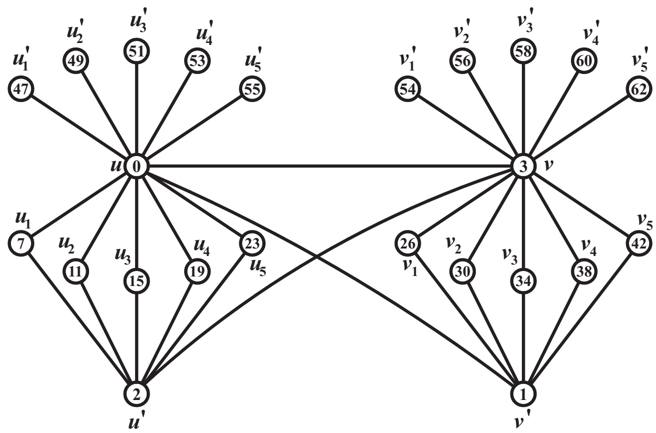

Fig. 2

Theorem 2.5 *Arbitrary supersubdivision of path P_n is an odd harmonious graph.*

Proof Let P_n be the path with n vertices and v_i $(1 \leqslant i \leqslant n)$ be the vertices of P_n. Arbitrary supersubdivision of P_n is obtained by replacing every edge e_i of P_n with K_{2,m_i} and we denote this graph by G. Let u_{ij} be the vertices of m_i-vertices part of K_{2,m_i} where $1 \leqslant i \leqslant n-1$ and $1 \leqslant j \leqslant max\{m_i\}$. Let $\alpha = \sum\limits_{i=1}^{n-1} m_i$ and $q = 2\alpha$. We define vertex labeling $f : V(G) \rightarrow \{0, 1, 2, 3, \cdots, 2q-1\}$ as follows.

$$\begin{aligned}
&f(v_{i+1}) = 2i,\ 0 \leqslant i \leqslant n-1,\\
&f(u_{1j}) = 1 + 4(j-1),\ 1 \leqslant j \leqslant m_1,\\
&f(u_{ij}) = f(u_{(i-1)n}) + 2 + 4(j-1),\ 1 \leqslant j \leqslant m_i, 2 \leqslant i \leqslant n.
\end{aligned}$$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \rightarrow \{1, 3, \cdots, 2q-1\}$. Thus f is an odd harmonious labeling of G. Hence arbitrary supersubdivision of path P_n is an odd harmonious graph. $\square$

Illustration 2.6 Odd harmonious labeling of arbitrary supersubdivision of path P_5 is shown in Fig. 3.

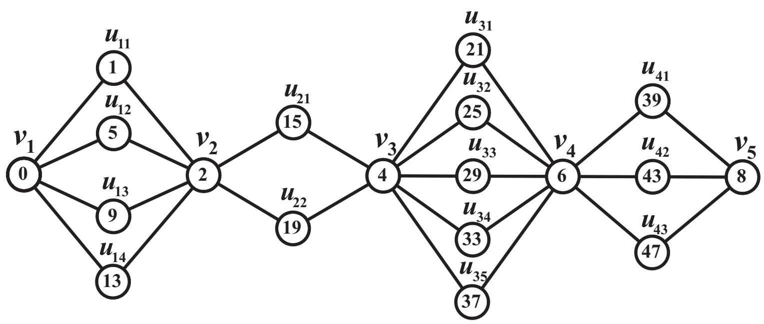

Fig. 3

Theorem 2.7 *Joint sum of two copies of C_n admits an odd harmonious labeling for $n \equiv 0(mod\ 4)$.*

Proof We denote the vertices of first copy of C_n by $v_1, v_2, \ldots, v_n$ and vertices of second copy by $v_{n+1}, v_{n+2}, \ldots, v_{2n}$. Join the two copies of C_n with a new edge and denote the resultant graph by G then $|V(G)| = 2n$ and $|E(G)| = 2n + 1$. Without loss of generality we assume that the new edge by $v_n v_{n+1}$ and $v_1, v_2, \cdots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n}$ will form a spanning path in G. Define $f : V(G) \to \{0, 1, 2, 3, \cdots, 4n + 1\}$ as follows.

$$f(v_{2i+1}) = 2i,\ 0 \leqslant i \leqslant \tfrac{3n}{4} - 1,$$
$$f\left(v_{\frac{3n}{2}+2i-1}\right) = \tfrac{3n}{2} + 2i,\ 1 \leqslant i \leqslant \tfrac{n}{4},$$
$$f(v_{2i}) = 2i - 1,\ 1 \leqslant i \leqslant \tfrac{n}{4},$$
$$f\left(v_{\frac{n}{2}+2i+2}\right) = \tfrac{n}{2} + 3 + 2i,\ 0 \leqslant i \leqslant \tfrac{3n}{4} - 1.$$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \to \{1, 3, \ldots, 4n + 1\}$. Thus f is an odd harmonious labeling of G. Hence joint sum of two copies of C_n admits odd harmonious labeling for $n \equiv 0(mod\ 4)$. □

Illustration 2.8 Odd harmonious labeling of the joint sum of two copies of C_{12} is shown in Fig. 4.

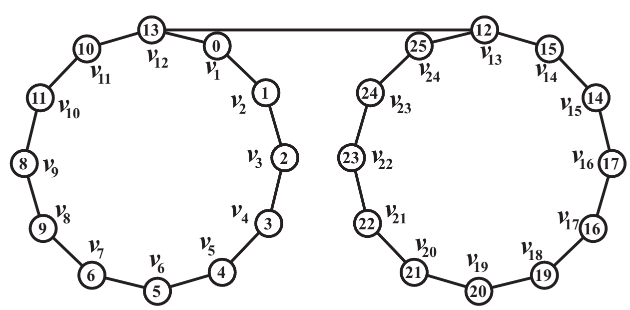

Fig. 4

Theorem 2.9 *The graph $H_{n,n}$ is on odd harmonious graph.*

Proof Let $V = \{v_1, v_2, \cdots, v_n\}$, $U = \{u_1, u_2, \cdots, u_n\}$ be the partition of $V(H_{n,n})$. Let $G = H_{n,n}$ then $|V(G)| = 2n$ and $|E(G)| = \dfrac{n(n+1)}{2}$. We define odd harmonious labeling $f : V(G) \to \{0, 1, 2, 3, \cdots, (n^2 + n - 1)\}$ as follows.

$$f(v_i) = i(i-1),\ 1 \leqslant i \leqslant n,$$
$$f(u_i) = (2n+1) - 2i,\ 1 \leqslant i \leqslant n.$$

The vertex function f defined above induces a bijective edge function $f^* : E(G) \to \{1, 3, \cdots, n^2 + n - 1\}$. Thus f is an odd harmonious labeling of G. Hence the graph $H_{n,n}$ is on odd harmonious graph. $\square$

Illustration 2.10 Odd harmonious labeling of the graph $H_{5,5}$ is shown in Fig. 5.

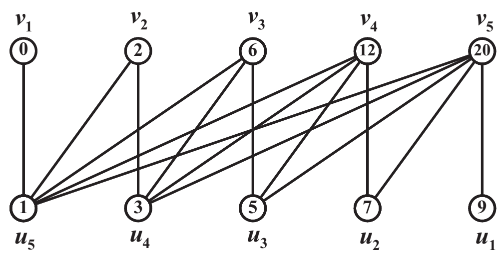

Fig. 5.

§3. Concluding Remarks

Liang and Bai have proved that P_n, $B_{n,n}$ are odd harmonious graphs for all n and C_n is odd harmonious graph for $n \equiv 0(mod\ 4)$ while we proved that the shadow and the splitting graphs of $B_{n,n}$ admit odd harmonious labeling. Thus odd harmoniousness remains invariant for the shadow graph and splitting graph of $B_{n,n}$. It is also invariant under arbitrary supersubdivision of P_n. To investigate similar results for other graph families and in the context of various graph labeling problems is a potential area of research.

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A Note on 1-Edge Balance Index Set

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Abstract: Let G be a graph with vertex set V and edge set E, and $Z_2 = \{0, 1\}$. Let f be a labeling from E to Z_2, so that the labels of the edges are 0 or 1. The edges labelled 1 are called 1-edges and edges labelled 0 are called 0-edges. The edge labeling f induces a vertex labeling $f^* : V \longrightarrow Z_2$ defined by

$$f^*(v) = \begin{cases} 1 & \text{if the number of 1-edges incident on } v \text{ is odd,} \\ 0 & \text{if the number of 1-edges incident on } v \text{ is even.} \end{cases}$$

For $i \in Z_2$ let $e_f(i) = e(i) = |\{e \in E : f(e) = i\}|$ and $v_f(i) = v(i) = |\{v \in V : f^*(v) = i\}|$. A labeling f is said to be edge-friendly if $| \; e(0) - e(1) \; | \leq 1$. The 1- edge balance index set ($OEBI$) of a graph G is defined by $\{| \; v_f(0) - v_f(1) \; | :$ the edge labeling f is edge-friendly$\}$. The main purpose of this paper is to completely determine the 1-edge balance index set of wheel and Mycielskian graph of a path.

Key Words: Mycielskian graph, edge labeling, edge-friendly, 1-edge balance index set, Smarandachely induced vertex labeling, Smarandachely edge-friendly graph.

AMS(2010): 05C78

§1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. Varieties of graph labeling have been investigated by many authors [2], [3] [5] and they serve as useful models for broad range of applications.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and $Z_2 = \{0, 1\}$. Let f be a labeling from $E(G)$ to Z_2, so that the labels of the edges are 0 or 1. The edges labelled 1 are called 1-edges and edges labelled 0 are called 0-edges. The edge labeling f induces a vertex labeling $f^* : V(G) \longrightarrow Z_2$, defined by

$$f^*(v) = \begin{cases} 1 & \text{if the number of 1-edges incident on } v \text{ is odd,} \\ 0 & \text{if the number of 1-edges incident on } v \text{ is even.} \end{cases}$$

For $i \in Z_2$, let $e_f(i) = e(i) = |\{e \in E(G) : f(e) = i\}|$ and $v_f(i) = v(i) = |\{v \in V(G) : f^*(v) = i\}|$. Generally, let $f : E(G) \rightarrow Z_p$ be a labeling from $E(G)$ to Z_p for an integer

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$p \geq 2$. A *Smarandachely induced vertex labeling* on G is defined by $f^v = (l_1, l_2, \cdots, l_p)$ with $n_k(v) \equiv l_k(\mathrm{mod} p)$, where $n_k(v)$ is the number of k-edges, i.e., edges labeled with an integer k incident on v. Let

$$e_k(G) = \frac{1}{2} \sum_{e \in E(G)} n_k(v)$$

for an integer $1 \leq k \leq p$. Then a Smarandachely edge-friendly graph is defined as follows.

Definition 1.1 *A graph G is said to be Smarandachely edge-friendly if $| e_k(G) - e_{k+1}(G) | \leq 1$ for integers $1 \leq k \leq p$. Particularly, if $p = 2$, such a Smarandachely edge-friendly graph is abbreviated to an edge-friendly graph.*

Definition 1.2 *The* 1*-edge balance index set of a graph G, denoted by $OEBI(G)$, is defined as $\{| v_f(1) - v_f(0) | : f$ is edge-friendly$\}$.*

For convenience, a vertex is called 0-vertex if its induced vertex label is 0 and 1-vertex, if its induced vertex label is 1.

In the mid 20^{th} century there was a question regarding the construction of triangle-free k-chromatic graphs, where $k \leq 3$. In this search Mycielski [4] developed an interesting graph transformation known as the Mycielskian which is defined as follows:

Definition 1.3 *For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set consisting of the disjoint union $V \cup V' \cup \{v_0\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{x'y : xy \in E\} \cup \{x'v_0 : x' \in V'\}$.*

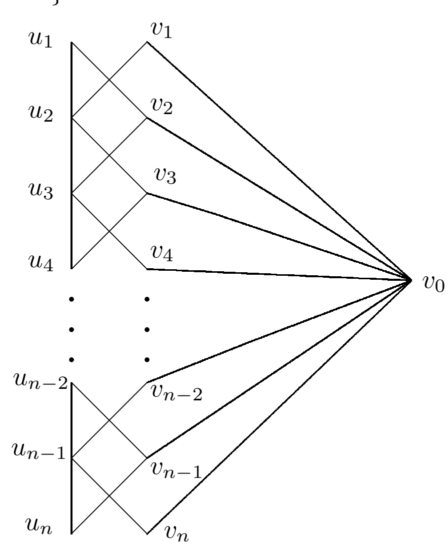

Figure 1 Mycielskian graph of the path P_n

Recently Chandrashekar Adiga et al. [1] have introduced and studied the 1-edge balance index set of several classes of graphs. In Section 2, we completely determine the 1-edge balance index set of the Mycielskian graph of path P_n. In Section 3, we establish that $OEBI(W_n) = \{0, 4, 8 \ldots, n\}$ if $n \equiv 0(mod\ 4)$, $OEBI(W_n) = \{2, 6, 10 \ldots, n\}$ if $n \equiv 2(mod\ 4)$ and $OEBI(W_n) = \{1, 2, 5 \ldots, n\}$ if n is odd.

§2. The 1-Edge Balance Index Set of $\mu(P_n)$

In this section we consider the Mycielskian graph of the path P_n $(n \geq 2)$, which consists of $2n+1$ vertices and $4n-3$ edges. To determine the $OEBI(\mu(P_n))$ we need the following theorem, whose proof is similar to the proof of the Theorem 1 in [6].

Theorem 2.1 *If the number of vertices in a graph G is even(odd) then the* 1*-edge balance index set contains only even(odd)numbers.*

Now we divide the problem of finding $OEBI(\mu(P_n))$ into two cases, viz,

$$n \equiv 0(mod\ \ 2) \qquad \text{and} \qquad n \equiv 1(mod\ \ 2),$$

Denoted by $max\{OEBI(\mu(P_n))\}$ the largest number in the 1-edge balance index set of $\mu(P_n)$. Then we get the follpowing result.

Theorem 2.2 *If $n \equiv 0(mod\ \ 2)$ i.e, $n = 2k(k \in N)$, then $OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n+1\}$.*

Proof Let f be an edge-friendly labeling on $\mu(P_n)$. Since the graph contains $2n+1 = 4k+1$ vertices, $4n-3 = 8k-3$ edges, we have two possibilities: i) $e(0) = 4k-1$, $e(1) = 4k-2$ ii) $e(0) = 4k-2$, $e(1) = 4k-1$. Now we consider the first case namely $e(0) = 4k-1$ and $e(1) = 4k-2$. Denote the vertices of $\mu(P_n)$ as in the Figure 1. Now we label the edges $u_{2q-1}v_{2q}$, $u_{2q+1}v_{2q}$ for $1 \leq q \leq k-1$, $u_q u_{q+1}$ for $1 \leq q \leq 2k-3$, $u_{2k-2}v_{2k-1}$, $u_{2k}v_{2k-1}$ and $u_{2k-1}u_{2k}$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k+1$ and there is no 1-vertex in the graph. Thus $\mid v(1) - v(0) \mid = 4k+1 = 2n+1 = max\{OEBI(\mu(P_n))\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $u_{2q}u_{2q+1}$ and $u_{2q}v_{2q+1}$ for $1 \leq q \leq k-2$, we get, $\mid v(0) - v(1) \mid = 4k+1-4q$. Further interchanging $u_{2k-1}u_{2k}$ and $u_{2k-1}v_{2k}$, we get $\mid v(0) - v(1) \mid = 5$.

In the next four steps we interchange two pairs of edges as follows to see that $1, 3, 7, 11 \in OEBI(\mu(P_n))$

$$u_1v_2 \text{ and } v_1v_0,\ u_2v_3 \text{ and } v_2v_0.$$
$$u_3v_2 \text{ and } v_3v_0,\ u_3v_4 \text{ and } v_4v_0.$$
$$u_4v_5 \text{ and } v_5v_0,\ u_5v_4 \text{ and } v_6v_0.$$
$$u_5v_6 \text{ and } v_7v_0,\ u_6v_7 \text{ and } v_8v_0.$$

Now we interchange $u_{2\lfloor\frac{q-1}{2}\rfloor+7}\ v_{2\lceil\frac{q-1}{2}\rceil+6}$ and $v_{2q+7}\ v_0$, $u_{2q+6}\ v_{2q+7}$ and $v_{2q+8}\ v_0$ for $1 \leq q \leq k-5$ to obtain $\mid v(0) - v(1) \mid = 4q+11$. Finally by interchanging the labels of the edges $u_{2\lfloor\frac{k-5}{2}\rfloor+7}\ v_{2\lceil\frac{k-5}{2}\rceil+6}$ and $u_{2k-2}\ u_{2k-1}$ we get $\mid v(0) - v(1) \mid = 4k-5$ and $u_{2\lfloor\frac{k-4}{2}\rfloor+7}\ v_{2\lceil\frac{k-4}{2}\rceil+6}$ and $u_{2k-1}\ v_0$ we get $\mid v(0) - v(1) \mid = 4k-1$.

Proof of the second case follows similarly. Thus

$$OEBI(\mu(P_n)) = \{1, 3, 5, \cdots, 2n+1\}.$$ □

Theorem 2.3 *If $n \equiv 1(mod\ \ 2)$ i.e, $n = 2k+1(k \in N)$, then $OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n+1\}$.*

Proof Let f be an edge-friendly labeling on $\mu(P_n)$. Since the graph contains $2n+1 = 4k+3$ vertices, $4n-3 = 8k+1$ edges, we have two possibilities: i) $e(0) = 4k+1$, $e(1) = 4k$ ii) $e(0) = 4k$, $e(1) = 4k+1$. Now we consider the first case namely $e(0) = 4k+1$ and $e(1) = 4k$. Denote the vertices of $\mu(P_n)$ as in the Figure 1. Now we label the edges $u_{2q-1}v_{2q}$, $u_{2q+1}v_{2q}$ for $1 \leq q \leq k$ and $u_q u_{q+1}$ for $1 \leq q \leq 2k$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k+3$ and there is no 1-vertex in the graph. Thus $\mid v(1) - v(0) \mid = 4k+3 = 2n+1 = max\{OEBI(\mu(P_n))\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $u_{2q}u_{2q+1}$ and $u_{2q}v_{2q+1}$ for $1 \leq q \leq k$ we get $\mid v(0) - v(1) \mid = 4k+3-4q$. Further interchanging $u_{2k}v_{2k+1}$ and $v_{2k+1}v_0$ we get $\mid v(0) - v(1) \mid = 1$.

In the next four steps we interchange two pairs of edges as follows to see that $5, 9, 13.17 \in OEBI(\mu(P_n))$

$$u_1v_2 \text{ and } v_1v_0,\ u_2v_3 \text{ and } v_2v_0.$$
$$u_3v_2 \text{ and } v_3v_0,\ u_3v_4 \text{ and } v_4v_0.$$
$$u_4v_5 \text{ and } v_5v_0,\ u_5v_4 \text{ and } v_6v_0.$$
$$u_5v_6 \text{ and } v_7v_0,\ u_6v_7 \text{ and } v_8v_0.$$

And finally by interchanging the labels of edges $u_{2\lfloor \frac{q-1}{2} \rfloor + 7}\ v_{2\lceil \frac{q-1}{2} \rceil + 6}$ and $v_{2q+7}\ v_0$, $u_{2q+6}\ v_{2q+7}$ and $v_{2q+8}\ v_0$ for $1 \leq q \leq k-4$, we Obtain $\mid v(0) - v(1) \mid = 4q+17$.

Proof of the second case follows similarly. Thus

$$OEBI(\mu(P_n)) = \{1, 3, 5, \ldots, 2n+1\}. \qquad \square$$

§3. The 1-Edge Balance Index Set of Wheel

In this section we consider the wheel, denoted by W_n which consists of n vertices and $2n-2$ edges. To determine the $OEBI(W_n)$ we consider four cases, namely,

$$n \equiv 0(mod\ \ 4), \qquad n \equiv 1(mod\ \ 4),$$

$$n \equiv 2(mod\ \ 4), \qquad n \equiv 3(mod\ \ 4).$$

Theorem 3.1 *If $n \equiv 0(mod\ \ 4)$ i.e, $n = 4k(k \in N)$, then $OEBI(W_n) = \{0, 4, 8, \ldots, n\}$.*

Proof Let f be an edge-friendly labeling on W_n. Since the graph contains $n = 4k$ vertices, $2n-2 = 8k-2$ edges, we must have $e(0) = e(1) = 4k-1$. Denote the vertices on the rim of the wheel by $v_0, v_1, v_2, \cdots, v_{4k-1}$ and denote the center by v_0. Now we label the edges $v_q v_{q+1}$ for $1 \leq q \leq 4k-2$ and $v_{4k-1}v_1$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k$ and there is no 1-vertex in the graph. Thus $\mid v(1) - v(0) \mid = 4k = n = max\{OEBI(W_n)\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $v_{2q-1}v_{2q}$ and $v_{2q-1}v_0$, $v_{2q}v_{2q+1}$ and $v_{2q}v_0$ for $1 \leq q \leq k$ we get $\mid v(0) - v(1) \mid = 4k - 4q$. Thus $0, 4, 8, \cdots, n$ are elements of $OEBI(W_n)$.

Let $a_i = card\{v \in V \mid \text{number of 1-edges incident on } v \text{ is equal to } i\}$, $i = 1, 2, 3, \ldots, 4k-1$. Then we have

$$\sum_{i=1}^{4k-1} ia_i = a_1 + 2a_2 + 3a_3 +, \ldots, +(4k-1)a_{4k-1} = 8k - 2$$

implies that $a_1 + 3a_3 + 5a_5 +, \ldots, +(4k-1)a_{4k-1}$ is even, which is possible if and only if, $a_1 + a_3 + a_5 +, \ldots, +a_{4k-1}$ is even, that is, the number of 1-vertices is even and hence the number of 0-vertices is also even. Therefore, the numbers $2, 6, 10, \ldots, n-2$ are not elements of $OEBI(W_n)$. $\square$

Theorem 3.2 *If $n \equiv 1(mod\ \ 4)$ i.e, $n = 4k+1 (k \in N)$, then $OEBI(W_n) = \{1, 3, 5, \ldots, n\}$.*

Proof Let f be an edge-friendly labeling on W_n. Since the graph contains $n = 4k+1$ vertices, $2n - 2 = 8k$ edges, we must have $e(0) = e(1) = 4k$. Denote the vertices on the rim of the wheel by $v_0, v_1, v_2, \cdots, v_{4k}$ and denote the center by v_0. Now we label the edges $v_q v_{q+1}$ for $1 \leq q \leq 4k-1$ and $v_{4k}v_1$ by 1 and label the remaining edges by 0. Then it is easy to observe that $v(0) = 4k+1$ and there is no 1-vertex in the graph. Thus $\mid v(1) - v(0) \mid = 4k+1 = n = max\{OEBI(W_n)\}$.

Now we interchange the labels of the edges to get the remaining 1-edge balance index numbers. By interchanging the labels of edges $v_{2q-1}v_{2q}$ and $v_{2q-1}v_0$, $v_{2q}v_{2q+1}$ and $v_{2q}v_0$ for $1 \leq q \leq 2k-1$, we get $\mid v(0) - v(1) \mid = \mid 4k+1-4q \mid$ and by interchanging the labels of edges $v_{4k-1}v_{4k}$ and $v_{4k-1}v_0$, $v_{4k}v_1$ and $v_{4k}v_0$, we get $\mid v(0) - v(1) \mid = 4k-1$. Thus

$$OEBI(W_n) = \{1, 3, 5, \ldots, n\}. \qquad \square$$

Similarly one can prove the following results.

Theorem 3.3 *If $n \equiv 2(mod\ \ 4)$ i.e, $n = 4k+2 (k \in N)$, then $OEBI(W_n) = \{2, 6, 10, \ldots, n\}$.*

Theorem 3.4 *If $n \equiv 3(mod\ \ 4)$ i.e, $n = 4k+3 (k \in N)$, then $OEBI(W_n) = \{1, 3, 5, \ldots, n\}$.*

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A New Proof of Menelaus's Theorem of Hyperbolic Quadrilaterals in the Poincaré Model of Hyperbolic Geometry

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Abstract: In this study, we present a proof of the Menelaus theorem for quadrilaterals in hyperbolic geometry, and a proof for the transversal theorem for triangles.

Key Words: Hyperbolic geometry, hyperbolic quadrilateral, Menelaus theorem, the transversal theorem, gyrovector.

AMS(2010): 51K05, 51M10

§1. Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disc and ball models, the Poincaré half-plane model, and the Beltrami-Klein disc and ball models [3] etc. Following [6] and [7] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model. Menelaus of Alexandria was a Greek mathematician and astronomer, the first to recognize geodesics on a curved surface as natural analogs of straight lines. The well-known Menelaus theorem states that if l is a line not through any vertex of a triangle ABC such that l meets BC in D, CA in E, and AB in F, then $\frac{DB}{DC} \cdot \frac{EC}{EA} \cdot \frac{FA}{FB} = 1$ [2]. Here, in this study, we give hyperbolic version of Menelaus theorem for quadrilaterals in the Poincaré disc model. Also, we will give a reciprocal hyperbolic version of this theorem. In [1] has been given proof of this theorem, but to use Klein's model of hyperbolic geometry.

We begin with the recall of some basic geometric notions and properties in the Poincaré disc. Let D denote the unit disc in the complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

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The most general Möbius transformation of D is

$$z \to e^{i\theta}\frac{z_0+z}{1+\overline{z_0}z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disc to be viewed as a Möbius left gyro-translation

$$z \to z_0 \oplus z = \frac{z_0+z}{1+\overline{z_0}z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1+a\overline{b}}{1+\overline{a}b},$$

then is true gyro-commutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyro-vector space $(G, \oplus, \otimes)$ is a gyro-commutative gyro-group $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\ gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$;

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$;

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$;

(G4) $\frac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$;

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$;

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = \mathit{1}$;

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$;

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \le \|\mathbf{a}\| \oplus \|\mathbf{b}\|$.

Definition 1. *The hyperbolic distance function in D is defined by the equation*

$$d(a, b) = |a \ominus b| = \left|\frac{a-b}{1-\overline{a}b}\right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

For further details we refer to the recent book of A.Ungar [7].

Theorem 2(The Menelaus's Theorem for Hyperbolic Gyrotriangle) *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and with gyroangles α, β, and γ at the vertices A, B, and C. If l is a gyroline not through any vertex of an gyrotriangle ABC such that l meets BC in D, CA in E, and AB in F, then*

$$\frac{(AF)_\gamma}{(BF)_\gamma} \cdot \frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CE)_\gamma}{(AE)_\gamma} = 1.$$

where $v_\gamma = \frac{v}{1 - \frac{v^2}{s^2}}$ [6].

§2. Main Results

In this section, we prove Menelaus's theorem for hyperbolic quadrilateral.

Theorem 3(The Menelaus's Theorem for Gyroquadrilateral) *If l is a gyroline not through any vertex of a gyroquadrilateral $ABCD$ such that l meets AB in X, BC in Y, CD in Z, and DA in W, then*

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BY)_\gamma}{(CY)_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1. \tag{1}$$

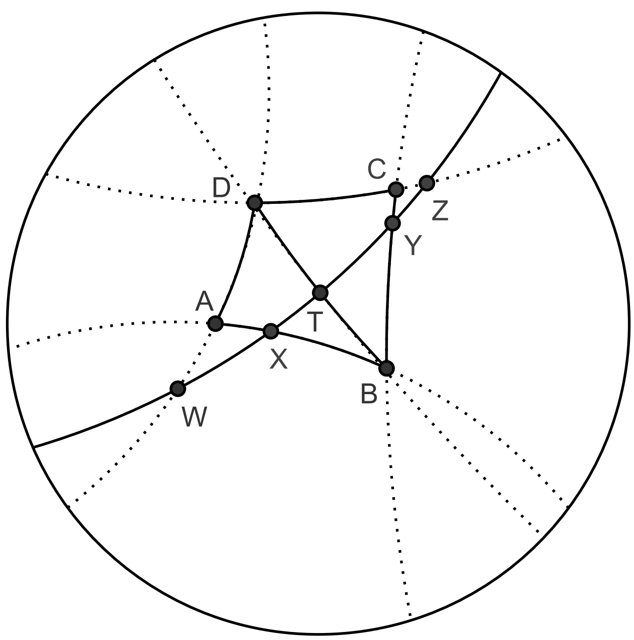

Figure 1

Proof Let T be the intersection point of the gyroline DB and the gyroline XYZ (See

Figure 1). If we use Theorem 2 in the gyrotriangles ABD and BCD respectively, then

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BT)_\gamma}{(DT)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1 \tag{2}$$

and

$$\frac{(DT)_\gamma}{(BT)_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(BY)_\gamma}{(CY)_\gamma} = 1. \tag{3}$$

Multiplying relations (2) and (3) member with member, we obtain

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BY)_\gamma}{(CY)_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1.$$

□

Naturally, one may wonder whether the converse of Menelaus theorem for hyperbolic quadrilateral exists. Indeed, a partially converse theorem does exist as we show in the following theorem.

Theorem 4(Converse of Menelaus's Theorem for Gyroquadrilateral) *Let $ABCD$ be a gyroquadrilateral. Let the points $X, Y, Z,$ and W be located on the gyrolines $AB, BC, CD,$ and DA respectively. If three of four gyropoints X, Y, Z, W are collinear and*

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BY)_\gamma}{(CY)_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1,$$

then all four gyropoints are collinear.

Proof Let the points W, X, Z are collinear, and gyroline WXZ cuts gyroline BC, at Y' say. Using the already proven equality (1), we obtain

$$\frac{(AX)_\gamma}{(BX)_\gamma} \cdot \frac{(BY')_\gamma}{(CY')_\gamma} \cdot \frac{(CZ)_\gamma}{(DZ)_\gamma} \cdot \frac{(DW)_\gamma}{(AW)_\gamma} = 1,$$

then we get

$$\frac{(BY)_\gamma}{(CY)_\gamma} = \frac{(BY')_\gamma}{(CY')_\gamma}. \tag{4}$$

This equation holds for $Y = Y'$. Indeed, if we take $x := |\ominus B \oplus Y'|$ and $b := |\ominus B \oplus C|$, then we get $b \ominus x = |\ominus Y' \oplus C|$. For $x \in (-1, 1)$ define

$$f(x) = \frac{x}{1 - x^2} : \frac{b \ominus x}{1 - (b \ominus x)^2}. \tag{5}$$

Because $b \ominus x = \dfrac{b - x}{1 - bx}$, then $f(x) = \dfrac{x(1 - b^2)}{(b - x)(1 - bx)}$. Since the following equality holds

$$f(x) - f(y) = \frac{b(1 - b^2)(1 - xy)}{(b - x)(1 - bx)(b - y)(1 - by)}(x - y), \tag{6}$$

we get $f(x)$ is an injective function. This implies $Y = Y'$, so $W, X, Z,$ and Y are collinear. □

We have thus obtained in (1) the following.

Theorem 5(Transversal theorem for gyrotriangles) *Let D be on gyroside BC, and l is a gyroline not through any vertex of a gyrotriangle ABC such that l meets AB in M, AC in N, and AD in P, then*

$$\frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CA)_\gamma}{(NA)_\gamma} \cdot \frac{(NP)_\gamma}{(MP)_\gamma} \cdot \frac{(MA)_\gamma}{(BA)_\gamma} = 1. \tag{7}$$

Proof If we use a theorem 2 for gyroquadrilateral $BCNM$ and collinear gyropoints D, A, P, and A (See Figure 2), we obtain the conclusion. □

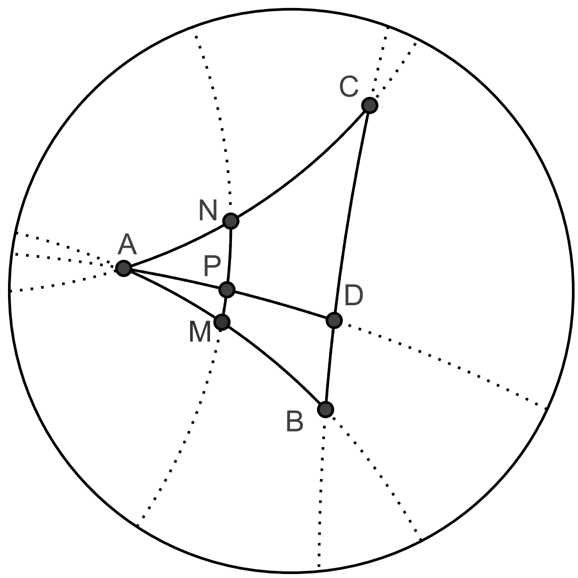

Figure 2

The Einstein relativistic velocity model is another model of hyperbolic geometry. Many of the theorems of Euclidean geometry are relatively similar form in the Poincaré model, Menelaus's theorem for hyperbolic gyroquadrilateral and the transversal theorem for gyrotriangle are an examples in this respect. In the Euclidean limit of large s, $s \to \infty$, gamma factor v_γ reduces to v, so that the gyroinequalities (1) and (7) reduces to the

$$\frac{AX}{BX} \cdot \frac{BY}{CY} \cdot \frac{CZ}{DZ} \cdot \frac{DW}{AW} = 1$$

and

$$\frac{BD}{CD} \cdot \frac{CA}{NA} \cdot \frac{NP}{MP} \cdot \frac{MA}{BA} = 1,$$

in Euclidean geometry. We observe that the previous equalities are identical with the equalities of theorems of euclidian geometry.

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By Len Evans, an mathematician of the United States.

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Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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