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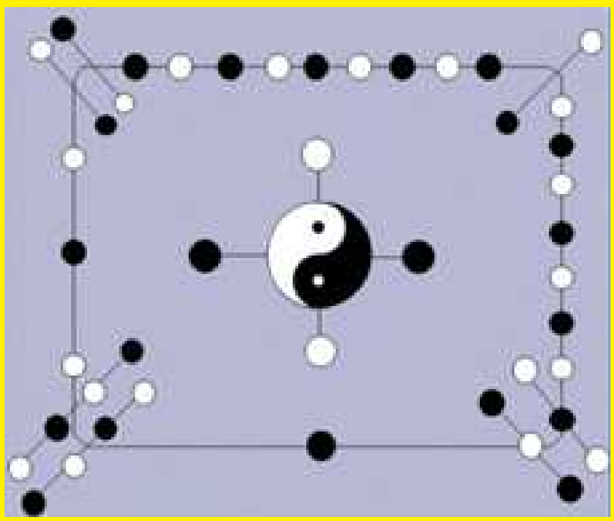

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Aims and Scope: The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly comprising 110-160 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

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Mathematics, rightly viewed, posses not only truth, but supreme beauty – a beauty cold and austere8, like that of sculpture.

By Bertrand Russell, a British philosopher, logician, mathematician.

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Generalized abc-Block Edge Transformation Graph $Q^{abc}(G)$ When $abc = +0-$

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Abstract: The generalized abc-block edge transformation graph $Q^{+0-}(G)$ is a graph whose vertex set is the union of the edges and blocks of G, in which two vertices are adjacent whenever corresponding edges of G are adjacent or one corresponds to an edge and other to a block of G are not incident with each other. In this paper, we study the girth, covering invariants and the domination number of $Q^{+0-}(G)$. We present necessary and sufficient conditions for $Q^{+0-}(G)$ to be planar, outerplanar, minimally nonouterplanar and maximal outerplanar. Further, we establish a necessary and sufficient condition for the generalized abc-block edge transformation graph $Q^{+0-}(G)$ have crossing number one.

Key Words: Line graph, abc-block edge transformation, generalized abc-block edge transformation graph, Smarandachely block-edge H-graph.

AMS(2010): 05C10, 05C40.

§1. Introduction

Throughout the paper, we only consider simple graphs without isolated vertices. Definitions not given here may be found in [5]. A *cut vertex* of a connected graph is the one whose removal increases the number of components. A *nonseparable* graph is connected, nontrivial and has no cut vertices. A *block* of a graph is a maximal nonseparable subgraph. Let $G = (V, E)$ be a graph with block set $U(G)$=$\{B_i$; B_i is a block of $G\}$. If a block $B \in U(G)$ with the edge set $\{e_1, e_2, \cdots, e_m; m \geq 1\}$, then we say that the edge e_i and block B are incident with each other, where $1 \leq i \leq m$. The *girth* of a graph G, denoted by $g(G)$, is the length of the shortest cycle if any in G. Let $\lceil x \rceil (\lfloor x \rfloor)$ denote the least (greatest) integer greater (less) than or equal to x.

A vertex and an edge are said to *cover* each other if they are incident. A set of vertices in a graph G is a *vertex covering set*, which covers all the edges of G. The *vertex covering number* $\alpha_0(G)$ of G is the minimum number of vertices in a vertex covering set of G. A set of edges in a graph G is an *edge covering set*, which covers all vertices of G. The *edge covering number*

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$\alpha_1(G)$ of G is the minimum number of edges in an edge covering set of G. A set of vertices in a graph G is *independent* if no two of them are adjacent. The maximum number of vertices in such a set is called the *vertex independence number* of G and is denoted by $\beta_0(G)$. The set of edges in a graph G is *independent* if no two of them are adjacent. The maximum number of edges in such a set is called the *edge independence number* of G and is denoted by $\beta_1(G)$.

The *line graph* $L(G)$ of a graph G is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common [5]. The *plick graph* $P(G)$ of a graph G is the graph whose set of vertices is the union of the set of edges and blocks of G and in which two vertices are adjacent if and only if the corresponding edges of G are adjacent or one is corresponds to an edge and other is corresponds to a block are incident [8]. In [2], we generalized the concept of plick graph and were termed as generalized abc-block edge transformation graphs $Q^{abc}(G)$ of a graph G and obtained 64 kinds of graphs. In this paper, we consider one among those 64 graph which is defined as follows:

Definition 1.1 *The generalized abc-block edge transformation graph $Q^{+0-}(G)$ is a graph whose vertex set is the union of the edges and blocks of G, in which two vertices are adjacent whenever corresponding edges of G are adjacent or one corresponds to an edge and other to a block of G are not incident with each other.*

Generally, a *Smarandachely block-edge H-graph* is such a graph with vertex set $E(G)\bigcup B(G)$ and two vertices $e_1, e_2 \in E(G)\bigcup B(G)$ are adjacent if $e_1, e_2 \in E(H)$ are adjacent, or at least one of e_1, e_2 not in $E(H)$ and they are non-adjacent, or one in $E(H)$ and other in $B(G)$ which are not incident, where H is a subgraph of G with property $\mathscr{P}$. Clearly, a Smarandachely block-edge $E(G)\bigcup B(G)$-graph is nothing else but a generalized abc-block edge transformation graph.

In this paper, we study the girth, covering invariants and the domination number of $Q^{+0-}(G)$. We present necessary and sufficient conditions for $Q^{+0-}(G)$ to be planar, outerplanar, minimally nonouterplanar and maximal outerplanar. Further, we establish a necessary and sufficient condition for the generalized abc-block edge transformation graph $Q^{+0-}(G)$ have crossing number one. Some other graph valued functions were studied in [3, 4, 7, 8, 9, 11, 12]. In Figure 1, a graph G and its generalized abc-block edge transformation graph $Q^{+0-}(G)$ are shown.

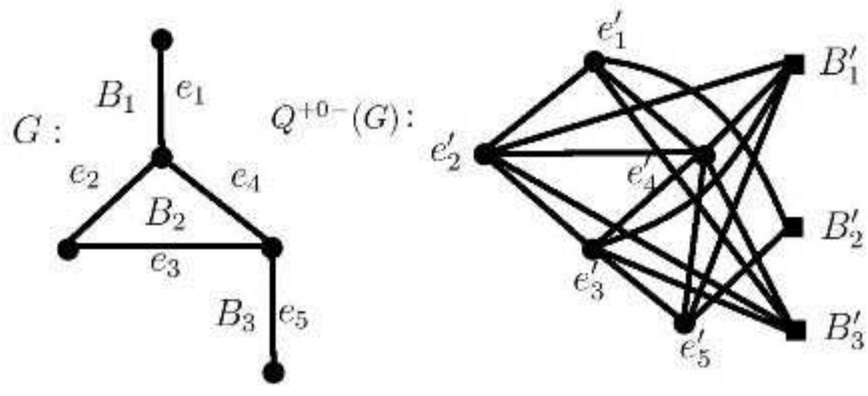

Figure 1. Graph G and its $Q^{+0-}(G)$.

In $Q^{+0-}(G)$, the vertices correspond to edges of G denoted by circles and vertices correspond to blocks of G denoted by squares. The vertex $e_i^{'}$ ($B_i^{'}$) of $Q^{+0-}(G)$ corresponding to edge e_i (block B_i) of G and is refereed as edge (block)-vertex.

The following theorems will be useful in the proof of our results.

Theorem 1.1([8]) *If G is a nontrivial connected (p, q) graph whose vertices have degree d_i and if b_i the number of blocks to which vertex v_i belongs in G, then $P(G)$ has $q - p + 1 + \sum\limits_{i=1}^{p} b_i$ vertices and $\frac{1}{2}\sum\limits_{i=1}^{p} d_i^2$ edges.*

Theorem 1.2([5]) *For any nontrivial connected graph G with p vertices,*

$$\alpha_0(G) + \beta_0(G) = p = \alpha_1(G) + \beta_1(G).$$

Theorem 1.3([6]) *If $L(G)$ is the line graph of a nontrivial connected graph G with q edges, then*

$$\alpha_1(L(G)) = \lceil \frac{q}{2} \rceil.$$

§2. Basic Results on $Q^{+0-}(G)$

We start with preliminary remarks.

Remark 2.1 $L(G)$ is an induced subgraph of $Q^{+0-}(G)$.

Remark 2.2 If G is a block, then $Q^{+0-}(G) = L(G) \cup K_1$.

Remark 2.3 Let G be a graph with edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$ and r blocks. Then $d_{Q^{+0-}(G)} e_i' = d_G e_i + r - 1$.

Remark 2.4 Let G be a (p, q)-graph with block set $U(G) = \{B_1, B_2, \cdots, B_r\}$ such that $| E(B_i) |= n_i$. Then $d_{Q^{+0-}(G)} B_i' = q - n_i$.

Theorem 2.1 *Let G be a $(p, q)-$connected graph whose vertices have degree d_i with $r \geq 1$ blocks and b_i $(1 \leq i \leq p)$ the number of blocks to which vertex v_i belongs in G. Then*

(1) *The order of $Q^{+0-}(G)$ $= q - p + 1 + \sum\limits_{i=1}^{p} b_i$;*

(2) *The size of $Q^{+0-}(G)$ $= q(r - 2) + \frac{1}{2}\sum\limits_{i=1}^{p} d_i^2$.*

Proof It is shown in [5] that for a connected graph G with p vertices and b_i number of blocks to which vertex v_i $(1 \leq i \leq p)$ belongs in G. Then the number of blocks of G is given by $b(G) = 1 + \sum\limits_{i=1}^{p} (b_i - 1)$. The order of $Q^{+0-}(G)$ is the sum of the number of edges of G and number of blocks of G. Hence the order of $Q^{+0-}(G)$

$$= \quad q+1+\sum_{i=1}^{p}(b_i-1)=q-p+1+\sum_{i=1}^{p}b_i.$$

The total number of edges formed by joining each of the r block-vertices to all the q edge-vertices is rq. The number of edges in line graph $L(G)$ is $-q+\frac{1}{2}\sum_{i=1}^{p}d_i^2$. Thus, the size of

$$Q^{+0-}(G)=rq-q-q+\frac{1}{2}\sum_{i=1}^{p}d_i^2=q(r-2)+\frac{1}{2}\sum_{i=1}^{p}d_i^2. \qquad \square$$

An immediate consequence of the above theorem is the following corollary.

Corollary 2.2 *Let G be a (p,q) graph whose vertices have degree d_i with r blocks and m components. If b_i $(1\le i\le p)$ is the number of blocks to which vertex v_i belongs in G, then*

(1) *The order of* $Q^{+0-}(G)\ =q-p+m+\sum_{i=1}^{p}b_i$;

(2) *The size of* $Q^{+0-}(G)\ =q(r-2)+\frac{1}{2}\sum_{i=1}^{p}d_i^2$.

Theorem 2.3 *Let G be a graph. The graphs $Q^{+0-}(G)$ and $P(G)$ are isomorphic if and only if G has two blocks.*

Proof Let G be a (p,q) graph with $r\ge 1$ blocks. Suppose $Q^{+0-}(G)=P(G)$. Then $|E(Q^{+0-}(G))|=|E(P(G))|$. By Theorems 1.1 and 2.1, we have

$$\begin{aligned} q(r-2)+\frac{1}{2}\sum_{i=1}^{p}d_i^2 &= \frac{1}{2}\sum_{i=1}^{p}d_i^2 \\ q(r-2) &= 0. \end{aligned}$$

Since G has at least one edge and hence equality holds only when $r=2$. Therefore G has two blocks.

Conversely, suppose G has two blocks B_1 and B_2. Then by definitions of $Q^{+0-}(G)$ and $P(G)$, $L(G)$ is induced subgraph of $Q^{+0-}(G)$ and $P(G)$. In $Q^{+0-}(G)$, block-vertex B_1' is adjacent all the edge-vertices corresponding to edges of B_2 and block-vertex B_2' is adjacent to all the edge-vertices corresponding to edges of B_1. In $P(G)$, block-vertex B_1' is adjacent all the edge-vertices corresponding to edges of B_1 and block-vertex B_2' is adjacent to all the edge-vertices corresponding to edges of B_2. This implies that there exist a one-to-one correspondence between vertices of $Q^{+0-}(G)$ and $P(G)$ which preserves adjacency. Therefore the graphs $Q^{+0-}(G)$ and $P(G)$ are isomorphic. $\square$

The following theorem gives the girth of $Q^{+0-}(G)$.

Theorem 2.4 *For a graph $G \neq 2K_2, K_2, P_3$,*

$$g(Q^{+0-}(G)) = \begin{cases} 3 \ if \ G \ contains \ K_{1,3} \ or \ K_3 \ or \ G = P_n; \ n \geq 4 \ or \ G \ is \ union \ of \ at \ least \\ two \ cycles \ or \ paths \ or \ G \ is \ union \ of \ paths \ and \ cycles, \\ 4 \ if \ G = mK_2, m \geq 4, \\ 6 \ if \ G = 3K_2, \\ n \ if \ G = C_n, n \geq 4. \end{cases}$$

Proof If G contains a triangle or $K_{1,3}$, then the line graph $L(G)$ of G contains triangle. By Remark 2.1, it follows that girth of $Q^{+0-}(G)$ is 3. Assume that G is triangle free and $K_{1,3}$ free. Then we have the following cases:

Case 1. Assume G has every vertex of degree is 2. We have two subcases:

Subcase 1.1 If G is connected, then clearly $G = C_n$; $n \geq 4$, we have $Q^{+0-}(G) = C_n \bigcup K_1$, $n \geq 4$. Therefore girth of $Q^{+0-}(G)$ is n.

Subcase 1.2 If G is disconnected, then G is union of at least two cycles and $Q^{+0-}(G)$ contains at least two wheels. Therefore girth of $Q^{+0-}(G)$ is 3.

Case 2. Assume that $G \neq 2K_2, K_2$ has every vertex of degree is one. It is easy to see that

$$g(Q^{+0-}(G))=\begin{cases} 6 \text{ if } G = 3K_2, \\ 4 \text{ if } G = mK_2; \ m \geq 4. \end{cases}$$

Case 3. Assume that $G \neq P_3$ has vertices of degree one or two. Then G is either union of paths P_n or union of paths and cycles. Therefore girth of $Q^{+0-}(G)$ is 3. □

§3. Covering Invariants of $Q^{+0-}(G)$

Theorem 3.1 *For a connected (p, q)-graph G with r blocks, if $Q^{+0-}(G)$ is connected, then $\alpha_0(Q^{+0-}(G)) = q$ and $\beta_0(Q^{+0-}(G)) = r$.*

Proof Let G be a connected (p, q)-graph. By Remark 2.1, $L(G)$ is an induced subgraph of $Q^{+0-}(G)$. Therefore by definition of $Q^{+0-}(G)$, the edge-vertices covers all the edges of $L(G)$. Since $Q^{+0-}(G)$ is connected, it follows that for each block-vertex B' of $Q^{+0-}(G)$, there exists a edge-vertex e' such that e' and B are adjacent in $Q^{+0-}(G)$. Therefore the vertex set of $L(G)$ covers all the edges of $Q^{+0-}(G)$ and this is minimum covering. Hence $\alpha_0(Q^{+0-}(G)) = q$. Since $Q^{+0-}(G)$ is connected. By Theorem 1.2, we have $\alpha_0(Q^{+0-}(G)) + \beta_0(Q^{+0-}(G)) = q + r$. Thus $\beta_0(Q^{+0-}(G)) = r$. □

Theorem 3.2 *Let G be a connected (p, q)-graph with r blocks. If $Q^{+0-}(G)$ is connected, then*

$$\alpha_1(Q^{+0-}(G))=\begin{cases} r & if \ G \ is \ a \ tree, \\ r + \lceil \frac{q-r}{2} \rceil & otherwise. \end{cases}$$

and

$$\beta_1(Q^{+0-}(G))= \begin{cases} q & if\ G\ is\ a\ tree, \\ q - \lceil \frac{q-r}{2} \rceil & otherwise. \end{cases}$$

Proof Let T be the set of minimum edges covering all block-vertices of $Q^{+0-}(G)$. i.e., $|T| = r$. Let S be the set of minimum edge cover of $L(G)$. By Theorem 1.3, $|S| = \lceil \frac{q}{2} \rceil$. We consider the following two cases:

Case 1. If G is a tree, then $q = r$. By the definition, T covers all block-vertices and edge-vertices of $Q^{+0-}(G)$. Thus $\alpha_1(Q^{+0-}(G)) = r$.

Case 2. If G is not a tree, then $q > r$. By the definition, T covers all block-vertices and only r edge-vertices of $Q^{+0-}(G)$. Therefore there exists a set of edge-vertices F, say of $Q^{+0-}(G)$ such that no element of T is incident with any element of F in $Q^{+0-}(G)$. i.e., $|F| = q - r$. Since each element of S covers two elements of $L(G)$ and $F \subset V(Q^{+0-}(G))$, it follows that we need $\left\lceil \frac{|F|}{2} \right\rceil$ elements from S to cover all elements of F. Thus $\alpha_1(Q^{+0-}(G))= r + \lceil \frac{q-r}{2} \rceil$.

Therefore, $\alpha_1(Q^{+0-}(G))= \begin{cases} r & \text{if } G \text{ is a tree,} \\ r + \lceil \frac{q-r}{2} \rceil & \text{otherwise.} \end{cases}$

Since $Q^{+0-}(G)$ is connected. By Theorem 1.2, we have $\alpha_1(Q^{+0-}(G)) + \beta_1(Q^{+0-}(G)) = q + r$. Thus

$$\beta_1(Q^{+0-}(G))= \begin{cases} q & \text{if } G \text{ is a tree,} \\ q - \lceil \frac{q-r}{2} \rceil & \text{otherwise.} \end{cases} \qquad \square$$

§4. Domination Number of $Q^{+0-}(G)$

A set D of vertices in a graph $G = (V, E)$ is called a *dominating set* of G if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D is called *minimal dominating set* if no proper subset of D is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G ([10]).

The following result is immediate from Remark 2.2.

Theorem 4.1 *If G is a block, then $\gamma(Q^{+0-}(G)) = \gamma(L(G)) + 1$.*

Theorem 4.2 *If G has two blocks, then $\gamma(Q^{+0-}(G)) = 2$.*

Proof Suppose G has two blocks B_1 and B_2. Then B_1' dominates all the edge-vertices in $Q^{+0-}(G)$ corresponding to edges of B_2 and B_2' dominates all the edge-vertices in $Q^{+0-}(G)$ corresponding to edges of B_1. Therefore $\gamma(Q^{+0-}(G))= |\{B_1', B_2'\}| = 2$ where $\{B_1', B_2'\}$ is a minimal dominating set in $Q^{+0-}(G)$. $\square$

Theorem 4.3 *For any graph G with at least three blocks,*

$$\gamma(Q^{+0-}(G))=\begin{cases} 2 & if\ G\ contain\ an\ edge\ is\ adjacent\ to\ every\ other\ edge\ of\ its\ block, \\ 3 & otherwise. \end{cases}$$

Proof Let G be a graph having at least three blocks. We consider following two cases:

Case 1. If G contain an edge e is adjacent to every other edge of its block B, then block-vertex B' dominates the edge-vertices corresponding to the edges not in B. And edge-vertex e' dominates the block-vertices except B' and dominates the edge-vertices corresponding to edges of B. Therefore $\gamma(Q^{+0-}(G)) = |\{e', B'\}| = 2$ where $\{e', B'\}$ is a minimal dominating set in $Q^{+0-}(G)$.

Case 2. If G contain no edge is adjacent to every other edge of its block, then there exist two block-vertices B', B'_1 and one edge-vertex e', where e is in B in G, such that B' dominates the edge-vertices corresponding to the edges not in B and edge-vertex e' dominates all the block-vertices except B' and block vertex B'_1 dominates the edge-vertices which are not dominated from e' and B'. Therefore $\gamma(Q^{+0-}(G))= |\{e', B', B'_1\}| = 3$ where $\{e', B', B'_1\}$ is a minimal dominating set in $Q^{+0-}(G)$. □

§5. Planarity of Graphs $Q^{+0-}(G)$

A graph is *planar* if it can be drawn on the plane in such a way that no two of its edges intersect. A planar graph is *outerplanar* if it can be embedded in the plane so that all its vertices lie on the exterior region. In [1], Kulli introduced the concept of a minimally nonouterplanar graph. The *inner vertex number* $i(G)$ of a planar graph G is the minimum possible number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. Obviously G is outerplanar if and only if $i(G) = 0$. A graph G is *minimally nonouterplanar* if $i(G) = 1$. An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing outerplanarity. The *crossing number* $Cr(G)$ of a graph G is the minimum number of pairwise intersections of its edges when G is drawn in the plane. Obviously, $Cr(G) = 0$ if and only if G is planar. A *cactus* is a connected graph in which every block is an edge or a cycle. If G and H are graphs with the property that the identification of any vertex of G with an arbitrary vertex of H results in a unique graph, then we write $G \cdot H$ for this graph.

The condition for the planar, outerplanar, minimally nonouterplanar, maximal outerplanar and crossing number of line graph of G and generalized abc-block edge transformation graph $Q^{+0-}(G)$ are same when G is a block. So that in this section we assume graph G under consideration is not a block in what follows.

Lemma 5.1 *If G is not a tree having more than two blocks, then $Q^{+0-}(G)$ is nonplanar.*

Proof Let G be not a tree having more than two blocks, i.e., G has a block B contains a cycle C. Then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(2K_2 \cup K_3)$, and $Cr(Q^{+0-}(2K_2 \cup K_3)) = 1$. Therefore $Q^{+0-}(G)$ is nonplanar. □

Theorem 5.2 *Let G be a connected graph with more than one block. Then generalized abc-*

block edge transformation graph $Q^{+0-}(G)$ is planar if and only if G satisfies one of the following conditions:

(1) *G is a cactus having two blocks;*

(2) *G is a tree of order ≤ 5.*

Proof Suppose $Q^{+0-}(G)$ is planar. Assume a connected graph G has atleast 5 blocks. We consider the following cases:

Case 1. If G is not a tree, then by Lemma 5.1, $Q^{+0-}(G)$ is nonplanar, a contradiction.

Case 2. If G is a tree, i.e., every block of G is K_2, then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(5K_2)$ and $Cr(Q^{+0-}(5K_2)) = 4$. Therefore $Q^{+0-}(G)$ is nonplanar, a contradiction.

In either case we arrive at a contradiction. Hence G contains at most four blocks. We discuss two possibilities on number of blocks:

Subcase 2.1 If G is not a cactus having two blocks, i.e., some block B of G contains a subgraph homeomorphic to $C_n + e$, then edge-vertices corresponding to edges of $C_n + e$ and block-vertex corresponding to block other than B forms a subgraph with at least one crossing in $Q^{+0-}(G)$. Therefore $Q^{+0-}(G)$ is nonplanar, a contradiction. This proves (1).

Subcase 2.2 If G is not a tree having 3 or 4 blocks, then by Lemma 5.1, $Q^{+0-}(G)$ is nonplanar, a contradiction. This proves (2).

Conversely, suppose G satisfies (1) or (2). Then $G = C_n \cdot K_2$ or $C_n \cdot C_m$ or P_4 or $K_{1,3}$ or $K_{1,3} \cdot K_2$ or P_3 or P_5. Therefore it is easy to check that $Q^{+0-}(G)$ is planar. □

Theorem 5.3 *Let G be a connected graph with more than one block. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ is outerplanar if and only if G is a tree of order ≤ 4.*

Proof Suppose $Q^{+0-}(G)$ is outerplanar. Then $Q^{+0-}(G)$ is planar. By Theorem 5.2, we have, G is a cactus having two blocks or G is a tree of order ≤ 5. Assume G is a tree of order 5. Then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(4K_2)$ and $i(Q^{+0-}(4K_2)) = 4$. Therefore $Q^{+0-}(G)$ is nonouterplanar, a contradiction. Assume $G = C_m \cdot C_m$ or $C_n \cdot K_2$. Then $Q^{+0-}(G)$ is nonouterplanar, a contradiction. In either case we arrive at a contradiction. Hence G is a tree of order ≤ 4.

Assume G is not a tree of order ≤ 4, i.e., G has a block B contains a cycle C. Then edge-vertices corresponding to edges of C and a block-vertex corresponding to block other than B forms a subgraph wheel in $Q^{+0-}(G)$. Therefore $Q^{+0-}(G)$ is nonouterplanar, a contradiction. Hence G is a tree of order ≤ 4.

Conversely, suppose G is a tree of order ≤ 4. Then $G = P_3$ or P_4 or $K_{1,3}$. Therefore $Q^{+0-}(G)$ is outerplanar. □

Theorem 5.4 *Let G be a connected graph with more than one block. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ is minimally nonouterplanar if and only if $G = C_n \cdot K_2$.*

Proof Suppose $Q^{+0-}(G)$ is minimally nonouterplanar. Then $Q^{+0-}(G)$ is planar. By Theorem 5.2, we have, G is either cactus having two blocks or tree of order ≤ 5. If G is a tree of order ≤ 4, then by Theorem 5.3, $Q^{+0-}(G)$ is outerplanar, a contradiction. If G is a tree of order 5, then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(4K_2)$, and $i(Q^{+0-}(4K_2)) = 4$. Therefore $Q^{+0-}(G)$ is not minimally outerplanar, a contradiction.

Suppose $G \neq C_n \cdot K_2$ is cactus having two blocks. Then $G = P_3$ or $C_n \cdot C_m$. Therefore $Q^{+0-}(G)$ is not minimally nonouterplanar, a contradiction. Thus G is $C_n \cdot K_2$.

Conversely, suppose $G = C_n \cdot K_2$. Then $Q^{+0-}(G)$ is minimally nonouterplanar. □

Theorem 5.5 *Let G be a connected graph with more than one block. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ is maximal outerplanar if and only if $G = K_{1,3}$.*

Proof Suppose $Q^{+0-}(G)$ is maximal outerplanar. Then $Q^{+0-}(G)$ is outerplanar. By Theorem 5.3, we have, G is a tree of order ≤ 4. Assume $G \neq K_{1,3}$ is a tree of order ≤ 4. Then $G = P_3$ or P_4. Therefore $Q^{+0-}(G)$ is not maximal outerplanar, a contradiction. Hence $G = K_{1,3}$.

Conversely, suppose $G = K_{1,3}$. Then $Q^{+0-}(G)$ is maximal outerplanar. □

§6. Graphs $Q^{+0-}(G)$ and Crossing Number One

Lemma 6.1 *Let G be a connected graph having two blocks. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ has crossing number one if and only if G is either $C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$.*

Proof Suppose $Q^{+0-}(G)$ has crossing number one. Assume $G \neq C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$. Then we have the following cases:

Case 1. If G is a cactus, then by Theorem 5.2, $Q^{+0-}(G)$ is planar, a contradiction.

Case 2. If G is not a cactus, then G is homeomorphic to $K_2 \cdot (C_t + 2e)$ or $K_2 \cdot (\overline{K_2 \cup K_3})$ or $(C_t + e) \cdot (C_s + e)$ or $C_t \cdot (C_s + e)$ with $\triangle(G) = 5$. Therefore $Cr(K_2 \cdot (C_t + 2e)) \geq 2$, $Cr(K_2 \cdot (\overline{K_2 \cup K_3})) \geq 2$, $Cr((C_t + e) \cdot (C_s + e)) \geq 2$, $Cr(C_t \cdot (C_s + e)) = 2$. Hence $Cr(Q^{+0-}(G)) \geq 2$, a contradiction.

Conversely, suppose G is either $C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$. Then $Cr(Q^{+0-}(G)) = 1$. □

Theorem 6.2 *Let G be a connected graph with more than one block. Then generalized abc-block edge transformation graph $Q^{+0-}(G)$ has crossing number one if and only if $G = C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$ or $C_n \cdot P_3$.*

Proof Suppose $Q^{+0-}(G)$ has crossing number one. Assume $G \neq C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$ or $C_n \cdot P_3$. We consider the following cases:

Case 1. If G is a tree, then we consider following subcases:

Subcase 1.1 If G is a tree of order ≤ 5, then by Theorem 5.2, $Q^{+0-}(G)$ is planar, a contradiction.

Subcase 1.2 If G is a tree of order at least 6, then $Q^{+0-}(G)$ has a subgraph homoemorphic to $Q^{+0-}(5K_2)$ and $Cr(Q^{+0-}(5K_2)) = 4$. Therefore $Cr(Q^{+0-}(G)) \geq 4$, a contradiction.

Case 2. If G is not a tree, then G contains at least one cycle. We consider the following subcases:

Subcase 2.1 If G has more than 3 blocks, then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}(3K_2 \cup K_3)$ and $Cr(Q^{+0-}(3K_2 \cup K_3)) = 5$. Therefore $Cr(Q^{+0-}(G)) \geq 5$, a contradiction.

Subcase 2.2 If G has three blocks, then $Q^{+0-}(G)$ has a subgraph homeomorphic to $Q^{+0-}((C_4+e) \cdot P_3)$ or G_1 where $G_1 = K_3^+ - e$, e is pendant edge, and $Cr(Q^{+0-}((C_4+e) \cdot P_3)) \geq 4$, $Cr(Q^{+0-}(G_1)) = 2$. Therefore $Cr(Q^{+0-}(G)) \geq 2$, a contradiction.

Subcase 2.3 If G has two blocks, then by Lemma 6.1, crossing number of $Q^{+0-}(G)$ is not equal to one, a contradiction.

Conversely, suppose G=$C_t \cdot (C_s + e)$ with $\triangle(G) \leq 4$ or $K_2 \cdot (C_s + e)$ or $C_n \cdot P_3$. Then $Q^{+0-}(G)$ has crossing number one. $\square$

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Isotropic Curves and Their Characterizations in Complex Space $\mathbb{C}^4$

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Abstract: In this study, we investigate the classical differential geometry of isotropic curves in the complex space $\mathbb{C}^4$. We examine the constant breadth of isotropic curves and obtain some results regarding these isotropic curves. We express some characterizations of these curves via the É. Cartan derivative formula. We also indicate that the isotropic vector of these curves and pseudo curvature satisfy a third order vector differential equation with variable coefficients. We study this differential equation in some special cases. We dene evolute and involute of the isotropic curve and express some characterizations of these curves in terms of É. Cartan equations. The isotropic rectifying curve and isotropic helix are characterized in $\mathbb{C}^4$. Finally, we present the conditions for an isotropic curve to be an isotropic helix.

Key Words: Complex spaces, isotropic helix, isotropic curve of constant breath, Bertrand curves, iotropic rectifying curves.

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§1. Introduction

At the beginning of the nineteenth century, V. Pancelet's isotropic curve opened a door for a number of new concepts. The imaginary curve in the complex space was pioneered by Cartan. He defined his moving frame and the Cartan equations in $\mathbb{C}^3$. Altınışık extended the Cartan apparatus of isotropic curves to $\mathbb{C}^4$. Furthermore, isotopic Bertrand curves and isotropic helices in $\mathbb{C}^3$ were characterized, [9], [10], [16]. Also, the concept of a slant helix in the complex space in $\mathbb{C}^4$ was offered by Yılmaz [13].

Curves of constant breadth were introduced by Euler [3]. The curves have been studied in different spaces by researchers. For instance, Izumiya and Takeuchi defined slant helices [5]. Ali and Lopez gave some characterizations of slant helices in Minkowski 3-space [1]. Yılmaz

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studied spherical indicatrices of curves in Euclidean 4-space and Lorenzian 4-space [14], [15]. In [7], Mağden and Yılmaz extended the well known properties of constant breadth of the curves in four dimensional Galilean space $\mathbb{G}^4$.

Many researchers have studied involute-evolute curves in other spaces. The Frenet apparatus of involute-evolute curves couple in the space $\mathbb{E}^3$ and $\mathbb{E}^4$ is given [4], [8]. In [12], Turgut and Yılmaz studied involute-evolute curve couple in Minkowksi space-time. Şemin mentioned involute-evolute isotropic curve in [11]. In Euclidean 4-space, rectifying curves are introduced by İlarslan and Nesoviç in [6] as space curves whose position vector always lies in its rectifying plane, spanned by tangent, the first binormal and second binormal vector fields T, B_1 and B_2.The position vector of a rectifying curve α in $\mathbb{E}^4$ according to chosen origin satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \varphi(s)B_1(s) + \mu(s)B_2(s),$$

where λ, φ and μ are some differentiable functions of the pseudo arc-length parameter s.

Thus, the main goal of this paper is to define some isotropic curves in the four dimensional complex space $\mathbb{C}^4$. In the present paper, we first study isotropic curves of constant breadth and the involute-evolute of the curve in $\mathbb{C}^4$. Then we introduce the Bertrand curve and present some characterizations of the mentioned curves in terms of É. Cartan equations. Also, we give a new characterization of the isotropic helix. Throughout this study some complex curves are characterized in the complex space $\mathbb{C}^4$.

§2. Preliminaries

To meet the requirements in the next sections, the basic elements of the theory of imaginary curves in the space $\mathbb{C}^4$ are briefly presented (a more complete elementary operation can be found in [11]).

Let x_p be a complex analytic function of a complex variable t. Then the vector function

$$\mathbf{x}(t) = \sum_{p=1}^{4} x_p(t)\mathbf{k}_p,$$

is called an imaginary curve, where $t = t_1 + it_2$, $\mathbf{x} : \mathbb{C} \to \mathbb{C}^4$ and $\mathbf{k}_p$ are standard basis unit vectors of $\mathbb{E}^4$, $i^2 = -1$. An arbitrary vector $\mathbf{x} \in \mathbb{C}^4$, is called an isotropic vector if and only if $\mathbf{x}^2 = 0$, $(\mathbf{x} \neq \mathbf{0})$. In this space, the curves for which the square of the distance between any two points equal to zero, are called minimal or isotropic curves [11]. Let s denote pseudo arc-length (for details, see [10] or [11]). Then, a curve is an isotropic curve if and only if

$$ds^2 = d\mathbf{x}^2 = 0.$$

The complex four dimensional space $\mathbb{C}^4$, is the real vector space $\mathbb{E}^4$ endowed with the standard flat Euclidean metric given by

$$g = dx_1^2 + 2dx_1dx_3 - dx_4^2,$$

where (x_1, x_2, x_3, x_4) is the complex coordinate system of $\mathbb{C}^4$.

The É. Cartan frame moving along the isotropic curve $\mathbf{x}$ in the space $\mathbb{C}^4$ is denoted by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$. This frame is defined ([11]) as

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{x}' \\ \mathbf{e}_2 &= i\mathbf{x}'' \\ \mathbf{e}_3 &= -\frac{\beta}{2}\mathbf{x}' + \mathbf{x}''' \\ \mathbf{e}_4 &= \mu(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) \end{aligned} \tag{2.1}$$

where $\beta = (\mathbf{x}''')^2$, μ is taken as ± 1. If μ is taken as $+1$, the determinant of matrix $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4]$, the É. Cartan frame becomes positively oriented. Here, the triple vector product is cross product expressed as in [2]. The inner products of these frame vectors are given by

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & \text{if } i+j \equiv 1, 2, 3 \bmod 4 \\ 1 & \text{if } i+j = 4 \\ -1 & \text{if } i+j = 8 \end{cases}$$

where the vectors $\mathbf{e}_1$ and $\mathbf{e}_3$ are isotropic vectors; $\mathbf{e}_2$ is real and $\mathbf{e}_4$ is a complex vector. É. Cartan derivative formulas can be expressed as follows:

$$\begin{aligned} \mathbf{e}_1' &= -i\mathbf{e}_2 \\ \mathbf{e}_2' &= ik\mathbf{e}_1 + i\mathbf{e}_3 \\ \mathbf{e}_3' &= -ik\mathbf{e}_2 \qquad (2.2) \\ \mathbf{e}_4' &= -\xi(k'' + \xi k)\mathbf{e}_1 - \xi k\mathbf{e}_3 + \frac{\xi'}{\xi}\mathbf{e}_4 \end{aligned}$$

where $k(s) = \frac{1}{2}\beta(s)$ is the pseudo curvature of the isotropic curve in the class C^5 and $\xi(s) = \pm\frac{1}{\sqrt{\beta^2(s)+\gamma(s)}}$, where $\gamma(s) = (\mathbf{x}^{(iv)})^2$, the derivative being taken with respect to the pseudo arc-length s. In the rest of the paper, we shall suppose pseudo curvature is non-vanishing except in the case of an isotropic cubic.

An isotropic hypersphere with centre $\mathbf{m}$ and radius $r > 0$ in $\mathbb{C}^4$ is defined as

$$S^3 = \{\mathbf{p} = (p_1, p_2, p_3, p_4) \in \mathbb{C}^4 : (\mathbf{p} - \mathbf{m})^2 = r^2\}.$$

Definition 2.1 *An isotropic curve $\boldsymbol{x} = \boldsymbol{x}(s)$ in $\mathbb{C}^4$ is called an isotropic cubic if the pseudo curvature $k(s) = 0$, where s is the pseudo arc-length parameter of the curve.*

Definition 2.2 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be a complex curve in $\mathbb{C}^4$. If the pseudo curvature of the curve is constant, then $\boldsymbol{x}(s)$ is called a pseudo helix or isotropic helix in $\mathbb{C}^4$.*

Definition 2.3 *An isotropic curve $\boldsymbol{x} = \boldsymbol{x}(s)$ in $\mathbb{C}^4$ is called an isotropic helix if inner product of its tangent vector $\mathbf{e}_1$ is constant with some fixed isotropic vector $\mathbf{v}$, that is, $\mathbf{e}_1.\mathbf{v} = constant$.*

Definition 2.4 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be an isotropic curve in $\mathbb{C}^4$. If there exists another isotropic curve $\boldsymbol{x}^* = \boldsymbol{x}^*(s)$ in $\mathbb{C}^4$ such that principal normal vector field $\boldsymbol{x}^*$ coincides with that normal vector field of $\boldsymbol{x}$, then $\boldsymbol{x}$ is called a Bertrand curve and $\boldsymbol{x}^*$ is called the Bertrand mate of $\boldsymbol{x}$ and vice versa, where $\boldsymbol{x}(s)$ and $\boldsymbol{x}^*(s)$ are opposite points of the curve.*

Definition 2.5 *Let φ and δ be two unit speed complex curves in $\mathbb{C}^4$. If the tangent vector of the curve φ at the point $\varphi(s_0)$ is orthogonal to the tangent vector of the curve δ at the $\delta(s_0)$ then curve δ is called the involute of the curve φ as follows:*

$$g(\boldsymbol{e}_{1\varphi}, \boldsymbol{e}_{1\delta}) = 0,$$

where $\{\boldsymbol{e}_{1\varphi}, \boldsymbol{e}_{2\varphi}, \boldsymbol{e}_{3\varphi}, \boldsymbol{e}_{4\varphi}\}$ and $\{\boldsymbol{e}_{1\delta}, \boldsymbol{e}_{2\delta}, \boldsymbol{e}_{3\delta}, \boldsymbol{e}_{4\delta}\}$ are Frenet frames of φ and δ, respectively. Also, the curve φ is called the evolute of the curve δ. This definition suffices to define this curve mate as $\delta = \varphi + \lambda \boldsymbol{e}_{1\varphi}$.

Definition 2.6 *Let α be a complex curve in $\mathbb{C}^4$. A rectifying curve is defined in $\mathbb{C}^4$ as an α isotropic curve whose position vector always lies in orthogonal complement $\boldsymbol{e}_2^{\perp}$ of its principal normal vector field $\boldsymbol{e}_2$.*

§3. Isotropic Curves of Constant Breadth and Their Characterizations

Let $\mathbf{x}(s)$ and $\mathbf{x}^*(s)$ be isotropic curves in $\mathbb{C}^4$. If the tangent isotropic vector $\mathbf{e}_1$ of $\mathbf{x}(s)$ coincides with the tangent isotropic vector $\mathbf{e}_1^*$ of $\mathbf{x}^*(s)$ opposite directions at the corresponding points and the distance between these points is always constant, then $\mathbf{x}(s)$ is a constant breadth of the isotropic curve. Suppose that $\mathbf{x}(s)$ and $\mathbf{x}^*(s)$ are isotropic curves of constant breadth. Then $\mathbf{e}_1^*$ can be expressed by

$$\mathbf{e}_1 = -\mathbf{e}_1^*$$

where $\mathbf{e}_1$ and $\mathbf{e}_1^*$ are inverse direction and parallel vectors.

Let $\mathbf{x}(s)$ and $\mathbf{x}^*(s)$ be isotropic curves of constant breadth in $\mathbb{C}^4$. Taking into account the Cartan equations, it can be decomposed by

$$\mathbf{X}^*(s) = \mathbf{X}(s) + m_1(s)\mathbf{e}_1 + m_2(s)\mathbf{e}_2 + m_3(s)\mathbf{e}_3 + m_4(s)\mathbf{e}_4, (0 \leq s \leq 1), \tag{3.1}$$

where $\mathbf{X}(s)$ and $\mathbf{X}^*(s)$ are opposite points and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ denote the É. Cartan frame in $\mathbb{C}^4$.

Differentiating the equation (3.1) with respect to s, we get

$$\begin{aligned} \frac{d\mathbf{X}^*}{ds} &= \frac{d\mathbf{X}^*}{ds^*} \cdot \frac{ds^*}{ds} = \mathbf{e}_1^* \frac{ds^*}{ds} \\ &= (\frac{dm_1}{ds} + m_2 ik + m_4\eta_1)\mathbf{e}_1 + (-m_1 i + \frac{dm_2}{ds} - m_3 ik)\mathbf{e}_2 \\ &\quad +(m_2 i + \frac{dm_3}{ds} + m_4\eta_2)\mathbf{e}_3 + (\frac{dm_4}{ds} + m_4\eta_3)\mathbf{e}_4, \end{aligned} \tag{3.2}$$

where $\eta_1(s) = -\xi(k'' + \xi k), \eta_2(s) = -\xi k, \eta_3(s) = \frac{\xi'}{\xi}$ and $k = \frac{1}{2}\beta$ is a pseudo curvature of the isotropic curve in the class C^5. Since $\mathbf{e}_1^* = -\mathbf{e}_1$, we obtain

$$
\begin{aligned}
&1 + \frac{dm_1}{ds} + m_2 ik + m_4\eta_1 = -\frac{d\mathbf{s}^*}{ds}\\
&-m_1 i + \frac{dm_2}{ds} - m_3 ik = 0\\
&m_2 i + \frac{dm_3}{ds} - m_4\eta_2 = 0\\
&\frac{dm_4}{ds} + m_4\eta_3 = 0.
\end{aligned}
\tag{3.3}
$$

Putting $f(s) = -1 - \frac{ds^*}{ds}$, in the equation (3.3), it can be written as

$$
\begin{aligned}
&\frac{dm_1}{ds} = -m_2 ik - m_4\eta_1 + f(s)\\
&\frac{dm_2}{ds} = m_1 i + m_3 ik\\
&\frac{dm_3}{ds} = -m_2 i - m_4\eta_2\\
&\frac{dm_3}{ds} = -m_4\eta_3.
\end{aligned}
\tag{3.4}
$$

By virtue of the equation $(3.4)_4$ (i.e. the fourth expression of the equation (3.4)) we have $m_4 = c$ is constant. Rearranging the equation (3.4) we get

$$
\begin{aligned}
&\frac{dm_1}{ds} = -m_2 ik - c(k'' + \xi k) + f(s)\\
&\frac{dm_2}{ds} = m_1 i + m_3 ik\\
&\frac{dm_3}{ds} = -m_2 i - ck.
\end{aligned}
\tag{3.5}
$$

The following corollary is a consequence of the equations (3.4) and (3.5).

Corollary 3.1 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be an isotropic cubic. The isotropic position vector of $\boldsymbol{x}$ with respect to É. Cartan frame can be formed by the equations (3.5) and can be obtained as*

$$
\begin{aligned}
\mathbf{x}(s) \quad = \quad & \mathbf{x}^*(s) + \left(\int f(s)ds + k_1(s)\right)\mathbf{e}_1\\
& + \left(\left[\int\left(\int f(s)ds\right) + k_1(s)ds\right] + k_2(s)\right)\mathbf{e}_2\\
& + \left(\int((\int f(s)ds)ds) + k_1(s)\frac{s^2}{2}ds + ik_2(s) + k_3(s)\right)\mathbf{e}_3 + c\mathbf{e}_4.
\end{aligned}
$$

Proof Let $\mathbf{x} = \mathbf{x}(s)$ be an isotropic cubic. Then, $k = 0$ from Definition 2.1. From equation $(3.5)_1$ we get $\frac{dm_1}{ds} = f(s)$. Integrating this expression we have,

$$
m_1 = \int f(s)ds + k_1,
$$

where k_1 is a complex constant, from equations (3.4),$(3.5)_2$ and $(3.5)_3$,

$$\begin{aligned} m_2 &= i(\int(\int f(s)ds) + k_1(s)ds) + k_2(s) \\ m_3 &= \int((\int(\int f(s)ds) + k_1(s)\frac{s^2}{2})ds) + ik_2(s) + k_3(s) \end{aligned}$$

and $m_4 = c$ is constant. After m_1, m_2, m_3 and m_4 are substituted into the isotropic position vector $\mathbf{x} = \mathbf{x}(s)$, the proof is completed. □

Theorem 3.1 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be complex curve of constant breadth with pseudo arc-length in $\mathbb{C}^4$. If $\boldsymbol{x} = \boldsymbol{x}(s)$ lies fully in the $\boldsymbol{e}_3\boldsymbol{e}_4$ subspace, then $\boldsymbol{x} = \boldsymbol{x}(s)$ is an isotropic helix.*

Proof Let $\mathbf{x} = \mathbf{x}(s)$ be the pseudo arc-length parameter of constant breadth of complex curve in $\mathbb{C}^4$. From equations (3.5), if we take $m_1 = m_3 = 0$, then we have $m_2 = c_1$ (where c_1 is a constant). Using this expression in the third equation of (3.5), we obtain $k = \frac{c_1}{c}i$ is constant. From Definition 2.3), it is clear that the curve $\mathbf{x} = \mathbf{x}(s)$ is an isotropic helix. □

Theorem 3.2 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be complex curve of constant breadth with pseudo arc-length in $\mathbb{C}^4$. There is no constant breadth of isotropic curve that lies fully in the $\boldsymbol{e}_1\boldsymbol{e}_2$ subspace.*

Proof Let $\mathbf{x} = \mathbf{x}(s)$ be the pseudo arc-length parameter of constant breadth of complex curve in $\mathbb{C}^4$. If we take $m_3 = m_4 = 0$ in equation (3.5), we get $m_1 = 0$ and $m_2 = cki$. So $\mathbf{x} = \mathbf{x}(s)$ lies fully in the $\mathbf{e}_1\mathbf{e}_2$ subspace. □

Theorem 3.3 *Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be complex curve of constant breadth with pseudo arc-length in $\mathbb{C}^4$. There is no constant breadth of complex curve which lies fully in the $\boldsymbol{e}_1\boldsymbol{e}_4$ subspace, and $\boldsymbol{x} = \boldsymbol{x}(s)$ is isotropic cubic.*

Proof Let $\mathbf{x} = \mathbf{x}(s)$ be the pseudo arc-length parameter of constant breadth of complex curve in $\mathbb{C}^4$. From equation (3.5), we get $m_1 = 0$, $m_4 = c$ and $k = 0$. So $\mathbf{x} = \mathbf{x}(s)$ lies fully in the $\mathbf{e}_1\mathbf{e}_4$ subspace. From Definition 2.1, $\mathbf{x} = \mathbf{x}(s)$ is an isotropic cubic. □

Theorem 3.4 *A pseudo arc-length isotropic $\boldsymbol{x} = \boldsymbol{x}(s)$ in $\mathbb{C}^4$ is of constant breadth if and only if it satisfies the following third order differential equation.*

Proof From equation $(3.5)_1$, we get

$$m_2 = \frac{ck'' + c\xi k - f(s) + \frac{dm_1}{ds}}{-ik}.$$

Substituting into $(3.5)_2$, this expression m_3 is obtained

$$m_3 = \frac{\frac{d}{ds}\left[\frac{ck''+c\xi k-f(s)+\frac{dm_1}{ds}}{-ik}\right] - m_1 i}{ik}.$$

Taking the derivative of this expression, we obtain

$$\frac{dm_3}{ds} = \frac{d}{ds}\left[\frac{\frac{d}{ds}\left(\frac{ck''+c\xi k-f(s)+\frac{dm_1}{ds}}{-ik}\right)-m_1}{k}\right].$$

Substituting into equation $(3.5)_3$, this expression, we have a differential equation of third order with complex variable coefficients as follows:

$$\frac{d}{ds}\left[-\frac{1}{ik}\frac{d}{ds}\left(\frac{ck''+c\xi k-f(s)+\frac{dm_1}{ds}}{-ik}\right)\right]+\frac{d}{ds}\left(\frac{m_1}{k}\right)$$
$$-\frac{1}{k}\left(ck''+c\xi k-f\left(s\right)+\frac{dm_1}{ds}\right)+ck=0. \tag{3.6}$$

The differential equation of third order with variable coefficients in equation (3.6) is characterized for the constant breadth of isotropic curve $\mathbf{x}=\mathbf{x}(s)$.

Now, we characterize the distance between opposite points of the curves of constant breadth in $\mathbb{C}^4$. Remember the equation (3.1)

$$\mathbf{X}^*(s)=\mathbf{X}(s)+m_1(s)\mathbf{e}_1+m_2(s)\mathbf{e}_2+m_3(s)\mathbf{e}_3$$
$$+m_4(s)\mathbf{e}_4,(0\le s\le 1).$$

If the distance between opposite points of (C) and (C^*) is constant, then we can write that

$$\|x^*-x\|=m_1^2+2m_1m_3-m_4^2=l^2=constant. \tag{3.7}$$

Hence, we write

$$m_2\frac{dm_2}{ds}+m_3\frac{dm_1}{ds}+m_1\frac{dm_3}{ds}-m_4\frac{dm_4}{ds}=0 \tag{3.8}$$

from equations (3.5) since $m_4=c$ is constant. Rearranging the equation (3.8), we obtain

$$m_2\frac{dm_2}{ds}+m_3\frac{dm_1}{ds}+m_1\frac{dm_3}{ds}=0. \tag{3.9}$$

Considering equations (3.5), we have

$$m_3\left[\mu\left(s\right)-k^2i-\frac{m_2'ck}{m_3}\right]=0. \tag{3.10}$$

We write $m_3=0$ or $\mu\left(s\right)-k^2i-\frac{m_2'ck}{m_3}=0$, obviously, $m_3\neq 0$. Then it can be expressed in the following cases:

Case 1. Let us suppose $m_3=c_1\neq 0$ constant. From equations $(3.5)_2$ and $(3.5)_3$ we easily have $m_2=cki, m_1=-c_1k$. Then the isotropic position vector of φ^* can be written as follows:

$$\varphi^*=\varphi+c_1k\mathbf{e}_1+cki\mathbf{e}_2+c_1\mathbf{e}_3+c\mathbf{e}_4.$$

Case 2. Let us suppose that m_3 is constant and φ is isotropic helix. Thus, the equation (3.6) takes the form

$$\frac{d^2g(s)}{ds^2} - kh(s) + ck^3 = 0, \tag{3.11}$$

where $h(s) = c\xi k - f(s)$. The solution of the equation (3.11) is

$$h(s) = L_1 e^{\sqrt{k}x} + L_2 e^{-\sqrt{k}x} + \frac{1}{2} - \frac{1}{\sqrt{2}}, \tag{3.12}$$

where L_1 and L_2 are real numbers.

Case 3. Let us suppose

$$\mu(s) - k^2 i - \frac{m_2' ck}{m_3} = 0. \tag{3.13}$$

In this case, (C^*) is transformed by the constant vector $\eta = m_1\mathbf{e}_1 + m_2\mathbf{e}_2 + m_3\mathbf{e}_3 + m_4\mathbf{e}_4$ of (C). Now, let us investigate the solution to Case 3.

Suppose that μ is an isotropic cubic. Then, we get from equation (3.13) $\mu(s) = 0$ and from equation (3.5) we get m_1 =constant, $m_2 = 0, m_3 = -\frac{c}{k}$. □

§4. Involute and Evolute of Isotropic Curves in $\mathbb{C}^4$

Theorem 4.1 *Let φ and δ be complex curves and φ be an evolute of δ. The Cartan apparatus of $\varphi\{\boldsymbol{e}_{1\varphi}, \boldsymbol{e}_{2\varphi}, \boldsymbol{e}_{3\varphi}, \boldsymbol{e}_{4\varphi}, k_\varphi\}$ can be formed according to the Cartan apparatus of $\delta\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, k\}$.*

Proof Let φ and δ be complex curves and φ be an evolute of δ. According to the property of involute-evolute curve couples, we have

$$\varphi = \delta + \lambda\mathbf{e}_1. \tag{4.1}$$

Differentiating both sides of the equation (4.1) with respect to s, we obtain

$$\frac{d\varphi}{ds_\varphi}.\frac{ds_\varphi}{ds} = \mathbf{e}_1 + \frac{d\lambda}{ds}\mathbf{e}_1 + \lambda(-i\mathbf{e}_2). \tag{4.2}$$

Rearranging equation (4.2), we have

$$\frac{d\varphi}{ds_\varphi}\frac{ds_\varphi}{ds} = (1 + \frac{d\lambda}{ds})\mathbf{e}_1 - \lambda i\mathbf{e}_2. \tag{4.3}$$

Similarly, based on the definition of involute and evolute curves, we can say $\mathbf{e}_{1\varphi} \perp \mathbf{e}_1$. Obviously, we get

$$1 + \frac{d\lambda}{ds} = 0. \tag{4.4}$$

We get $\lambda = c - s$, where c is constant. Rearranging the equation (4.1), we get

$$\varphi = \delta + (c - s)\mathbf{e}_1. \tag{4.5}$$

By differentiating the equation (4.5), we have the following equation

$$\varphi' = \mathbf{e}_{1\varphi}.\frac{ds_\varphi}{ds} = (c-s)(-i\mathbf{e}_2). \tag{4.6}$$

Taking the norm of both sides, we get

$$\mathbf{e}_{1\varphi} = -\mathbf{e}_2 \tag{4.7}$$

and

$$\frac{ds_\varphi}{ds} = (c-s)i. \tag{4.8}$$

Differentiating the equation (4.6) two times with respect to s, we get

$$\varphi'' = -(c-s)k\mathbf{e}_1 + i\mathbf{e}_2 + (c-s)\mathbf{e}_3 \tag{4.9}$$

and

$$\varphi''' = \left[-2k + (c-s)k'\right]\mathbf{e}_1 + \left[-2i(c-s)k\right]\mathbf{e}_2 - 2\mathbf{e}_3. \tag{4.10}$$

Thus, we have the following expressions for $\mathbf{e}_{2\varphi}, \mathbf{e}_{3\varphi}$ and k_φ.

$$\begin{aligned} \mathbf{e}_{2\varphi} &= (c-s)ki\mathbf{e}_1 - \mathbf{e}_2 + (c-s)i\mathbf{e}_3 \\ \mathbf{e}_{3\varphi} &= \left[-2k + (c-s)k'\right]\mathbf{e}_1 + i(c-s)(\frac{\beta}{2} - 2k)\mathbf{e}_2 - 2\mathbf{e}_3 \end{aligned} \tag{4.11}$$

and

$$k_\varphi = -2\left[-2k + (c-s)k'\right] + \left[-2(c-s)k\right]^2. \tag{4.12}$$

Using the exterior product $\sigma(\mathbf{e}_{1\varphi} \wedge \mathbf{e}_{2\varphi} \wedge \mathbf{e}_{3\varphi})$, we get

$$\mathbf{e}_{4\varphi} = \sigma\left[2(c-s)ik(1 + (c-s)k)\mathbf{e}_4\right], \tag{4.13}$$

where $\sigma = \pm 1$. □

Since from equation (4.7), it follows that $\mathbf{e}_{1\varphi}$ is not an isotropic vector, we can state the following.

Remark 4.1 Let φ be an evolute of a complex curve in $\mathbb{C}^4$. The curve φ cannot be an isotropic curve.

Theorem 4.2 *Let φ and δ be complex curve and φ be an evolute of δ in $\mathbb{C}^4$. The evolute φ cannot be an isotropic helix in $\mathbb{C}^4$.*

Proof Considering the definition of isotropic helix, we write

$$\mathbf{e}_{1\varphi}.\mathbf{v} = \text{constant}, \tag{4.14}$$

where $\mathbf{v}$ is a constant isotropic vector. From equation (4.7), we easily have

$$-\mathbf{e}_2.\mathbf{v} = \text{constant}, \tag{4.15}$$

Differentiating both sides of equation (4.15), we get

$$-(ik\mathbf{e}_1 + i\mathbf{e}_3).\mathbf{v} = 0. \tag{4.16}$$

Therefore $\mathbf{v}\perp\mathbf{e}_1$ and $\mathbf{v}\perp\mathbf{e}_3$. Let us decompose $\mathbf{v}$ as

$$\mathbf{v} = t_1\mathbf{e}_2 + t_2\mathbf{e}_4. \tag{4.17}$$

Differentiating equation (4.17) consecutively and using Cartan equations, we have $t_1 = 0$ and $t_2 = 0$. According to the result, we write

$$\mathbf{v} = 0. \tag{4.18}$$

Equations (4.14) and (4.18) yield a contradiction. Therefore, evolute φ cannot be an isotropic helix in space $\mathbb{C}^4$. □

§5. Bertrand Couple Curves of Isotropic Curves in $\mathbb{C}^4$

Theorem 5.1 *Let α^* and α be Bertrand curves in complex space $\mathbb{C}^4$. The Cartan apparatus of $\alpha^*\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*, \boldsymbol{e}_3^*, \boldsymbol{e}_4^*, k^*\}$ can be formed by the Cartan apparatus of $\alpha\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4, k\}$.*

Proof Suppose that $\{\alpha(s), \alpha^*(s^*)\}$ is an isotropic Bertrand pair of curves. Then $\alpha^*(s^*)$ can be expressed by

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)\mathbf{e}_2, \tag{5.1}$$

where $\lambda(s)$ is the non zero analytic function and s^* is the pseudo arc-length parameter of $\alpha^*(s^*)$. Differentiating both sides of the equation (5.1) with respect to s, we get

$$\alpha^* = \frac{d\alpha^*}{ds^*}\frac{ds^*}{ds} = \mathbf{e}_1^*\frac{ds^*}{ds} = (1 + \lambda ki)\,\mathbf{e}_1 + \frac{d\lambda}{ds}\mathbf{e}_2 + \lambda i\mathbf{e}_3. \tag{5.2}$$

The definition of Bertrand curves yields $\mathbf{e}_1^*\perp\mathbf{e}_2$. Multiplying both sides of equation (5.2) with $\mathbf{e}_2$ we have

$$\frac{d\lambda}{ds} = 0 \tag{5.3}$$

which implies that λ is constant. Using this in the equation (5.2) and taking the norm of the both sides, we get

$$\frac{ds^*}{ds} = \sqrt{2(1 + \lambda ki)\lambda i}$$

and the tangent vector $\mathbf{e}_1^*$ is equal to

$$\mathbf{e}_1^* = \frac{(1+\lambda ki)}{\sqrt{2\,(1+\lambda ki)\,\lambda i}}\mathbf{e}_1 + \frac{\lambda i}{\sqrt{2\,(1+\lambda ki)\,\lambda i}}\mathbf{e}_3. \tag{5.4}$$

Taking the derivative of the equation (5.2) two times with respect to s, we get

$$\alpha^{*''} = (1+\lambda k' i)\mathbf{e}_1 + (-1-\lambda ki+\lambda k)\mathbf{e}_2 \tag{5.5}$$

and

$$\alpha^{*'''} = (1+\lambda k'' i - ki + \lambda k^2 + \lambda k^2 i)\mathbf{e}_1 + (-1-\lambda k' i+\lambda k')\mathbf{e}_2 + (-i-\lambda k+\lambda ki)\mathbf{e}_3. \tag{5.6}$$

Using the equation (5.6), we get the vectors $\mathbf{e}_2^*$, $\mathbf{e}_3^*$ and pseudo curvature k^*, as follows:

$$\mathbf{e}_2^* = \frac{1}{i}[(1+\lambda k' i)\mathbf{e}_1 + (-1-\lambda ki+\lambda k)\mathbf{e}_2],$$

$$\begin{aligned}\mathbf{e}_3^* = \frac{1}{2}\{[&-1-\lambda k' i + 2(1+\lambda k'' i - ki + \lambda k^2 i)(-i-\lambda k+\lambda ki)]\mathbf{e}_1 \\ &+(-1-\lambda k' i+\lambda k')\mathbf{e}_2 + [-1-\lambda ki+\lambda k]\mathbf{e}_3\}\end{aligned}$$

and

$$k^* = \frac{1}{2}\{-1-\lambda k' i+\lambda k' + 2(1+\lambda k'' i - ki + \lambda k^2 + \lambda k^2 i)(-i-\lambda k+\lambda ki)\}.$$

So, the pseudo curvature $k^*(s)$ is a non zero constant. □

Remark 5.1 Obviously, $\mathbf{e}_1^*$ isn't an isotropic vector from equation (5.4). So, the Bertrand curve α^* cannot be an isotropic curve.

Remark 5.2 Let α^* and α be Bertrand curves in $\mathbb{C}^4$. If one of the Bertrand curves lies fully in $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ subspace, then the Bertrand mate also lies fully in the same subspace of $\mathbb{C}^4$.

Theorem 5.2 *Let* $\mathbf{x} = \mathbf{x}(\mathbf{s})$ *be an isotropic curve in* $\mathbb{C}^4$. *Then,* $\mathbf{x}(\mathbf{s})$ *is a pseudo isotropic helix if and only if the following statements are equivalent:*

(a) $\det(\mathbf{x}''(s), \mathbf{x}'''(s), \mathbf{x}^{(iv)}(s)) = 0$*;*

(b) $\det(\mathbf{e}_1'(s), \mathbf{e}_1''(s), \mathbf{e}_1'''(s)) = 0$

c) $\det(\mathbf{e}_3'(s), \mathbf{e}_3''(s), \mathbf{e}_3'''(s)) = 0$.

Proof Taking the 2^{nd}, 3^{rd} and 4^{th} derivatives of equation (2.1), we obtain

$$\mathbf{x}' = \mathbf{e}_1, \quad \mathbf{x}'' = -i\mathbf{e}_2, \quad \mathbf{x}''' = k\mathbf{e}_1 + \mathbf{e}_3, \quad \mathbf{x}^{(iv)} = k'\mathbf{e}_1 - 2ik\mathbf{e}_2. \tag{5.7}$$

We calculate that

$$\det(\mathbf{x}''(s), \mathbf{x}'''(s), \mathbf{x}^{(iv)}(s)) = \begin{vmatrix} 0 & -i & 0 \\ k & 0 & 1 \\ k' & -2ik & 0 \end{vmatrix} = -ik'.$$

Since $\mathbf{x} = \mathbf{x}(s)$ is an isotropic helix, then $\det(\mathbf{x}''(s), \mathbf{x}'''(s), \mathbf{x}^{(iv)}(s)) = 0$. Conversely, let the statement "**a)**" be true. Then $\det(\mathbf{x}''(s), \mathbf{x}'''(s), \mathbf{x}^{(iv)}(s)) = -ik' = 0$. Thus, k is constant and $\mathbf{x}(s)$ is an isotropic helix. This completes the proof "**a)**". Similarly, denoting $\mathbf{x}' = \mathbf{e}_1, \mathbf{x}'' = \mathbf{e}_1', \mathbf{x}''' = \mathbf{e}_1''$ and $\mathbf{x}^{(iv)} = \mathbf{e}_1'''$, we easily see that "**a)**" and "**b)**" are equivalent. Also, because of the fact that the equations

$$\begin{aligned} \mathbf{e}_3' &= -ik\mathbf{e}_2 \\ \mathbf{e}_3'' &= k^2\mathbf{e}_1 - ik'\mathbf{e}_2 + k\mathbf{e}_3 \\ \mathbf{e}_3''' &= 3kk'\mathbf{e}_1 - (2k^2 + ik'')\mathbf{e}_2 + 2k'\mathbf{e}_3, \end{aligned}$$

are hold, we can calculate that

$$\det(\mathbf{e}_3'(s), \mathbf{e}_3''(s), \mathbf{e}_3'''(s)) = -k^3k' = 0.$$

Since $\mathbf{x} = \mathbf{x}(s)$ is an isotropic helix, then $\det(\mathbf{e}_3'(s), \mathbf{e}_3''(s), \mathbf{e}_3'''(s)) = 0$. Conversely, let us say that in the determinant,

$$\det(\mathbf{e}_3'(s), \mathbf{e}_3''(s), \mathbf{e}_3'''(s)) = \begin{vmatrix} 0 & ik & 0 \\ k^2 & -ik' & k \\ 3kk' & -\left(2k^2 + ik''\right) & 2k' \end{vmatrix} = -k^3k' = 0, \qquad (5.8)$$

we get $\frac{dk}{ds} = 0$ then k is a constant. As an immediate consequence of Definition 2.2, $\mathbf{x} = \mathbf{x}(s)$ is an isotropic helix. □

§6. Isotropic rectifying curves in $\mathbb{C}^4$

In this section, we firstly characterize the rectifying curves in $\mathbb{C}^4$ in terms of their pseudo curvature. In analogy with Euclidean four dimensional case, we define the rectifying curves in complex space $\mathbb{C}^4$ as a curve whose position vector always lies in the orthogonal complement $\mathbf{e}_2^\perp$ of its principal normal vector field $\mathbf{e}_2$. Hence, $\mathbf{e}_2^\perp$ is a three dimensional subspace of $\mathbb{C}^4$, spanned by vector filed $\mathbf{e}_1, \mathbf{e}_3$ and $\mathbf{e}_4$. Therefore the position vector with respect to some chosen origin of a rectifying curve α in $\mathbb{C}^4$, satisfies the equation

$$\alpha(s) = \lambda(s)\mathbf{e}_1(s) + \mu\mathbf{e}_3(s) + \delta(s)\mathbf{e}_4(s) \qquad (6.1)$$

for differentiable functions $\lambda(s), \mu(s)$ and $\delta(s)$ with pseudo arc-length parameter s. Firstly, let us characterize the rectifying curve α in $\mathbb{C}^4$ in terms of its pseudo curvature. Let $\alpha = \alpha(s)$ be a unit speed complex rectifying curve in $\mathbb{C}^4$, with non zero pseudo curvature $k(s)$. By definition, the position vector of complex curve α satisfies equation (6.1) for some differentiable functions $\lambda(s), \mu(s)$ and $\delta(s)$. Differentiating the equation (6.1) and using Cartan derivative formulas (2.2), we get

$$[\lambda' - 1 - \delta\xi(k'' + \xi k)]\mathbf{e}_1 + [\lambda i - \mu i k]\mathbf{e}_2 + [\lambda' - \delta\xi k]\mathbf{e}_3 + [\delta' + \delta\frac{\xi'}{\xi}]\mathbf{e}_4 = 0.$$

It follows that

$$\begin{aligned} \lambda'(s) - \delta(s)\xi(s)(k''(s) + \xi(s)k(s)) &= 1 \\ \lambda(s)i - \mu(s)k(s)i &= 0 \\ \lambda'(s) - \delta(s)\xi(s)k(s) &= 0 \\ \delta'(s) + \delta(s)\frac{\xi'(s)}{\xi(s)} &= 0 \end{aligned} \tag{6.2}$$

and thus

$$\begin{aligned} \lambda(s) &= c\int_0^s k(s)ds \\ \mu(s) &= \frac{c}{k(s)}\int_0^s k(s)ds \\ \delta(s) &= \frac{c}{\xi(s)}. \end{aligned} \tag{6.3}$$

Conversely, assuming that the pseudo curvature $k(s)$ of an arbitrary unit speed complex curve α in $\mathbb{C}^4$, satisfied the following equation

$$\alpha(s) = \left(c\int_0^s k(s)ds\right)\mathbf{e}_1(s) + \left(\frac{c}{k(s)}\int_0^s k(s)ds\right)\mathbf{e}_3(s) + \left(\frac{c}{\xi(s)}\right)\mathbf{e}_4(s)$$

Remark 6.1 (i) α cannot be an isotropic cubic, since $\frac{c}{k(s)} \neq 0$;

(ii) If α is a helix, then $\alpha(s) = s\left[(ck)\mathbf{e}_1 + (c)\mathbf{e}_3 + \left(\frac{c}{\xi(s)}\right)\mathbf{e}_4\right]$.

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On Hyper Generalized Quasi Einstein Manifolds

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Abstract: In this paper its proved three theorems about global properties of hyper generalized quasi-Einstein manifolds.

Key Words: Non-flat Riemannian manifold, hyper generalized quasi Einstein manifold $(HGQE)_n$, compact orientable.

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§1. Introduction

The notion of quasi Einstein manifold was introduced in a paper [8] by M.C.Chaki and R.K.Maity. According to them a non-flat Riemannian manifold (M^n, g),$(n \geq 3)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{1}$$

and is not identically zero, where a, b are scalars $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, \xi_1) = A(X), \qquad \forall X \ \in TM, \tag{2}$$

where, ξ_1 is a unit vector field.

In such a case a, b are called the associated scalars. A is called the associated 1-form. Such an n-dimensional manifold is denoted by the symbol$(QE)_n$.

Again, U.C.De and G.C.Ghosh defined generalized quasi Einstein manifold. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold if its Ricci-tensor S of type (0,2) is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y), \tag{3}$$

where a, b, c are non-zero scalars and A, B are two 1-forms such that

$$g(X, \xi_1) = A(X) \quad \text{and} \quad g(X, \xi_2) = B(X) \tag{4}$$

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with ξ_1, ξ_2 unit vectors which are orthogonal, i.e,

$$g(\xi_1, \xi_2) = 0. \tag{5}$$

This type of manifold are denoted by $G(QE)_n$.

In [16], H.G. Nagaraja introduced the concept of $N(k)$-mixed quasi Einstein manifold and mixed quasi constant curvature.A non flat Riemannian manifold (M^n, g) is called a $N(k)$-mixed quasi Einstein manifold if its Ricci tensor of type $(0, 2)$ is non zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)B(Y) + cB(X)A(Y), \tag{6}$$

where a, b, c are smooth functions and A, B are non zero 1-forms such that

$$g(X, \xi_1) = A(X) \quad \text{and} \quad g(X, \xi_2) = B(X) \qquad \forall \quad X, \tag{7}$$

with ξ_1, ξ_2 the orthogonal unit vector fields. Such aa manifold is denoted by the symbol $N(k)-(MQE)_n$.

The notion of hyper-generalized quasi Einstein manifold has been introduced by A.A.Shaikh, C. Özgür and A.Patra[17] in 2011. An n-dimensional Riemannian manifold (M^n, g) $(n > 2)$ is called a hyper generalized quasi-Einstein manifold if its Ricci tensor of type $(0, 2)$ is non zero and satisfies the following condition

$$\begin{aligned} S(X, Y) &= ag(X, Y) + bA(X)A(Y) + c[A(X)B(Y) + A(Y)B(X)] \\ &\quad + d[A(X)D(Y) + A(Y)D(X)] \end{aligned} \tag{8}$$

for all $X, Y \in \chi(M)$, where a, b, c and d are real valued, non-zero scalars functions on (M^n, g). A, B and D are non zero 1-forms such that

$$g(X, \xi_1) = A(X), \quad g(X, \xi_2) = B(X) \quad and \quad g(X, \xi_3) = D(X), \tag{9}$$

wheer ξ_1, ξ_2, ξ_3 are three unit vector fields mutually orthogonal to each other at every point on M.Here a, b, c, d are called the associated scalars, A, B, D are called the associated main and auxiliary 1-forms. We denote this type of manifold $(HGQE)_n$.

§2. **Preliminaries**

From (8) and (9), we get

$$S(X, X) = a|X|^2 + b|g(X, \xi_1)|^2 + 2c|g(X, \xi_1)g(X, \xi_2)| + 2d|g(X, \xi_1)g(X, \xi_3)|, \quad \forall \quad X. \tag{10}$$

Let θ_1 be the angle between ξ_1 and any vector X; θ_2 be the angle between ξ_2 and any vector X; θ_3 be the angle between ξ_3 and any vector X. Then

$$\cos\theta_1 = \frac{g(X,\xi_1)}{\sqrt{g(\xi_1,\xi_1)}\sqrt{g(X,X)}} = \frac{g(X,\xi_1)}{\sqrt{g(X,X)}}$$

as $g(\xi_1,\xi_1) = 1$), and

$$\cos\theta_2 = \frac{g(X,\xi_2)}{\sqrt{g(X,X)}} \quad \text{and} \quad \cos\theta_3 = \frac{g(X,\xi_3)}{\sqrt{g(X,X)}}.$$

If $b > 0$, $c > 0$, we have from (10)

$$\begin{aligned}(a+b+2c+2d)|X|^2 \quad \geq \quad & a|X|^2 + b|g(X,\xi_1)|^2 + 2c|g(X,\xi_1)g(X,\xi_2)| \\ & +2d|g(X,\xi_1)g(X,\xi_3)| = S(X,X). \end{aligned} \tag{11}$$

Now, contracting (8) over X and Y, we get

$$r = na, \tag{12}$$

where r is the scalar curvature. Since ξ_1, ξ_2 and ξ_3 are orthogonal unit vector fields, therefore $g(\xi_1,\xi_1) = 1$, $g(\xi_2,\xi_2) = 1$, $g(\xi_3,\xi_3) = 1$, $g(\xi_1,\xi_2) = 0$, $g(\xi_1,\xi_3) = 0$ and $g(\xi_2,\xi_3) = 0$.

Again, putting $X = Y = \xi_1$ in (8) we get $S(\xi_1,\xi_1) = a + b$. Putting $X = Y = \xi_2$ in (8) we get $S(\xi_2,\xi_2) = a$. Putting $X = Y = \xi_3$ in (8) we get $S(\xi_3,\xi_3) = a$.

If X is a unit vector field, then $S(X,X)$ is the Ricci-curvature in the direction of X.

Notice that Q is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S, where

$$g(QX,Y) = S(X,Y) \ \ \forall \ X,Y \in TM. \tag{13}$$

Let l^2 denote the squares of the lengths of the Ricci-tensor S. Then

$$l^2 = \sum_{i=1}^{n} S(Qe_i,e_i), \tag{14}$$

where $\{e_i\}$, $\ i = 1,2,\cdots,n$ is an orthonormal basis of the tangent space at a point of $(HGQE)_n$.

Now from (8) we get

$$\begin{aligned}S(Qe_i,e_i) \quad = \quad & ag(Qe_i,e_i) + bA(Qe_i)A(e_i) + c[A(Qe_i)B(e_i) + A(e_i)B(Qe_i)] \\ & +d[A(Qe_i)D(e_i) + A(e_i)D(Qe_i)],\end{aligned}$$

i.e,

$$l^2 = (n-2)a^2 + (a+b)^2 + 2c^2 + 2d^2. \tag{15}$$

These result will be used in the sequel.

§3. Ricci Semi-symmetric $(HGQE)_n (n > 3)$

Chaki and Maity proved that $(QE)_n (n > 3)$ is Ricci Semi-symmetric if and only if

$$A(R(X, Y)Z = 0.$$

Let us suppose that $(HGQE)_n$ $(n > 3)$ is Ricci-Semi symmetric. Then

$$A(R(X, Y)Z = 0. \tag{16}$$

From (16) we get

$$A(Q(X)) = 0, \tag{17}$$

where Q be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S. Then

$$g(QX, Y) = S(X, Y). \tag{18}$$

Then from (8) we get

$$A(Q(X)) = (a + b)A(X) + cB(X) + dD(X). \tag{19}$$

From (17) and (19) it follows that

$$(a + b)A(X) + cB(X) + dD(X) = 0. \tag{20}$$

Thus we can state the following.

Theorem 3.1 *If a $(HGQE)_n$ is Ricci Semi symmetric than (a+b)A(X)+cB(X)+dD(X)=0.*

§4. Sufficient Condition for a Compact Orientable $(HGQE)_n$(n≥3) Without Boundary to be Isometric to a Sphere

In this section we consider a compact, orientable $(HGQE)_n$ without boundary having constant associated scalars a, b, c and d. Then from (11) and (15), it follows that the scalar curvature is constant and so also is the length of the Ricci-tensor.

We further suppose that $(HGQE)_n$ under consideration admits a non-isometric conformal motion generated by a vector field X. Since l^2 is constant, it follows that

$$\pounds_X l^2 = 0, \tag{21}$$

where $\pounds_X$ denotes Lie differentiation with respect to X.

Now, it is known ([2], [4], [5], [9], [12], [13], [14], [15]) that if a compact Riemannian manifold M of dimension $n > 2$ with constant scalar curvature admits an infinitesimal non-isometric conformal transformation X such that $\pounds_X l^2 = 0$ then M is isometric to a sphere. But a sphere is Einstein so that b, c and d vanish which is a contradiction.This leads to the following theorem.

Theorem 4.1 *A compact orientable hyper generalized quasi Einstein manifold* $(HGQE)_n$ $(n \geq 3)$ *without boundary does not admit a non-isometric conformal vector field.*

§5. Killing Vector Field in a Compact Orientable $(HGQE)_n(n \geq$3) Without Boundary

In this section, we consider a compact, orientable $(HGQE)_n(n \geq 3)$ without boundary with a, b, c and d as associated scalars.

It is known [4] that in such a manifold M, the following relation holds

$$\int_M [S(X,X) - |\nabla X|^2 - (divX)^2]dv \leq 0 \qquad \forall X. \tag{22}$$

If X is a killing vector field, then $divX = 0$ ([4]). Hence (22) takes the form

$$\int_M [S(X,X) - |\nabla X|^2]dv = 0. \tag{23}$$

Let $b > 0$, $c > 0$, $d > 0$. Then by (11)

$$(a + b + 2c + 2d)|X|^2 \geq S(X,X). \tag{24}$$

Therefore,

$$(a + b + 2c + 2d)|X|^2 - |\nabla X|^2 \geq S(X,X) - |\nabla X|^2. \tag{25}$$

Consequently,

$$\int_M [(a + b + 2c + 2d)|X|^2 - |\nabla X|^2]dv \geq \int_M [S(X,X) - |\nabla X|^2]dv, \tag{26}$$

and by (23)

$$\int_M [(a + b + 2c + 2d)|X|^2 - |\nabla X|^2]dv \geq 0. \tag{27}$$

If $a + b + 2c + 2d < 0$, then

$$\int_M [(a + b + 2c + 2d)|X|^2 - |\nabla X|^2]dv = 0. \tag{28}$$

Therefore, $X = 0$. This leads to the following.

Theorem 5.1 *If in a compact orientable* $(HGQE)_n(n \geq 3)$ *without boundary and the associated scalars are such that* $b > 0, c > 0, d > 0$ *and* $a + b + 2c + 2d < 0$*, then there exists no non-zero killing vector field in this manifold.*

Corollary 5.1 *If in a compact orientable* $(HGQE)_n(n \geq 3)$ *without boundary, and each of the associated scalars* a, b, c, d*, is greater than zero, then any harmonic vector field* X *in the* $(HGQE)_n$ *is parallel and orthogonal to one of the generators of the manifold which makes greatest angle with the vector* X.

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Mechanical Quadrature Methods from Fitting Least Squire Interpolation Polynomials

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Abstract: In this paper, we are developing Quadrature Methods (*numerical integration method*) of continuous function $f(x)$ on a compact interval $[a, b]$ and deriving a polynomial $P_m(x)$ of degree m such that integration of $P_m(x)$ from a to b is equal to integration of $f(x)$ from a to b. We are using least square method to fit the polynomial $P_m(x)$. Also derive Newton-Cotes formulas and composite formula from this method, estimate errors and given MATLAB codes.

Key Words: Numerical integration, Newton-cotes method, quadrature method.

AMS(2010): 65D30, 65D32, 26B15.

§1. Introduction

With the advent of the modern high speed electronic digital computers, the Numerical Integration have been successfully applied to study problems in Mathematics, Engineering, Computer Science and Physical Sciences. Numerical integration, also called *Quadrature*, is the study of how the numerical value of an integral can be found. The purpose of this paper is quadrature methods for approximate calculation of definite integrals

$$I[f] = \int_a^b f(x)dx \tag{1.1}$$

where $f(x)$ is integrable, in the Riemann sense on $[a, b]$. The limit of the integration may be finite. Numerical integration is always carried out by mechanical quadrature and its basic scheme is as follows:

$$\int_a^b f(x) = \sum_{i=0}^{n-1} A_i f_i + R[f], \tag{1.2}$$

where $f_i = f(x_i)$ is continuous function in$[a, b]$. $A_i$and x_i are called *Coefficients(Weights)* and *nodes* for Numerical Quadrature, respectively, and R[f] is error of Quadrature method. Once the coefficients and nodes are set down, the scheme (1) can be determined.

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§2. Preliminaries

2.1 Order of Quadrature Method

Order of accuracy, or precision, of a Quadrature formula is the largest positive integer n such that the formula is exact for x^k, for each $k = 0, 1, \cdots, n$.

2.2 Error of Quadrature Method

The integration (1.1) is approximated by a finite linear combination of value of $f(x)$ in the form (1.2). The error of approximation of (1.2) is given as

$$R_n = \frac{C}{(m+1)!} f^{(m+1)}(\xi), \tag{2.1}$$

where $\xi = (a, b)$, $m \geq n$ is order of (1.2) and error constant of (1.2) is

$$C = \int_a^b x^{m+1} - \sum_{i=0}^{n-1} A_i x_i^{m+1}. \tag{2.2}$$

2.3 Interpolation Polynomial

Let $f(x)$ be a continuous function defined on some interval $[a, b]$, and be prescribed at $n+1$ distinct tabular points $x_0, x_1, \cdots, x_n$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. The distinct tabular points $x_0, x_1, \cdots, x_n$ are equispaced, that is $x_{k+1} - x_k = h$, $k = 0, 1, 2, \cdots, n-1$. The problem of polynomial approximation is to find a polynomial $P_n(x)$, of degree$\leq n$, which fits the given data exactly, that is,

$$P_n(x_i) = f(x_i), i = 0, 1, 2, \cdots, n. \tag{2.3}$$

The polynomial $P_n(x)$ is called the interpolating polynomial. The conditions given in (5)are called the interpolating conditions.

2.4 Least Squares Interpolation Polynomial

Let the polynomial of the m^{th} degree

$$P_m(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

be fitted to the data points $(x_i, f(x_i))$ $i = 0, 1, 2, \cdots, n$, where $m < n$ and $a_i's$ are satisfy the the system of equations

$$(n+1)a_0 + a_1 \sum_{i=0}^{n} x_i + a_2 \sum_{i=0}^{n} x_i^2 + \cdots + a_m \sum_{i=0}^{n} x_i^m = \sum_{i=0}^{n} f(x_i), \tag{2.4}$$

$$a_0 \sum_{i=0}^{n} x_i + a_1 \sum_{i=0}^{n} x_i^2 + \cdots + a_m \sum_{i=0}^{n} x_i^{m+1} = \sum_{i=0}^{n} x_i f(x_i)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_0 \sum_{i=0}^{n} x_i^m + a_1 \sum_{i=0}^{n} x_i^{m+1} + \cdots + a_m \sum_{i=0}^{n} x_i^{2m} = \sum_{i=0}^{n} x_i^m f(x_i).$$

There are $m+1$ equations in $m+1$ unknowns.

Lemma 2.1 *Let $P_m(x)$ be the least squares interpolation equation of $f(x)$ on $[a,b]$. Then*

$$\sum_{i=0}^{n} P_m(x_i) \approx \sum_{i=0}^{n} f(x_i), \tag{2.5}$$

where $x_0 = a, x_n = b$, $x_i = a + ih$ and $h = (b-a)/n$.

Proof Let the $P_m(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$ is least squares interpolation equation of $f(x)$ on $[a,b]$. Then $P_m(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_m x_0^m$ and $P_m(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_m x_1^m$, and so on $P_m(x_n) = a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_m x_n^m$. Adding all we get

$$\sum_{i=0}^{n} P_m(x_i) = (n+1)a_0 + a_1 \sum_{i=0}^{n} x_i + a_2 \sum_{i=0}^{n} x_i^2 + \cdots + a_m \sum_{i=0}^{n} x_i^m$$

apply Equation (2.4), we get

$$\sum_{i=0}^{n} P_m(x_i) \approx \sum_{i=0}^{n} f(x_i). \qquad \square$$

Theorem 2.2 *Let $P_m(x)$ is least squares interpolation equation of the integrable function $f(x)$ on finite interval $[a,b]$ and*

$$\sum_{i=0}^{n} P_m(x_i) \cong \sum_{i=0}^{n} f(x_i),$$

where $x_0 = a, x_n = b$ if and only if

$$\int_{x_0}^{x_n} P_m(x)dx \cong \int_{x_0}^{x_n} f(x)dx. \tag{2.6}$$

Proof Multiplying with $h = (b-a)/n$ and take limit $h \longrightarrow 0$ in (2.5), we get

$$\lim_{h \longrightarrow 0} h \sum_{n=0}^{n} P_m(x_n) = \lim_{h \longrightarrow 0} h \sum_{n=0}^{n} f(x_n).$$

This completes the theorem. $\square$

§3. Least Square Quadrature Method

Consider the integral in the form (1.2) for each $i = 0, 1, 2, \cdots, n$. Now we are dividing the interval $[a,b]$ into n (finite) equal sub interval and take the nodes $x's$ are equispaced points such that $x_i = x_0+ih \in [a,b]$, $i = 0,1,2,\cdots,n$, where $x_0 = a, x_n = b$ and $h = (b-a)/(n)$. So we have data points $(x_i, f(x_i))$ $i = 0,1,2,\cdots,n$ for fit a polynomial $P_m(x) = a_0+a_1x+a_2x^2+\cdots+a_mx^m$. we have

$$\begin{aligned}\int_{x_0}^{x_n} f(x)dx &= \int_{x_0}^{x_n} P_m(x)dx \\ &= a_0(x_n - x_0) + a_1\frac{x_n^2 - x_0^2}{2} + a_2\frac{x_n^3 - x_0^3}{3} + \cdots + a_m\frac{x_n^{m+1} - x_0^{m+1}}{m+1}. \qquad (3.1)\end{aligned}$$

This method is called L_m^n -Quadrature method($L_m^n - rule$), here m is donate degree of polynomial and n is donate number of data points. To solve the least square Quadrature method we have at least m+1 points. Order of this method is greater then or equal to m, since it's exact for polynomial of degree m. The error constant of (3.1) is

$$C = \int_{x_0}^{x_n} x^k - a_0 + \sum_{i=1}^{n} \frac{x_n^i - x_0^i}{i} a_i$$

and error

$$R = \frac{C}{k!} f^{(k)}(\xi),$$

where $k \geqslant m, a \leqslant \xi \leqslant b$. Now following cases arise:

Case 1. $m = 0$, that is P_0 is a constant function.

From (2.4) we have $a_0(n+1) = \sum_{i=0}^{n} f(x_i)$ and $a_1 = a_2 = \cdots = a_m = 0$, substituting this values in (9) web get

$$\int_{x_0}^{x_n} f(x)dx = \frac{(x_n - x_0)}{n+1} \sum_{i=1}^{n} f(x_i). \qquad (3.2)$$

Case 2. $m = 1$, that is P_1 is a linear polynomial.

From (2.4) we have

$$a_0(n+1) + a_1\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} f_i, a_0\sum_{i=0}^{n} x_i + a_1\sum_{i=0}^{n} x_i^2 = \sum_{i=0}^{n} x_i f_i$$

and $a_2 = a_3 = \cdots = a_m = 0$. Solving for a_1 and a_2 we get

$$a_0 = \frac{\sum_{i=0}^n f_i \sum_{i=0}^n x_i^2 - \sum_{i=0}^n x_i \sum_{i=0}^n x_i f_i}{(n+1)\sum_{i=0}^n x_i^2 - (\sum_{i=0}^n x_i)^2},$$

$$a_1 = \frac{(n+1)\sum_{i=0}^n x_i f_i - \sum_{i=0}^n x_i \sum_{i=0}^n f_i}{(n+1)\sum_{i=0}^n x_i^2 - (\sum_{i=0}^n x_i)^2}.$$

After simplification we get

$$a_0 = \frac{2}{nh(n+1)(n+2)} \left[n(3x_0 + h(n+1)) \sum_{i=0}^{n} f_i - 3(x_0 + x_n) \sum_{i=0}^{n} i f_i \right],$$

$$a_1 = \frac{6}{nh(n+1)(n+2)}\left[2\sum_{i=0}^{n} if_i - i\sum_{i=0}^{n} f_i\right].$$

Substituting this values in (3.1), and simplification we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{nh}{n+1}\sum_{i=1}^{n} f(x_i).$$

This is same as $m = 0$. The method (3.2) is called L_1^n- Quadrature method and the error constant of (3.2) is

$$C = \int_{x_0}^{x_n} x^2 dx - \frac{nh}{n+1}\sum_{i=0}^{n}(x+ih)^2 = \frac{-h^3n^2}{6} = \frac{-(x_n - x_a)^3}{6n} = -\frac{(b-a)^3}{6n}$$

and error of (3.2) is

$$R = \frac{-(b-a)^3}{6n\cdot 2!}f^{(2)}(\xi) = \frac{-(b-a)^3}{12n}f^{(2)}(\xi), \tag{3.3}$$

where $x_0 \leqslant \xi \leqslant x_n$. To solve this method, we have at least 2 data points and the order of (3.2) is 2.

Case 3. $m = 2$, that is P_2 is a polynomial of degree two.

From (2.4) we have

$$(n+1)a_0 + a_1\sum_{i=0}^{n} x_i + a_2\sum_{i=0}^{n} x_i^2 = \sum_{i=0}^{n} f_i = A,$$

$$a_0\sum_{i=0}^{n} x_i + a_1\sum_{i=0}^{n} x_i^2 + a_2\sum_{i=0}^{n} x_i^3 = \sum_{i=0}^{n}(x_0+ih)f_i = Ax_0 + hB,$$

$$a_0\sum_{i=0}^{n} x_i^2 + a_1\sum_{i=0}^{n} x_i^3 + a_2\sum_{i=0}^{n} x_i^4 = \sum_{i=0}^{n}(x_0+ih)^2 f_i = Ax_0^2 + 2Bhx_0 + Ch^2,$$

where $A = \sum_{i=0}^{n} f_i$, $B = \sum_{i=0}^{n} if_i$, and $C = \sum_{i=0}^{n} i^2 f_i$. we have $a_3 = a_4 = \cdots = a_m = 0$.

Solving the three linear system of equation for a_0, a_1 and a_2 by MATLAB, we get

$$\begin{aligned} a_0 \quad = \quad & \frac{3}{(n+1)(n^3+4n^2+n-6)h^2n} \\ & \times(3Ah^2n^4 + 12Ahn^3x_0 - 12Bh^2n^3 - Ah^2n^2 - 6Ahn^2x_0 + 10An^2x_0^2 \\ & +6Bh^2n^2 - 64Bhn^2x_0 + 10Ch^2n^2 - 2Ah^2n - 6Ahnx_0 - 10Anx_0^2 \\ & +6Bh^2n - 8Bhnx_0 - 60Bnx_0^2 - 10Ch^2n + 60Chnx_0 + 12Bhx_0 + 60Cx_0^2) \end{aligned}$$

$$\begin{aligned} a_1 \quad = \quad & -\{6(6Ahn^3 - 3Ahn^2 + 10An^2x - 32Bhn^2 - 3Ahn - 10Anx \\ & -4Bhn - 60Bnx + 30Chn + 6Bh + 60Cx)\}/h^2n(n^2+3n+2)(n^2+2n-3) \end{aligned}$$

and

$$a_2 = \frac{30(An^2 - An - 6Bn + 6C)}{h^2 n(n^4 + 5n^3 + 5n^2 - 5n - 6)}.$$

Substituting these values in (3.1), and simplification we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{hn(An^3 - An^2 + 6An + 30Bn - 6A - 30C)}{(n-1)(n+3)(n+2)(n+1)}.$$

Substituting A, B and C we get

$$\int_{x_0}^{x_n} f(x)dx = \frac{hn}{(n-1)(n+3)(n+2)(n+1)} \sum_{i=0}^{n} (n^3 - n^2 + 6n - 6 + 30ni - 30i^2) f_i. \tag{3.4}$$

This method is called L_2^n-Quadrature method. To solve this method, we have at least 3 data points.

Case 4. $m = 3$, that is P_3 is a polynomial of degree three.

Following previous case we get the same as (3.3). The error constant of (3.4) is

$$\begin{aligned} C &= \int_{x_0}^{x_n} x^4 dx - \frac{hn}{(n-1)(n+3)(n+2)(n+1)} \sum_{i=0}^{n} (n^3 - n^2 + 6n - 6 + 30ni - 30i^2)(x+ih)^4 \\ &= -\frac{(3n^2 - 8n + 18)n^2 h^5}{210} = -\frac{(3n^2 - 8n + 18)(x_n - x_0)^5}{210n^3} = -\frac{(3n^2 - 8n + 18)(b-a)^5}{210n^3}. \end{aligned}$$

The error of (3.4) is

$$R = -\frac{(3n^2 - 8n + 18)(b-a)^5}{210n^3 \cdot 4!} f^{(4)}(\xi), \tag{3.5}$$

where $a \leqslant \xi \leqslant b$. The order of (3.4) is 4.

Note 3.1 If $m \geqslant 0$ is even number then L_m^n method same as L_{m+1}^n method.

§4. Newton-Cotes Formulas from Least Square Method

We can derive trapezoidal rule, Simpson 1-3rd rule and Simpson 3-8th rule from least square method.

Taking $n = 1$ in (3.2) we get

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f_0 + f_1).$$

This formula is called trapezoidal rule. The error of trapezoidal rule is, from (3.3)

$$R = \frac{-(b-a)^3}{12} f^{(2)}(\xi), \quad a \leqslant \xi \leqslant b.$$

Taking $n = 2$ in (3.4) we get

$$\begin{aligned}\int_{x_0}^{x_2} f(x)dx &= \frac{2h}{1\cdot 5\cdot 4\cdot 3}\sum_{i=0}^{2}(10+60i-30i^2)f_i \\ &= \frac{h}{30}(10f_0+40f_1+10f_2) = \frac{h}{3}(f_0+4f_1+f_2).\end{aligned}$$

This formula is called Simpson 1-3rd rile rule. The error Simpson 1-3rd rule of is, from (3.5)

$$R = \frac{-(b-a)^5}{90}f^{(4)}(\xi), a \leqslant \xi \leqslant b.$$

Similarly, Simpson 3-8th rule come from (3.4) with $n = 3$, that is

$$\int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}(f_0+3f_1+3f_2+f_3)$$

and error come from (3.5), $R = -(3/80)h^5 f^{(4)}(\xi), a \leqslant \xi \leqslant b$.

The weights of the integration method of (3.4) with equispaced point for $n \le 6$ are given in Table 1.

n	comman ratio	Newton-Cotes weight	common ratio	L_2^n Method
1	1/2	1 1	—	—
2	1/3	1 4 1	1/3	1 4 1
3	3/8	1 3 3 1	3/8	1 3 3 1
4	2/45	7 32 12 32 7	4/105	11 26 31 26 11
5	5/288	19 75 50 50 75 19	5/336	31 61 78 78 61 31
6	1/140	41 216 27 272 27 216 41	1/14	7 12 15 16 15 12 7

Table 1. Weight of Newton-cote rules and Weights of L_2^n Quadrature Method

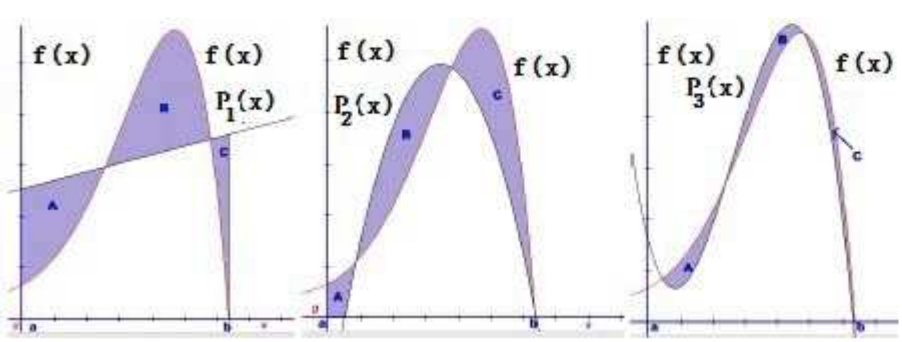

Figure 1 a, b, c

§5. Graphically Meaning of Least Square Integration Method

Let the polynomial $P_m(x)$ of degree m is fitted by least square interpolation method by using data points (x_i, f_i) $i = 0, 1, 2 \cdots, n,$. If m=1, take n is large number then the the polynomial $P_1(x)$ is going to exact fit polynomial such that the area A+C=B $(fib : 1(a))$. That's way the integration of $P_1(x)$ on $[a, b]$ is gives exact value of integration of $f(x)$ on $[a, b]$. Similarly $P_2(x)$ (or $P_3(x)$) is best interpolation polynomial such that the area A+C=B, such as those shown in Figure 1.

§6. Problems

Problem 6.1 Find approximate value of

$$I = \int_1^3 sin(x)e^x dx$$

fit a straight line $y(x)$ such that $\int_1^3 y(x)dx = I$.

Solution Let $f(x) = sin(x)e^x$ and y_n be the straight line by fit n+1 data points $(x_i, f(x_i))$, $i = 0, 1, 2, \cdots, n$. Now we divide the interval [1 3] into two equal subinterval, that is $n = 2$ or $h = 1$. then 3 data points are $(1, f(1))$, $(2, f(2))$ and $(3, f(3))$. we fit a straight line y_2 by normal equation (5) we get

$$y_2 = 0.27x + 3.4$$

following this we get

$$y_4 = 0.78x + 3.15,$$

$$y_8 = 1.17x + 2.77$$

$$y_{16} = 1.39x + 2.51$$

$$y_{32} = 1.51x + 2.36$$

and

$$y_{64} = 1.57x + 2.28.$$

But we know if $n \to \infty$ then $\int_1^3 y_n(x)dx \to \int_1^3 f(x)dx$. Therefore, $I = \int_1^3 (1.57x + 2.28)dx = 10.84$.

Problem 6.2 Fit quadratic equation $P_2(x)$ such that

$$\int_0^1 P_2(x)dx = \int_0^1 x\sqrt{x+1}dx$$

and find approximate value of $\int_0^1 x\sqrt{x+1}dx$.

Solution Let P_{2_n} be the quadratic equation by fit n equal space data points in [0, 1]. By least square method we have

$$\begin{aligned} P_{2_3}(x) &= 0.37893738x^2 + 1.03527618x + 3.61400724(E-20), \\ P_{2_{11}}(x) &= 0.37892845x^2 + 1.03956285x - 0.00227848, \\ P_{2_{51}}(x) &= 0.37839273x^2 + 1.04141576x - 0.00304322, \\ P_{2_{101}}(x) &= 0.3783134x^2 + 1.0416701x - 0.00314653. \end{aligned}$$

Let $I_n = \int_0^1 P_{2_n}(x)dx$ then $I_3 = 0.643950551$, $I_{11} = 0.643812428$, $I_{51} = 0.643795564$ and $I_{101} = 0.643792992$. The exact value of $\int_0^1 x\sqrt{x+1}dx$ upto five decimal is 0.64379.

Problem 6.3 Find the approximate value of

$$I = \int_0^1 \frac{1}{2+x}dx,$$

using L_1^n and L_2^n rules with different equal subintervals. Using the exact solution, find the absolute errors.

Solution Results for the L_1^n and L_2^n rules to estimate the integral of $f(x) = 1/(2+x)$ from x = 0 to 1. The exact value is $I_{exact} = \int_0^1 1/(2+x)dx = ln(x+2)]_0^1 = ln(3) - ln(2)$=0.4054651. We get

n	$I_1^n = L_1^n$ method	Error=$I_1^n - I_{exact}$
1	0.4167	0.0112
2	0.4111	0.0056
4	0.4083	0.0028
8	0.4069	0.0014
16	0.4062	0.0007
32	0.4058	0.0003
64	0.4056	0.0001

n	$I_2^n = L_2^n$ method	Error=$I_2^n - I_{exact}$
2	0.4055556	0.0000905
4	0.4054930	0.0000279
8	0.4054801	0.0000150
16	0.4054735	0.0000084
32	0.4054696	0.0000045
64	0.4054675	0.0000024
128	0.4054663	0.0000012

§7. Conclusion

We develop this new method for easy to solve Definite Integral of finite interval with equispaced nodes and derived Simpson 1/3rd rule and Simpson 3/8th rule from L_2^n Quadrature Method. In this method (L_2^n) weights are increasing from a to midpoint(i.e $(a+b)/2$) of interval and decreasing from midpoint to b. The advances is the weights of $L_2^n - method$ are positive (since($n^3 - n^2 + 6n - 6 + 30ni - 30i^2) \geq 0$ for all $n \geq 2$ for all i) . We have given the MATLAB code also, give any continuous function $f(x)$ on $[a, b]$ that will be give an approximation integration value of $f(x)$ from a to b. Also, we are developing this concept to high degree polynomials and high dimension.

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Blaschke Approach to the Motion of a Robot End-Effector

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Abstract: In this paper, we examine the motion of a robot end-effector by using the Blaschke approach of a ruled surface generated by a line fixed in the robot end-effector. In this way, we determine time dependent linear and angular differential properties of motion such as velocity and acceleration which play important roles in robot trajectory planning. Moreover, motion of a robot end-effector which can be represented by a right conoid and an additional parameter called spin angle is investigated as a practical example.

Key Words: Blaschke frame, curvature theory, robot end-effector, robot trajectory planning, ruled surface.

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§1. Introduction

In robotics, a robot end-effector is a device at the end of a robotic arm. Robot end-effectors are widely used in transportation, welding industry, medical science, military and many other areas. Recently, they can be used in the research areas which have critical importance of accurate motion such as surgical operations and bomb disposal. So accurate trajectory planning of a robot end-effector becomes an important research area of robotics. In this area, one of the most interesting problems is determining time dependent differential properties of motion of a robot end-effector which are linear and angular velocities and accelerations. These differential properties play important roles in robot trajectory planning.

As a robot end-effector moves on a specified trajectory in space, a line fixed in the end-effector generates a ruled surface [13]. There is an important relationship between time dependent properties of motion of the robot end-effector and differential geometry of the ruled surface. By using this relationship, Ryuh and Pennock proposed a method based on the curvature theory of a ruled surface generated by a line fixed in the end-effector to determine linear and angular properties of motion [12, 13, 14]. After that, this research area was also studied in Lorentzian space. Ekici et al. examined motion of a robot end-effector in Lorentzian space by using the curvature theory of timelike ruled surface with timelike ruling [7]. Ayyıldız and Turhan also

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determined differential properties of motion of a robot end-effector whose trajectory is a null curve [3].

On the other hand, there is also an efficient relationship between directed lines and dual unit vectors. This relationship known as "E. Study mapping" or "transference principle" which can be stated as: "there exists one-to-one correspondence between the directed lines in line space and dual unit vectors in dual space" [11, 16]. By the aid of this correspondence, W. Blaschke defined a frame called Blaschke frame on a ruled surface by taking directed lines pass through striction curve of the ruled surface instead of real unit vectors used in Frenet frame of ruled surface. He also gave some invariants which characterize the shape of a ruled surface. Several authors used Blaschke frame in their researches concerning with kinematics, spatial mechanisms and many other areas [1, 2, 18].

In this paper, we use the relationships between kinematic, ruled surfaces and dual vector algebra. First, we represent motion of a robot end-effector on a specified trajectory in space as a ruled surface generated by a line fixed in the end-effector and an additional parameter called spin angle. We define a dual frame called dual tool frame on robot end-effector in order to obtain a relationship between Blaschke frame of the ruled surface, which is used to study the differential geometry of a ruled surface by means of dual quantities, and time dependent differential properties of robot end-effector. By using this relation, we determine time dependent differential properties of motion of a robot end-effector which are linear (translational) and angular (rotational) velocities and accelerations. These differential properties have important roles in robot trajectory planning. In this method, we use just a dual vector called dual instantaneous rotation vector of dual tool frame to determine the differential properties. So, this method has more advantages than traditional methods which based on matrix representations in terms of being simple and systematic.

§2. Preliminaries

In this section, we give a brief summary of basic concepts for the reader who is not familiar with dual numbers, dual vectors and dual space.

As introduced by W. Clifford, a dual number can be defined as $\bar{a} = a + \varepsilon a^*$, where a and a^* are real numbers and called real part and dual part of dual number $\bar{a}$, respectively, and ε is dual unit which satisfies the condition $\varepsilon^2 = 0$, [17]. The set of all dual numbers can be denoted by $\mathbb{D}$. Addition and multiplication of two dual numbers $\bar{a} = a + \varepsilon a^*$ and $\bar{b} = b + \varepsilon b^*$ can be defined as

$$\bar{a} + \bar{b} = (a + b) + \varepsilon(a^* + b^*)$$

and

$$\bar{a}\,\bar{b} = ab + \varepsilon(ab^* + a^*b)$$

respectively [4, 10]. The set $\mathbb{D}$ is a commutative ring, not a field. A function of a dual number $f(\bar{a})$ can be expanded in a Maclaurin series as

$$f(\bar{a}) = f(a + \varepsilon a^*) = f(a) + \varepsilon a^* f'(a),$$

where the prime indicates derivation of $f(a)$ with respect to a [5].

A dual vector can also be defined as $\tilde{a}= a+\varepsilon a^*$, where a and a^* are three dimensional vectors in real space and $\varepsilon^2 = 0$. The set of all dual vectors is a module over the ring $\mathbb{D}$ and is called dual space or $\mathbb{D}$-module, denoted by $\mathbb{D}^3$, [15]. Dual scalar and vector products of two dual vectors $\tilde{a}= a+\varepsilon a^*$ and $\tilde{b}= b+\varepsilon b^*$ can be defined as

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \langle a, b\rangle + \varepsilon \left(\langle a, b^*\rangle + \langle a^*, b\rangle \right)$$

and

$$\tilde{a} \times \tilde{b} = a \times b + \varepsilon \left(a \times b^* + a^* \times b \right)$$

respectively [16]. The norm of a dual vector $\tilde{a}$ can also be given by [10, 17]

$$\|\tilde{a}\| = \|a\| + \varepsilon \frac{\langle a, a^*\rangle}{\|a\|}, \quad (a \neq 0).$$

If $\|\tilde{a}\| = 1$, then $\tilde{a}$ is called a dual unit vector. The set

$$S^2 = \left\{ \tilde{a}= a+\varepsilon a^* \mid \ \|\tilde{a}\| = 1; \ \ a, a^* \in \mathbb{R}^3 \right\}$$

is called dual unit sphere.

Theorem 2.1([8]) *The set of all directed straight lines in $\mathbb{R}^3$ are in one-to-one correspondence with the set of all points of the dual unit sphere in $\mathbb{D}^3$.*

A dual angle between two oriented lines in three dimensional real space can be defined as $\bar{\theta} = \theta + \varepsilon\ \theta^*$, where θ and θ^* are the real angle and the shortest distance between these lines, respectively, [4].

§3. A Robot End-Effector and its Dual Tool Frame

In this section, we introduce tool frame of a robot end-effector which consists of three mutually perpendicular unit vectors described by Ryuh and Pennock [13] in detail. Then, we represent motion of a robot end-effector by using a ruled surface generated by a line fixed in the end-effector and an additional parameter called spin angle. By taking three lines instead of three unit vectors, we define a dual frame called dual tool frame on robot end-effector which will be used to study the motion.

The tool frame consists of three orthogonal unit vectors strictly attached to robot end-effector. These are; orientation vector O which is a unit vector in the direction of the gripper motion as it opens and closes, approach vector A which is a unit vector in the direction normal to the palm of robot end-effector, and normal vector N which is a unit vector in the direction perpendicular to the plane of the gripper (see Figure 1), [12]. The origin of the tool frame is called tool center point and denoted by TCP. By using tool frame and tool center point, location and orientation of a robot end-effector can be described completely.

As a robot end-effector moves on a specified trajectory in space, a line called tool line fixed in the end-effector which passes through TCP and whose direction vector is parallel to the orientation vector O generates a ruled surface [12]. This ruled surface can be expressed as

$$X(t, v) = \alpha(t) + v\, u(t),$$

where α is the specified trajectory which robot end-effector follows (directrix of the ruled surface), u is a unit vector called ruling parallel to the orientation vector O, t is the parameter of time, and v is an arbitrary parameter.

During motion, the approach vector A may not be always perpendicular to the ruled surface. As seen in Figure 1, there may be an angle between the approach vector A and the surface normal vector on the directrix which is denoted by S_n. This angle is called spin angle and denoted by η [12]. Thus, a robot end-effector motion which has six degrees of freedom in space can be completely described by a ruled surface generated by a line in robot end-effector which provides five independent parameters and a spin angle.

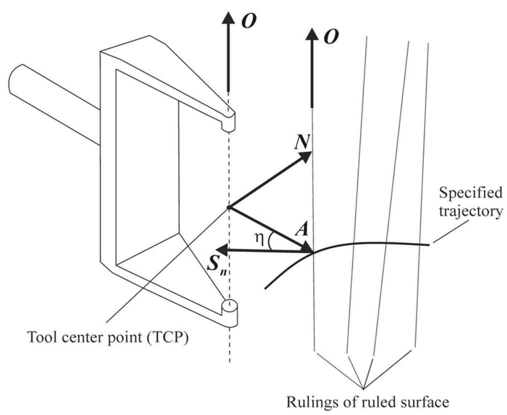

Figure 1 Robot end-effector and spin angle

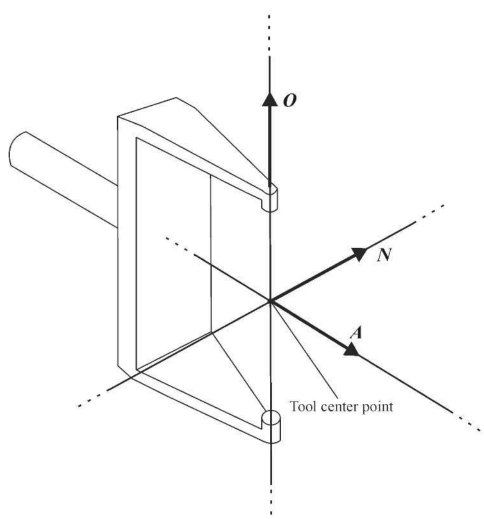

Figure 2 Dual tool frame of a robot end-effector

Now, we define dual tool frame by taking three directed lines instead of three unit vectors of the tool frame. These lines pass through the TCP and their direction vectors are the orientation vector O, the approach vector A and the normal vector N, respectively. From Theorem 2.1, these lines correspond to three dual unit vector which can be called dual orientation vector, dual approach vector and dual normal vector and can be denoted by $\tilde{O}$, $\tilde{A}$ and $\tilde{N}$, respectively (see Figure 2).

§4. Blaschke Approach to the Motion

In this section, we give Blaschke frame of a ruled surface generated by a line fixed in the robot end-effector. By relating Blaschke frame and dual tool frame, we determine linear and angular differential properties of motion. Furthermore, we give corollaries for some special cases of motion.

From Theorem 2.1, it can be said that a ruled surface can be represented by a dual unit vector based on a real parameter. So, we can consider the ruled surface generated by motion of robot end-effector as a dual unit vector $\tilde{u}(t) = u(t) + \varepsilon u^*(t)$, where u is ruling of the ruled surface, u^* is moment vector of u about the origin, t is the parameter of time, and $\varepsilon^2 = 0$. The moment vector can be found as $u^* = c \times u$, where c is striction curve of the ruled surface satisfies the condition that $\langle c', u' \rangle = 0$, [6]. In this paper, we consider the case without $u(t) = c_1$ which means ruled surface is a cylinder and $u^*(t) = c_1$ which means ruled surface is a cone, where c_1 is a constant. In order to simplify formulations, arc-length parameter of the striction curve denoted by s can be used instead of the parameter of time t and it can be obtained as

$$s(t) = \int_0^t \left\| \frac{dc}{dt} \right\| dt.$$

The Blaschke frame of a ruled surface is defined on striction curve and it consists of three orthogonal dual unit vectors given as follows [4]:

$$\tilde{u}_1 = \tilde{u} \ , \ \ \tilde{u}_2 = \frac{\tilde{u}_1'}{\bar{p}} \ , \ \ \tilde{u}_3 = \tilde{u}_1 \times \tilde{u}_2,$$

where $\bar{p} = p + \varepsilon p^* = \|\tilde{u}_1'\|$, $\tilde{u}_2$ and $\tilde{u}_3$ are normal line and tangent line of the ruled surface on the striction curve, respectively, and the prime indicates the derivation with respect to s, [4]. The derivative formulae of Blaschke frame can be given as

$$\frac{d}{ds}\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix} = \begin{bmatrix} 0 & \bar{p} & 0 \\ -\bar{p} & 0 & \bar{q} \\ 0 & -\bar{q} & 0 \end{bmatrix}\begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix}, \tag{1}$$

where $\bar{q} = q + \varepsilon q^* = \dfrac{\det(\tilde{u}_1, \tilde{u'}_1, \tilde{u''}_1)}{\|\tilde{u}_1'\|^2}$. $\bar{p}$ and $\bar{q}$ which are called the Blaschke's invariants characterize the shape of a ruled surface. If $p = 0$, ruled surface is a cylinder; if $p^* = 0$, ruled surface is a developable ruled surface which is a surface that can be flattened onto a plane without distortion; if $q = 0$, all rulings of ruled surface are parallel to a plane; if $\bar{q} = 0$, ruled surface consists of binormal vectors of a curve, [4].

Let $\bar{\varphi} = \varphi + \varepsilon\, \varphi^*$ be a dual angle between dual unit vectors $\tilde{A}$ and $\tilde{u}_2$, where $\varphi = \eta + \sigma$ is real angle, where η is the spin angle mentioned in Section 3 and σ is an angle between two normal vectors of ruled surface, one is on the directrix and other is on the striction curve, and φ^* is the shortest distance from striction curve to directrix, i.e., $\varphi^* = \dfrac{\langle \alpha', u' \rangle}{\|u'\|^2}$ (see Figure 3).

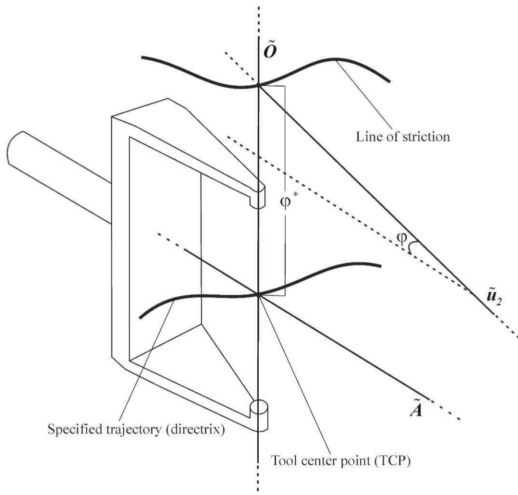

Figure 3 Dual angle between the dual unit vectors $\tilde{A}$ and $\tilde{u}_2$

By the aid of dual angle $\bar{\varphi}$, we can give dual tool frame relative to Blaschke frame in matrix form as

$$\begin{bmatrix} \tilde{O} \\ \tilde{A} \\ \tilde{N} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\bar{\varphi} & \sin\bar{\varphi} \\ 0 & -\sin\bar{\varphi} & \cos\bar{\varphi} \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix}. \tag{2}$$

By differentiating equation (2) and substituting equation (1) into the result, we have

$$\begin{bmatrix} \tilde{O}' \\ \tilde{A}' \\ \tilde{N}' \end{bmatrix} = \begin{bmatrix} 0 & \bar{p} & 0 \\ -\bar{p}\cos\bar{\varphi} & -\bar{\delta}\sin\bar{\varphi} & \bar{\delta}\cos\bar{\varphi} \\ \bar{p}\sin\bar{\varphi} & -\bar{\delta}\cos\bar{\varphi} & -\bar{\delta}\sin\bar{\varphi} \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix},$$

where $\bar{\delta} = \bar{\varphi}' + \bar{q}$. By using equation (2), derivative formulas of the dual tool frame can be obtained in terms of itself in matrix form as

$$\begin{bmatrix} \tilde{O}' \\ \tilde{A}' \\ \tilde{N}' \end{bmatrix} = \begin{bmatrix} 0 & \bar{p}\cos\bar{\varphi} & -\bar{p}\sin\bar{\varphi} \\ -\bar{p}\cos\bar{\varphi} & 0 & \bar{\delta} \\ \bar{p}\sin\bar{\varphi} & -\bar{\delta} & 0 \end{bmatrix} \begin{bmatrix} \tilde{O} \\ \tilde{A} \\ \tilde{N} \end{bmatrix}.$$

From the above matrix equality, dual instantaneous rotation vector of the dual tool frame which plays an important role to determine both linear and angular differential properties of motion of a robot end-effector can be obtained as

$$\tilde{w}_O = \bar{\delta}\ \tilde{O} + \bar{p}\sin\bar{\varphi}\ \tilde{A} + \bar{p}\cos\bar{\varphi}\ \tilde{N}.$$

By using equation (2), the dual instantaneous rotation vector can also be expressed in terms of the Blaschke frame as

$$\tilde{w}_O = \bar{\delta}\ \tilde{u}_1 + \bar{p}\ \tilde{u}_3. \tag{3}$$

This dual vector is similar to dual Pfaff vector in terms of playing role in motion. The dual Pfaff vector is considered as dual velocity vector in dual spherical motion (see ref. [9]). So, we can consider the dual instantaneous rotation vector of dual tool frame $\tilde{w}_O$ as dual velocity vector of the motion of robot end-effector.

The dual tool frame attached to robot end-effector moves along unit direction $\dfrac{\tilde{w}_O}{\|\tilde{w}_O\|}$ with dual angle $\|\tilde{w}_O\|$. This dual motion contains both rotational and translational motion in real space. The real and dual parts of the dual vector $\tilde{w}_O$ correspond to instantaneous angular velocity and instantaneous linear velocity, respectively. By separating equation (3) into the real and dual parts, these velocity vectors can be found as follows

$$w_O = \delta\ u_1 + p\ u_3, \tag{4}$$

and

$$w_O^* = \delta u_1^* + \delta^* u_1 + p u_3^* + p^* u_3. \tag{5}$$

In order to find dual acceleration vector of the motion, we should differentiate dual velocity vector. By differentiating equation (3) and using equation (1), the dual acceleration vector can be obtained in terms of the Blaschke frame as

$$\tilde{w}_O^{'} = \bar{\delta}'\ \tilde{u}_1 + \bar{\varphi}'\bar{p}\ \tilde{u}_2 + \bar{p}'\tilde{u}_3, \tag{6}$$

where the prime indicates differentiation with respect to s. By separating equation (6) into the real and dual parts, instantaneous angular acceleration vector and instantaneous linear acceleration vector can be found as

$$w_O^{'} = \delta'\ u_1 + \varphi' p\ u_2 + p' u_3 \tag{7}$$

and

$$w_O^{*'} = \delta'\ u_1^* + \delta^{*'} u_1 + \varphi' p\ u_2^* + (\varphi' p^* + \varphi^{*'} p) u_2 + p' u_3^* + p^{*'} u_3, \tag{8}$$

respectively. Thus, linear and angular velocities and accelerations which are important differential properties of motion of a robot end-effector are found in terms of the parameter s which is the arc-length parameter of striction curve of the generating ruled surface. In order to determine time dependent differential properties, the vectors given in equations (4), (5), (7) and (8) should be related to the parameter of time. Now, we give time dependent linear and angular differential properties of motion of a robot end-effector as corollaries.

Corollary 4.1 *Let the motion of a robot end-effector be represented by a ruled surface* $X(t,v) = \alpha(t) + v\ u(t)$ *and a spin angle* η*, where* α *is specified trajectory of robot end-effector,* u *is a unit vector parallel to the orientation vector* O*, and* t *is the parameter of time. Angular and linear velocities of robot end-effector can be given, respectively, as*

$$v_A = w_O\ \dot{s} \tag{9}$$

and

$$v_L = w_O^*\ \dot{s}, \tag{10}$$

where w_O *and* w_O^* *are given by equations (4) and (5), respectively, and the dot indicates differentiation with respect to time, i.e.,* $\dot{s} = \dfrac{ds}{dt}$.

Corollary 4.2 *Let the motion of a robot end-effector be represented by a ruled surface* $X(t,v) = \alpha(t) + v\ u(t)$ *and a spin angle* η*, where* α *is specified trajectory of robot end-effector,* u *is a unit vector parallel to the orientation vector* O*, and* t *is the parameter of time. Angular and linear accelerations of the robot end-effector can be given, respectively, as*

$$a_A = w_O\ \ddot{s} + w_O^{'}\dot{s}^2 \tag{11}$$

and

$$a_L = w_O^*\ \ddot{s} + w_O^{*'}\dot{s}^2, \tag{12}$$

where $w_O^{'}$ *and* $w_O^{*'}$ *are as given by equations (7) and (8), respectively.*

Now, we consider some special cases of motion of a robot end-effector and give some corollaries about these cases.

Case 1. As a robot end-effector moves on a specified trajectory in real space, spin angle η may be constant. Then, the derivative of the spin angle is equal to zero. For this case, by substituting the value of spin angle into equations (4), (5), (7), and (8), and by rearranging these equations, we can give time dependent linear and angular differential properties of the motion of a robot end-effector as in the following corollaries.

Corollary 4.3 *Let the motion of a robot end-effector be represented by a ruled surface* $X(t,v) = \alpha(t) + v\ u(t)$ *and a spin angle* η*, where* α *is specified trajectory of the robot end-effector,* u *is a unit vector parallel to the orientation vector* O*, and* t *is the parameter of time. If the spin angle* η *is a constant, then angular and linear velocities of robot end-effector can be given as*

$$v_A = ((\sigma' + q)\ u_1 + p\ u_3)\ \dot{s}$$

and

$$v_L = ((\sigma' + q)u_1^* + \delta^* u_1 + p u_3^* + p^* u_3)\ \dot{s},$$

respectively.

Corollary 4.4 *Let the motion of a robot end-effector be represented by a ruled surface* $X(t,v) =$

$\alpha(t) + v\ u(t)$ and a spin angle η, where α is the specified trajectory of the robot end-effector, u is a unit vector parallel to the orientation vector O, and t is the parameter of time. If the spin angle η is constant, then angular and linear accelerations of robot end-effector can be respectively given as

$$\begin{aligned} a_A &= ((\sigma' + q)\ u_1 + p\ u_3)\ \ddot{s} + ((\sigma'' + q')\ u_1 + \sigma' p u_2 + p' u_3)\ \dot{s}^2, \\ a_L &= ((\sigma' + q) u_1^* + \delta^* u_1 + p u_3^* + p^* u_3)\ \ddot{s} \\ &\quad + ((\sigma'' + q')\ u_1^* + \delta^{*'} u_1 + \sigma' p\ u_2^* + (\sigma' p^* + \varphi^{*'} p) u_2 + p' u_3^* + p^{*'} u_3) \dot{s}^2. \end{aligned}$$

Case 2. A specified trajectory which robot end-effector follows may be striction curve of ruled surface generated by a line fixed in the robot end-effector. Namely, directrix and striction curve of generating ruled surface may be the same curve. Then, the angle σ which is the angle between two normal vectors on directrix and striction curve and the distance between these curves are equal to zero. For this case, by rearranging equations (4), (5), (7), and (8), we can give time dependent linear and angular differential properties of the motion of a robot end-effector as in the following corollaries.

Corollary 4.5 *Let the motion of a robot end-effector be represented by a ruled surface $X(t, v) = \alpha(t) + v\ u(t)$ and a spin angle η, where α is specified trajectory of the robot end-effector, u is a unit vector parallel to the orientation vector O, and t is the parameter of time. If the specified trajectory is also the striction curve of the ruled surface, then angular and linear velocities of robot end-effector can be given as*

$$v_A = ((\eta' + q)\ u_1 + p\ u_3)\ \dot{s}$$

and

$$v_L = ((\eta' + q) u_1^* + q^* u_1 + p u_3^* + p^* u_3)\ \dot{s},$$

respectively.

Corollary 4.6 *Let the motion of a robot end-effector be represented by a ruled surface $X(t, v) = \alpha(t) + v\ u(t)$ and a spin angle η, where α is specified trajectory of robot end-effector, u is a unit vector parallel to the orientation vector O, and t is the parameter of time. If the specified trajectory is also the striction curve of the ruled surface, then angular and linear accelerations of robot end-effector can be given as*

$$a_A = ((\eta' + q)\ u_1 + p\ u_3)\ \ddot{s} + ((\eta'' + q')\ u_1 + \eta' p u_2 + p' u_3)\ \dot{s}^2$$

and

$$\begin{aligned} a_L &= ((\eta' + q) u_1^* + q^* u_1 + p u_3^* + p^* u_3)\ \ddot{s} \\ &\quad + ((\eta'' + q')\ u_1^* + q^{*'} u_1 + \eta' p\ u_2^* + \eta' p^* u_2 + p' u_3^* + p^{*'} u_3) \dot{s}^2, \end{aligned}$$

respectively.

Case 3. Ruled surface generated by a line fixed in a robot end-effector may be a developable ruled surface (except a cylinder and a cone). So, the dual part of Blaschke's invariant $\bar{p}$ is equal to zero, i.e., $p^* = 0$. For this case, by making the necessary arrangement in equations (4), (5), (7), and (8), we can give time dependent linear and angular differential properties of the motion of a robot end-effector as in the following corollaries.

Corollary 4.7 *Let the motion of a robot end-effector be represented by a ruled surface $X(t,v) = \alpha(t) + v\ u(t)$ and a spin angle η, where α is specified trajectory of robot end-effector, u is a unit vector parallel to the orientation vector O, and t is the parameter of time. If the ruled surface is developable, then angular and linear velocities of robot end-effector can be given as*

$$v_A = ((\eta' + q)\ u_1 + p\ u_3)\ \dot{s}$$

and

$$v_L = ((\eta' + q)u_1^* + \delta^* u_1 + pu_3^*)\ \dot{s},$$

respectively.

Corollary 4.8 *Let the motion of a robot end-effector be represented by a ruled surface $X(t,v) = \alpha(t) + v\ u(t)$ and a spin angle η, where α is specified trajectory of robot end-effector, u is a unit vector parallel to the orientation vector O, and t is the parameter of time. If the ruled surface is a developable, then angular and linear accelerations of the robot end-effector can be given as*

$$a_A = ((\eta' + q)\ u_1 + p\ u_3)\ \ddot{s} + ((\eta'' + q')\ u_1 + \eta' p u_2 + p' u_3)\ \dot{s}^2$$

and

$$a_L = ((\eta' + q)u_1^* + \delta^* u_1 + pu_3^*)\ \ddot{s} + ((\eta'' + q')\ u_1^* + {\delta^*}'u_1 + \eta' p\ u_2^* + {\varphi^*}'\ p\ u_2 + p' u_3^*)\dot{s}^2,$$

respectively.

§5. An Example

Let the motion of a robot end-effector be represented a right conoid given by the equation $X(t,v) = (v\cos t,\ v\sin t,\ 2\sin t)$ and a spin angle η, where t is the parameter of time (see Figure 4). Directrix and ruling of the right conoid are $\alpha(t) = (0,\ 0,\ 2\sin t)$ and $u(t) = (\cos t,\ \sin t,\ 0)$, respectively. Since $\langle \alpha', u' \rangle = 0$, directrix and striction curve of the ruled surface are the same curve, i.e., $c = \alpha$. The right conoid can be expressed as a dual unit vector

$$\tilde{u}(s) = u(s) + \varepsilon u^*(s) = (\cos t,\ \sin t,\ 0) + \varepsilon(-2\sin^2 t,\ \sin 2t,\ 0)$$

where s is the arc-length parameter of striction curve. The first dual unit vector of Blaschke frame is $\tilde{u}_1(s) = \tilde{u}(s)$. The second and third dual unit vectors of Blaschke frame can be found as

$$\tilde{u}_2(s) = (-\sin t,\ \cos t,\ 0) + \varepsilon(-\sin 2t,\ -2\sin^2 t,\ 0)$$

and

$$\tilde{u}_3(s) = (0,\ 0,\ 1)\,,$$

respectively. The Blaschke's invariants can be obtained as $\bar{p} = p + \varepsilon p^* = 1 + \varepsilon 2\cos t$ and $\bar{q} = q + \varepsilon q^* = 0 + \varepsilon 0$. Let $\bar{\varphi} = \varphi + \varepsilon\varphi^*$ be a dual angle between dual unit vectors $\tilde{A}$ and $\tilde{u}_2$, where φ and φ^* are the real angle and the shortest distance between the lines correspond to the dual vectors $\tilde{A}$ and $\tilde{u}_2$, respectively. Since directrix is also striction curve, the distance between these curves equals to zero, i.e., $\varphi^* = 0$, and the angle between two normal vectors on directrix and on striction curve equals to zero, i.e., $\sigma = 0$. Thus, we have $\bar{\varphi} = \eta + \varepsilon 0$. Dual instantaneous rotation vector of dual tool frame can be found as

$$\tilde{w}_O = w_O + \varepsilon w_O^* = (\eta' \cos s,\ \eta' \sin s,\ 1) + \varepsilon(-2\eta' \sin^2 s,\ \eta' \sin 2s,\ 2\cos s).$$

Angular and linear velocities of the robot end-effector can be obtained by substituting w_O and w_O^* into equations (9) and (10), respectively. By differentiating the dual instantaneous rotation vector, we get

$$\begin{aligned}\tilde{w}_O' = w_O^{'} + \varepsilon w_O^{*'} \quad = \quad & (\eta'' \cos s - \eta' \sin s,\ \eta'' \sin s + \eta' \cos s)\\ & + \varepsilon(-2\eta'' \sin^2 s - 2\eta' \sin 2s,\ \eta'' \sin 2s + 2\eta' \cos 2s,\ -2\sin s).\end{aligned}$$

Angular and linear accelerations of the robot end-effector can also be obtained by substituting $w_O^{'}$ and $w_O^{*'}$ into equations (11) and (12), respectively.

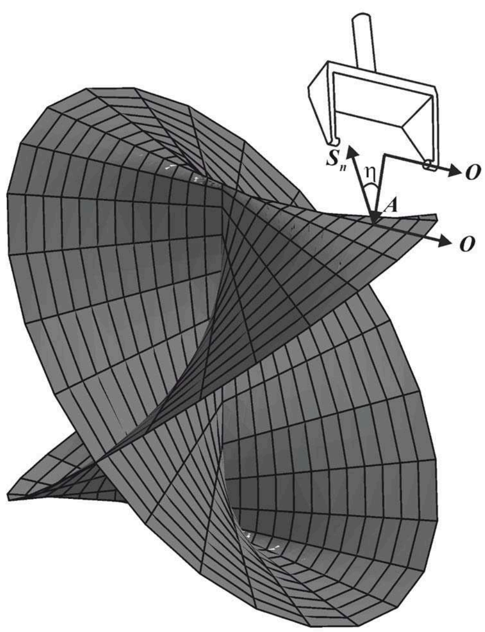

Figure 4 Motion of a robot end-effector
which can be represented by a right conoid and a spin angle η

§6. Conclusions

In this paper, time dependent differential properties which are linear and angular velocities and accelerations of the motion of a robot end-effector are determined by using Blaschke approach of a ruled surface generated by a line fixed in the end-effector. These differential properties are important information in robot trajectory planning. By the aid of Blaschke approach which uses dual numbers and dual vectors as basic tool, both linear and angular differential properties can be determined. This is achieved only by using a dual vector which is dual instantaneous rotation vector of dual tool frame. Thus, Blaschke approach presents a simple and systematic method to study motion of a robot end-effector without redundant parameter. This paper does not contain a computer program which compares Blaschke approach and conventional method of scalar curvature theory of ruled surfaces in real space. This is the subject of ongoing research works. However, it is believed that the presented method based on Blaschke approach will reduce computation time in computer programming for determining differential properties of motion and contribute to research area of robot trajectory planning.

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Domination Stable Graphs

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Abstract: In this paper, we study the domination polynomials of some graph and its square. We discuss nonzero real domination roots of these graphs. We also investigate whether all the domination roots of some graphs lying left half plane or not.

Key Words: Dominating set, Smarandachely k-dominating set, domination number, domination polynomial, domination root, d-number, stable.

AMS(2010): 05C25.

§1. Introduction

Let $G(V,E)$ be a simple finite graph. The order of G is the number of vertices of G. A set $S \subseteq V$ is a dominating set if every vertex $v \in V - S$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of the dominating sets in G. Generally, a dominating set S is said to be a *Smarandachely k-dominating set* if each vertex of G is dominated by at least k vertices of S. Let $\mathcal{D}(G,i)$ be the family of dominating sets of G with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$. The polynomial

$$D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$$

is defined as domination polynomial of G. For more information on this polynomial the reader may refer to [8]. A root of $D(G,x)$ is called a domination root of G. It is easy to see that the domination polynomial is monic with no constant term. Consequently, 0 is a root of every domination polynomial (in fact, 0 is a root whose multiplicity is the domination number of the graph).

§2. d-Number

In this section we mainly focus on the number of real domination roots of some specific graphs. So we introduce a new definition as follows.

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Definition 2.1 *Let G be a graph. The number of distinct real domination roots of the graph G is called* $\mathtt{d}$*-number of G and is denoted by* $\mathtt{d}(G)$.

Theorem 2.1 *For any graph G,* $\mathtt{d}(G) \geq 1$.

Proof It follows from the fact that 0 is a domination root of any graph. □

Theorem 2.1 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then*

$$\mathtt{d}(G) \leq \sum_{i=1}^{m} \mathtt{d}(G_i) - m + 1.$$

Proof It follows from the fact that $D(G, x) = \prod_{i=1}^{m} D(G_i, x)$. □

Theorem 2.3 *If G and H are isomorphic, then* $\mathtt{d}(G) = \mathtt{d}(H)$.

Proof It follows from the fact that if G and H are isomorphic, then $D(G, x) = D(H, x)$. □

Theorem 2.4 *If G has exactly two distinct domination roots, then* $\mathtt{d}(G) = 2$.

Proof It follows from the fact that 0 is a domination root and complex roots occurs in conjugate pairs. □

Theorem 2.5 *Let G be a graph without pendent vertices. If G has exactly three distinct domination roots, then* $\mathtt{d}(G) = 1$.

Proof It follows from the fact that with the given conditions in theorem, $\mathbb{Z}(D(G, x)) \subseteq \{0, -2 \pm i\sqrt{2}, \frac{-3 \pm i\sqrt{3}}{2}\}$ ([8]). □

Theorem 2.6 *For all n we have the following :*

$$\mathtt{d}(K_n) = \begin{cases} 1 & ;\ \text{if } n \text{ is odd}, \\ 2 & ;\ \text{if } n \text{ is even}. \end{cases}$$

Proof We have known the domination polynomial of K_n is

$$D(K_n, x) = (1 + x)^n - 1. \tag{1}$$

The result follows from the transformation $y = 1 + x$ in equation (1). □

Theorem 2.7 *For any graph G,* $\mathtt{d}(G \circ K_1) = 2$.

Proof Notice that $D(G \circ K_1, x) = x^n (x + 2)^n$ ([8]), where n is the order of G. Therefore $\mathtt{d}(G \circ K_1) = 2$. □

Theorem 2.8 *For any graph G,* $\mathtt{d}(G \circ \overline{K_2}) = 3$.

Proof Notice that $D(G \circ \overline{K_2}, x) = x^{\frac{n}{3}}(x^2 + 3x + 1)^{\frac{n}{3}}$ ([8]), where n is the order of G. Therefore $\mathbb{Z}(D(G, x)) = \{0, \frac{-3\pm\sqrt{5}}{2}\}$. This implies that $\mathtt{d}(G \circ \overline{K_2}) = 3$. □

Theorem 2.9 *For all n the* $\mathtt{d}$*-number of the star graph S_n is*

$$\mathtt{d}(S_n) = \begin{cases} 2 & ; \textit{if } n \textit{ is odd,} \\ 3 & ; \textit{if } n \textit{ is even.} \end{cases}$$

Proof We have known the domination polynomial of S_n is

$$D(S_n, x) = x(1 + x)^n + x^n. \tag{2}$$

Therefore it suffices to prove that $f(x) = (1 + x)^n + x^{n-1}$ has exactly one real root if n is odd and two real roots if n is even. But the number of real roots of $f(x)$ is equal to the number of real roots of $g(x) = \left(1 + \frac{1}{x}\right)^n + \frac{1}{x}$. Again the number of real roots of $g(x)$ is equal to the number of real roots of $g(\frac{1}{x}) = (1+x)^n + x$. Consider $g(\frac{1}{y-1}) = y^n + y - 1$, we find the number of real roots of $h(y) = y^n + y - 1$. We have $h(0) = -1 < 0$ and $h(1) = 1 > 0$. Therefore by the intermediate value theorem, $h(y)$ has at least one real root in $(0, 1)$. Also by De Gua's rule [11] for imaginary roots, there are at least $n - 1$ complex roots for odd n and there are at least $n - 2$ complex roots for even n. Therefore we can conclude that $h(y)$ has exactly one real root for odd n and two real roots for even n. It remains to show that all the real roots of $f(x)$ are distinct. Suppose $a \in \mathbb{R}$ is a double root of $f(x)$. Whence,

$$(1 + a)^n + a^{n-1} = 0, \tag{3}$$

$$n(1 + a)^{n-1} + (n - 1)a^{n-2} = 0. \tag{4}$$

From equation (3) we get

$$(1 + a)^{n-1} = -\frac{a^{n-1}}{1 + a} \quad (\text{ since } a \neq -1). \tag{5}$$

Putting the value of $(1 + a)^{n-1}$ in (4) and simplify, we obtain $a = n - 1$. Which is a contradiction since $a < 0$. □

Theorem 2.10 *For all n the* $\mathtt{d}$*-number of $K_{2n,2n}$ is* 1.

Proof Notice that the domination polynomial of $K_{2n,2n}$ is

$$D(K_{2n,2n}, x) = \left((1 + x)^{2n} - 1\right)^2 + 2x^{2n}. \tag{6}$$

Suppose for $a \in \mathbb{R}$, $\left((1 + a)^{2n} - 1\right)^2 + 2a^{2n} = 0$, then $\left((1 + a)^{2n} - 1\right)^2 = -2a^{2n}$. But this is true only if $a = 0$, hence $\mathtt{d}(K_{2n,2n}) = 1$. □

Theorem 2.11 *The* $\mathtt{d}$*-number of $K_{2n+1,2n+1}$ is greater than or equal to* 3 *for all n.*

Proof We have known the domination polynomial of $K_{2n+1,2n+1}$ is

$$D(K_{2n+1,2n+1},x) = \left((1+x)^{2n+1}-1\right)^2 + 2x^{2n+1}. \tag{7}$$

It is easy to verify that

$$\begin{aligned} D\left(K_{2n+1,2n+1},-\frac{1}{2}\right) &= 1+\frac{1}{2^{2n-1}}\left(\frac{1}{2^{2n+3}}-1\right) > 0 \\ D\left(K_{2n+1,2n+1},-1\right) &= -1 < 0 \\ D\left(K_{2n+1,2n+1},-2\right) &= 2^2(1-2^{2n}) < 0 \\ D\left(K_{2n+1,2n+1},-3\right) &= (2^{2n+1}+1)^2 - 2\times 3^{2n+1} > 0 \end{aligned}$$

Therefore by the intermediate value theorem, $K_{2n+1,2n+1}$ has at least one real domination root in $(-1,-\frac{1}{2})$ and at least one in $(-3,-2)$, hence $\mathtt{d}(K_{2n+1,2n+1}) \geq 3$. □

The Dutch-Windmill graph G_3^n is the graph obtained by selecting one vertex in each of n triangles and identifying them.

Theorem 2.12 *For $n \geq 2$ the domination polynomial of the Dutch-Windmill graph G_3^n is*

$$D(G_3^n,x) = x(1+x)^{2n} + (2x+x^2)^n.$$

Proof Let v be the center vertex of G_3^n. It is clear that $\{v\}$ is the only dominating set of cardinality 1. Therefore $\gamma(G_3^n) = 1$ and $d(G_3^n,1) = 1$. The number of ways of selecting dominating set of cardinality which containing the center is $\binom{2n}{i-1}$. Also there are 2^n dominating sets of cardinality n which does not contain the center vertex v. Similarly there are $\binom{n}{i}2^{n-i}$ ways to select a dominating set of cardinality $n+i$ which does not contain the center vertex v. Therefore $D(G_3^n,x) = x(1+x)^{2n} + (2x+x^2)^n$. □

Theorem 2.13 *For all n the $\mathtt{d}$-number of the Dutch windmill graph G_3^{2n+1} is* 1.

Proof We have known the domination polynomial of the Dutch windmill graph G_{2n+1}^3 is

$$D(G_3^{2n+1},x) = x(1+x)^{4n+2} + (2x+x^2)^{2n+1}.$$

Suppose there is a number $a \in \mathbb{R}$ with $a \neq 0$ such that $a(1+a)^{4n+2} + (2a+a^2)^{2n+1} = 0$. Then we have $a < 0$ and by a simple calculation we have

$$a = -\left(1-\frac{1}{(1+a)^2}\right). \tag{7}$$

Suppose $-2 < a < 0$, then the left side of the equation (7) is negative but the right side is positive, a contradiction. Now suppose $a \leq -2$. Then the left side of the equation (7) is less than or equal to -2 but the right side is greater than -1, a contradiction. Therefore there is no nonzero real domination root for G_3^{2n+1} and hence $\mathtt{d}(G_3^{2n+1}) = 1$. □

Theorem 2.14 *The* $\mathtt{d}$*-number of* G_3^{2n} *is greater than or equal to* 3 *for all* n.

Proof Notice that the domination polynomial of the Dutch windmill graph G_3^{2n} is

$$D(G_3^{2n},x) = x(1+x)^{4n} + (2x+x^2)^{2n}.$$

It is easy to verify that $D(G_3^{2n},-1) > 0$ and $D(G_3^{2n},-2) < 0$. Also if a is a negative real number near to 0, then $D(G_3^{2n},a) < 0$. Therefore by the intermediate value theorem, we have G_3^{2n} has a real domination root in $(-2,-1)$ and a real domination root in $(-1,0)$ and hence $\mathtt{d}(G_3^{2n}) \geq 3$. □

The lollipop graph $L_{n,1}$ is the graph obtained by joining a complete graph K_n to a path P_1, with a bridge.

Theorem 2.15 *For* $n \geq 2$ *the domination polynomial of the lollipop graph* $L_{n,1}$ *is*

$$D(L_{n,1},x) = x\left((1+x)^n + (1+x)^{n-1} - 1\right).$$

Proof Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of the complete graph K_n and v be the path P_1 and let v is adjacent to v_1. Clearly, $\gamma(L_{n,1}) = 1$ and $d(L_{n,1},1) = 1$. For $2 \leq i \leq n-1$, the only non dominating sets of i vertices of $L_{n,1}$ are the subset of $\{v_2, v_3, \cdots, v_n\}$. Therefore $d(L_{n,1},i) = \binom{n+1}{i} - \binom{n-1}{i}$. Also $d(L_{n,1},n) = n+1$ and $d(L_{n,1},n+1) = 1$. □

Theorem 2.16 *For all* $n \geq 2$ *the* $\mathtt{d}$*-number of the lollipop graph* $L_{n,1}$ *is*

$$\mathtt{d}(L_{n,1}) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even.} \end{cases}$$

Proof By Theorem 2.15 it is enough to prove that $f(y) = y^n + y^{n-1} - 1$ has only one real root if n is odd and has exactly two real roots if n is even. By De Gua's rule for imaginary roots, there are at least $n-1$ complex roots if n is odd and there are at least $n-2$ complex roots if n is even. Now, $f(0) = -1 < 0$ and $f(1) = 2 > 0$ for all n and $f(-1) = -1 < 0$ and $f(-2) = 2^{n-1} - 1 > 0$ for all even n. Therefore by the intermediate value theorem, we have the result. □

The generalized barbell graph $B_{m,n,1}$ is the simple graph obtained by connecting two complete graphs K_m and K_n by a path P_1.

Theorem 2.17 *For* $m \leq n$, *the domination polynomial of generalized barbell graph* $B_{m,n,1}$ *is*

$$D(B_{m,n,1},x) = [(1+x)^m - 1]\,[(1+x)^n - 1].$$

Proof Let $V = \{v_1, v_2, \cdots, v_m\}$ and $U = \{u_1, u_2, \cdots, u_n\}$ be the vertices of $B_{m,n,1}$ such that if $i \neq j$ every vertices V are adjacent, every vertices U are adjacent and v_m and u_n is adjacent. There is no one element dominating set and $\{v_i, u_j\}$ is a dominating set of cardinality 2 of $B_{m,n,1}$. Therefore $\gamma(B_{m,n,1}) = 2$ and $d(B_{m,n,1}, 2) = mn$. Also observe that for $2 \leq i \leq 2n$, a subset S of vertices $B_{m,n,1}$ of cardinality i is not a dominating set if either $S \subset V$ or $S \subset U$. Therefore $d(B_{m,n,1}, i) = \binom{2n}{i} - \binom{n}{i} - \binom{m}{i}$; for $2 \leq i \leq m$, $d(B_{m,n,1}, i) = \binom{2n}{i} - \binom{n}{i}$; for $m+1 \leq i \leq n$ and $d(B_{m,n,1}, i) = \binom{2n}{i}$; for $n+1 \leq i \leq 2n$. This implies that $D(B_{m,n,1}, x) = [(1+x)^m - 1]\,[(1+x)^n - 1]$. □

Theorem 2.18 *For all m, n the* $\mathtt{d}$*-number of the generalized barbell graph $B_{m,n,1}$ is*

$$\mathtt{d}(B_{m,n,1}) = \begin{cases} 1 & \textit{if both } m \textit{ and } n \textit{ are odd,} \\ 2 & \textit{otherwise.} \end{cases}$$

Proof The result follows from the transformation $y = 1 + x$ in the domination polynomial of $B_{m,n,1}$. □

The $n-$ barbell graph $B_{n,1}$ is the simple graph obtained by connecting two copies of complete graph K_n by a bridge.

Corollary 2.19 *The domination polynomial of the n-barbell graph $B_{n,1}$ is*

$$D(B_{n,1}, x) = ((1+x)^n - 1)^2 .$$

Proof It follows from the fact that the n-barbell graph $B_{n,1}$ and the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

Corollary 2.20 *For all n, the* $\mathtt{d}$*-number of the n-barbell graph $B_{n,1}$ is*

$$\mathtt{d}(B_{n,1}) = \begin{cases} 1 & \textit{if } n \textit{ is odd,} \\ 2 & \textit{if } n \textit{ is even.} \end{cases}$$

A bi-star graph $B_{(m,n)}$ is a tree obtained from the graph K_2 with two vertices u and v by attaching m pendant edges in u and n pendant edges in v.

Theorem 2.21 *The domination polynomial of the bi-star graph $B_{(m,n)}$ is*

$$D(B_{(m,n)}, x) = x^{m+n} + x^2(1+x)^{m+n} + x^{m+1}(1+x)^n + x^{n+1}(1+x)^m.$$

Proof Let $\{u, v\}$, $U = \{u_1, u_2, \cdots, u_n\}$ and $V = \{v_1, v_2, \cdots, v_m\}$ be the vertices of $B_{m,n}$ such that u and v are adjacent, every vertices U are adjacent to u and every vertices V are adjacent to v. Clearly there is no one element dominating set. The set $\{u, v\}$ is the only dominating set of cardinality 2 of $B_{m,n}$. Therefore $\gamma(B_{m,n}) = 2$ and $d(B_{m,n}, 2) = 1$. For $3 \le i \le m$, the $i-$element dominating set of $B_{m,n}$ must contain $\{u, v\}$, and the $i-2$ elements can have $\binom{m+n}{i-2}$ choice. For $m+1 \le i \le n$, there are $\binom{m+n}{i-2}$ $i-$element dominating set of $B_{m,n}$ containing $\{u, v\}$ and $\binom{n}{i-m-1}$ $i-$element dominating set of $B_{m,n}$ containing $V \bigcup \{u\}$. For $n+1 \le i \le m+n-1$, there are $\binom{m+n}{i-2}$ $i-$element dominating set of $B_{m,n}$ containing $\{u, v\}$, $\binom{n}{i-m-1}$ $i-$element dominating set of $B_{m,n}$ containing $V \bigcup \{u\}$ and $\binom{m}{i-n-1}$ $i-$element dominating set of $B_{m,n}$ containing $U \bigcup \{v\}$. For $i = m+n$, there are $\binom{m+n}{i-2}$ $(m+n)-$element dominating set of $B_{m,n}$ containing $\{u, v\}$, n $(m+n)-$element dominating set of $B_{m,n}$ containing $V \bigcup \{u\}$, m $(m+n)-$element dominating set of $B_{m,n}$ containing $U \bigcup \{v\}$ and one $(m+n)-$element dominating set of $B_{m,n}$ not containing $\{u, v\}$. Also $d(B_{m,n}, m+n+1) = m+n+2$ and $d(B_{m,n}, m+n+2) = 1$. That is,

$$d(B_{m,n}, i) = \begin{cases} 1 & \text{if } i = 2, m+n+2 \\ \binom{m+n}{i-2} & \text{if } 3 \le i \le m \\ \binom{m+n}{i-2} + \binom{n}{i-m-1} & \text{if } m+1 \le i \le n \\ \binom{m+n}{i-2} + \binom{n}{i-m-1} + \binom{m}{i-n-1} & \text{if } n+1 \le i \le m+n-1 \\ \binom{m+n}{i-2} + n + m + 1 & \text{if } i = m+n \\ m+n+2 & \text{if } i = m+n+1 \end{cases}.$$

Hence

$$D(B_{m,n}) = x^{m+n} + x^2(1+x)^{m+n} + x^{m+1}(1+x)^n + x^{n+1}(1+x)^m. \qquad \square$$

Corollary 2.22 *The domination polynomial of the bi-star graph $B_{(n,n)}$ is*

$$D(B_{(n,n)}, x) = (x(1+x)^n + x^n)^2.$$

Theorem 2.23 *For the bi-star graph $B_{(m,n)}$, $m \ne n$ we have the following :*

$$\mathtt{d}(B_{(m,n)}) = \begin{cases} 3 & \textit{if both } m \textit{ and } n \textit{ are odd,} \\ 5 & \textit{if both } m \textit{ and } n \textit{ are even,} \\ 4 & \textit{if } m \textit{ and } n \textit{ have opposite parity.} \end{cases}$$

Proof By Theorem 2.21 we have,

$$\begin{aligned} D(B_{(m,n)}, x) &= x^{m+n} + x^2(1+x)^{m+n} + x^{m+1}(1+x)^n + x^{n+1}(1+x)^m \\ &= x^2\left(x^{m+n-2} + (1+x)^{m+n} + x^{m-1}(1+x)^n + x^{n-1}(1+x)^m\right) \\ &= x^2\left(x^{m-1}\left((1+x)^n + x^{n-1}\right) + (1+x)^m\left((1+x)^n + x^{n-1}\right)\right) \\ &= x^2\left((1+x)^m + x^{m-1}\right)\left((1+x)^n + x^{n-1}\right). \end{aligned}$$

We have known that there is no real number satisfying both the equations $(1+x)^m+x^{m-1}=0$ and $(1+x)^n+x^{n-1}=0$ simultaneously. Therefore it suffices to prove that $(1+x)^m+x^{m-1}$ has exactly one real root for odd m and two real roots for even m. The remaining proof is similar to the proof of Theorem 2.9. □

Theorem 2.24 *For bi-star graph $B_{(n,n)}$, we have the following :*

$$\mathtt{d}(B_{(n,n)}) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 3 & \text{if } n \text{ is even.} \end{cases}$$

Proof The proof similar to the proof of Theorem 2.9. □

The corona $H \circ G$ of two graphs H and G is the graph formed from one copy of H and $|V(H)|$ copies of G, where the i^{th} vertex of H is adjacent to every vertex in the i^{th} copy of G.

Lemma 2.25([9]) *Let G and H be nonempty graphs of order m and n respectively. Then*

$$D(G \circ H, x) = (x(1+x)^n + D(H,x))^m .$$

Theorem 2.26 *If K_m and K_n be the complete graphs with m and n vertices respectively. Then the domination polynomial of $K_m \circ K_n$ is*

$$D(K_m \circ K_n, x) = \left((1+x)^{n+1} - 1\right)^m .$$

Theorem 2.27 *For the corona $K_m \circ K_n$, we have the following :*

$$\mathtt{d}(K_m \circ K_n) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof It follows from the transformation $y = 1 + x$ in the domination polynomial $D(K_m \circ K_n, x)$. □

Consider the graph K_m and m copies of K_n . The graph $Q(m,n)$ is obtained by identifying each vertex of K_m with a vertex of a unique K_n.

Corollary 2.28 *For $m \geq 2$, the domination polynomial of $Q(m,n)$ is*

$$D(Q(m,n), x) = ((1+x)^n - 1)^m .$$

Proof It follows from the fact that $Q(m,n)$ and $K_m \circ K_{n-1}$ are isomorphic. □

Corollary 2.29 *For the graph $Q(m, n)$, we have the following :*

$$\mathtt{d}(Q(m,n)) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

§3. Domination Stable Graph

In this section we introduce $\mathtt{d}$-stable and $\mathtt{d}$-unstable graphs. We obtained some examples of $\mathtt{d}$-stable and $\mathtt{d}$-unstable graphs.

Definition 3.1 *Let $G = (V(G), E(G))$ be a graph. The graph G is said to be a domination stable graph or simply $\mathtt{d}$-stable graph if all the nonzero domination roots of G lie in the left open half-plane, that is, if real part of the nonzero domination roots is negative. If G is not $\mathtt{d}$-stable graph, then G is said to be a domination unstable graph or simply $\mathtt{d}$-unstable graph.*

Theorem 3.1 *If G and H are isomorphic graphs, then G is $\mathtt{d}$-stable if and only if H is $\mathtt{d}$-stable.*

Proof It follows from the fact that if G and H are isomorphic graphs then $D(G, x) = D(H, x)$. □

Corollary 3.2 *If G and H are isomorphic graphs then G is $\mathtt{d}$-unstable if and only if H is $\mathtt{d}$-unstable.*

Theorem 3.3 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}$-stable if and only if each G_i is $\mathtt{d}$-stable.*

Proof It follows from the fact that

$$D(G, x) = \prod_{i=1}^{m} D(G_i, x).$$ □

Corollary 3.4 *If a graph G consists of m components $G_1, G_2, \ldots, G_m$, then G is $\mathtt{d}$-unstable if and only if one of the G_i is $\mathtt{d}$-unstable.*

Theorem 3.5 *Let G be a connected graph of order $n > 2$ without pendent vertices. If G is $\mathtt{d}$-stable, then*

$$n < 1 + 2\,\mathtt{d}(G, n-3).$$

Proof Suppose G is $\mathtt{d}$-stable. Then by Routh-Hurwitz criteria, we have Routh-Hurwitz matrix $H_2 > 0$. This implies that

$$\mathtt{d}(G, n-1)\mathtt{d}(G, n-3) - \mathtt{d}(G, n-2) > 0.$$

Since G is connected and without pendent vertices we have

$$\mathtt{d}(G, n-1) = n \text{ and } \mathtt{d}(G, n-2) = \frac{1}{2}n(n-1).$$

This completes the proof. $\square$

Theorem 3.6 *The complete graph K_n is $\mathtt{d}$-stable graph for all n.*

Proof The domination polynomial of K_n is

$$D(K_n, x) = (1+x)^n - 1.$$

Therefore

$$\mathbb{Z}(D(K_n, x)) = \left\{ \exp\left(\frac{2k\pi i}{n}\right) - 1 | k = 0, 1, \ldots, n-1 \right\}.$$

Clearly, real part of all the roots are non-positive. This implies that K_n is $\mathtt{d}$-stable for all n. $\square$

Theorem 3.7 *The complement of the complete graph K_n is $\mathtt{d}$-stable graph for all n.*

Proof It follows from the fact that the graph $\overline{K_n}$ has no nonzero domination roots. $\square$

We use the following definitions and results to prove some graphs which are $\mathtt{d}$-unstable. These definitions and theorems are taken from [10].

Definition 3.2 *If $f_n(x)$ is a family of complex polynomials, we say that a number $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\mathbb{Z}(f_n(x))$, $\mathbb{Z}(f_n(x))$ is the set of the roots of the family $f_n(x)$.*

Now, a family $f_n(x)$ of polynomials is a recursive family of polynomials if $f_n(x)$ satisfy a homogeneous linear recurrence

$$f_n(x) = \sum_{i=1}^{k} a_i(x) f_{n-i}(x), \tag{8}$$

where the $a_i(x)$ are fixed polynomials, with $a_k(x) \neq 0$. The number k is the order of the recurrence. We can form from equation (8) its associated characteristic equation

$$\lambda^k - a_1(x)\lambda^{k-1} - a_2\lambda^{k-2} - \cdots - a_k(x) = 0 \tag{9}$$

whose roots $\lambda = \lambda(x)$ are algebraic functions, and there are exactly k of them counting multiplicity.

If these roots, say $\lambda_1(x), \lambda_2(x), \cdots, \lambda_k(x)$, are distinct, then the general solution to equation (8) is known to be

$$f_n(x) = \sum_{i=1}^{k} \alpha_i(x) \lambda_i(x)^n \tag{10}$$

with the usual variant if some of the $\lambda_i(x)$ are repeated. The functions

$$\alpha_1(x), \alpha_2(x), \cdots, \alpha_k(x)$$

are determined from the initial conditions, that is, the k linear equations in the α_i obtained by letting $n = 0, 1, \cdots, k-1$ in equation (10) or its variant. The details are available in [10]. Beraha, Kahane and Weiss [10] proved the following results on recursive families of polynomials and their roots.

Theorem 3.8 *If $f_n(x)$ is a recursive family of polynomials, then a complex number z is a limit of roots of $f_n(x)$ if and only if there is a sequence (z_n) in $\mathbb{C}$ such that $f_n(z_n) = 0$ for all n and $z_n \to z$ as $n \to \infty$.*

Theorem 3.9 *Under the non-degeneracy requirements that in equation* (10) *no $\alpha_i(x)$ is identically zero and that for no pair $i \neq j$ is it true that $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some complex number ω of unit modulus, then $z \in \mathbb{C}$ is a limit of roots of $f_n(x)$ if and only if either*

(1) *two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater (in modulus) than the others; or*

(2) *for some j, $\lambda_j(z)$ has modulus strictly greater than all the other $\lambda_i(z)$, and $\alpha_j(z) = 0$.*

Corollary 3.10([6]) *Suppose $f_n(x)$ is a family of polynomials such that*

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \ldots + \alpha_k(x)\lambda_k(x)^n, \tag{11}$$

where the $\alpha_i(x)$ and the $\lambda_i(x)$ are fixed non-zero polynomials, such that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus. Then the limits of roots of $f_n(x)$ are exactly those z satisfying (1) *or* (2) *of Theorem* 3.9.

Theorem 3.11 *The generalized barbell graph $B_{m,n,1}$ is* d*-stable for all m, n.*

Proof We have known by Theorem 2.17 that the domination polynomial of $B_{m,n,1}$ is

$$D(B_{m,n,1}, x) = [(1+x)^m - 1]\,[(1+x)^n - 1]\,.$$

Therefore

$$\begin{aligned} \mathbb{Z}(D(B_{m,n,1}, x)) &= \left\{\exp\left(\frac{2k\pi i}{m}\right) - 1 | k = 0, \cdots, m-1\right\} \\ &\quad \bigcup \left\{\exp\left(\frac{2k\pi i}{n}\right) - 1 | k = 0, \cdots, n-1\right\}. \end{aligned}$$

Clearly, real part of all the roots are non-positive. This implies that the generalized barbell graph $B_{m,n,1}$ is d-stable for all m, n. □

The domination roots of the generalized barbell graph $B_{m,n,1}$ for $1 \leq m, n \leq 30$ are shown in Figure 1.

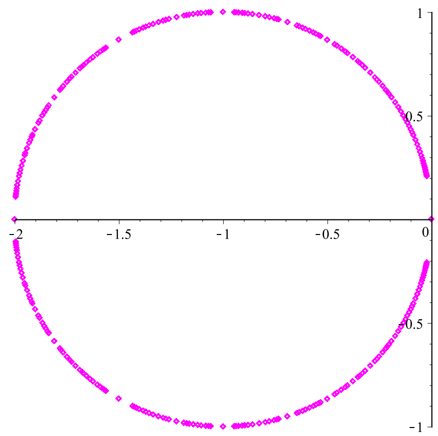

Figure 1 Domination roots of $B_{m,n,1}$ for $1 \leq m, n \leq 30$.

Corollary 3.12 *The n-barbell graph $B_{n,1}$ is* d*-stable for all n.*

Proof It follows from the fact that the n-barbell graph $B_{n,1}$ and the generalized barbell graph $B_{n,n,1}$ are isomorphic. □

The domination roots of the n-barbell graph $B_{n,1}$ for $1 \leq n \leq 60$ are shown in Figure 2.

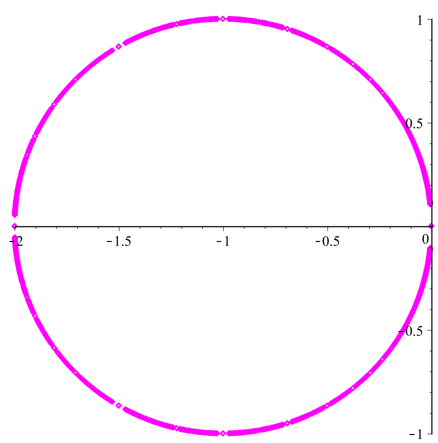

Figure 2 Domination roots of $B_{n,1}$ for $1 \leq n \leq 60$.

Theorem 3.13 *The corona $K_m \circ K_n$ is* d*-stable for all m, n.*

Proof Notice that the domination polynomial of $K_m \circ K_n$ is

$$D(K_m \circ K_n, x) = \left((1+x)^{n+1} - 1\right)^m.$$

Therefore

$$\mathbb{Z}(D(K_m \circ K_n, x)) = \left\{ \exp\left(\frac{2k\pi i}{n+1}\right) - 1 | k = 0, 1, \cdots, n \right\}.$$

Clearly, real part of all the roots are non-positive. This implies that the corona $K_m \circ K_n$ is $\mathtt{d}$-stable for all m, n. $\square$

Corollary 3.14 *The graph $Q(m,n)$ is $\mathtt{d}$-stable for all m, n.*

Proof It follows from the fact that the graph $Q(m,n)$ and $K_m \circ K_{n-1}$ are isomorphic. $\square$

Theorem 3.15 *Let G be a connected graph of order n and $D(G,x)$ be its domination polynomial. If $D(G,x)$ has exactly two distinct domination roots, then G is $\mathtt{d}$-stable for all n.*

Proof It follows from the fact that the two distinct roots are real. $\square$

Theorem 3.16 *Let G be a graph of order n, then the corona $G \circ K_1$ is $\mathtt{d}$-stable for all n.*

Proof We have known the domination polynomial of $G \circ K_1$ is

$$D(G \circ K_1, x) = x^n(x+2)^n.$$

Therefore $\mathbb{Z}(D(G \circ K_1, x) = \{0, -2\}$, that is, $G \circ K_1$ is $\mathtt{d}$-stable for all n. $\square$

Theorem 3.17 *Let G be a graph of order n, then the corona $G \circ \overline{K_2}$ is $\mathtt{d}$-stable for all n.*

Proof Notice that the domination polynomial of $G \circ \overline{K_2}$ is

$$D(G \circ \overline{K_2}, x) = x^{\frac{n}{3}}(x^2+3x+1)^{\frac{n}{3}}.$$

Therefore $\mathbb{Z}(D(G \circ \overline{K_2}, x) = \left\{0, \frac{-3\pm\sqrt{5}}{2}\right\}$, That is, $G \circ \overline{K_2}$ is $\mathtt{d}$-stable for all n. $\square$

Theorem 3.18 *Let G be a graph without pendent vertices and let $D(G,x)$ be its domination polynomial. If $D(G,x)$ has exactly three distinct roots, then G is $\mathtt{d}$-stable.*

Proof Notice that

$$\mathbb{Z}(D(G,x)) \subset \left\{0, -2 \pm i\sqrt{2}, \frac{-3 \pm i\sqrt{3}}{2}\right\}.$$

This implies that G is $\mathtt{d}$-stable. $\square$

Theorem 3.19 *Any graph G with three distinct domination roots is $\mathtt{d}$-stable.*

Proof Notice that

$$\mathbb{Z}(D(G,x)) \subset \left\{-2, 0, \frac{-3 \pm \sqrt{5}}{2}, -2 \pm i\sqrt{2}, \frac{-3 \pm i\sqrt{3}}{2}\right\}.$$

This implies that G is $\mathtt{d}$-stable. $\square$

Theorem 3.20 *The Dutch windmill graph G_3^n is not $\mathtt{d}$-stable graph for all but finite values of n.*

Proof Using maple, we find that the Dutch windmill graph G_3^n is d-stable for $n \leq 6$. Notice that $D(G_3^n, x) = x(1+x)^{2n} + (2x+x^2)^n$. We rewrite $f_n(x) = D(G_3^n, x)$ as

$$f_n(x) = x\left((1+x)^2\right)^n + (1)(2x+x^2)^n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n,$$

where, $\alpha_1 = x,\ \lambda_1 = (1+x)^2,\ \alpha_2 = 1,\ \lambda_2 = 2x + x^2$.

Clearly, 1 and x are not identically zero and $\lambda_1 \neq \omega\lambda_2$ for any complex number ω of modulus 1. Therefore the initial conditions of Theorem .19 are satisfied. Now, for $z = a + ib \in \mathbb{C}$, $|\lambda_1(z)| = |\lambda_2(z)|$ holds if and only if $|(1+z)^2| = |2z + z^2|$. That is, $|(1+a+ib)^2| = |2(a+ib) + (a+ib)^2|$. By a simple calculation we have $(a+1)^2 + b^2 = \frac{1}{2}$. Therefore 0 and the complex numbers z such that $(1+\mathcal{R}(z))^2 + (\mathcal{I}(z))^2 = \frac{1}{2}$ are limits of domination roots of G_3^n. This implies that the domination roots of G_3^n have unbounded positive real part. Therefore the Dutch windmill graph G_3^n is not d-stable for all but finite values of n. □

The domination roots of the Dutch windmill graph G_3^n for $1 \leq n \leq 6$ and for $1 \leq n \leq 30$ are shown in Figures 3 and 4, respectively.

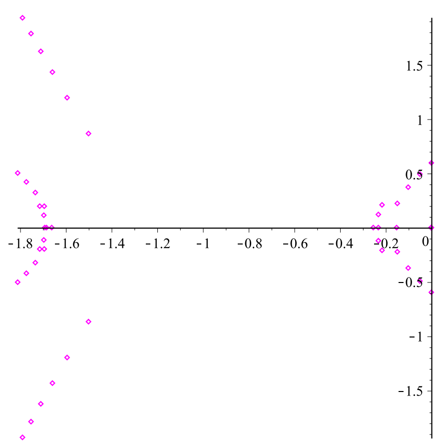

Figure 3 Domination roots of G_3^n for $1 \leq n \leq 6$.

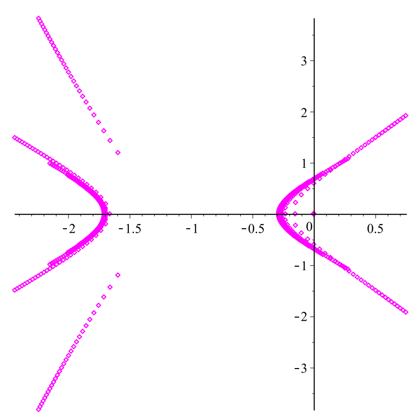

Figure 4 Domination roots of G_3^n for $1 \leq n \leq 30$.

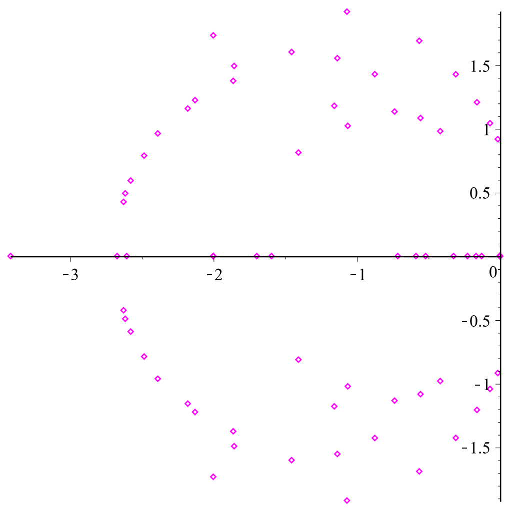

Figure 5 Domination roots of B_n for $1 \leq n \leq 9$.

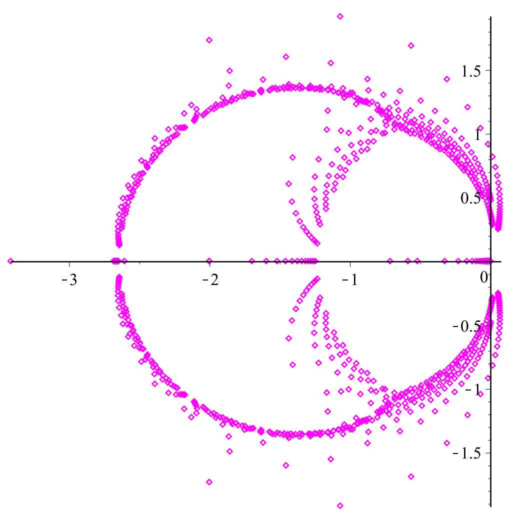

Figure 6 Domination roots of B_n for $1 \leq n \leq 30$.

The domination roots of the bipartite cocktail party graph B_n for $1 \leq n \leq 9$ and for $1 \leq n \leq 30$ are shown in Figures 5 and 6, respectively.

Remark 3.21 The domination polynomial of S_n is

$$\begin{aligned} D(S_n, x) &= x^n + x(1+x)^n \\ &= 1(x)^n + x(1+x)^n \\ &= \alpha_1\lambda_1^n + \alpha_2\lambda_2^n, \end{aligned}$$

where $\alpha_1 = 1$, $\lambda_1 = x$, $\alpha_2 = x$ and $\lambda_2 = 1+x$. Clearly 1 and x are not identically zero and

$\lambda_1 \neq \omega\lambda_2$ for any complex number ω of modulus 1. Therefore the initial conditions of Theorem 3.9 are satisfied. Now, $|\lambda_1| = |\lambda_2|$ holds if and only if $|x - 0| = |x - (-1)|$, that is, if and only if x is equidistant from 0 and -1. This holds if and only if real part of x is $-\frac{1}{2}$. Also α_1 is never 0 and $\alpha_2 = 0$ if and only if $x = 0$ and in this case $|\lambda_2(0)| = 1 > 0 = |\lambda_1(0)|$. By these arguments we have 0 and the complex numbers z such that $\mathcal{R}(z) = -\frac{1}{2}$ are the limits of roots of $D(S_n, x)$. Therefore we think that that there is no complex number z with positive real part is a root of $D(S_n, x)$. We conjectured that the star graph S_n is d-stable graph for all n.

The domination roots of the star graph S_n for $1 \leq n \leq 60$ are shown in Figure 7.

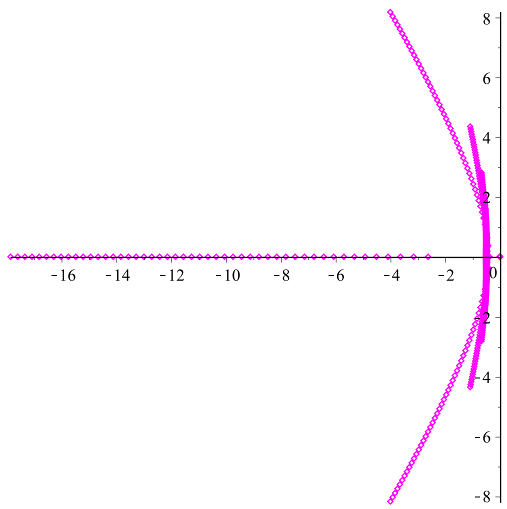

Figure 7 Domination roots of S_n for $1 \leq n \leq 60$.

Remark 3.22 The domination polynomial of $K_{m,n}$ is

$$D(K_{m,n}, x) = ((1+x)^m - 1)((1+x)^n - 1) + x^m + x^n.$$

Let m be fixed and rewrite $D(K_{m,n}, x)$ as :

$$\begin{aligned} D(K_{m,n}, x) &= ((1+x)^m - 1)(1+x)^n + ((1 + x^m - (1+x)^m))(1)^n + 1(x)^n \\ &= \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n, \end{aligned}$$

where $\alpha_1 = (1+x)^m - 1$, $\lambda_1 = 1 + x$, $\alpha_2 = 1 + x^m - (1+x)^m$, $\lambda_2 = 1$, $\alpha_3 = 1$ and $\lambda_3 = x$. Clearly α_1,α_2 and α_3 are not identically zero and $\lambda_i \neq \omega\lambda_j$ for $i \neq j$ and any complex number ω of modulus 1. Therefore the initial conditions of Theorem 3.9 are satisfied. Now, applying part(i) of Theorem 3.9, we consider the following four different cases:

(*i*) $|\lambda_1| = |\lambda_2| = |\lambda_3|$,
(*ii*) $|\lambda_1| = |\lambda_2| > |\lambda_3|$,
(*iii*) $|\lambda_1| = |\lambda_3| > |\lambda_2|$,
(*iv*) $|\lambda_2| = |\lambda_3| > |\lambda_1|$.

Case 1. Assume that $|1+x| = |1| = |x|$. Then $|x-(-1)| = |x-0|$ implies that x lies on the vertical line $z = -\frac{1}{2}$, $|x-(-1)| = 1$ implies that x lies on the unit circle centered at $(-1,0)$ and $1 = |x-0|$ implies that x lies on the unit circle centered at the origin. Therefore the two points of intersection, $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ are limits of roots.

Case 2. Assume that $|1+x| = |1| > |x|$. Then $|x-(-1)| = 1$ implies that x lies on the unit circle centered at $(-1,0)$, $|x-(-1)| > |x-0|$ implies that x lies to the right of the vertical line $z = -\frac{1}{2}$. Therefore the complex numbers x that satisfy $|x-(-1)| = 1$ and $\mathcal{R}(x) > -\frac{1}{2}$ are limits of roots.

Case 3. Assume that $|1+x| = |x| > |1|$. Then $|x-(-1)| = |x-0|$ implies that x lies on the vertical line $x = -\frac{1}{2}$ and $|x-0| > 1$ implies that x lies outside the unit circle centered at the origin. Therefore the complex numbers x that satisfy $|x| > 1$ and $\mathcal{R}(x) > -\frac{1}{2}$ are limits of roots.

Case 4. Assume that $|1| = |x| > |1+x|$. Then $1 = |x-0|$ implies that x lies on the unit circle centered at the origin and $|x-0| > |x-(-1)|$ implies that x lies to the left of the vertical line $x = -\frac{1}{2}$. Therefore the complex numbers x that satisfy $|x| = 1$ and $\mathcal{R}(x) < -\frac{1}{2}$ are limits of roots.

Also there may be some additional isolated limits of roots, being roots of α_2 inside $|1+x| = 1$ and $|x| = 1$. The union of the curves and points above yield that for m fixed, the limits of roots of the domination polynomial of the complete bipartite graph $K_{m,n}$ consists of the part of the circle $|z| = 1$ with real part at most $-\frac{1}{2}$, the part of the circle $|z+1| = 1$ with real part at least $-\frac{1}{2}$ and the part of the line $\mathcal{R}(z) = -\frac{1}{2}$ with modulus at least 1. So we conjectured that the complete bipartite graph $K_{m,n}$ is $\mathtt{d}$-stable for all m, n.

The domination roots of the complete bipartite graphs $K_{m,n}$ for $1 \le m \le 15$, $1 \le n \le 30$ and $K_{n,n}$ for $1 \le n \le 30$ are respectively shown in Figures 8 and 9.

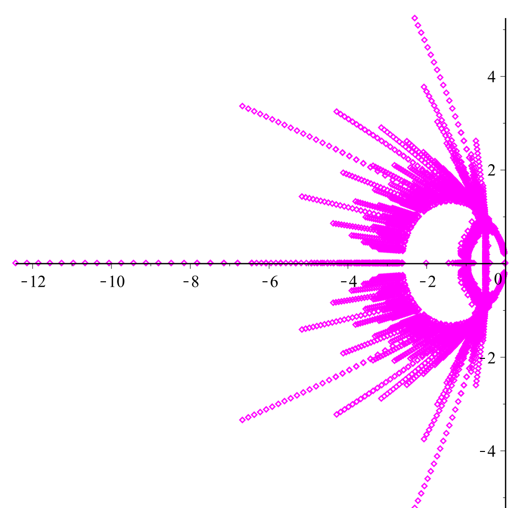

Figure 8 Domination roots of $K_{m,n}$ for $1 \le m \le 15$ and $1 \le n \le 30$.

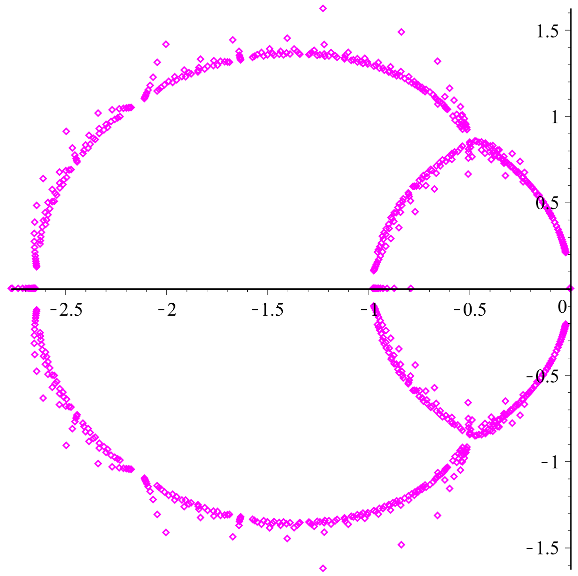

Figure 9 Domination roots of $K_{n,n}$ for $1 \leq n \leq 30$.

Remark 3.23 We have that $D(B_{(m,n)}, x) = x^2 \left((1+x)^m + x^{m-1}\right)\left((1+x)^n + x^{n-1}\right)$. Let m be fixed, we rewrite $D(B_{(m,n)}, x)$ as $f_n(x)$:

$$\begin{aligned} f_n(x) &= \left(x^{m+1} + x^2(1+x)^m\right)(1+x)^n + \left(x^m + x(1+x)^m\right)x^n \\ &= \alpha_1\lambda_1^n + \alpha_2\lambda_2^n, \end{aligned}$$

where

$$\alpha_1 = \left(x^{m+1} + x^2(1+x)^m\right),\ \lambda_1 = 1+x,\ \alpha_2 = \left(x^m + x(1+x)^m\right) \text{ and } \lambda_2 = x.$$

Clearly $\left(x^{m+1} + x^2(1+x)^m\right)$ and $\left(x^m + x(1+x)^m\right)$ are not identically zero and $\lambda_1 \neq \omega\lambda_2$ for any complex number ω of modulus 1. Therefore the initial conditions of Theorem 3.9 are satisfied. Now, $|\lambda_1| = |\lambda_2|$ holds if and only if $|x - (-1)| = |x - 0|$, that is, if and only if x is equidistant from -1 and 0. The latter holds if and only if $\mathcal{R}(x) = -\frac{1}{2}$. Notice that $\alpha_1(0) = 0$ and $\alpha_1(0) = 1 + 0 = 1$ has modulus strictly greater than $\lambda_2(0) = 0$.

Note that there may be some additional limits of roots, being roots of α_1 and α_2. But from the Remark 3.21, we can conclude that α_1 and α_2 have no roots in the right-half plane. By these arguments we have 0 and the complex numbers z that satisfy $\mathcal{R}(z) = -\frac{1}{2}$ are the limits of roots of $D(B_{(m,n)}, x)$. So we conjectured that the bi-star graph $B_{(m,n)}$ is $\mathtt{d}$-stable for all m, n.

The domination roots of the bi-star graph $B_{(n,n)}$ for $1 \leq n \leq 50$ are shown in Figure 10.

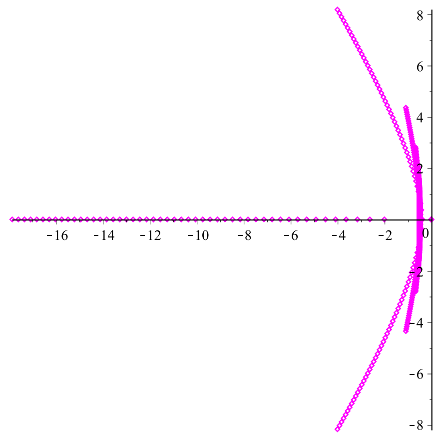

Figure 10 Domination roots of bi-star graph $B_{(n,n)}$ for $1 \leq n \leq 50$.

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International J.Math. Combin. Vol.3(2018), 74-80

Energy, Wiener index and Line Graph of Prime Graph of a Ring

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Abstract: Let $\mathbb{Z}_n$ be the commutative ring of residue classes modulo n, $PG(\mathbb{Z}_n)$ be the prime graph of a ring over a ring $\mathbb{Z}_n$. In this paper we study Energy and Wiener index of $PG(\mathbb{Z}_n)$ and give some results of line graph of prime graph of a ring over a ring $\mathbb{Z}_n$, denote it by $L(PG(\mathbb{Z}_n))$.

Key Words: Prime graph of a ring $PG(R)$, line graph, energy, Wiener index.

AMS(2010): 05C25, 05C15, 13E15.

§1. Introduction

Prime graph of a ring first introduced by Satyanarayana et al. [3]. Prime graph of a ring is defined as a graph whose vertices are all elements of the ring and any two distinct vertices $x, y \in R$ are adjacent if and only if $xRy = 0$ or $yRx = 0$. This graph is denoted by $PG(R)$. The concept of energy and Wiener index of zero divisor graph was introduced by Mohammad Reza and Reza Jahani in [4]. Motivated from the article in [4] in Section 2 of this paper we discuss energy of prime graph of a ring and give general MATLAB code for our calculation. In section 3, We calculate Wiener index of $PG(\mathbb{Z}_n)$, for $n = p$, $n = p^2$ and $n = p^3$. In last section of paper, we introduce Line Graph of Prime Graph of a Ring denoted by $L(PG(\mathbb{Z}_n))$ and discuss Planerity, Girth and degree of all vertices in $L(PG(\mathbb{Z}_n))$. Also, we find center, eccentricity, point covering number, independence number, Energy, Wiener index and Chromatic number of $L(PG(\mathbb{Z}_n))$, where $n = p$, p prime. Here, we also discuss complement of line graph of prime graph of a ring over a ring $\mathbb{Z}_n$, denote it by $L(PG(\mathbb{Z}_n))^c$. We study Girth of $L(PG(\mathbb{Z}_n))^c$ and also find Eulerianity and degree of all vertices in $L(PG(\mathbb{Z}_n))^c$, where $n = p$, p prime.

For more preliminary definitions and Notations the reader is referred to [5]-[8].

§2. Energy of Prime Graph of a Ring

In this section we give some examples and calculate the Energy of prime graph of a ring.

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Definition 2.1 *The energy of the prime graph of a ring $PG(\mathbb{Z}_n)$ is defined as the sum of the absolute values of all the eigen values of its adjacency matrix $M(PG[R])$. i.e. if $\lambda_1, \lambda_2, \cdots, \lambda_n$ are n eigen values of $M(PG[R])$, then the energy of $PG(\mathbb{Z}_n)$ is -*

$$E(PG[R]) = \sum_{i=1}^{n} |\lambda_i|.$$

Example 2.2 For $p = 2$, the adjacency matrix of $PG(\mathbb{Z}_2)$ is

$$M(PG[\mathbb{Z}_2]) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The characteristic polynomial is $\lambda^2 - 1$.The eigen values are $\lambda_1 = 1, \lambda_2 = -1$. Therefore, $E(PG[\mathbb{Z}_2]) = 2$.

Example 2.3 For $p = 3$, the adjancency matrix of $PG(\mathbb{Z}_3)$ is

$$M(PG[\mathbb{Z}_3]) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial is $\lambda^3 - 2\lambda$. The eigen values are $\lambda_1 = -1.4142, \lambda_2 = 1.4142, \lambda_3 = 0$. Therefore, $E(PG[\mathbb{Z}_3]) = 2.8284$.

Example 2.4 For $p = 4$, the adjancency matrix of $PG(\mathbb{Z}_4)$ is

$$M(PG[\mathbb{Z}_4]) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial is $\lambda^4 - 3\lambda^2$. The eigen values are $\lambda_1 = 1.7321, \lambda_2 = -1.7321, \lambda_3 = 0, \lambda_4 = 0$. Therefore, $E(PG[\mathbb{Z}_4]) = 3.4641$.

Example 2.5 For $p = 5$, the adjancency matrix of $PG(\mathbb{Z}_5)$ is

$$M(PG[\mathbb{Z}_5]) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial is $\lambda^5 - 4\lambda^3$. The eigen values are $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 =$

$0, \lambda_4 = 0,\ \lambda_5 = 0$. Therefore, $E(PG[\mathbb{Z}_5]) = 4$.

From the above Discussion we conclude the following theorem.

Theorem 2.6 *If p is a prime number then energy of $PG(\mathbb{Z}_p)$ is $2\sqrt{p-1}$.*

General MATLAB code to find Energy of a Graph

syms λ To create Symbolic Variables;
$A = [\cdots ; \cdots ; \cdots ; \cdots]$ To create a matrix that has multiple rows, separate the rows with semicolons;
$charpoly(A, \lambda)$ Returns the characteristic polynomial of A in terms of variable λ;
$p = [\quad]$ To input the coefficients of characteristic polynomial;
$r = roots(p)$ Gives the eigen Values of matrix A;
$s = sum(abs(r))$ Gives the energy of a graph.

The values of $E(PG[\mathbb{Z}_n])$ for $n = 2, 3, 4, 5, 6, 9$ and 10 are given in table below.

Sr.No.	n	Characteristic Polynomial	Energy
1	2	$\lambda^2 - 1$	2
2	3	$\lambda^3 - 2\lambda$	2.8284
3	4	$\lambda^4 - 3\lambda^2$	3.4641
4	5	$\lambda^5 - 4\lambda^3$	4
5	6	$\lambda^6 - 7\lambda^4 - 4\lambda^3 + 4\lambda^2$	6.6858
6	9	$\lambda^9 - 9\lambda^7 - 2\lambda^6 + 6\lambda^5$	7.4641
7	10	$\lambda^{10} - 13\lambda^8 - 8\lambda^7 + 16\lambda^6$	9.2058

§3. Wiener Index of Prime Graph of a Ring

In this section, We calculate Wiener index of $PG(\mathbb{Z}_n)$, for $n = p$, $n = p^2$ and $n = p^3$.

Definition 3.1 *Let $PG(R)$ be a Prime Graph of a Ring with vertex set V. We denote the length of the shortest path between every pair of vertices $x, y \in V$ with $d(x, y)$. Then the Wiener index of $PG(R)$ is the sum of the distances between all pair of vertices of $PG(R)$, i.e.*

$$W(PG[R]) = \sum_{x,y \in V} d(x, y).$$

The following results can be easily verified.

Theorem 3.2 $W(PG[\mathbb{Z}_p]) = (p-1)^2$ *if p is a prime.*

Theorem 3.3 $W(PG[\mathbb{Z}_{p^2}]) = \frac{p\cdot(p-1)}{2} \cdot \left[2p^2 - 2p + 1\right]$ *if p is a prime.*

Theorem 3.4 $W(PG[\mathbb{Z}_{p^3}]) = \frac{p\cdot(p-1)}{2} \left[2p^4 + 2p^3 - 2p - 3\right]$ *if p is a prime.*

§4. Line Graph of Prime Graph of a Ring

In this section we define line graph of prime graph of a ring, presented some examples and give some results.

Definition 4.1 *The line graph $L(PG(\mathbb{Z}_n))$ of the graph $PG(\mathbb{Z}_n)$ is defined to the graph whose set of vertices constitutes of the edges of $PG(\mathbb{Z}_n)$, where two vertices are adjacent if the corresponding edges have a common vertex in $PG(\mathbb{Z}_n)$.*

Consider $\mathbb{Z}_n$, the ring of integers modulo n.

Example 4.2 $L(PG(\mathbb{Z}_2))$ is a single vertex graph, there is no edge in $L(PG(\mathbb{Z}_2))$.

Example 4.3 In $L(PG(\mathbb{Z}_3))$, there is an edge between the vertices [0,1] to [0,2], as shown in figure below.

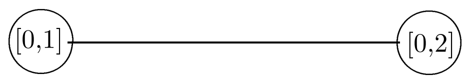

Figure 1

Example 4.4 In $L(PG(\mathbb{Z}_4))$, there is an edge between the vertices [0,1] to [0,2], [0,2] to [0,3] and [0,3] to [0,1] as shown in figure below.

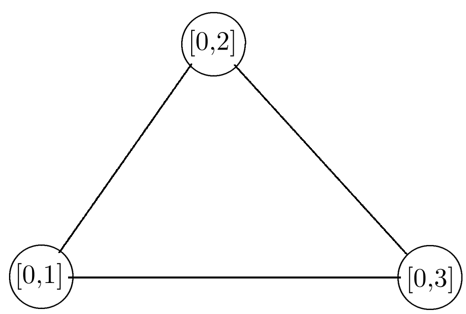

Figure 2

i.e. $L(PG(\mathbb{Z}_4))$ is a complete graph k_3.

Example 4.5 In $L(PG(\mathbb{Z}_5))$, there is an edge between the vertices [0,1] to [0,2], [0,2] to [0,3], [0,3] to [0,4], [0,4] to [0,1], [0,1] to [0,3] and [0,2] to [0,4] as shown in figure below.

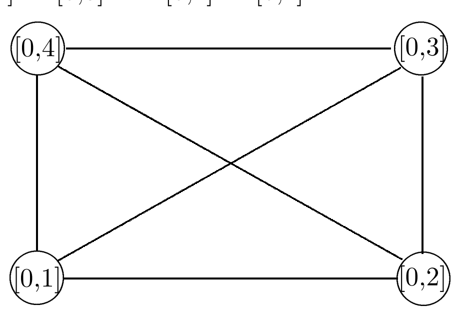

Figure 3

i.e. $L(PG(\mathbb{Z}_5))$ is a complete graph k_4.

Example 4.6 Let us construct $L(PG(\mathbb{Z}_6))$.

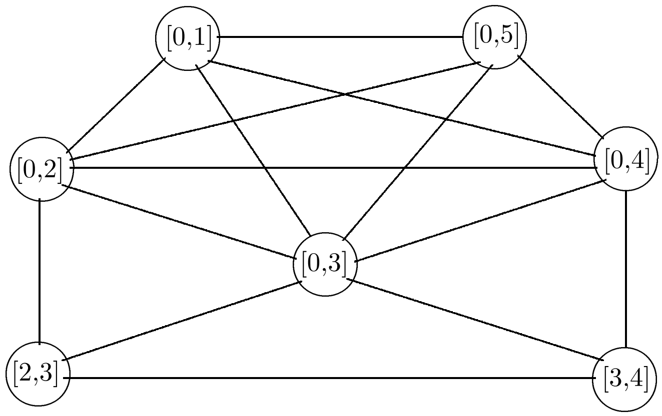

Figure 4

i.e. $L(PG(\mathbb{Z}_6))$ contains a complete subgraph k_5.

Example 4.7 Let us construct $L(PG(\mathbb{Z}_7))$.

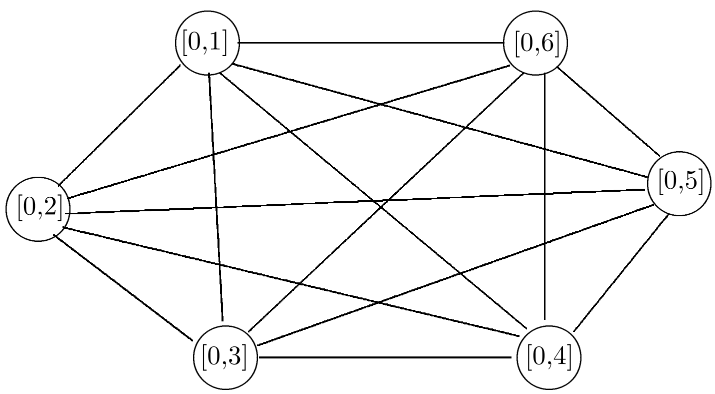

Figure 5

i.e. $L(PG(\mathbb{Z}_7))$ is a complete graph k_6.

Observations 4.8 Every $L(PG(\mathbb{Z}_n))$ contains a complete subgraph on $n-1$ vertices.

Observations 4.9 If $\mathbb{Z}_n$ is a prime ring then $L(PG(\mathbb{Z}_n))$ is a regular graph.

Observations 4.10 If $n = p$, a prime number then $PG(\mathbb{Z}_n)$ is a star graph. So, its line graph $L(PG(\mathbb{Z}_n))$ is a complete graph and hence its eccentricity $e(v) = 1, \forall v \in V(L(PG(\mathbb{Z}_n)))$. Therefore, centre is $L(PG(\mathbb{Z}_n))$.

Theorem 4.11 *The graph $L(PG(\mathbb{Z}_n))$ is Hamiltonian if and only if $n = p$, a prime number and $n \geq 4$.*

Proof When $n = 2$, $L(PG(\mathbb{Z}_n))$ is a single vertex graph, hence there is no cycle. For $n = 3$, $L(PG(\mathbb{Z}_n))$ is a single edge graph, hence there is no cycle exist. For $n = 4$, $L(PG(\mathbb{Z}_n))$ is a triangle graph and there exist a cycle which containing every vertex. So, $L(PG(\mathbb{Z}_4))$ is a

Hamiltonian graph. Now, for $n = p$, a prime number then $L(PG(\mathbb{Z}_n))$ is Hamiltonian graph because there exist a cycle containing every vertex. Hence, the graph $L(PG(\mathbb{Z}_n))$ is Hamiltonian if and only if $n = p$, a prime number and $n \geq 4$. □

Theorem 4.12 *Let $L(PG(\mathbb{Z}_n))$ be a line graph of prime graph of a ring, where $n = p$ and p is an odd prime number then point covering number and independence number of $L(PG(\mathbb{Z}_n))$ both are one.*

Proof When $n = p$, $PG(\mathbb{Z}_n)$ is a star graph. So, there is a common vertex which is adjacent to all other vertices and that vertex is called center of the graph. When we draw the line graph of $PG(\mathbb{Z}_n)$, for $n = p$, and let $a_1 = 0$ be the common vertex of $PG(\mathbb{Z}_n)$ which is the end point of every edge of $PG(\mathbb{Z}_n)$. Then a_1 appears in every vertex of the line graph. $[a_1, v_i] \in V(L(PG(\mathbb{Z}_n)))$, where $i = 1, 2, 3, \cdots, (p-1)$ forms a complete line graph of $PG(\mathbb{Z}_n)$ and here, $[a_1, v_1]$ is adjacent with all other vertices of line graph. In other words, we can say that single vertex cover all other vertices of line graph of $PG(\mathbb{Z}_n)$. Thus, the point cover is one and from that vertex an independence number is also one. □

The following results can be immediately verified.

Theorem 4.13 *The general formula for degree of vertex in $L(PG(\mathbb{Z}_n))$ is:*

$$\begin{aligned} deg[u,v] &= gcd(u,n) + gcd(v,n) - 2, && \text{if } u^2 \neq 0 \text{ and } v^2 \neq 0 \\ &= gcd(u,n) + gcd(v,n) - 3, && \text{if either } u^2 = 0\ ,\ v^2 = 0 \\ &= gcd(u,n) + gcd(v,n) - 4, && \text{if } u^2 = 0 \text{ and } v^2 = 0 \end{aligned}$$

Theorem 4.14 *$L(PG(\mathbb{Z}_n))$ is planer if and only if $n = 2, 3, 4, 5$ and is non-planer for $n \geq 6$.*

Theorem 4.15 *The girth $gr(L(PG(\mathbb{Z}_n))) = 3$ if and only if $n \geq 4$. If $n = 2, 3$ then $gr(L(PG(\mathbb{Z}_n))) = \infty$.*

Theorem 4.16 *The chromatic number $\chi(L(PG(\mathbb{Z}_p))) = p - 1$ for $p = 2, 3, 5, \cdots$.*

Theorem 4.17 *The chromatic number $\chi(L(PG(\mathbb{Z}_{p^n}))) = p^n - 1$, p prime.*

Theorem 4.18 *The energy $E(L(PG(\mathbb{Z}_p))) = 2p - 4$, for $p = 3, 5, \cdots$ and $n = 4$.*

Theorem 4.19 *The Wiener index $W(L(PG(\mathbb{Z}_p))) = \frac{p(p-1)}{2}$, for $p = 3, 5, \cdots$ and $n = 4$.*

Theorem 4.20 *The graph $L(PG(\mathbb{Z}_n))^c$ is Eulerian if and only if $n = p$, a prime number and $n \geq 4$.*

Proof When $n = 2$, there is no graph, as there is no edge between the vertices 0 and 1 in $(PG(\mathbb{Z}_n))^c$. For $n = 3$, $L(PG(\mathbb{Z}_n))^c$is a single vertex graph. For $n = 4$, $L(PG(\mathbb{Z}_n))^c$ is triangle graph and every vertex is of even degree. Now, For $n = p$, a prime number, every vertex of $L(PG(\mathbb{Z}_n))^c$ have even degree. Hence, the graph $L(PG(\mathbb{Z}_n))^c$ is Eulerian if and only if $n = p$, a prime number and $n \geq 4$. □

Theorem 4.21 *The general formula for degree of vertex in* $L(PG(\mathbb{Z}_n))^c$, *where* $n = p$ *a prime number and* $n \geq 5$ *is:*

$$deg[u, v] = n + \phi(n) - 5$$

Theorem 4.22 *The girth* $gr(L(PG(\mathbb{Z}_n))^c) = 3$ *if and only if* $n \geq 4$. *If* $n = 2, 3$ *then* $gr(L(PG(\mathbb{Z}_n))^c) = \infty$.

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International J.Math. Combin. Vol.3(2018), 81-86

Steiner Domination Number of Splitting and Degree Splitting Graphs

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Abstract: A tree T contained in graph G is a Steiner tree with respect to $W \subseteq V(G)$ if T is a tree of minimum order with $W \subseteq V(T)$. The set $S(W)$ consists of all the vertices of G which lie on some Steiner tree with respect to W. The set W is a Steiner set for G if $S(W) = V(G)$. The minimum cardinality among the Steiner sets of G is the Steiner number of G, denoted as $s(G)$. The set W is called Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality among such sets is a Steiner domination number, denoted as $\gamma_s(G)$. We investigate Steiner domination number of some splitting and degree splitting graphs.

Key Words: Steiner distance, Steiner set, Steiner number, domination number, Steiner domination number.

AMS(2010): 05C69, 05C76.

§1. Introduction

We consider simple, finite, connected and undirected graph G with vertex set V and edge set E. For the standard graph theoretic terminology and notation we follow Chatrand and Lesniak [2] while the terms related to the theory of domination are used in the sense of Haynes et al. [6].

Definition 1.1 *The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of the shortest $u - v$ path in G.*

Definition 1.2 *The Steiner distance $sd(W)$ of a subset W of vertices of a connected graph G is the minimum number of edges in a connected subgraph of G that contains W. If H is a subgraph of minimum size that contains a set W, then H is necessarily a tree, called a Steiner tree for W or a Steiner W-tree.*

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Chartrand et al. have introduced a generalization of distance in [3]. The sharp upper and lower bounds for the Steiner k-diameter of G and $\overline{G}$ are given by Mao [9] while the same author have identified some graph classes attaining these bounds. Let n be an integer such that $2 \leq n \leq |V(G)|$, then the n diameter of G, $diam_n(G)$, is defined to be the maximum Steiner distance of any n-subset(subset with n elements) of vertices of G. If G be any graph of order p with minimum degree $\delta \geq 2$ and $2 \leq n \leq p$ then $diam_n(G) \leq \dfrac{p}{\delta+1} + 2n - 5$, is proved by Ali et al. [1].

Definition 1.3 *The set of all vertices of G that lie on some Steiner W-tree is denoted by $S(W)$. If $S(W) = V(G)$, then W is called a Steiner set for G. A Steiner set of minimum cardinality is a minimum Steiner set and this cardinality is the Steiner number $s(G)$.*

The concept of Steiner number was introduced by Chartrand and Zhang [4]. In the same paper authors have proved many results on this newly defined concept. This concept was further studied by Santhakumaran and John [8]. For the graph G of Figure 1, there are three Steiner trees related to $W = \{w_1, w_2\}$ which are shown in the same figure. Since $S(W) = V(G)$, W is a Steiner set of G.

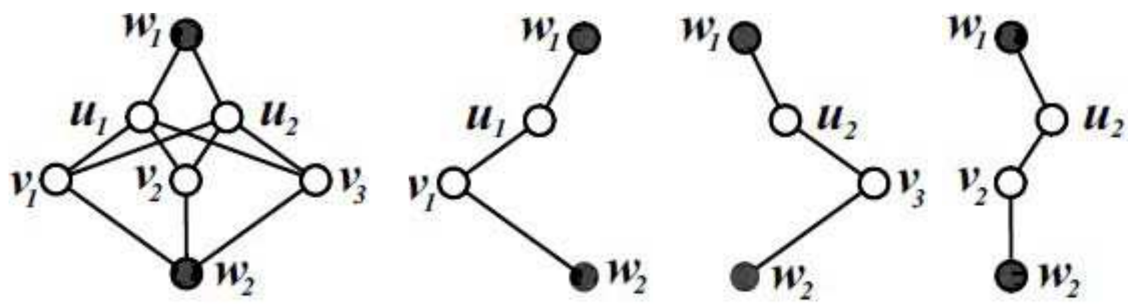

Figure 1 The graph G and its Steiner trees

Definition 1.4 *A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is a minimal dominating set if no proper subset $S' \subset S$ is a dominating set. The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in graph G.*

Definition 1.5 *Let G be a connected graph with vertex set $V(G)$. A set of vertices W in G is called a Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of G is called its Steiner domination number, denoted by $\gamma_s(G)$.*

The concept of Steiner domination number was introduced by John et al. [7]. It is very interesting to investigate Steiner domination number of graph or graph families as it is known only for handful number of graphs. Vaidya and Mehta [11] have investigated the Steiner domination number of W_n, H_n and Fl_n and the same authors [12] have established some characterizations for Steiner domination in graphs while Steiner domination number for $S'(P_n)$, $S'(C_n)$, $M(P_n)$,

$M(C_n)$ and F_n are obtained by Vaidya and Karkar [10].

For the graph G of Figure 1, $W = \{w_1, w_2\}$ is a Steiner dominating set of minimum cardinality. Therefore, $\gamma_s(G) = 2$.

Definition 1.6 *A vertex v is an extreme vertex of a graph G if the subgraph induced by neighbors of v is complete.*

Definition 1.7([5]) *A systematic visit of each vertex of a tree is called a tree traversal.*

Definition 1.8 *The bistar $B_{m,n}$ is the graph obtained by joining the center(apex) vertices of $K_{1,m}$ and $K_{1,n}$ by an edge.*

Definition 1.9 *Let G be a graph with $V(G) = S_1 \cup S_2 \cup S_3 \cup \cdots S_t \cup T$ where each S_i is a set of all vertices of the same degree with at least two elements and $T = V(G) \setminus \bigcup\limits_{i=1}^{t} S_i$. The degree splitting of G denoted by $DS(G)$ is obtained from G by adding vertices $w_1, w_2, w_3, \cdots, w_t$ and joining w_i to each vertex of S_i for $1 \leqslant i \leqslant t$. Note that if $V(G) = \mathop{\cup}\limits_{i=1}^{t} S_i$ then $T = \emptyset$.*

Definition 1.10 *For a graph G the splitting graph $S'(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.*

Definition 1.11 *A friendship graph F_n is a one point union of n copies of cycle C_3.*

§2. Main Results

Observation 2.1 $\gamma(B_{m,n}) = m + n$.

Theorem 2.2 $\gamma_s(S'(B_{m,n})) = m + n + 2$.

Proof Let $u, u_1, u_2, \cdots u_m, v, v_1, v_2, \cdots v_n$ be $m+n+2$ vertices of $B_{m,n}$ and $u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots, v'_n$ be the corresponding vertices which are added to obtain $S'(B_{m,n})$. Then $V(S'(B_{m,n})) = \{u, u_1, u_2, \cdots, u_m, v_1, v_2, \cdots v_n, v, u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n\}$. Now $u'_1, u'_2, \cdots u'_m, v'_1, v'_2, \cdots v'_n$ are extreme vertices as the subgraph induced by their neighbors is complete, namely, the complete graph K_1. Therefore, they must be in Steiner dominating set W. If $u'_1, u'_2, \cdots u'_m, v'_1, v'_2, \cdots v'_n \in W$ then $u'_1, u'_2, \cdots u'_m, v'_1, v'_2, \cdots v'_n, u, v \in S(W)$. Now there some trees between u' and v' which include remaining vertices $u_1, u_2, \cdots u_m, v_1, v_2, \cdots v_n$. So if $u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n \in W$ then there are four Steiner W-trees which include all the vertices of the graph. That is, if $u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n \in W$ then $u_1, u_2, \cdots u_m, v_1, v_2, \cdots v_n, u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n \in S(W)$. Therefore, $W = \{u', u'_1, u'_2, \cdots u'_m, v', v'_1, v'_2, \cdots v'_n\}$ becomes a Steiner set of minimum cardinality $m + n + 2$ and it is also a dominating set. Hence

$$\gamma_s(S'(B_{m,n})) = m + n + 2. \qquad \square$$

Theorem 2.3 $\gamma_s(DS(B_{m,n})) = 2$.

Proof Let $u, u_1, u_2, \cdots u_m, v, v_1, v_2, \cdots v_n$ be $m + n + 2$ vertices of $B_{m,n}$ and x_1, x_2 be the corresponding vertices which are added in order to obtain $DS(B_{m,n})$. Then, $V(DS(B_{m,n})) = \{u, u_1, u_2, \cdots u_m, v, v_1, v_2, \cdots v_n, x_1, x_2\}$. Now if G is a connected graph of order $n \geq 2$ then $2 \leq S(G) \leq n$. Without loss of generality let $x_1, x_2 \in W$ then there are four Steiner W-tree traversal between x_1 and x_2 which together include all the vertices of $DS(B_{m,n})$. Therefore, $W = \{x_1, x_2\}$ becomes a Steiner set of minimum cardinality and it is also a dominating set. Therefore, $W = \{x_1, x_2\}$ becomes a Steiner dominating set of minimum cardinality. Hence

$$\gamma_s(DS(B_{m,n})) = 2. \qquad \Box$$

Lemma 2.4 *$S(DS(P_n)) = n - 5$, $n \geq 7$.*

Proof Consider P_n with $V(P_n) = \{v_1, v_2, \cdots v_n\}$ with partition $V_1 = \{v_2, v_3, \cdots v_{n-1}\}$ and $V_2 = \{v_1, v_n\}$. Now in order to obtain $DS(P_n)$ from P_n we add x_1 and x_2 corresponding to V_1 and V_2. Thus, $V(DS(P_n)) = \{x_1, x_2, v_1, v_2, \cdots v_n\}$. Let $x_1, v_4 \in W$ then there are some Steiner W-trees which include the vertices $x_1, v_1, v_2, v_3, v_4, x_2$. So, if $x_1, v_4 \in W$ then $x_1, v_1, v_2, v_3, v_4, x_2 \in S(W)$. Let $x_1, v_4, v_{n-3} \in W$ then $x_1, v_1, v_2, v_3, v_4, x_2, v_{n-3}, v_{n-2}, v_{n-1}, v_n \in S(W)$. Then, there does not exist tree traversal containing x_1, v_4, v_{n-3} which includes $v_5, v_6, \cdots, v_{n-4}$. The vertices $v_5, v_6, \cdots, v_{n-4}$ must be included in W to obtain Steiner tree of minimum size which include $v_5, v_6, \cdots, v_{n-4}$. Therefore, if $x_1, v_4, v_5, \cdots v_{n-4}, v_{n-3} \in W$. Then there are following four Steiner W- trees as listed below:

(1) $x_1 v_1 v_2 v_3 \cdots v_{n-4} v_{n-3}$,
(2) $x_1 v_1 v_2 x_2 v_4 v_5 v_6 \cdots v_{n-3}$,
(3) $x_1 v_n v_{n-1} v_{n-2} v_{n-3} \cdots v_5 v_4$,
(4) $x_1 v_n v_{n-1} x_2 v_{n-2} v_{n-3} v_{n-4} \cdots v_5, v_4$,

which include all the vertices of the graph. Thus $W = \{x_1, v_4, v_5, \cdots v_{n-4}, v_{n-3}\}$ becomes a Steiner set of minimum size which include $n - 6$ vertices of P_n and a vertex x_1. Hence

$$S(DS(P_n)) = n - 5. \qquad \Box$$

Theorem 2.5 *$\gamma_s(DS(P_n)) = n - 3$, $n \geq 7$.*

Proof From the Theorem 2.4 $W = \{x_1, v_4, v_5, \cdots v_{n-4}, v_{n-3}\}$ is a Steiner set of minimum cardinality. But it is not a dominating set as v_2 and v_{n-1} are not dominated by any of the vertices. Therefore, these two vertices must be in Steiner dominating set W. So, $\{x_1, v_2, v_4, v_5, \cdots v_{n-4}, v_{n-3}, v_{n-1}\}$ is a Steiner dominating set of minimum cardinality. Hence

$$\gamma_s(DS(P_n)) = n - 3. \qquad \Box$$

Proposition 2.6([7]) *$\gamma_s(K_{m,n}) = min\{m, n\}$ if $m, n \geq 2$.*

Theorem 2.7 *$\gamma_s(S'(K_{m,n})) = m + n$.*

Proof Let $v_1, v_2, \cdots v_m, u_1, u_2, \cdots, u_n$ be $m+n$ vertices of $K_{m,n}$. Now $v'_1, v'_2, \cdots v'_m, u'_1, u'_2, \cdots, u'_n$ be the corresponding vertices which are added in order to obtain $S'(K_{m,n})$ with partitions $W = \{v'_1, v'_2, \cdots v'_m, u'_1, u'_2, \cdots u'_n\}$ and $X = \{v_1, v_2, \cdots v_m, u_1, u_2, \cdots u_n\}$. It is very clear that W is a Steiner set as there are $\max\{m, n\}$ number of Steiner trees which include all the vertices of the graph. Here W dominates all the vertices of the graph. Therefore, it is also a dominating set. Thus, W is a Steiner dominating set. We claim that W is a Steiner dominating set with minimum cardinality. If possible let U be any Steiner set such that $|U| < |W|$ and $U \subset W$. Then, there exists a vertex $v'_i \in W$ such that $v'_i \notin U$. But as the vertices of W are mutually non adjacent, the Steiner U-tree containing v'_j and v'_k ($j \neq i$, $k \neq i$, $1 \leq j, k \leq n$) will not contain v'_i. Therefore, U is not Steiner set. If $U \subset X$ then some vertices of W and some vertices of X which are not included in U are not in any Steiner U- trees. Therefore, U is not Steiner set. Let $U \subset W \cup X$ such that U contain at least one vertex from each of W and X then some vertices of W and X do not lie on any Steiner U-tree. Thus, U is not a Steiner set. So, W is a Steiner dominating set of minimum cardinality $m + n$. Hence

$$\gamma_s(S'(K_{m,n})) = m + n. \qquad \square$$

Proposition 2.8([4]) *Let G be a connected graph of order $p \geq 2$. Then $\gamma_s(G) = 2$ if and only if there exists a Steiner dominating set $S = \{u, v\}$ of G such that $d(u, v) \leq 3$.*

Theorem 2.9 *$\gamma_s(DS(K_{m,n})) = 2$, $m \neq n$, $m, n \geq 2$.*

Proof Let $v_1, v_2, \cdots v_m, u_1, u_2, \cdots u_n$ be $m + n$ vertices of $K_{m,n}$ with partitions $W = \{v_1, v_2, \cdots v_m\}$ and $X = \{u_1, u_2, \cdots u_n\}$. In order to construct $DS(K_{m,n})$ we add w_1 and w_2. If we consider w_1 and w_2 in Steiner set W then $S(W) = V(G)$ and W is also a dominating set. Therefore W becomes a Steiner dominating set and $d(w_1, w_2) = 3$. Hence by Proposition 2.8,

$$\gamma_s(DS(K_{m,n})) = 2. \qquad \square$$

Proposition 2.10([7]) *Each extreme vertex of a connected graph G belongs to every Steiner dominating set of G.*

Theorem 2.11 *$\gamma_s(S'(F_n)) = 2n + 1$.*

Proof Let $v_0, v_1, v_2, \ldots v_n, v_{n+1}, \cdots v_{2n}$ be the $2n + 1$ vertices of F_n where v_0 is the apex vertex. Now $v'_0, v'_1, v'_2, \ldots v'_n, v'_{n+1}, \cdots v'_{2n}$ be the vertices which are added to obtain $S'(F_n)$. The vertices $v'_0, v'_1, v'_2, \ldots v'_n, v'_{n+1}, \cdots v'_{2n}$ must be in Steiner dominating set W as they are extreme vertices. But $W = \{v'_0, v'_1, v'_2, \ldots v'_n, v'_{n+1}, \cdots v'_{2n}\}$ is not a Steiner dominating set as it is neither a Steiner set nor a dominating set. Therefore, we must include some more vertices to obtain a Steiner dominating set. Let $v_0 \in W$ then $S(W) = V(S'(F_n))$ and

$$W = \{v'_0, v'_1, v'_2, \cdots v'_n, v'_{n+1}, \cdots v'_{2n}\}$$

is a dominating set of minimum cardinality. Hence

$$\gamma_s(S'(F_n)) = 2n + 1. \qquad \square$$

§3. Concluding Remarks

The Steiner domination in graphs is one of the interesting domination models. It is always challenging to investigate Steiner domination number of a graph. We have obtained Steiner domination number of larger graphs which are obtained by means of various graph operations.

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On Certain Coloring Parameters of Graphs

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Abstract: Coloring the vertices of a graph G according to certain conditions can be considered as a random experiment and a discrete random variable X can be defined as the number of vertices having a particular color in the proper coloring of G. In this paper, we extend the concepts of mean and variance, two important statistical measures, to the theory of graph coloring and determine the values of these parameters for a number of standard graphs.

Key Words: Graph coloring, Smarandachely Λ-coloring, coloring sum of graphs, coloring mean, coloring variance, χ-chromatic mean, χ^+-chromatic.

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§1. Introduction

Investigations on graph coloring problems have attracted wide interest among researchers since its introduction in the second half of the nineteenth century. The vertex coloring or simply a coloring of a graph is an assignment of colors or labels to the vertices of a graph subject to certain conditions. For example, Smarandachely Λ-coloring of graph G by colors in $\mathscr{C}$ such that $\varphi(u) \neq \varphi(v)$ if u and v are elements of a subgraph isomorphic to graph Λ in G. In a proper coloring of a graph, its vertices are colored in such a way that no two adjacent vertices in that graph have the same color.

Different types of graph colorings have been introduced during several subsequent studies. Many practical and real life situations paved paths to different graph coloring problems.

Several researchers have also introduced various parameters related to different types of graph coloring and studied their properties extensively. The first and the most important parameter in the theory of graph coloring is the *chromatic number* of graphs which is defined as the minimum number of colors required in a proper coloring of the given graph. The concept of chromatic number has been extended to almost all types of graph colorings.

The notion of chromatic sums of graphs related to various graph colorings have been

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introduced and studied extensively. Some of these studies can be found in [9, 10, 11]. The notion of a general coloring sum of a graph has been explained in [9] as follows:

Let $\mathcal{C} = \{c_1, c_2, c_3, \cdots, c_k\}$ be a particular type of proper k-coloring of a given graph G and $\theta(c_i)$ denotes the number of times a particular color c_i is assigned to vertices of G. Then, the *coloring sum* of a coloring $\mathcal{C}$ of a given graph G, denoted by $\omega_{\mathcal{C}}(G)$, is defined to be $\omega_{\mathcal{C}}(G) = \sum\limits_{i=1}^{k} i\,\theta(c_i)$.

Motivated by the studies on different types of graph coloring problems, corresponding parameters and their applications, we discuss the concepts of mean and variance, two important statistical parameters, to the theory of graph coloring in this paper.

For all terms and definitions, not defined specifically in this paper, we refer to [2, 3, 4, 6, 15, 16] and for the terminology of graph coloring, we refer to [5, 7, 8]. For the concepts in Statistics, please see [12, 13]. Unless mentioned otherwise, all graphs considered in this paper are simple, finite, connected and non-trivial.

§2. Coloring Mean and Variance of Graphs

We can identify the coloring of the vertices of a given graph G with a random experiment. Let $\mathcal{C} = \{c_1, c_2, c_3, \cdots, c_k\}$ be a proper k-coloring of G and let X be the random variable $(r.v)$ which denotes the color of an arbitrarily chosen vertex in G. Since the sum of all weights of colors of G is the order of G, the real valued function $f(i)$ defined by

$$f(i) = \begin{cases} \frac{\theta(c_i)}{|V(G)|}; & i = 1, 2, 3, \cdots, k \\ 0; & \text{elsewhere} \end{cases}$$

is the probability mass function $(p.m.f)$of the $r.v$ X. If the context is clear, we can also say that $f(i)$ is the $p.m.f$ of the graph G with respect to the given coloring $\mathcal{C}$.

Hence, analogous to the definitions of the mean and variance of random variables, the mean and variance of a graph G, with respect to a general coloring of G can be defined as follows.

Definition 2.1 *Let $\mathcal{C} = \{c_1, c_2, c_3, \cdots, c_k\}$ be a certain type of proper k-coloring of a given graph G and $\theta(c_i)$ denotes the number of times a particular color c_i is assigned to vertices of G. Then, the coloring mean of a coloring $\mathcal{C}$ of a given graph G, denoted by $\mu_{\mathcal{C}}(G)$, is defined to be*

$$\mu_{\mathcal{C}}(G) = \frac{\sum\limits_{i=1}^{k} i\,\theta(c_i)}{\sum\limits_{i=1}^{k} \theta(c_i)}.$$

Definition 2.2 *For a positive integer r, the r-th moment of the coloring $\mathcal{C}$ is denoted by $\mu_{\mathcal{C}^r}(G)$*

and is defined as

$$\mu_{\mathcal{C}^r}(G) = \frac{\sum\limits_{i=1}^{k} i^r\,\theta(c_i)}{\sum\limits_{i=1}^{k} \theta(c_i)}.$$

Definition 2.3 *The coloring variance of a coloring $\mathcal{C}$ of a given graph G, denoted by $\sigma^2_{\mathcal{C}}(G)$, is defined to be*

$$\sigma^2_{\mathcal{C}}(G = \frac{\sum\limits_{i=1}^{k} i^2\,\theta(c_i)}{\sum\limits_{i=1}^{k} \theta(c_i)} - \left(\frac{\sum\limits_{i=1}^{k} i^2\,\theta(c_i)}{\sum\limits_{i=1}^{k} \theta(c_i)}\right)^2.$$

2.1 χ-Chromatic Mean and Variance of Graphs

Coloring mean and variance corresponding to a particular type of minimal proper coloring of the vertices of G are defined as follows.

Definition 2.4 *A coloring mean of a graph G, with respect to a proper coloring $\mathcal{C}$ is said to be a χ-chromatic mean of G, if $\mathcal{C}$ is the minimum proper coloring of G and the coloring sum ω_G is also minimum. The χ-chromatic mean of a graph G is denoted by μ_χ.*

Definition 2.5 *The χ-chromatic variance of G, denoted by $\sigma^2_\chi(G)$, is a coloring variance of G with respect to a minimal proper coloring $\mathcal{C}$ of G which yields the minimum coloring sum.*

Let us now determine the χ-chromatic mean and variance of certain standard graph classes. The following result discusses the χ-chromatic mean and variance of a complete graph K_n.

Proposition 2.6 *The χ-chromatic mean of a complete graph K_n is $\frac{n+1}{2}$ and its χ-chromatic variance is $\frac{n^2-1}{12}$.*

Proof Note that all vertices of a complete graph K_n must have different colors as they are all adjacent to each other. That is, $\theta(c_i) = 1$ for color c_i, $1 \leqslant i \leqslant n$. Therefore,

$$\mu_\chi(K_n) = \frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}$$

and

$$\sigma^2_\chi(K_n) = \frac{1}{n}\sum_{i=1}^{n} i^2 - \left(\frac{n+1}{2}\right)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{2} = \frac{n^2-1}{12}. \qquad \square$$

The following theorem gives the probability distribution of a proper coloring of a complete graph.

Theorem 2.7 *Any proper coloring of a complete graph K_n has the discrete uniform distribution on $\{1, 2, \cdots, k\}(DU(k))$.*

Proof Let X be the *r.v* representing the number of colors in a proper k-coloring of a complete graph K_n. For any proper k-coloring $\mathcal{C}$ of the complete graph K_n, $\theta(c_i) = 1$ and $k = n$. Hence, the corresponding *p.m.f* is

$$f(i) = \begin{cases} \frac{1}{n}; & n = 1, 2, 3, \ldots, n, \\ 0; & \text{elsewhere} \end{cases}$$

which is that of the discrete uniform distribution on $\{1, 2, \cdots, k\}$. Hence, $X \sim DU(k)$. $\square$

The following result determines the χ-chromatic mean and variance for a path P_n.

Proposition 2.8 *The χ-chromatic mean of a path P_n is*

$$\mu_\chi(P_n) = \begin{cases} \frac{3}{2}; & \textit{if } n \textit{ is even}, \\ \frac{3n-1}{2n}; & \textit{if } n \textit{ is odd}, \end{cases}$$

and the χ-chromatic variance of P_n is

$$\sigma^2_\chi(P_n) = \begin{cases} \frac{1}{4}; & \textit{if } n \textit{ is even}, \\ \frac{n^2-1}{4n^2}; & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof Consider a path P_n on n vertices. Being a bipartite graph, the vertices of P_n can be colored using two colors, say c_1 and c_2. Then, we have the following cases.

(i) If n is even, exactly $\frac{n}{2}$ vertices of P_n have color c_1 and $\frac{n}{2}$ vertices have color c_2. Then, the *p.m.f* of the corresponding *r.v* X is

$$f(i) = \begin{cases} \frac{1}{2}; & i = 1, 2, \\ 0; & \text{elsewhere.} \end{cases}$$

Hence, the χ-chromatic mean is

$$\mu_\chi(P_n) = \sum_{i=1}^{2} i\,\frac{1}{2} = \frac{3}{2}$$

and the χ-chromatic variance is

$$\sigma^2_\chi(P_n) = \sum_{i=1}^{2} i^2\,\frac{1}{2} - (\mu_\chi)^2 = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{1}{4}.$$

(ii) If n is odd, then the *p.m.f* of the corresponding *r.v* X is

$$f(i) = \begin{cases} \frac{n+1}{2n}; & i = 1, \\ \frac{n-1}{2n}; & i = 2, \\ 0; & \text{elsewhere.} \end{cases}$$

Then, the χ-chromatic mean of P_n is

$$\mu_\chi(P_n) = 1 \cdot \frac{n+1}{2n} + 2 \cdot \frac{n-1}{2n} = \frac{3n-1}{2n}$$

and its χ-chromatic variance is

$$\sigma^2_\chi(P_n) = 1^2 \cdot \frac{n+1}{2n} + 2^2 \cdot \frac{n-1}{2n} - \left(\frac{3n-1}{2n}\right)^2 = \frac{n^2-1}{4n^2}.$$

□

The following result determines the values of these parameters for a cycle C_n.

Proposition 2.9 *The χ-chromatic mean of a cycle C_n is*

$$\mu_\chi(C_n) = \begin{cases} \frac{3}{2}; & \text{if } n \text{ is even,} \\ \frac{3n+3}{2n}; & \text{if } n \text{ is odd,} \end{cases}$$

and the χ-chromatic variance of C_n is

$$\sigma^2_\chi(C_n) = \begin{cases} \frac{1}{4}; & \text{if } n \text{ is even,} \\ \frac{n^2-8n+9}{4n^2}; & \text{if } n \text{ is odd.} \end{cases}$$

Proof Consider a cycle C_n on n vertices. Then, we have the following cases.

(i) If n is even, then C_n is bipartite and is 2-colorable. Then, exactly $\frac{n}{2}$ vertices of C_n also have color c_1 and c_2 each. Then, as explained in the first part of previous theorem, we have $\mu_\chi(C_n) = \frac{3}{2}$ and $\sigma^2_\chi(C_n) = \frac{1}{4}$.

(ii) If n is odd, then C_n is 3-colorable. Let $\mathcal{C} = \{c_1, c_2, c_3\}$ be the minimal proper coloring of C_n. Then, the *p.m.f* of the *r.v* X is given by

$$f(i) = \begin{cases} \frac{n-1}{2n}; & \text{if } \ i = 1, 2, \\ \frac{1}{n}; & \text{if } \ i = 3, \\ 0; & \text{elsewhere.} \end{cases}$$

Then, the χ-chromatic mean of G is

$$\mu_\chi(C_n) = 1 \cdot \frac{n-1}{2n} + 2 \cdot \frac{n-1}{2n} + 3 \cdot \frac{1}{n} = \frac{3n+3}{2n}$$

and the χ-chromatic variance of C_n is

$$\sigma^2_\chi(C_n) = \left(1^2 \cdot \frac{n-1}{2n} + 2^2 \cdot \frac{n-1}{2n} + 3^2 \cdot \frac{1}{n}\right) - \left(\frac{3n+3}{2n}\right)^2 = \frac{n^2-8n+9}{4n^2}.$$

□

In the following theorem, we determine the χ-chromatic mean and variance of a wheel graph $W_n = K_1 + C_{n-1}$.

Proposition 2.10 *The χ-chromatic mean of a wheel graph W_n is*

$$\mu_\chi(W_n) = \begin{cases} \frac{3n+3}{2n}; & \text{if } n \text{ is odd,} \\ \frac{3n+1}{2n+2}; & \text{if } n \text{ is even,} \end{cases}$$

and the χ-chromatic variance of W_n is

$$\sigma^2_\chi(W_n) = \begin{cases} \frac{n^2+8n-9}{4n^2}; & \text{if } n \text{ is odd,} \\ \frac{n^2+32n-64}{4n^2}; & \text{if } n \text{ is even.} \end{cases}$$

Proof Note that the wheel graph W_n is 3-colorable, when n is odd and 4-colorable when is even. Then, we have the following cases.

(i) First, assume that n is an odd integer. Then, the outer cycle C_{n-1} of W_n is an even cycle. Hence, $\frac{n-1}{2}$ vertices of C_{n-1} have color c_1, $\frac{n-1}{2}$ vertices of C_{n-1} have color c_2 and the central vertex of W_n has color c_3. Hence the corresponding $p.m.f$ for W_n is given by

$$f(i) = \begin{cases} \frac{n-1}{2n}; & \text{if} \quad i = 1, 2, \\ \frac{1}{n}; & \text{if} \quad i = 3, \\ 0; & \text{elsewhere.} \end{cases}$$

Hence, the corresponding χ-chromatic mean is

$$\mu_\chi(W_n) = 1 \cdot \frac{n-1}{2n} + 2 \cdot \frac{n-1}{2n} + 3 \cdot \frac{1}{n} = \frac{3n+3}{2n}.$$

Now, the χ-chromatic variance is

$$\sigma^2_\chi(W_n) = (1^2+2^2) \cdot \frac{n-1}{2n} + 3^2 \cdot \frac{1}{n} - (\mu_\chi(W_n))^2 = \left(\frac{5(n-1)}{2n} + \frac{9}{n}\right) - \left(\frac{3n+3}{2n}\right)^2 = \frac{n^2+8n-9}{4n^2}.$$

(ii) Next, assume that n is an even integer. Then, the outer cycle C_{n-1} of W_n is an odd cycle. Hence, $\frac{n-2}{2}$ vertices of the outer cycle C_{n-1} have color c_1, $\frac{n-2}{2}$ vertices of C_{n-1} have color c_2 and one vertex of C_{n-1} has color c_3 and the central vertex of W_n has the c_4. Hence, the $p.m.f$ for W_n is given by

$$f(i) = \begin{cases} \frac{n-2}{2n}; & \text{if} \quad i = 1, 2, \\ \frac{1}{n}; & \text{if} \quad i = 3, 4 \\ 0; & \text{elsewhere.} \end{cases}$$

Hence, the corresponding χ-chromatic mean is

$$\mu_\chi(W_n) = 1 \cdot \frac{n-2}{2n} + 2 \cdot \frac{n-2}{2n} + 3 \cdot \frac{1}{n} + 4 \cdot \frac{1}{n} = \frac{3n+8}{2n}$$

and the χ-chromatic variance is

$$\begin{aligned}
\sigma^2_\chi(W_n) &= (1^2+2^2)\cdot\frac{n-2}{2n}+(3^2+4^2)\cdot\frac{1}{n}-(\mu_\chi(W_n))^2\\
&= \left(\frac{5(n-2)}{2n}+\frac{3^2+4^2}{n}\right)-\left(\frac{3n+8}{2n}\right)^2=\frac{n^2+32n-64}{4n^2}. \qquad \square
\end{aligned}$$

Remark 2.1 From the above discussion, we observe that the minimum proper coloring of bipartite graph follows a two-point distribution. In general, for a bipartite graph $G(V_1,V_2,E)$, with $|V_1|=m_1>|V_2|=m_2, m_1+m_2=n$, the *p.m.f* can be defined as

$$f(i)=\begin{cases}\frac{m_1}{n}; & \text{if} \quad i=1,\\ \frac{m_2}{n}; & \text{if} \quad i=2,\\ 0; & \text{elsewhere.}\end{cases}$$

Hence, we have $\mu_\chi(G)=\frac{m_1+2m_2}{n}=1+\frac{m_2}{n}$ and $\sigma^2_\chi(G)=\frac{m_1+4m_2}{n}-\left(1+\frac{m_2}{n}\right)^2=\frac{1}{n^2}\left[(n-1)m_1+2(2n-1)m_2\right]$.

Remark 2.2 If G is a regular bipartite graph on n vertices, then there will be $\frac{n}{2}$ vertices in each partition and hence with respect to a minimal proper coloring, exactly $\frac{n}{2}$ vertices having the colors c_1 and c_2 each. Hence the *p.m.f* is

$$f(i)=\begin{cases}\frac{1}{2}; & i=1,2,\\ 0; & \text{elsewhere.}\end{cases}$$

Hence, $\mu_\chi(G)=\frac{3}{2}$ and $\sigma^2_\chi(G)=\frac{1}{4}$ as mentioned in Proposition 2.9.

2.2 χ^+-Chromatic Mean and Variance of Graphs

Coloring mean and variance corresponding to another type of a minimal proper coloring of the vertices of G are defined as follows.

Definition 2.11 *A coloring mean of a graph G, with respect to a proper coloring $\mathcal{C}$ is said to be a χ^+-chromatic mean of G, if $\mathcal{C}$ is a minimum proper coloring of G such that the corresponding coloring sum ω_G is maximum. The χ^+-chromatic number of a graph G is denoted by $\mu_{\chi^+}(G)$.*

Definition 2.12 *The χ^+-chromatic variance of G, denoted by $\sigma^2_{\chi^+}(G)$, is a coloring variance of G with respect to a minimal proper coloring $\mathcal{C}$ of G such that the corresponding coloring sum is maximum.*

Invoking the definitions of two types of chromatic means and variances mentioned above, we can infer the following.

Remark 2.3 For any arbitrary minimal proper coloring $\mathcal{C}$ of a given graph G, we have $\mu_\chi(G)\leqslant \mu_{\mathcal{C}}(G)\leqslant \mu_{\chi^+}(G)$ and $\sigma^2_\chi(G)\leqslant\sigma^2_{\mathcal{C}}(G)\leqslant\sigma^2_{\chi^+}(G)$.

Remark 2.4 Since all vertices of a complete graph have different colors, the χ-chromatic mean and the χ^+-chromatic mean are equal and the χ-chromatic variance and the χ^+-chromatic variance are equal.

Let us now discuss the χ^+-chromatic mean and variance of the graph classes mentioned in the previous section.

Proposition 2.13 *The χ^+-chromatic mean of a path P_n is*

$$\mu_{\chi^+}(P_n) = \begin{cases} \frac{3}{2}; & \text{if } n \text{ is even}, \\ \frac{3n-1}{2n}; & \text{if } n \text{ is odd}, \end{cases}$$

and the χ^+-chromatic variance of P_n is

$$\sigma^2_{\chi^+}(P_n) = \begin{cases} \frac{1}{4}; & \text{if } n \text{ is even}, \\ \frac{n^2-1}{4n^2}; & \text{if } n \text{ is odd}. \end{cases}$$

Proof As in Proposition 2.8, we consider the following cases.

(i) If n is even, as mentioned in Proposition 2.8, exactly $\frac{n}{2}$ vertices of P_n have color c_1 and $\frac{n}{2}$ vertices have color c_2. Then, the *p.m.f* of the corresponding *r.v* X is also as defined there. Hence, the χ^+-chromatic mean is $\mu_{\chi^+}(P_n) = \frac{3}{2}$ and the χ^+-chromatic variance is $\sigma^2_{\chi^+}(P_n) = \frac{1}{4}$.

(ii) If n is odd, χ^+-coloring assigns color c_1 to $\frac{n-1}{2n}$ vertices and color c_2 to the remaining $\frac{n+1}{2n}$ vertices. Then the *p.m.f* is

$$f(i) = \begin{cases} \frac{n-1}{2n}; & i = 1, \\ \frac{n+1}{2n}; & i = 2, \\ 0; & \text{elsewhere}. \end{cases}$$

Then, the χ^+-chromatic mean of P_n is given by

$$\mu_{\chi^+}(P_n) = 1 \cdot \frac{n-1}{2n} + 2 \cdot \frac{n+1}{2n} = \frac{3n+1}{2n}$$

and its χ^+-chromatic variance is given by

$$\sigma^2_{\chi^+}(P_n) = 1^2 \cdot \frac{n-1}{2n} + 2^2 \cdot \frac{n+1}{2n} - \left(\frac{3n+1}{2n}\right)^2 = \frac{n^2+1}{4n^2}. \qquad \square$$

The following proposition discusses the χ^+-chromatic mean and variance of a cycle on n vertices.

Proposition 2.14 *The χ^+-chromatic mean of a cycle C_n is*

$$\mu_{\chi^+}(C_n) = \begin{cases} \frac{3}{2}; & \text{if } n \text{ is even,} \\ \frac{5n-3}{2n}; & \text{if } n \text{ is odd,} \end{cases}$$

and the χ^+-chromatic variance of P_n is

$$\sigma^2_{\chi^+}(C_n) = \begin{cases} \frac{1}{4}; & \text{if } n \text{ is even,} \\ \frac{n^2+8n-9}{4n^2}; & \text{if } n \text{ is odd.} \end{cases}$$

Proof Here, we have to consider the following two cases.

(i) If n is even, as mentioned in Proposition 2.13, exactly $\frac{n}{2}$ vertices of C_n have color c_1 and color c_2 each. Then, exactly as explained there, we have, $\mu_{\chi^+}(C_n) = \frac{3}{2}$ and $\sigma^2_{\chi^+}(C_n) = \frac{1}{4}$.

(ii) If n is odd, χ^+-coloring assigns color c_1 to one vertex, color c_2 to $\frac{n-1}{2n}$ vertices and color c_3 to the remaining $\frac{n-1}{2n}$ vertices of the cycle C_n. Then the *p.m.f* is

$$f(i) = \begin{cases} 1; & i = 1, \\ \frac{n-1}{2n}; & i = 2, 3 \\ 0; & \text{elsewhere.} \end{cases}$$

Then, the χ^+-chromatic mean of C_n is

$$\mu_{\chi^+}(C_n) = 1 \cdot \frac{1}{2n} + 2 \cdot \frac{n-1}{2n} + 3 \cdot \frac{n-1}{2n} = \frac{5n-3}{2n}$$

and its χ^+-chromatic variance is

$$\sigma^2_{\chi^+}(C_n) = 1^2 \cdot \frac{1}{n} + 2^2 \cdot \frac{n+1}{2n} + 3^2 \cdot \frac{n-1}{2n} - \left(\frac{5n-3}{2n}\right)^2 = \frac{n^2+8n-9}{4n^2}. \qquad \square$$

The following proposition discusses the χ^+-chromatic mean and variance of a wheel graph on n vertices.

Proposition 2.15 *The χ^+-chromatic mean of a wheel graph W_n is*

$$\mu_{\chi^+}(W_n) = \begin{cases} \frac{5n-3}{2n}; & \text{if } n \text{ is odd,} \\ \frac{3n+1}{2n+2}; & \text{if } n \text{ is even,} \end{cases}$$

and the χ^+-chromatic variance of W_n is

$$\sigma^2_{\chi^+}(W_n) = \begin{cases} \frac{n^2+30n-31}{4n^2}; & \text{if } n \text{ is odd,} \\ \frac{n^2+32n-64}{4n^2}; & \text{if } n \text{ is even.} \end{cases}$$

Proof As mentioned in Proposition 1.10, the wheel graph W_n is 3-colorable, when n is odd and 4-colorable when is even. Then, we have to consider the following cases.

(i) First, assume that n is an odd integer. Then, the outer cycle C_{n-1} of W_n is an even cycle. Hence, we can assign color c_1 to the central vertex of W_n, color c_2 to $\frac{n-1}{2}$ vertices of C_{n-1} and color c_3 to the remaining $\frac{n-1}{2}$ vertices of C_{n-1}. Hence the corresponding $p.m.f$ for W_n is given by

$$f(i) = \begin{cases} \frac{1}{n}; & \text{if} \quad i = 1, \\ \frac{n-1}{2n}; & \text{if} \quad i = 2, 3, \\ 0; & \text{elsewhere.} \end{cases}$$

Hence, the χ^+-chromatic mean is

$$\mu_{\chi^+}(W_n) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{n-1}{2n} + 3 \cdot \frac{n-1}{2n} = \frac{5n-3}{2n}$$

and the χ^+-chromatic variance is

$$\begin{aligned} \sigma^2_{\chi^+}(W_n) &= 1^2 \cdot \frac{1}{n} + (2^2 + 3^2) \cdot \frac{n-1}{2n} - (\mu_{chi}(W_n))^2 \\ &= \left(\frac{13(n-1)}{2n} + \frac{1}{n}\right) - \left(\frac{5n-3}{2n}\right)^2 = \frac{n^2 + 30n - 31}{4n^2}. \end{aligned}$$

(ii) Let n be an even integer. Then, the outer cycle C_{n-1} of W_n is an odd cycle. Hence, we can assign color c_1 to the central vertex of W_n, color c_2 to one vertex of the outer cycle C_{n-1}, color c_3 to $\frac{n-2}{2}$ vertices of C_{n-1} and color c_4 to the remaining $\frac{n-2}{2}$ vertices of C_{n-1}. Therefore, the corresponding $p.m.f$ for W_n is given by

$$f(i) = \begin{cases} \frac{1}{n}; & \text{if} \quad i = 1, 2 \\ \frac{n-2}{2n}; & \text{if} \quad i = 3, 4, \\ 0; & \text{elsewhere.} \end{cases}$$

Hence, the corresponding χ^+-chromatic mean is

$$\mu_{\chi^+}(W_n) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{n-2}{2n} + 4 \cdot \frac{n-2}{2n} = \frac{7n-8}{2n}$$

and the χ^+-chromatic variance is

$$\begin{aligned} \sigma^2_{\chi^+}(W_n) &= (1^2 + 2^2) \cdot \frac{1}{n} + (3^2 + 4^2) \cdot \frac{n-2}{2n} - (\mu_{\chi}(W_n))^2 \\ &= \left(5 \cdot \frac{1}{n} + 25 \cdot \frac{n-2}{2n}\right) - \left(\frac{7n-8}{2n}\right)^2 = \frac{n^2 + 32n - 64}{4n^2}. \end{aligned}$$ □

2.3 Some Interpretations

A *block graph* or *clique tree* G is an undirected graph in which every biconnected component (block) is a clique. By Theorem 2.7, minimum proper coloring of every component of G follows uniform distribution. Hence, we have

Theorem 2.16 *The probability distribution of a block graph G is mixture of discrete uniform distributions.*

An n-partite graph is a graph whose set of vertices can be partitioned in to n subsets such that no two vertices in the same partitions are adjacent. Then, we have the following result.

Theorem 2.17 *Let G be a regular k-partite graph on vertices. Then, any minimal proper coloring of G follows uniform distribution (in each partition).*

proof Any minimal proper coloring of a k-partite graph contains k-colors. Let G be an r-regular k-partite graph. Then, $rk = n$. Then, the $p.m.f$ of G is

$$f(i) = \begin{cases} \frac{1}{k}; & i = 1, 2, 3, \ldots, k, \\ 0; & \text{elsewhere.} \end{cases}$$

which is that of the $DU(k)$ distribution. □

Corollary 2.18 *Let G be a k-partite graph. Then, the χ-chromatic mean (and χ^+-chromatic mean) of G is $\frac{k+1}{2}$ and the χ-chromatic variance (and χ^+-chromatic variance) of G is $\frac{k^2-1}{12}$.*

Proof The proof follows immediately from the fact that the minimal proper coloring of a k-partite graph follows uniform distribution. □

Certain areas where these notions can be made use of are: nodes in communication and traffic networks.

§3. Scope for Further Studies

In this paper, we have extended the notions of mean and variance to the theory of graph coloring and determined their values for certain graphs and graph classes. More problems in this area are still open.

The χ-chromatic mean and variance of many other graph classes are yet to be studied. Determining the sum, mean and variance corresponding to the coloring of certain generalized graphs like generalized Petersen graphs, fullerence graphs etc. are some of the promising open problems. Studies on the sum, mean and variance corresponding to different types of edge colorings, map colorings, total colorings etc. of graphs also offer much for future studies.

We can associate many other parameters to graph coloring and other notions like covering, matching etc. All these facts highlight a wide scope for future studies in this area.

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On Status Indices of Some Graphs

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Abstract: Ramane, Yalnaik recently defined another molecular structural descriptor on the lines of Weiner index, Zagreb Index, etc. Here we construct new graphs of fixed diameter and compute the status indices as well as harmonic status indices of those graphs.

Key Words: Status of vertex, first status connectivity index, second status connectivity index, harmonic status index.

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§1. Introduction

There are several molecular structural graph descriptors such as Weiner Index, Zagreb Index, Hosoya Index etc which strongly correlate studies in graph theory with chemistry. Most of these indices are based on the distance between vertices in a graph.Motivated by harmonic mean we have harmonic index of a graph defined by Fajtlowicz [5]. For more work one can refer [6]. Further motivated by the same, Ramane and Yalnaik introduced the harmonic status index of graphs [4].

Definition 1.1([1]) *The status of a vertex $u \in V(G)$ is defined as the sum of its distance from every other vertex in $V(G)$ and is denoted by $\sigma(u)$. That is*

$$\sigma(u) = \sum_{u \in V(G)} d(u, v).$$

Definition 1.2 *The first status connectivity index $S_1(G)$and second status connectivity index $S_2(G)$ of a connected graph G are defined respectively as*

$$S_1(G) = \sum_{uv \in E(G)} [\sigma(u) + \sigma(v)] \text{ and } S_2(G) = \sum_{uv \in E(G)} [\sigma(u)\sigma(v)].$$

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Similarly the first and second status connectivity coindices of a connected graph G are defined as

$$\overline{\mathrm{S_1}}(G) = \sum_{uv \notin E(G)} [\sigma(u) + \sigma(v)] \text{ and } \overline{\mathrm{S_2}}(G) = \sum_{uv \notin E(G)} [\sigma(u)\sigma(v)].$$

Definition 1.3([5]) *The Harmonic index of a graph G is defined as*

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}.$$

The harmonic status index of a connected graph G as ([4])

$$HS(G) = \sum_{uv \in E(G)} \frac{2}{\sigma(u) + \sigma(v)}.$$

Similarly the harmonic status coindex of a connected graph G is defined as

$$\overline{\mathrm{HS}}(G) = \sum_{uv \notin E(G)} \frac{2}{\sigma(u) + \sigma(v)}.$$

§2. Status Connectivity Indices and Coindices of Some Graphs

In what follows, we consider a class of graphs constructed by first joining a path of length $l(\geq 1)$ to each vertex of G and then attaching k pendent vertices to each end vertex of the path attached. Such a graph can be called l level thorn graph denoted by $G^{l(+k)}$. The usual thorny graph G^{+k} can be regarded as 0 level thorn graph. If $l = 1$ we get first level thorn graph $G^{1(+k)}$.

Example 2.1 A graph G and it's first level thorn graph $G^{\wedge 1(+3)}$ are as shown below.

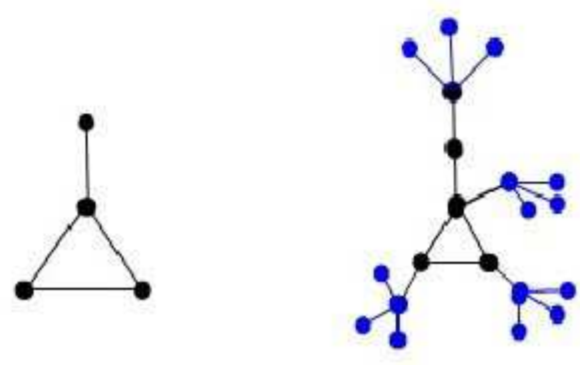

Figure 1

First we evaluate the status connectivity index and coindex of 0 level thorn graphs denoted by G^{+k}.To obtain the harmonic status connectivity index and coindex of this graph we need to calculate status of each vertex and number of pairs of adjacent vertices and pairs of non adjacent vertices in G^{+k}.If G is a r regular graph then,with respect to degree there are two types of vertices in G^{+k},nk pendent vertices (external),n vertices of degree $'r+nk'$ we call them as internal.

Theorem 2.1 *The first and second status connectivity index of thorn graph K_n^{+k} are given by*

$$\begin{aligned} S_1(K_n^{+k}) &= n(n-1)(2nk+n-k-1)+nk(5nk+3n-2k-4) \\ S_2(K_n^{+k}) &= (nk)C_2 \times (3nk+2n-k-3)^2 + nC_2 \times (3nk+2n-k-3)(2nk+n-k-1) \end{aligned}$$

Proof The graph K_n^{+k} is of diameter3and there are two types of vertices in it. A set of $'nk'$ pendent vertices and n vertices of degree $'n+1'$. Let u_i, $i=1,2,\cdots,nk$ denote the pendent vertices and v_i,$i=1,2,\cdots,n$ denote the vertices of degree $'n+1'$. Then the status of pendent vertex is

$$\sigma(u_i) = 1+2(k-1)+2(n-1)+3k(n-1) = 3nk+2nk-3$$

and the status of the internal vertex v_i is

$$\sigma(v_i) = 1(n-1)+k+2k(n-1) = (2nk+n-k-1).$$

Now in K_n^{+k} there are $\frac{n(n-1)}{2}$ adjacent pairs internal vertices and nk pairs of vertices forming edges formed by one internal and one external vertex. Hence by definition the status connectivity index of K_n^{+k} is

$$\begin{aligned} S_1(K_n^{+k}) &= \frac{n(n-1)2(2nk+n-k-1)}{2} + nk(3nk+2n-k-3+2nk+n-k-1) \\ &= n(n-1)(2nk+n-k-1)+nk(5nk+3n-2k-4). \end{aligned}$$

Also in K_n^{+k} there are $(nk)C_2$ pairs of nonadjacent pendent vertices and $nk(n-1)$ pairs of nonadjacent pairs of vertices formed by one pendant and one internal vertex. So that status connectivity coindex of K_n^{+k} is

$$S_2(K_n^{+k}) = (nk)C_2 \times (3nk+2n-k-3)^2 + nC_2 \times (3nk+2n-k-3)(2nk+n-k-1). \quad \square$$

Theorem 2.2 *The harmonic status index and coindex of thorn graph K_n^{+k} are given by*

$$\begin{aligned} HS(K_n^{+k}) &= \frac{n(n-1)}{2}\frac{1}{(2nk+n-k-1)} + nk\frac{2}{(5nk+3n-2k-4)}, \\ \overline{HS}(K_n^{+k}) &= (nk)C_2\frac{1}{(3nk+2n-2k-4)} + nk(n-1)\frac{2}{(5nk+3n-k-3)}. \end{aligned}$$

Proof The graph K_n^{+k} is of diameter 4 and there are two types of vertices in it. A set of $'nk'$ pendent vertices and n vertices of degree $'n+1'$. Let $u_i, i=1,2,\cdots,nk$ denote the pendent vertices and $v_i, i = 1, 2, \cdots, n$ denote the vertices of degree $'n+1'$. Then the status of pendent vertex is

$$\sigma(u_i) = 1 + 2(k-1) + 2(n-1) + 3k(n-1) = 3nk + 2n - k - 3$$

and the status of the internal vertex v_i is

$$\sigma(u_i) = 1(n-1) + k + 2k(n-1) = 2nk + n - k - 1.$$

Now in K_n^{+k} there are $\frac{n(n-1)}{2}$ adjacent pairs internal vertices and nk pairs of vertices forming edges formed by one internal and one external vertex. Hence by definition the harmonic status index of K_n^{+k} is

$$\begin{aligned} HS(K_n^{+k}) &= \frac{n(n-1)}{2}\frac{2}{2(2nk+n-k-1)} + nk\frac{2}{(3nk+2n-k-3+2nk+n-k-1)} \\ &= \frac{n(n-1)}{2}\frac{1}{(2nk+n-k-1)} + nk\frac{2}{(5nk+3n-2k-4)}. \end{aligned}$$

Also in K_n^{+k} there are $(nk)C_2$ pairs of nonadjacent pendent vertices and $nk(n-1)$ pairs of nonadjacent pairs of vertices formed by one pendant and one internal vertex. So that harmonic status coindex of K_n^{+k} is

$$\begin{aligned} \overline{HS}(K_n^{+k}) &= (nk)C_2\frac{2}{2(3nk+2n-2k-4)} + nk(n-1)\frac{2}{(3nk+2n-k-3+2nk+n-k-1)} \\ &= nkC_2\frac{1}{(3nk+2n-2k-4)} + nk(n-1)\frac{2}{(5nk+3n-k-3)}. \end{aligned}$$ □

Now, we discuss the status connectivity indices and the coindices of regular graphs with diameter 2.

Theorem 2.3 *If G is 'r' regular graph of diameter 2 then the first and second status connectivity index of G^{+k} are given by*

$$\begin{aligned} S_1(G^{+k}) &= nr(2n+2kr+k-r-2) + nk(5n+5kr+3k-3r-6), \\ S_2(G^{+k}) &= \frac{nr}{2}(2n+2kr+k-r-2)^2 + nk(3n+3kr+2k-r-2). \end{aligned}$$

Proof The proof follows by direct counting. □

Theorem 2.4 *If G is 'r' regular graph of diameter 2 then the first and second status connectivity co index of G^{+k} are given by*

$$\begin{aligned} \overline{\mathrm{S}_1}(G^{+k}) &= \frac{nk(nk-1)}{2}2(3n+3kr+2k-r-4) + nk(n-1)(5n+5kr+3k-3r-6) \\ &\quad +(nC_2 - \frac{nr}{2})(4n+4kr+2k-2r-4) \end{aligned}$$

$$\begin{aligned} &= nk(nk-1)(3n+3kr+2k-r-4)+nk(n-1)(5n+5kr+3k-3r-6) \\ &\quad +(nC_2-\frac{nr}{2})(4n+4kr+2k-2r-4), \\ \overline{\mathrm{S}_2}(G^{+k}) &= \frac{nk(nk-1)}{2}(3n+3kr+2k-r-4)^2 \\ &\quad +nk(n-1)(3n+3kr+2k-2r-4)(2n+2kr+k-r-2) \\ &\quad +(nC_2-\frac{nr}{2})(2n+2kr+k-r-2)^2. \end{aligned}$$

Proof The proof follows by direct counting. □

Theorem 2.5 *If G is 'r' regular graph of diameter 2 then the harmonic status index of G^{+k} is*

$$HS(G^{+k})=\frac{nr}{2}\frac{1}{(2n+2kr+k-r-2)}+nk\frac{2}{(5n+5kr+3k-2r-6)}.$$

Proof First, we observe that if G has diameter 2 then G^{+k} has diameter 4. Hence from the structure we have the status of each internal vertex v_i as

$$\sigma(v_i)=1.(k+r)+2kr+2.(n-1-r)=2n+2kr+k-r-2.$$

Also the status of each pendant vertex u_i as

$$\sigma(u_i)=1+2.r+2(k-1)+3(n-1-r)=3n+3rk+2k-r-4.$$

There are $\frac{nr}{2}$ internal edges giving harmonic status contribution

$$\frac{nr}{2}\frac{2}{2(2n+2kr+k-r-2)}=\frac{nr}{2}\frac{1}{(2n+2kr+k-r-2)}.$$

Similarly the pendent $'nk'$ vertices adjacent to 'n' internal vertices contribute,

$$nk\frac{2}{(5n+5kr+3k-2r-6)}.$$

Hence the harmonic status index of G^{+k} is

$$HS(G^{+k})=\frac{nr}{2}\frac{1}{(2n+2kr+k-r-2)}+nk\frac{2}{(5n+5kr+3k-2r-6)}.$$ □

Theorem 2.6 *If G is 'r' regular graph of diameter 2 then the harmonic status coindex of G^{+k} is*

$$\overline{\mathrm{HS}}(G^{+k})=(nkC_2)\frac{1}{(3n+3kr+2k-r-4)}+nk(n-1)\frac{2}{(5n+5rk+3k-2r-6)}+(nC_2-\frac{nr}{2}).$$

Proof We note that there are $n(k+1)C_2-(\frac{nr}{2}+nk)$ non adjacent pairs of vertices in

G^{+k}. There are $(nk)C_2$ pendent nonadjacent pendent vertices, $nk(n-1)$ pairs of nonadjacent vertices combining one pendant and one internal vertex and finally $(nC_2 - \frac{nr}{2})$ nonadjacent internal vertices. Taking contribution from each of them we have status connectivity coindex of G^{+k} as

$$\begin{aligned}
\overline{\mathrm{HS}}(G^{+k}) &= (nk)C_2 \frac{2}{6n+6kr+4k-4r-8} + nk(n-1)\frac{2}{5n+5rk+3k-2r-6} \\
&\quad + \left(nC_2 - \frac{nr}{2}\right)\frac{2}{2(2n+2kr+k-r-2)} \\
&= (nk)C_2 \frac{1}{(3n+3kr+2k-2r-4)} + nk(n-1)\frac{2}{(5n+5rk+3k-2r-6)} \\
&\quad +(nC_2 - \frac{nr}{2})\frac{1}{(2n+2kr+k-r-2)}. \qquad \square
\end{aligned}$$

§3. Status Connectivity Indices and Coindices of First Level Thorn Graphs

Now we discuss the harmonic status index and coindex of first level thorn graphs.We need to calculate status of each vertex and number of pairs of adjacent vertices and pairs of non adjacent vertices in $G^{\wedge 1(+k)}$.With respect to degree there are three types of vertices in $G^{\wedge 1(+k)}$.nk pendent vertices,n vertices of degree $'k+1'$ we call them as internal and lastly $'n'$ vertices having degree sequence added by 1.We call them external, in particular if G is 'r'' regular their degrees will become '$r+1'$'.

Theorem 3.1 *The first and second status connectivity index and coindex of first level thorn graph of a 'r'' regular graph of order 'n'' and diameter* 2 *are given by*

$$\begin{aligned}
S_1(G^{1(+k)}) &= \frac{nr}{2} \times (5n+4nk-rk-2r-2k-4) + n(12n+9nk-2rk-4r-4k-10) \\
&\quad +2n^2 \times k(7n+5nk-rk-2r-3k-6), \\
S_2(G^{1(+k)}) &= \frac{nr}{2} \times (5n+4nk-rk-2r-2k-4)^2 \\
&\quad +n(5n+4nk-rk-2r-2k-4)(7n+5nk-2r-4k-rk-6), \\
\overline{\mathrm{S_1(G^{1(+k)})}} &= (nk)C_2 \times 2(9n+6nk-rk-4k-2r-8) \\
&\quad +(nC_2 - \frac{nr}{2}) \times 2(5n+4nk-rk-2r-2) \\
&\quad +2(nC_2)(7n+5nk-rk-2r-3k-6) + n^2k(7n+5nk-rk-2r-3k-6) \\
&\quad +2n(n-1)(12n+9nk-2rk-4r-6k-10) \\
\overline{\mathrm{S_2(G^{1(+k)})}} &= (nk)C_2 \times (9n+6nk-rk-4k-2r-8)^2 \\
&\quad +(nC_2 - \frac{nr}{2})(5n+4nk-rk-2r-2)^2 \\
&\quad +nC_2(7n+5nk-rk-2r-3k-6)^2 + n^2k(7n+5nk-rk-2r-3k-6)^2 \\
&\quad +n(n-1)(12n+9nk-2rk-4r-6k-10)^2.
\end{aligned}$$

Proof The proof follows by direct counting. $\square$

Theorem 3.2 *If G is 'r' regular graph of diameter 2 then the harmonic status connectivity index of $G^{\wedge 1(+k)}$ is*

$$\begin{aligned}
HS(G^{\wedge 1(+k)}) &= \frac{nr}{2}\frac{1}{(5n+4nk-rk-2r-2k-4)} \\
&+n\frac{2}{(5n+4nk-rk-2r-2k-4+7n+5nk-2r-4k-rk-6)} \\
&+n^2k\frac{2}{(5n+4nk-rk-2r-2k-4+9n+6nk-rk-4k-2r-8)}.
\end{aligned}$$

Proof First, we observe that if G has diameter2 then $G^{\wedge 1(+k)}$ has diameter 6. Hence from the structure we have the status of each internal vertex v_i as

$$\begin{aligned}
\sigma(v_i) &= 1(r+1)+2(r+k)+2.(n-1-r)+3rk+3(n-1-r)+4k(n-1-r) \\
&= 5n+4nk-rk-2r-2k-4.
\end{aligned}$$

Also the status of each external vertex u_i as

$$\begin{aligned}
\sigma(u_i) &= 1(k+1)+2r+3r+3(n-1-r)+4(n-1-r)+4rk+5k(n-1-r) \\
&= 7n+5nk-2r-4k-rk-6.
\end{aligned}$$

Finally the pendent vertices being the only vertices on the diametrical path have the status

$$\begin{aligned}
\sigma(w_i) &= 1+2\times 1+2(k-1)+3r+4(n-1-r)+4r+5kr+5(n-1-r)+6k(n-1-r) \\
&= 9n+6nk-rk-4k-2r-8.
\end{aligned}$$

In $G^{\wedge 1(+k)}$ there are $\frac{nr}{2}$ pairs of internal adjacent vertices, n pair of adjacent vertices formed of one internal and one external vertex and finally n^2k pairs of adjacent vertices formed of one internal and one pendant vertex.Hence the harmonic status index of $G^{\wedge 1(+k)}$ is given by

$$\begin{aligned}
HS(G^{\wedge 1(+k)}) &= \frac{nr}{2}\frac{1}{(5n+4nk-rk-2r-2k-4)} \\
&+n\frac{2}{(5n+4nk-rk-2r-2k-4+7n+5nk-2r-4k-rk-6)} \\
&+n^2k\frac{2}{(5n+4nk-rk-2r-2k-4+9n+6nk-rk-4k-2r-8)} \\
&= \frac{nr}{2}\frac{1}{(5n+4nk-rk-2r-2k-4)} \\
&+n\frac{2}{(12n+9nk-2rk-4r-4k-10)}+n^2k\frac{2}{(14n+10nk-2rk-4r-6k-12)} \\
&= \frac{nr}{2}\frac{1}{(5n+4nk-rk-2r-2k-4)}+n\frac{2}{(12n+9nk-2rk-4r-4k-10)} \\
&+n^2k\frac{1}{(7n+5nk-rk-2r-3k-6)}. \qquad \square
\end{aligned}$$

Theorem 3.3 *The harmonic status coindex of $G^{\wedge 1(+k)}$ is given by*

$$\begin{aligned}\overline{\mathrm{HS}}(G^{\wedge 1(+k)}) &= (nk)C_2\frac{1}{(9n+6nk-rk-4k-2r-8)}\\&+(nC_2-\frac{nr}{2})\frac{1}{(5n+4nk-rk-2r-2k-4)}\\&+nC_2\frac{1}{(7n+5nk-rk-2r-3k-6)}+n^2k\frac{1}{(7n+5nk-rk-2r-3k-6)}\\&+n(n-1)\frac{2}{(12n+9nk-2rk-4r-6k-10)}.\end{aligned}$$

Proof In $G^{\wedge 1(+k)}$ there are $(nk)C_2$ pairs of nonadjacent pendent vertices, $(nC_2-\frac{nr}{2})$ pairs of nonadjacent vertices formed by internal vertices, nC_2 pairs of nonadjacent vertices formed by external vertices, n^2k nonadjacent pair of vertices formed by one pendant and one internal vertex and finally $n(n-1)$ pairs of nonadjacent vertices formed by one internal and one external vertex.Hence the harmonic status connectivity coindex is given by

$$\begin{aligned}\overline{\mathrm{HS}}(G^{\wedge 1(+k)}) &= (nk)C_2\frac{2}{2(9n+6nk-rk-4k-2r-8)}\\&+(nC_2-\frac{nr}{2})\frac{2}{2(5n+4nk-rk-2r-2k-4)}\\&+nC_2\frac{2}{2(7n+5nk-rk-2r-3k-6)}\\&+n^2k\frac{2}{(5n+4nk-rk-2r-2k-4+9n+6nk-rk-4k-2r-8)}\\&+n(n-1)\frac{2}{(5n+4nk-rk-2r-2k-4+7n+5nk-2r-4k-rk-6)}\\&= (nk)C_2\frac{1}{(9n+6nk-rk-4k-2r-8)}\\&+(nC_2-\frac{nr}{2})\frac{1}{(5n+4nk-rk-2r-2k-4)}\\&+nC_2\frac{1}{(7n+5nk-rk-2r-3k-6)}+n^2k\frac{1}{(7n+5nk-rk-2r-3k-6)}\\&+n(n-1)\frac{2}{(12n+9nk-2rk-4r-6k-10)}. \qquad \square\end{aligned}$$

§4. Conclusion

We considered general l level thorn graphs and obtained in particular, status connectivity indices and coindices as well as Harmonic status indices and coindices of 0 level and first level thorn graphs for some class of graphs.

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Various Domination Energies in Graphs

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Abstract: Representing a subset of vertices in a graph by means of a matrix was introduced by E. Sampath Kumar. Let $G(V,E)$ be a graph and $S \subseteq V$ be a set of vertices. We can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if, $v_i \in S$. In this paper we study the set S being dominating set and corresponding domination energy of some class of graphs.

Key Words: Adjacency matrix, Smarandachely k-dominating set, eigenvalues, energy of graph, distance energy, Laplacian energy.

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§1. Introduction

A set $D \subseteq V$ of G is said to be a Smarandachely k-dominating set if each vertex of G is dominated by at least k vertices of S and the Smarandachely k-domination number $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if $k = 1$, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the domination number of G and denoted by $\gamma(G)$ in general.

The concept of graph energy arose in theoretical chemistry where certain numerical quantities like the heat of formation of a hydrocarbon are related to total π electron energy that can be calculated as the energy of corresponding molecular graph. The molecular graph is a representation of the molecular structure of a hydrocarbon whose vertices are the position of carbon atoms and two vertices are adjacent if there is a bond connecting them.

Eigen values and eigenvectors provide insight into the geometry of the associated linear transformation. The energy of a graph is the sum of the absolute values of the Eigen values of its adjacency matrix. From the pioneering work of Coulson []1 there exists a continuous interest towards the general mathematical properties of the total π electron energy ε as calculated within the framework of the Huckel Molecular Orbital (HMO) model. These efforts enabled one to get an insight into the dependence of ε on molecular structure. The properties of $\varepsilon(G)$ are discussed in detail in [2, 3, 4].

The importance of Eigen values is not only used in theoretical chemistry but also in analyzing structures. Car designers analyze Eigen values in order to damp out the noise to reduce

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the vibration of the car due to music. Eigen values can be used to test for cracks or deformities in a solid. Oil companies frequently use Eigen value analysis to explore land for oil. Eigen values are also used to discover new and better designs for the future.

§2. Definitions and Notations

Representation of a subset of vertices of a graph by means of a matrix was first introduced by E.Sampath Kumar [5]. Let $G(V,E)$ be a graph and $S \subseteq V$ be a set of vertices. We can represent the set S by means of a matrix as follows:

In the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S denoted by $A_S(G)$. The energy $E(G)$ obtained from the matrix $A_S(G)$ is called the set energy denoted by $E_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix as domination matrix denoted by $A_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the domination matrix $A_\gamma(G)$ is defined as domination energy denoted by $E_\gamma(G)$.

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The domination matrix of G is defined to be the square matrix $A_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the domination matrix denoted by $\kappa_1, \kappa_2, \kappa_3, \cdots, \kappa_n$ are said to be the A_γ Eigen values of G. Since the A_γ matrix is symmetric, its Eigen values are real and can be ordered $\kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \cdots \geqslant \kappa_n$. Therefore, the domination energy

$$E_\gamma = E_\gamma(G) = \sum_{i=1}^{n} |\kappa_i|. \tag{1}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy ([2]).

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \tag{2}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$ are the Eigen values of the adjacency matrix $A(G)$. Recall that in the last few years, the graph energy $E(G)$ and domination energy [9,10] or covering energy ([6]) has been extensively studied in the mathematics ([6,7]) and mathematic-chemical literature ([8,12]).

Definition 2.1(Minimal domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The domination energy $E_\gamma(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{\gamma-\min}(G)$.*

Definition 2.2(Maximal domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The domination energy $E_\gamma(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{\gamma-\max}(G)$.*

Similarly to domination energy of graph G, *distance domination energy, Laplacian domi-*

nation energy and Laplacian distance domination energy can also be defined as follows.

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The *distance matrix* of G, denoted by $D(G)$ is defined to be the square matrix $D(G) = [d_{ij}]$, where d_{ij} is the shortest distance between the vertex v_i and v_j in G. The Eigen values of the distance matrix denoted by $\mu_1, \mu_2, \mu_3, \cdots, \mu_n$ are said to be the D Eigen values of G. Since the $D(G)$ matrix is symmetric, its Eigen values are real and can be ordered $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots \geqslant \mu_n$. Therefore, the distance energy

$$E_D = E_D(G) = \sum_{i=1}^{n} |\mu_i|. \tag{3}$$

In the distance matrix $D(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the distance matrix can be considered as the *distance matrix of the set S* denoted by $D_S(G)$. The energy $E(G)$ obtained from the matrix $D_S(G)$ is called the *distance set energy* denoted by $D_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix is *distance domination matrix* denoted by $D_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the distance domination matrix $D_\gamma(G)$ is defined as *distance domination energy* denoted by $E_{D\gamma}(G)$.

The distance domination matrix of G is defined to be the square matrix $D_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the distance domination matrix denoted by $\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n$ are said to be the D_γ Eigen values of G. Since the $D_\gamma(G)$ matrix is symmetric, its D-Eigen values are real and can be ordered as $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant \cdots \geqslant \sigma_n$. Therefore, the distance domination energy

$$E_{D\gamma} = E_{D\gamma}(G) = \sum_{i=1}^{n} |\sigma_i|. \tag{4}$$

Definition 2.3(Minimal distance domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The distance domination energy $E_{D\gamma}(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{D\gamma-\min}(G)$.*

Definition 2.4(Maximal distance domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The distance domination energy $E_{D\gamma}(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{D\gamma-\max}(G)$.*

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The *Laplacian matrix* of G is denoted by $L(G)$ is defined to be the square matrix $L(G) = d(G) - A(G)$, where $A(G)$ and $d(G)$ are the adjacency matrix and diagonal matrix with vertex degree of G on the principal diagonal element respectively. The Eigen values of the Laplacian matrix denoted by $\psi_1, \psi_2, \psi_3, \cdots, \psi_n$ are said to be the L Eigen values of G. Since the $L(G)$ matrix is symmetric, its Eigen values are real and can be ordered $\psi_1 \geqslant \psi_2 \geqslant \psi_3 \geqslant \cdots \geqslant \psi_n$. Therefore, the Laplacian energy

$$E_L = E_L(G) = \sum_{i=1}^{n} |\psi_i|. \tag{5}$$

The energy $E_{L\gamma}(G)$ obtained from the matrix $L_S(G) = d(G) - A_S(G)$ is called the *Laplacian set energy* denoted by $L_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix is *Laplacian domination matrix* denoted by $L_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the Laplacian domination matrix $L_\gamma(G)$ is defined as *Laplacian domination energy* denoted by $E_{L\gamma}(G)$.

The Laplacian domination matrix of G is defined to be the square matrix $L_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the Laplacian domination matrix denoted by $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ are said to be the L_γ Eigen values of G. Since the $L_\gamma(G)$ matrix is symmetric, its L-Eigen values are real and can be ordered as $\alpha_1 \geqslant \alpha_2 \geqslant \alpha_3 \geqslant \cdots \geqslant \alpha_n$. Therefore, the Laplacian domination energy

$$E_{L\gamma} = E_{L\gamma}(G) = \sum_{i=1}^{n} |\alpha_i|. \tag{6}$$

Definition 2.5(Minimal lapalcian domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The Laplacian domination energy $E_{L\gamma}(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{L\gamma-\min}(G)$.*

Definition 2.6(Maximal lapalcian domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The Laplacian domination energy $E_{L\gamma}(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{L\gamma-\max}(G)$.*

The energy $E_{LD\gamma}(G)$ obtained from the matrix $LD_S(G) = d(G) - D_S(G)$ is called the *Laplacian distance set energy* denoted by $LD_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix is *Laplacian distance domination matrix* denoted by $LD_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the Laplacian distance domination matrix $LD_\gamma(G)$ is defined as *Laplacian distance domination energy* denoted by $E_{LD\gamma}(G)$.

The Laplacian distance domination matrix of G is defined to be the square matrix $LD_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the Laplacian distance domination matrix denoted by $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$ are said to be the LD_γ Eigen values of G. Since the $LD_\gamma(G)$ matrix is symmetric, its L-Eigen values are real and can be ordered as $\beta_1 \geqslant \beta_2 \geqslant \beta_3 \geqslant \cdots \geqslant \beta_n$. Therefore, the Laplacian distance domination energy

$$E_{LD\gamma} = E_{LD\gamma}(G) = \sum_{i=1}^{n} |\beta_i|. \tag{7}$$

Definition 2.7(Minimal Lapalcian distance domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The Laplacian dis-*

tance domination energy $E_{LD\gamma}(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{LD\gamma-\min}(G)$.

Definition 2.8(Maximal Lapalcian distance domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The Laplacian distance domination energy $E_{LD\gamma}(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{LD\gamma-\max}(G)$.*

§3. Various Domination Energies

Definition 3.1 *A book graph (B_m) consists of m quadrilaterals sharing a common edge. That is, it is a Cartesian product S_{m+1} and P_2, where S_m is a star graph and P_2 is the path graph on two nodes. Some book graphs are shown in Figure 1.*

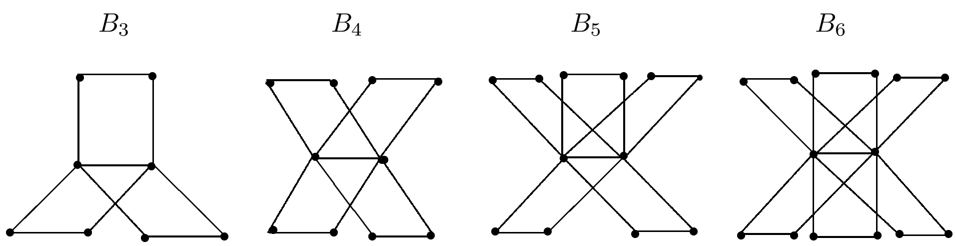

Figure 1 Book graph B_m, $3 \leqslant m \leqslant 6$

Theorem 3.1 *For $m \geqslant 3$, the minimum dominating energy of a book graph (B_m) is*

$$2\left(\sqrt{4m+1}+m-1\right).$$

Proof Calculation enables one to find the characteristic polynomial of B_m for $m \geqslant 3$ directly.

For $m = 3$, B_3 is a book graph with 8 vertices. The minimum dominating set is $S = \{v_1, v_2\}$.

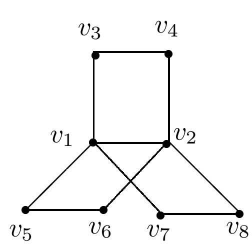

Figure 2 Book graph B_3

Calculation shows that the domination matrix and the characteristic polynomial of B_3 are respectively given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^8-2\kappa^7-9\kappa^6+12\kappa^5+18\kappa^4-18\kappa^3-13\kappa^2+8\kappa+3 = (\kappa-1)^2(\kappa+1)^2(\kappa^2-3\kappa-1)(\kappa^2+\kappa-3)$.

And calculation shows that the domination matrix and the characteristic polynomial of B_4 are respectively given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^{10}-2\kappa^9-12\kappa^8+16\kappa^7+38\kappa^6-36\kappa^5-52\kappa^4+32\kappa^3+33\kappa^2-10\kappa-8 = (\kappa-1)^3(\kappa+1)^3(\kappa^2-3\kappa-2)(\kappa^2+\kappa-4)$.

Similarly, the domination matrix and the characteristic polynomial of B_5 are respectively given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $(\kappa-1)^4(\kappa+1)^4(\kappa^2-3\kappa-3)(\kappa^2+\kappa-5)$, respectively.

And the characteristic polynomial of B_6 is given by

$$(\kappa-1)^5(\kappa+1)^5(\kappa^2-3\kappa-4)(\kappa^2+\kappa-6)$$

Generally, the characteristic polynomial of B_m using domination adjacency matrix is

$$(\kappa-1)^{m-1}(\kappa+1)^{m-1}(\kappa^2-3\kappa-(m-2))(\kappa^2+\kappa-m).$$

Solving the equation we get

$(\kappa-1)^{m-1}=0$, or $(\kappa+1)^{m-1}=0$, or $(\kappa^2-3\kappa-(m-2))=0$ or $(\kappa^2+\kappa-m)=0$. So $\kappa=1,1,1,\cdots,1$ ($(m-1)$times), or $\kappa=-1,-1,-1,\cdots,-1((m-1)\text{times})$.

By $(\kappa^2-3\kappa-(m-2))=0$, we get

$$\kappa_1=\frac{1}{2}\left(3-\sqrt{4m+1}\right) \text{ and}$$
$$\kappa_2=\frac{1}{2}\left(3+\sqrt{4m+1}\right) \quad \text{here } m\geqslant 3.$$

By $(\kappa^2+\kappa-m)=0$ we know that

$$\kappa_3=\frac{1}{2}\left(-1-\sqrt{4m+1}\right) \text{ and}$$
$$\kappa_4=\frac{1}{2}\left(-1+\sqrt{4m+1}\right)$$

Hence,

$$\begin{aligned}E_{\gamma-\min}=E_{\gamma-\min}(G)&=\sum_{i=1}^{n}|\kappa_i|\\&=(m-1)+(m-1)+\left|\frac{1}{2}\left(3-\sqrt{4m+1}\right)\right|\\&\quad+\left|\frac{1}{2}\left(3+\sqrt{4m+1}\right)\right|+\left|\frac{1}{2}\left(-1-\sqrt{4m+1}\right)\right|\\&\quad+\left|\frac{1}{2}\left(-1+\sqrt{4m+1}\right)\right|.\end{aligned}$$

Therefore,

$$E_{\gamma-\min}=E_{\gamma-\min}(B_m)=2\left(\sqrt{4m+1}+m-1\right).$$

This completes the proof. □

Theorem 3.2 *For $m\geqslant 3$, the minimum distance domination energy of a book graph (B_m) is* $4(m-1)+\sqrt{25m^2-24m+36}+\sqrt{m}\sqrt{m+4}$.

Proof Calculation enables one to find the characteristic polynomial of B_m for $m\geqslant 3$ directly.

For $m = 3$, B_3 is a book graph with 8 vertices. The minimum dominating set is $S = \{v_1, v_2\}$. Calculation shows that the distance domination matrix and the characteristic polynomial of B_3 are respectively given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 1 & 0 & 3 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 & 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 2 & 1 & 0 & 3 & 2 \\ 1 & 2 & 2 & 3 & 2 & 3 & 0 & 1 \\ 2 & 1 & 3 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^8 - 2\sigma^7 - 111\sigma^6 - 512\sigma^5 - 545\sigma^4 + 504\sigma^3 + 240\sigma^2 = \sigma^2 (\sigma+4)^2 \left(\sigma^2 - 13\sigma - 5\right)\left(\sigma^2 + 3\sigma - 3\right).$

Similarly, calculation shows that the distance domination matrix and the characteristic polynomial of B_4 are respectively given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 & 2 & 3 & 2 & 3 & 2 & 3 \\ 2 & 1 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 & 0 & 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 3 & 2 & 1 & 0 & 3 & 2 & 3 & 2 \\ 1 & 2 & 2 & 3 & 2 & 3 & 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 2 & 3 & 2 & 1 & 0 & 3 & 2 \\ 1 & 2 & 2 & 3 & 2 & 3 & 2 & 3 & 0 & 1 \\ 2 & 1 & 3 & 2 & 3 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^{10} - 2\sigma^9 - 200\sigma^8 - 1512\sigma^7 - 4048\sigma^6 - 2240\sigma^5 + 4352\sigma^4 + 1024\sigma^3 = \sigma^3 (\sigma+4)^3 \left(\sigma^2 - 18\sigma - 4\right)\left(\sigma^2 + 4\sigma - 4\right)$.

And the characteristic polynomial of B_5 and B_6 are respectively given by

$$\sigma^4 (\sigma+4)^4 \left(\sigma^2 - 23\sigma - 3\right)\left(\sigma^2 + 5\sigma - 5\right),$$
$$\sigma^5 (\sigma+4)^5 \left(\sigma^2 - 28\sigma - 2\right)\left(\sigma^2 + 6\sigma - 6\right).$$

Generally, the characteristic polynomial of B_m using the distance domination matrix is

$$\sigma^{m-1} (\sigma+4)^{m-1} \left[\sigma^2 - (5m-2)\sigma + (m-8)\right]\left(\sigma^2 + m\sigma - m\right) = 0.$$

Solving the equation we get

$\sigma^{m-1} = 0$, or $(\sigma+4)^{m-1} = 0$, or $(\sigma^2 - (5m-2)\sigma + (m-8)) = 0$, or $(\sigma^2 + m\sigma - m) = 0$. So $\sigma = 0, 0, 0, \cdots, 0$ ($(m-1)$times), or $\sigma = -4, -4, -4, \cdots, -4$ ($(m-1)$times), and $(\sigma^2 - (5m-2)\sigma + (m-8)) = 0$,

$$\begin{aligned}
\sigma_1 &= \frac{1}{2}\left(5m - 2 - \sqrt{25m^2 - 24m + 36}\right) \text{ and} \\
\sigma_2 &= \frac{1}{2}\left(5m - 2 + \sqrt{25m^2 - 24m + 36}\right) \quad \text{here } m \geqslant 3, \\
(\sigma^2 + m\sigma - m) &= 0, \\
\sigma_3 &= \frac{1}{2}\left(-m - \sqrt{m}\sqrt{m+4}\right) \text{ and} \\
\sigma_4 &= \frac{1}{2}\left(-m + \sqrt{m}\sqrt{m+4}\right) \\
E_{D\gamma-\min} &= E_{D\gamma-\min}(G) \\
&= \sum_{i=1}^{n} |\sigma_i| \\
&= 4(m-1) + \left|\frac{1}{2}\left(2\sqrt{25m^2 - 24m + 36}\right)\right| + \left|\frac{1}{2}\left(2\sqrt{m}\sqrt{m+4}\right)\right|.
\end{aligned}$$

Therefore,

$$E_{D\gamma-\min} = E_{D\gamma-\min}(G) = 4(m-1) + \sqrt{25m^2 - 24m + 36} + \sqrt{m}\sqrt{m+4}.$$

This completes the proof. □

Theorem 3.3 *For $m \geqslant 3$, the minimum Laplacian domination energy of a book graph (B_m) is $5m + \sqrt{m^2+4}$.*

Proof Calculation enables one to find the characteristic polynomial of B_m for $m \geqslant 3$ directly.

For $m = 3$, B_3 is a book graph with 8 vertices. The minimum dominating set is $S = \{v_1, v_2\}$. The Laplacian domination matrix and the characteristic polynomial of B_3 are respectively calculated by

$$L_\gamma(G) = \begin{bmatrix} 3 & -1 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & 3 & 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

and $\alpha^8-18\alpha^7+131\alpha^6-496\alpha^5+1038\alpha^4-1154\alpha^3+543\alpha^2+36\alpha-81=(\alpha-1)^2(\alpha-3)^2(\alpha^2-7\alpha+9)(\alpha^2-3\alpha-1)$.

Similarly, the Laplacian domination matrix and the characteristic polynomial of B_4 are respectively given by

$$L_\gamma(G)=\begin{bmatrix} 4 & -1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0\\ -1 & 4 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1\\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

and $\alpha^{10}-24\alpha^9+243\alpha^8-1360\alpha^7+4618\alpha^6-9792\alpha^5+12774\alpha^4-9520\alpha^3+3141\alpha^2+216\alpha-297=(\alpha-1)^3(\alpha-3)^3(\alpha^2-8\alpha+11)(\alpha^2-4\alpha-1)$

And the characteristic polynomial of B_5 and B_6 is given by $(\alpha-1)^4(\alpha-3)^4(\alpha^2-9\alpha+13)(\alpha^2-5\alpha-1)$, $(\alpha-1)^5(\alpha-3)^5(\alpha^2-10\alpha+15)(\alpha^2-6\alpha-1)$, respectively.

Generally, the characteristic polynomial of B_m using the Laplacian domination matrix is

$$(\alpha-1)^{m-1}(\alpha-3)^{m-1}(\alpha^2-(m+4)\alpha+(2m+3))(\alpha^2-m\alpha-1)=0.$$

solving the equation we get

$(\alpha-1)^{m-1}=0$, or $(\alpha-3)^{m-1}=0$, or $(\alpha^2-(m+2)\alpha+(2m+3))=0$, or $(\alpha^2-m\alpha-1)=0$. So $\alpha=1,1,1,\cdots,1$ ($(m-1)$times), or $\alpha=3,3,3,\cdots,3$ ($(m-1)$times), and $(\alpha^2-(m+2)\alpha+(2m+3))=0$,

$$\begin{aligned} \alpha_1 &= \frac{1}{2}\left(m+4-\sqrt{m^2+28}\right) \text{ and}\\ \alpha_2 &= \frac{1}{2}\left(m+4+\sqrt{m^2+28}\right) \qquad \text{here } m\geqslant 3,\\ (\alpha^2-m\alpha-1) &= 0,\\ \alpha_3 &= \frac{1}{2}\left(m-\sqrt{m^2+4}\right) \text{ and}\\ \alpha_4 &= \frac{1}{2}\left(m+\sqrt{m^2+4}\right), \end{aligned}$$

$$\begin{aligned} E_{L\gamma-\min} &= E_{L\gamma-\min}(G) = \sum_{i=1}^{n} |\alpha_i| \\ &= (m-1) + 3(m-1) + \left|\frac{1}{2}\left(2\sqrt{m^2+4}\right)\right| + \left|\frac{1}{2}\left(2(m+4)\right)\right| \end{aligned}$$

Therefore, $E_{L\gamma-\min} = E_{L\gamma-\min}(G) = 5m + \sqrt{m^2+4}$. This completes the proof. □

Theorem 3.4 *For $m \geqslant 3$, the minimum Laplacian distance domination energy of a Book Graph (B_m) is $10m - 5 + \sqrt{36m^2 - 48m + 49}$.*

Proof The characteristic polynomial of B_m for $m \geqslant 3$ can be found directly.

For $m = 3$, B_3 is a book graph with 8 vertices. The minimum dominating set is $S = \{v_1, v_2\}$. The Laplacian distance domination matrix and the characteristic polynomial of B_3 are respectively calculated by

$$LD_\gamma(G) = \begin{bmatrix} 3 & -1 & -1 & -2 & -1 & -2 & -1 & -2 \\ -1 & 3 & -2 & -1 & -2 & -1 & -2 & -1 \\ -1 & -2 & 2 & -1 & -2 & -3 & -2 & -3 \\ -2 & -1 & -1 & 2 & -3 & -2 & -3 & -2 \\ -1 & -2 & -2 & -3 & 2 & -1 & -2 & -3 \\ -2 & -1 & -3 & -2 & -1 & 2 & -3 & -2 \\ -1 & -2 & -2 & -3 & -2 & -3 & 2 & -1 \\ -2 & -1 & -3 & -2 & -3 & -1 & -1 & 2 \end{bmatrix}$$

and $\beta^8 - 18\beta^7 + 29\beta^6 + 1612\beta^5 - 16629\beta^4 + 75536\beta^3 - 181032\beta^2 + 222336\beta - 110160 = (\beta-2)^2(\beta-6)^2(\beta^2 - 9\beta + 17)(\beta^2 + 7\beta - 45)$.

Similarly, calculation shows that the Laplacian distance domination matrix and the characteristic polynomial of B_4 are respectively given by

$$LD_\gamma(G) = \begin{bmatrix} 4 & -1 & -1 & -2 & -1 & -2 & -1 & -2 & -1 & -2 \\ -1 & 4 & -2 & -1 & -2 & -1 & -2 & -1 & -2 & -1 \\ -1 & -2 & 2 & -1 & -2 & -3 & -2 & -3 & -2 & -3 \\ -2 & -1 & -1 & 2 & -3 & -2 & -3 & -2 & -3 & -2 \\ -1 & -2 & -2 & -3 & 2 & -1 & -2 & -3 & -2 & -3 \\ -2 & -1 & -3 & -2 & -1 & 2 & -3 & -2 & -3 & -2 \\ -1 & -2 & -2 & -3 & -2 & -3 & 2 & -1 & -2 & -3 \\ -2 & -1 & -3 & -2 & -3 & -2 & -1 & 2 & -3 & -2 \\ -1 & -2 & -2 & -3 & -2 & -3 & -2 & -3 & 2 & -1 \\ -2 & -1 & -3 & -2 & -3 & -2 & -3 & -2 & -1 & 2 \end{bmatrix}$$

$\beta^{10} - 24\beta^9 + 55\beta^8 + 4208\beta^7 - 66192\beta^6 + 494272\beta^5 - 2178656\beta^4 + 5934336\beta^3 - 9801216\beta^2 + 8985600\beta - 3504384 = (\beta-2)^3(\beta-6)^3(\beta^2 - 11\beta + 26)(\beta^2 + 11\beta - 78)$.

The characteristic polynomial of B_5 is given by $(\beta-2)^4(\beta-6)^4(\beta^2-13\beta+37)(\beta^2+15\beta-121)$, and the characteristic polynomial of B_6 is given by $(\beta-2)^5(\beta-6)^5(\beta^2-15\beta+50)(\beta^2+19\beta-174)$.

Generally, the characteristic polynomial of B_m using the Laplacian distance domination matrix is

$$(\beta-2)^{m-1}(\beta-6)^{m-1}(\beta^2-(2m+3)\beta+(m+1)^2+1)(\beta^2+(4m-5)\beta-(5m^2-2m+6)).$$

Solving the equation we get $(\beta-2)^{m-1}=0$, or $(\beta-6)^{m-1}=0$, or $\beta^2-(2m+3)\beta+(m+1)^2+1=0$, or $\beta^2+(4m-5)\beta-(5m^2-2m+6)=0$. So $\beta=2,2,2,\cdots,2$ ($(m-1)$times), or $\beta=6,6,6,\cdots,6$ ($(m-1)$times), and $\beta^2-(2m+3)\beta+(m+1)^2+1=0$,

$$\begin{aligned}
\beta_1 &= \frac{1}{2}\left(2m+3-\sqrt{4m+1}\right) \text{ and}\\
\beta_2 &= \frac{1}{2}\left(2m+3+\sqrt{4m+1}\right) \quad \text{here } m\geqslant 3,\\
\beta^2 &+ (4m-5)\beta-(5m^2-2m+6)=0,\\
\beta_3 &= \frac{1}{2}\left(-\sqrt{36m^2-48m+49}-4m+5\right) \text{ and}\\
\beta_4 &= \frac{1}{2}\left(\sqrt{36m^2-48m+49}-4m+5\right),\\
E_{LD\gamma-\min} &= E_{LD\gamma-\min}(G)\\
&= \sum_{i=1}^{n}|\beta_i| = 8(m-1)+\left|\frac{1}{2}(4m+6)\right|+\left|\frac{1}{2}\left(2\sqrt{36m^2-48m+49}\right)\right|.
\end{aligned}$$

Whence, $E_{LD\gamma-\min}=E_{LD\gamma-\min}(G)=10m-5+\sqrt{36m^2-48m+49}$. This completes the proof. □

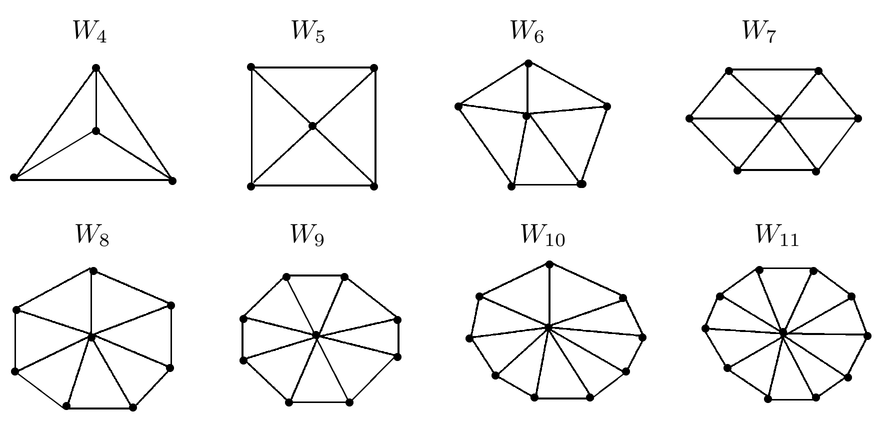

Figure 3 Wheel graph W_n, $4\leqslant n\leqslant 11$

Definition 3.2 *A wheel graph W_n of order n, sometimes simply called an n-wheel, is a graph* *that contains a cycle of order $n-1$, and for which every graph vertex in the cycle is connected to one other graph vertex (which is known as the hub). The edges of a wheel which include the hub are called spokes. The wheel W_n can be defined as the graph $K_1 + C_{n-1}$, where K_1 is the singleton graph and C_n is the cycle graph. Some wheel graphs are shown in Figure* 3.

Theorem 3.5 *For $n \geqslant 4$, the minimum dominating energy of a wheel graph (W_n) is $> \sqrt{4n-3}$.*

Proof We can find the characteristic polynomial of W_n for $n \geqslant 4$ by calculation directly.

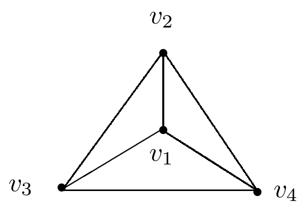

Figure 4 W_4

For $n = 4$, W_4 is a wheel graph with 4 vertices. The minimum dominating sets are $S = \{v_1\}$ or $S = \{v_2\}$ or $S = \{v_3\}$.

For $S = \{v_1\}$ the domination matrix and the characteristic polynomial of W_4 are respectively calculated by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^4 - \kappa^3 - 6\kappa^2 - 5\kappa - 1 = (\kappa^2 - 3\kappa - 1)(\kappa^2 + 2\kappa + 1)$. The characteristic polynomial is found to be same when $S = \{v_2\}$ or $S = \{v_3\}$.

For $n = 5$, W_5 is a wheel graph with 5 vertices. The minimum dominating sets is $S = \{v_1\}$. Calculation shows that the domination matrix and the characteristic polynomial of W_5 are respectively given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^5 - \kappa^4 - 8\kappa^3 - 4\kappa^2 = (\kappa^2 - 3\kappa - 2)(\kappa^3 + 2\kappa^2)$.

Similarly the characteristic polynomial of W_6, W_7 and W_8 are given by $(\kappa^2 - 3\kappa - 3)(\kappa^2 + \kappa - 1)^2$, $(\kappa^2 - 3\kappa - 4)(\kappa - 1)^2(\kappa + 1)^2(\kappa + 2)$ and $(\kappa^2 - 3\kappa - 5)(\kappa^3 + \kappa^2 - 2\kappa - \kappa)^2$, respectively.

Generally, the characteristic polynomial of W_n for $n \geqslant 4$ using domination matrix is

$$\left[\kappa^2 - 3\kappa - (n-3)\right] P(\kappa).$$

Solving the equation $(\kappa^2 - 3\kappa - (n-3)) = 0$ we get $\kappa_1 = \frac{1}{2}\left(3 - \sqrt{4n-3}\right)$ and $\kappa_2 = \frac{1}{2}\left(3 + \sqrt{4n-3}\right)$. $E_{\gamma-\min} = E_{\gamma-\min}(G) > \sum\limits_{i=1}^{2} |\kappa_i|$, $E_{\gamma-\min}(G) > \sqrt{4n-3}$. This completes the proof. □

Theorem 3.6 *For $n \geqslant 4$, the minimum distance dominating energy of a wheel graph (W_n) is $> \sqrt{4n^2 - 24n + 45}$.*

Proof The characteristic polynomial of W_n for $n \geqslant 4$ can be obtained by calculation directly.

For $n = 4$, W_4 is a wheel graph with 4 vertices. The minimum dominating sets are $S = \{v_1\}$ or $S = \{v_2\}$ or $S = \{v_3\}$. For $S = \{v_1\}$, Calculation shows that the distance domination matrix and the characteristic polynomial of W_4 are respectively given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\sigma^4 - \sigma^3 - 6\sigma^2 - 5\sigma - 1 = (\sigma^2 - 3\sigma - 1)(\sigma^2 + 2\sigma + 1)$.

The characteristic polynomial is found to be same when $S = \{v_2\}$ or $S = \{v_3\}$.

For $n = 5$, W_5 is a wheel graph with 5 vertices. The minimum dominating sets is $S = \{v_1\}$. Calculation shows that the distance domination matrix and the characteristic polynomial of W_5 are respectively given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix}$$

and $\sigma^5 - \sigma^4 - 16\sigma^3 - 20\sigma^2 = \sigma(\sigma^2 - 5\sigma + 0)(\sigma + 2)^2$.

Similarly, the characteristic polynomial of W_6, W_7 and W_8 are given by $(\sigma^2 - 7\sigma + 1)(\sigma^2 + 3\sigma + 1)^2, \sigma(\sigma^2 - 9\sigma + 2)(\sigma + 1)^2(\sigma + 3)^2$ and $(\sigma^2 - 11\sigma + 3)(\sigma^3 + 5\sigma^2 + 6\sigma + 1)^2$, respectively.

Generally, the characteristic polynomial of W_n for $n \geqslant 4$ using distance domination matrix is

$$\left[\sigma^2 - (2n-5)\sigma + (n-5)\right] P(\sigma).$$

Solving the equation $(\sigma^2 - (2n-5)\sigma + (n-5)) = 0$ we get

$$\begin{aligned} \sigma_1 &= \frac{1}{2}\left(2n - 5 - \sqrt{4n^2 - 24n + 45}\right), \\ \sigma_2 &= \frac{1}{2}\left(2n - 5 + \sqrt{4n^2 - 24n + 45}\right) \end{aligned}$$

and $E_{D\gamma-\min} = E_{D\gamma-\min}(G) > \sum_{i=1}^{2} |\sigma_i|$, $E_{D\gamma-\min}(G) > \sqrt{4n^2 - 24n + 45}$. Hence, we complete the proof. $\square$

Theorem 3.7 *For $n \geqslant 4$, the minimum Laplacian domination energy of a wheel graph (W_n) is $> \sqrt{n^2 - 2n + 5}$.*

Proof Calculation enables one to find the characteristic polynomial of W_n for $n \geqslant 4$ directly.

For $n = 4$, W_4 is a wheel graph with 4 vertices. The minimum dominating sets are $S = \{v_1\}$ or $S = \{v_2\}$ or $S = \{v_3\}$. For $S = \{v_1\}$, Calculation shows that the Laplacian domination matrix and the characteristic polynomial of W_4 are respectively given by

$$L_\gamma(G) = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

and $\alpha^4 - 11\alpha^3 + 39\alpha^2 - 40\alpha - 16 = (\alpha^2 - 3\alpha - 1)(\alpha - 4)^2$.

The characteristic polynomial is found to be same when $S = \{v_2\}$ or $S = \{v_3\}$.

For $n = 5$, W_5 is a wheel graph with 5 vertices. The minimum dominating sets is $S = \{v_1\}$. The Laplacian domination matrix and the characteristic polynomial of W_5 are respectively calculated by

$$L_\gamma(G) = \begin{bmatrix} 3 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ -1 & -1 & 0 & 3 & -1 \\ -1 & 0 & -1 & -1 & 3 \end{bmatrix}$$

and $\alpha^5 - 15\alpha^4 + 82\alpha^3 - 190\alpha^2 + 141\alpha + 45 = (\alpha^2 - 4\alpha - 1)(\alpha - 3)^2(\alpha - 5)$.

Similarly, the characteristic polynomial of W_6, W_7 and W_8 are given by $(\alpha^2 - 5\alpha - 1)\left(\alpha^2 - 7\alpha + 11\right)^2$, $(\alpha^2 - 6\alpha - 1)(\alpha - 2)^2(\alpha - 4)^2(\alpha - 5)$ and $(\alpha^2 - 7\alpha - 1)\left(\alpha^3 - 10\alpha^2 + 31\alpha - 29\right)^2$, respectively.

Generally, the characteristic polynomial of W_n for $n \geqslant 4$ using Laplacian domination matrix is

$$\left[\alpha^2 - (n-1)\alpha - 1\right] P(\alpha).$$

Solving the equation $(\alpha^2 - (n-1)\alpha - 1) = 0$ we get

$$\alpha_1 = \frac{1}{2}\left(n - 1 - \sqrt{n^2 - 2n + 5}\right)$$

and

$$\alpha_2 = \frac{1}{2}\left(n - 1 + \sqrt{n^2 - 2n + 5}\right),$$

$$E_{L\gamma-\min} = E_{L\gamma-\min}(G) > \sum_{i=1}^{2} |\alpha_i| = \sqrt{n^2 - 2n + 5}.$$

Hence the proof is completed. □

Theorem 3.8 *For $n \geqslant 4$, the minimum Laplacian distance dominating energy of a wheel graph (W_n) is $> \sqrt{9n^2 - 62n + 117}$.*

Proof The characteristic polynomial of W_n for $n \geqslant 4$ can be obtained by calculation directly.

For $n = 4$, W_4 is a wheel graph with 4 vertices. The minimum dominating sets are $S = \{v_1\}$ or $S = \{v_2\}$ or $S = \{v_3\}$. For $S = \{v_1\}$, Calculation shows that the Laplacian distance domination matrix and the characteristic polynomial of W_4 are respectively given by

$$LD_\gamma(G) = \begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

and $\beta^4 - 11\beta^3 + 39\beta^2 - 40\beta - 16 = (\beta^2 - 3\beta - 1)(\beta - 4)^2$.

The characteristic polynomial is found to be same when $S = \{v_2\}$ or $S = \{v_3\}$.

For $n = 5$, W_5 is a wheel graph with 5 vertices. The minimum dominating sets is $S = \{v_1\}$. The Laplacian distance domination matrix and the characteristic polynomial of W_5 are respectively given by

$$LD_\gamma(G) = \begin{bmatrix} 3 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & -2 \\ -1 & -1 & 3 & -2 & -1 \\ -1 & -1 & -2 & 3 & -1 \\ -1 & -2 & -1 & -1 & 3 \end{bmatrix}$$

and

$$\beta^5 - 15\beta^4 + 74\beta^3 - 94\beta^2 - 235\beta + 525 = (\beta^2 - 2\beta - 7)(\beta - 5)^2(\beta - 3).$$

Similarly, the characteristic polynomial of W_6, W_7 and W_8 are given respectively by

$$(\beta^2 - \beta - 17)\left(\beta^2 - 9\beta + 19\right)^2,$$
$$(\beta^2 + 0\beta - 31)(\beta - 6)^2(\beta - 4)^2(\beta - 3)$$

and

$$(\beta^2 + \beta - 49)\left(\beta^3 - 14\beta^2 + 63\beta - 91\right)^2.$$

Generally, the characteristic polynomial of W_n for $n \geqslant 4$ using Laplacian distance domination matrix is

$$\left[\beta^2 + (n - 7)\beta - (2n^2 - 12n + 17)\right] p(\beta).$$

Solving the equation $\beta^2 + (n-7)\beta - (2n^2 - 12n + 17) = 0$ we get

$$\begin{aligned}\beta_1 &= \frac{1}{2}\left(-\sqrt{9n^2 - 62n + 117} - n + 7\right),\\ \beta_2 &= \frac{1}{2}\left(\sqrt{9n^2 - 62n + 117} - n + 7\right)\end{aligned}$$

and

$$E_{LD\gamma-\min} = E_{LD\gamma-\min}(G) > \sum_{i=1}^{2} |\beta_i| = \sqrt{9n^2 - 62n + 117}.$$

Hence the proof is completed. □

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C-Geometric Mean Labeling of Some Ladder Graphs

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Abstract: A function f is called a C-geometric mean labeling of a graph $G(V, E)$ if $f : V(G) \to \{1, 2, 3, \cdots, q+1\}$ is injective and the induced function $f^* : E(G) \to \{2, 3, 4, \cdots, q+1\}$ defined by $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$ for all $uv \in E(G)$ is bijective. A graph that admits a C-geometric mean labeling is called a C-geometric mean graph. In this paper, we have discussed the C-geometric meanness of some ladder graphs.

Key Words: Labeling, C-geometric mean labeling, C-Geometric mean graph, Smarandache k-mean graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [5]. For a detailed survey on graph labeling we refer to [4].

Path on n vertices is denoted by P_n. $G \odot S_m$ is the graph obtained from G by attaching m pendant vertices at each vertex of G. Let G_1 and G_2 be any two graphs with p_1 and p_2 vertices respectively. Then the cartesian product $G_1 \times G_2$ has $p_1 p_2$ vertices which are $\{(u, v) : u \in G_1, v \in G_2\}$. The edges are obtained as follows: (u_1, v_1) and (u_2, v_2) are adjacent in $G_1 \times G_2$ if either $u_1 = u_2$ and v_1 and v_2 are adjacent in G_2 or u_1 and u_2 are adjacent in G_1 and $v_1 = v_2$. The ladder graph L_n is a graph obtained from the cartesian product of P_2 and P_n. The triangular ladder $TL_n, n \geq 2$ is a graph obtained by completing the ladder L_n by the edges $u_i v_{i+1}$ for $1 \leq i \leq n-1$, where L_n is the graph $P_2 \times P_n$. The slanting ladder SL_n is a graph that consists of two copies of P_n having vertex set $\{u_i : 1 \leq i \leq n\} \bigcup \{v_i : 1 \leq i \leq n\}$ and edge set is formed by adjoining u_{i+1} and v_i for all $1 \leq i \leq n-1$ ([2]).

Let P_n be a path on n vertices denoted by $u_{1,1}, u_{1,2}, u_{1,3}, \cdots, u_{1,n}$ and with $n-1$ edges denoted by $e_1, e_2, \cdots, e_{n-1}$ where e_i is the edge joining the vertices $u_{1,i}$ and $u_{1,i+1}$. On each edge e_i, erect a ladder with $n-(i-1)$ steps including the edge e_i, for $i = 1, 2, 3, \ldots, n-1$. The graph thus obtained is called a one sided step graph and it is denoted by ST_n. Let P_{2n} be

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a path on $2n$ vertices $u_{1,1}, u_{1,2}, u_{1,3}, \cdots, u_{1,2n}$ and with $2n-1$ edges $e_1, e_2, \ldots, e_{2n-1}$ where e_i is the edge joining the vertices $u_{1,i}$ and $u_{1,i+1}$. On each edge e_i, we erect a ladder with '$i+1$' steps including the edge e_i, for $i = 1, 2, 3, \ldots, n$ and on each e_i erect a ladder with $2n+1-i$ steps including e_i, for $i = n+1, n+2, \cdots, 2n-1$. The graph thus obtained is called a double sided step graph and it is denoted by $2ST_{2n}$.

The study of graceful graphs and graceful labeling methods was first introduced by Rosa [7]. The concept of mean labeling was first introduced by S. Somasundaram and R. Ponraj [8] and it was developed in [6] and [9]. In [11], R. Vasuki et al. discussed the mean labeling of cyclic snake and armed crown. In [1, 3], some graph labelings of step graphs were analyzed.

In a study of traffic, the labeling problems in graph theory can be used by considering the crowd at every junctions as the weights of a vertex and expected average traffic in each street as the weight of the corresponding edge. If we assume the expected traffic at each street as the arithmetic mean of the weight of the end vertices, that eases mean labeling of the graph. When we consider a geometric mean instead of arithmetic mean in a large population of a city, the rate of growth of traffic in each street will be more accurated. Motivated by this, we establish the geometric mean labeling on graphs.

Motivated by the works of so many authors in the area of graph labeling, we introduced a new type of labeling called C-geometric mean labeling. A function f is called a C-geometric mean labeling of a graph G if $f : V(G) \to \{1, 2, 3, \cdots, q+1\}$ is injective and the induced function $f^* : E(G) \to \{2, 3, 4, \cdots, q+1\}$ defined as

$$f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil \quad \text{for all } uv \in E(G)$$

is bijective. A graph that admits a C-geometric mean labeling is called a C-geometric mean graph.

In [10], S. Somasundaram et al. defined the geometric mean labeling as follows.

A graph $G = (V, E)$ with p vertices and q edges is said to be a geometric mean graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \cdots, q+1$ in such way that when each edge $e = uv$ is labeled with $f(uv) = \left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ or $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ then the edge labels are distinct.

In the above definition, the readers will get some confusion in finding the edge labels which edge is assigned by flooring function and which edge is assigned by ceiling function. Generally, a graph $G = (V, E)$ with p vertices and q edges is said to be a *Smarandache k-mean graph* for an integer $k \geq 2$ if it is labeled vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, \cdots, q+1$ in such way that edge $e = uv$ is labeled with $f(uv) = \left\lfloor \sqrt[k]{f(u)^k f(v)^k} \right\rfloor$ or $\left\lceil \sqrt[k]{f(u)^k f(v)^k} \right\rceil$ then the edge labels are distinct. Clearly, a Smarandache 2-mean graph is nothing else but a geometric mean labeling graph.

In [10], S. Somasundaram et al. have given the geometric mean labeling of the graph $C_5 \cup C_7$ as shown in Figure 1.

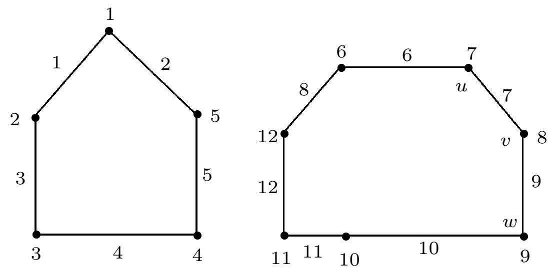

Figure 1 A geometric mean labeling of $C_5 \cup C_7$.

From the above figure, for the edge uv, they have used flooring function $\left\lfloor \sqrt{f(u)f(v)} \right\rfloor$ and for the edge vw, they have used ceiling function $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ for fulfilling their requirement. To avoid the confusion of assigning the edge labels in their definition, we just consider the ceiling function $\left\lceil \sqrt{f(u)f(v)} \right\rceil$ for our discussion. Based on our definition, the C-geometric mean labeling of the same graph $C_5 \cup C_7$ is given in Figure 2.

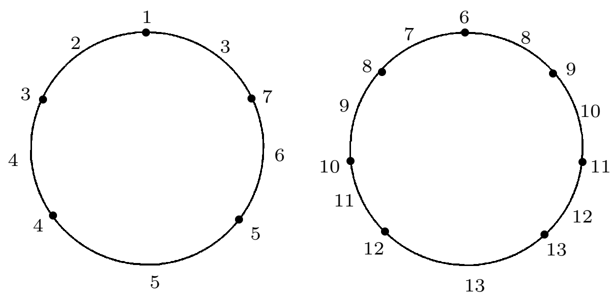

Figure 2 A C-geometric mean labeling of $C_5 \cup C_7$

In this paper, we have discussed the C-geometric mean labeling of the ladder graphs L_n for $n \geq 2$, $L_n \odot S_m$ for $n \geq 2$ and $m \leq 2$, TL_n for $n \geq 2$, $TL_n \odot S_m$ for $n \geq 2$ and $m \leq 2$, SL_n for $n \geq 2$, $SL_n \odot S_m$ for $n \geq 2$ and $m \leq 2$, step graph ST_n and double sided step graph $2ST_{2n}$.

§2. Main Results

Theorem 2.1 *The graph L_n is a C-geometric mean graph for $n \geq 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of $L_n = P_n \times P_2$. Then the ladder graph L_n having $2n$ vertices and $3n-2$ edges.

Define $f : V(L_n) \to \{1, 2, 3, \ldots, 3n-1\}$ as follows:

$$\begin{aligned} &f(u_1) = 1, \\ &f(u_i) = 3i-1, \text{ for } 2 \leq i \leq n\ , \\ &f(v_1) = 3 \text{ and} \\ &f(v_i) = 3i-2, \text{ for } 2 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$\begin{aligned} f^*(u_1u_2) &= 3, \\ f^*(u_iu_{i+1}) &= 3i+1,\ \text{for } 2 \leq i \leq n-1, \\ f^*(v_1v_2) &= 4, \\ f^*(v_iv_{i+1}) &= 3i,\ \text{for } 2 \leq i \leq n-1 \text{ and} \\ f^*(u_iv_i) &= 3i-1,\ \text{for } 1 \leq i \leq n. \end{aligned}$$

Hence, f is a C-geometric mean labeling of the ladder $P_n \times P_2$. Thus the ladder $P_n \times P_2$ is a C-geometric mean graph for $n \geq 2$. $\square$

Theorem 2.2 *The graph $L_n \odot S_m$ is a C-geometric mean graph for $n \geq 2$ and $m \leq 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of $L_n = P_n \times P_2$. Let $w_1^{(i)}, w_2^{(i)}, \cdots, w_m^{(i)}$ and $x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)}$ be the pendent vertices attached at each vertex u_i and v_i of the ladder L_n, for $1 \leq i \leq n$.

Case 1. $m = 1$.

Define $f : V(L_n \odot S_1) \to \{1, 2, 3, \cdots, 5n-1\}$ as follows:

$$\begin{aligned} f(u_1) &= 3, \\ f(u_i) &= 5i-3,\ \text{for } 2 \leq i \leq n\ , \\ f(v_1) &= 4, \\ f(v_i) &= 5i-2,\ \text{for } 2 \leq i \leq n\ , \\ f(w_1^{(i)}) &= 5i-4,\ \text{for } 1 \leq i \leq n\ , \\ f(x_1^{(1)}) &= 2 \text{ and} \\ f(x_1^{(i)}) &= 5i-1,\ \text{for } 2 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$\begin{aligned} f^*(u_iu_{i+1}) &= 5i,\ \text{for } 1 \leq i \leq n-1\ , \\ f^*(v_iv_{i+1}) &= 5i+1,\ \text{for } 1 \leq i \leq n-1\ , \\ f^*(u_1v_1) &= 4, \\ f^*(u_iv_i) &= 5i-2,\ \text{for } 2 \leq i \leq n\ , \\ f^*(u_iw_1^{(i)}) &= 5i-3,\ \text{for } 1 \leq i \leq n\ , \\ f^*(v_1x_1^{(1)}) &= 3 \text{ and} \\ f^*(v_ix_1^{(i)}) &= 5i-1,\ \text{for } 2 \leq i \leq n. \end{aligned}$$

Case 2. $m = 2$.

Define $f : V(L_n \odot S_2) \to \{1, 2, 3, \cdots, 7n-1\}$ as follows:

$$f(u_i) = \begin{cases} 3 & i = 1 \\ 7i-2 & 2 \le i \le n \text{ and } i \text{ is even} \\ 7i-5 & 2 \le i \le n \text{ and } i \text{ is odd }, \end{cases}$$

$$f(v_i) = \begin{cases} 5 & i = 1 \\ 7i-4 & 2 \le i \le n \text{ and } i \text{ is even} \\ 7i-1 & 2 \le i \le n \text{ and } i \text{ is odd }, \end{cases}$$

$$f(w_1^{(i)}) = \begin{cases} 1 & i = 1 \\ 7i-3 & 2 \le i \le n \text{ and } i \text{ is even} \\ 7i-6 & 2 \le i \le n \text{ and } i \text{ is odd }, \end{cases}$$

$$f(x_1^{(i)}) = \begin{cases} 3i+1 & 1 \le i \le 2 \\ 7i-6 & 3 \le i \le n \text{ and } i \text{ is even} \\ 7i-3 & 3 \le i \le n \text{ and } i \text{ is odd} \end{cases}$$

$$\text{and } f(x_2^{(i)}) = \begin{cases} 8 & i = 1 \\ 7i-5 & 2 \le i \le n \text{ and } i \text{ is even} \\ 7i-2 & 2 \le i \le n \text{ and } i \text{ is odd }. \end{cases}$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 6 & i = 1 \\ 7i & 2 \le i \le n-1, \end{cases}$$

$$f^*(v_i v_{i+1}) = 7i+1, \text{ for } 1 \le i \le n-1 \text{ },$$

$$f^*(u_i v_i) = 7i-3, \text{ for } 1 \le i \le n \text{ },$$

$$f^*(u_i w_1^{(i)}) = \begin{cases} 2 & i = 1 \\ 7i-2 & 2 \le i \le n \text{ and } i \text{ is even} \\ 7i-5 & 2 \le i \le n \text{ and } i \text{ is odd }, \end{cases}$$

$$f^*(u_i w_2^{(i)}) = \begin{cases} 7i-1 & 1 \le i \le n \text{ and } i \text{ is even} \\ 7i-4 & 1 \le i \le n \text{ and } i \text{ is odd }, \end{cases}$$

$$f^*(v_i x_1^{(i)}) = \begin{cases} 7i-5 & 1 \le i \le n \text{ and } i \text{ is even} \\ 7i-2 & 1 \le i \le n \text{ and } i \text{ is odd} \end{cases}$$

$$\text{and } f^*(v_i x_2^{(i)}) = \begin{cases} 7 & i = 1 \\ 7i - 4 & 2 \le i \le n \text{ and } i \text{ is even} \\ 7i - 1 & 2 \le i \le n \text{ and } i \text{ is odd .} \end{cases}$$

Hence, f is a C-geometric mean labeling of the graph $L_n \odot S_m$. Thus the graph $L_n \odot S_m$ is a C-geometric mean graph for $n \ge 2$ and $m \le 2$. □

Theorem 2.3 *The graph TL_n is a C-Geometric mean graph for $n \ge 2$.*

Proof Let the vertex set of TL_n be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and the edge set of TL_n be $\{u_i u_{i+1}; 1 \le i \le n-1\} \cup \{v_i v_{i+1}; 1 \le i \le n-1\} \cup \{u_i v_i; 1 \le i \le n\} \cup \{v_i u_{i+1}; 1 \le i \le n-1\}$. Then TL_n has $2n$ vertices and $4n-3$ edges.

Define $f : V(TL_n) \to \{1, 2, 3, \cdots, 4n-2\}$ as follows:

$$\begin{aligned} f(u_i) &= 4i - 3, \text{ for } 1 \le i \le n, \\ f(v_i) &= 4i - 1, \text{ for } 1 \le i \le n-1 \text{ and} \\ f(v_n) &= 4n - 2. \end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$\begin{aligned} f^*(u_i u_{i+1}) &= 4i - 1, \text{ for } 1 \le i \le n-1, \\ f^*(u_i v_i) &= 4i - 2, \text{ for } 1 \le i \le n, \\ f^*(v_i v_{i+1}) &= 4i + 1, \text{ for } 1 \le i \le n-1 \text{ and} \\ f^*(v_i u_{i+1}) &= 4i, \text{ for } 1 \le i \le n-1. \end{aligned}$$

Hence f is a C-geometric mean labeling of TL_n. Thus the triangular ladder TL_n is a C-geometric mean graph for $n \ge 2$. □

Theorem 2.4 *The graph $TL_n \odot S_m$ is a C-geometric mean graph for $n \ge 2$ and $m \le 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of TL_n. Let $w_1^{(i)}, w_2^{(i)}, \cdots, w_m^{(i)}$ *and* $x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)}$ be the pendent vertices attached at each vertex u_i and v_i of the ladder L_n, for $1 \le i \le n$.

Case 1. $m = 1$.

Define $f : V(TL_n \odot S_1) \to \{1, 2, 3, \cdots, 6n-2\}$ as follows:

$$\begin{aligned} f(u_1) &= 3, \\ f(u_i) &= 6i - 4, \text{ for } 2 \le i \le n \ , \end{aligned}$$

$$\begin{aligned}
f(v_i) &= 6i-2, \text{ for } 1 \le i \le n\ ,\\
f(w_1^{(i)}) &= 6i-5, \text{ for } 1 \le i \le n\ ,\\
f(x_1^{(1)}) &= 2 \text{ and}\\
f(x_1^{(i)}) &= 6i-3, \text{ for } 2 \le i \le n.
\end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$\begin{aligned}
f^*(u_iu_{i+1}) &= 6i-1, \text{ for } 1 \le i \le n-1\ ,\\
f^*(v_iv_{i+1}) &= 6i+1, \text{ for } 1 \le i \le n-1\ ,\\
f^*(v_iu_{i+1}) &= 6i, \text{ for } 1 \le i \le n-1\ ,\\
f^*(u_1v_1) &= 4,\\
f^*(u_iv_i) &= 6i-3, \text{ for } 2 \le i \le n\ ,\\
f^*(u_iw_1^{(i)}) &= 6i-4, \text{ for } 1 \le i \le n\ ,\\
f^*(v_1x_1^{(1)}) &= 3 \text{ and}\\
f^*(v_ix_1^{(i)}) &= 6i-2, \text{ for } 2 \le i \le n.
\end{aligned}$$

Case 2. $m = 2$.

Define $f : V(TL_n \odot S_2) \to \{1, 2, 3, \cdots, 8n-2\}$ as follows:

$$\begin{aligned}
f(u_1) &= 3,\\
f(u_i) &= 8i-3, \text{ for } 2 \le i \le n\ ,\\
f(v_1) &= 5,\\
f(v_i) &= 8i-5, \text{ for } 2 \le i \le n\ ,\\
f(w_1^{(1)}) &= 1,\\
f(w_1^{(i)}) &= 8i-4, \text{ for } 2 \le i \le n\ ,\\
f(w_2^{(1)}) &= 2,\\
f(w_2^{(i)}) &= 8i-2, \text{ for } 2 \le i \le n\ ,\\
f(x_1^{(1)}) &= 4,\\
f(x_1^{(i)}) &= 8i-7, \text{ for } 2 \le i \le n\ ,\\
f(x_2^{(1)}) &= 6 \text{ and}\\
f(x_2^{(i)}) &= 8i-6, \text{ for } 2 \le i \le n.
\end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$\begin{aligned}
f^*(u_1u_2) &= 7,\\
f^*(u_iu_{i+1}) &= 8i+1, \text{ for } 2 \le i \le n-1\ ,
\end{aligned}$$

$$
\begin{aligned}
f^*(v_1v_2) &= 8,\\
f^*(v_iv_{i+1}) &= 8i-1, \text{ for } 2 \le i \le n-1 \ ,\\
f^*(u_iv_i) &= 8i-4, \text{ for } 1 \le i \le n \ ,\\
f^*(v_1u_2) &= 9,\\
f^*(v_iu_{i+1}) &= 8i, \text{ for } 2 \le i \le n-1 \ ,\\
f^*(u_1w_1^{(1)}) &= 2,\\
f^*(u_iw_1^{(i)}) &= 8i-3, \text{ for } 2 \le i \le n \ ,\\
f^*(u_1w_2^{(1)}) &= 3,\\
f^*(u_iw_2^{(i)}) &= 8i-2, \text{ for } 2 \le i \le n \ ,\\
f^*(v_1x_1^{(1)}) &= 5,\\
f^*(v_ix_1^{(i)}) &= 8i-6, \text{ for } 2 \le i \le n \ ,\\
f^*(v_1x_2^{(1)}) &= 6 \text{ and}\\
f^*(v_ix_2^{(i)}) &= 8i-5, \text{ for } 2 \le i \le n.
\end{aligned}
$$

Hence, f is a C-geometric mean labeling of the graph $TL_n \odot S_m$. Thus the graph $TL_n \odot S_m$ is a C-geometric mean graph for $n \geq 2$ and $m \leq 2$. □

Theorem 2.5 *The graph SL_n is a C-geometric mean graph for $n \geq 2$.*

Proof Let the vertex set of SL_n be $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and the edge set of SL_n be $\{u_iu_{i+1}; 1 \le i \le n-1\} \cup \{v_iv_{i+1}; 1 \le i \le n-1\} \cup \{v_iu_{i+1}; 1 \le i \le n-1\}$. Then SL_n has $2n$ vertices and $3n-3$ edges.

Define $f : V(SL_n) \to \{1, 2, 3, \cdots, 3n-2\}$ as follows:

$$
\begin{aligned}
f(u_1) &= 1,\\
f(u_i) &= 3i-4, \text{ for } 2 \le i \le n,\\
f(v_i) &= 3i, \text{ for } 1 \le i \le n-1 \text{ and}\\
f(v_n) &= 3n-2.
\end{aligned}
$$

Then the induced edge labeling is obtained as follows:

$$
\begin{aligned}
f^*(u_1u_2) &= 2,\\
f^*(u_iu_{i+1}) &= 3i-2, \text{ for } 2 \le i \le n-1,\\
f^*(v_iv_{i+1}) &= 3i+2, \text{ for } 1 \le i \le n-2,\\
f^*(v_{n-1}v_n) &= 3n-2 \text{ and}\\
f^*(v_iu_{i+1}) &= 3i, \text{ for } 1 \le i \le n-1.
\end{aligned}
$$

Hence f is a C-geometric mean labeling of SL_n. Thus the slanting ladder SL_n is a C-geometric mean graph for $n \geq 2$. □

Theorem 2.6 *The graph $SL_n \odot S_m$ is a C-geometric mean graph for $n \geq 2$ and $m \leq 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of SL_n. Let $w_1^{(i)}, w_2^{(i)}, \cdots, w_m^{(i)}$ *and* $x_1^{(i)}, x_2^{(i)}, \cdots, x_m^{(i)}$ be the pendent vertices attached at each vertex u_i and v_i of the ladder L_n, for $1 \leq i \leq n$.

Case 1. $m = 1$ and $n \geq 3$.

Define $f : V(SL_n \odot S_1) \to \{1, 2, 3, \cdots, 5n-2\}$ as follows:

$$\begin{aligned}
f(u_1) &= 2,\\
f(u_i) &= 5i-6, \text{ for } 2 \leq i \leq n\ ,\\
f(v_1) &= 6,\\
f(v_i) &= 5i, \text{ for } 2 \leq i \leq n-1\ ,\\
f(v_n) &= 5n-2,\\
f(w_1^{(1)}) &= 1,\\
f(w_1^{(i)}) &= 5i-7, \text{ for } 2 \leq i \leq n\ ,\\
f(x_1^{(1)}) &= 7,\\
f(x_1^{(i)}) &= 5i+1, \text{ for } 2 \leq i \leq n-1 \text{ and}\\
f(x_1^{(n)}) &= 5n-3.
\end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$\begin{aligned}
f^*(u_iu_{i+1}) &= \begin{cases} 3i & 1 \leq i \leq 2 \\ 5i-3 & 3 \leq i \leq n-1, \end{cases}\\
f^*(v_iv_{i+1}) &= 5i+3, \text{ for } 1 \leq i \leq n-2\ ,\\
f^*(v_{n-1}v_n) &= 5n-3,\\
f^*(v_iu_{i+1}) &= 5i, \text{ for } 1 \leq i \leq n-1\ ,\\
f^*(u_1w_1^{(1)}) &= 2,\\
f^*(u_iw_1^{(i)}) &= 5i-6, \text{ for } 2 \leq i \leq n\ ,\\
f^*(v_1x_1^{(1)}) &= 7,\\
f^*(v_ix_1^{(i)}) &= 5i+1, \text{ for } 2 \leq i \leq n-1 \text{ and}\\
f^*(v_nx_1^{(n)}) &= 5n-2.
\end{aligned}$$

Case 2. $m = 2$ and $n \geq 3$.

Define $f : V(SL_n \odot S_2) \rightarrow \{1, 2, 3, \cdots, 7n-2\}$ as follows:

$$f(u_i) = \begin{cases} 2i+1 & 1 \leq i \leq 2 \\ 7i-6 & 3 \leq i \leq n-1 \text{ and } i \text{ is even} \\ 7i-9 & 3 \leq i \leq n-1 \text{ and } i \text{ is odd} , \end{cases}$$

$$f(u_n) = \begin{cases} 7n-10 & n \text{ is even} \\ 7n-9 & n \text{ is odd} , \end{cases}$$

$$f(v_i) = \begin{cases} 9 & i = 1 \\ 7i+2 & 2 \leq i \leq n-3 \text{ and } i \text{ is even} \\ 7i-1 & 2 \leq i \leq n-3 \text{ and } i \text{ is odd} , \end{cases}$$

$$f(v_{n-2}) = \begin{cases} 7n-13 & n \text{ is even} \\ 7n-15 & n \text{ is odd} , \end{cases}$$

$$f(v_{n-1}) = 7n-5,$$

$$f(v_n) = 7n-3,$$

$$f(w_1^{(i)}) = \begin{cases} 1 & i = 1 \\ 6i-8 & 2 \leq i \leq 3 \\ 7i-7 & 4 \leq i \leq n-1 \text{ and } i \text{ is even} \\ 7i-10 & 4 \leq i \leq n-1 \text{ and } i \text{ is odd} , \end{cases}$$

$$f(w_1^{(n)} = \begin{cases} 7n-11 & n \text{ is even} \\ 7n-10 & n \text{ is odd} , \end{cases}$$

$$f(w_2^{(i)}) = \begin{cases} 4i-2 & 1 \leq i \leq 2 \\ 7i-5 & 3 \leq i \leq n-1 \text{ and } i \text{ is even} \\ 7i-8 & 3 \leq i \leq n-1 \text{ and } i \text{ is odd} , \end{cases}$$

$$f(w_2^{(n)} = \begin{cases} 7n-7 & n \text{ is even} \\ 7n-8 & n \text{ is odd} , \end{cases}$$

$$f(x_1^{(i)}) = \begin{cases} 8 & i = 1 \\ 7i & 2 \leq i \leq n-3 \text{ and } i \text{ is even} \\ 7i-3 & 2 \leq i \leq n-3 \text{ and } i \text{ is odd} , \end{cases}$$

$$f(x_1^{(n-2)}) = \begin{cases} 7n-12 & n \text{ is even} \\ 7n-17 & n \text{ is odd} , \end{cases}$$

$$f(x_1^{(n-1)}) = \begin{cases} 7n-8 & n \text{ is even} \\ 7n-7 & n \text{ is odd} , \end{cases}$$

$$f(x_1^{(n)}) = 7n-4,$$

$$f(x_2^{(i)}) = \begin{cases} 11 & i = 1 \\ 7i+1 & 2 \le i \le n-3 \text{ and } i \text{ is even} \\ 7i-2 & 2 \le i \le n-3 \text{ and } i \text{ is odd}\ , \end{cases}$$

$$f(x_2^{(n-2)}) = \begin{cases} 7n-9 & n \text{ is even} \\ 7n-16 & n \text{ is odd}\ , \end{cases}$$

$$f(x_2^{(n-1)}) = 7n-6$$

and $f(x_2^{(n)}) = 7n-2.$

Then the induced edge labeling is obtained as follows:

$$f^*(u_iu_{i+1}) = \begin{cases} 4i & 1 \le i \le 2 \\ 7i-4 & 3 \le i \le n-2\ , \end{cases}$$

$$f^*(u_{n-1}u_n) = \begin{cases} 7n-13 & n \text{ is even} \\ 7n-11 & n \text{ is odd}\ , \end{cases}$$

$$f^*(v_iv_{i+1}) = \begin{cases} 12 & i = 1 \\ 7i+4 & 2 \le i \le n-3\ , \end{cases}$$

$$f^*(v_{n-2}v_{n-1}) = \begin{cases} 7n-9 & n \text{ is even} \\ 7n-10 & n \text{ is odd}\ , \end{cases}$$

$$f^*(v_{n-1}v_n) = 7n-4,$$

$$f^*(v_iu_{i+1}) = 7i, \text{ for } 1 \le i \le n-1\ ,$$

$$f^*(u_iw_1^{(i)}) = \begin{cases} 2 & i = 1 \\ 6i-7 & 2 \le i \le 3 \\ 7i-6 & 4 \le i \le n-1 \text{ and } i \text{ is even} \\ 7i-9 & 4 \le i \le n-1 \text{ and } i \text{ is odd}\ , \end{cases}$$

$$f^*(u_nw_1^{(n)}) = \begin{cases} 7n-10 & n \text{ is even} \\ 7n-9 & n \text{ is odd}\ , \end{cases}$$

$$f^*(u_iw_2^{(i)}) = \begin{cases} 3i & 1 \le i \le 2 \\ 7i-5 & 3 \le i \le n-1 \text{ and } i \text{ is even} \\ 7i-8 & 3 \le i \le n-1 \text{ and } i \text{ is odd}\ , \end{cases}$$

$$f^*(u_nw_2^{(n)}) = 7n-8,$$

$$f^*(v_ix_1^{(i)}) = \begin{cases} 9 & i = 1 \\ 7i+1 & 2 \le i \le n-3 \text{ and } i \text{ is even} \\ 7i-2 & 2 \le i \le n-3 \text{ and } i \text{ is odd}\ , \end{cases}$$

$$f^*(v_{n-2}x_1^{(n-2)}) = \begin{cases} 7n-12 & n \text{ is even} \\ 7n-16 & n \text{ is odd}\ , \end{cases}$$

$$f^*(v_{n-1}x_1^{(n-1)}) = 7n-6,$$

$$f^*(v_n x_1^{(n)}) = 7n-3,$$

$$f^*(v_i x_2^{(i)}) = \begin{cases} 10 & i=1 \\ 7i+2 & 2 \le i \le n-3 \text{ and } i \text{ is even} \\ 7i-1 & 2 \le i \le n-3 \text{ and } i \text{ is odd}\ , \end{cases}$$

$$f^*(v_{n-2}x_2^{(n-2)}) = \begin{cases} 7n-11 & n \text{ is even} \\ 7n-15 & n \text{ is odd}\ , \end{cases}$$

$$f^*(v_{n-1}x_2^{(n-1)}) = 7n-5$$

and $f^*(v_n x_2^{(n)}) = 7n-2.$

Case 3. $m = 1, 2$ and $n = 2$.

The C-geometric mean labeling of $SL_2 \odot S_1$ and $SL_2 \odot S_2$ is given in Figure 3.

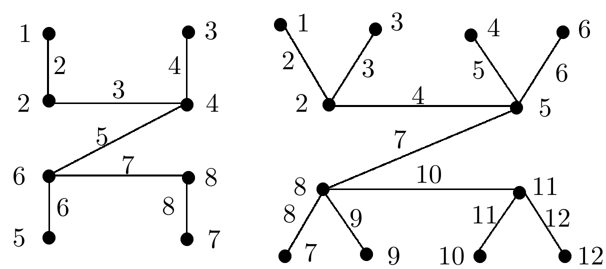

Figure 3 The C-geometric mean labeling of $SL_2 \odot S_1$ and $SL_2 \odot S_2$.

Hence, f is a C-geometric mean labeling of the graph $SL_n \odot S_m$. Thus the graph $SL_n \odot S_m$ is a C-geometric mean graph for $n \ge 2$ and $m \le 2$. □

Theorem 2.7 *The graph ST_n is a C-geometric mean graph for $n \ge 2$.*

Proof Let $u_{1,1}, u_{1,2}, u_{1,3}, \cdots, u_{1,n}, u_{2,1}, u_{2,2}, u_{2,3}, \cdots, u_{2,n}, u_{3,1}, u_{3,2}, u_{3,3}, \cdots, u_{3,n-1}, u_{4,1}, u_{4,2}, u_{4,3}, \cdots, u_{4,n-2}, \cdots, u_{n,1}, u_{n,2}$ be the vertices of the step graph ST_n.

In $u_{i,j}$, i denotes the row (counted from bottom to top) and j denotes the column (counted from left to right) in which the vertex occurs.

Define $f : V(ST_n) \to \{1, 2, 3, \cdots, n^2 + n - 1\}$ as follows: For $1 \le i \le n-1$,

$$f(u_{i,j}) = \begin{cases} (n+1-i)^2 + 2(j-1) & 1 \le j \le \lfloor \frac{n+2-i}{2} \rfloor \\ (n+1-i)(n+3-i) - 2j + 1 & \lfloor \frac{n+2-i}{2} \rfloor + 1 \le j \le n+1-i, \end{cases}$$

$$f(u_{i,n+2-i}) = (n+1-i)(n+3-i), \text{ for } 2 \le i \le n-1 \ ,$$
$$f(u_{n,1}) = 3 \text{ and}$$
$$f(u_{n,2}) = 1.$$

Then the induced edge labeling is obtained as follows:

For $1 \le i \le n-2$,

$$f^*(u_{i,j}u_{i,j+1}) = \begin{cases} (n+1-i)^2 + 2j - 1 & 1 \le j \le \lfloor \frac{n+2-i}{2} \rfloor - 1 \\ (n+1-i)^2 + 2j - 1 & j = \lfloor \frac{n+2-i}{2} \rfloor \text{ and } i \text{ is odd} \\ (n+1-i)(n+3-i) - 2j & j = \lfloor \frac{n+2-i}{2} \rfloor \text{ and } i \text{ is even} \\ (n+1-i)(n+3-i) - 2j & \lfloor \frac{n+2-i}{2} \rfloor + 1 \le j \le n-i, \end{cases}$$

$$f^*(u_{n-1,1}u_{n-1,2}) = 5,$$
$$f^*(u_{n,1}u_{n,2}) = 2,$$
$$f^*(u_{i,n+1-i}u_{i+1,n+2-i}) = (n+1-i)(n+2-i), \text{ for } 2 \le i \le n-1,$$
$$f^*(u_{i,1}u_{i+1,1}) = (n+1-i)(n-i), \text{ for } 1 \le i \le n-2,$$
$$f^*(u_{n-1,1}u_{n,1}) = 4,$$

For $1 \le i \le n-3$,

$$f^*(u_{i,j}u_{i+1,j}) = \begin{cases} (n+1-i)(n-i) + 2j - 1 & 2 \le j \le \lfloor \frac{n+2-i}{2} \rfloor - 1 \\ (n+1-i)(n-i) + 2j - 1 & j = \lfloor \frac{n+2-i}{2} \rfloor \text{ and } i \text{ is odd} \\ (n+1-i)(n+2-i) - 2j & j = \lfloor \frac{n+2-i}{2} \rfloor \text{ and } i \text{ is even} \\ (n+1-i)(n+2-i) - 2j & \lfloor \frac{n+2-i}{2} \rfloor + 1 \le j \le n-i, \end{cases}$$

$$f^*(u_{n-2,2}u_{n-1,2}) = 8,$$
$$f^*(u_{n-1,2}u_{n,2}) = 3,$$
$$f^*(u_{i,n+1-i}u_{i+1,n+1-i}) = (n+1-i)^2, \text{ for } 1 \le i \le n-2$$
$$\text{and } f^*(u_{n-1,2}u_{n,2}) = 3.$$

Hence, f is a C-geometric mean labeling of ST_n. Thus the step graph ST_n is a C-geometric mean graph, for $n \ge 2$. □

Theorem 2.8 *The graph $2ST_{2n}$ is a C-geometric mean graph, for $n \ge 2$.*

Proof Let $u_{1,1}, u_{1,2}, u_{1,3}, \cdots, u_{1,n}, u_{2,1}, u_{2,2}, u_{2,3}, \cdots, u_{2,2n}, u_{3,1}, u_{3,2}, u_{3,3}, \cdots, u_{3,2n-2}, u_{4,1}, u_{4,2}, u_{4,3}, \cdots, u_{4,2n-4}, \cdots, u_{n+1,1}, u_{n+1,2}$ be the vertices of the double sided step graph $2ST_{2n}$.

In $u_{i,j}$, i denotes the row (counted from bottom to top) and j denotes the column (counted from left to right) in which the vertex occurs.

Define $f : V(2ST_{2n}) \rightarrow \{1, 2, 3, \cdots, 2n^2 + 3n\}$ as follows:

$$f(u_{1,j}) = \begin{cases} 2n^2 + n + 1 + 2(j-1) & 1 \leq j \leq n \\ 2n^2 + 3n - 2(j - n - 1) & n + 1 \leq j \leq 2n, \end{cases}$$

for $2 \leq i \leq n$ and $2 \leq j \leq n + 2 - i$,

$$f(u_{i,j}) = 2(n + 1 - i)^2 + (n + 2 - i) + 2(j - 2),$$

for $2 \leq i \leq n$ and $n + 3 - i \leq j \leq 2n + 3 - 2i$,

$$\begin{aligned} f(u_{i,j}) &= 2(n + 1 - i)^2 + 3(n + 1 - i) - 2(i + j - n - 3), \\ f(u_{2,1}) &= 2n^2 + n - 2, \\ f(u_{1,1}) &= 3, \\ f(u_{1,2}) &= 1, \\ f(u_{i,1}) &= 2(n + 2 - i)^2 + n - i, \text{ for } 3 \leq i \leq n \text{ and} \\ f(u_{i,2n+4-2i}) &= 2(n + 2 - i)^2 + n + 1 - i, \text{ for } 2 \leq i \leq n. \end{aligned}$$

Then the induced edge labeling is obtained as follows:

$$f^*(u_{1,j}u_{1,j+1}) = \begin{cases} 2n^2 + n + 2 + 2(j-1) & 1 \leq j \leq n \\ 2n^2 + 3n + 1 - 2(j - n) & n + 1 \leq j \leq 2n - 1, \end{cases}$$

for $2 \leq i \leq n - 1$ and $2 \leq j \leq n + 2 - i$,

$$f^*(u_{i,j}u_{i,j+1}) = 2(n + 1 - i)^2 + (n + 3 - i) + 2(j - 2),$$

for $2 \leq i \leq n - 1$ and $n + 3 - i \leq j \leq 2n + 2 - 2i$,

$$\begin{aligned} f^*(u_{i,j}u_{i,j+1}) &= 2(n + 1 - i)^2 + 3(n + 1 - i) + 1 - 2(i + j - n - 2), \\ f^*(u_{i,2n+3-2i}u_{i+1,2n+2-2i}) &= 2(n + 1 - i)^2 + (n + 2 - i), \text{ for } 2 \leq i \leq n - 1, \\ f^*(u_{n,3}u_{n+1,2}) &= 3, \\ f^*(u_{n,2}u_{n,3}) &= 5, \\ f^*(u_{n+1,1}u_{n+1,2}) &= 2, \\ f^*(u_{1,1}u_{2,1}) &= 2n^2 + n, \\ f^*(u_{1,2n}u_{2,2n}) &= 2n^2 + n + 1, \\ f^*(u_{i,2}u_{i+1,1}) &= 2(n + 1 - i)^2 + n + 1 - i, \text{ for } 2 \leq i \leq n - 1, \\ f^*(u_{n,2}u_{n+1,1}) &= 4, \end{aligned}$$

$$f^*(u_{1,j}u_{2,j}) = \begin{cases} 2n^2 - n + 2 + 2(j-2) & 2 \le j \le n \\ 2n^2 + n - 1 - 2(j-n-1) & n+1 \le j \le 2n-1, \end{cases}$$

for $2 \le i \le n-1$ and $3 \le j \le n+2-i$,

$$f^*(u_{i,j}u_{i+1,j-1}) = 2(n+1-i)^2 - (n+1-i) + 2(j-2),$$

for $2 \le i \le n-1$ and $n+3-i \le j \le 2n+2-2i$,

$$f^*(u_{i,j}u_{i+1,j-1}) = 2(n+1-i)^2 + (n+4-i) - 2(i+j-n-1),$$
$$f^*(u_{i,1}u_{i,2}) = 2(n+1-i)^2 + 3(n+1-i) + 1, \text{ for } 2 \le i \le n \text{ and}$$
$$f^*(u_{i,2n+3-2i}u_{i,2n+4-2i}) = 2(n+1-i)^2 + 3(n+1-i) + 2, \text{ for } 2 \le i \le n.$$

Hence, f is a C-geometric mean labeling of $2ST_{2n}$. Thus the double sided step graph $2ST_{2n}$ is a C-geometric mean graph, for $n \ge 2$. □

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Edge Hubtic Number in Graphs

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Abstract: The maximum order of partition of the edge set $E(G)$ into edge hub sets is called edge hubtic number of G and denoted by $\xi_e(G)$. In this paper, we determine the edge hubtic number of some standard graphs. Also we obtain bounds for $\xi_e(G)$. In addition we characterize the class of all (p, q) graphs for which $\xi_e(G) = q$.

Key Words: Edge hubtic number, edge hub number, partition.

AMS(2010): 05C40, 05C99.

§1. Introduction

By a graph $G = (V, E)$, we mean a finite and undirected graph without loops and multiple edges. A graph G with p vertices and q edges is called a (p, q) graph, the number p is referred to as the order of a graph G and q is referred to as the size of a graph G. In general, the degree of a vertex v in a graph G denoted by $deg(v)$ is the number of edges of G incident with v. The degree of an edge uv is defined to be $deg(u) + deg(v) - 2$. Also $\Delta^{\backprime}(G)$ denotes the maximum degree among the edges of G, and $\delta^{\backprime}(G)$ denotes the minimum degree among the edges of G. $\lfloor x \rfloor$ is the greatest integer less than or equal to x. In a tree, a leaf is a vertex of degree one, a leaf edge is an edge incident to a leaf. We refer to [6] for terminology and notations not defined here.

Introduced by Walsh [13], a hub set in a graph G is a set H of vertices in G such that any two vertices outside H are connected by a path whose internal vertices lie in H. The hub number of G, denoted by $h(G)$, is the minimum size of a hub set in G. A connected hub set in G is a vertex hub set F such that the subgraph of G induced by F (*denoted* $G[F]$) is connected.

Let G be a graph, let $e = (u, v)$ and $f = (u_1, v_1)$, a path between two edges e and f is a path between one end vertex from e and another end vertex from f such that $d(e, f) = min\{d(u, u_1), (u, v_1), (v, u_1), (v, v_1)\}$. Internal edges of a path between two edges e and f are all the edges of the path except e and f [11]. A subset $H_e \subseteq E(G)$ is called an edge hub set of G if every pair of edges $e, f \in E \setminus H_e$ are connected by a path where all internal edges are from H_e. The minimum cardinality of an edge hub set is called edge hub number of G, and is denoted by $h_e(G)$ [11]. An edge hub set $H_e \subseteq E(G)$ is called a connected edge hub set, if the subgraph $[H_e]$ is connected. The minimum cardinality of a connected edge hub set of G

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is called a connected edge hub number and is denoted by $h_{ce}(G)$ [1]. For more details on the hub studies we refer to [10]. Graphs G_1, and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Their union, $G = G_1 \cup G_2$ has, as expected, $V = V_1 \cup V2$ and $E = E_1 \cup E_2$ [6].

A set D of vertices in a graph G is called dominating set of G if every vertex in $V \setminus D$ is adjacent to some vertex in D, the minimum cardinality of a dominating set in G is called the domination number $\gamma(G)$ of a graph G ([7].

A set B of edges in a graph G is called an edge dominating set of G if every edge in $E \setminus B$ is adjacent to some edge in B, the minimum cardinality of an edge dominating set in G is called the edge domination number $\gamma'(G)$ of a graph G ([7]). An edge-domatic partition of G is a partition of $E(G)$, all of whose classes are edge-dominating sets in G. The maximum number of classes of an edge-domatic partition of G is called the edge-domatic number of G and denoted by $ed(G)$ ([1]).

A double star $S_{n,m}$ is the tree obtained from two disjoint stars $K_{1,n-1}$ and $K_{1,m-1}$ by connecting their centers [5]. The line graph $L(G)$ of G has the edges of G as it is vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G [6]. A friendship graph, is the graph obtained by taking m copies of the cycle graph C_3 with a vertex in common and denoted by F_m. The following results will be useful in the proof of our results.

Theorem 1.1([10]) *For any graph G, $h_e(G) \leq q - \Delta'(G)$, and the inequality is sharp for any path P_p, $p \geq 4$.*

Proposition 1.1([10]) *For any graph G, $h_e(G) \leq p - 3$.*

Theorem 1.2([10]) *For any tree T with $p \geq 3$ vertices and l leaves,*
$h_e(T) = h_{ce}(T) = p - (l + 1)$.

Proposition 1.2([9]) *For any graph G, $\xi(G) \leq \delta(G) + 2$.*

§2. Main Results

Definition 2.1 *The maximum order of partition of the edge set $E(G)$ into edge hub sets is called edge hubtic number of G and denoted by $\xi_e(G)$. The maximum order of partition of the edge set $E(G)$ into connected edge hub sets is called connected edge hubtic number of G and denoted by $\xi_{ce}(G)$.*

It is obvious that $\xi_e(G) \geq \xi_{ce}(G)$, since $h_e(G) \leq h_{ce}(G)$. We first determine the edge hubtic number of some standard graphs.

Observation 2.1 (1) For any cycle C_p,

$$\xi_e(C_p) = \begin{cases} 3, & \text{if } p = 3 ; \\ 4, & \text{if } p = 4 ; \\ 2, & \text{if } p = 5, 6 ; \\ 1, & \text{if } p \geq 7. \end{cases}$$

(2) For any path P_p,

$$\xi_e(P_p) = \begin{cases} 3, & \text{if } p = 4 ; \\ 2, & \text{if } p = 3, 5 ; \\ 1, & \text{if } p \geq 6. \end{cases}$$

(3) For the wheel graph $W_{1,p-1}$, $p \geq 4$,

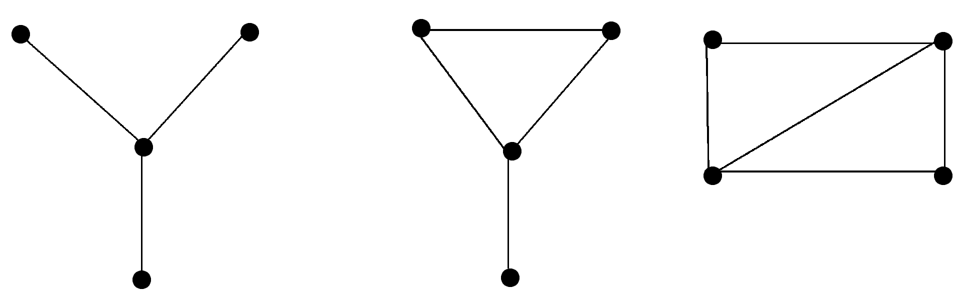
$$\xi_e(W_{1,p-1}) = \begin{cases} 6, & \text{if } p = 4 ; \\ 4, & \text{if } p = 5 ; \\ 3, & \text{if } p \geq 6. \end{cases}$$

(4) For the star $K_{1,p-1}$, $\xi_e(K_{1,p-1}) = p - 1$.

(5) For the double star $S_{n,m}$, $\xi_e(S_{n,m}) = 3$.

(6) For the complete bipartite graph $K_{n,m}$, $\xi_e(K_{n,m}) = max\{n, m\}$.

We will check that if the edge hubtic number is a suitable measure of stability?. Now we ask, does the edge hubtic number discriminate between graphs. There are many examples of graphs which propose that $\xi_e(G)$ is a suitable measure of stability which is able to discriminate between graphs. For example, consider the graphs G_1, G_2 and G_3 in Figure 1.



G_1 G_2 G_3

Figure 1: G_1, G_2, and G_3.

It is clear from Figure 1, that $ed(G_1) = ed(G_2) = ed(G_3) = 3$, the edge domatic number does not discriminate between graphs G_1, G_2 and G_3, but $\xi_e(G_1) = 3$, $\xi_e(G_2) = 4$ and $\xi_e(G_3) = 5$, therefore $\xi_e(G_1) \neq \xi_e(G_2) \neq \xi_e(G_3)$. So the edge hubtic number discriminates between graphs G_1, G_2 and G_3.

Observation 2.2 For any graph G, $0 \leq \xi_e(G) \leq q$.

Theorem 2.1 *If a graph G is a tree with at least 3 non-leaf edges and the induced sub graph $G[(E \setminus L)]$ is not a star where L is the set of all leaf edges in G, then $\xi_e(G) = 1$.*

Proof Let a graph G be a tree with at least 3 non-leaf edges and the induced sub graph $G[(E \setminus L)]$ is not a star, we discuss the following cases:

Case 1. Suppose that H_e is a set of all non-leaf edges, clearly any path between two leaf edges does not pass through another leaf edge. So, H_e is an edge hub set of G, and by Theorem 1.2 it is minimum edge hub set. Now, suppose $Z_e \subseteq E \setminus H_e$ be an edge hub set of G. Since G is a tree with at least 3 non-leaf edges and the induced sub graph $G[(E \setminus L)]$ is not a star, then the induced subgraph $G[E \setminus Z_e]$ is not complete. Also any path in a tree never passes through a leaf edge. Therefore there are at least two non adjacent edges $e, f \in E \setminus Z_e$ such that no path between them is in Z_e, this is a contradiction. Hence H_e is the only edge hub set.

Case 2. Suppose that H_e is an edge hub set of G but not containing all non-leaf edges. Since G has at least three non-leaf edges, let $\{e_1, e_2, e_3\}$ be non-leaf edges where e_1 and e_3 not adjacent, let l_1, l_3 be two leaf edges adjacent to e_1 and e_3, respectively. Clearly, $G[\{l_1, e_1, e_2, e_3, l_3\}]$ is a path P_6. As $h_e(P_6) = 3$, then H_e contains at least three edges from P_6. Therefore any other edge hub set of G must intersects H_e since size of P_6 is 5. Then $\xi_e(G) = 1$. □

Proposition 2.1 *For any (p, q)-graph G, $\xi_e(G) \leq \frac{q}{h_e(G)}$, where $h_e(G) \neq 0$.*

Proof Let $H = \{H_1, H_2, H_3, \cdots, H_t\}$, be the edge hubtic partition of G and $\xi_e(G) = t$. Clearly $|H_i| \geq h_e(G), i = 1, 2, \cdots, t$ and we get $q = \sum_{i=1}^{t} |H_i| \geq t h_e(G)$, hence the result. □

Observation 2.4 Let G' be a subgraph of G, then is not necessary $\xi_e(G') \leq \xi_e(G)$.

For example, $G = K_1 + P_4$, and $G' = K_1 + P_3$, $\xi_e(G') = 5 \nleq 3 = \xi_e(G)$.

Proposition 2.2 *For any (p, q)-graph G of order $p \geq 5$,*

$$\xi_e(G) \leq \delta'(G) + 2.$$

Proof By the definition of edge hub number it is obvious that $h_e(G) = h(L(G))$, so $\xi_e(G) = \xi(L(G))$. By Proposition 1.2, $\xi_e(G) = \xi(L(G)) \leq \delta(L(G)) + 2$, since $\delta'(G) = \delta(L(G))$, the result follows. □

Corollary 2.1 *For any (p, q)-graph G of order $p \geq 5$,*

$$\xi_e(G) + h_e(G) \leq \delta'(G) + p - 1.$$

Proof By Proposition 1.1 and Proposition 2.2, we get the result. □

Theorem 2.2 *For any (p,q)-graph G of order p, $\xi_e(G)+\xi_e(\overline{G}) \leq \frac{p(p-1)}{2}$, and the inequality is sharp for stars $K_{1,3}$, and $K_{1,4}$.*

Proof By Observation 2.2, $\xi_e(G) \leq q$ and $\xi_e(\overline{G}) \leq \overline{q}$. Then

$$\xi_e(G)+\xi_e(\overline{G}) \leq q+\overline{q} = \frac{p(p-1)}{2}. \qquad \square$$

Theorem 2.3 *Let G be a (p,q)-graph. Then*

$$\xi_e(G)+h_e(G) \leq q+2.$$

Proof By Theorem 1.1, $h_e(G) \leq q-\Delta'(G)$. Hence $h_e(G) \leq q-\delta'(G)$. Proposition 2.2, completes the proof. $\square$

Observation 2.5 If $\xi_e(G_1)=\xi_e(G_2)$,then not necessary $h_e(G_1)=h_e(G_2)$.

For example, $G_1=K_{1,3}$, and $G_2=F_3$ such that $\xi_e(G_1)=\xi_e(G_2)=3$, and $h_e(G_1)=0 \neq 3=h_e(G_2)$.

Theorem 2.4 *Let G be a graph of size q. Then $\xi_e(G)=q$ if and only if G with $\delta' \geq q-2$.*

Proof Assume that $\xi_e(G)=q$, then there is a q partition of $E(G)$ into edge hub sets and every partite set consists of one edge, we have the following cases:

Case 1. All edges of G are adjacent, so any edge of G is an edge hub set of G. So $\delta'=q-1$.

Case 2. Any edge of degree $q-1$, is adjacent to all edges and hence it constitute an edge hub set of G, and since any edge of degree $q-2$, is adjacent to all edges of G except one, so every edge of them must be an edge hub set for G, hence $\delta'(G)=q-2$, if we consider any edge f such that $deg(f) < q-2$, in this case let $deg(f)=q-3$, so there is two edges e_1, e_2 not adjacent to f, now if the set $\{f\}$ is an edge hub set for G then e_1 must be adjacent to e_2, but by this assumption $\{e_1\}$ is not edge hub set for G, since e_2 not adjacent to f and e_1 not a path between them. So $\xi_e(G)=q$ only if the graph G satisfies $\delta'(G) \geq q-2$. Converse is obvious. $\square$

Proposition 2.3 *For any two connected graphs G_1 and G_2,*

$$\xi_e(G_1 \cup G_2) = \begin{cases} 1, & \text{if } G_1 \text{ or } G_2 \text{ is with } \delta' < q-1; \\ 2, & \text{if } G_1 \text{ and } G_2 \text{ are with } \delta' = q-1. \end{cases}$$

Proof Let G_1, G_2 be two graphs both with $\delta'=q-1$, clearly $E(G_1)$ is an edge hub set for $G_1 \cup G_2$ and $E(G_2)$ is an edge hub set of the same graph, therefore $\xi_e(G_1 \cup G_2)=2$. Suppose that G_1 or G_2 is with $\delta' < q-1$, then any edge hub set of $G_1 \cup G_2$ must contain all of the edges of G_1 and any edge hub set of G_2, therefore $\xi_e(G_1 \cup G_2)=1$. $\square$

Corollary 2.2 *For any disconnected graph G with $m \geq 3$ components, $\xi_e(G)=1$.*

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MATHEMATICAL REVIEW

Mathematical 4th Crisis: to Reality

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Abstract: There are 3 crises in the development of mathematics from its internal, and particularly, the 3th crisis extensively made it to be consistency in logic, which finally led to its more and more abstract, but getting away the reality of things. It should be noted that the original intention of mathematics is servicing other sciences to hold on the reality of things but today's mathematics is no longer adequate for the needs of other sciences such as those of theoretical physics, complex system and network, cytology, biology and economy developments change rapidly as the time enters the 21st century. Whence, a new crisis appears in front of mathematicians, i.e., *how to keep up mathematics with the developments of other sciences?* I call it the 4th crisis of mathematics from the external, i.e., the original intention of mathematics because it is the main topic of human beings.

Key Words: Mathematical crisis, reality, contradiction, TAO TEH KING, mathematical universe hypothesis, Smarandachely denied axiom, Smarandache multispace, mathematical combinatorics, traditional Chinese medicine.

AMS(2010): 03A05.

§1. Introduction

As we known, one or the main function of mathematics in science is it can establish exact mathematical expressions for scientific models on things. Certainly, a theory can not be without the practice, and it can be only from the practice. By this view, the creating source of mathematics can be only from solving problems appeared in practice of human beings, and then move its method and technique upward a mathematica theory for understanding the reality of things in the world.

Usually, a thing is complex, even hybrid with other things sometimes. Then, *what is the reality of a thing?* The reality of a thing is its state of existed, exists, or will exist in the world,

[1]An extended version of the preface of my book *Mathematical Reality – My Philosophy on Mathematics with Reality*, Published in USA, August, 2018.

[2]Received March 16, 2018, Accepted August 26, 2018.

independent on the understanding of human beings, which implies that the reality holds on by human beings maybe local or gradual, not the reality of a thing. Hence, to hold on the reality of things is the main objective of science in the history of human development.

But, *a mathematical conclusion really reflects the reality of a thing?* The answer is not certain because the practice of human beings shows the mathematical conclusion do not correspond to the reality of a thing sometimes, for instance the *Ames Room*. Usually, the understanding of a thing is by observation of human beings, which is dependent on the observable model, data collection by scientific instruments with data processing by mathematics. Such an observation brings about a unilateral, or an incomplete knowledge on a thing. In this case, the mathematical conclusion reflects partial datum, not all the collection, and in fact, all collection data (by different observers with different model) with data processed is not a mathematical system, even with contradictions in usual unless a data set, which implies that there are no mathematical subfields applicable.

We all know that it appeared 3 crises in mathematical development. In each time, mathematics itself was enriched, improved and completed. However, along with the solving process of the 3th mathematical crisis, the trend of mathematical developing in 19th and 20th centuries shows that a mathematical system is more concise, and its conclusion is more extended, then farther to the reality of things because it abandons more and more characters of things. Besides, more and more researchers only pay attention on questions or problems in himself branch along with the mathematical branch divided, and few peoples consider his research whether is or not valuable for developing the whole mathematics, for understanding the nature and beneficial to human progress, which finally results in mathematics father to the practice of human beings, weaker for hold on the true face of things in the world.

As the time enters the 21st century, science developments change rapidly, and meanwhile, a few global questions constantly emerge, such as those of local war, food safety, epidemic spreading network, environmental protection, multilateral trade dispute, more and more questions accompanied with the overdevelopment and applying the internet, $\cdots$, etc. Clearly, today's mathematics is no longer adequate for the needs of other sciences. It is far falling behind the development of society. A new crisis appears in front of mathematicians, i.e., *how to keep up mathematics with the developments of other sciences for hold on the reality of things?* I think this is a big and more important problem in the development of mathematics in the 21th century, and call it the *mathematical 4th crisis* because holding on the reality of things is the central objective of human beings.

The main purpose of the review is analyzing this crisis and points out the way of one how to out this crisis by establishing new mathematical theory, also provides an envelope theory, i.e., *mathematical combinatorics* as the candidate for the way.

§2. Be Understood or Not

For reality of things, an elementary but fundamental question should be answered first. That is, *can one really holds on the reality of things?* For this question, there are two but quite opposite answers. One is the reality of things can not completely understanding, i.e., one can only holds on the approximate reality of things. Another is one can finally understanding the reality of all things, i.e., *Theory of Everything*. We respectively discuss them following.

Not Understood. There is a well-known philosophical book: *TAO TEH KING* written by an ideologist *Lao Zi* in ancient China. In this book, it discussed extensively on the relation of *TAO*, a more general object than the reality with name and things, and shown in its first but central chapter ([8]):

The Tao that experienced is not the eternal Tao;
The Name named is not the eternal Name;
The unnamable is the eternally real and naming is the origin of all particular things;
Freely desire, you realize the mystery but caught in desire, you see only the manifestations;
The mystery and manifestations arise from the same source called darkness;
The darkness within darkness, the gateway to all understanding.

For explaining the relation of Tao with knowing ability of human beings respectively in his Chapter 42:

Tao gives birth to One, One gives birth to Two, Two gives birth to Three and Three gives birth to all things;

All things have their backs to the female and stand facing the male. When male and female combine, all things achieve harmony.

and also in Chapter 25:

Human beings follow the earth, Earth follows the universe,

The universe follows the Tao and the Tao follows only itself.

By the view of Lao Zi, the reality of things is not understood because the Tao that experienced is not the eternal Tao, the Name named is not the eternal Name, and human beings is born after Three along with Three gives birth to all things, particularly, the reality. I agree Lao Zi's notion, i.e., it is difficult to know the reality of all things, and all mathematical reality is only approximate reality, not the reality. For Tao, One and Two before Three, we can only analyze their various possibility by science, can not really hold on their true faces.

Be Understood. The notion is the supporting and main trending in scientific community today, i.e., the reality of all things can be understood by human beings and one can finally holds on and become the dominate of the world. Particularly, the physical world is nothing else but a mathematical structure ([12], [13]) by Tegmark Max, a famous Swedish-American physicist and cosmologist in MIT now.

Here, I would like to analyze 2 hypotheses, i.e., the Big Bang and mathematical universe hypothesis on the physical world.

1. Big Bang Hypothesis. The Big Bang model states that the earliest state of the Universe was an extremely hot and dense one, and that the Universe subsequently expanded and cooled, which is based on general relativity following:

Applying his principle of general relativity, i.e. *all the laws of physics take the same form in any reference system* and the equivalence principle, i.e., *there are no difference for physical effects of the inertial force and the gravitation in a field small enough*, Einstein got the equation of gravitational field

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu},$$

where $R_{\mu\nu} = R_{\nu\mu} = R^{\alpha}_{\mu i \nu}$,

$$R^{\alpha}_{\mu i \nu} = \frac{\partial \Gamma^i_{\mu i}}{\partial x^{\nu}} - \frac{\partial \Gamma^i_{\mu\nu}}{\partial x^i} + \Gamma^{\alpha}_{\mu i}\Gamma^i_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^i_{\alpha i},$$

$$\Gamma^g_{mn} = \frac{1}{2} g^{pq}(\frac{\partial g_{mp}}{\partial u^n} + \frac{\partial g_{np}}{\partial u^m} - \frac{\partial g_{mn}}{\partial u^p})$$

and $R = g^{\nu\mu}R_{\nu\mu}$.

Combining the Einstein's equation of gravitational field with the cosmological principle, i.e., *there are no difference at different points and different orientations at a point of a universe on the metric* $10^4 l.y.$, Friedmann got a standard model of universe. The metrics of the standard universe are

$$ds^2 = -c^2dt^2 + a^2(t)[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]$$

and

$$g_{tt} = 1, \ g_{rr} = -\frac{R^2(t)}{1-Kr^2}, g_{\phi\phi} = -r^2R^2(t)\sin^2\theta.$$

The standard model of cosmos finally enables the birth of Big Bang model for the cosmos in thirties of the 20th century, and finally, the NASA's explorer mission WMAP(Wilkinson Microwave Anisotropy Probe) determined the radius of the universe was $13.7b.l.y$ on Big Bang hypothesis.

Mathematical Universe Hypothesis. The mathematical universe hypothesis proposed by Max Tegmark, is a speculative *Theory of Everything*, which claims that *our external physical reality is a mathematical structure*([12], [13]), i.e., the physical universe is not merely described by mathematics, but is mathematics (specifically, a mathematical structure), which implies the mathematical existence equals to that the physical existence, and all structures that exist mathematically exist physically as well. And observers, including humans ourself, are *self-aware substructures (SASs)*, and in any mathematical structure complex enough to contain such a substructure, it will subjectively perceive itself as existing in a physically real' world.

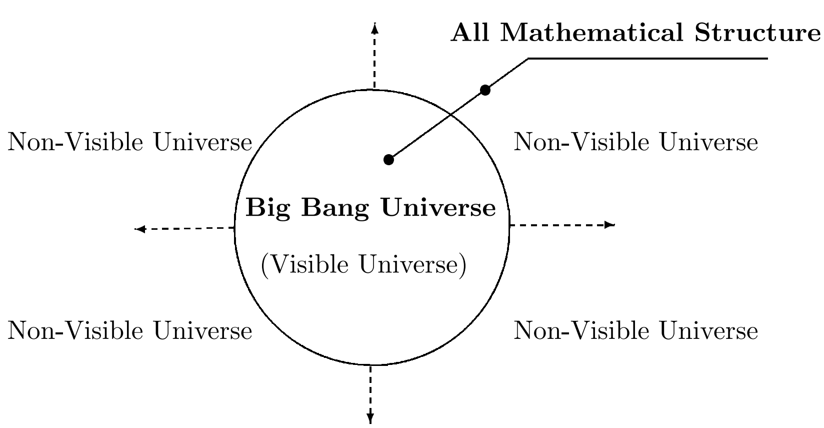

Fig.1

According to Lao Zi's birth ruler, the WMAP is essentially determined the radius of visible universe by human beings is $13.7b.l.y$, but we can not claim the radius of universe is $13.7b.l.y$ just by the Big Bang hypothesis. Otherwise, we are in an awkward situation and can not answer what is the outer of the sphere of radius $13.7b.l.y$ unless its radius is finite. The advantage of Max Tegmark's hypothesis is it avoids the finite or not of the universe but claims its physical reality is a mathematical structure. These 2 hypotheses are simply shown in Fig.1.

Therefore, the Big Bang hypothesis is only a notion locally on the universe. But why various experimental of human beings verifies it maybe right just because our human beings are after Three, i.e., after the Big Bang by Lao Zi, and the Friedmann's standard model of universe is a special solution of Einstein's gravitational equations, which is essentially to explain the general by special cases. However, we have many solutions on Einstein's gravitational equations, even with constant $\lambda = 0$ ([2]). Certainly, the Max Tegmark's hypothesis is on the whole universe but it also contains lethal deficiency following:

(1) If the Big Bang hypothesis is right, i.e., we can only hold on the reality of the visible universe, how can we verify the external universe, i.e., non-visible unverse is mathematics or not;

(2) *Is our mathematical theory can already be used for understanding the reality of all things in the world?* The answer is not certain because mathematics is homogenous without contradictions, i.e. a compatible one in logic but contradictions exist everywhere in the world by philosophy. Thus, the reality known by mathematics on things can be only a subset of the reality set ([4], [5]), i.e., the mathematical structure is not equal to the physical reality.

All of these show that even if the Big Bang and Max Tegmark's hypotheses are both right, we also need to establish a new mathematical theory so that the mathematical structure is equal to the physical reality, i.e., a mathematical crisis is confronted with mathematicians.

§3. Mathematical Crisis in 21th Century

3.1 Brief Review 3 Crises of Mathematics in History

As we known, there are 3 crises in the development of mathematics following, each of them motivates mathematics itself constantly enriched, improved and completed..

First Crisis. The early Pythagorean mathematics was based on the so-called *commensurability principle*, i.e., Pythagorean's assertion: "*everything is a number*". According to this principle two geometric values Q and V have common measure, divisible by it, i.e., their ratio can be expressed as the ratio of the relative prime numbers m and n. However, Hippasu, a member of Pythagorean's found the length of the diagonal of a unit square is $\sqrt{2}$, which can not be as a ratio of two relative prime numbers, i.e., it is an irrational number. This discovery became a *turning point* in mathematics development, which ruined the former system of Pythagoreans, extended the rational to real numbers and finally resulted in new mathematical theories.

Second Crisis. Even at present, calculus is a subject with the most widely applying to other science for hold on reality of things. However, its foundations refers to the rigorous development of the subject in its early time. The cause was the unrigorous use of infinitesimal quantities in that time, which resulted in the second crisis of mathematics, i.e., the foundation of calculus. Certainly, there are many mathematicians work hard for going out this crisis, formed new mathematical theories. For example, the limitation of Weierstrass eventually became common of calculus base, instead of infinitesimal quantities as the rigorous approach, and established real analysis which included full definitions, theorems with rigorous proof of calculus.

Third Crisis. The third crisis of mathematics came from the foundation of mathematics, i.e., set theory by Russell paradox following:

Let R be the set of all sets that are not members of themselves. If R is not a member of itself, then its definition dictates that it must contain itself, and if it contains itself, then it contradicts its own definition as the set of all sets that are not members of themselves, i.e.,

$$R = \{x|x \notin x\}, \text{ then } R \in R \Leftrightarrow R \notin R.$$

Russell paradox finally resulted in the establishing of axiomatic set theory, i.e., Russell's type theory and the Zermelo set theory in 1908.

3.2 Mathematical Crisis in 21th Century

The mathematical crisis, or the 4th crisis in 21th century does not come from its internal but in the external needs or in its original intension. As we discussed, the axiomatic and abstract on mathematics in the 19th and 20th centuries finally results in mathematics away from practice. This trend also found by physicists in 20th century. Einstein once complain mathematics: *as far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.* Besides, more and more problems appeared in the practice can not find an applicable mathematics and don't know how to hold on their characters. In fact, there are more examples supporting this claim with social development in 21th century.

Elementary Particle. We have known matters consist of two classes particles, i.e., bosons with integer spin n, fermions with fractional spin $n/2$, $n \equiv 1(\text{mod}2)$, and by a widely held view, the elementary particles consists of quarks, leptons with interaction quanta including photons and other particles of mediated interactions ([6], [7]), which constitute hadrons, i.e., mesons, baryons and their antiparticles.

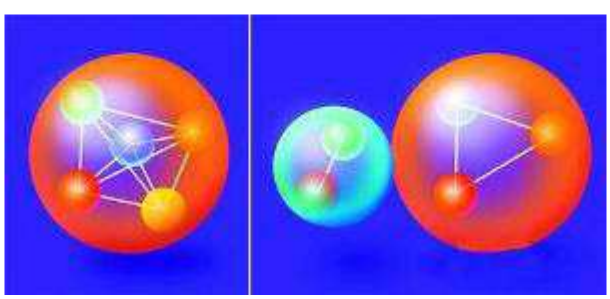

Fig.2

Although quark model is a formal classifying scheme for hadrons, i.e., the quarks and antiquarks of Sakata, or Gell-Mann and Ne'eman, it appeared subconscious in the multiverse interpretation of H.Everett on the superposition of particle. It should be noted that the multiverse interpretation or quark is proposed by physicist for explaining behavior of particles without an applicable mathematics. However, it completely changed the usual notion that a particle is an geometrical point or a subset of space, and opened a new way for understanding the reality of a hadron in notion, i.e., we are not need to insist again that a hadron is a geometrical point or a subset of space such as those of assumptions in determinable science. For example, a baryon is predominantly formed by three quarks, and a meson is mainly composed of a quark and an antiquark in quark models, such as those shown in the right of Fig.2, where a particle consists of 5 quarks can be also found on the left.

Biological Population. The biological populations are dependent each other by food web, i.e., a natural interconnection of food chains and a graphical representation of what-eats-what in an ecological community on the earth. For example, a food chain starts from producer organisms (such as grass or trees which use radiation from the sun to make their food) and end at apex predator species (like grizzly bears or killer whales), detritivores (like earthworms or woodlice), or decomposer species (such as fungi or bacteria). Usually, a model of a biological system is converted into a system of equations. The solution of the equations, by either analytical or numerical means, describes how the biological system behaves either over time or at equilibrium. In fact, a food web is an interaction system in physics which can be mathematically characterized by the strength of what action on what. For a biological 2-system, let x, y be the two species with the action strength $F'(x \to y)$, $F(y \to x)$ of x to y and y to x on their growth rate, ([1]). Such a biological 2-system can be quantitatively characterized by differential equations

$$\begin{cases} \dot{x} = F(y \to x) \\ \dot{y} = F'(x \to y) \end{cases}$$

on the populations of species x and y. However, this method can be only applied to the small number (≤ 3) of populations in this web. If the number m of populations≥ 4, such as those shown birds in Fig.3,

Fig.3

a natural way for characterizing the behavior of m birds is to collect all dynamic equations of cells, i.e.,

$$\begin{cases} \dot{\overline{x}}_1 = F_1(t, \overline{x}_1) + \sum\limits_{j \neq 1} H_j(\overline{x}_j \to \overline{x}_1) \\ \dot{\overline{x}}_2 = F_2(t, \overline{x}_2) + \sum\limits_{j \neq 2} H_j(\overline{x}_j \to \overline{x}_2) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{\overline{x}}_m = F_m(t, \overline{x}_m) + \sum\limits_{j \neq m} H_j(\overline{x}_j \to \overline{x}_m) \end{cases},$$

where $F_i : \mathbb{R}^3 \to \mathbb{R}^3$ is generally a nonlinear function characterizing the external appearance of ith cell and $H_j(x_j \to x_i)$ is the action strength of the jth cell to the ith cell in this system for integers $1 \leq i, j \leq m$.

However, this system maybe non-solvable, i.e., one can not characterize the behavior of birds in the hope that of solutions. Thus, even in mathematical biology one has no a mathematical branch applicable for the reality of biology unless partially by differential equations and statistical analysis.

Regional Economy. Today, the regional trade regulation enables each one of his member develops extensively on other members. Achieving mutual benefit, and finally striving for trade balance is the purpose of the regional trade organization. This situation appears both in the global or area economy because there are few countries or areas still in self-sufficient today. The trade surplus and deficit usually result in the trade disputing in members, processes the multilateral negotiations, and then reaches a new regulation for members in the international trade. Usually, one can obtains statistical data published by customs or statistical services in a country or an area, but there are no a mathematical subfield for characterizing the global or local changing in economy.

Furthermore, we can easily get other fields that there are no mathematical subfields applicable. For example, the complex network, including community network, epidemic spreading network with their behaviors.

Why I say it to be the 4th crisis of mathematics? The reason is that mathematics is gradually ignored and replaced by computer simulation in society development. It already become a subject of indulge in self-admiration by mathematician ourselves. If we do not turn it around, it will be abandoned sooner or later by the society finally.

§4. Out of the Crisis

The essence of out of the 4th crisis of mathematics is return to the original intension, i.e., unveiling the reality of things. For such a objective, two main things should be processed first on today's mathematics. One is the contradiction between things such that different things should be in equal rights, and another is the dependence of things because there is a universal connection between things by philosophy, which implies one should redefine elements in mathematics.

Let us discuss the two things flowing.

1. Smarandache Multispace. Today, we have known a kind of geometry breaking through the non-contradiction in classical mathematics, i.e., *Smarandache geometry* (1969) by introducing a new type axiom for space. An axiom is said to be *Smarandachely denied* if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways. A Smarandache geometry [10] is a geometry which has at least one Smarandachely denied axiom (1969). If $\mathscr{A}$ is a Smarandache denied axiom on space $\mathscr{T}$, then all points in $\mathscr{T}$ with $\mathscr{A}$ validated or invalided consist of points sets $T^{H(\mathscr{A})}$ and $T^{N(\mathscr{A})}$, and if it is in multiple distinct ways invalided, without loss of generality, let s be its multiplicity. Then all points of $\mathscr{T}$ are classified into $T_1^{\mathscr{A}}, T_2^{\mathscr{A}}, \cdots, T_s^{\mathscr{A}}$. Hence, we get a partition on points of space $\mathscr{T}$ as follow:

$$\mathscr{T} = T^{H(\mathscr{A})} \bigcup T^{N(\mathscr{A})}, \quad \text{or} \quad \mathscr{T} = T_1^{\mathscr{A}} \bigcup T_2^{\mathscr{A}} \bigcup \cdots \bigcup T_s^{\mathscr{A}}.$$

This shows that $\mathscr{T}$ should be a Smarandace multispace.

Generally, let $(\Sigma_1; \mathcal{R}_1)$, $(\Sigma_2; \mathcal{R}_2)$, $\cdots$, $(\Sigma_m; \mathcal{R}_m)$ be m mathematical spaces, different two by two. A *Smarandache multispace* $\widetilde{\Sigma}$ is a union $\bigcup\limits_{i=1}^{m} \Sigma_i$ with rules $\widetilde{\mathcal{R}} = \bigcup\limits_{i=1}^{m} \mathcal{R}_i$ on $\widetilde{\Sigma}$, i.e., the rule $\mathcal{R}_i$ on Σ_i for integers $1 \leq i \leq m$ ([3],[9]-[10]). Thus, the reality of things, whatever its accurate or approximate should be characterized or found out on Smarandache multispaces. Whence, the Smarandache multispace solved better the contradiction in classical mathematics. However, an abstract Smarandache multispace is nothing else but an algebraic or set problem ([11]), which worked out finely the equal rights, but

(1) To be also new conceptions accumulation;

(2) Not solve the universal connection of things;

(3) Can not extensively applies achievements in today's mathematics, $\cdots$, etc..

Thus, for understanding the reality of things, a new envelope theory should be established on Smarandache multispace, i.e., *mathematical combinatorics.*

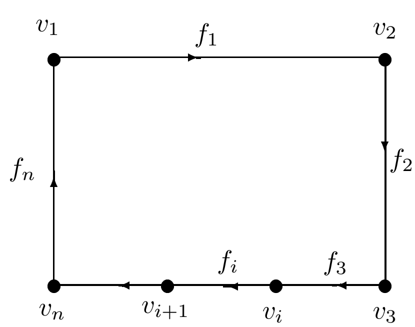

Fig.4

2. Mathematical Combinatorics. *What is mathematical combinatorics?* The mathematical combinatorics is such a mathematics over topological graphs $\overrightarrow{G}$, i.e., establish an envelope mathematics on the elements of universal connection of things, which worked out finely both the equal rights and universal connection of things. And *how to combine classical mathematics with topological graphs $\overrightarrow{G}$?* I found a typical set of labeled graphs $\overrightarrow{G}^L$, called *continuity flows* can be viewed as mathematical elements, i.e., labeling their edges by elements in a Banach space $\mathscr{B}$ with two end-operators on $\mathscr{B}$ and holding on the continuity equation on each vertex in $\overrightarrow{G}$. For example, such a continuity flow over $\overrightarrow{C}_n$ is shown in Fig.4, where, $A^+_{v_iv_{i+1}} = 1$, $A^+_{v_iv_{i-1}} = 2$ and

$$f_i = \frac{f_1 + \left(2^{i-1} - 1\right) F(t, \overline{x})}{2^{i-1}}$$

for integers $1 \leq i \leq n$ Then, such a set of labeled graphs $\overrightarrow{G}^L$ inherits the character of today's mathematics, i.e., if $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ are oriented topological graphs and $\mathscr{B}$ a Banach space, then all such labeled graphs $\overrightarrow{G}^L$ with linear end-operators is also a Banach space, and furthermore, if $\mathscr{B}$ is a Hilbert space, all such labeled graphs $\overrightarrow{G}^L$ with linear end-operators is a Hilbert space too.

Now, there are 2 kinds of problems on continuity flows $\overrightarrow{G}^L$:

(1) Globally, given a graph family $\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n\}, n \geq 1$ and a Banach space $\mathscr{B}$, whether there exists continuity flows over graphs $\overrightarrow{G}^L_1, \overrightarrow{G}^L_2, \cdots, \overrightarrow{G}^L_n$ to be elements form a mathematical space;

(2) Locally, for a continuity flow $\overrightarrow{G}^L$, if some vertices are no longer conserved by outside interference, how to make it conserved again such that it is still a continuity flow.

The first problem has been solved by a series papers of mine (See references of [5] in details), but for the second problems, there are only a few local or partially results. In fact, an independent energy system, including computer, car and human body, cell tissue, biological populations, $\cdots$, etc. adaptive system is nothing else but a continuity flow, and furthermore, conservation flow. Thus, we can use continuity flows to characterize behavior of these systems for reality.

Here, I would like to introduce the twelve meridians theory in traditional Chinese medicine ([14]), which can be viewed as a typical example of continuity flows, particularly, in treating an illness. It is in fact to make the patient balance in Yin and Yang on acupoints of meridians, i.e., conservation, where Yin (Y^-) or Yang (Y^+) can be viewed as negative or positive energy, tendency, $\cdots$, etc. are basic conceptions in traditional Chinese culture, i.e., Y^+ and Y^- are everywhere with that Y^+ in Y^- and Y^- in Y^+, such as those shown in Fig.5, where the black and white areas respectively represent Y^- and Y^+.

Fig.5

According to the characteristics of human body, the traditional Chinese medicine proposed

12 meridian theory, i.e., there 12 meridians in human body completely reflects the physical condition. They are respectively *Hand Tai Yang small intestine meridian* (H_1), *Hand Shao Yang Tri-Jiao meridian* (H_2), *Hand Yang Ming large intestine meridian* (H_3), *Hand Tai Yin lungs meridian* (H_4), *Hand Shao Yin heart meridian* (H_5), *Hand Jue Yin pericardium meridian* (H_6), *Foot Yang Ming stomach meridian* (F_1), *Foot Jue Yin liver meridian* (F_2), *Foot Tai Yin spleen meridian* (F_3), *Foot Shao Yin kidney meridian* (F_4), *Foot Shao Yang gallbladder meridian* (F_5), *Foot Tai Yang bladder meridian* (F_6), such as those shown in Fig.6(these red lines in human bodies without acupoint).

Fig.6

The balance of $\{Y^-, Y^+\}$ at points on the 12 meridians is the basic ruler for human body in traditional Chinese medicine. If there exists a point in one of the 12 meridians in which $\{Y^-, Y^+\}$ is imbalance, this person must be ill, and in turn, for a patient there are must be points on the 12 meridians in which $\{Y^-, Y^+\}$ are imbalance. This is the healing theory of traditional Chinese medicine, and by thousands of years of testing, there are no counterexamples appeared in China.

Certainly, the healing theory of traditional Chinese medicine is nothing else but continuity flows. Notice that the 12 meridians are in fact 12 directed pathes $H_1, H_2, H_3, H_4, H_5, H_6, F_1, F_2, F_3, F_4, F_5, F_6$ with vertices of acupoints. Define

$$\overrightarrow{G} = \left(\bigcup_{i=1}^{6} H_i\right) \bigcup \left(\bigcup_{i=1}^{6} F_i\right)$$

with $L: V(\overrightarrow{G}) \to \{Y^-, Y^+\}$, then, $\overrightarrow{G}^L$ should be conserved on its vertices in $\{Y^-, Y^+\}$ for a person, i.e., a continuity flow.

For a patient, i.e., there are points to be imbalance on the 12 meridians, the doctor detects the points on which meridians, at which acupoints and the imbalance is Y^- more than Y^+, or Y^+ more than Y^-, and then by a natural ruler of the universe in traditional Chinese culture, i.e., *reducing the excess with supply the insufficient*, the doctor regulates these related acupoints by acupuncture or drugs so that the acupoints balance in $\{Y^-, Y^+\}$ again. Clearly, this implies a mathematical process for a continuity flow $\overrightarrow{G}^L$ again.

Certainly, there are no specific amount for the action strength $H(x_i \to x_0)$, where x_0 is the acupoint with $\{Y^-, Y^+\}$ imbalance, x_i is the related acupoints, $1 \leq i \leq s$, which completely depends on the judgement of the doctor, and continuous regulation based on the actual situation of the patient, i.e., a process of response. This also implies that getting a continuity flow $\overrightarrow{G}^L$ again maybe by repeatedly regulation of the flows on conditions.

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we know nothing of what will happen in future, but by the analogy of past experience.

By Abraham Lincoln, an American president.

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[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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