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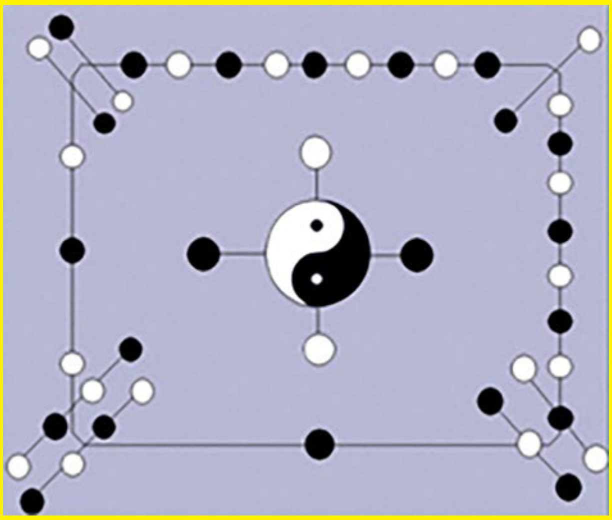

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In terms of truth and knowledge, any person acting the authority, is bound to collapse in God laugh!

By *Albert Einstein*, an American theoretical physicist.

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Fuzzy Product Rule for Solving Fully Fuzzy Linear Systems

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Abstract: In this paper we construct solutions of the fuzzy matrix equation $\widehat{A}\widehat{x} = \widehat{b}$ for $\widehat{x}$ when the elements in $\widehat{A}$ and $\widehat{b}$ are MMCE-triangular fuzzy numbers. Here we apply the product rule to solve the equation without any restriction on the signs of multiplied fuzzy numbers. Then we give two examples of the fuzzy product rule.

Key Words: Neutrosophic fuzzy set, fuzzy number, fully fuzzy linear system (FFLS), fuzzy product, MMCE-representation.

AMS(2010): 34A07, 34L10.

§1. Introduction

The systems of linear equations play important role in various areas of mathematics, statistic and engineering systems. Fuzzy systems represented by fuzzy numbers rather then crisp numbers have a major role for fuzzy modelling which can formulate uncertainty in real world problems. Fuzzy arithmetic operations have an essential role for treat linear systems whose parameters are all or partially fuzzy numbers. Firstly, the basic arithmetic structure for fuzzy numbers was introduced by Zadeh [11] and later this was developed many researcher such as Mizumoto and Tanaka [15], Dubois and Prade [7], Klir [9]. The fuzzy addition operation is practically easy to use. But, the other three fuzzy operations see various difficulties. Here we consider the multiplication operation for use the system of linear equations. A main disadvantage of this operation is that the shape of type fuzzy numbers (L-R, triangular or trapezoidal numbers) is not preserved. For this reason the researchers sought alternative ways for the product of fuzzy numbers.

Ma et al. introduced a new multiplicative operation of product type in [4]. They defined easily computable arithmetic operations based on split representation. But it has a drawback about the support. This problem has been solved by using middle-core-ecart representation (MCE-representation) of fuzzy numbers. Later, Zeinali and Maheri [6] introduced the modified MCE-product (MMCE, for short).

The system of linear equations $A\widehat{x} = \widehat{b}$, where A is a crisp matrix and $\widehat{b}$ is a fuzzy number vector, is called a fuzzy system of linear equation (FSLE) have been solved firstly Friedman et al. [12]. Following general model for solving such a fuzzy linear systems was proposed by many

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researchers ([13], [16], [4]). The linear system $\widehat{A}\widehat{x} = \widehat{b}$, where $\widehat{A}$ fuzzy matrix and $\widehat{b}$ is a fuzzy number vector, is called a fully fuzzy linear system (FFLS). A lot of works have been done this area with different methods ([[17], [14], [8], [10]).

In this paper we will investigate the solutions of FFLS with MMCE-representation of triangular fuzzy numbers. This paper is organized as follows. In Section 2, we briefly present the necessary preliminaries on fuzzy theory and MMCE-representation. In section 3, we summarise the definition and some properties of the FFLS. Then the solution of FFLS is constructed via MMCE-representation of fuzzy numbers. The proposed method is illustrated by solving some examples. Section 4 conclusion and some suggestions for future works are given.

§2. **Preliminaries**

In this section, we recall the basic notation of fuzzy numbers, the cross product and FFLS.

Definition 2.1([2]) *Let E be a universal set. A fuzzy subset $\widehat{A}$ of E is given by its membership function $\mu_{\widehat{A}} : E \to [0,1]$, where $\mu_{\widehat{A}}(t)$ represents the degree to which $t \in E$ belongs to $\widehat{A}$. We denote the class of the fuzzy subsets of E by the sembol $F(E)$.*

Generally, a neutrosophic fuzzy set $A(NFSA)$ is characterized by truth membership function $T_A(x)$, an indeterminacy membership functions $I_A(x)$ and a falsity membership function $F_A(x)$.

Definition 2.2([9]) *The $\alpha-$level of a fuzzy set $\widehat{A} \subseteq E$, denoted by $\left[\widehat{A}\right]^{\alpha}$, is defined as*

$$\left[\widehat{A}\right]^{\alpha} = \left\{x \in E : \widehat{A}(t) \geq \alpha\right\}, \quad \forall \alpha \in (0,1].$$

Furthermore, if E is also topological space, then the $0-$level is defined as the closure of the support of $\widehat{A}$. That is,

$$\left[\widehat{A}\right]^{0} = \overline{\left\{x \in E : \widehat{A}(t) > 0\right\}}.$$

The $1-$level of a fuzzy subset $\widehat{A}$ is also called as core of $\widehat{A}$ and denoted by $\left[\widehat{A}\right]^{1} = core\left(\widehat{A}\right)$.

Definition 2.3([9]) *A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy interval), whose $\alpha-$ cut set is denoted by $[\widehat{u}]^{\alpha}$, i.e., $[\widehat{u}]^{\alpha} = \{x : \widehat{u}(t) \geq 0\}$, if it satisfies two axioms:*

(i) There exists $r \in \mathbb{R}$ such that $\widehat{u}(r) = 1$;

(ii) For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u_{\alpha}^{-} \leq u_{\alpha}^{+} < +\infty$ such that $[\widehat{u}]^{\alpha}$ is equal to the closed interval $[u_{\alpha}^{-}, u_{\alpha}^{+}]$.

Similarly, we can also define a neutrosophic fuzzy real number and fuzzy interval.

Definition 2.4([2]) *A fuzzy number $\widehat{A}$ is said to be triangular if the parametric representation of its $\alpha-$level is of the form*

$$\left[\widehat{A}\right]^{\alpha} = [(a_2 - a_1)\alpha + a_1, a_3 - (a_3 - a_2)\alpha]$$

for all $\alpha \in [0,1]$*, where* $\left[\widehat{A}\right]^0 = [a_1, a_3]$ *and core* $\left(\widehat{A}\right) = a_2$*. A triangular fuzzy number is denoted by the triple* (a_1, a_2, a_3)*.*

The set of all fuzzy real numbers (fuzzy intervals) and triangular fuzzy numbers are denoted by $\mathbb{R}_F$ and $\mathbb{R}_T$, respectively.

Definition 2.5([1]) *An arbitrary fuzzy number* $\widehat{u}$ *in the parametric form is represented by an ordered pair of functions* $[u_\alpha^-, u_\alpha^+]$*,* $0 \le \alpha \le 1$*, which satisfy the following requirements*

(i) u_α^- *is a bounded non-decreasing left continuous function on* $(0,1]$ *and right- continuous for* $\alpha = 0$*;*

(ii) u_α^+ *is bounded non- increasing left continuous function on* $(0,1]$ *and right- continuous for* $\alpha = 0$*;*

(iii) $u_\alpha^- \le u_\alpha^+$*,* $0 < \alpha \le 1$*.*

Definition 2.6([4],[3]) *For* $\widehat{u} \in \mathbb{R}_F$*, consider the functions* $\theta_u^-, \theta_u^+ \to \mathbb{R}_+$ *defined by*

$$\theta_u^-(\alpha) = m_u - u_\alpha^-, \qquad \theta_u^+(\alpha) = u_\alpha^+ - m_u$$

where $m_u = \frac{u_1^- + u_1^+}{2}$*. Then,* $\widehat{u} = (m_u; \theta_u^-; \theta_u^+)$ *is MCE-representation of* $\widehat{u}$*. Note that the semicolon symbol makes this different from the well-known notation of a general triangular fuzzy number denoted by* (a, b, c)*. From now on, this notation is used for fuzzy numbers.*

From reference [5], clearly, $(m_u; \theta_u^-; \theta_u^+)$ represents a fuzzy number if and only if θ_u^-, θ_u^+ are bounded, positive, non-increasing, left-continuous on $(0; 1]$ and right-continuous at 0.

Although the MCE-product is easy to use, it doesn't preserve the shapes of triangular and trapezoidal fuzzy numbers in reference [4].

First, we note that for a triangular fuzzy number $\widehat{u} = (a; b; c)$, MCE-representation is in the form

$$\widehat{u} = (b; (b-a)(1-\alpha); (c-b)(1-\alpha)),$$

which means that if $\widehat{u} \in \mathbb{R}_T$, then $\widehat{u}$ can be presented by $(m_u; k_u^-(1-\alpha); k_u^+(1-\alpha))$, where $k_u^-, k_u^+ \in \mathbb{R}_+$. Now, the modification of MCE-product can be done as follows:

Definition 2.7([6]) *Let* $\widehat{u} = (m_u; k_u^-(1-\alpha); k_u^+(1-\alpha))$ *and* $\widehat{v} = (m_v; k_v^-(1-\alpha); k_v^+(1-\alpha))$ *be two triangular fuzzy numbers. The modified MCE-product (denoted by MMCE-product for short) is defined by*

$$\widehat{u} \circledast \widehat{v} = (m_u m_v; k_u^- k_v^-(1-\alpha); k_u^+ k_v^+(1-\alpha)),$$

the $\alpha-$*cut of* $\widehat{u} \circledast \widehat{v}$ *is*

$$(\widehat{u} \circledast \widehat{v})_\alpha = [m_u m_v - k_u^- k_v^-(1-\alpha); m_u m_v + k_u^+ k_v^+(1-r)]$$

and its support is

$$\sup p \widehat{u} \circledast \widehat{v} = [m_u m_v - k_u^- k_v^-; m_u m_v + k_u^+ k_v^+].$$

The triple representation for $\widehat{u} \circledast \widehat{v}$ as a triangular fuzzy number is

$$\widehat{u} \circledast \widehat{v} = \left(m_u m_v - k_u^- k_v^-, m_u m_v, k_u^+ k_v^+ \right).$$

For $\widehat{u}, \widehat{v} \in \mathbb{R}_T$ and $t \in \mathbb{R}$, the sum and scalar multiplication are defined by

$$\begin{aligned} \widehat{u} \oplus \widehat{v} &= \left(m_u + m_v; k_u^-(1-\alpha) + k_v^-(1-\alpha); k_u^+(1-\alpha) + k_v^+(1-\alpha) \right) \\ tu &= \begin{cases} (tm_u; tk_u^-(1-\alpha); tk_u^+(1-\alpha)), & t \geq 0, \\ (tm_u; -tk_u^+(1-\alpha); -tk_u^-(1-\alpha)), & t < 0 \end{cases} \end{aligned}$$

and the MMCE-product is invertible and the inverse of $\widehat{u} = (m_u; k_u^-(1-\alpha); k_u^+(1-\alpha))$ is

$$\widehat{u}^{-1} = \left(\frac{1}{m_u}; \frac{1}{k_u^-}(1-\alpha); \frac{1}{k_u^+}(1-\alpha) \right).$$

Example 2.8 Consider $\widehat{A} = (-1, 1, 3)$ and $\widehat{B} = (1, 3, 6)$, then $\widehat{A} = (1; 2(1-\alpha), 2(1-\alpha))$ and $\widehat{B} = (3; 2(1-\alpha), 3(1-\alpha))$ are MMCE triangular fuzzy numbers. The below algebraic operations can be easily done.

$$\begin{aligned} A \oplus B &= (4; 4(1-\alpha), 5(1-\alpha)) = (0, 4, 9); \\ A \ominus B &= (-2; 5(1-\alpha), 4(1-\alpha)) = (-7, -2, 2); \\ A \circledast B &= (3; 4(1-\alpha), 6(1-\alpha)) = (-1, -3, 9); \\ A \setminus B &= A \circledast B^{-1} = (-3; 2(1-\alpha), 2(1-\alpha)) = (-5, -3, -1). \end{aligned}$$

Definition 2.9([7]) *A matrix $\widehat{C} = (\widehat{c}_{ij})$ is called a fuzzy matrix,if each element of $\widehat{C}$ is a MMCE triangular fuzzy number.*

Up to rest of this paper, we represent $n \times m$ fuzzy matrix $\widehat{C} = (\widehat{c}_{ij})_{n \times m}$ with $\widehat{c}_{ij} = ((m_c)_{ij}; (k_u^-)_{ij}(1-\alpha); (k_u^+)_{ij}(1-\alpha))$ and a new notation $\widehat{C} = (C, E, F)$, where $C = (m_c)_{ij}$, $E = (k_u^-)_{ij}(1-\alpha)$ and $F = (k_u^+)_{ij}(1-\alpha)$ are three $n \times m$ crisp matrix.

Definition 2.10([7]) *Consider the $n \times n$ linear system of equations:*

$$\begin{cases} (\widehat{c}_{11} \circledast \widehat{x}_1) \oplus (\widehat{c}_{12} \circledast \widehat{x}_2) \ldots (\widehat{c}_{1n} \circledast \widehat{x}_n) = \widehat{b}_1 \\ (\widehat{c}_{21} \circledast \widehat{x}_1) \circledast (\widehat{c}_{22} \circledast \widehat{x}_2) \cdots (\widehat{c}_{2n} \circledast \widehat{x}_n) = \widehat{b}_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\widehat{c}_{n1} \otimes \widehat{x}_1) \circledast (\widehat{c}_{n2} \circledast \widehat{x}_2) \ldots (\widehat{c}_{nn} \circledast \widehat{x}_n) = \widehat{b}_n. \end{cases}$$

The matrix form of the above equations is

$$\widehat{C} \circledast \widehat{x} = \widehat{b} \tag{1}$$

where the coefficient matrix $\widehat{C} = (\widehat{c}_{ij})$ *is a* $n \times n$ *fuzzy matrix for integers* $1 \leq i, j \leq n$ *and* $\widehat{x}_i, \widehat{b}_i$ *are MMCE triangular fuzzy numbers,* $1 \leq i \leq n$*. Such a system is called a fully fuzzy linear system (FFLS).*

§3. The Solution of FFLS

In this section we solve a FFLS $\widehat{C} \circledast \widehat{x} = \widehat{b}$ using Computational methods given by [8]. However, in this fuzzy system we use MMCE triangular fuzzy numbers instead of LR fuzzy numbers. Thus, a FFLS will be solved not only for positive fuzzy numbers but also for all fuzzy numbers. In this paper, we suppose that all fuzzy numbers are MMCE triangular fuzzy numbers.

Definition 3.1([8]) *We say* $\widehat{x}$ *is a fuzzy approximate solution or more shortly, fuzzy solution of* $\widehat{C} \circledast \widehat{x} = \widehat{b}$ *with the left and right shape functions similar to that* $L(.)$ *and* $R(.)$ *which used in* $\widehat{C}$ *and* $\widehat{b}$ *if and only if* $\widehat{C} \circledast \widehat{x} = \widehat{b}$ *with approximate operators as mentioned above, i.e.* $\widehat{x} = (x, y, z)$ *is said to be fuzzy solution of* $(C, E, F) \otimes \widehat{x} = (b, q, s)$ *iff*

$$Cx = b, \qquad Cy + Ex = q, \qquad Cz + Fx = s \tag{2}$$

where the membership function of each element of $\{x \mid \mu_{\widehat{x}} > 0\}$ *can be defined with the same functions* L *and* R *which used in* $\widehat{C}$ *and* $\widehat{b}$.

Note that we use MMCE triangular fuzzy number with semicolon notation which is p-resented by $(m_u; k_u^-(1-\alpha); k_u^+(1-\alpha)) = (C; E; F)$ instead of LR fuzzy number denoted by (C, E, F) in (2).

Now, we use the Eq (1) as follow

$$\begin{aligned}&((m_c)_{ij}\,; \left(k_c^-\right)_{ij}(1-r); \left(k_c^+\right)_{ij}(1-\alpha)) \circledast ((m_x)_j\,; \left(k_x^-\right)_j(1-\alpha); \left(k_x^+\right)_j(1-\alpha))\\ &= ((m_b)_j\,; \left(k_b^-\right)_j(1-\alpha); \left(k_b^+\right)_j(1-\alpha)).\end{aligned}$$

If we rearrange the Eqs in (2), we get the following equations

$$\begin{aligned}\sum_{j=1}^{n} (m_c)_{ij} \,.\, (m_x)_j &= (m_b)_i\\ \sum_{j=1}^{n} (m_c)_{ij} \,.\, \left(k_x^-\right)_j(1-\alpha) + \sum_{j=1}^{n} \left(k_c^-\right)_{ij}(1-\alpha).\, (m_x)_j &= \left(k_b^-\right)_j(1-\alpha)\\ \sum_{j=1}^{n} (m_c)_{ij} \,.\, \left(k_x^+\right)_j(1-\alpha) + \sum_{j=1}^{n} \left(k_c^+\right)_{ij}(1-\alpha).\, (m_x)_j &= \left(k_x^+\right)_j(1-\alpha)\end{aligned}$$

where C is a nonsingular crisp matrix $(1 \leq j \leq n)$.

If we assume that C is a nonsingular crisp matrix, we can write similarly from reference [8] that

$$(Cx, Cy + Ex, Cz + Fx) = (b, q, s)\,.$$

So, we have

$$\begin{cases} Cx = b, \\ Cy = q - Ex, \\ Cz = s - Fx. \end{cases} \quad (3)$$

Thus, we can easily get

$$x = C^{-1}b, \quad (4)$$
$$y = C^{-1}q - C^{-1}Ex, \quad (5)$$
$$z = C^{-1}s - C^{-1}Fx, \quad (6)$$

by using the inverse matrix of C, which enables us to get the following result.

Theorem 3.2 *Let $\widehat{A} = (A; M; N)$ and $\widehat{b} = (b; g; h)$ be non-negative fuzzy matrix and non-negative fuzzy vector, respectively, and $\widehat{A}$ be the product of a permutation matrix by a diagonal matrix with positive diagonal entries. Moreover let $h \geq MA^{-1}b$, $g \geq NA^{-1}b$ and $\left(MA^{-1} + I\right) b \geq h$. Then the system $\widehat{A}\widehat{x} = \widehat{b}$ has a positive fuzzy solution.*

Proof See [8] for its proof. □

Now we consider two examples, one consisting of positive triangular fuzzy numbers and for the other example it does not matter the sign of the triangular fuzzy numbers.

Example 3.3 Consider the following FFLS for positive fuzzy numbers (Test 3.2 in [8])

$$\begin{cases} \widehat{5}\widehat{x}_1 + \widehat{6}\widehat{x}_2 = \widehat{50}, \\ \widehat{7}\widehat{x}_1 + \widehat{4}\widehat{x}_2 = \widehat{48} \end{cases}$$

where $\widehat{4} = (4, 4, 5)$, $\widehat{5} = (4, 5, 6)$, $\widehat{6} = (5, 6, 8)$, $\widehat{7} = (6, 7, 7)$, $\widehat{48} = (43, 48, 55)$ and $\widehat{50} = (40, 50, 67)$ are triangular fuzzy numbers. Using MMCE triangular fuzzy numbers instead of triangular fuzzy numbers we mean

$$\begin{cases} (5; (1-\alpha); (1-\alpha)) \circledast (x_1; y_1(1-\alpha); z_1(1-\alpha)) \oplus (6; (1-\alpha); 2\,(1-\alpha)) \\ \circledast (x_2; y_2(1-\alpha); z_2(1-\alpha)) = (50; 10(1-\alpha); 17\,(1-\alpha)) \\ (7; (1-\alpha); 0) \circledast (x_1; y_1(1-\alpha); z_1(1-\alpha)) \oplus (4; 0; (1-\alpha)) \circledast (x_2; y_2(1-\alpha); z_2(1-\alpha)) \\ = (48; 5(1-\alpha); 7\,(1-\alpha)) \end{cases}$$

So with Eq. (4) we get

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 48 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$$

Similarly by Eqs. (5) and (6) we have

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} y_1\,(1-\alpha) \\ y_2\,(1-\alpha) \end{bmatrix} = \begin{bmatrix} 10\,(1-\alpha) \\ 5\,(1-\alpha) \end{bmatrix} - \begin{bmatrix} 1\,(1-\alpha) & 1\,(1-\alpha) \\ 1\,(1-\alpha) & 0\,(1-\alpha) \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/11 \\ 1/11 \end{bmatrix}$$

and

$$\begin{bmatrix} 5 & 6 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} z_1\,(1-\alpha) \\ z_2\,(1-\alpha) \end{bmatrix} = \begin{bmatrix} 17\,(1-\alpha) \\ 7\,(1-\alpha) \end{bmatrix} - \begin{bmatrix} 1\,(1-\alpha) & 2\,(1-\alpha) \\ 0\,(1-\alpha) & 1\,(1-\alpha) \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}.$$

So, the solution is

$$\widehat{x} = \begin{bmatrix} \left(4; \frac{1}{11}\,(1-\alpha)\,; 0\right) \\ \left(5; \frac{1}{11}\,(1-\alpha)\,; \frac{1}{2}\,(1-\alpha)\right) \end{bmatrix}$$

where this solution is a fuzzy solution; also we consider that our solution is the same as the solution in [8].

Example 3.4 Consider the following FFLS for all fuzzy numbers (Example 2 in [10])

$$\begin{cases} \widehat{-1}\widehat{x}_1 + \widehat{2}\widehat{x}_2 = \widehat{-2}, \\ \widehat{-3}\widehat{x}_1 + \widehat{3}\widehat{x}_2 = \widehat{4} \end{cases}$$

where,

$$\begin{aligned} \widehat{-3} &= (-4,-3,-2), \quad \widehat{-1} = (-3,-1,2), \\ \widehat{2} &= (1,2,4), \quad \widehat{3} = (1,3,6), \\ \widehat{-2} &= (-3,-2,-1), \quad \widehat{4} = (1,4,5) \end{aligned}$$

are triangular fuzzy numbers. Using MMCE triangular fuzzy numbers instead of triangular fuzzy numbers we mean

$$\begin{cases} (-1;2(1-\alpha);3\,(1-\alpha)) \circledast (x_1;y_1(1-\alpha);z_1(1-\alpha)) + (2;(1-\alpha);2\,(1-\alpha)) \\ \circledast\,(x_2;y_2(1-\alpha);z_2(1-\alpha)) = (-2;(1-\alpha);(1-\alpha)) \\ (-3;(1-\alpha);(1-\alpha)) \circledast (x_1;y_1(1-\alpha);z_1(1-\alpha)) + (3;2\,(1-\alpha)\,;3\,(1-\alpha)) \\ \circledast\,(x_2;y_2(1-\alpha);z_2(1-\alpha)) = (4;3(1-\alpha);(1-\alpha)) \end{cases}$$

So with Eq. (4) we get

$$\begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -14/3 \\ -10/3 \end{bmatrix}.$$

Similarly by Eq. (5) and (6) we have

$$\begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} y_1 (1-\alpha) \\ y_2 (1-\alpha) \end{bmatrix} = \begin{bmatrix} 1(1-\alpha) \\ 3(1-\alpha) \end{bmatrix} - \begin{bmatrix} 2(1-\alpha) & 1(1-\alpha) \\ 1(1-\alpha) & 2(1-\alpha) \end{bmatrix} \begin{bmatrix} -14/3 \\ -10/3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1/11 \\ 1/11 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 2 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} z_1 (1-\alpha) \\ z_2 (1-\alpha) \end{bmatrix} = \begin{bmatrix} 1(1-\alpha) \\ 1(1-\alpha) \end{bmatrix} - \begin{bmatrix} 5(1-\alpha) & 2(1-\alpha) \\ 1(1-\alpha) & 3(1-\alpha) \end{bmatrix} \begin{bmatrix} -14/3 \\ -10/3 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 11.22 \\ 16.44 \end{bmatrix}.$$

So, the solution of system is

$$\widehat{x} = \begin{bmatrix} (-4.66; -8.77(1-\alpha); 11.22(1-\alpha)) \\ (-3.33; 8.88(1-\alpha); 16.44(1-\alpha)) \end{bmatrix} = \begin{bmatrix} (-8.77, -4.66, 15.33) \\ (-12.21, -3.33, 13.11) \end{bmatrix}$$

where this solution is a fuzzy solution; but the solution in [10] is not a fuzzy solution. So the method we used in this paper is more convenient.

§4. **Conclusion**

In this study, we introduced the Direct method in [8] for finding the solution of fully fuzzy linear system (FFLS) by using the MMCE triangular fuzzy numbers with the product rule instead of LR fuzzy numbers. We presented two examples to implement the given method. We verified that the sign of fuzzy numbers does not matter in the fuzzy solution of the system.

This product rule easy to use for multiplication of fuzzy numbers which are not depend on the signs . For this reason, it provides a great advantage in solving fuzzy equation systems. So, for future work, we can apply this new method to find fuzzy eigenvalues and fuzzy eigenvectors of the the system of linear equations $\widehat{A}\widehat{x} = \widehat{\lambda}\widehat{x}$.

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Geometry of Chain of Spheres Inside an Ellipsoidal Fragment

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Abstract: The objective of this article is to establish a condition by which we are able to state that an ellipsoidal fragment formed by a plane cutting the ellipsoid can always contain a sphere in any position inside in it. A method to construct a chain of mutually tangent spheres inscribed in the ellipsoidal segment has been proposed. The locus of the centroid as well as the radii of the mutually tangent spheres have been computed. The prime concern of our work is to explore some geometrical properties of such a chain of spheres which includes the condition of inscribability of a sphere in any position inside the ellipsoid along with the computation of points of tangency between consecutive spheres.

Key Words: Spherical chain, ellipsoidal segment, ellipsoid.

AMS(2010): 51M04, 51M05.

§1. Introduction

The proposed problem can be considered as a novel problem as there is not so much information present about this in the literature. The proposed problem is the enhancement of the same type of problem in 2-dimensions in which a chain of circles was considered in an elliptical segment. The purpose of this article is to extend the same problem to 3-dimensions in which a chain of spheres are considered to be inscribed in an ellipsoidal segment formed by a cutting plane to the ellipsoid. Lucca (2009) described the properties of the chain of mutually tangent circles inside a circular segment. In this paper, the authors discussed the locus of the centers of mutually tangent circles inside a circle. They also explored the points of tangency of the these circles and later they derived the recursive and non recursive formula for centers and radii of the circles in the chain. In their paper, Poelaert et al. (2011) discussed about the surface area and curvature of a general ellipsoid. They also derived the expressions for mean and Gaussian curvature of the ellipsoid. Pal et al. (2016) explored the properties related to the chain of mutually tangent spheres inside a spherical segment. In this article all the properties that has been found for a circular chain is recalled for a chain of spheres. Finally, Lucca (2021) enhanced his previous work to explore the properties of mutually tangent circles inside an elliptical segment.

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The organization of the proposed article is as follows. The introduction is given in section 1. Section 3 contains the basic concepts used to formulate the results. In section 4, radii and centers of the chain spheres are derived. In section 5, the condition for inscribing the chain of spheres inside the ellipsoidal segment has been obtained. Some geometrical properties are derived in section 6. Section 7 contains the conclusion.

§2. Motivation of Work

The research work done in the above articles motivates us to extend this idea for a 3-dimensional objects like ellipsoid. The novelty of our work is the extension of the geometrical properties of the objects inscribed in a conic to the properties in a conicoid. In this paper, we have considered an ellipsoid cutting by a plane vertically to form an ellipsoidal segment. A vertical chain of mutually tangent spheres are considered inside the ellipsoidal fragment to describe various properties like point of tangency, locus of centroid of the spheres.

§3. Basic Concepts

Let us consider a chain of spheres inscribed in an ellipsoid. It is assumed that a plane cutting the ellipsoid to form an ellipsoidal fragment MQN to which the spheres are inscribed. Now our aim is to explore some geometrical properties of the chain of mutually tangent spheres in an ellipsoidal segment. For this, it is better to deal with the problem in spherical coordinates. Therefore we consider the coordinate system as

$$\begin{aligned} x &= \rho(\theta,\phi)\cos\theta\sin\phi \\ y &= \rho(\theta,\phi)\sin\theta\sin\phi \\ z &= \rho(\theta,\phi)\cos\theta. \end{aligned}$$

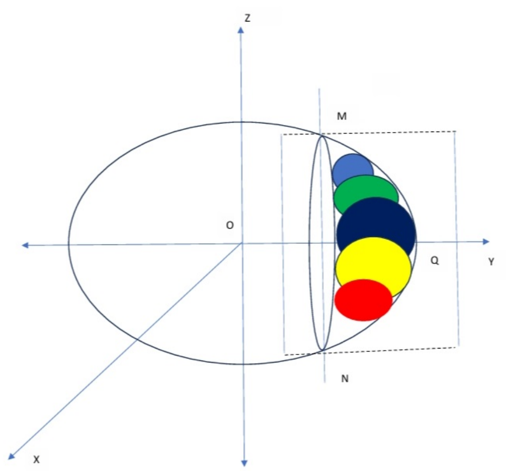

Figure 1. A chain of spheres inscribed in an ellipsoidal fragment

The equation of ellipsoid with principle semi axes a, b and $c(a \geqslant b \geqslant c)$ and with eccentric anomalies $(0 \leqslant \theta \leqslant 180)$ and $(0 \leqslant \phi < 360)$ is given by

$$\rho_e(\theta, \phi) = \frac{abc}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}}. \tag{1}$$

The equation of a plane cutting the ellipsoid in spherical coordinates is given by

$$\rho_r(\theta, \phi) = \frac{p}{l\sin\theta\cos\phi + m\sin\theta\sin\phi + n\cos\theta}, \tag{2}$$

where l, m, n be the direction cosines of the line perpendicular to the plane and p be the distance of the plane from the origin.

Equating equations (1) and (2) and simplifying, we get the expression for p as

$$p = \frac{abc(l\cos\phi + m\sin\phi + n\cot\theta)}{\sqrt{b^2c^2\cos^2\phi + c^2a^2\sin^2\phi + a^2b^2\cot^2\theta}}. \tag{3}$$

§4. Radii and Centers of the Spheres Under Two Tangent Planes

In order to inscribe a generic sphere inside an ellipsoidal segment, it is obvious to determine its radius and center. For this, it is mandatory that the centers of the spheres must lie on the bisector of the angle formed by the plane intersecting the ellipsoid and the tangent plane to the ellipsoid in the point of tangency between the spheres and the ellipsoid.

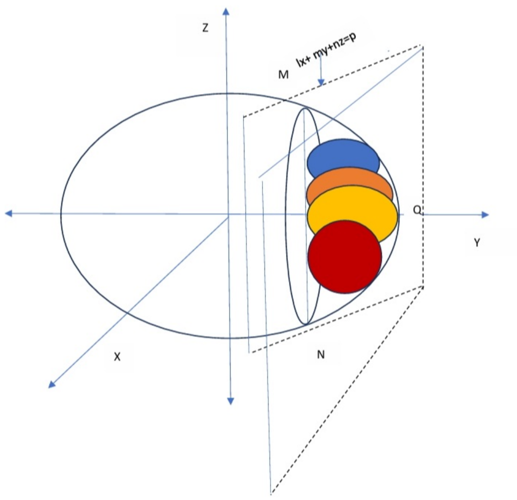

Figure 2. Spherical chain inside the ellipsoidal fragment under two planes

Theorem 1 *The radii $r_i(\theta, \phi)$ and centers $[X_c(\theta, \phi), Y_c(\theta, \phi), Z_c(\theta, \phi)]$ of spheres inscribed in an ellipsoidal segment under two tangent planes are*

$$\frac{kG}{W} \quad \textit{and} \quad \left[\frac{(kb^2c^2+1)abc\sin\theta\cos\phi}{W}, \frac{(kc^2a^2+1)abc\sin\theta\sin\phi}{W}, \frac{(ka^2b^2+1)abc\cos\theta)}{W}\right]$$

respectively, where

$$\begin{aligned}
k &= \frac{GM+N}{abc(GS+T)},\\
G &= abc\sqrt{b^4c^4\sin^2\theta\cos^2\phi + a^4c^4\sin^2\theta\sin^2\phi + a^4b^4\cos^2\theta},\\
M &= pW - abc(l\sin\theta\cos\phi + m\sin\theta\sin\phi + n\cos\theta),\\
N &= a^2b^2c^2(W^2 - b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta),\\
S &= lb^2c^2\sin\theta\cos\phi + mc^2a^2\sin\theta\sin\phi + na^2b^2\cos\theta,\\
T &= abc(b^4c^4 - a^4c^4 - a^4b^4),\\
W &= \sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}.
\end{aligned}$$

Proof Let us consider a point Q be the generic tangancy point of the sphere with the ellipsoid. The coordinates of Q are

$$\begin{aligned}
x_e(\theta,\phi) &= \frac{abc\sin\theta\cos\phi}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}},\\
y_e(\theta,\phi) &= \frac{abc\sin\theta\sin\phi}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}},\\
z_e(\theta,\phi) &= \frac{abc\cos\theta}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}}.
\end{aligned}$$

The equation of tangent plane to the ellipsoid at Q is given by

$$\frac{xx_e(\theta,\phi)}{a^2} + \frac{yy_e(\theta,\phi)}{b^2} + \frac{zz_e(\theta,\phi)}{c^2} = 1. \tag{4}$$

The equation of a plane cutting the ellipsoid is

$$lx + my + nz = p. \tag{5}$$

The equation of angle bisector between the planes (4) and (5) is given by

$$lx + my + nz - p - \frac{a^2b^2c^2 - b^2c^2xx_e(\theta,\phi) - c^2a^2yy_e(\theta,\phi) - a^2b^2zz_e(\theta,\phi)}{\sqrt{b^4c^4x_e^2(\theta,\phi) + c^4a^4y_e^2(\theta,\phi) + a^4b^4z_e^2(\theta,\phi)}} = 0. \tag{6}$$

The equation of normal to the ellipsoid at Q given by equation

$$\frac{x - x_e(\theta,\phi)}{b^2c^2x_e(\theta,\phi)} = \frac{y - y_e(\theta,\phi)}{a^2c^2y_e(\theta,\phi)} = \frac{z - z_e(\theta,\phi)}{a^2b^2z_e(\theta,\phi)} = k(say). \tag{7}$$

Now, substituting the values of $x_e(\theta,\phi)$, $y_e(\theta,\phi)$ and $z_e(\theta,\phi)$ in equation (7), we get the coordinates of the centers $[X_c(\theta,\phi), Y_c(\theta,\phi), Z_c(\theta,\phi)]$ of the spheres inside the the ellipsoidal segment. Next using the distance formula between the points $(x_e(\theta,\phi), y_e(\theta,\phi), z_e(\theta,\phi))$ and $(X_c(\theta,\phi), Y_c(\theta,\phi), Z_c(\theta,\phi))$, we get the radii $r_i(\theta,\phi)$ of the spheres inside the ellipsoidal segment. □

§5. Inscribability Condition

In this section, we derived the condition for inscribability of a sphere inside an ellipsoidal segment.

Theorem 2 *A generic sphere can always be inscribed in an ellipsoidal segment formed by a vertical plane cutting the ellipsoid if*

$$A^2 \sin^2\theta + B^2\cos^2\theta = (C\sin\theta + D\cos\theta - U)^2$$

where,

$$\begin{aligned} A^2 &= k^2[b^4c^4\cos^2\phi + c^4a^4\sin^2\phi], \\ B^2 &= k^2a^4b^4, \\ C &= l(kb^2c^2+1)\cos\phi + m(kc^2a^2+1))\sin\phi, \\ D &= n(ka^2c^2+1), \\ U &= \frac{pW}{abc}. \end{aligned}$$

Proof Notice that the equation of sphere having center $(X_c(\theta,\phi), Y_c(\theta,\phi), Z_c(\theta,\phi))$ and radius $r(\theta,\phi)$ is

$$(x - X_c(\theta,\phi))^2 + (y - Y_c(\theta,\phi))^2 + (z - Z_c(\theta,\phi))^2 = r^2(\theta,\phi). \tag{8}$$

and the equation of the ellipsoid circumscribing the sphere is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{9}$$

Considering a generic sphere touches the ellipsoid at the point $(x_e(\theta,\phi), y_e(\theta,\phi), z_e(\theta,\phi))$ of Q in Figure 2 and $lx + my + nz = p$ be the plane cutting the ellipsoid and also touching the sphere. It is obvious that a sphere will be completely inscribed inside the ellipsoid if the distance between the center of the sphere from the point Q is equal to the length of the perpendicular from the center to the plane $lx + my + nz = p$.

Now, the distance between the center of the sphere and point Q is

$$\sqrt{(X_c(\theta,\phi) - x_e(\theta,\phi))^2 + (Y_c(\theta,\phi) - y_e(\theta,\phi))^2 + (Z_c(\theta,\phi) - z_e(\theta,\phi))^2}$$

and the length of perpendicular on the given plane from the center of the sphere is

$$\frac{lX_c(\theta,\phi) + mY_c(\theta,\phi) + nZ_c(\theta,\phi) - p}{\sqrt{l^2+m^2+n^2}}.$$

Equating the above two expressions and squaring both the sides, we have

$$\begin{aligned}(X_c(\theta,\phi) - x_e(\theta,\phi))^2 + (Y_c(\theta,\phi) &- y_e(\theta,\phi))^2 + (Z_c(\theta,\phi) - z_e(\theta,\phi))^2 \\ &= (lX_c(\theta,\phi) + mY_c(\theta,\phi) + nZ_c(\theta,\phi) - p)^2, \qquad (10)\end{aligned}$$

where $l^2 + m^2 + n^2 = 1$.

Substituting the values of $X_c(\theta,\phi), Y_c(\theta,\phi), Z_c(\theta,\phi)$ and $x_e(\theta,\phi), y_e(\theta,\phi), z_e(\theta,\phi)$ in the above expression, we have

$$A^2 \sin^2\theta + B^2 \cos^2\theta = (C \sin\theta + D\cos\theta - U)^2, \qquad (11)$$

which is the desired result. □

§6. Geometrical Properties of a Spherical Chain Inside an Ellipsoidal Segment

In this section we have explored some of the properties of a spherical chain inside an ellipsoidal segment.

Theorem 3 *The locus of the centers of mutually tangent spheres inscribed in an ellipsoidal fragment formed by a plane cutting the ellipsoid is*

$$(I + t^2 + z^2)^2 = 4(I + J)$$

where, $I = a^2 \sin^2\theta \cos^2\phi + b^2 \sin^2\theta \sin^2\phi + c^2 \cos^2\theta$ *and* $J = t^2 + y^2 + z^2$.

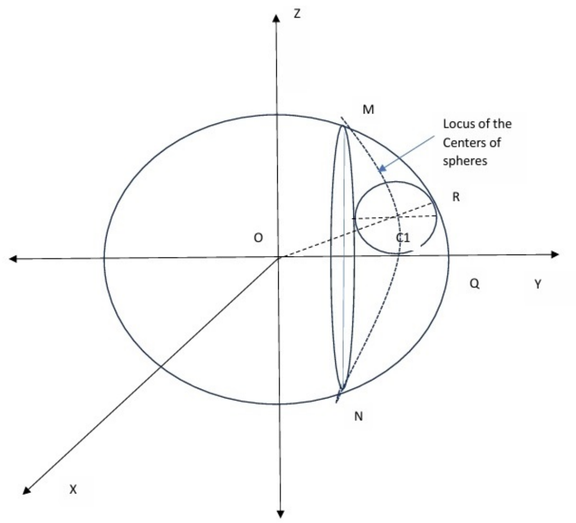

Figure 3. Locus of centers of Chain of spheres inscribed in an ellipsoidal fragment

Proof Let us consider a chain of mutually tangent spheres inscribed inside an ellipsoidal fragment formed by plane cutting the ellipsoid and tangent to the spheres. Let the origin O be the center of the ellipsoid. Now a generic point R on ellipsoid will be $(a\sin\theta\cos\phi, b\sin\theta\sin\phi, c\cos\theta)$. Let (t, y, z) be the center C_1 of spheres inside the ellipsoidal fragment MNQ. Now the line OR can be defined as

$$OR = \sqrt{a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta}.$$

Similarly, the line $OC_1 = \sqrt{t^2 + y^2 + z^2}$.

Using the geometry, we have

$$\sqrt{a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta} - y = \sqrt{t^2 + y^2 + z^2},$$

i.e.,

$$y = \sqrt{a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta} - \sqrt{t^2 + y^2 + z^2}.$$

Squaring both the sides, we have

$$\begin{aligned} y^2 &= a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta + t^2 + y^2 + z^2 \\ &\quad -2\sqrt{(a^2\sin^2\theta\cos^2\phi + b^2\sin^2\theta\sin^2\phi + c^2\cos^2\theta)(t^2 + y^2 + z^2)}. \end{aligned}$$

Again, squaring both the sides and simplifying the above expression, we have the required result. □

Theorem 4 *The locus of points of tangency between consecutive spheres of the chain lie on*

$$PT_i^2 - t^2 - b^2 - 2r_ib - z_i^2 = 0,$$

where r_i be the radii of the spheres inscribed in the ellipsoidal segment and P is a point on ellipsoid in y-axis and T be the point of tangency.

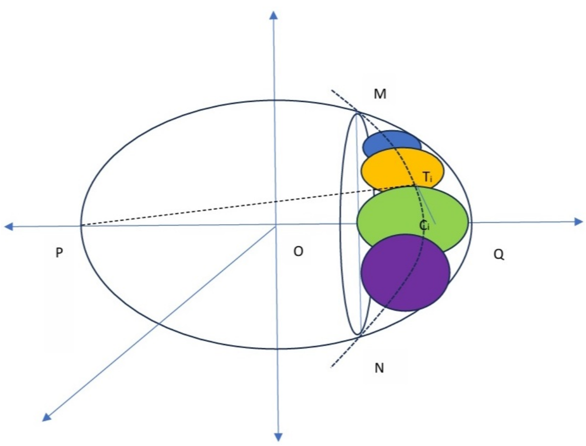

Figure 4. Point of tangancy of the spheres inside ellipsoidal fragment

Proof Let us assume that the two neighbouring spheres having centers $C_i(t, y_i, z_i)$ and

$T_i(t, y_i + 1, z_i + 1)$ with respective radii r_i and $r_i + 1$ and tangent to each other at T_i and also touching the ellipsoidal fragment and the plane Y=0. From the above figure it is observed that the coordinate of point P is $(0, -b, 0)$. Then we have,

$$PC_i^2 \ = \ t^2 + (y_i + b)^2 + z_i^2 = t^2 + b^2 + y_i^2 + 2y_i b + z_i^2.$$

But it is obvious that $r_i^2 = y_i^2$ and hence using it we can write

$$PC_i^2 \ = \ t^2 + b^2 + r_i^2 + 2r_i b + z_i^2.$$

Now, using the Pythagoras theorem in the right angled triangle PC_iT_i, we have

$$PT_i^2 \ = \ PC_i^2 - r_i^2 = t^2 + b^2 + 2r_i b + z_i^2.$$

This proves the theorem. □

§7. Conclusion

In this paper, we have analyzed various properties of a chain of spheres inscribed in an ellipsoidal segment formed by a vertical plane cutting the ellipsoid. We have derived the radii and coordinates of centers of mutually tangent spheres inside the ellipsoidal segment. An inscribability condition for the vertical chain of spheres along with the locus of the centers of such a chain has been also derived. Finally some geometrical properties are also developed for such an arrangement. From a very short literature review, it has been observed that not so much work has been done so far in this field. A symmetrical extension has been done by Pal et al. (2016) of the work done by Lucca (2009) which pulls the properties of chain of circles inside a circular segment to the chain of spheres inside spherical segment. In this article, we have accomplished the task of unsymmetrical extension which extends the properties of chain of circle inside an ellipse to the chain of spheres inside an ellipsoidal fragment.

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Appendix A

Substituting the values of $x_e(\theta,\phi)$, $y_e(\theta,\phi)$, $z_e(\theta,\phi)$ in *equation*(7), we have

$$\begin{aligned}
x &= \frac{(kb^2c^2+1)abc\sin\theta\cos\phi}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi+a^2c^2\sin^2\theta\sin^2\phi+a^2b^2\cos^2\theta}} = \frac{(kb^2c^2+1)abc\sin\theta\cos\phi}{W},\\
y &= \frac{(kc^2a^2+1)abc\sin\theta\sin\phi}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi+a^2c^2\sin^2\theta\sin^2\phi+a^2b^2\cos^2\theta}} = \frac{(kc^2a^2+1)abc\sin\theta\sin\phi}{W},\\
z &= \frac{(ka^2b^2+1)abc\cos\theta}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi+a^2c^2\sin^2\theta\sin^2\phi+a^2b^2\cos^2\theta}} = \frac{(ka^2b^2+1)abc\cos\theta}{W}.
\end{aligned}$$

Substituting the above values of x, y and z along with $x_e(\theta,\phi)$, $y_e(\theta,\phi)$ and $z_e(\theta,\phi)$ in equation (6), we have

$$\begin{aligned}
&\frac{l(kb^2c^2+1)abc\sin\theta\cos\phi}{W}+\frac{m(kc^2a^2+1)abc\sin\theta\sin\phi}{W}\\
&+\frac{n(ka^2b^2+1)abc\cos\theta}{W}-\frac{Wabc}{\sqrt{b^4c^4\sin^2\theta\cos^2\phi+c^4a^4\sin^2\theta\sin^2\phi+a^4b^4\cos^2\theta}}\\
&+\frac{b^2c^2k(b^2c^2+1)abc\sin^2\theta\cos^2\phi}{W\sqrt{b^4c^4\sin^2\theta\cos^2\phi+c^4a^4\sin^2\theta\sin^2\phi+a^4b^4\cos^2\theta}}\\
&+\frac{a^2c^2k(a^2c^2+1)abc\sin^2\theta\sin^2\phi}{W\sqrt{b^4c^4\sin^2\theta\cos^2\phi+c^4a^4\sin^2\theta\sin^2\phi+a^4b^4\cos^2\theta}}\\
&+\frac{a^2b^2k(a^2b^2+1)abc\cos^2\theta}{W\sqrt{b^4c^4\sin^2\theta\cos^2\phi+c^4a^4\sin^2\theta\sin^2\phi+a^4b^4\cos^2\theta}}=p.
\end{aligned}$$

Simplifying the above expression for k, we have the desired value of k.

Appendix B

Substituting the values of $(X_c(\theta,\phi),Y_c(\theta,\phi),Z_c(\theta,\phi))$ and $((x_e(\theta,\phi),y_e(\theta,\phi),z_e(\theta,\phi))$ in equation (10), we have

$$\begin{aligned}
&\frac{k^2b^4c^4a^2b^2c^2\sin^2\theta\cos^2\phi}{W^2}+\frac{k^2a^4c^4a^2b^2c^2\sin^2\theta\sin^2\phi}{W^2}+\frac{k^2a^4b^4a^2b^2c^2\cos^2\theta}{W^2}\\
&=\left[\frac{l(kb^2c^2+1)abc\sin\theta\cos\phi}{W}+\frac{m(ka^2c^2+1)abc\sin\theta\sin\phi)}{W}+\frac{n(ka^2b^2+1)abc\cos\theta}{W}-p\right]^2\\
&\frac{k^2a^2b^2c^2}{W^2}((b^4c^4\cos^2\phi+c^4a^4\sin^2\phi)\sin^2\theta+a^4b^4\cos^2\theta)\\
&=\frac{a^2b^2c^2}{W^2}\left[(l(kb^2c^2+1)\cos\phi+m(kc^2a^2+1)\sin\phi)\sin\theta+n(ka^2b^2+1)\cos\theta-\frac{pW}{abc}\right]^2\\
&k^2((b^4c^4\cos^2\phi+c^4a^4\sin^2\phi)\sin^2\theta+a^4b^4\cos^2\theta)\\
&=\left[(l(kb^2c^2+1)\cos\phi+m(kc^2a^2+1)\sin\phi)\sin\theta+n(ka^2b^2+1)\cos\theta-\frac{pW}{abc}\right]^2.
\end{aligned}$$

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On Derivative of Eta Quotients of Levels 12 and 16

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Abstract: Z. S. Aygin and P. C. Toh have deduced a technique using the theory of modular forms to determine all eta quotients whose derivative is also an eta quotient up to level 36. This paper aims to find a technique without using the theory of modular forms to deduce all the identities of Aygin and Toh of levels 12 and 16.

Key Words: Eisenstein series, Dedekind eta function, eta quotients.

AMS(2010): 11M36, 11F20.

§1. Introduction

The Dedekind eta function is defined by

$$\eta(\tau) \;=\; q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n),$$

where $q = e^{2\pi i\tau}$, with $Im(\tau) > 0$. A Dedekind eta function identity is said to be of level n, if it involves Dedekind eta functions $\eta(d_1\tau), \eta(d_2\tau), \ldots, \eta(d_k\tau)$, where the least common multiple of $d_1, d_2, \cdots, d_k$ is n.

Recently Z. S. Aygin and P. C. Toh [2] determined all eta quotients whose derivative is also an eta quotient up to level 36 by employing the theory of modular forms. In fact, they have obtained one hundred of level 12 and four of level 16 of above said type. Further they have conjectured that these are the only identities of level 12 and 16 of this nature. Also they have shown application of these identities to the theory of partitions, integral representation of eta functions and many more. Some of the identities of Aygin and Toh [2] exist before their discovery, see for example [6], [5] and [10]. The purpose of this article is to give an elementary proof for level 12 and 16 eta quotient identities by using the theory developed in [5] and [10].

In Section 2, we provide alternative proof for level 12 identities. In Section 3, we give proof for level 16 identities. We close this section by recalling the definitions, notations and certain existing eta function identities which are required to prove the above said identities.

Let

$$k \;=\; \frac{\eta_2^5\eta_3^2\eta_{12}^2}{\eta_1^2\eta_4^2\eta_6^5}.$$

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We require following eta function identities:

$$k^2 - 1 = 4\frac{\eta_2^3\eta_3^3\eta_{12}^6}{\eta_1\eta_4^2\eta_6^9}, \tag{1.1}$$

$$k^2 + 1 = 2\frac{\eta_2^3\eta_3^6\eta_{12}^3}{\eta_1^2\eta_4\eta_6^9}, \tag{1.2}$$

$$3 - k^2 = 2\frac{\eta_1^2\eta_2\eta_3^2\eta_{12}^3}{\eta_4\eta_6^7} \tag{1.3}$$

and

$$3 + k^2 = 4\frac{\eta_2\eta_3^3\eta_4^2\eta_{12}^2}{\eta_1\eta_6^7}, \tag{1.4}$$

where $\eta_k = \eta(k\tau)$. The proofs of the above four identities are found in [4], [8], [3]. S. Ramanujan recorded two of the above four theta function identities in the form of modular equations in his notebook [7, p.230]. We denote

$$\eta_n[k_1, k_2, \cdots, k_l] = \eta_1^{k_1}\eta_{d_1}^{k_2}\eta_{d_2}^{k_3}\eta_{d_3}^{k_4}, \cdots, \eta_{d_s}^{k_{l-1}}\eta_n^{k_l},$$

where $d_1, d_2, \cdots, d_s$ are proper divisors of n and $k_1, k_2, \ldots, k_l \in \mathbb{Z}$.

§2. Level 12 Identities

Theorem 2.1 *Let* $k = \dfrac{\eta_2^5\eta_3^2\eta_{12}^2}{\eta_1^2\eta_4^2\eta_6^5}$. *Then we have*

$$q\frac{d}{dq}(\log k) = 2\frac{\eta_1^2\eta_4^2\eta_3^2\eta_{12}^2}{\eta_2^2\eta_6^2}, \tag{2.1}$$

$$\frac{\eta_2^6\eta_3^9\eta_{12}^9}{\eta_1^3\eta_4^3\eta_6^{18}} = \frac{k^4-1}{8}, \tag{2.2}$$

$$\frac{\eta_1^4\eta_4^4\eta_6^4}{\eta_2^4\eta_3^4\eta_{12}^4} = \frac{9-k^4}{k^4-1} \tag{2.3}$$

and

$$\frac{\eta_2^{24}\eta_3^{12}\eta_{12}^{12}}{\eta_1^{12}\eta_4^{12}\eta_6^{24}} = \frac{k^4(k^4-1)}{9-k^4}. \tag{2.4}$$

The parameter k is almost the same as the p defined in [1]. In fact $k = 2p+1$, where $p = \frac{1}{2}\left[\frac{\eta_2^{10}\eta_3^4\eta_{12}^4}{\eta_1^4\eta_4^4\eta_6^{10}} - 1\right]$. The (2.1) is due to Ramanujan and the proof of the same was given by B. C. Berndt [4]. The proof of (2.2) – (2.4) are found in [5].

From the above, one can easily deduce that

$$\frac{\eta_2^{48}}{\eta_1^{24}\eta_4^{24}} = \frac{8^4k^{12}}{(k^4-1)(9-k^4)^3}, \tag{2.5}$$

$$\frac{\eta_6^{48}}{\eta_3^{24}\eta_{12}^{24}} = \frac{8^4k^4}{(k^4-1)^3(9-k^4)}, \tag{2.6}$$

and

$$\frac{\eta_2^6}{\eta_6^6} = \frac{k^2(9-k^4)}{k^4-1}. \tag{2.7}$$

Also, from [9], we have

$$x := \frac{\eta_1^4\eta_6^2}{\eta_2^2\eta_3^4} = \frac{3-k^2}{1+k^2}. \tag{2.8}$$

From [5], we have

$$\frac{\eta_1^{12}\eta_6^{12}}{\eta_2^{12}\eta_3^{12}} = \frac{x^2(1-x^2)}{9-x^2} \tag{2.9}$$

and

$$\frac{\eta_1^9\eta_6^3}{\eta_2^9\eta_3^3} = \frac{8x^2}{9-x^2}. \tag{2.10}$$

From (2.8), (2.9) and (2.10), one can easily deduce that

$$\frac{\eta_1^{24}}{\eta_2^{24}} = \frac{(1+k^2)^2(3-k^2)^6}{k^6(3+k^2)^3(k^2-1)} \tag{2.11}$$

and

$$\frac{\eta_3^{24}}{\eta_6^{24}} = \frac{(3-k^2)^2(1+k^2)^6}{k^2(3+k^2)(k^2-1)^3}. \tag{2.12}$$

From (2.5), (2.6), (2.11) and (2.12), we have

$$\frac{\eta_2^{24}}{\eta_4^{24}} = \frac{8^4k^6(1+k^2)(3-k^2)^3}{(3+k^2)^6(k^2-1)^2} \tag{2.13}$$

and

$$\frac{\eta_6^{24}}{\eta_{12}^{24}} = \frac{8^4k^2(1+k^2)^3(3-k^2)}{(k^2-1)^6(3+k^2)^2}. \tag{2.14}$$

Now, we prove two out of one hundred level-12 identities.

Theorem 2.2 *If* $X = \eta_{12}[10,-36,18,8,0,0]$*, then*

$$q\frac{d}{dq}(\log(X)) = \eta_{12}[10,-7,-6,1,9,-3].$$

Proof By the definition of X, we have

$$X^{12} = \left(\frac{\eta_1^{24}}{\eta_2^{24}}\right)^5\left(\frac{\eta_6^6}{\eta_2^6}\right)^{36}\left(\frac{\eta_4^{24}}{\eta_2^{24}}\right)^4\left(\frac{\eta_3^{24}}{\eta_6^{24}}\right)^9. \tag{2.15}$$

Employing (2.11), (2.7), (2.13) and (2.12) in the above, we find that

$$X = \frac{(k^2-1)(1+k^2)^8}{16\,k^{12}(3+k^2)^3}. \tag{2.16}$$

Taking logarithm on both sides and differentiating with respect to q, we obtain

$$q\frac{d}{dq}(\log(X)) = \frac{4(3-k^2)^2}{(1+k^2)(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}. \tag{2.17}$$

Using $(1.1), (1.2), (1.3), (1.4)$ and (2.1) in the right hand side of the above, we find that

$$q\frac{d}{dq}(\log(X)) = \eta_{12}[10, -7, -6, 1, 9, -3]. \tag{2.18}$$

This completes the proof. □

Theorem 2.3 *If* $Y = \eta_{12}[-18, 0, -10, 0, 36, -8]$*, then*

$$q\frac{d}{dq}(\log(Y)) = 3\eta_{12}[-6, 9, 10, -3, -7, 1].$$

Proof By the definition of Y, we have

$$Y^{12} = \left(\frac{\eta_2^{24}}{\eta_1^{24}}\right)^9 \left(\frac{\eta_6^{24}}{\eta_3^{24}}\right)^5 \left(\frac{\eta_6^{24}}{\eta_{12}^{24}}\right)^4 \left(\frac{\eta_6^{6}}{\eta_2^{6}}\right)^{36}. \tag{2.19}$$

Employing $(2.7), (2.11), (2.12)$ and (2.14) in the above, we find that

$$Y = \frac{16(k^2-1)^3}{(3-k^2)^8(3+k^2)}. \tag{2.20}$$

Taking logarithm on both sides and differentiating with respect to q, we obtain

$$q\frac{d}{dq}(\log(Y)) = \frac{12k^2(1+k^2)^2}{(k^2-1)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}. \tag{2.21}$$

Using $(1.1), (1.2), (1.3), (1.4)$ and (2.1) in the right hand side of the above, we find that

$$q\frac{d}{dq}(\log(Y)) = 3\eta_{12}[-6, 9, 10, -3, -7, 1]. \tag{2.22}$$

This completes the proof. □

We proved the remaining 98 identities of level 12 [2], in the same way. Let

$$f(\tau) = \eta_n(k_1, k_2, \cdots, k_l).$$

We first express $f(\tau)$ in terms of product of powers of $k, k^2 \pm 1, k^2 \pm 3$ and then, we display the q times of logarithmic differentiation of $f(\tau)$ in terms of $k, k^2 \pm 1, k^2 \pm 3$ and $q\frac{dk}{dq}$, and finally we represent

$$q\frac{d}{dq}log(f)$$

in terms of $\eta_n(k_1, k_2, \cdots, k_l)$ in the following Table 1- Table 7.

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
1	$\eta_{12}[4,-18,0,5,0,9]$	$\frac{(k^2-1)^4\sqrt{(k^2+1)}}{2^7k^6(3-k^2)^{3/2}}$	$\frac{4(k^2+3)^2}{(k^4-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[1,-7,-3,10,9,-6]$
2	$\eta_{12}[0,0,-4,-9,18,-5]$	$\frac{2^7(1+k^2)^{3/2}}{\sqrt{(3-k^2)(3+k^2)^4}}$	$\frac{6k^2(k^2-1)^2}{(k^2+1)(3-k^2)(k^2+3)}\frac{q}{k}\frac{dk}{dq}$	$12\eta_{12}[-3,9,1,-6,-7,10]$
3	$\eta_{12}[-2,4,6,0,-16,8]$	$\frac{(k^2-1)(3+k^2)}{2^4}$	$\frac{4k^2(k^2+1)}{(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-2,7,6,-3,-5,1]$
4	$\eta_{12}[6,-16,-2,8,4,0]$	$\frac{(k^2-1)(3+k^2)}{2^4k^4}$	$\frac{4(k^2-3)}{(k^2-1)(k^2+3)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[6,-5,-2,1,7,-3]$
5	$\eta_{12}[-4,8,0,-3,-2,1]$	$\frac{2k^2}{\sqrt{(1+k^2)(3-k^2)}}$	$\frac{4(k^2+3)}{(k^2+1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[1,-5,-3,6,7,-2]$
6	$\eta_{12}[0,-2,-4,1,8,-3]$	$\frac{2}{\sqrt{(1+k^2)(3-k^2)}}$	$\frac{4k^2(k^2-1)}{(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[-3,7,1,-2,-5,6]$
7	$\eta_{12}[2,0,-6,-8,12,0]$	$\frac{(k^2-1)}{(3+k^2)^3}$	$\frac{4k^2(3-k^2)}{(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[2,5,2,-3,-3,1]$
8	$\eta_{12}[6,-12,-2,0,0,8]$	$\frac{(k^2-1)^3}{2^4k^4(3+k^2)}$	$\frac{12(1+k^2)}{(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[2,-3,2,1,5,-3]$
9	$\eta_{12}[-4,0,0,1,6,-3]$	$2\sqrt{\frac{(1+k^2)}{(3-k^2)^3}}$	$\frac{4k^2(3+k^2)}{(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[-3,5,1,2,-3,2]$
10	$\eta_{12}[0,-6,4,3,0,-1]$	$\sqrt{\frac{(1+k^2)^3}{2^2k^4(3-k^2)}}$	$\frac{(k^2-1)}{(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$12\eta_{12}[1,-3,-3,2,5,2]$
11	$\eta_{12}[6,-12,-18,24,0,0]$	$\frac{(k^2-1)(3+k^2)^9}{8^4(1+k^2)^8}$	$\frac{4k^2(3-k^2)^2}{(k^2-1)(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[6,3,-2,-3,-1,1]$
12	$\eta_{12}[-6,0,2,0,-4,8]$	$\sqrt[3]{\frac{(k^2-1)^9(3+k^2)}{k^4(3-k^2)^8}}$	$\frac{12(1+k^2)^2}{(k^2-1)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-2,-1,6,1,3,-3]$
13	$\eta_{12}[-12,6,0,-3,0,9]$	$\sqrt{\frac{(k^2-1)^8}{8^2(1+k^2)(3-k^2)^9}}$	$\frac{2k^2(3+k^2)^2}{(k^2-1)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[-3,3,1,6,-1,-2]$
14	$\eta_{12}[0,0,-4,3,2,-1]$	$\sqrt[6]{\frac{k^4(3-k^2)(3+k^2)^8}{8^4(1+k^2)^9}}$	$\frac{36(k^2-1)^2}{(3-k^2)(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$12\eta_{12}[1,-1,-3,-2,3,6]$
15	$\eta_{12}[9,-30,9,12,0,0]$	$\frac{(1+k^2)^4(k^2-1)}{k^9}$	$\frac{(3-k^2)^2}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[9,-6,-3,3,2,-1]$

Table 1

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
16	$\eta_{12}[-3,0,-3,0,10,-4]$	$\sqrt[3]{\frac{8^2k}{(3-k^2)^4(3+k^2)}}$	$\frac{3(1+k^2)^2}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-3,2,9,-1,-6,3]$
17	$\eta_{12}[12,-30,0,9,0,9]$	$\frac{(1+k^2)(k^2-1)^4}{k^9}$	$\frac{(3+k^2)^2}{(k^2-1)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[3,-6,-1,9,2,-3]$
18	$\eta_{12}[0,0,-4,-3,10,-3]$	$\sqrt[3]{\frac{8^3k}{(3-k^2)(3+k^2)^4}}$	$\frac{3(k^2-1)^2}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$12\eta_{12}[-1,2,3,-3,-6,9]$
19	$\eta_{12}[-8,-2,-8,0,26,-8]$	$\frac{8^4(k^2-1)^3}{(3-k^2)^{12}(3+k^2)^3}$	$\frac{8k^4(1+k^2)}{(k^2-1)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-8,16,8,-6,-8,2]$
20	$\eta_{12}[8,-26,8,8,2,0]$	$\frac{(1+k^2)^4(k^2-1)}{k^8(3+k^2)}$	$\frac{8(3-k^2)}{(1+k^2)(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[8,-8,-8,2,16,-6]$
21	$\eta_{12}[4,-13,0,4,1,4]$	$\sqrt{\frac{(k^2-1)^4(1+k^2)}{k^8(3-k^2)}}$	$\frac{4(3+k^2)}{(k^2-1)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[2,-8,-6,8,16,-8]$
22	$\eta_{12}[0,-1,-4,-4,13,-4]$	$\sqrt{\frac{8^4(1+k^2)^3}{(3-k^2)^3(3+k^2)^{12}}}$	$\frac{4k^4(k^2-1)}{(1+k^2)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-6,16,2,-8,-8,8]$
23	$\eta_{12}[-4,6,12,4,-30,12]$	$\frac{(k^2-1)(3+k^2)^3}{2^8}$	$\frac{8k^4}{(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-4,14,4,-6,-6,2]$
24	$\eta_{12}[-2,-3,-6,2,15,-6]$	$\frac{1}{\sqrt{(1+k^2)(3-k^2)^3}}$	$\frac{4k^4}{(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-6,14,2,-4,-6,4]$
25	$\eta_{12}[12,-30,-4,12,6,4]$	$\frac{(3+k^2)(k^2-1)^3}{2^8k^8}$	$\frac{24}{(3+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[4,-6,-4,2,14,-6]$
26	$\eta_{12}[-6,15,-2,-6,-3,2]$	$\sqrt{\frac{k^8}{(1+k^2)^3(3-k^2)}}$	$\frac{12}{(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[2,-6,-6,4,14,-4]$
27	$\eta_{12}[-1,-3,-9,-5,27,-9]$	$\sqrt{\frac{2^{14}(1+k^2)}{(3-k^2)^3(3+k^2)^6}}$	$\frac{8k^6}{(1+k^2)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-9,23,3,-10,-9,6]$
28	$\eta_{12}[-10,-6,-18,-2,54,-18]$	$\frac{2^{10}(k^2-1)}{(3-k^2)^6(3+k^2)^3}$	$\frac{16k^6}{(k^2-1)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-10,23,6,-9,-9,3]$
29	$\eta_{12}[18,-54,10,18,6,2]$	$\frac{(1+k^2)^6(k^2-1)^3}{k^{16}(3+k^2)}$	$\frac{48}{(1+k^2)(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[6,-9,-10,3,23,-9]$

Table 2

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
30	$\eta_{12}[9,-27,1,9,3,5]$	$\sqrt{\frac{(1+k^2)^3(k^2-1)^6}{2^{14}k^{16}(3-k^2)}}$	$\frac{24}{(1+k^2)(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[3,-9,-9,6,23,-10]$
31	$\eta_{12}[-6,6,18,18,-54,18]$	$\frac{(k^2-1)(3+k^2)^9}{(1+k^2)^2}$	$\frac{16k^6}{(k^2-1)(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-6,21,2,-9,-7,3]$
32	$\eta_{12}[-9,-3,-9,3,27,-9]$	$\sqrt[4]{\frac{8^4(k^2-1)^2}{(1+k^2)(3-k^2)^9}}$	$\frac{8k^6}{(k^2-1)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-9,21,3,-6,-7,2]$
33	$\eta_{12}[-3,9,-3,-3,-1,1]$	$\sqrt[3]{\frac{8^2k^{16}(3+k^2)^2}{(1+k^2)^9(3-k^2)}}$	$\frac{144}{(3+k^2)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[3,-7,-9,2,21,-6]$
34	$\eta_{12}[6,-18,-2,6,2,6]$	$\sqrt[3]{\frac{(3+k^2)(k^2-1)^9}{8^6k^{16}(3-k^2)^2}}$	$\frac{48}{(3+k^2)(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[2,-7,-6,3,21,-9]$
35	$\eta_{12}[-12,-12,-36,-12,108,-36]$	$\frac{8^8(k^2-1)(1+k^2)}{(3+k^2)^9(3-k^2)^9}$	$\frac{32k^8}{(k^2-1)(1+k^2)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-12,30,4,-12,-10,4]$
36	$\eta_{12}[12,-36,4,12,4,4]$	$\sqrt[3]{\frac{(1+k^2)^9(k^2-1)^9}{8^8k^{32}(3+k^2)(3-k^2)}}$	$\frac{(1+k^2)^9(k^2-1)^9}{8^8k^{32}(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[4,-10,-12,4,30,-12]$
37	$\eta_{12}[-1,6,-9,4,0,0]$	$\frac{k^3(3+k^2)^3}{2^2(1+k^2)^4}$	$\frac{(3-k^2)^2}{(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[9,-4,-3,-1,0,3]$
38	$\eta_{12}[-9,0,-1,0,6,4]$	$\frac{(k^2-1)^3}{k(3-k^2)^4}$	$\frac{3(1+k^2)^2}{(3-k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-3,0,9,3,-4,-1]$
39	$\eta_{12}[-4,-6,0,1,0,9]$	$\frac{(k^2-1)^4}{2^5k^3(3-k^2)^3}$	$\frac{(3+k^2)^2}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[-1,-4,3,9,0,-3]$
40	$\eta_{12}[0,0,-4,9,-6,1]$	$\frac{k(3+k^2)^4}{2^5(1+k^2)^3}$	$\frac{3(k^2-1)^2}{(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$12\eta_{12}[3,0,-1,-3,-4,9]$
41	$\eta_{12}[7,-21,3,8,3,0]$	$\frac{(1+k^2)^2(k^2-1)}{2^4k^6}$	$\frac{2(3-k^2)}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[7,-7,-5,4,9,-4]$
42	$\eta_{12}[8,-21,0,7,3,3]$	$\frac{(1+k^2)(k^2-1)^2}{2^5k^6}$	$\frac{2(3+k^2)}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[4,-7,-4,7,9,-5]$
43	$\eta_{12}[-3,-3,-7,0,21,-8]$	$\frac{2^4}{(3-k^2)^2(3+k^2)}$	$\frac{6k^2(1+k^2)}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-5,9,7,-4,-7,4]$

Table 3

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
44	$\eta_{12}[0,-3,-8,-3,21,-7]$	$\frac{2^5}{(3+k^2)^2(3-k^2)}$	$\frac{4k^4}{(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[-4,9,4,-5,-7,7]$
45	$\eta_{12}[-1,4,-1,-4,2,0]$	$\frac{2k}{(3+k^2)}$	$\frac{(3-k^2)}{(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[5,-2,1,-1,-2,3]$
46	$\eta_{12}[1,-2,1,0,-4,4]$	$\frac{(k^2-1)}{k}$	$\frac{(1+k^2)}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[1,-2,5,3,-2,-1]$
47	$\eta_{12}[0,-2,4,1,-4,1]$	$\frac{(1+k^2)}{2^2k}$	$\frac{(k^2-1)}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[3,-2,-1,1,-2,5]$
48	$\eta_{12}[-4,4,0,-1,2,-1]$	$\frac{2^2k}{(3-k^2)}$	$\frac{(3+k^2)}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[-1,-2,3,5,-2,1]$
49	$\eta_{12}[5,-12,-3,4,6,0]$	$\frac{(k^2-1)}{2^2k^3}$	$\frac{(3-k^2)}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[5,-4,1,3,0,-1]$
50	$\eta_{12}[-3,6,5,0,-12,4]$	$\frac{k(3+k^2)}{2^2}$	$\frac{3(1+k^2)}{(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[1,0,5,-1,-4,3]$
51	$\eta_{12}[-4,12,0,-5,-6,3]$	$\frac{2k^3}{(1+k^2)}$	$\frac{(3+k^2)}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[3,-4,-1,5,0,1]$
52	$\eta_{12}[0,-6,-4,3,12,-5]$	$\frac{2^2}{k(3-k^2)}$	$\frac{3(k^2-1)}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$12\eta_{12}[-1,0,3,1,-4,5]$
53	$\eta_{12}[-7,0,-3,1,12,-3]$	$\frac{2(k^2-1)}{(3-k^2)^3}$	$\frac{4k^4}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-7,14,5,-3,-6,1]$
54	$\eta_{12}[-1,0,3,7,-12,3]$	$\frac{(3+k^2)^2}{2^5(1+k^2)}$	$\frac{4k^4}{(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-3,14,1,-7,-6,5]$
55	$\eta_{12}[3,-12,-1,3,0,7]$	$\frac{(k^2-1)^3}{2^5k^4(3-k^2)^2}$	$\frac{12}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[1,-6,-3,5,14,-7]$
56	$\eta_{12}[-3,12,-7,-3,0,1]$	$\frac{2^2k^4(3+k^2)}{(1+k^2)^3}$	$\frac{12}{(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[5,-6,-7,1,14,-3]$
57	$\eta_{12}[-1,3,3,-8,3,0]$	$\frac{2^4(1+k^2)^2}{(3+k^2)^3}$	$\frac{2k^2(3-k^2)}{(1+k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[3,5,-1,-4,-3,4]$

Table 4

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
58	$\eta_{12}[-8,3,0,-1,3,3]$	$\frac{(k^2-1)^2}{2(3-k^2)^3}$	$\frac{2(3+k^2)k^2}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-4,5,4,3,-3,-1]$
59	$\eta_{12}[-3,-3,1,0,-3,8]$	$\frac{(k^2-1)^3}{2^4k^2(3-k^2)}$	$\frac{6(1+k^2)}{(3-k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-1,-3,3,4,5,-4]$
60	$\eta_{12}[0,-3,8,-3,-3,1]$	$\frac{(1+k^2)^3}{k^2(3+k^2)^2}$	$\frac{6(k^2-1)}{(1+k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[4,-3,-4,-1,5,3]$
61	$\eta_{12}[-1,-2,-5,-1,14,-5]$	$\frac{8}{(3-k^2)(3+k^2)}$	$\frac{4k^4}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-7,16,5,-7,-8,5]$
62	$\eta_{12}[5,-14,1,5,2,1]$	$\frac{(1+k^2)(k^2-1)}{8k^4}$	$\frac{4}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[5,-8,-7,5,16,-7]$
63	$\eta_{12}[-1,3,3,-2,-9,6]$	$\frac{(k^2-1)}{4}$	$\frac{2k^2}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-1,5,3,0,-3,0]$
64	$\eta_{12}[-2,3,6,-1,-9,3]$	$\frac{(1+k^2)}{16}$	$\frac{2k^2}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[0,5,0,-1,-3,3]$
65	$\eta_{12}[-3,9,1,-6,-3,2]$	$\frac{4k^2}{(3+k^2)}$	$\frac{6}{(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[3,-3,-1,0,5,0]$
66	$\eta_{12}[-6,9,2,-3,-3,1]$	$\frac{2k^2}{(3-k^2)}$	$\frac{6}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[0,-3,0,3,5,-1]$
67	$\eta_{12}[-3,6,9,-3,-18,9]$	$\frac{(1+k^2)(k^2-1)}{8}$	$\frac{4k^4}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-3,12,1,-3,-4,1]$
68	$\eta_{12}[-3,6,1,-3,-2,1]$	$\sqrt[3]{\frac{8k^4}{(3-k^2)(3+k^2)}}$	$\frac{12}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[1,-4,-3,1,12,-3]$
69	$\eta_{12}[-4,-1,0,0,1,4]$	$\frac{(k^2-1)^2}{2^6k(3-k^2)^2}$	$\frac{(3+k^2)(1+k^2)}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-2,-2,6,6,-2,-2]$
70	$\eta_{12}[0,-1,4,-4,1,0]$	$\frac{2^6(1+k^2)^2}{k(3+k^2)^2}$	$\frac{(k^2-1)(3-k^2)}{(1+k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[6,-2,-2,-2,-2,6]$

Table 5

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
71	$\eta_{12}[4,-9,0,2,-3,6]$	$\frac{(k^2-1)^2}{2^4k^3}$	$\frac{(3+k^2)}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[2,-4,2,6,0,-2]$
72	$\eta_{12}[-2,9,-6,-4,3,0]$	$\frac{2^2k^3}{(1+k^2)^2}$	$\frac{(3-k^2)}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[6,-4,-2,2,0,2]$
73	$\eta_{12}[-6,3,-2,0,9,-4]$	$\sqrt{\frac{2^4k}{(3-k^2)^2}}$	$\frac{3(1+k^2)}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[-2,0,6,2,-4,2]$
74	$\eta_{12}[0,-3,4,6,-9,2]$	$\frac{(3+k^2)^2}{2^4k}$	$\frac{3(k^2-1)}{(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[2,0,2,-2,-4,6]$
75	$\eta_{12}[4,-6,-12,8,6,0]$	$\frac{(3+k^2)^3(k^2-1)}{2^4(1+k^2)^4}$	$\frac{8k^2(3-k^2)}{(3+k^2)(k^2-1)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[4,2,-4,-2,6,-2]$
76	$\eta_{12}[-4,3,0,-2,-3,6]$	$\frac{(k^2-1)^4}{2^4(1+k^2)(3-k^2)^3}$	$\frac{4(3+k^2)k^2}{(k^2-1)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-2,2,-2,4,6,-4]$
77	$\eta_{12}[-12,6,4,0,-6,8]$	$\frac{(k^2-1)^3(3+k^2)}{2^8(3-k^2)^4}$	$\frac{24k^2(1+k^2)}{(k^2-1)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-4,6,4,-2,2,-2]$
78	$\eta_{12}[0,-3,-4,6,3,-2]$	$\frac{(3+k^2)^4}{2^4(3-k^2)(1+k^2)^3}$	$\frac{12k^2(k^2-1)}{(3+k^2)(3-k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[-2,6,-2,-4,2,4]$
79	$\eta_{12}[12,-33,0,12,9,0]$	$\frac{(1+k^2)^2(k^2-1)^2}{8^2k^9}$	$\frac{(3-k^2)(3+k^2)}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[6,-6,-2,6,2,-2]$
80	$\eta_{12}[0,-3,-4,0,11,-4]$	$\sqrt[3]{\frac{8^2}{k(3-k^2)^2(3+k^2)^2}}$	$\frac{3(k^2-1)(1+k^2)}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[-2,2,6,-2,-6,6]$
81	$\eta_{12}[1,0-3,-1,0,3]$	$\frac{(k^2-1)}{2(1+k^2)}$	$\frac{4k^2}{(k^2-1)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[1,2,-3,1,6,-3]$
82	$\eta_{12}[-3,0,1,3,0,-1]$	$\frac{(3+k^2)}{(3-k^2)}$	$\frac{12k^2}{(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-3,6,1,-3,2,1]$
83	$\eta_{12}[-1,1,-1,-1,-1,3]$	$\sqrt{\frac{(k^2-1)^2}{4(1+k^2)(3-k^2)}}$	$\frac{8k^2}{(k^2-1)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-1,1,-5,2,13,-6]$
84	$\eta_{12}[2,-2,-6,2,2,2]$	$\frac{(3+k^2)(k^2-1)}{(1+k^2)^2}$	$\frac{16k^2}{(3+k^2)(k^2-1)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[2,1,-6,-1,13,-5]$

Table 6

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
85	$\eta_{12}[-1,-1,-1,3,1,-1]$	$\sqrt{\frac{(3+k^2)^2}{2^2(1+k^2)(3-k^2)}}$	$\frac{4k^4}{(3+k^2)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-5,13,-1,-6,1,2]$
86	$\eta_{12}[-6,2,2,2,-2,2]$	$\frac{(k^2-1)(3+k^2)}{(3-k^2)^2}$	$\frac{16k^4}{(k^2-1)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-6,13,2,-5,1,-1]$
87	$\eta_{12}[-2,6,6,-10,-6,6]$	$\frac{2^2(1+k^2)^2(k^2-1)}{(3+k^2)^3}$	$\frac{16k^4}{(1+k^2)(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-2,11,-2,-5,3,-1]$
88	$\eta_{12}[-5,3,3,-1,-3,3]$	$\sqrt{\frac{2^4(1+k^2)(k^2-1)^2}{(3-k^2)^3}}$	$\frac{8k^4}{(1+k^2)(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-5,11,-1,-2,3,-2]$
89	$\eta_{12}[-3,3,5,-3,-3,1]$	$\sqrt{\frac{2^2(1+k^2)^3}{(3-k^2)(3+k^2)^2}}$	$\frac{24k^2}{(1+k^2)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-1,3,-5,-2,11,-2]$
90	$\eta_{12}[-6,6,2,-6,-6,10]$	$\frac{(k^2-1)^3}{(3-k^2)^2(3+k^2)}$	$\frac{48k^2}{(k^2-1)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-2,3,-2,-1,11,-5]$
91	$\eta_{12}[-4,4,4,-4,-4,4]$	$\frac{(1+k^2)(k^2-1)}{(3-k^2)(3+k^2)}$	$\frac{32k^4}{(1+k^2)(k^2-1)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-4,10,-4,-4,10,-4]$
92	$\eta_{12}[3,-7,-1,2,1,2]$	$\frac{(k^2-1)}{4k^2}$	$\frac{2}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[3,-5,-1,4,7,-4]$
93	$\eta_{12}[-1,1,3,2,-7,2]$	$\frac{(3+k^2)}{4}$	$\frac{k^2}{2(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-1,7,3,-4,-5,4]$
94	$\eta_{12}[-2,-1,-2,1,7,-3]$	$\frac{2}{(3-k^2)}$	$\frac{2k^2}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-4,7,4,-1,-5,3]$
95	$\eta_{12}[-2,7,-2,-3,-1,1]$	$\frac{2k^2}{(1+k^2)}$	$\frac{2}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[4,-5,-4,3,7,-1]$
96	$\eta_{12}[-3,2,1,-1,-2,3]$	$\frac{(k^2-1)}{2(3-k^2)}$	$\frac{4k^2}{(3-k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-3,4,1,1,4,-3]$
97	$\eta_{12}[-1,2,3,-3,-2,1]$	$\frac{2(1+k^2)}{(3+k^2)}$	$\frac{4k^2}{(1+k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[1,4,-3,-3,4,1]$
98	$\eta_{12}[-2,5,2,-2,-5,2]$	k	$\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[2,-2,2,2,-2,2]$

Table 7

§3. Level 16 Identities

Let

$$h = \frac{\eta_2\eta_{16}^2}{\eta_1^2\eta_8}. \tag{3.1}$$

From [10], we have

$$z = q\frac{d}{dq}(\log(h)) = \frac{\eta_2\eta_4^6\eta_8}{\eta_1^2\eta_{16}^2}, \tag{3.2}$$

$$1+2h = \frac{\eta_2\eta_8^5}{\eta_1^2\eta_4^2\eta_{16}^2}, \tag{3.3}$$

$$1+4h = \frac{\eta_2^6}{\eta_1^4\eta_4^2}, \tag{3.4}$$

$$1+6h+8h^2 = \frac{\eta_2^7\eta_8^5}{\eta_1^6\eta_4^4\eta_{16}^2}, \tag{3.5}$$

$$1+4h+8h^2 = \frac{\eta_4^{10}}{\eta_1^4\eta_2^2\eta_8^4}, \tag{3.6}$$

$$\eta_1^{24} = z^6\frac{h}{(1+2h)^5(1+4h)^2(1+4h+8h^2)^5}, \tag{3.7}$$

$$\eta_2^{24} = z^6\frac{h^2(1+4h)^2}{(1+2h)^4(1+4h+8h^2)^4}, \tag{3.8}$$

$$\eta_4^{24} = z^6\frac{h^4}{(1+2h)^2(1+4h)^2(1+4h+8h^2)^2}, \tag{3.9}$$

$$8_8^{24} = z^6\frac{h^8(1+2h)^2}{(1+4h)^4(1+4h+8h^2)^4}, \tag{3.10}$$

and

$$\eta_{16}^{24} = z^6\frac{h^{16}}{(1+2h)^2(1+4h)^5(1+4h+8h^2)^5}. \tag{3.11}$$

Now, we prove one out of four level-16 identities.

Theorem 3.1 *Let* $Z = \eta_{16}[2,-5,2,-1,2]$ *then, prove that*

$$q\frac{d}{dq}(\log(Z)) = \eta_{16}[2,-5,8,1,-2].$$

Proof By the definition of Z, we have

$$Z^{24} = \frac{\eta_1^{48}\eta_4^{48}\eta_{16}^{48}}{\eta_2^{120}\eta_8^{24}}. \tag{3.12}$$

Employing $(3.7), (3.8), (3.9), (3.10)$ and (3.11) in the above, we find that

$$Z = \frac{h}{1+4h}. \tag{3.13}$$

Taking logarithm on both sides and differentiating with respect to q, we obtain

$$q\frac{d}{dq}(\log(Z)) \;=\; \frac{8h^2}{1+4h}\frac{q}{h}\frac{dh}{dq}. \tag{3.14}$$

Using (3.1), (3.2) and (3.4) in the right hand side of the above, we find that

$$q\frac{d}{dq}(\log(Z)) \;=\; \eta_{12}[10,-7,-6,1,9,-3]. \tag{3.15}$$

This completes the proof. □

We proved the remaining 3 identities of level 16 [10], in the same way. Let $g(\tau) = \eta_n(k_1, k_2, \cdots, k_l)$. We first express $f(\tau)$ in terms of product of powers of $h, 1+2h, 1+4h$, and then we display the q times of logarithmic differentiation of $g(\tau)$ in terms of $h, 1+2h, 1+4h$and $q\frac{dh}{dq}$, and finally we represent $q\frac{d}{dq}log(g)$ in terms of $\eta_n(k_1, k_2, \cdots, k_l)$ in the following Table 8.

SI.No	eta quotient (f)	h-parameter representation [f(h)]	$q\frac{d}{dq}(\log f(h))$	logarithmic derivative of f
1	$\eta_{16}[-2,1,-2,5,-2]$	$1+2h$	$\frac{2h}{1+2h}\frac{q}{h}\frac{dh}{dq}$	$2\eta_{16}[-2,1,8,-5,2]$
2	$\eta_{16}[-2,1,0,-1,2]$	h	$\frac{q}{h}\frac{dh}{dq}$	$\eta_{16}[-2,1,6,1,-2]$
3	$\eta_{16}[-2,5,0,-5,2]$	$\frac{1+4h}{1+2h}$	$\frac{2h}{(1+2h)(1+4h)}\frac{q}{h}\frac{dh}{dq}$	$2\eta_{16}[2,-5,10,-5,2]$

Table 8

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General Connectivity Entropies of Certain Interconnection Networks

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Abstract: The general connectivity indices (including the general Randić index, the general sum-connectivity index, the general ABC index and the general ABS index) are an important degree-based topological index in chemical informatics. In this paper, the general connectivity entropies of a graph are defined as the Shannon's entropy based on the information functional that associates the general connectivity indices. We compute the general connectivity entropies for certain interconnection networks like butterfly networks, Benes networks, and mesh derived networks, which can be helpful to understand their underlying topologies and structural complexity.

Key Words: Shannon's entropy, interconnection network, connectivity index.

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§1. Introduction

In 1948, Shannon [17] introduced the concept of entropy in communication theory to measure the uncertainty of a system, which is defined as follows:

Definition 1.1 *Let* $p = (p_1, p_2, \cdots, p_n)$ *be a probability vector, namely,* $0 \le p_i \le 1$ *and* $p_1 + p_2 + \cdots + p_n = 1$. *The Shannon's entropy of* p *is defined as*

$$I(p) = \sum_{i=1}^{n} p_i \log \frac{1}{p_i} = -\sum_{i=1}^{n} p_i \log p_i,$$

where the notation log *denotes the logarithm based on 2.*

Due to the ubiquitous uncertainty, the Shannon's entropy has found extensive applications in various disciplines such as discrete mathematics, computer science, information theory, statistics, chemistry, biology, etc., see [2, 9, 19]. On this basis, the concept of graph entropy introduced by Rashevsky [16] and Trucco [18] has been used to measure the structural complexity of graphs (or networks) [7, 8]. An extensive overview on graph entropy measures can be found in [10]. A statistical analysis of topological graph measures has been performed by Emmert-Streib and Dehmer [11].

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In chemical informatics, many degree-based topological indices have been introduced and extensively studied. Let $f(x, y)$ be the information function with the property $f(x, y) = f(y, x)$. Then, their general formula is

$$DTI(G) = \sum_{uv \in E(G)} f(d_u, d_v),$$

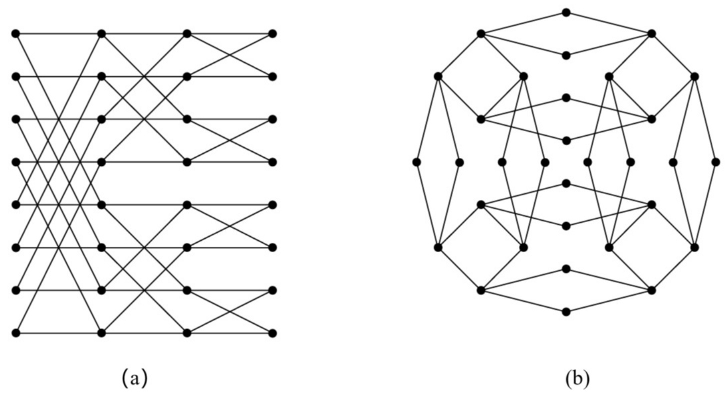
where $E(G)$ is the edge set of a graph G, and d_u is the degree of the vertex u. From the definition of Shannon's entropy, we can obtain the degree-based graph entropy [5, 6, 14] as follows:

$$I(G) = \log\, DTI(G) - \frac{1}{DTI(G)} \sum_{uv \in E(G)} f(d_u, d_v) \log\, f(d_u, d_v). \tag{1}$$

In this paper, we use the following classic general connectivity indices as the information function:

- The general Randić index [3]: $R_\alpha(G) = \sum_{uv \in E(G)} (d_u d_v)^\alpha$;
- The general sum-connectivity index [20]: $H_\alpha(G) = \sum_{uv \in E(G)} (d_u + d_v)^\alpha$;
- The general ABC index [12]: $ABC_\alpha(G) = \sum_{uv \in E(G)} \left(\frac{d_u + d_v - 2}{d_u d_v}\right)^\alpha$;
- The general ABS index [1]: $ABS_\alpha(G) = \sum_{uv \in E(G)} \left(\frac{d_u + d_v - 2}{d_u + d_v}\right)^\alpha$.

Further, the Shannon's entropy corresponding to these general connectivity indices is called the general connectivity entropies of a graph, denoted by $I_{R_\alpha}(G)$, $I_{H_\alpha}(G)$, $I_{ABC_\alpha}(G)$ and $I_{ABS_\alpha}(G)$, respectively. The main purpose of this paper is to provide quantitative calculation formulas for the general connectivity entropies of certain important interaction networks (including the butterfly networks, Benes networks, and mesh derived networks, see [4, 13, 15]), which can be helpful to understand their underlying topologies and structural complexity.



(a)Normal representation of butterfly $BF(3)$; (b)Diamond representation of butterfly $BF(3)$.

Figure 1

§2. The General Connectivity Entropies of Butterfly Networks

Certainly, the most popular bounded-degree derivative network of hypercube is the butterfly network. The set V of vertices of an r-dimensional butterfly network correspond to pairs $[w, i]$, where i is the dimension or level of a node ($0 \leq i \leq r$) and w is an r-bit binary number that denotes the row of the node. Two nodes $[w, i]$ and $[w', i']$ are linked by an edge if and only if $i' = i + 1$ and either: w and w' are identical, or w and w differ in precisely the ith bit. A r-dimensional butterfly network is denoted by $BF(r)$. Manuel et el. [15] proposed the diamond representations of these networks. The normal and diamond representations of 3-dimensional butterfly network are given in Figure 1, in which the number of vertices and number of edges in a butterfly network are

$$2^r(r+1) \quad \text{and} \quad r2^{r+1}.$$

Theorem 2.1 *For an r-dimensional butterfly network, the general connectivity entropies are equal to*

$$\begin{aligned}
I_{R_\alpha}(BF(r)) &= \log[2^{3\alpha+r+2}(1+2^{\alpha-1}(r-2))] - \frac{\alpha \log 2[3+2^{\alpha+1}(r-2)]}{1+2^{\alpha-1}(r-2)}, \\
I_{H_\alpha}(BF(r)) &= \log[2^{\alpha+r+2}(3^\alpha+2^{2\alpha-1}(r-2))] \\
&\quad -\frac{\alpha[3^\alpha \log 6 + 2^{2\alpha-1}(r-2)\log 8]}{3^\alpha+2^{2\alpha-1}(r-2)}, \\
I_{ABC_\alpha}(BF(r)) &= \log[2^{r+2-\alpha}(1+2^{-2\alpha-1}(r-2)3^\alpha)] \\
&\quad -\frac{\alpha[\log\frac{1}{2}+2^{-2\alpha-1}(r-2)3^\alpha \log\frac{3}{8}]}{1+2^{-2\alpha-1}(r-2)3^\alpha}, \\
I_{ABS_\alpha}(BF(r)) &= \log[2^{\alpha+r+2}\times 3^{-\alpha}(1+2^{-3\alpha-1}(r-2)3^{2\alpha})] \\
&\quad -\frac{\alpha[\log\frac{2}{3}+2^{-3\alpha-1}(r-2)3^{2\alpha}\log\frac{3}{4}]}{1+2^{-3\alpha-1}(r-2)3^{2\alpha}}.
\end{aligned}$$

Proof Let m_{d_u,d_v} be the number of edges in $BF(r)$ joining vertices of degree d_u and d_v. From the definition of r-dimensional butterfly network, we know that there are two types of edges in $BF(r)$ based on degrees of end vertices of each edge, see Table 1.

m_{d_u,d_v}	$m_{2,4}$	$m_{4,4}$
Number of edges	2^{r+2}	$2^{r+1}(r-2)$

Table 1. The basic information on $BF(r)$.

Thus, we have

$$\begin{aligned}
R_\alpha(BF(r)) &= 2^{r+2}(2\times 4)^\alpha + 2^{r+1}(r-2)(4\times 4)^\alpha \\
&= 2^{3\alpha+r+2}[1+2^{\alpha-1}(r-2)], \\
H_\alpha(BF(r)) &= 2^{r+2}(2+4)^\alpha + 2^{r+1}(r-2)(4+4)^\alpha \\
&= 2^{\alpha+r+2}[3^\alpha+2^{2\alpha-1}(r-2)], \\
ABC_\alpha(BF(r)) &= 2^{r+2}\left(\frac{2+4-2}{2\times 4}\right)^\alpha + 2^{r+1}(r-2)\left(\frac{4+4-2}{4\times 4}\right)^\alpha
\end{aligned}$$

$$
\begin{aligned}
&= 2^{r+2-\alpha}[1+2^{-2\alpha-1}(r-2)3^{\alpha}], \\
ABS_{\alpha}(BF(r)) &= 2^{r+2}(\frac{2+4-2}{2+4})^{\alpha} + 2^{r+1}(r-2)\left(\frac{4+4-2}{4+4}\right)^{\alpha} \\
&= 2^{\alpha+r+2} \times 3^{-\alpha}[1+2^{-3\alpha-1}(r-2)3^{2\alpha}].
\end{aligned}
$$

By Formula (1), we get

$$
\begin{aligned}
I_{R_{\alpha}(BF(r))} &= \log[2^{3\alpha+r+2}(1+2^{\alpha-1}(r-2))] - \frac{\alpha \log 2[3+2^{\alpha+1}(r-2)]}{1+2^{\alpha-1}(r-2)}, \\
I_{H_{\alpha}(BF(r))} &= \log[2^{\alpha+r+2}(3^{\alpha}+2^{2\alpha-1}(r-2))] - \frac{\alpha[3^{\alpha}\log 6 + 2^{2\alpha-1}(r-2)\log 8]}{3^{\alpha}+2^{2\alpha-1}(r-2)}, \\
I_{ABC_{\alpha}(BF(r))} &= \log[2^{r+2-\alpha}(1+2^{-2\alpha-1}(r-2)3^{\alpha})] \\
&\quad - \frac{\alpha[\log\frac{1}{2} + 2^{-2\alpha-1}(r-2)3^{\alpha}\log\frac{3}{8}]}{1+2^{-2\alpha-1}(r-2)3^{\alpha}}, \\
I_{ABS_{\alpha}(BF(r))} &= \log[2^{\alpha+r+2}\cdot 3^{-\alpha}(1+2^{-3\alpha-1}(r-2)3^{2\alpha})] \\
&\quad - \frac{\alpha[\log\frac{2}{3} + 2^{-3\alpha-1}(r-2)3^{2\alpha}\log\frac{3}{4}]}{1+2^{-3\alpha-1}(r-2)3^{2\alpha}}.
\end{aligned}
$$

This completes the proof. □

By Theorem 2.1, we obtain the following corollaries.

Corollary 2.2 *For a 3-dimensional butterfly network, the general connectivity entropies are equal to*

$$
\begin{aligned}
I_{R_{\alpha}(BF(3))} &= \log[2^{3\alpha+5}(1+2^{\alpha-1})] - \frac{\alpha\log 2[3+2^{\alpha+1}]}{1+2^{\alpha-1}}, \\
I_{H_{\alpha}(BF(3))} &= \log[2^{\alpha+5}(3^{\alpha}+2^{2\alpha-1})] - \frac{\alpha[3^{\alpha}\log 6 + 2^{2\alpha-1}\log 8]}{3^{\alpha}+2^{2\alpha-1}}, \\
I_{ABC_{\alpha}(BF(3))} &= \log[2^{5-\alpha}(1+2^{-2\alpha-1}\cdot 3^{\alpha})] - \frac{\alpha[\log\frac{1}{2} + 2^{-2\alpha-1}\cdot 3^{\alpha}\log\frac{3}{8}]}{1+2^{-2\alpha-1}\cdot 3^{\alpha}}, \\
I_{ABS_{\alpha}(BF(3))} &= \log[2^{\alpha+5}\cdot 3^{-\alpha}(1+2^{-3\alpha-1}\cdot 3^{2\alpha})] - \frac{\alpha[\log\frac{2}{3} + 2^{-3\alpha-1}\cdot 3^{2\alpha}\log\frac{3}{4}]}{1+2^{-3\alpha-1}\cdot 3^{2\alpha}}.
\end{aligned}
$$

Corollary 2.3 *For a 3-dimensional butterfly network with $\alpha = \frac{1}{2}$, the general connectivity entropies are equal to*

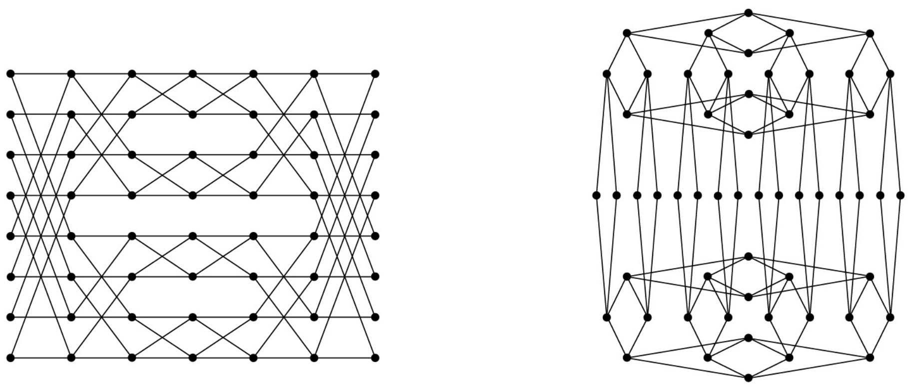
$$
\begin{aligned}
I_{R_{\frac{1}{2}}(BF(3))} &= \log[2^{\frac{13}{2}}(1+2^{-\frac{1}{2}})] - \frac{\log 2[3+2^{\frac{3}{2}}]}{2+2^{\frac{1}{2}}}, \\
I_{H_{\frac{1}{2}}(BF(3))} &= \log[2^{\frac{11}{2}}(3^{\frac{1}{2}}+1)] - \frac{3^{\frac{1}{2}}\log 6 + \log 8}{2\cdot 3^{\frac{1}{2}}+2},
\end{aligned}
$$

$$I_{ABC_{\frac{1}{2}}(BF(3))} = \log[2^{\frac{9}{2}}(1+2^{-2}\cdot 3^{\frac{1}{2}})] - \frac{\log\frac{1}{2}+\frac{1}{4}\cdot 3^{\frac{1}{2}}\log\frac{3}{8}}{2+2^{-1}\cdot 3^{\frac{1}{2}}},$$

$$I_{ABS_{\frac{1}{2}}(BF(3))} = \log[2^{\frac{11}{2}}\cdot 3^{-\frac{1}{2}}(1+2^{-\frac{5}{2}}\cdot 3)] - \frac{\log\frac{2}{3}+2^{-\frac{5}{2}}\cdot 3\log\frac{3}{4}}{2+2^{-\frac{3}{2}}\cdot 3}.$$

§3. The General Connectivity Entropies of Benes Networks

A r-dimensional Benes network is nothing but back-to-back butterflies. A r-dimensional Benes network has $2r+1$ levels, each level with $2r$ nodes. The level 0 to level r nodes in the network form an r-dimensional butterfly. The middle level of the Benes network is shared by these butterflies. An r-dimensional Benes is denoted by $B(r)$. Manuel et al. [15] proposed the diamond representation of the Benes network. The normal representation and diamond representation of $B(3)$ network is shown in Figure 2. The number of vertices and number of edges in a r-dimensional Benes network are $2^r(2r+1)$ and $r2^{r+2}$, respectively.



(a) (b)

(c)Normal representation of Benes network $B(3)$; (d)Diamond representation of Benes network $B(3)$.

Figure 2

Theorem 3.1 *For an r-dimensional Benes network, the general connectivity entropies are equal to*

$$I_{R_\alpha}(B(r)) = \log[2^{3\alpha+r+2}(1+2^\alpha(r-1))] - \frac{\alpha\log 2[3+2^{\alpha+2}(r-1)]}{1+2^\alpha(r-1)},$$

$$I_{H_\alpha}(B(r)) = \log[2^{\alpha+r+2}(3^\alpha+2^{2\alpha}(r-1))] - \frac{\alpha[3^\alpha\log 6+2^{2\alpha}(r-1)\log 8]}{3^\alpha+2^{2\alpha}(r-1)},$$

$$I_{ABC_\alpha}(B(r)) = \log[2^{r+2-\alpha}(1+2^{-2\alpha}(r-1)3^\alpha)] - \frac{\alpha[\log\frac{1}{2}+2^{-2\alpha}(r-1)3^\alpha\log\frac{3}{8}]}{1+2^{-2\alpha}(r-1)3^\alpha},$$

$$I_{ABS_\alpha}(B(r)) = \log[2^{\alpha+r+2}\cdot 3^{-\alpha}(1+2^{-3\alpha}(r-1)3^{2\alpha})] - \frac{\alpha[\log\frac{2}{3}+2^{-3\alpha}(r-1)3^{2\alpha}\log\frac{3}{4}]}{1+2^{-3\alpha}(r-1)3^{2\alpha}}.$$

Proof Let m_{d_u,d_v} be the number of edges in $B(r)$ joining vertices of degree d_u and d_v. From the definition of r-dimensional Benes network, we know that there are two types of edges in $B(r)$ based on degrees of end vertices of each edge, see Table 2.

$m_{d(u),d(v)}$	$m_{2,4}$	$m_{4,4}$
Number of edges	2^{r+2}	$2^{r+2}(r-1)$

Table 2. The basic information on $B(r)$.

Thus, we have

$$\begin{aligned}
R_\alpha(B(r)) &= 2^{r+2}\cdot(2\cdot 4)^\alpha + 2^{r+2}(r-1)\cdot(4\cdot 4)^\alpha = 2^{3\alpha+r+2}[1+2^\alpha(r-1)],\\
H_\alpha(B(r)) &= 2^{r+2}\cdot(2+4)^\alpha + 2^{r+2}(r-1)\cdot(4+4)^\alpha = 2^{\alpha+r+2}[3^\alpha+2^{2\alpha}(r-1)],\\
ABC_\alpha(B(r)) &= 2^{r+2}\cdot(\frac{2+4-2}{2\cdot 4})^\alpha + 2^{r+2}(r-1)\cdot(\frac{4+4-2}{4\cdot 4})^\alpha\\
&= 2^{r+2-\alpha}[1+2^{-2\alpha}(r-1)3^\alpha],\\
ABS_\alpha(B(r)) &= 2^{r+2}\cdot(\frac{2+4-2}{2+4})^\alpha + 2^{r+2}(r-1)\cdot(\frac{4+4-2}{4+4})^\alpha\\
&= 2^{\alpha+r+2}\cdot 3^{-\alpha}[1+2^{-3\alpha}(r-1)3^{2\alpha}].
\end{aligned}$$

By Formula (1), we get

$$\begin{aligned}
I_{R_\alpha}(B(r)) &= \log[2^{3\alpha+r+2}(1+2^\alpha(r-1))] - \frac{\alpha\log 2[3+2^{\alpha+2}(r-1)]}{1+2^\alpha(r-1)},\\
I_{H_\alpha}(B(r)) &= \log[2^{\alpha+r+2}(3^\alpha+2^{2\alpha}(r-1))] - \frac{\alpha[3^\alpha\log 6+2^{2\alpha}(r-1)\log 8]}{3^\alpha+2^{2\alpha}(r-1)},\\
I_{ABC_\alpha}(B(r)) &= \log[2^{r+2-\alpha}(1+2^{-2\alpha}(r-1)3^\alpha)] - \frac{\alpha[\log\frac{1}{2}+2^{-2\alpha}(r-1)3^\alpha\log\frac{3}{8}]}{1+2^{-2\alpha}(r-1)3^\alpha},\\
I_{ABS_\alpha}(B(r)) &= \log[2^{\alpha+r+2}\cdot 3^{-\alpha}(1+2^{-3\alpha}(r-1)3^{2\alpha})] - \frac{\alpha[\log\frac{2}{3}+2^{-3\alpha}(r-1)3^{2\alpha}\log\frac{3}{4}]}{1+2^{-3\alpha}(r-1)3^{2\alpha}}.
\end{aligned}$$

This completes the proof. □

By Theorem 3.1, we obtain the following corollaries.

Corollary 3.2 *For a* 3*-dimensional Benes network, the general connectivity entropies are equal to*

$$\begin{aligned}
I_{R_\alpha}(B(3)) &= \log[2^{3\alpha+5}(1+2^{\alpha+1})] - \frac{\alpha\log 2[3+2^{\alpha+3}]}{1+2^{\alpha+1}},\\
I_{H_\alpha}(B(3)) &= \log[2^{\alpha+5}(3^\alpha+2^{2\alpha+1})] - \frac{\alpha[3^\alpha\log 6+2^{2\alpha+1}\log 8]}{3^\alpha+2^{2\alpha+1}},
\end{aligned}$$

$$I_{ABC_\alpha}(B(3)) = \log[2^{5-\alpha}(1+2^{-2\alpha+1}\cdot 3^\alpha)] - \frac{\alpha[\log\frac{1}{2} + 2^{-2\alpha+1}\cdot 3^\alpha \log\frac{3}{8}]}{1+2^{-2\alpha+1}\cdot 3^\alpha},$$

$$I_{ABS_\alpha}(B(3)) = \log[2^{\alpha+5}\cdot 3^{-\alpha}(1+2^{-3\alpha+1}\cdot 3^{2\alpha})] - \frac{\alpha[\log\frac{2}{3} + 2^{-3\alpha+1}\cdot 3^{2\alpha}\log\frac{3}{4}]}{1+2^{-3\alpha+1}\cdot 3^{2\alpha}}.$$

Corollary 3.3 *For a* 3*-dimensional Benes network with* $\alpha = \frac{1}{2}$*, the general connectivity entropies are equal to*

$$I_{R_{\frac{1}{2}}}(B(3)) = \log[2^{\frac{13}{2}}(1+2^{\frac{3}{2}})] - \frac{\log 2[3+2^{\frac{7}{2}}]}{2+2^{\frac{5}{2}}},$$

$$I_{H_{\frac{1}{2}}}(B(3)) = \log[2^{\frac{11}{2}}(3^{\frac{1}{2}}+4)] - \frac{3^{\frac{1}{2}}\log 6 + 4\log 8}{2\cdot 3^{\frac{1}{2}}+8},$$

$$I_{ABC_{\frac{1}{2}}}(B(3)) = \log[2^{\frac{9}{2}}(1+3^{\frac{1}{2}})] - \frac{\log\frac{1}{2} + 3^{\frac{1}{2}}\log\frac{3}{8}}{2+2\cdot 3^{\frac{1}{2}}},$$

$$I_{ABS_{\frac{1}{2}}}(B(3)) = \log[2^{\frac{11}{2}}\cdot 3^{-\frac{1}{2}}(1+2^{-\frac{1}{2}}\cdot 3)] - \frac{\log\frac{2}{3} + 2^{-\frac{1}{2}}\cdot 3\log\frac{3}{4}}{2+2^{\frac{1}{2}}\cdot 3}.$$

§4. The General Connectivity Entropies of Mesh Derived Networks

The dual of a planar graph G, denoted by G^*, is a graph whose vertex set is the set of faces of G, where two vertices f^* and g^* in G^* are joined by an edge e^* if the faces f and g are separated by the edge e. Clearly, the number of vertices of G^* is equal to the number of faces of G and the number of edges of G^* is equal to the number of edges in G. Since every planar graph has exactly one unbounded face. By deleting the vertex placed in unbounded face, we get the bounded dual of that graph. The medial of a planar graph, denoted by G^{**} is obtained from graph G in a special way: Add a vertex at the middle of each edge in G, i.e. barycentric subdivision of G and then join two such newly added vertices whose original edges span an angle in G. By deleting the vertex placed in unbounded face, we get the bounded medial of that graph as shown in Figure 3.

Now, we introduce two new architectures using $m \times n$ mesh network in which defining parameters m and n are number of vertices in any row and column respectively. It can be easily observed that the bounded dual of $m \times n$ mesh is $m-1 \times n-1$ mesh. We apply medial operation on $m \times n$ mesh and then by deleting vertex placed on unbounded face we get bounded medial of $m \times n$ mesh. By taking union of $m \times n$ mesh and its bounded medial in a way that the vertices of bounded medial are placed in the middle of each edge of $m \times n$ mesh, the resulting architecture will be the planar named as mesh derived network of first type i.e. MDN1(m,n) network as depicted in Figure 4(e). The vertex and edge cardinalities of MDN1(m,n) network are

$$3mn - m - n \quad \text{and} \quad 8mn - 6(m+n) + 4,$$

respectively.

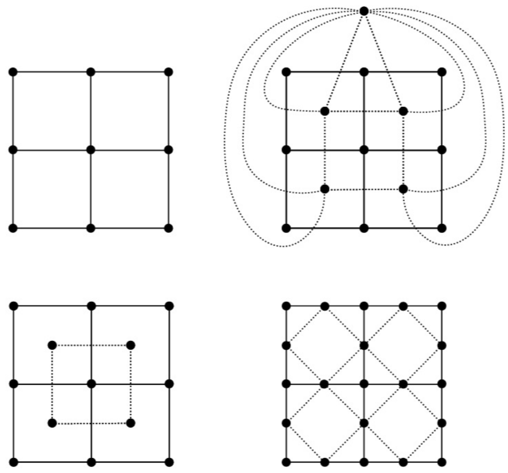


Figure 3. Bounded dual and bounded medial of graphs.

The second architecture is obtained from the union of $m \times n$ mesh and its bounded dual $m-1 \times n-1$ mesh by joining each vertex of $m-1 \times n-1$ mesh to each vertex of corresponding face of $m \times n$ mesh. The resulting architecture will be mesh derived network of second type i.e. MDN2(m,n) network as depicted in Figure 4(f). This non planar graph has number of vertices and edges are $2mn - m - n + 1$ and $8(mn - m - n + 1)$ respectively. Some other types of mesh derived networks are defined and studied in [4]. The important graph parameter which is discussed in [4] for mesh derived networks is the metric dimension/location number of networks.

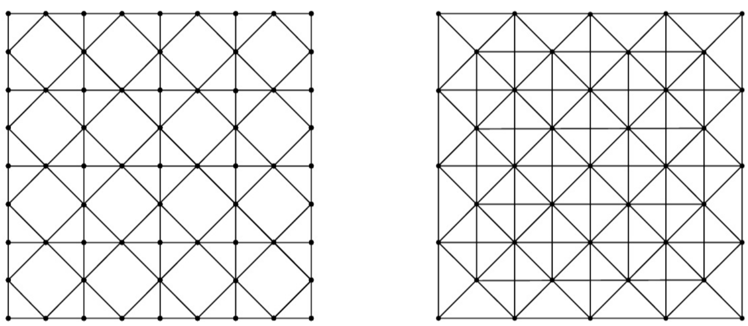


Figure 4. Mesh derived networks $MDN1(m,n)$ and $MDN2(m,n)$ with $m = n = 5$.

Theorem 4.1 *For a Mesh derived network $MDN1(m,n)$, the general connectivity entropies are equal to*

$$\begin{aligned}
I_{R_\alpha}(MDN1(m,n)) = \log R_\alpha(MDN1(m,n)) \\
&- \frac{8 \cdot (8)^\alpha \log(8)^\alpha + 4(m+n-4) \cdot (12)^\alpha \log(12)^\alpha}{R_\alpha(MDN1(m,n))} \\
&- \frac{2(m+n-4) \cdot (18)^\alpha \log(18)^\alpha + 4(mn-m-n) \cdot (24)^\alpha \log(24)^\alpha}{R_\alpha(MDN1(m,n))} \\
&- \frac{4 \cdot (16)^\alpha \log(16)^\alpha + 4(mn-2m-2n+4) \cdot (36)^\alpha \log(36)^\alpha}{R_\alpha(MDN1(m,n))},
\end{aligned}$$

$$\begin{aligned}
I_{H_\alpha}(MDN1(m,n)) &= \log H_\alpha(MDN1(m,n)) \\
&- \frac{8\cdot(6)^\alpha\log(6)^\alpha+4(m+n-4)\cdot(7)^\alpha\log(7)^\alpha}{H_\alpha(MDN1(m,n))} \\
&- \frac{2(m+n-4)\cdot(9)^\alpha\log(9)^\alpha+4(mn-m-n)\cdot(10)^\alpha\log(10)^\alpha}{H_\alpha(MDN1(m,n))} \\
&- \frac{4\cdot(8)^\alpha\log(8)^\alpha+4(mn-2m-2n+4)\cdot(12)^\alpha\log(12)^\alpha}{H_\alpha(MDN1(m,n))},
\end{aligned}$$

$$\begin{aligned}
I_{ABC_\alpha}(MDN1(m,n)) &= \log ABC_\alpha(MDN1(m,n)) \\
&- \frac{8\cdot(\frac{1}{2})^\alpha\log(\frac{1}{2})^\alpha+4(m+n-4)\cdot(\frac{5}{12})^\alpha\log(\frac{5}{12})^\alpha}{ABC_\alpha(MDN1(m,n))} \\
&- \frac{2(m+n-4)\cdot(\frac{7}{18})^\alpha\log(\frac{7}{18})^\alpha+4(mn-m-n)\cdot(\frac{1}{3})^\alpha\log(\frac{1}{3})^\alpha}{ABC_\alpha(MDN1(m,n))} \\
&- \frac{4\cdot(\frac{3}{8})^\alpha\log(\frac{3}{8})^\alpha+4(mn-2m-2n+4)\cdot(\frac{5}{18})^\alpha\log(\frac{5}{18})^\alpha}{ABC_\alpha(MDN1(m,n))},
\end{aligned}$$

$$\begin{aligned}
I_{ABS_\alpha}(MDN1(m,n)) &= \log ABS_\alpha(MDN1(m,n)) \\
&- \frac{8\cdot(\frac{2}{3})^\alpha\log(\frac{2}{3})^\alpha+4(m+n-4)\cdot(\frac{5}{7})^\alpha\log(\frac{5}{7})^\alpha}{ABS_\alpha(MDN1(m,n))} \\
&- \frac{2(m+n-4)\cdot(\frac{7}{9})^\alpha\log(\frac{7}{9})^\alpha+4(mn-m-n)\cdot(\frac{4}{5})^\alpha\log(\frac{4}{5})^\alpha}{ABS_\alpha(MDN1(m,n))} \\
&- \frac{4\cdot(\frac{3}{4})^\alpha\log(\frac{3}{4})^\alpha+4(mn-2m-2n+4)\cdot(\frac{5}{6})^\alpha\log(\frac{5}{6})^\alpha}{ABS_\alpha(MDN1(m,n))}.
\end{aligned}$$

Proof Let m_{d_u,d_v} be the number of edges in MDN1(m,n) joining vertices of degree d_u and d_v. From the definition of mesh derived network $MDN1(m,n)$, we know that the types of edges in MDN1(m,n) based on degrees of end vertices of each edge, see Table 3.

m_{d_u,d_v}	$m_{2,4}$	$m_{3,4}$	$m_{3,6}$	$m_{4,6}$	$m_{4,4}$	$m_{6,6}$
Number of edges	8	$4(m+n-4)$	$2(m+n-4)$	$4(mn-m-n)$	4	$4(mn-2m-2n+4)$

Table 3. The basic information on MDN1(m,n).

Thus, we have

$$\begin{aligned}
R_\alpha(MDN1(m,n)) &= 8(2\times4)^\alpha+4(m+n-4)(3\times4)^\alpha+2(m+n-4)(3\times6)^\alpha \\
&\quad+4(mn-m-n)(4\times6)^\alpha+4(4\times4)^\alpha+4(mn-2m-2n+4)(6\times6)^\alpha \\
&= 8\times8^\alpha+4(m+n-4)12^\alpha+2(m+n-4)18^\alpha \\
&\quad+4(mn-m-n)24^\alpha+4\times16^\alpha+4(mn-2m-2n+4)36^\alpha,
\end{aligned}$$

$$H_\alpha(MDN1(m,n)) = 8(2+4)^\alpha+4(m+n-4)(3+4)^\alpha+2(m+n-4)(3+6)^\alpha$$

$$
\begin{aligned}
&\quad + 4(mn-m-n)(4+6)^{\alpha} + 4(4+4)^{\alpha} + 4(mn-2m-2n+4)(6+6)^{\alpha} \\
&= 8 \times 6^{\alpha} + 4(m+n-4)7^{\alpha} + 2(m+n-4)9^{\alpha} \\
&\quad + 4(mn-m-n)10^{\alpha} + 4 \times 8^{\alpha} + 4(mn-2m-2n+4)12^{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
ABC_{\alpha}(MDN1(m,n)) &= 8\left(\frac{2+4-2}{2\times 4}\right)^{\alpha} + 4(m+n-4)\left(\frac{3+4-2}{3\times 4}\right)^{\alpha} \\
&\quad + 2(m+n-4)\left(\frac{3+6-2}{3\times 6}\right)^{\alpha} + 4(mn-m-n)\left(\frac{4+6-2}{4\times 6}\right)^{\alpha} \\
&\quad + 4\left(\frac{4+4-2}{4\times 4}\right)^{\alpha} + 4(mn-2m-2n+4)\left(\frac{6+6-2}{6\times 6}\right)^{\alpha} \\
&= 8\left(\frac{1}{2}\right)^{\alpha} + 4(m+n-4)\left(\frac{5}{12}\right)^{\alpha} + 2(m+n-4)\left(\frac{7}{18}\right)^{\alpha} \\
&\quad + 4(mn-m-n)\left(\frac{1}{3}\right)^{\alpha} + 4\left(\frac{3}{8}\right)^{\alpha} + 4(mn-2m-2n+4)\left(\frac{5}{18}\right)^{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
ABS_{\alpha}(MDN1(m,n)) &= 8\left(\frac{2+4-2}{2+4}\right)^{\alpha} + 4(m+n-4)\left(\frac{3+4-2}{3+4}\right)^{\alpha} \\
&\quad + 2(m+n-4)\left(\frac{3+6-2}{3+6}\right)^{\alpha} + 4(mn-m-n)\left(\frac{4+6-2}{4+6}\right)^{\alpha} \\
&\quad + 4\left(\frac{4+4-2}{4+4}\right)^{\alpha} + 4(mn-2m-2n+4)\left(\frac{6+6-2}{6+6}\right)^{\alpha} \\
&= 8\left(\frac{2}{3}\right)^{\alpha} + 4(m+n-4)\left(\frac{5}{7}\right)^{\alpha} + 2(m+n-4)\left(\frac{7}{9}\right)^{\alpha} \\
&\quad + 4(mn-m-n)\left(\frac{4}{5}\right)^{\alpha} + 4\left(\frac{3}{4}\right)^{\alpha} + 4(mn-2m-2n+4)\left(\frac{5}{6}\right)^{\alpha}.
\end{aligned}
$$

By Formula (1), we get

$$
\begin{aligned}
I_{R_{\alpha}}(MDN1(m,n)) &= \log R_{\alpha}(MDN1(m,n)) \\
&\quad - \frac{8\times 8^{\alpha}\log 8^{\alpha} + 4(m+n-4)12^{\alpha}\log 12^{\alpha}}{R_{\alpha}(MDN1(m,n))} \\
&\quad - \frac{2(m+n-4)18^{\alpha}\log 18^{\alpha} + 4(mn-m-n)24^{\alpha}\log 24^{\alpha}}{R_{\alpha}(MDN1(m,n))} \\
&\quad - \frac{4\times 16^{\alpha}\log 16^{\alpha} + 4(mn-2m-2n+4)36^{\alpha}\log 36^{\alpha}}{R_{\alpha}(MDN1(m,n))},
\end{aligned}
$$

$$
\begin{aligned}
I_{H_{\alpha}}(MDN1(m,n)) &= \log H_{\alpha}(MDN1(m,n)) \\
&\quad - \frac{8\times 6^{\alpha}\log 6^{\alpha} + 4(m+n-4)7^{\alpha}\log 7^{\alpha}}{H_{\alpha}(MDN1(m,n))} \\
&\quad - \frac{2(m+n-4)9^{\alpha}\log 9^{\alpha} + 4(mn-m-n)10^{\alpha}\log 10^{\alpha}}{H_{\alpha}(MDN1(m,n))} \\
&\quad - \frac{4\times 8^{\alpha}\log 8^{\alpha} + 4(mn-2m-2n+4)12^{\alpha}\log 12^{\alpha}}{H_{\alpha}(MDN1(m,n))},
\end{aligned}
$$

$$
\begin{aligned}
I_{ABC_\alpha}(MDN1(m,n)) = \log ABC_\alpha(MDN1(m,n)) \\
&- \frac{8\left(\frac{1}{2}\right)^\alpha \log\left(\frac{1}{2}\right)^\alpha + 4(m+n-4)\left(\frac{5}{12}\right)^\alpha \log\left(\frac{5}{12}\right)^\alpha}{ABC_\alpha(MDN1(m,n))} \\
&- \frac{2(m+n-4)\left(\frac{7}{18}\right)^\alpha \log\left(\frac{7}{18}\right)^\alpha + 4(mn-m-n)\left(\frac{1}{3}\right)^\alpha \log\left(\frac{1}{3}\right)^\alpha}{ABC_\alpha(MDN1(m,n))} \\
&- \frac{4\left(\frac{3}{8}\right)^\alpha \log\left(\frac{3}{8}\right)^\alpha + 4(mn-2m-2n+4)\left(\frac{5}{18}\right)^\alpha \log\left(\frac{5}{18}\right)^\alpha}{ABC_\alpha(MDN1(m,n))},
\end{aligned}
$$

$$
\begin{aligned}
I_{ABS_\alpha}(MDN1(m,n)) = \log ABS_\alpha(MDN1(m,n)) \\
&- \frac{8\left(\frac{2}{3}\right)^\alpha \log\left(\frac{2}{3}\right)^\alpha + 4(m+n-4)\left(\frac{5}{7}\right)^\alpha \log\left(\frac{5}{7}\right)^\alpha}{ABS_\alpha(MDN1(m,n))} \\
&- \frac{2(m+n-4)\left(\frac{7}{9}\right)^\alpha \log\left(\frac{7}{9}\right)^\alpha + 4(mn-m-n)\left(\frac{4}{5}\right)^\alpha \log\left(\frac{4}{5}\right)^\alpha}{ABS_\alpha(MDN1(m,n))} \\
&- \frac{4\left(\frac{3}{4}\right)^\alpha \log\left(\frac{3}{4}\right)^\alpha + 4(mn-2m-2n+4)\left(\frac{5}{6}\right)^\alpha \log\left(\frac{5}{6}\right)^\alpha}{ABS_\alpha(MDN1(m,n))}.
\end{aligned}
$$

This completes the proof. □

Theorem 4.2 *For Mesh derived network $MDN2(m,n)$, the general connectivity entropies are equal to*

$$
\begin{aligned}
I_{R_\alpha}(MDN2(m,n)) = \log R_\alpha(MDN2(m,n)) \\
&- \frac{4\times 18^\alpha \log 18^\alpha + 8\times 15^\alpha \log 15^\alpha}{R_\alpha(MDN2(m,n))} \\
&- \frac{8\times 30^\alpha \log 30^\alpha + 2(m+n-6)25^\alpha \log 25^\alpha + 4\times 48^\alpha \log 48^\alpha}{R_\alpha(MDN2(m,n))} \\
&- \frac{2(m+n-4)\times 40^\alpha \log 40^\alpha + 4(m+n-6)35^\alpha \log 35^\alpha}{R_\alpha(MDN2(m,n))} \\
&- \frac{2(m+n-8)\times 49^\alpha \log 49^\alpha + 8\times 42^\alpha \log 42^\alpha}{R_\alpha(MDN2(m,n))} \\
&- \frac{6(m+n-6)56^\alpha \log 56^\alpha + [8mn-24(m+n)+72]64^\alpha \log 64^\alpha}{R_\alpha(MDN2(m,n))},
\end{aligned}
$$

$$
\begin{aligned}
I_{H_\alpha}(MDN2(m,n)) = \log H_\alpha(MDN2(m,n)) \\
&- \frac{4\times 9^\alpha \log 9^\alpha + 8\times 8^\alpha \log 8^\alpha + 8\times 11^\alpha \log 11^\alpha}{H_\alpha(MDN2(m,n))} \\
&- \frac{2(m+n-6)10^\alpha \log 10^\alpha + [8mn-24(m+n)+72]16^\alpha \log 16^\alpha}{H_\alpha(MDN2(m,n))} \\
&- \frac{2(m+n)13^\alpha \log 13^\alpha + 4(m+n-6)12^\alpha \log 12^\alpha}{H_\alpha(MDN2(m,n))}
\end{aligned}
$$

$$- \frac{2(m+n-6)14^\alpha \log 14^\alpha + 6(m+n-6)15^\alpha \log 15^\alpha}{H_\alpha(MDN2(m,n))},$$

$$\begin{aligned}
I_{ABC_\alpha}(MDN2(m,n)) &= \log ABC_\alpha(MDN2(m,n)) \\
&- \frac{4\left(\frac{7}{18}\right)^\alpha \log\left(\frac{7}{18}\right)^\alpha + 8\left(\frac{2}{5}\right)^\alpha \log\left(\frac{2}{5}\right)^\alpha}{ABC_\alpha(MDN2(m,n))} \\
&- \frac{8\left(\frac{3}{10}\right)^\alpha \log\left(\frac{3}{10}\right)^\alpha + 2(m+n-6)\left(\frac{8}{25}\right)^\alpha \log\left(\frac{8}{25}\right)^\alpha + 4\left(\frac{1}{4}\right)^\alpha \log(\frac{1}{4})^\alpha}{ABC_\alpha(MDN2(m,n))} \\
&- \frac{2(m+n-4)\left(\frac{11}{40}\right)^\alpha \log\left(\frac{11}{40}\right)^\alpha + 4(m+n-6)\left(\frac{2}{7}\right)^\alpha \log\left(\frac{2}{7}\right)^\alpha}{ABC_\alpha(MDN2(m,n))} \\
&- \frac{2(m+n-8)\left(\frac{12}{49}\right)^\alpha \log\left(\frac{12}{49}\right)^\alpha + 8\left(\frac{11}{42}\right)^\alpha \log\left(\frac{11}{42}\right)^\alpha}{ABC_\alpha(MDN2(m,n))} \\
&- \frac{6(m+n-6)\left(\frac{13}{56}\right)^\alpha \log\left(\frac{13}{56}\right)^\alpha + [8mn-24(m+n)+72]\left(\frac{7}{32}\right)^\alpha \log\left(\frac{7}{32}\right)^\alpha}{ABC_\alpha(MDN2(m,n))},
\end{aligned}$$

$$\begin{aligned}
I_{ABS_\alpha}(MDN2(m,n)) &= \log ABS_\alpha(MDN2(m,n)) \\
&- \frac{4\left(\frac{7}{9}\right)^\alpha \log\left(\frac{7}{9}\right)^\alpha + 8\left(\frac{3}{4}\right)^\alpha \log\left(\frac{3}{4}\right)^\alpha}{ABS_\alpha(MDN2(m,n))} \\
&- \frac{2(m+n-6)\left(\frac{4}{5}\right)^\alpha \log\left(\frac{4}{5}\right)^\alpha + [8mn-24(m+n)+72]\left(\frac{7}{8}\right)^\alpha \log\left(\frac{7}{8}\right)^\alpha}{ABS_\alpha(MDN2(m,n))} \\
&- \frac{2(m+n)\left(\frac{11}{13}\right)^\alpha \log\left(\frac{11}{13}\right)^\alpha + 4(m+n-6)\left(\frac{5}{6}\right)^\alpha \log\left(\frac{5}{6}\right)^\alpha + 8\left(\frac{9}{11}\right)^\alpha \log\left(\frac{9}{11}\right)^\alpha}{ABS_\alpha(MDN2(m,n))} \\
&- \frac{2(m+n-6)\left(\frac{6}{7}\right)^\alpha \log\left(\frac{6}{7}\right)^\alpha + 6(m+n-6)\left(\frac{13}{15}\right)^\alpha \log\left(\frac{13}{15}\right)^\alpha}{ABS_\alpha(MDN2(m,n))}.
\end{aligned}$$

Proof Let m_{d_u,d_v} be the number of edges in MDN2(m,n) joining vertices of degree d_u and d_v. From the definition of mesh derived network $MDN2(m,n)$, we know that the types of edges in MDN2(m,n) based on degrees of end vertices of each edge, see Table 4.

m_{d_u,d_v}	$m_{3,6}$	$m_{3,5}$	$m_{5,6}$	$m_{5,5}$	$m_{6,8}$	$m_{5,8}$	$m_{5,7}$
Number of edges	4	8	8	$2(m+n-6)$	4	$2(m+n-4)$	$4(m+n-6)$

m_{d_u,d_v}	$m_{7,7}$	$m_{6,7}$	$m_{7,8}$	$m_{8,8}$
Number of edges	$2(m+n-8)$	8	$6(m+n-6)$	$8mn-24(m+n)+72$

Table 4. The basic information on $MDN2(m,n)$.

Thus, we have

$$\begin{aligned}
R_\alpha(MDN2(m,n)) &= 4\times(3\cdot 6)^\alpha + 8\times(3\cdot 5)^\alpha + 8\times(5\cdot 6)^\alpha \\
&+ 2(m+n-6)(5\times 5)^\alpha + 4\times(6\times 8)^\alpha + 2(m+n-4)(5\times 8)^\alpha
\end{aligned}$$

$$
\begin{aligned}
&+4(m+n-6)(5\times 7)^{\alpha}+2(m+n-8)(7\times 7)^{\alpha}+8(6\times 7)^{\alpha}\\
&+6(m+n-6)(7\times 8)^{\alpha}+[8mn-24(m+n)+72](8\times 8)^{\alpha}\\
&=4\times 18^{\alpha}+8\times 15^{\alpha}+8\times 30^{\alpha}\\
&+2(m+n-6)25^{\alpha}+4\times 48^{\alpha}+2(m+n-4)40^{\alpha}\\
&+4(m+n-6)35^{\alpha}+2(m+n-8)49^{\alpha}+8\times 42^{\alpha}\\
&+6(m+n-6)56^{\alpha}+[8mn-24(m+n)+72]64^{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
H_{\alpha}(MDN2(m,n)) &= 4(3+6)^{\alpha}+8(3+5)^{\alpha}+8(5+6)^{\alpha}\\
&+2(m+n-6)(5+5)^{\alpha}+4(6+8)^{\alpha}+2(m+n-4)(5+8)^{\alpha}\\
&+4(m+n-6)(5+7)^{\alpha}+2(m+n-8)(7+7)^{\alpha}+8(6+7)^{\alpha}\\
&+6(m+n-6)(7+8)^{\alpha}+[8mn-24(m+n)+72](8+8)^{\alpha}\\
&=4\times 9^{\alpha}+8\times 8^{\alpha}+8\times 11^{\alpha}+2(m+n-6)10^{\alpha}+2(m+n)13^{\alpha}\\
&+4(m+n-6)12^{\alpha}+2(m+n-6)14^{\alpha}+6(m+n-6)15^{\alpha}\\
&+[8mn-24(m+n)+72]16^{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
ABC_{\alpha}(MDN2(m,n)) &= 4\left(\frac{3+6-2}{3\cdot 6}\right)^{\alpha}+8\left(\frac{3+5-2}{3\times 5}\right)^{\alpha}+8\left(\frac{5+6-2}{5\times 6}\right)^{\alpha}\\
&+2(m+n-6)\left(\frac{5+5-2}{5\times 5}\right)^{\alpha}+4\left(\frac{6+8-2}{6\times 8}\right)^{\alpha}\\
&+2(m+n-4)\left(\frac{5+8-2}{5\times 8}\right)^{\alpha}+4(m+n-6)\left(\frac{5+7-2}{5\times 7}\right)^{\alpha}\\
&+2(m+n-8)\left(\frac{7+7-2}{7\times 7}\right)^{\alpha}+8\left(\frac{6+7-2}{6\times 7}\right)^{\alpha}\\
&+6(m+n-6)\left(\frac{7+8-2}{7\times 8}\right)^{\alpha}+[8mn-24(m+n)+72]\left(\frac{8+8-2}{8\times 8}\right)^{\alpha}\\
&=4\left(\frac{7}{18}\right)^{\alpha}+8\left(\frac{2}{5}\right)^{\alpha}+8\left(\frac{3}{10}\right)^{\alpha}+2(m+n-6)\left(\frac{8}{25}\right)^{\alpha}\\
&+4\left(\frac{1}{4}\right)^{\alpha}+2(m+n-4)\left(\frac{11}{40}\right)^{\alpha}+4(m+n-6)\left(\frac{2}{7}\right)^{\alpha}\\
&+2(m+n-8)\left(\frac{12}{49}\right)^{\alpha}+8\left(\frac{11}{42}\right)^{\alpha}+6(m+n-6)\left(\frac{13}{56}\right)^{\alpha}\\
&+[8mn-24(m+n)+72]\left(\frac{7}{32}\right)^{\alpha},
\end{aligned}
$$

$$
\begin{aligned}
ABS_{\alpha}(MDN2(m,n)) &= 4\left(\frac{3+6-2}{3+6}\right)^{\alpha}+8\left(\frac{3+5-2}{3+5}\right)^{\alpha}+8\left(\frac{5+6-2}{5+6}\right)^{\alpha}\\
&+2(m+n-6)\left(\frac{5+5-2}{5+5}\right)^{\alpha}+4\left(\frac{6+8-2}{6+8}\right)^{\alpha}\\
&+2(m+n-4)\left(\frac{5+8-2}{5+8}\right)^{\alpha}+4(m+n-6)\left(\frac{5+7-2}{5+7}\right)^{\alpha}
\end{aligned}
$$

$$+2(m+n-8)\left(\frac{7+7-2}{7+7}\right)^{\alpha}+8\left(\frac{6+7-2}{6+7}\right)^{\alpha}$$
$$+6(m+n-6)\left(\frac{7+8-2}{7+8}\right)^{\alpha}+[8mn-24(m+n)+72]\left(\frac{8+8-2}{8+8}\right)^{\alpha}$$
$$=4\left(\frac{7}{9}\right)^{\alpha}+8\left(\frac{3}{4}\right)^{\alpha}+8\left(\frac{9}{11}\right)^{\alpha}+2(m+n-6)\left(\frac{4}{5}\right)^{\alpha}$$
$$+2(m+n)\left(\frac{11}{13}\right)^{\alpha}+4(m+n-6)\left(\frac{5}{6}\right)^{\alpha}+2(m+n-6)\left(\frac{6}{7}\right)^{\alpha}$$
$$+6(m+n-6)\left(\frac{13}{15}\right)^{\alpha}+[8mn-24(m+n)+72]\left(\frac{7}{8}\right)^{\alpha}.$$

By Formula (1), we get

$$I_{R_\alpha}(MDN2(m,n))=\log R_\alpha(MDN2(m,n))$$
$$-\frac{4\times 18^{\alpha}\log 18^{\alpha}+8\times 15^{\alpha}\log 15^{\alpha}}{R_\alpha(MDN2(m,n))}$$
$$-\frac{8\times 30^{\alpha}\log(30)^{\alpha}+2(m+n-6)25^{\alpha}\log 25^{\alpha}+4\times 48^{\alpha}\log(48)^{\alpha}}{R_\alpha(MDN2(m,n))}$$
$$-\frac{2(m+n-4)40^{\alpha}\log 40^{\alpha}+4(m+n-6)35^{\alpha}\log 35^{\alpha}}{R_\alpha(MDN2(m,n))}$$
$$-\frac{2(m+n-8)49^{\alpha}\log 49^{\alpha}+8\times 42^{\alpha}\log 42^{\alpha}}{R_\alpha(MDN2(m,n))}$$
$$-\frac{6(m+n-6)56^{\alpha}\log 56^{\alpha}+[8mn-24(m+n)+72]64^{\alpha}\log 64^{\alpha}}{R_\alpha(MDN2(m,n))},$$

$$I_{H_\alpha}(MDN2(m,n))=\log H_\alpha(MDN2(m,n))$$
$$-\frac{4\times 9^{\alpha}\log 9^{\alpha}+8\times 8^{\alpha}\log 8^{\alpha}+8\times 11^{\alpha}\log(11)^{\alpha}}{H_\alpha(MDN2(m,n))}$$
$$-\frac{2(m+n-6)10^{\alpha}\log 10^{\alpha}+[8mn-24(m+n)+72]16^{\alpha}\log 16^{\alpha}}{H_\alpha(MDN2(m,n))}$$
$$-\frac{2(m+n)13^{\alpha}\log 13^{\alpha}+4(m+n-6)12^{\alpha}\log 12^{\alpha}}{H_\alpha(MDN2(m,n))}$$
$$-\frac{2(m+n-6)14^{\alpha}\log 14^{\alpha}+6(m+n-6)15^{\alpha}\log 15^{\alpha}}{H_\alpha(MDN2(m,n))},$$

$$I_{ABC_\alpha}(MDN2(m,n))=\log ABC_\alpha(MDN2(m,n))$$
$$-\frac{4\left(\frac{7}{18}\right)^{\alpha}\log\left(\frac{7}{18}\right)^{\alpha}+8\left(\frac{2}{5}\right)^{\alpha}\log\left(\frac{2}{5}\right)^{\alpha}}{ABC_\alpha(MDN2(m,n))}$$
$$-\frac{8\left(\frac{3}{10}\right)^{\alpha}\log\left(\frac{3}{10}\right)^{\alpha}+2(m+n-6)\left(\frac{8}{25}\right)^{\alpha}\log\left(\frac{8}{25}\right)^{\alpha}+4\left(\frac{1}{4}\right)^{\alpha}\log\left(\frac{1}{4}\right)^{\alpha}}{ABC_\alpha(MDN2(m,n))}$$

$$
\begin{aligned}
&- \frac{2(m+n-4)\left(\frac{11}{40}\right)^{\alpha}\log\left(\frac{11}{40}\right)^{\alpha} + 4(m+n-6)\left(\frac{2}{7}\right)^{\alpha}\log\left(\frac{2}{7}\right)^{\alpha}}{ABC_{\alpha}(MDN2(m,n))}\\
&- \frac{2(m+n-8)\left(\frac{12}{49}\right)^{\alpha}\log\left(\frac{12}{49}\right)^{\alpha} + 8\left(\frac{11}{42}\right)^{\alpha}\log\left(\frac{11}{42}\right)^{\alpha}}{ABC_{\alpha}(MDN2(m,n))}\\
&- \frac{6(m+n-6)\left(\frac{13}{56}\right)^{\alpha}\log\left(\frac{13}{56}\right)^{\alpha} + [8mn-24(m+n)+72]\left(\frac{7}{32}\right)^{\alpha}\log\left(\frac{7}{32}\right)^{\alpha}}{ABC_{\alpha}(MDN2(m,n))},
\end{aligned}
$$

$$
\begin{aligned}
I_{ABS_{\alpha}}(MDN2(m,n)) &= \log ABS_{\alpha}(MDN2(m,n))\\
&- \frac{4\left(\frac{7}{9}\right)^{\alpha}\log\left(\frac{7}{9}\right)^{\alpha} + 8\left(\frac{3}{4}\right)^{\alpha}\log\left(\frac{3}{4}\right)^{\alpha}}{ABS_{\alpha}(MDN2(m,n))}\\
&- \frac{2(m+n-6)\left(\frac{4}{5}\right)^{\alpha}\log\left(\frac{4}{5}\right)^{\alpha} + [8mn-24(m+n)+72]\left(\frac{7}{8}\right)^{\alpha}\log\left(\frac{7}{8}\right)^{\alpha}}{ABS_{\alpha}(MDN2(m,n))}\\
&- \frac{2(m+n)\left(\frac{11}{13}\right)^{\alpha}\log\left(\frac{11}{13}\right)^{\alpha} + 4(m+n-6)\left(\frac{5}{6}\right)^{\alpha}\log\left(\frac{5}{6}\right)^{\alpha} + 8\left(\frac{9}{11}\right)^{\alpha}\log\left(\frac{9}{11}\right)^{\alpha}}{ABS_{\alpha}(MDN2(m,n))}\\
&- \frac{2(m+n-6)\left(\frac{6}{7}\right)^{\alpha}\log\left(\frac{6}{7}\right)^{\alpha} + 6(m+n-6)\left(\frac{13}{15}\right)^{\alpha}\log\left(\frac{13}{15}\right)^{\alpha}}{ABS_{\alpha}(MDN2(m,n))}.
\end{aligned}
$$

This completes the proof. □

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Pair Mean Cordial Graphs Paired with Ladder

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Abstract: Let a graph $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \dfrac{p}{2}, & p \text{ is even} \\ \dfrac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G with a pair mean cordial labeling is called a pair mean cordial graph. In this paper, we discuss here the pair mean cordial labeling of union graphs like $L_m \cup L_n$, $P_m \cup L_n$, $C_m \cup L_n$, $W_m \cup L_n$, $S_m \cup L_n$.

Key Words: Pair mean cordial labeling, Smarandachely pair mean cordial labeling, S-marandachely pair mean cordial labeling graph, path, cycle, shell, wheel and ladder.

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§1. Introduction

In this paper, we will deal with finite, simple, connected and undirected graphs. We follow Harary [3] for basic terms and notations of graph theory and see Gallian [2] for more details on graph labeling. The concept of graph labeling was introduced by A. Rosa in 1967 [16]. The concept of cordial graphs was introduced by Cahit [1] and also studied some cordial related graphs in [4-7,13-15,17-20]. We have introduced the notion of pair mean cordial labeling of

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graphs in [8] and studied their properties in [9-12]. In this paper, we discuss here the pair mean cordial labeling of union graphs like $L_m \cup L_n$, $P_m \cup L_n$, $C_m \cup L_n$, $W_m \cup L_n$, $S_m \cup L_n$.

§2. Preliminaries

Definition 2.1 *A graph labeling is an assignment of integers to the vertices or edges, or both, subject to certain conditions.*—vskip3mm

Definition 2.2 *The union of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.*

Definition 2.3 *The shell S_n is the graph obtained by taking $n-3$ concurrent chord in cycle C_n. The vertex at which all the chords are concurrent is called the apex vertex.*

Definition 2.4 *A wheel W_n is a graph with $n+1$ vertices, formed by connecting a single vertex to all the vertices of the cycle C_n. It is denoted by $W_n = C_n + K_1$.*

Definition 2.5 *A ladder graph L_n is a planar, undirected graph with $2n$ vertices and $3n-2$ edges.*

§3. Pair Mean Cordial Labeling

Definition 3.1 *Let a graph $G = (V, E)$ be a (p, q) graph. Define*

$$\rho = \begin{cases} \dfrac{p}{2}, & p \text{ is even} \\ \dfrac{p-1}{2}, & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u)+\lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u)+\lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G with a pair mean cordial labeling is called a pair mean cordial graph.

Otherwise, if $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \geq 2$, such a labeling on G is said to be Smarandachely pair mean cordial labeling and G is called a Smarandachely pair mean cordial labeling graph.

Theorem 3.2 *$L_m \cup L_n$ is pair mean cordial for all $m, n \geq 2$.*

Proof Let $V(L_m \cup L_n) = \{u_i, v_i, x_j, y_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(L_m \cup L_n) = \{u_i v_i, x_j y_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{u_i u_{i+1}, v_i v_{i+1}, x_j x_{j+1}, y_j y_{j+1} : 1 \leq i \leq m-1, 1 \leq j \leq n-1\}$. Note that $L_m \cup L_n$ has $2m + 2n$ vertices and $3m + 3n - 4$ edges.

Case 1. $m \equiv 0 \pmod 4$

Let us assign the labels $2, 6, \ldots, m-2$ to the vertices $u_1, u_5, \cdots, u_{m-3}$ respectively and $-2, -6, \cdots, -m+2$ respectively to the vertices $u_2, u_6, \cdots, u_{m-2}$. Next we assign the labels $-3, -7, \cdots, -m+1$ to the vertices $u_3, u_7, \cdots, u_{m-1}$ respectively and $5, 9, \cdots, m+1$ respectively to the vertices $u_4, u_8, \cdots, u_m$. Now we give the labels $-1, -5, \cdots, -m+3$ to the vertices $v_1, u_5, \cdots, v_{m-3}$ respectively and $3, 7, \cdots, m-1$ respectively to the vertices $v_2, v_6, \cdots, v_{m-2}$. Then, we give the labels $4, 8, \cdots, m$ to the vertices $v_3, v_7, \cdots, v_{m-1}$ respectively and $-4, -8, \cdots,$ $-m$ respectively to the vertices $v_4, v_8, \cdots, v_m$. There are four subcases arise:

Subcase 1.1 $n \equiv 0 \pmod 4$

First we assign the labels $m+2, m+6, \cdots, m+n-2$ to the vertices $x_1, x_5, \cdots, x_{n-3}$ respectively and $m+3, m+7, \cdots, m+n-1$ respectively to the vertices $x_2, x_6, \cdots, x_{n-2}$. Now we assign the labels $-m-3, -m-7, \cdots, -m-n+1$ to the vertices $x_3, x_7, \cdots, x_{n-1}$ respectively and $-m-4, -m-8, \cdots, -m-n$ respectively to the vertices $x_4, x_8, \cdots, x_n$. Then we give the labels $-m-1, -m-5, \cdots, -m-n+3$ to the vertices $y_1, y_5, \cdots, y_{n-3}$ respectively and $-m-2, -m-6, \cdots, -m-n+2$ respectively to the vertices $y_2, y_6, \cdots, y_{n-2}$. Also we give the labels $m+4, m+6, \cdots, m+n$ to the vertices $y_3, y_7, \cdots, y_{n-1}$ respectively and $m+5, m+$ $9, \cdots, m+n-3$ respectively to the vertices $y_4, y_8, \cdots, y_{n-4}$. Fix the label 1 to the vertex y_n.

Subcase 1. $n \equiv 1 \pmod 4$

In this case, we assign the labels $m+2, m+6, \cdots, m+n-3$ respectively to the vertices $x_1, x_5, \cdots, x_{n-4}$ and $m+3, m+7, \cdots, m+n-2$ to the vertices $x_2, x_6, \cdots, x_{n-3}$ respectively. Also we assign the labels $-m-3, -m-7, \cdots, -m-n+2$ respectively to the vertices $x_3, x_7, \cdots, x_{n-2}$ and $-m-4, -m-8, \cdots, -m-n+1$ to the vertices $x_4, x_8, \cdots, x_{n-1}$ respectively. Fix the label 1 to the vertex x_n. Further more we give the labels $-m-1, -m-5, \cdots, -m-n$ respectively to the vertices $y_1, y_5, \cdots, y_n$ and $-m-2, -m-6, \cdots, -m-n+3$ to the vertices $y_2, y_6, \cdots, y_{n-3}$ respectively. We give the labels $m+4, m+8, \cdots, m+n-1$ respectively to the vertices $y_3, y_7, \cdots, y_{n-2}$ and $m+5, m+9, \cdots, m+n$ to the vertices $y_4, y_8, \cdots, y_{n-1}$ respectively.

Subcase 1.3 $n \equiv 2 \pmod 4$

We now assign the labels $m+2, m+6, \ldots, m+n$ respectively to the vertices $x_1, x_5, \cdots, x_{n-1}$ and $m+3, m+7, \cdots, m+n-3$ to the vertices $x_2, x_6, \cdots, x_{n-4}$ respectively. Then we assign the labels $-m-3, -m-7, \cdots, -m-n+3$ respectively to the vertices $x_3, x_7, \cdots, x_{n-3}$ and $-m-4, -m-8, \cdots, -m-n+2$ to the vertices $x_4, x_8, \cdots, x_{n-2}$ respectively. Fix the label 1 to the vertex x_n. More over we give the labels $-m-1, -m-5, \cdots, -m-n+1$ respectively to the vertices $y_1, y_5, \cdots, y_{n-1}$ and $-m-2, -m-6, \cdots, -m-n$ to the vertices $y_2, y_6, \cdots, y_n$ respectively. Finally we give the labels $m+4, m+8, \cdots, m+n-2$ respectively to the vertices $y_3, y_7, \cdots, y_{n-3}$ and $m+5, m+9, \cdots, m+n-1$ to the vertices $y_4, y_8, \cdots, y_{n-2}$ respectively.

Subcase 1.4 $n \equiv 3 \pmod 4$

Let us assign the labels $m+2, m+6, \cdots, m+n-1$ to the vertices $x_1, x_5, \cdots, x_{n-2}$ respectively and $m+3, m+7, \cdots, m+n$ respectively to the vertices $x_2, x_6, \cdots, x_{n-1}$. Next we assign the labels $-m-3, -m-7, \cdots, -m-n$ to the vertices $x_3, x_7, \cdots, x_n$ respectively and $-m-4, -m-8, \cdots, -m-n+3$ respectively to the vertices $x_4, x_8, \cdots, x_{n-3}$. We give the labels $-m-1, -m-5, \cdots, -m-n+2$ to the vertices $y_1, y_5, \cdots, y_{n-2}$ respectively and $-m-2, -m-6, \cdots, -m-n+1$ respectively to the vertices $y_2, y_6, \cdots, y_{n-1}$. We give the labels $m+4, m+8, \cdots, m+n-3$ to the vertices $y_3, y_7, \cdots, y_{n-4}$ respectively and $m+5, m+9, \cdots, m+n-2$ respectively to the vertices $y_4, y_8, \cdots, y_{n-3}$. Fix the label 1 to the vertex y_n.

Case 2. $m \equiv 1 \pmod 4$

First we assign the labels $2, 6, \cdots, m+1$ to the vertices $u_1, u_5, \cdots, u_m$ respectively and $-2, -6, \cdots, -m+3$ respectively to the vertices $u_2, u_6, \cdots, u_{m-3}$. We now assign the labels $-3, -7, \cdots, -m+2$ to the vertices $u_3, u_7, \cdots, u_{m-2}$ respectively and $5, 9, \cdots, m$ respectively to the vertices $u_4, u_8, \cdots, u_{m-1}$. Then we give the labels $-1, -5, \cdots, -m$ to the vertices $v_1, u_5, \cdots, v_m$ respectively and $3, 7, \cdots, m-2$ respectively to the vertices $v_2, v_6, \cdots, v_{m-3}$. Also, we give the labels $4, 8, \cdots, m-1$ to the vertices $v_3, v_7, \cdots, v_{m-2}$ respectively and $-4, -8, \cdots, -m+1$ respectively to the vertices $v_4, v_8, \cdots, v_{m-1}$. There are four subcases arise:

Subcase 2.1 $n \equiv 0 \pmod 4$

Now we assign the labels $-m-1, -m-5, \cdots, -m-n+3$ to the vertices $x_1, x_5, \cdots, x_{n-3}$ respectively and $m+3, m+7, \cdots, m+n-1$ respectively to the vertices $x_2, x_6, \cdots, x_{n-2}$. Now we assign the labels $m+4, m+8, \cdots, m+n$ to the vertices $x_3, x_7, \cdots, x_{n-1}$ respectively and $-m-4, -m-8, \cdots, -m-n$ respectively to the vertices $x_4, x_8, \cdots, x_n$. Then we give the labels $m+2, m+6, \cdots, m+n-2$ to the vertices $y_1, y_5, \ldots, y_{n-3}$ respectively and $-m-2, -m-6, \cdots, -m-n+2$ respectively to the vertices $y_2, y_6, \cdots, y_{n-2}$. Also we give the labels $-m-3, -m-7, \cdots, -m-n+1$ to the vertices $y_3, y_7, \cdots, y_{n-1}$ respectively and $m+5, m+9, \cdots, m+n-3$ respectively to the vertices $y_4, y_8, \cdots, y_{n-4}$. Fix the label 1 to the vertex y_n.

Subcase 2.2 $n \equiv 1 \pmod 4$

Also we assign the labels $-m-1, -m-5, \cdots, -m-n$ respectively to the vertices $x_1, x_5, \cdots, x_n$ and $m+3, m+7, \cdots, m+n-2$ to the vertices $x_2, x_6, \cdots, x_{n-3}$ respectively. Next we assign the labels $m+4, m+8, \cdots, m+n-1$ respectively to the vertices $x_3, x_7, \cdots, x_{n-2}$ and $-m-4, -m-8, \cdots, -m-n+1$ to the vertices $x_4, x_8, \cdots, x_{n-1}$ respectively. More over we give the labels $m+2, m+6, \cdots, m+n-3$ respectively to the vertices $y_1, y_5, \cdots, y_{n-4}$ and $-m-2, -m-6, \cdots, -m-n+3$ to the vertices $y_2, y_6, \cdots, y_{n-3}$ respectively. Then we give the labels $-m-3, -m-7, \cdots, -m-n+2$ respectively to the vertices $y_3, y_7, \cdots, y_{n-2}$ and $m+5, m+9, \cdots, m+n$ to the vertices $y_4, y_8, \cdots, y_{n-1}$ respectively. Fix the label 1 to the vertex y_n.

Subcase 2.3 $n \equiv 2 \pmod 4$

We now assign the labels $-m-1, -m-5, \cdots, -m-n+1$ respectively to the vertices $x_1, x_5, \cdots, x_{n-1}$ and $m+3, m+7, \cdots, m+n-3$ to the vertices $x_2, x_6, \cdots, x_{n-4}$ respectively. Then we assign the labels $m+4, m+8, \cdots, m+n-2$ respectively to the vertices $x_3, x_7, \cdots, x_{n-3}$ and $-m-4, -m-8, \cdots, -m-n+2$ to the vertices $x_4, x_8, \cdots, x_{n-2}$ respectively. Fix the label 1 to the vertex x_n. More over we give the labels $m+2, m+6, \cdots, m+n$ respectively to the vertices $y_1, y_5, \cdots, y_{n-1}$ and $-m-2, -m-6, \cdots, -m-n$ to the vertices $y_2, y_6, \cdots, y_n$

respectively. Finally we give the labels $-m-3, -m-7, \cdots, -m-n+3$ respectively to the vertices $y_3, y_7, \cdots, y_{n-3}$ and $m+5, m+9, \cdots, m+n-1$ to the vertices $y_4, y_8, \cdots, y_{n-2}$ respectively.

Subcase 2.4 $n \equiv 3 \pmod 4$

Then we assign the labels $-m-1, -m-5, \cdots, -m-n+2$ to the vertices $x_1, x_5, \cdots, x_{n-2}$ respectively and $m+3, m+7, \cdots, m+n$ respectively to the vertices $x_2, x_6, \cdots, x_{n-1}$. Next we assign the labels $m+4, m+8, \cdots, m+n-3$ to the vertices $x_3, x_7, \cdots, x_{n-4}$ respectively and $-m-4, -m-8, \cdots, -m-n+3$ respectively to the vertices $x_4, x_8, \cdots, x_{n-3}$. Fix the label 1 to the vertex x_n. We give the labels $m+2, m+6, \cdots, m+n-1$ to the vertices $y_1, y_5, \cdots, y_{n-2}$ respectively and $-m-2, -m-6, \ldots, -m-n+1$ respectively to the vertices $y_2, y_6, \cdots, y_{n-1}$. We give the labels $-m-3, -m-7, \cdots, -m-n$ to the vertices $y_3, y_7, \cdots, y_n$ respectively and $m+5, m+9, \cdots, m+n-2$ respectively to the vertices $y_4, y_8, \cdots, y_{n-3}$.

Case 3. $m \equiv 2 \pmod 4$

Assign the labels $2, 6, \cdots, m$ to the vertices $u_1, u_5, \cdots, u_{m-1}$ respectively and $-2, -6, \cdots, -m$ respectively to the vertices $u_2, u_6, \cdots, u_m$. Then we assign the labels $-3, -7, \cdots, -m+3$ to the vertices $u_3, u_7, \cdots, u_{m-2}$ respectively and $5, 9, \cdots, m-1$ respectively to the vertices $u_4, u_8, \cdots, u_{m-3}$. Also we give the labels $-1, -5, \cdots, -m+1$ to the vertices $v_1, u_5, \cdots, v_{m-1}$ respectively and $3, 7, \cdots, m+1$ respectively to the vertices $v_2, v_6, \cdots, v_m$. Next we give the labels $4, 8, \cdots, m-2$ to the vertices $v_3, v_7, \cdots, v_{m-2}$ respectively and $-4, -8, \cdots, -m+2$ respectively to the vertices $v_4, v_8, \cdots, v_{m-3}$.

Subcase 3.1 $n \equiv 0 \pmod 4$

Assign the labels to the vertices x_j and y_j for $1 \leq j \leq n$ as in Subcase 1.1 of Case 1.

Subcase 3.2 $n \equiv 1 \pmod 4$

Let us assign the labels to the vertices x_j and y_j for $1 \leq j \leq n$ as in Subcase 1.2 of Case 1.

Subcase 3.3 $n \equiv 2 \pmod 4$

Also, we assign the labels to vertices x_j and y_j for $1 \leq j \leq n$ as in Subcase 1.3 of Case 1.

Subcase 3.4 $n \equiv 3 \pmod 4$

Now, we assign the labels to vertices x_j and y_j for $1 \leq j \leq n$ as in Subcase 1.4 of Case 1.

Case 4: $m \equiv 3 \pmod 4$

Let us assign the labels $2, 6, \cdots, m-1$ to the vertices $u_1, u_5, \cdots, u_{m-2}$ respectively and $-2, -6, \cdots, -m$ respectively to the vertices $u_2, u_6, \cdots, u_{m-1}$. Then we assign the labels $-3, -7, \cdots, -m$ to the vertices $u_3, u_7, \cdots, u_m$ respectively and $5, 9, \cdots, m-2$ respectively to the vertices $u_4, u_8, \cdots, u_{m-3}$. Also we give the labels $-1, -5, \cdots, -m+2$ to the vertices $v_1, u_5, \cdots, v_{m-2}$ respectively and $3, 7, \cdots, m$ respectively to the vertices $v_2, v_6, \cdots, v_{m-1}$. Next we give the labels $4, 8, \cdots, m+1$ to the vertices $v_3, v_7, \cdots, v_m$ respectively and $-4, -8, \cdots, -m+3$ respectively to the vertices $v_4, v_8, \cdots, v_{m-3}$.

In this case, there are 4 subcases should be discussed.

Subcase 4.1 $n \equiv 0 \pmod 4$

Assign the labels to the vertices x_j and y_j for $1 \le j \le n$ as in Subcase 2.1 of Case 2.

Subcase 4.2 $n \equiv 1 \pmod 4$

Let us assign the labels to the vertices x_j and y_j for $1 \le j \le n$ as in Subcase 2.2 of Case 2.

Subcase 4.3 $n \equiv 2 \pmod 4$

Also, we assign the labels to vertices x_j and y_j for $1 \le j \le n$ as in Subcase 2.3 of Case 2.

Subcase 4.4 $n \equiv 3 \pmod 4$

Now, we assign the labels to vertices x_j and y_j for $1 \le j \le n$ as in Subcase 2.4 of Case 2.

The following table shows that this vertex labeling λ is a pair mean cordial of $L_m \cup L_n$ for all $m, n \ge 2$.

m	n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
$m \equiv 0, 2 \pmod 4$	$n \equiv 0 \pmod 4$	$\frac{3m+3n-4}{2}$	$\frac{3m+3n-4}{2}$
	$n \equiv 1 \pmod 4$	$\frac{3m+3n-5}{2}$	$\frac{3m+3n-3}{2}$
	$n \equiv 2 \pmod 4$	$\frac{3m+3n-4}{2}$	$\frac{3m+3n-4}{2}$
	$n \equiv 3 \pmod 4$	$\frac{3m+3n-5}{2}$	$\frac{3m+3n-3}{2}$
$m \equiv 1, 3 \pmod 4$	$n \equiv 0 \pmod 4$	$\frac{3m+3n-5}{2}$	$\frac{3m+3n-3}{2}$
	$n \equiv 1 \pmod 4$	$\frac{3m+3n-4}{2}$	$\frac{3m+3n-4}{2}$
	$n \equiv 2 \pmod 4$	$\frac{3m+3n-5}{2}$	$\frac{3m+3n-3}{2}$
	$n \equiv 3 \pmod 4$	$\frac{3m+3n-4}{2}$	$\frac{3m+3n-4}{2}$

Table 1

This completes the proof. □

Example 3.3 A pair mean cordial labeling of $L_5 \cup L_6$ is shown in Figure 1.

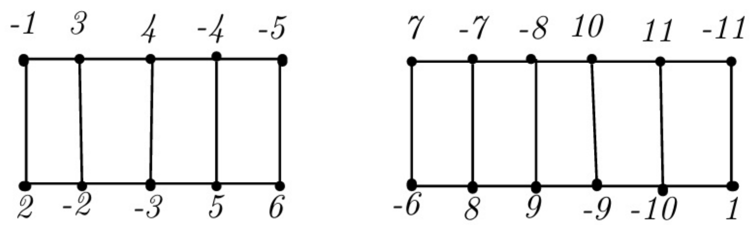

Figure 1

Theorem 3.4 *$W_m \cup L_n$ is pair mean cordial for all $m \ge 3$ and $n \ge 2$.*

Proof Let $V(W_m \cup L_n) = \{u_0, u_i, v_j, w_j : 1 \le i \le m, 1 \le j \le n\}$ and $E(W_m \cup L_n) = \{u_0u_i, v_jw_j : 1 \le i \le m, 1 \le j \le n\} \cup \{u_iu_{i+1}, u_mu_1, v_jv_{j+1}, w_jw_{j+1} : 1 \le i \le m-1, 1 \le j \le n-1\}$. Note that $W_m \cup L_n$ has $m + 2n + 1$ vertices and $2m + 3n - 2$ edges.

Case 1. m is odd.

Fix the label 2 to the vertex u_0. Let $\lambda(v_1) = \frac{m+5}{2}$ and $\lambda(w_1) = \frac{-m-3}{2}$. Then we assign the labels $-1, -2, \cdots, \frac{-m-1}{2}$ to the vertices $u_1, u_3, \ldots, u_m$ respectively and $3, 4, \cdots, \frac{m+3}{2}$ respectively to the vertices $u_2, u_4, \cdots, u_{m-1}$. Hence there are four subcases arise:

Subcase 1.1 $n \equiv 0 \pmod 4$

Next we assign the labels $\frac{-m-5}{2}, \frac{-m-13}{2}, \cdots, -m-n+4$ to the vertices $v_2, v_6, \cdots, v_{n-2}$ respectively and $\frac{m+9}{2}, \frac{m+17}{2}, \cdots, m+n-2$ respectively to the vertices $v_3, v_7, \cdots, v_{n-1}$. Then we give the labels $\frac{m+11}{2}, \frac{m+19}{2}, \cdots, m+n-5$ to the vertices $v_4, v_8, \cdots, v_{n-4}$ respectively and $\frac{-m-11}{2}, \frac{-m-19}{2}, \cdots, -m-n+5$ respectively to the vertices $v_5, v_9, \cdots, v_{n-3}$. Fix the label 1 to the vertex v_n. More over we assign the labels $\frac{m+7}{2}, \frac{m+15}{2}, \cdots, m+n-3$ to the vertices $w_2, w_6, \cdots, w_{n-2}$ respectively and $\frac{-m-7}{2}, \frac{-m-15}{2}, \cdots, -m-n+3$ respectively to the vertices $w_3, w_7, \cdots, w_{n-1}$. Also we give the labels $\frac{-m-9}{2}, \frac{-m-17}{2}, \cdots, -m-n+2$ to the vertices $w_4, w_8, \cdots, w_n$ respectively and $\frac{m+13}{2}, \frac{m+21}{2}, \cdots, m+n-4$ respectively to the vertices $w_5, w_9, \cdots, w_{n-3}$.

Subcase 1.2 $n \equiv 1 \pmod 4$

Now we assign the labels $\frac{-m-5}{2}, \frac{-m-13}{2}, \cdots, -m-n+5$ to the vertices $v_2, v_6, \cdots, v_n$ respectively and $\frac{m+9}{2}, \frac{m+17}{2}, \cdots, m+n-3$ respectively to the vertices $v_3, v_7, \cdots, v_{n-1}$. Therefore we give the labels $\frac{m+11}{2}, \frac{m+19}{2}, \cdots, m+n-2$ to the vertices $v_4, v_8, \cdots, v_{n-2}$ respectively and $\frac{-m-11}{2}, \frac{-m-19}{2}, \cdots, -m-n+2$ respectively to the vertices $v_5, v_9, \cdots, v_n$. More over we assign the labels $\frac{m+7}{2}, \frac{m+15}{2}, \cdots, m+n-4$ to the vertices $w_2, w_6, \cdots, w_{n-3}$ respectively and $\frac{-m-7}{2}, \frac{-m-15}{2}, \cdots, -m-n+4$ respectively to the vertices $w_3, w_7, \cdots, w_{n-2}$. Also we give the labels $\frac{-m-9}{2}, \frac{-m-17}{2}, \cdots, -m-n+3$ to the vertices $w_4, w_8, \cdots, w_{n-1}$ respectively and $\frac{m+13}{2}, \frac{m+21}{2}, \cdots, m+n-5$ respectively to the vertices $w_5, w_9, \cdots, w_{n-4}$. Finally fix the label 1 to the vertex w_n.

Subcase 1.3 $n \equiv 2 \pmod 4$

In this case, we assign the labels $\frac{-m-5}{2}, \frac{-m-13}{2}, \cdots, -m-n+2$ to the vertices $v_2, v_6, \cdots, v_n$ respectively and $\frac{m+9}{2}, \frac{m+17}{2}, \cdots, m+n-4$ respectively to the vertices $v_3, v_7, \cdots, v_{n-3}$. Next we give the labels $\frac{m+11}{2}, \frac{m+19}{2}, \cdots, m+n-3$ to the vertices $v_4, v_8, \cdots, v_{n-2}$ respectively and $\frac{-m-11}{2}, \frac{-m-19}{2}, \cdots, -m-n+3$ respectively to the vertices $v_5, v_9, \cdots, v_{n-1}$. Further more we assign the labels $\frac{m+7}{2}, \frac{m+15}{2}, \cdots, m+n-5$ to the vertices $w_2, w_6, \cdots, w_{n-4}$ respectively and $\frac{-m-7}{2}, \frac{-m-15}{2}, \cdots, -m-n+5$ respectively to the vertices $w_3, w_7, \cdots, w_{n-3}$. Also we give the labels $\frac{-m-9}{2}, \frac{-m-17}{2}, \cdots, -m-n+4$ to the vertices $w_4, w_8, \cdots, w_{n-2}$ respectively and $\frac{m+13}{2}, \frac{m+21}{2}, \cdots, m+n-2$ respectively to the vertices $w_5, w_9, \cdots, w_{n-1}$. Finally fix the label 1 to the vertex w_n.

Subcase 1.4 $n \equiv 3 \pmod 4$

In this case, we assign the labels $\frac{-m-5}{2}, \frac{-m-13}{2}, \cdots, -m-n+3$ to the vertices $v_2, v_6, \cdots, v_{n-1}$ respectively and $\frac{m+9}{2}, \frac{m+17}{2}, \cdots, m+n-5$ respectively to the vertices $v_3, v_7, \cdots, v_{n-4}$. Then we give the labels $\frac{m+11}{2}, \frac{m+19}{2}, \cdots, m+n-4$ to the vertices $v_4, v_8, \cdots, v_{n-3}$ respectively and $\frac{-m-11}{2}, \frac{-m-19}{2}, \cdots, -m-n+4$ respectively to the vertices $v_5, v_9, \cdots, v_{n-2}$. Fix the label

1 to the vertex w_n. More over we assign the labels $\frac{m+7}{2}, \frac{m+15}{2}, \cdots, m+n-2$ to the vertices $w_2, w_6, \cdots, w_{n-1}$ respectively and $\frac{-m-7}{2}, \frac{-m-15}{2}, \cdots, -m-n+2$ respectively to the vertices $w_3, w_7, \cdots, w_n$. Also we give the labels $\frac{-m-9}{2}, \frac{-m-17}{2}, \cdots, -m-n+5$ to the vertices $w_4, w_8, \cdots, w_{n-3}$ respectively and $\frac{m+13}{2}, \frac{m+21}{2}, \cdots, m+n-3$ respectively to the vertices $w_5, w_9, \cdots, w_{n-2}$. Finally fix the label 1 to the vertex w_n.

Case 2. m is even.

Fix the label 3 to the vertex u_0. Then we assign the labels $2, \cdots, \frac{m+2}{2}$ to the vertices $u_1, u_3, \cdots, u_{m-1}$ respectively and $-1, -2, \cdots, \frac{-m}{2}$ respectively to the vertices $u_2, u_4, \cdots, u_m$. Hence there are four subcases arise:

Subcase 2.1 $n \equiv 0 \pmod 4$

In this case, we assign the labels $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, m+n-5$ to the vertices $v_1, v_5, \cdots, v_{n-3}$ respectively and $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, m+n-4$ respectively to the vertices $v_2, v_6, \cdots, v_{n-2}$. Now we give the labels $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, -m-n+4$ to the vertices $v_3, v_7, \cdots, v_{n-1}$ respectively and $\frac{-m-8}{2}, \frac{-m-16}{2}, \cdots, -m-n+3$ respectively to the vertices $v_4, v_8, \cdots, v_n$. More over we assign the labels $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, -m-n+6$ to the vertices $w_1, w_5, \cdots, w_{n-3}$ respectively and $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, -m-n+5$ respectively to the vertices $w_2, w_6, \cdots, w_{n-2}$. We also give the labels $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, m+n-3$ to the vertices $w_3, w_7, \cdots, w_{n-1}$ respectively and $\frac{m+10}{2}, \frac{m+18}{2}, \cdots, m+n-6$ respectively to the vertices $w_4, w_8, \cdots, w_{n-4}$. Finally fix the label 1 to the vertex w_n.

Subcase 2.2 $n \equiv 1 \pmod 4$

We now assign the labels $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, m+n-6$ to the vertices $v_1, v_5, \cdots, v_{n-4}$ respectively and $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, m+n-5$ respectively to the vertices $v_2, v_6, \cdots, v_{n-3}$. Therefore we give the labels $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, -m-n+5$ to the vertices $v_3, v_7, \cdots, v_{n-2}$ respectively and $\frac{-m-8}{2}, \frac{-m-16}{2}, \cdots, -m-n+4$ respectively to the vertices $v_4, v_8, \cdots, v_{n-1}$. Fix the label 1 to the vertex v_n. Furthermore we assign the labels $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, -m-n+3$ to the vertices $w_1, w_5, \cdots, w_{n-3}$ respectively and $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, -m-n+6$ respectively to the vertices $w_2, w_6, \cdots, w_{n-3}$. We also give the labels $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, m+n-4$ to the vertices $w_3, w_7, \cdots, w_{n-2}$ respectively and $\frac{m+10}{2}, \frac{m+18}{2}, \cdots, m+n-3$ respectively to the vertices $w_4, w_8, \cdots, w_{n-1}$.

Subcase 2.3 $n \equiv 2 \pmod 4$

In this case, let us assign the labels $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, m+n-3$ to the vertices $v_1, v_5, \cdots, v_{n-1}$ respectively and $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, m+n-6$ respectively to the vertices $v_2, v_6, \cdots, v_{n-4}$. Now we give the labels $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, -m-n+6$ to the vertices $v_3, v_7, \cdots, v_{n-3}$ respectively and $\frac{-m-8}{2}, \frac{-m-16}{2}, \cdots, -m-n+5$ respectively to the vertices $v_4, v_8, \cdots, v_{n-2}$. Fix the label 1 to the vertex v_n. More over we assign the labels $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, -m-n+4$ to the vertices $w_1, w_5, \cdots, w_{n-1}$ respectively and $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, -m-n+3$ respectively to the vertices $w_2, w_6, \cdots, w_n$. We also give the labels $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, m+n-5$ to the vertices $w_3, w_7, \cdots, w_{n-3}$ respectively and $\frac{m+10}{2}, \frac{m+18}{2}, \cdots, m+n-4$ respectively to the vertices $w_4, w_8, \cdots, w_{n-2}$.

Subcase 2.4 $n \equiv 3 \pmod 4$

In this case, we assign the labels $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, m+n-4$ to the vertices $v_1, v_5, \cdots, v_{n-2}$ respectively and $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, m+n-3$ respectively to the vertices $v_2, v_6, \cdots, v_{n-1}$. Next we give the labels $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, -m-n+3$ to the vertices $v_3, v_7, \cdots, v_n$ respectively and $\frac{-m-8}{2}, \frac{-m-16}{2}, \cdots, -m-n+6$ respectively to the vertices $v_4, v_8, \cdots, v_{n-3}$. Then we assign the labels $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, -m-n+5$ to the vertices $w_1, w_5, \cdots, w_{n-2}$ respectively and $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, -m-n+4$ respectively to the vertices $w_2, w_6, \cdots, w_{n-1}$. We also give the labels $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, m+n-6$ to the vertices $w_3, w_7, \cdots, w_{n-4}$ respectively and $\frac{m+10}{2}, \frac{m+18}{2}, \cdots, m+n-5$ respectively to the vertices $w_4, w_8, \cdots, w_{n-3}$. Fix the label 1 to the vertex w_n.

The following table shows that this vertex labeling λ is a pair mean cordial of $W_m \cup L_n$ for all $m \geq 3$ and $n \geq 2$.

m	n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
m is odd	$n \equiv 0 \pmod 4$	$\frac{2m+3n-2}{2}$	$\frac{2m+3n-2}{2}$
	$n \equiv 1 \pmod 4$	$\frac{2m+3n-3}{2}$	$\frac{2m+3n-1}{2}$
	$n \equiv 2 \pmod 4$	$\frac{2m+3n-2}{2}$	$\frac{2m+3n-2}{2}$
	$n \equiv 3 \pmod 4$	$\frac{2m+3n-3}{2}$	$\frac{2m+3n-1}{2}$
m is even	$n \equiv 0 \pmod 4$	$\frac{2m+3n-2}{2}$	$\frac{2m+3n-2}{2}$
	$n \equiv 1 \pmod 4$	$\frac{2m+3n-3}{2}$	$\frac{2m+3n-1}{2}$
	$n \equiv 2 \pmod 4$	$\frac{2m+3n-2}{2}$	$\frac{2m+3n-2}{2}$
	$n \equiv 3 \pmod 4$	$\frac{2m+3n-3}{2}$	$\frac{2m+3n-1}{2}$

Table 2

This completes the proof. □

Example 3.5 A pair mean cordial labeling of $W_8 \cup L_7$ is shown in Figure 2.

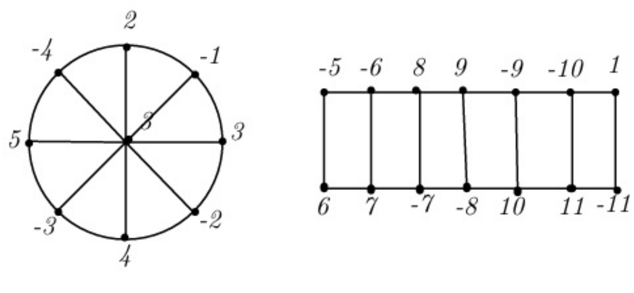

Figure 2

Theorem 3.6 *$P_m \cup L_n$ is pair mean cordial for all $m, n \geq 2$.*

Proof Let P_m be the path $u_1 u_2 \cdots u_n$. Note that $P_m \cup L_n$ has $m+2n$ vertices and $m+3n-3$ edges.

Case 1. m is even.

First we assign the labels $1, 2, \cdots, \frac{m}{2}$ to the vertices $u_1, u_3, \cdots, u_{m-1}$ respectively and $-1, -2, \cdots, \frac{-m+2}{2}$ respectively to the vertices $u_2, u_4, \cdots, u_{m-2}$. Fix the label $\frac{-m-2n}{2}$ to the vertex u_m. Hence there are four subcases that arise:

Subcase 1.1 $n \equiv 0 \pmod 4$

In this case, we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n-6}{2}$ to vertices $v_1, v_5, \cdots, v_{n-3}$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n-4}{2}$ respectively to vertices $v_2, v_6, \cdots, v_{n-2}$. Next, we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+4}{2}$ to vertices $v_3, v_7, \cdots, v_{n-1}$ respectively and the labels $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+2}{2}$ respectively to vertices $v_4, v_8, \cdots, v_n$. Moreover, we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+8}{2}$ to vertices $w_1, w_5, \cdots, w_{n-3}$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+6}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_{n-2}$. Also, we give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n-2}{2}$ to the vertices $w_3, w_7, \cdots, w_{n-1}$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{m+2n}{2}$ respectively to vertices $w_4, w_8, \cdots, w_n$.

Subcase 1.2 $n \equiv 1 \pmod 4$

Furthermore we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n}{2}$ to the vertices $v_1, v_5, \cdots, v_n$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n-6}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_{n-3}$. Therefore we assign the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+6}{2}$ to the vertices $v_3, v_7, \cdots, v_{n-2}$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+4}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_{n-1}$. More over we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+2}{2}$ to the vertices $w_1, w_5, \cdots, w_n$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+8}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_{n-3}$. Also we give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n-4}{2}$ to the vertices $w_3, w_7, \cdots, w_{n-2}$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{m+2n-2}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_{n-1}$.

Subcase 1.3 $n \equiv 2 \pmod 4$

In this case, we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n-2}{2}$ to the vertices $v_1, v_5, \cdots, v_{n-1}$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_n$. Therefore we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+8}{2}$ to the vertices $v_3, v_7, \cdots, v_{n-3}$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+6}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_{n-2}$. Furthermore we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+4}{2}$ to the vertices $w_1, w_5, \cdots, w_{n-1}$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+2}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_n$. We give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n-6}{2}$ to the vertices $w_3, w_7, \cdots, w_{n-3}$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{m+2n-4}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_{n-2}$.

Subcase 1.4 $n \equiv 3 \pmod 4$

In this case, we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n-4}{2}$ to the vertices $v_1, v_5, \cdots, v_{n-2}$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n-2}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_{n-1}$. Therefore we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+2}{2}$ to the vertices $v_3, v_7, \cdots, v_n$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+8}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_{n-3}$. More over we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+6}{2}$ to the vertices $w_1, w_5, \cdots, w_{n-2}$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+4}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_{n-1}$. Then we give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n}{2}$ to the vertices $w_3, w_7, \cdots, w_n$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots,$

$\frac{m+2n-6}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_{n-3}$.

Case 2. m is odd.

In this case, assign the labels to the vertices $u_i, 1 \leq i \leq m-2$ and $v_j, w_j, 1 \leq j \leq n$ as in Case 1, $\lambda(u_{m-1}) = \frac{-m+3}{2}$ and $\lambda(u_m) = \frac{-m-2n}{2}$.

In general, we shows that this vertex labeling λ is a pair mean cordial of $P_m \cup L_n$ for all $m, n \geq 2$ in Table 3.

m	n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
m is odd	$n \equiv 0 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-3}{2}$
	$n \equiv 1 \pmod 4$	$\frac{m+3n-4}{2}$	$\frac{m+3n-2}{2}$
	$n \equiv 2 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-3}{2}$
	$n \equiv 3 \pmod 4$	$\frac{m+3n-4}{2}$	$\frac{m+3n-2}{2}$
m is even	$n \equiv 0 \pmod 4$	$\frac{m+3n-4}{2}$	$\frac{m+3n-2}{2}$
	$n \equiv 1 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-3}{2}$
	$n \equiv 2 \pmod 4$	$\frac{m+3n-4}{2}$	$\frac{m+3n-2}{2}$
	$n \equiv 3 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-3}{2}$

Table 3

This completes the proof. □

Example 3.7 A pair mean cordial labeling of $P_7 \cup L_5$ is shown in Figure 3.

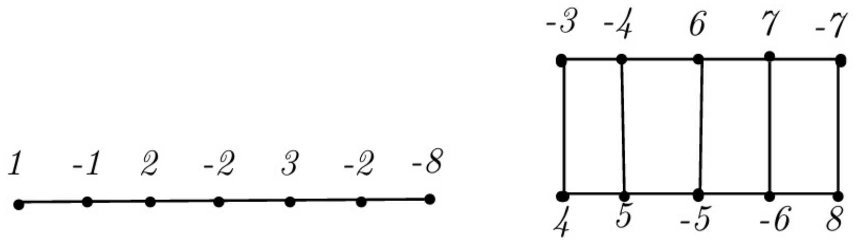

Figure 3

Theorem 3.8 *$C_m \cup L_n$ is pair mean cordial for all $m \geq 3$ and $n \geq 2$.*

Proof Let C_m be the cycle $u_1 u_2 \cdots u_m u_1$. Note that $C_m \cup L_n$ has $m + 2n$ vertices and $m + 3n - 2$ edges.

Case 1. m is even.

In this case, we assign the labels to the vertices $u_i, 1 \leq i \leq m$ as in Case 1 of Theorem 3.7. Hence there are four subcases that arise:

Subcase 1.1 $n \equiv 0 \pmod 4$

In this case, we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n-6}{2}$ to the vertices $v_1, v_5, \cdots, v_{n-3}$

respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+6}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_{n-2}$. Next we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+4}{2}$ to the vertices $v_3, v_7, \cdots, v_{n-1}$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{m+2n}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_n$. More over we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+8}{2}$ to the vertices $w_1, w_5, \cdots, w_{n-3}$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n-4}{2}$ respectively to the vertices $w_2, w_6, \ldots, w_{n-2}$. Also we give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n-2}{2}$ to the vertices $w_3, w_7, \cdots, w_{n-1}$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+2}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_n$.

Subcase 1.2 $n \equiv 1 \pmod 4$

In this case, we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n}{2}$ to the vertices $v_1, v_5, \cdots, v_n$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_{n-3}$. Therefore we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+6}{2}$ to the vertices $v_3, v_7, \cdots, v_{n-2}$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{m+2n-2}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_{n-1}$. More over we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+2}{2}$ to the vertices $w_1, w_5, \cdots, w_n$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n-6}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_{n-3}$. More over we give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n-4}{2}$ to vertices $w_3, w_7, \cdots, w_{n-2}$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+4}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_{n-1}$.

Subcase 1.3 $n \equiv 2 \pmod 4$

Furthermore assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n-2}{2}$ to the vertices $v_1, v_5, \cdots, v_{n-1}$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+2}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_n$. Therefore we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+8}{2}$ to the vertices $v_3, v_7, \cdots, v_{n-3}$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{m+2n-4}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_{n-2}$. Next we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+4}{2}$ to the vertices $w_1, w_5, \cdots, w_{n-1}$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_n$. Finally we give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n-6}{2}$ to the vertices $w_3, w_7, \cdots, w_{n-3}$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+6}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_{n-2}$.

Subcase 1.4 $n \equiv 3 \pmod 4$

In this case, we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n-4}{2}$ to the vertices $v_1, v_5, \cdots, v_{n-2}$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+4}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_{n-1}$. Also we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+2}{2}$ to the vertices $v_3, v_7, \cdots, v_n$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{-m-2n+6}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_{n-3}$. More over we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+6}{2}$ to the vertices $w_1, w_5, \cdots, w_{n-2}$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n-2}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_{n-1}$. Then we give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n}{2}$ to vertices $w_3, w_7, \cdots, w_n$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+8}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_{n-3}$.

Case 2. m is even.

In this case, assign the labels to the vertices $u_i, 1 \leq i \leq m$ and $v_j, w_j, 1 \leq j \leq n$ as in Case 1. If $m = 3$, $\lambda(u_{m-1}) = 1$ and Table 4 shows that this vertex labeling λ is a pair mean cordial of $C_m \cup L_n$ for all $m \geq 3$ and $n \geq 2$.

m	n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
m is odd	$n \equiv 0 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-1}{2}$
	$n \equiv 1 \pmod 4$	$\frac{m+3n-2}{2}$	$\frac{m+3n-2}{2}$
	$n \equiv 2 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-1}{2}$
	$n \equiv 3 \pmod 4$	$\frac{m+3n-2}{2}$	$\frac{m+3n-2}{2}$
m is even	$n \equiv 0 \pmod 4$	$\frac{m+3n-2}{2}$	$\frac{m+3n-2}{2}$
	$n \equiv 1 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-1}{2}$
	$n \equiv 2 \pmod 4$	$\frac{m+3n-2}{2}$	$\frac{m+3n-2}{2}$
	$n \equiv 3 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-1}{2}$

Table 4

This completes the proof. □

Example 3.9 A pair mean cordial labeling of $C_7 \cup L_8$ is shown in Figure 4.

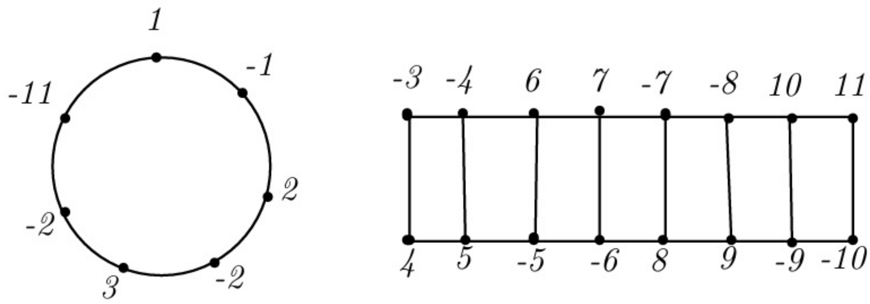

Figure 4

Theorem 3.10 *$S_m \cup L_n$ is pair mean cordial for all $m \geq 3$ and $n \geq 2$.*

Proof Let us define

$$V(S_m \cup L_n) = \{u_i, v_j, w_j : 1 \leq i \leq m \text{ and } 1 \leq \text{j} \leq \text{n}\},$$

$$E(S_m \cup L_n) = \{u_i u_{i+1}, u_m u_1, v_j w_j : 1 \leq i \leq m-1 \text{ and } 1 \leq \text{j} \leq \text{n}\}$$
$$\bigcup \{u_1 u_{i+2}, v_j v_{j+1}, w_j w_{j+1} : 1 \leq i \leq m-3 \text{ and } 1 \leq \text{j} \leq \text{n}-1\}.$$

Clearly, the graph $S_m \cup L_n$ has $m+2n$ vertices and $m+3n-2$ edges.

Case 1. m is even.

In this case, we assign the labels to the vertices $u_i, 1 \leq i \leq m$ as in case (i) of theorem 3.9. Hence there are four subcases that arise:

Subcase 1.1 $n \equiv 0 \pmod 4$

In this case, we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n-6}{2}$ to the vertices $v_1, v_5, \cdots, v_{n-3}$

respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+6}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_{n-2}$. Next we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+4}{2}$ to the vertices $v_3, v_7, \cdots, v_{n-1}$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{m+2n}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_n$. More over we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+8}{2}$ to the vertices $w_1, w_5, \cdots, w_{n-3}$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n-4}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_{n-2}$. Also we give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n-2}{2}$ to the vertices $w_3, w_7, \cdots, w_{n-1}$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+2}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_n$.

Subcase 1.2 $n \equiv 1 \pmod 4$

Assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n}{2}$ to the vertices $v_1, v_5, \cdots, v_n$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_{n-3}$. Then we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+6}{2}$ to the vertices $v_3, v_7, \cdots, v_{n-2}$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{m+2n-2}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_{n-1}$. More over we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+2}{2}$ to the vertices $w_1, w_5, \cdots, w_n$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n-6}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_{n-3}$. We give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n-4}{2}$ to the vertices $w_3, w_7, \cdots, w_{n-2}$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+4}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_{n-1}$.

Subcase 1.3 $n \equiv 2 \pmod 4$

In this case, we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n-2}{2}$ to the vertices $v_1, v_5, \cdots, v_{n-1}$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+2}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_n$. Then we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+8}{2}$ to the vertices $v_3, v_7, \cdots, v_{n-3}$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{m+2n-4}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_{n-2}$. We also assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+4}{2}$ to vertices $w_1, w_5, \cdots, w_{n-1}$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_n$. Furthermore we give labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n-6}{2}$ to the vertices $w_3, w_7, \cdots, w_{n-3}$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+6}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_{n-2}$.

Subcase 1.4 $n \equiv 3 \pmod 4$

More over we assign the labels $\frac{m+2}{2}, \frac{m+10}{2}, \cdots, \frac{m+2n-4}{2}$ to the vertices $v_1, v_5, \cdots, v_{n-2}$ respectively and $\frac{-m-2}{2}, \frac{-m-10}{2}, \cdots, \frac{-m-2n+4}{2}$ respectively to the vertices $v_2, v_6, \cdots, v_{n-1}$. Therefore we give the labels $\frac{-m-4}{2}, \frac{-m-12}{2}, \cdots, \frac{-m-2n+2}{2}$ to the vertices $v_3, v_7, \cdots, v_n$ respectively and $\frac{m+8}{2}, \frac{m+16}{2}, \cdots, \frac{-m-2n+6}{2}$ respectively to the vertices $v_4, v_8, \cdots, v_{n-3}$. Next we assign the labels $\frac{-m}{2}, \frac{-m-8}{2}, \cdots, \frac{-m-2n+6}{2}$ to the vertices $w_1, w_5, \cdots, w_{n-2}$ respectively and $\frac{m+4}{2}, \frac{m+12}{2}, \cdots, \frac{m+2n-2}{2}$ respectively to the vertices $w_2, w_6, \cdots, w_{n-1}$. Furthermore we give the labels $\frac{m+6}{2}, \frac{m+14}{2}, \cdots, \frac{m+2n}{2}$ to the vertices $w_3, w_7, \cdots, w_n$ respectively and $\frac{-m-6}{2}, \frac{-m-14}{2}, \cdots, \frac{-m-2n+8}{2}$ respectively to the vertices $w_4, w_8, \cdots, w_{n-3}$.

Case 2. m is even.

In this case, assign the labels to the vertices $u_i, 1 \leq i \leq m$ and $v_j, w_j, 1 \leq j \leq n$ as in Case 1. If $m = 3$, $\lambda(u_{m-1}) = 1$ and Table 5 shows that this vertex labeling λ is a pair mean cordial of $S_m \cup L_n$ for all $m \geq 3$ and $n \geq 2$.

m	n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
m is odd	$n \equiv 0 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-1}{2}$
	$n \equiv 1 \pmod 4$	$\frac{m+3n-2}{2}$	$\frac{m+3n-2}{2}$
	$n \equiv 2 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-1}{2}$
	$n \equiv 3 \pmod 4$	$\frac{m+3n-2}{2}$	$\frac{m+3n-2}{2}$
m is even	$n \equiv 0 \pmod 4$	$\frac{m+3n-2}{2}$	$\frac{m+3n-2}{2}$
	$n \equiv 1 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-1}{2}$
	$n \equiv 2 \pmod 4$	$\frac{m+3n-2}{2}$	$\frac{m+3n-2}{2}$
	$n \equiv 3 \pmod 4$	$\frac{m+3n-3}{2}$	$\frac{m+3n-1}{2}$

Table 5

This completes the proof. □

Example 3.11 A pair mean cordial labeling of $S_9 \cup L_{12}$ is shown in Figure 5.

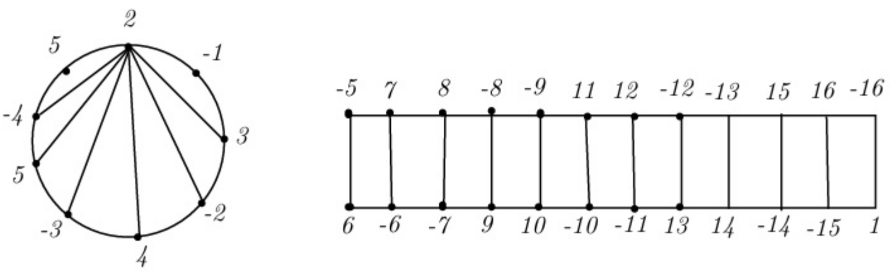

Figure 5

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On Generalized Integral Type $\alpha - \tilde{\mathcal{F}}$ Contraction Mappings in Partial Metric Spaces

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Abstract: This study introduces generalized integral type $\alpha - \tilde{\mathcal{F}}$-contraction mappings in partial metric spaces that combine $\tilde{\mathcal{F}}$-contraction, integral transformations, and α-admissible mappings. It also investigates the existence and uniqueness of fixed points in the context of partial metric spaces. We provide some examples to corroborate our findings.

Key Words: Generalized integral type $\alpha - \tilde{\mathcal{F}}$ contraction mapping, $\tilde{\mathcal{F}}$-contraction mapping, α-admissible mapping, partial metric spaces.

AMS(2010): 47H10, 54H25.

§1. Introduction

In 2002, Branciari [2] introduced the integral contraction as follows.

Theorem 1.1 *Let (Ω_s, d) be a complete metric space, $k \in (0,1)$ and let $\Upsilon : \Omega_s \to \Omega_s$ be a mapping such that for each $\gamma_s, \zeta_s \in \Omega_s$*

$$\int_0^{d(\Upsilon\gamma_s, \Upsilon\zeta_s)} \xi(t)\, dt \leq k \int_0^{d(\gamma_s, \zeta_s)} \xi(t)\, dt$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable map which is summable, (i.e., with finite integral) on each compact subset of $[0,\infty)$, nonnegative, and such that for each $\epsilon > 0$, $\int_0^{\epsilon} \xi(t)\, dt > 0$, then Υ has a unique fixed point.

For some motivated results on integral type contractions, see [10, 12, 5].

In 2012, Samet et al. [13] introduced $\alpha - \psi$ contractive type mappings and shown some fixed point results for them. Wardowski [15, 16, 17] identified a new sort of contraction mapping called $\tilde{\mathcal{F}}$-contraction and shown that this mapping is a Banach contraction. Wardowski's result has been generalized by many authors (see [9, 1, 4, 14, 6]).

We begin by recalling a few definitions and lemmas. In 1992, Matthews [7] presented the concept of partial metric space (PMS) as follows:

Definition 1.1 *Let Ω_s be a non-empty set. A function $\varrho_{pm} : \Omega_s \times \Omega_s \to [0,\infty)$ is said to be a*

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partial metric on Ω_s *if the following conditions hold:*

$(PMS1)$ $\gamma_s = \zeta_s \Leftrightarrow \varrho_{pm}(\gamma_s, \gamma_s) = \varrho_{pm}(\zeta_s, \zeta_s) = \varrho_{pm}(\gamma_s, \zeta_s)$;
$(PMS2)$ $\varrho_{pm}(\gamma_s, \gamma_s) \leq \varrho_{pm}(\gamma_s, \zeta_s)$;
$(PMS3)$ $\varrho_{pm}(\gamma_s, \zeta_s) = \varrho_{pm}(\zeta_s, \gamma_s)$;
$(PMS4)$ $\varrho_{pm}(\gamma_s, \zeta_s) \leq \varrho_{pm}(\gamma_s, \iota_s) + \varrho_{pm}(\iota_s, \zeta_s) - \varrho_{pm}(\iota_s, \iota_s)$. *for all* $\gamma_s, \zeta_s, \iota_s \in \Omega_s$.

Lemma 1.2([7]) *Let* (Ω_s, ϱ_{pm}) *be a partial metric space.*

(a) *A sequence* $\{\gamma_{s_n}\}$ *in* (Ω_s, ϱ_{pm}) *converges to a point* $\gamma_s \in \Omega_s \Leftrightarrow$

$$\varrho_{pm}(\gamma_s, \gamma_s) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_s).$$

(b) *A sequence* $\{\gamma_{s_n}\}$ *in* (Ω_s, ϱ_{pm}) *is a Cauchy sequence if* $\lim_{m,n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m})$ *exists and finite.*

(c) (Ω_s, ϱ_{pm}) *is complete if every Cauchy* $\{\gamma_{s_n}\}$ *in* Ω_s *converges to a point* $\gamma_s \in \Omega_s$*, such that*

$$\varrho_{pm}(\gamma_s, \gamma_s) = \lim_{m,n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_s) = \varrho_{pm}(\gamma_s, \gamma_s).$$

Lemma 1.3([7],[8]) *Let* ϱ_{pm} *be a partial metric on* Ω_s*, then the function* $d^{\varrho}_{pm} : \Omega_s \times \Omega_s \to \mathbb{R}^+$ *such that*

$$d^{\varrho}_{pm}(\gamma_s, \zeta_s) = 2\varrho_{pm}(\gamma_s, \zeta_s) - \varrho_{pm}(\gamma_s, \gamma_s) - \varrho_{pm}(\zeta_s, \zeta_s)$$

is metric on Ω_s*. Let* (Ω_s, ϱ_{pm}) *be a partial metric space. Then,*

(1) *A sequence* $\{\gamma_{s_n}\}$ *in* (Ω_s, ϱ_{pm}) *is a Cauchy sequence* $\Leftrightarrow$ $\{\gamma_{s_n}\}$ *is a Cauchy sequence in the metric space* $(\Omega_s, d^{\varrho}_{pm})$.

(2) (Ω_s, ϱ_{pm}) *is complete* $\Leftrightarrow$ $(\Omega_s, d^{\varrho}_{pm})$ *is complete. Moreover,*

$$\lim_{n\to\infty} d^{\varrho}_{pm}(\gamma_{s_n}, \gamma_s) = 0 \Leftrightarrow \varrho_{pm}(\gamma_s, \gamma_s) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_s) = \lim_{n,m\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}).$$

Lemma 1.4([11]) *Assume that* $\gamma_{s_n} \to \iota_s$ *as* $n \to \infty$ *in a partial metric space* (Ω_s, ϱ_{pm}) *such that* $\varrho_{pm}(\iota_s, \iota_s) = 0$ *Then* $\lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \zeta_s) = \varrho_{pm}(\iota_s, \zeta_s)$ *for every* $\zeta_s \in \Omega_s$.

Lemma 1.5([3]) *Let* (Ω_s, ϱ_{pm}) *be a partial metric space.*

(1) *if* $\varrho_{pm}(\gamma_s, \zeta_s) = 0$ *then* $\gamma_s = \zeta_s$.
(2) *If* $\gamma_s \neq \zeta_s$ *then* $\varrho_{pm}(\gamma_s, \zeta_s) > 0$.

Samet et al. [13] introduced α-admissible mapping as follows:

Definition 1.6 *Let* $\Upsilon : \Omega_s \to \Omega_s$ *and* $\alpha : \Omega_s \times \Omega_s \to [0, \infty)$. Υ *is said to* α*-admissible if*

$$\alpha(\gamma_s, \zeta_s) \ \geq \ 1 \ \Rightarrow \ \alpha(\Upsilon\gamma_s, \Upsilon\zeta_s) \geq 1$$

for all $\gamma_s, \zeta_s \in \Omega_s$.

Wardowski [15] presented a new class of contraction mappings as follows:

Definition 1.7 *Let $\Delta_{\mathfrak{F}}$ be family of all functions $\tilde{\mathcal{F}} : \mathbb{R}^+ \to \mathbb{R}$ satisfying*

$(F1)$ *$\tilde{\mathcal{F}}$ is strictly increasing, i.e. for all $\omega, \upsilon \in \mathbb{R}^+$ if $\omega < \upsilon$ then $\tilde{\mathcal{F}}(\omega) < \tilde{\mathcal{F}}(\upsilon)$;*
$(F2)$ *for each sequence $\{\omega_n\}$ of positive numbers,*

$$\lim_{n\to\infty} \omega_n = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \tilde{\mathcal{F}}(\omega_n) = -\infty;$$

$(F3)$ *there exists $\lambda \in (0, 1)$ such that*

$$\lim_{\omega\to 0^+} \omega^{\lambda} \tilde{\mathcal{F}}(\omega) = 0.$$

Wardowski [15] defined $\tilde{\mathcal{F}}$-contraction as follows:

Definition 1.8 *Let (Ω_s, d) be a metric space, then the mapping $\Upsilon : \Omega_s \to \Omega_s$ is said to be an $\tilde{\mathcal{F}}$-contraction, if there exist $F \in \Delta_{\mathfrak{F}}$ and $\tau > 0$ such that for all $\gamma_s, \zeta_s \in \Omega_s$ with $d(\Upsilon\gamma_s.\Upsilon\zeta_s) > 0$ we have*

$$\tau + \tilde{\mathcal{F}}(d(\Upsilon\gamma_s, \Upsilon\zeta_s)) \le \tilde{\mathcal{F}}(d(\gamma_s, \zeta_s)).$$

§2. Main Results

Let Φ be family of all functions $\varphi : [0, \infty) \to [0, \infty)$ such that φ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0, \infty)$, nonnegative and for each $\epsilon > 0$

$$\int_0^{\epsilon} \xi(t)\, dt > 0$$

Definition 2.1 *Let (Ω_s, ϱ_{pm}) be partial metric space and let $\Upsilon : \Omega_s \to \Omega_s$ be a self map.Then Υ is said to be generalized integral type $\alpha - \tilde{\mathcal{F}}$-contractive mapping if there exists two functions $\alpha : \Omega_s \times \Omega_s \to [0, \infty)$ and $\tilde{\mathcal{F}} \in \Delta_{\mathfrak{F}}$ such that for $\tau > 0$ with $\varrho_{pm}(\Upsilon\gamma_s.\Upsilon\zeta_s) > 0$*

$$\tau + \tilde{\mathcal{F}}\bigg(\alpha(\gamma_s, \zeta_s) \int_0^{\varrho_{pm}(\Upsilon\gamma_s, \Upsilon\zeta_s)} \xi(t)\, dt\bigg) \le \tilde{\mathcal{F}}\bigg(\int_0^{\Lambda(\gamma_s, \zeta_s)} \xi(t)\, dt\bigg), \tag{2.1}$$

where $\varphi \in \Phi$ and

$$\Lambda(\gamma_s, \zeta_s) = \max\{\varrho_{pm}(\gamma_s, \zeta_s), \varrho_{pm}(\gamma_s, \Upsilon\gamma_s), \varrho_{pm}(\zeta_s, \Upsilon\zeta_s)\}$$

Theorem 2.1 *Let (Ω_s, ϱ_{pm}) be a complete partial metric space and $\Upsilon : \Omega_s \to \Omega_s$ be self mapping. Suppose $\alpha : \Omega_s \times \Omega_s \to [0, \infty)$ be the mapping satisfying the conditions:*

(i) *Υ is α-admissible mapping;*
(ii) *Υ is generalized integral type $\alpha - \tilde{\mathcal{F}}$-contractive mapping;*
(iii) *There exists $\gamma_{s_0} \in \Omega_s$ such that $\alpha(\gamma_{s_0}, \Upsilon\gamma_{s_0}) \ge 1$;*
(iv) *Υ is continuous,*

then Υ has a fixed point in Ω_s.

Proof Let γ_{s_0} be an arbitrary point such that $\alpha(\gamma_{s_0}, \Upsilon\gamma_{s_0}) \geq 1$. Consider a sequence $\{\gamma_{s_n}\}$ in Ω_s such that $\gamma_{s_{n+1}} = \Upsilon\gamma_{s_n}$ for all $n \in \mathbb{N}$.

If $\gamma_{s_n} = \gamma_{s_{n+1}}$ for some $n \in \mathbb{N}$, γ_{s_n} is a fixed point of Υ, completing the existence proof. Assume $\gamma_{s_n} \neq \gamma_{s_{n+1}}$ for every $n \in \mathbb{N}$, Lemma 1.5 states that

$$\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) = \varrho_{pm}(\Upsilon\gamma_{s_{n-1}}, \Upsilon\gamma_{s_n}) > 0.$$

Now, since Υ is α-admissible, so

$$\alpha(\Upsilon\gamma_{s_0}, \Upsilon\gamma_{s_1}) = \alpha(\gamma_{s_1}, \gamma_{s_2}) \geq 1$$

$$\alpha(\Upsilon\gamma_{s_1}, \Upsilon\gamma_{s_2}) = \alpha(\gamma_{s_2}, \gamma_{s_3}) \geq 1$$

and using induction we have $\alpha(\gamma_{s_n}, \gamma_{s_{n+1}}) \geq 1$ for all $n \in \mathbb{N}$.

Now, Using the property $(F1)$ we get

$$\begin{aligned}\tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})} \xi(t)\, dt\right) &\leq \tau + \tilde{\mathcal{F}}\left(\alpha(\gamma_{s_n}, \gamma_{s_{n+1}}) \int_0^{\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})} \xi(t)\, dt\right)\\ &= \tau + \tilde{\mathcal{F}}\left(\alpha(\gamma_{s_n}, \gamma_{s_{n+1}}) \int_0^{\varrho_{pm}(\Upsilon\gamma_{s_{n-1}}, \Upsilon\gamma_{s_n})} \xi(t)\, dt\right)\\ &\leq \tilde{\mathcal{F}}\left(\int_0^{\Lambda(\gamma_{s_{n-1}}, \gamma_{s_n})} \xi(t)\, dt\right) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\Lambda(\gamma_{s_{n-1}}, \gamma_{s_n}) &= \max\{\varrho_{pm}(\gamma_{s_{n-1}},, \gamma_{s_n}), \varrho_{pm}(\gamma_{s_{n-1}}, \Upsilon\gamma_{s_{n-1}}), \varrho_{pm}(\gamma_{s_n}, \Upsilon\gamma_{s_n})\}\\ &= \max\{\varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n}), \varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n}), \varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})\}\\ &= \max\{\varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n}), \varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})\}. \end{aligned} \tag{2.3}$$

Now, using (2.3) in (2.2) we get that

$$\tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})} \xi(t)\, dt\right) \leq \tilde{\mathcal{F}}\left(\int_0^{\max\{\varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n}), \varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})\}} \xi(t)\, dt\right). \tag{2.4}$$

Now, if $\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) > \varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n})$, then we get

$$\tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})} \xi(t)\, dt\right) \leq \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})} \xi(t)\, dt\right),$$

which is a contradiction, Therefore

$$\Lambda(\gamma_{s_{n-1}}, \gamma_{s_n}) = \varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n}). \tag{2.5}$$

Again, Using (2.5) in (2.4) we get

$$\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg) \le \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_{n-1}},\gamma_{s_n})} \xi(t)\,dt\bigg) - \tau. \tag{2.6}$$

Continuing in the same way, we obtain

$$\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n-1}})} \xi(t)\,dt\bigg) \le \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_{n-1}},\xi_{p_{n-2}})} \xi(t)\,dt\bigg) - \tau. \tag{2.7}$$

Using (2.7) in (2.6) we get that

$$\begin{aligned}\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg) &\le \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_{n-1}},\gamma_{s_n})} \xi(t)\,dt\bigg) - \tau \\ &\le \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_{n-1}},\xi_{p_{n-2}})} \xi(t)\,dt\bigg) - 2\tau\end{aligned}$$

On generalizing

$$\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg) < \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\xi_{p_0},\xi_{p_1})} \xi(t)\,dt\bigg) - n\tau. \tag{2.8}$$

Letting the limit $n \to \infty$ in (**??**) and using the definition of $\tilde{\mathcal{F}}$ we get

$$\lim_{n\to\infty} \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg) = -\infty \Leftrightarrow \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}}) = 0. \tag{2.9}$$

Consequently, we get

$$\lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}}) = 0. \tag{2.10}$$

Now, we show that $\{\gamma_{s_n}\}$ is a Cauchy sequence in Ω_s, i.e., we prove that

$$\lim_{n,m\to\infty} \varrho_{pm}(\gamma_{s_n},\gamma_{s_m}) = 0.$$

Put $e_n = \varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})$ for $n \in \mathbb{N}$. Then, from the property (F_3) of $\tilde{\mathcal{F}}$ contraction there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} e_n^k \tilde{\mathcal{F}}(e_n) = 0. \tag{2.11}$$

Following (2.8) for all $n \in \mathbb{N}$ we obtain

$$e_n^k\bigg(\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg) - \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\xi_{p_0},\xi_{p_1})} \xi(t)\,dt\bigg)\bigg) \le -e_n^k n\tau \le 0. \tag{2.12}$$

Considering $(2.10), (2.11)$ and letting $n \to \infty$ in (2.12) we get

$$\lim_{n\to\infty} (n(\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})^k) = 0. \tag{2.13}$$

Since (2.13) holds, there exists $n_p \in \mathbb{N}$ such that $n(\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}))^k \leq 1$ for all $n \geq n_p$ or

$$\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) \leq \frac{1}{n^{\frac{1}{k}}} \tag{2.14}$$

for all $n \geq n_p$.

Using *PMS4* (triangular inequality) and (2.14) we obtain that for $m > n \geq n_p$,

$$\begin{aligned} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) &\leq \varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) + \varrho_{pm}(\gamma_{s_{n+1}}, \gamma_{s_{n+2}}) + \cdots + \varrho_{pm}(\gamma_{s_{m-1}}, \gamma_{s_m}) \\ &\quad - [\varrho_{pm}(\gamma_{s_{n+1}}, \gamma_{s_{n+1}}) + \varrho_{pm}(\gamma_{s_{n+2}}, \gamma_{s_{n+2}}) + \cdots + \varrho_{pm}(\gamma_{s_{m-1}}, \gamma_{s_{m-1}})] \\ &\leq \varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) + \varrho_{pm}(\gamma_{s_{n+1}}, \gamma_{s_{n+2}}) + \cdots + \varrho_{pm}(\gamma_{s_{m-1}}, \gamma_{s_m}) \\ &= \sum_{i=n}^{m-1} \varrho_{pm}(\gamma_{s_i}, \gamma_{s_{i+1}}) \leq \sum_{i=n}^{\infty} \varrho_{pm}(\gamma_{s_i}, \gamma_{s_{i+1}}) \leq \sum_{i=n}^{\infty} \frac{1}{n^{\frac{1}{k}}} \end{aligned}$$

Since $k \in (0, 1)$, the series $\sum_{i=n}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ is convergent, so

$$\lim_{n,m\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) = 0.$$

This implies that $\{\gamma_{s_n}\}$ is a Cauchy sequence in (Ω_s, ϱ_{pm}). Due to Lemma 1.3, $\{\gamma_{s_n}\}$ is a Cauchy sequence in $(\Omega_s, d^{\varrho}_{pm})$ which is complete. Therefore the sequence $\{\gamma_{s_n}\}$ is convergent in the space $(\Omega_s, d^{\varrho}_{pm})$ as a result there exist $\iota_s \in \Omega_s$ such that $\lim_{n\to\infty} d^{\varrho}_{pm}(\gamma_{s_n}, \iota_s) = 0$. Again from Lemma 1.2, we get

$$\varrho_{pm}(\iota_s, \iota_s) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \iota_s) = \lim_{m,n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) = 0. \tag{2.15}$$

Moreover, As Υ is continuous, we have

$$\iota_s = \lim_{n\to\infty} \gamma_{s_{n+1}} = \lim_{n\to\infty} \Upsilon\gamma_{s_n} = \Upsilon\iota_s$$

This completes the proof. □

Theorem 2.2 *Let (Ω_s, ϱ_{pm}) be a complete partial metric space and $\Upsilon : \Omega_s \to \Omega_s$ be self mapping. Suppose $\alpha : \Omega_s \times \Omega_s \to [0, \infty)$ be the mapping satisfying the conditions:*

(*i*) *Υ is α-admissible mapping;*

(*ii*) *Υ is integral type generalized $\alpha - \tilde{\mathcal{F}}$-contractive mapping;*

(*iii*) *There exists $\gamma_{s_0} \in \Omega_s$ such that $\alpha(\gamma_{s_0}, \Upsilon\gamma_{s_0}) \geq 1$;*

(*iv*) *If $\{\gamma_{s_n}\}$ s a sequence in Ω_s such that $\alpha(\gamma_{s_n}, \gamma_{s_{n+1}}) \geq 1$ for all n and $\gamma_{s_n} \to \iota_s \in \Omega_s$ as $n \to \infty$, then there exists a subsequence $\gamma_{s_{n(i)}}$ of $\{\gamma_{s_n}\}$ such that $\alpha(\gamma_{s_{n(i)}}, \iota_s) \geq 1$ for all i;*

(*v*) *$\tilde{\mathcal{F}}$ is continuous,*

then Υ has a fixed point in Ω_s.Further if ι_s, ι_t are fixed points of Υ with $\alpha(\iota_s, \iota_t) \geq 1$, then Υ has a unique fixed point in Ω_s.

Proof From the proof of the Theorem 2.1, the sequence $\{\gamma_{s_n}\}$ defined by $\gamma_{s_{n+1}} = \Upsilon\gamma_{s_n}$ is a Cauchy sequence in (Ω_s, ϱ_{pm}). Due to Lemma 1.3, $\{\gamma_{s_n}\}$ is a Cauchy sequence in $(\Omega_s, d^{\varrho}_{pm})$ which is complete. Therefore the sequence $\{\gamma_{s_n}\}$ is convergent in the space $(\Omega_s, d^{\varrho}_{pm})$ as a result there exist $\iota_s \in \Omega_s$ such that $\lim_{n\to\infty} d^{\varrho}_{pm}(\gamma_{s_n}, \iota_s) = 0$. Again from Lemma 1.2, we get

$$\varrho_{pm}(\iota_s, \iota_s) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \iota_s) = \lim_{m,n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) = 0. \tag{2.16}$$

We now prove that Υ has a fixed point.

On contrary we suppose that $(\Upsilon\iota_s, \iota_s) > 0$. Then from condition (iii) there exists a subsequence $\gamma_{s_{n(i)}}$ of $\{\gamma_{s_n}\}$ such that $\alpha(\gamma_{s_{n(i)}}, \iota_s) \geq 1$ for all i. By Using given contractive condition (2.1) for $\gamma_s = \gamma_{s_{n(i)}}$ and $\zeta_s = \iota_s$ and property of $\tilde{\mathcal{F}}$ we have

$$\begin{aligned}\tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\xi_{p_{n(i)+1}}, \Upsilon\iota_s)} \xi(t)\,dt\bigg) &= \tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\Upsilon\gamma_{s_{n(i)}}, \Upsilon\iota_s)} \xi(t)\,dt\bigg)\\ &\leq \tau + \tilde{\mathcal{F}}\bigg(\alpha(\gamma_{s_{n(i)}}, \iota_s)\int_0^{\varrho_{pm}(\Upsilon\gamma_{s_{n(i)}}, \Upsilon\iota_s)} \xi(t)\,dt\bigg)\\ &\leq \tilde{\mathcal{F}}\bigg(\int_0^{\Lambda(\gamma_{s_{n(i)}}, \iota_s)} \xi(t)\,dt\bigg) \end{aligned}\tag{2.17}$$

where

$$\begin{aligned}\Lambda(\gamma_{s_{n(i)}}, \iota_s) &= \max\{\varrho_{pm}(\gamma_{s_{n(i)}}, \iota_s), \varrho_{pm}(\gamma_{s_{n(i)}}, \Upsilon\gamma_{s_{n(i)}}), \varrho_{pm}(\iota_s, \Upsilon\iota_s)\}\\ &= \max\{\varrho_{pm}(\gamma_{s_{n(i)}}, \iota_s), \varrho_{pm}(\gamma_{s_{n(i)}}, \xi_{p_{n(i)+1}}), \varrho_{pm}(\iota_s, \Upsilon\iota_s)\}.\end{aligned}\tag{2.18}$$

Taking $n \to \infty$ in (2.18) and using (2.16) we get that

$$\lim_{n\to\infty} \Lambda(\gamma_{s_{n(i)}}, \iota_s) = \varrho_{pm}(\iota_s, \Upsilon\iota_s). \tag{2.19}$$

Now, Letting $n \to \infty$ in (2.17) and using (2.19) and the continuity of $\tilde{\mathcal{F}}$ we get that

$$\tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\iota_s, \Upsilon\iota_s)} \xi(t)\,dt\bigg) \leq \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\iota_s, \Upsilon\iota_s)} \xi(t)\,dt\bigg)$$

which is a contradiction since $\tau > 0$, Thus we have $\Upsilon\iota_s = \iota_s$. This shows that ι_s is a fixed point of Υ. Further, suppose ι_s and ι_t be two fixed point of Υ such that $\varrho_{pm}(\iota_s, \iota_t) > 0$. From (2.1) we have

$$\begin{aligned}\tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\iota_s, \iota_t)} \xi(t)\,dt\bigg) &= \tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\Upsilon\iota_s, \Upsilon\iota_t)} \xi(t)\,dt\bigg)\\ &\leq \tau + \tilde{\mathcal{F}}\bigg(\alpha(\iota_s, \iota_t)\int_0^{\varrho_{pm}(\Upsilon\iota_s, \Upsilon\iota_t)} \xi(t)\,dt\bigg)\\ &\leq \tilde{\mathcal{F}}\bigg(\int_0^{\Lambda(\iota_s, \iota_t)} \xi(t)\,dt\bigg),\end{aligned}\tag{2.20}$$

where

$$\Lambda(\iota_s,\iota_t) = \max\{\varrho_{pm}(\iota_s,\iota_t), \varrho_{pm}(\iota_s,\Upsilon\iota_s), \varrho_{pm}(\iota_t,\Upsilon\iota_t)\}$$
$$= \max\{\varrho_{pm}(\iota_s,\iota_t), \varrho_{pm}(\iota_s,\iota_s), \varrho_{pm}(\iota_t,\iota_t)\} = \varrho_{pm}(\iota_s,\iota_t). \tag{2.21}$$

Putting (2.21) in (2.20) we get

$$\tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\iota_s,\iota_t)} \xi(t)\,dt\bigg) \le \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\iota_s,\iota_t)} \xi(t)\,dt\bigg), \tag{2.22}$$

which is a contradiction. Hence Υ has a unique fixed point. □

Following are consequences of the theorems.

Corollary 2.3 *Let (Ω_s,ϱ_{pm}) be a complete partial metric space and let $\Upsilon:\Omega_s\to\Omega_s$ be a self map. Suppose that there exist $\tilde{\mathcal{F}}\in\Delta_{\mathfrak{F}}$ and $\tau>0$ with $\varrho_{pm}(\Upsilon\gamma_s.\Upsilon\zeta_s)>0$ be such that*

$$\tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)} \xi(t)\,dt\bigg) \le \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_s,\zeta_s)} \xi(t)\,dt\bigg) \tag{2.23}$$

for all $\gamma_s,\zeta_s\in\Omega_s$ and $\varphi:[0,\infty)\to[0,\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,\infty)$, nonnegative and for each $\epsilon>0$

$$\int_0^{\epsilon} \xi(t)\,dt > 0$$

and $\tilde{\mathcal{F}}$ or Υ is continuous. Then Υ has a unique fixed point in Ω_s.

Corollary 2.4 *Let (Ω_s,ϱ_{pm}) be a complete partial metric space and let $\Upsilon:\Omega_s\to\Omega_s$ be a continuous self map. Suppose that there exist $k\in(0,1)$ with $\varrho_{pm}(\Upsilon\gamma_s.\Upsilon\zeta_s)>0$ such that*

$$\int_0^{\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)} \xi(t)\,dt \le k\int_0^{\varrho_{pm}(\gamma_s,\zeta_s)} \xi(t)\,dt \tag{2.24}$$

and $\varphi:[0,\infty)\to[0,\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,\infty)$, nonnegative and for each $\epsilon>0$

$$\int_0^{\epsilon} \xi(t)\,dt > 0,$$

then Υ has a unique fixed point in Ω_s.

Example 2.5 Let $\Omega_s=[0,1]$ and define $\varrho_{pm}:\Omega_s\times\Omega_s\to\mathbb{R}^+$ by $\varrho_{pm}(\gamma_s,\zeta_s)=\max\{\gamma_s,\zeta_s\}$. Then (Ω_s,ϱ_{pm}) is a complete partial metric space. Consider the mapping $\Upsilon:\Omega_s\to\Omega_s$ defined by $\Upsilon(\iota_s)=\frac{\iota_s}{4}$. Suppose that $\xi(t)=2t$. Define the function $\tilde{\mathcal{F}}:\mathbb{R}^+\to\mathbb{R}$ by $\tilde{\mathcal{F}}(a)=\ln a$ for all $a\in\mathbb{R}^+>0$ and $\alpha:\Omega_s\times\Omega_s\to[0,\infty)$ by $\alpha(\gamma_s,\zeta_s)=4$ for all $\gamma_s,\zeta_s\in\Omega_s$.

We show that contractive condition of Theorem 2.1 is satisfied. Let $\gamma_s,\zeta_s\in\Omega_s$, without loss of generality we assume that $\gamma_s\ge\zeta_s$. Suppose that $\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)>0$ and let $\tau=\ln(2)$, then

$$\begin{aligned}\tau + \tilde{\mathcal{F}}\bigg(\alpha(\gamma_s,\zeta_s)\int_0^{\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)}\xi(t)\,dt\bigg) &= \tau + \tilde{\mathcal{F}}\bigg(4\int_0^{\varrho_{pm}(\frac{\gamma_s}{4},\frac{\zeta_s}{4})}2t\,dt\bigg) = \tau + \tilde{\mathcal{F}}(\frac{\gamma_s^2}{4})\\ &= \ln(2) + \ln(\frac{\gamma_s^2}{4}) = \ln(\frac{\gamma_s^2}{2})\\ &\le \ln(\gamma_s^2) = \tilde{\mathcal{F}}(\gamma_s^2) = \tilde{\mathcal{F}}\bigg(\int_0^{\Lambda(\gamma_s,\zeta_s)}\xi(t)\,dt\bigg). \qquad (2.25)\end{aligned}$$

Hence, Υ has a fixed point, which in this case is 0.

Example 2.6 Let $\Omega_s = [0,1]$ and define $\varrho_{pm} : \Omega_s \times \Omega_s \to \mathbb{R}^+$ by $\varrho_{pm}(\gamma_s,\zeta_s) = \max\{\gamma_s,\zeta_s\}$. Then (Ω_s,ϱ_{pm}) is a complete partial metric space. Consider the mapping $\Upsilon : \Omega_s \to \Omega_s$ defined by $\Upsilon(\iota_s) = \frac{\iota_s^2+0.045}{12}$. Suppose that $\tau = \ln(1.5)$ and $\xi(t) = 1$ for $t > 0$. Define the function $\tilde{\mathcal{F}} : \mathbb{R}^+ \to \mathbb{R}$ by $\tilde{\mathcal{F}}(a) = \ln(a)$ for all $a \in \mathbb{R}^+ > 0$.

We show that contractive condition of Corollary 2.3 is satisfied. Let $\gamma_s,\zeta_s \in \Omega_s$, without loss of generality we assume that $\gamma_s \ge \zeta_s$. Suppose that $\Upsilon\gamma_s \ne \Upsilon\zeta_s$, then

$$\begin{aligned}\tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)}\xi(t)\,dt\bigg) &= \tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}\left(\frac{\gamma_s^2+0.045}{12},\frac{\zeta_s^2+0.045}{12}\right)} dt\bigg)\\ &= \tau + \tilde{\mathcal{F}}\bigg(\frac{\gamma_s^2+0.045}{12}\bigg)\\ &= \ln(1.5) + \ln\left(\frac{\gamma_s^2+0.045}{12}\right)\\ &\le \ln(\gamma_s) = \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_s,\zeta_s)}\xi(t)\,dt\bigg). \qquad (2.26)\end{aligned}$$

Therefore, it satisfies the condition of Corollary 2.3. Hence Υ has a fixed point, which in this case is 0.003751.

§3. Conclusion

In this article, we prove fixed point theorems for generalized integral type $\alpha - \tilde{\mathcal{F}}$ contraction in complete partial metric spaces and provide corollaries of the results. We also provided some examples to validate the results. This article extends and generalises previous research findings.

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On Modified Maximum Degree Energy of Graph and HDR Energy of Graph

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Abstract: The spectral graph theory explores connections between combinatorial features of graphs and algebraic properties of associated matrices. In this paper, we introduce modified maximum degree matrix $M_M(\zeta)$ of a simple graph ζ and obtain a bound for eigenvalues of $M_M(\zeta)$. We also introduce modified maximum degree energy $E_{M_M}(\zeta)$ of a graph ζ and obtain bounds for $E_{M_M}(\zeta)$.

Key Words: Modified maximum degree matrix, modified maximum degree energy, eigenvalues, energy of a graph.

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§1. Introduction

The spectral graph theory plays an important role in analyzing the matrices of graphs with the help of matrix theory and linear algebra. Now, spectral graph theory has attracted the attention of both pure and applied mathematicians whose benefit lies far from the spectral graph theory, which may be surprised because graph energy is a special kind of matrix norm. They will then recognize that the concept of graph energy (under different names) is encountered in several seemingly unrelated areas of their own expertise.

The eigenvalues are closely related to almost all major invariants of a graph, linking one extremal property to another, they play a central role in the fundamental understanding of graph. In 1978, I. Gutman related the Graph energy and total π-electron energy in a molecular graph; it was defined as, the sum of absolute values of the eigenvalues of the associated adjacency matrix of a graph ζ. Later, many matrices were defined based on distance and adjacency among the vertices, degree of the vertices involved in forming the graph structure like: Zagreb matrix [5], Randic matrix [10], distance matrix [1], Seidel matrix [2], Laplacian matrix [6], Seidel Laplacian matrix [9], signless Laplacian matrix [3], Seidel signless Laplacian matrix [8], degree sum matrix [7], etc.

In the study of spectral graph theory, we use the spectra of certain matrix associated with the graph, such as the adjacency matrix, the Laplacian matrix and other related matrices. Some useful information about the graph can be obtained from the spectra of these various matrices.

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Throughout the paper, we consider a simple graph ζ, that is nonempty, finite, having no loops, no multiple and directed edges. Let $V(\zeta) = \{\delta_1, \delta_2, \cdots, \delta_{|V(\zeta)|}\}$. The adjacency matrix $A(\zeta)$ of the graph ζ is a square matrix of order $|V(\zeta)|$ whose (i, j)- entry is equal to unity if the vertices δ_i and δ_j are adjacent and is equal to zero otherwise. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{|V(\zeta)|}$ of $A(\zeta)$, assumed in non-increasing order, are the eigenvalues of the graph ζ. As we defined before the energy of ζ is

$$E(\zeta) = \sum_{i=1}^{n} |\lambda_i|.$$

The concept of graph energy arose in chemistry. An interesting quantity in Huckel theory is the sum of the energies of all the electrons in a molecule, the so-called total $\pi-$electron energy.

In this article, we introduce modified maximum degree Matrix $M_M(\zeta)$ of a simple graph ζ and obtain a bound for eigenvalues of $M_M(\zeta)$. We also introduce modified maximum degree energy $E_{M_M}(\zeta)$ of a graph ζ and obtain bounds for $E_{M_M}(\zeta)$. Also we define the concept of HDR energy of graph with some interesting results.

§2. Modified Maximum Degree Matrix of a Graph

Definition 2.1 *Let ζ be a simple graph with vertices $\delta_1, \delta_2, \cdots, \delta_{|V(\zeta)|}$ and let d_i be the degree of δ_i, $i = 1, 2, \cdots, |V(\zeta)|$. Define,*

$$b_{ij} = \begin{cases} max\{d_{\delta_i}, d_{\delta_j}\} + 1, & \text{if } \delta_i \text{ and } \delta_j \text{ are adjacent,;} \\ 0, & \text{otherwise.} \end{cases}$$

Then, the $|V(\zeta)| \times |V(\zeta)|$ matrix $M_M(\zeta) = [b_{ij}]$ is called the modified maximum degree matrix of graph ζ.

The characteristic polynomial of modified maximum degree matrix $M_M(\zeta)$ is defined by

$$\begin{aligned}\psi(\zeta; \lambda) &= Det(\lambda I - M_M(\zeta)) \\ &= \lambda^{|V(\zeta)|} + a_1\ \lambda^{|V(\zeta)|-1} + a_2\ \lambda^{|V(\zeta)|-2} + ... + a_{|V(\zeta)|},\end{aligned}$$

where I is the unit matrix of order $|V(\zeta)|$. The roots $\lambda_1, \lambda_2, \cdots, \lambda_{|V(\zeta)|}$ assumed in non-increasing order are the modified maximum degree eigenvalues of ζ. The modified maximum degree energy of a graph ζ is defined as

$$E_{M_M}(\zeta) = \sum_{i=1}^{|V(\zeta)|} |\lambda_i|.$$

Since $M_M(\zeta)$ is a real symmetric matrix with zero trace, these modified maximum degree eigenvalues are real numbers with sum equal to zero. Thus $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{|V(\zeta)|}$ and

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{|V(\zeta)|} = 0.$$

Example 2.2 The modified maximum degree matrix of the graph ζ_1 in Figure 1 is

$$M_M(\zeta_1) = \begin{bmatrix} 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \end{bmatrix}.$$

The characteristic polynomial of the modified maximum degree matrix $M_M(\zeta)$ is

$$\begin{aligned}
\psi(\zeta_1;\lambda) &= Det(\lambda I - M_M(\zeta_1)) \\
&= \begin{bmatrix} \lambda & -3 & 0 & -3 \\ -3 & \lambda & -3 & 0 \\ 0 & -3 & \lambda & -3 \\ -3 & 0 & -3 & \lambda \end{bmatrix} \\
&= \lambda \begin{bmatrix} \lambda & -3 & 0 \\ -3 & \lambda & -3 \\ 0 & -3 & \lambda \end{bmatrix} + 3 \begin{bmatrix} -3 & -3 & 0 \\ 0 & \lambda & -3 \\ -3 & -3 & \lambda \end{bmatrix} + 0 \begin{bmatrix} -3 & \lambda & 0 \\ 0 & -3 & -3 \\ -3 & 0 & \lambda \end{bmatrix} \\
&\quad + 3 \begin{bmatrix} -3 & \lambda & -3 \\ 0 & -3 & \lambda \\ -3 & 0 & -3 \end{bmatrix} \\
&= \lambda\Big(\lambda(\lambda^2-9) + 3(-3\lambda - 0) + 0\Big) \\
&\quad + 3\Big(-3(\lambda^2-9) + 3(0-9) + 0\Big) \\
&\quad + 3\Big(-3(9-0) - \lambda(0+3\lambda) - 3(0-9)\Big) \\
&= \lambda^4 - 36\lambda^2
\end{aligned}$$

and the modified maximum degree eigenvalues of ζ_1 are

$$\lambda_1 = 0,\ \ \lambda_2 = 0,\ \ \lambda_3 = 6,\ \ \lambda_4 = -6.$$

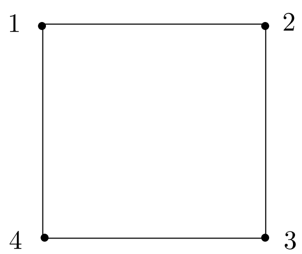

Figure 1. The graph ζ_1

Definition 2.3 *$d_{hr}(v) = |\{u, v \in V(\zeta) | d(u,v) = \lceil \frac{R}{2} \rceil\}|$ and $d(u,v)$ is the distance between the vertices u and v in $V(\zeta)$ and R is the radius of graph ζ.*

Definition 2.4 *Let ζ be a simple graph with n vertices $v_1, v_2, ..., v_n$ and let $d_{hr}i$ be the degree of v_i, $i = 1, 2, ..n$ defined*

$$d_{hr}ij = max\{d_{hr}i, d_{hr}j\}$$

if the $v_{hr}i, v_{hr}j$ adjacent and 0 otherwise. Then the $n \times n$ matrix $H(\zeta) = [d_{hr}ij]$ is called maximum HDR degree matrix of ζ. The characteristic polynomial of the maximum degree matrix $H(\zeta)$ is defined by

$$\begin{aligned} \alpha(\zeta; \epsilon) &= \det(\epsilon I - H(\zeta)) \\ &= \epsilon^n + a_1\epsilon^{n-1} + a_2\epsilon^{n-2} + \cdots + a_n, \end{aligned}$$

where I is the unit matrix of order n.

§3. Some Bounds of Modified Maximum Degree Energy

We now give the explicit expression for the coefficient a_i of $\lambda^{|V(\zeta)|-i}$ $(i = 1, 2, 3, \cdots, |V(\zeta)|)$ in the modified characteristic polynomial of the maximum degree matrix $M_M(\zeta)$. It is clear that $a_0 = 1$ and $a_1 = trace \ \ M_M(\zeta) = 0$. We have

$$a_2 = \sum_{1 \le L \le J \le |V(\zeta)|} \begin{vmatrix} 0 & \delta_{LJ} + 1 \\ \delta_{JL} + 1 & 0 \end{vmatrix}$$

and

$$\begin{vmatrix} 0 & d_{LJ} + 1 \\ d_{JL} + 1 & 0 \end{vmatrix} = \begin{cases} -(max\{d_J + 1, d_L + 1\})^2, & \text{if } \delta_J \text{ and } \delta_L \text{ are adjacent;} \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} a_2 &= -\sum_{k=1}^{|V(\zeta)|} (r_k + z_k)(d_k + 1)^2 \\ &= -\sum_{k=1}^{|V(\zeta)|} (r_k + z_k)(d(\delta_k) + 1)^2 \end{aligned}$$

where $r_k =$ the number of vertices in the neighborhood of δ_k whose degrees are less than $d(\delta_k)$ and $z_k =$ the number of vertices δ_j $j > k$ in the neighborhood of δ_k whose degrees are equal to

$d(\delta_k)$.

$$a_3 = (-1)^{-3} \sum_{1\le L\le J\le h\le |V(\zeta)|} \begin{vmatrix} d_{LL}+1 & d_{LJ}+1 & d_{Lh}+1 \\ d_{JL}+1 & d_{JJ}+1 & d_{Jh}+1 \\ d_{hL}+1 & d_{hJ}+1 & d_{hh}+1 \end{vmatrix}$$

$$= -2 \sum_{1\le L\le J\le h\le |V(\zeta)|} (d_{LJ}+1)(d_{Jh}+1)(d_{hL}+1)$$

$$= -2 \sum_{d(\delta_L)\le d(\delta_J)\le d(\delta_h)} (d(\delta_h)+1)^2(d(\delta_J)+1)$$

Example 3.1 For the graph ζ_1 in Figure 1, the coefficient a_2 of λ^2 in the characteristic polynomial of the modified maximum degree matrix $M_M(\zeta_1)$ is equal to

$$\begin{aligned} a_2 &= -\sum_{k=1}^{4}(r_k+z_k)(d(\delta_k)+1)^2 \\ &= -\Big((0+2)(2+1)^2+(0+1)(2+1)^2+(0+1)(2+1)^2+(0+0)(2+1)^2\Big) = -36. \end{aligned}$$

Theorem 3.2 *If $\lambda_1, \lambda_2, \cdots, \lambda_{|V(\zeta)|}$ are the modified maximum degree eigenvalues of a graph ζ, then*

$$\sum_{i=1}^{|V(\zeta)|} \lambda_i^2 = -2a_2.$$

Proof We have

$$\begin{aligned} \sum_{i=1}^{|V(\zeta)|} \lambda_i^2 = trace\ \ of\ \ M_M^2(\zeta) &= \sum_{i=1}^{|V(\zeta)|} \left(\sum_{j=1}^{|V(\zeta)|} d_{ij}d_{ji}\right) \\ &= 2\sum_{i=1}^{|V(\zeta)|}(r_i+z_i)(d(\delta_i)+1)^2 = -2a_2. \end{aligned}$$

This completes the proof. □

Theorem 3.3 *Let ζ be a graph. Then,*

$$\sqrt{2\sum_{i=1}^{|V(\zeta)|}(d(\delta_i)+1)^2 + |V(\zeta)|(|V(\zeta)|-1)\beta^{\frac{2}{|V(\zeta)|}}} \le E_{M_M}(\zeta) \le \sqrt{2|V(\zeta)|\sum_{i=1}^{|V(\zeta)|}(r_i+z_i)(d(\delta_i)+1)^2}.$$

Proof We have

$$E^2_{M_M}(\zeta) = \left(\sum_{i=1}^{|V(\zeta)|} |\lambda_i|\right)^2 = \sum_{i=1}^{|V(\zeta)|} |\lambda_i|^2 + \sum_{i\neq l} |\lambda_i||\lambda_l|$$
$$\geq 2\sum_{i=1}^{|V(\zeta)|} (r_i + z_i)(d(\delta_i)+1)^2 + |V(\zeta)|(|V(\zeta)-1|)\beta^{\frac{2}{|V(\zeta)|}},$$

where

$$\beta = \prod_{i=1}^{|V(\zeta)|} |\lambda_i|$$

and the last inequality is due to Theorem 3.2, the arithmetic mean, the geometric mean inequality. On employing Holder's inequality, we obtain

$$E_{M_M}(\zeta) = \sum_{i=1}^{|V(\zeta)|} |\lambda_i|$$
$$\leq \sqrt{\sum_{i=1}^{|V(\zeta)|} |\lambda_i|^2}.\ \sqrt{|V(\zeta)|}$$
$$= \sqrt{2|V(\zeta)| \sum_{i=1}^{|V(\zeta)|} (r_i + z_i)(d(\delta_i)+1)^2}$$

This completes the proof. □

Proposition 3.4 *Let ζ be a graph such that $\zeta \cong C_n, K_{n,m}, W_n, F_p$. Then, the maximum HDR degree energy of ζ is same as its maximum first degree energy.*

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On Grundy Coloring of Degree Splitting Graphs

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Abstract: A Grundy n-coloring of a graph G is a proper vertex coloring in which every vertex in $V(G)$ colored with C_n is adjacent with all C_{n-1} colors. The Grundy cloring or Grundy number $\Gamma(G)$ is the maximum number which can also be predicted by greedy coloring strategy by choosing some vertex order to obtain maximum colors. In this paper, we provide some exact values for Grundy coloring of degree splitting graph of wheel graph, helm graph, sunlet graph, crown graph and Friendship graph which are denoted by $[DS(W_n)]$, $[DS(H_n)]$, $[DS(S_n)]$, $[DS(H_{n,n})]$ and $\Gamma[DS(F_n)]$ respectively.

Key Words: Proper coloring, Grundy coloring, Smarandachely Grundy coloring, greedy algorithm, degree splitting graph.

AMS(2010): 05C15.

§1. Introduction

Throughout this, we consider only a simple, finite, undirected & connected graph. Graph coloring is the allocation of colors to the vertices of a graph G. A proper k-coloring is defined by the mapping $\sigma : V(G) \to C_s$ where $\sigma(f) \neq \sigma(g)$ for $\forall f \sim g$, $(f, g) \in V(G)$ [1, 10]. The Grundy k-coloring is a proper k-coloring in which $f \sim C_1$, $f \sim C_2$, $f \sim C_3$, $\cdots$, $f \sim C_{s-1}$ for $\forall$ $\sigma(f) = C_s$. This Grundy number $\Gamma(G)$ was initially studied by P.M.Grundy for directed version in 1939 but the undirected version was introduced by Christen and Selkow in 1979 [1, 2, 5]. This can also be predicted by using greedy algorithm which consider the vertices in some sequence and assign them its first available color. We know that, $\mu(G) \leq \chi(G) \leq \Gamma(G) \leq \triangle(G) + 1$ where $\mu(G)$ is the clique number [3].

§2. Preliminaries

A Grundy n-coloring of G is an n-coloring of G such that $\forall$ color C_t, every node colored with C_t is adjacent to at least one node colored with C_s, $\forall$ $C_s < C_t$ and the Grundy chromatic number $\Gamma(G)$ is the maximum number n such that G is Grundy n-coloring [3]. Generally, if $G \setminus H$ is Grundy n-colourable for a typical subgraph $H \prec G$ such as a path P_s, cycle C_s or $K_{1,s}$ for an integer $s \geq 1$, then G is said to be Smarandachely Grundy n-colourable on H. Clearly, such a

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Smarandachely Grundy n-colouring is nothing else but a Grundy n-colouring if $H = \emptyset$.

A graph with $V(G) = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of all vertices of same degree with at least two elements and $T = V(G) \backslash \{S_1 \cup S_2 \cup \cdots \cup S_t\}$. The degree splitting graph $DS(G)$, is obtained from G by adding vertices $w_1, w_2, \cdots, w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$ [8, 10].

For any integer $n \geq 4$, the wheel graph W_n is the n-vertex graph obtained by joining a vertex v_1 to each of the n-1 vertices $w_1, w_2, \cdots, w_{n-1}$ of the cycle graph C_{n-1} [11].

A helm graph H_n is a graph formed from a wheel W_n by attaching a pendant edge to each terminal vertex [7].

An n-sunlet graph on $2n$ vertices is obtained by attaching n-pendant edges to the cycle C_n and is denoted by S_n [11].

A crown graph(also known as a cocktail party graph) $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching [6] and the friendship graph F_n is the n-collection of cycle C_3 with a common vertex.

§3. Main Results

Here, we concentrate on exact values of Grundy Coloring of Degree Splitting graph of wheel graphs, helm graphs, sunlet graphs, crown graphs and friendship graphs which are symbolised by $[DS(W_n)]$, $[DS(H_n)]$, $[DS(S_n)]$, $[DS(H_{n,n})]$ and $\Gamma[DS(F_n)]$ respectively.

Theorem 3.1 *For $n \geq 4$, the grundy coloring for degree splitting graph of wheel graph W_n is given by*

$$\Gamma[DS(W_n)] = \begin{cases} n+1, & n = 4, \\ n-2, & n = 5, \\ 4, & n \geq 6. \end{cases}$$

Proof Consider a wheel graph W_n with vertex set

$$V(W_n) = \bigcup_{i=1}^{n} v_i$$

where v_1 is the hub vertex and edge set

$$E(W_n) = \{v_1 v_i : i \in (1, n]\} \cup \{v_i v_{i+1} : i \in (1, n)\} \bigcup \{v_2 v_n\}$$

such that $\mid V(W_n) \mid = n$ and $\mid E(W_n) \mid = 2n - 2$. Moreover, $\triangle(W_n) = n - 1$ and $\delta(W_n) = 3$. We have $T = \{v_i : i \in [1, n]\}$ for $n = 4$ otherwise $T = \{v_i : i \in (1, n]\}$.

Thus, by the construction of degree splitting graph, we introduce a new vertex w corresponding to the vertex set T and hve $V[DS(W_n)] = \{v_i : i \in [1, n]\} \cup \{w\}$ and $E[DS(W_n)] = \{v_2 v_n\} \cup \{v_1 v_i : i \in (1, n]\} \cup \{v_i v_{i+1} : i \in (1, n)\} \cup \{w v_i : i \in [1, n]\}$ for $n = 4$ otherwise $E[DS(W_n)] = \{v_2 v_n\} \cup \{v_1 v_i : i \in (1, n]\} \cup \{v_i v_{i+1} : i \in (1, n)\} \cup \{w v_i : i \in (1, n]\}$ where

$| V[DS(W_n)] |= n+1$ and

$$| E[DS(W_n)] |= \begin{cases} 3n-2, & n=4, \\ 3n-3, & n \neq 4 \end{cases}$$

provided $\delta[DS(W_n)] = 4$ and

$$\triangle[DS(W_n)] = \begin{cases} n, & n=4, \\ n-1, & n \neq 4. \end{cases}$$

Consider the colors $C_1, C_2, C_3, \cdots$ and assign the colors as follows.

Case 1. $n = 4$

In this case, assign the colors by using the mapping $\pi : V[DS(W_n)] \to \{C_k : 1 \leq k \leq 5\}$ such that

$$\bullet\ \pi(w) = C_5,$$

$$\bullet\ \pi(v_i) = \begin{cases} C_4, & i=1, \\ C_3, & i=2, \\ C_2, & i=3, \\ C_1, & i=4. \end{cases}$$

Thus, $\Gamma[DS(W_n)] = 5$ for $n = 4$ where $\Gamma[DS(W_n)] > 5$ is not possible since $\Gamma \leq \triangle + 1$ and suppose $\Gamma[DS(W_n)] < 5$, eventhough it satisfies the definition of Grundy coloring it is not maximum. Hence, $\Gamma[DS(W_n)] = n + 1$ for $n = 4$.

Case 2. $n = 5$

In this case, consider the mapping $\phi : V[DS(W_n)] \to \{C_k : 1 \leq k \leq 3\}$ and assign the colors as follows.

$$\bullet\ \phi(w) = C_3,$$

$$\bullet\ \phi(v_i) = \begin{cases} C_3, & i=1, \\ C_2, & i \equiv 0(\text{mod})2, \\ C_1, & i \equiv 1(\text{mod})2. \end{cases}$$

Thus, $\Gamma[DS(W_n)] = 3$ for $n = 5$.

Suppose $\Gamma[DS(W_n)] > 3$, then it makes the vertex v_3 colored with C_2 not adjacent with C_1 which contradicts grundy coloring for the mapping $\phi(w) = \phi(v_1) = C_4$,

$$\phi(v_i) = \begin{cases} C_3, & i \equiv 0(\text{mod})2, \\ C_2, & i=3, \\ C_1, & i=5 \end{cases}$$

and suppose $\Gamma[DS(W_n)] < 3$, it contradicts the definition of proper coloring. Therefore, $\Gamma[DS(W_n)] = n-2$ for $n = 5$.

Case 3. $n \geq 6$

Let us consider the mapping $\psi : V[DS(W_n)] \to \{C_k : 1 \leq k \leq 4\}$ and assign the colors as follows.

Subcase 3.1 $n \equiv 0(\text{mod})2$

$$\begin{aligned} \psi(w) &= \psi(v_1) = C_4, \\ \psi(v_2) &= C_3, \\ \psi(v_i) &= \begin{cases} C_2, & i \equiv 1(\text{mod})2, \\ C_1, & i \equiv 0(\text{mod})2. \end{cases} \end{aligned}$$

Subcase 3.2 $n \equiv 1(\text{mod})2$,

$$\begin{aligned} \psi(w) &= \psi(v_1) = C_4, \\ \psi(v_2) &= \psi(v_{n-1}) = C_3, \\ \psi(v_i) &= \begin{cases} C_2, & i \equiv 1(\text{mod})2, \\ C_1, & i \equiv 0(\text{mod})2 \text{ and } i = n. \end{cases} \end{aligned}$$

Thus, from the above subcases, $\Gamma[DS(W_n)] = 4$ for $n \geq 6$.

Suppose $\Gamma[DS(W_n)] > 4$, then it makes some vertex colored with C_k not adjacent with all C_{k-1} colors. For instance, $\Gamma[DS(W_6)] = 5$ in which the vertex v_2 colored with C_4 and v_3 colored with C_3 are not adjacent with the color C_1 for the mapping $\psi(w) = \psi(v_1) = C_5$, $\psi(v_2) = C_4$, $\psi(v_3) = C_3$, $\psi(v_4) = \psi(v_6) = C_2$ and $\psi(v_5) = C_1$. This contradicts Grundy coloring. Similarly $4 < \Gamma[DS(W_n)] \leq n$ leads to contradiction of grundy coloring. And suppose $\Gamma[DS(W_n)] < 4$, eventhough it satisfies grundy coloring it is not maximum. Therefore, $\Gamma[DS(W_n)] = 4$ for $n \geq 6$.

Thus, from all above cases, we have

$$\Gamma[DS(W_n)] = \begin{cases} n+1, & n = 4, \\ n-2, & n = 5, \\ 4, & n \geq 6 \end{cases} \qquad \square$$

Theorem 3.2 *For $n \geq 3$, the Grundy coloring for degree splitting graph of sunlet graph S_n is given by*

$$\Gamma[DS(S_n)] = \begin{cases} 4, & n = 4, \\ 5, & n \neq 4 \end{cases}$$

Proof Consider a sunlet graph S_n with vertex set $V(S_n) = \{u_i : i \in [1,n]\} \cup \{v_j : j \in [1,n]\}$

and edge set $E(S_n) = \{u_iu_{i+1} : i \in [1,n)\} \cup \{u_1u_n\} \cup \{u_iv_j : i,j \in [1,n]$ and $i = j\}$ such that $\mid V(S_n) \mid = \mid E(S_n) \mid = 2n$. Moreover, $\triangle(S_n) = 3$ and $\delta(S_n) = 1$. Hence, we have $T_1 = \{u_i : i \in [1,n]\}$ and $T_2 = \{v_j : j \in [1,n]\}$.

Thus, by the construction of degree splitting graph, we introduce new vertices $\{w_1, w_2\}$ corresponding to vertex set T_1 and T_2 and therefore, $V[DS(S_n)] = \{u_i : i \in [1,n]\} \cup \{v_j : j \in [1,n]\} \cup \{w_k : k \in [1,2]\}$ and $E[DS(S_n)] = \{u_iu_{i+1} : i \in [1,n)\} \cup \{u_1u_n\} \cup \{u_iv_j : i,j \in [1,n]$ and $i = j\} \cup \{u_iw_1 : i \in [1,n]\} \cup \{v_jw_2 : j \in [1,n]\}$ where $\mid V[DS(S_n)] \mid = 2n+2$ and $\mid E[DS(S_n)] \mid = 4n$ provided $\delta[DS(S_n)] = 2$ and

$$\triangle[DS(S_n)] = \begin{cases} n+1, & n = 3, \\ n, & n \neq 3. \end{cases}$$

Consider the colors $C_1, C_2, C_3, \cdots$ and assign the colors as follows.

Case 1. $n = 4$

Assign the colors by using the mapping $\rho : V[DS(S_n)] \to \{C_t : 1 \leq t \leq 4\}$ such that

- $\rho(w_k) = \begin{cases} C_4, & k = 1, \\ C_3, & k = 2; \end{cases}$
- $\rho(u_i) = C_3$ for $i \equiv 1(\text{mod}2)$;
- for $i \equiv 0(\text{mod}2)$, $\rho(u_i) = \begin{cases} C_2, & i = 2, \\ C_1, & i = 4; \end{cases}$;
- for $1 \leq i \leq n$, $\rho(v_j) = \begin{cases} C_2, & j = n, \\ C_1, & 1 \leq j \leq n-1 \end{cases}$.

Thus, $\Gamma[DS(S_n)] = 4$ for $n = 4$.

Suppose $\Gamma[DS(S_n)] > 4$ then it makes the vertex v_4 colored with C_2 not adjacent with C_1 which contradicts grundy coloring for the mapping $\rho(u_i) = C_i$, $\rho(v_1) = \rho(v_n) = C_2$, $\rho(v_2) = \rho(v_3) = C_1$, $\rho(w_1) = C_5$ and $\rho(w_2) = C_3$ and suppose $\Gamma[DS(S_n)] < 4$, Even through it satisfies the definition of Grundy coloring it is not maximum. Thus, $\Gamma[DS(S_n)] = 4$ for $n = 4$.

Case 2. $n \neq 4$

Let us consider the mapping $\lambda : V[DS(S_n)] \to \{C_t : 1 \leq t \leq 5\}$ and assign the colors as follows.

Subcase 2.1 $n = 3$

- $\lambda(u_i) = C_{i+2}, \forall 1 \leq i \leq n$;
- $\lambda(v_j) = C_2, \forall 1 \leq j \leq n$;
- $\lambda(w_k) = C_1, \forall k \in \left[1, \left\lceil \frac{n}{2} \right\rceil\right]$.

Subcase 2.2 $n \geq 5$

- $\lambda(w_1) = C_5$ and $\lambda(w_2) = C_3$;
- $\lambda(u_i) = C_{i+2}, \forall i \in [1, 2]$;
- for odd n, $\lambda(u_i) = \begin{cases} C_2, & i \equiv 1(\text{mod})2, \\ C_1, & i \equiv 0(\text{mod})2; \end{cases}$

$$\lambda(v_j) = \begin{cases} C_2, & j \equiv 0(\text{mod})2, \\ C_1, & j = 2 \text{ and } j \equiv 1(\text{mod})2; \end{cases}$$

- for even n, $\lambda(u_i) = \begin{cases} C_3, & i = n-1, \\ C_2, & i \equiv 1(\text{mod})2 \text{ and } i = n, \\ C_1, & i \equiv 0(\text{mod})2; \end{cases}$

$$\lambda(v_j) = \begin{cases} C_2, & j \equiv 0(\text{mod})2 \text{ and } 4 \leq j < n, \\ C_1, & j = 2, n \text{ and } j \equiv 1(\text{mod})2. \end{cases}$$

Thus, from all above subcases, $\Gamma[DS(S_n)] = 5$ for $n \neq 4$.

Suppose $\Gamma[DS(S_n)] > 5$, it is not possible for $n = 3$ since $\Gamma \leq \triangle + 1$ whereas for $n \geq 5$, it makes some vertex colored with C_t not adjacent with all C_{t-1} colors. For instance, $\Gamma[DS(S_5)] = 6$ in which the vertex w_1 colored with C_6 is not adjacent with the color C_5 for the mapping $\lambda(w_1) = C_6$, $\lambda(w_2) = C_3$,

$$\lambda(v_j) = \begin{cases} C_2, & j = 1, \\ C_1, & 2 \leq j \leq 5 \end{cases} \quad \text{and} \quad \lambda(u_i) = \begin{cases} C_i, & 1 \leq i \leq 4, \\ C_2, & i = 5. \end{cases}$$

This contradicts Grundy coloring. Similarly $7 \leq \Gamma[DS(S_n)] \leq \triangle[DS(S_n)] + 1$ for $n \geq 6$ leads to contradiction and suppose $\Gamma[DS(S_n)] < 5$. Even though it satisfies Grundy coloring it is not maximum. We get $\Gamma[DS(S_n)] = 5$ for $n \neq 4$.

Thus, from all above cases, we have

$$\Gamma[DS(S_n)] = \begin{cases} 4, & n = 4, \\ 5, & n \neq 4. \end{cases}$$

This completes the proof. □

Theorem 3.3 *For $n \geq 3$, the grundy coloring for degree splitting graph of helm graph H_n is given by*

$$\Gamma[DS(H_n)] = 5.$$

Proof Consider a helm graph H_n with vertex set $V(H_n) = \{v_i : i \in [0, n]\} \cup \{u_j : j \in [1, n]\}$

and edge set $E(H_n) = \{v_0v_i : i \in [1, n]\} \cup \{v_1v_n\} \cup \{v_iv_{i+1} : i \in [1, n)\} \cup \{v_iu_j : i, j \in [1, n] andi = j\}$ such that $| V(H_n) |= 2n + 1$ and $| E(H_n) |= 3n$. Moreover,

$$\triangle(H_n) = \begin{cases} 4, & n = 3, 4, \\ n, & n \geq 5 \end{cases} \quad \text{and} \quad \delta(H_n) = d(u_j : j \in [1, n]) = 1.$$

Hence, we have

$$T_1 = \{v_i : i \in [0, n]\} \text{ and } T_2 = \{u_j : j \in [1, n]\}.$$

Thus, by the construction of degree splitting graph, we introduce a new set of vertices $\{w_1, w_2\}$ corresponding to vertex set T_1 and T_2. Consequently, $V[DS(H_n)] = \{v_i : i \in [0, n]\} \cup \{u_j : j \in [1, n]\} \cup \{w_k : k \in [1, 2]\}$ and $E[DS(H_n)] = \{v_0v_i : i \in [1, n]\} \cup \{v_1v_n\} \cup \{v_iv_{i+1} : i \in [1, n)\} \cup \{v_iu_j : i, j \in [1, n]$ and $i = j\} \cup \{v_iw_1 : i \in [0, n]\} \cup \{u_jw_2 : j \in [1, n]\}$ for $n = 4$. Otherwise, $E[DS(H_n)] = \{v_0v_i : i \in [1, n]\} \cup \{v_1v_n\} \cup \{v_iv_{i+1} : i \in [1, n)\} \cup \{v_iu_j : i, j \in [1, n]$ and $i = j\} \cup \{v_iw_1 : i \in [1, n]\} \cup \{u_jw_2 : j \in [1, n]\}$ where $| V[DS(H_n)] |= 2n + 3$ and

$$| E[DS(H_n)] |= \begin{cases} 5n + 1, & n = 4, \\ 5n, & \text{Otherwise} \end{cases}$$

provided

$$\triangle[DS(H_n)] = \begin{cases} 5, & n = 3, 4, \\ n, & n \geq 5 \end{cases}$$

and $\delta[DS(H_n)] = 2$.

Consider the colors $C_1, C_2, C_3, \cdots$ and assign the colors by using the mapping $\eta : V[DS(H_n)] \rightarrow \{C_t : 1 \leq t \leq 5\}$.

Case 1. $n = 3$

- $\eta(v_0) = \eta(w_k : k \in [1, 2]) = C_1$;
- $\eta(u_j) = C_2, \forall j \in [1, n]$;
- $\eta(v_i) = C_{i+2}, \forall i \in [1, n]$.

Case 2. $n = 4$

- $\eta(w_1) = C_5$ and $\eta(w_2) = \eta(v_0) = C_1$;
- $\eta(u_j) = \begin{cases} C_3, & j = 2, \\ C_2, & \text{Otherwise, for } \forall j \in [1, n]; \end{cases}$
- $\eta(v_i) = \begin{cases} C_3, & i = 1, \\ C_i, & i \geq 2 \text{ for } \forall i \in [1, n]. \end{cases}$

Case 3. $n \geq 5$

- $\eta(v_0) = \eta(w_1) = C_5$ and $\eta(w_2) = C_3$;
- $\eta(u_j) = \begin{cases} C_2, & j = 1, \\ C_1, & j \geq 2 \ \forall j \in [1, n]; \end{cases}$
- $\eta(v_i) = \begin{cases} C_i, & 1 \leq i \leq 4, \\ C_2, & i \equiv 1(\text{mod})2, \\ C_3, & i \equiv 0(\text{mod})2 \text{ for } \forall i \in [1, n]. \end{cases}$

Thus, from all above cases, $\Gamma[DS(H_n)] = 5$.

Suppose $\Gamma[DS(H_n)] > 5$, then it makes some vertex v_i colored with C_t is not adjacent with all C_{t-1} colors. For instance, $\Gamma[DS(H_3)] = 6$ in which the vertex v_0 colored with C_3 is not adjacent with C_2 and C_1 for the mapping $\eta(w_k : k \in [1, 2]) = C_1$, $\eta(v_i : i \in [0, 3]) = C_{i+3}$ and $\eta(u_j : j \in [1, 3]) = C_2$. This leads to the contradiction of Grundy coloring. Similarly, $7 \leq \Gamma[DS(H_n)] \leq n + 1$ for $n \geq 5$ leads to contradiction and suppose $\Gamma[DS(H_n)] < 5$. Even though it satisfies the definition of Grundy coloring it is not maximum, i.e., $\Gamma[DS(H_n)] = 5$ for $n \geq 3$. □

Theorem 3.4 *For $n \geq 2$, the Grundy coloring for degree splitting graph of crown graph $H_{n,n}$ is given by*

$$\Gamma[DS(H_{n,n})] = n + 1.$$

Proof Consider a crown graph $H_{n,n}$ with vertex set $V(H_{n,n}) = \{u_i : i \in [1, n]\} \cup \{v_j : j \in [1, n]\}$ and edge set $E(H_{n,n}) = \{u_i v_j : i, j \in [1, n] \text{ and } i \neq j\}$ such that $| V(H_{n,n}) |= 2n$ and $| E(H_{n,n}) |= n(n - 1)$. Moreover, $\triangle(H_{n,n}) = \delta(H_{n,n}) = n - 1$, i.e., we have $T = u_i \cup v_j$ where $i, j \in [1, n]$.

Thus, by the construction of degree splitting graph, we introduce a new vertex w corresponding to the vertex set T, and so $V[DS(H_{n,n})] = \{u_i : i \in [1, n]\} \cup \{v_j : j \in [1, n]\} \cup \{w\}$ and $E[DS(H_{n,n})] = \{u_i v_j : i, j \in [1, n] \text{ and } i \neq j\} \cup \{u_i w : i \in [1, n]\} \cup \{v_j w : j \in [1, n]\}$ where $| V[DS(H_{n,n})] |= 2n + 1$ and $| E[DS(H_{n,n})] |= n(n + 1)$ provided $\triangle[DS(H_{n,n})] = 2n$ and $\delta[DS(H_{n,n})] = n$.

Consider the colors $C_1, C_2, \cdots$ and assign colors by using the mapping $\sigma : V[DS(H_{n,n})] \to \{C_k : k = 1, 2, 3, \cdots\}$ as follows:

- $\sigma(w) = C_{n+1}$;
- $\sigma(u_i) = \sigma(v_j) = C_i, \quad \forall i, j \in [1, n]$.

Thus, $\Gamma[DS(H_{n,n})] = n + 1$.

Suppose $\Gamma[DS(H_{n,n})] > n + 1$, then some vertex u_i or v_j colored with C_k is not adjacent with all C_{k-1} colors. For instance, $\Gamma[DS(H_{n,n})] = n+2$ for $n = 2$ in which the vertex u_2 colored with C_3 and the vertex v_1 colored with C_2 are not adjacent with C_1 for the mapping $\sigma(w) = C_4$,

$\sigma(u_1) = \sigma(v_1) = C_2$, $\sigma(u_n) = C_{n+1}$ and $\sigma(v_n) = C_{n-1}$. This leads to the contradiction of Grundy coloring. Similarly, $n+3 \leq \Gamma[DS(H_{n,n})] \leq 2n+1$ leads to contradiction. And suppose $\Gamma[DS(H_{n,n})] < n+1$, then it contradicts the definition of proper coloring, i.e., $\Gamma[DS(H_{n,n})] = n+1$ for $n \geq 2$. □

Theorem 3.5 *For $n \geq 1$, the Grundy coloring for degree splitting graph of friendship graph F_n is given by*

$$\Gamma[DS(F_n)] = \begin{cases} 4, & n = 1, \\ 3, & n \neq 1 \end{cases}$$

Proof Consider a friendship graph F_n with vertex set $V(F_n) = \bigcup_{i=0}^{2n}\{v_i\}$ and edge set $E(F_n) = \{v_0v_i : i \in [1, 2n]\} \cup \{v_iv_{i+1} : i \equiv 1(\text{mod})2\}$ such that $\mid V(F_n) \mid= 2n+1$ and $\mid E(F_n) \mid= 3n$. Moreover,

$$\triangle(F_n) = \begin{cases} 2, & n = 1, \\ 2n, & n \neq 1 \end{cases}$$

and $\delta(F_n) = 2$. We have $T = \bigcup_{i=0}^{2n}\{v_i\}$ for $n = 1$ otherwise $T = \bigcup_{i=1}^{2n}\{v_i\}$ for $n \geq 2$.

Thus, by the construction of degree splitting graph, we introduce a new vertex w corresponding to vertex set T, i.e., $V[DS(F_n)] = \{v_i : i \in [0, 2n]\} \cup \{w\}$ and $E[DS(F_n)] = \{v_0v_i : i \in [1, 2n]\} \cup \{v_iv_{i+1} : i \equiv 1(\text{mod})2\} \cup \{v_iw : i \in [0, 2n]\}$ for $n = 1$ otherwise $E[DS(F_n)] = \{v_0v_i : i \in [1, 2n]\} \cup \{v_iv_{i+1} : i \equiv 1(\text{mod}2\} \cup \{v_iw : i \in (0, 2n]\}$ for $n \geq 2$ where

$$\mid V[DS(F_n)] \mid= 2(n+1) \quad \text{and} \quad \mid E[DS(F_n)] \mid= \begin{cases} 3(n+1), & n = 1, \\ 5n, & n \neq 1 \end{cases}$$

provided

$$\triangle[DS(F_n)] = \begin{cases} 3, & n = 1, \\ 2n, & n \neq 1 \end{cases}$$

and $\delta[DS(F_n)] = 3$.

Consider the colors $C_1, C_2, C_3, \cdots$ and assign the colors as follows.

Case 1. $n = 1$

Let us consider the mapping $\zeta : V[DS(F_n)] \to \{C_s : 1 \leq s \leq 4\}$ such that

- $\zeta(w) = C_4$;
- $\zeta(v_i) = C_{i+1}$ for $0 \leq i \leq 2n$.

Obviously, $\Gamma[DS(F_n)] = 4$ for $n = 1$.

Case 2. $n \geq 1$

Assume the mapping $\tau : V[DS(F_n)] \to \{C_t : 1 \leq t \leq 3\}$ and assign the colors as follows.

- $\tau(w) = \tau(v_0) = C_3$;
- $\tau(v_i) = \begin{cases} C_2, & i \equiv 0(\text{mod})2, \\ C_1, & i \equiv 1(\text{mod})2 \text{ for } 1 \leq i \leq 2n. \end{cases}$

Thus, $\Gamma[DS(F_n)] = 3$. Suppose $\Gamma[DS(F_n)] > 3$ then the vertex v_i colored with C_t is not adjacent with all C_{t-1} colors. For instance, $\Gamma[DS(F_n)] = 4$, the vertex v_0 colored with C_4 is not adjacent with C_1 for the mapping $\tau(v_0) = C_4$,

$$\tau(v_i) = \begin{cases} C_3, & i \equiv 1(\text{mod})2, \\ C_2, & i \equiv 0(\text{mod})2 \end{cases}$$

for $\forall i \in [1, 2n]$ and $\tau(w) = C_1$. This leads to the contradiction of Grundy coloring. Similarly, $5 \leq \Gamma[DS(F_n)] \leq 2n + 1$ leads to contradiction. And suppose $\Gamma[DS(F_n)] < 3$, it contradicts the definition of proper coloring. Therefore, $\Gamma[DS(F_n)] = 3$ for $n \neq 1$.

From all above cases, we have

$$\Gamma[DS(F_n)] = \begin{cases} 4, & n = 1, \\ 3, & n \neq 1. \end{cases}$$

This completes the proof. □

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Generalized Perfect Neighborhood Number of a Graph

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Abstract: In this article, we generalize the perfect neighborhood number of a graph $G = (V, E)$ as a k - perfect neighborhood number $\eta_{kp}(G)$. Here many bounds and exact values of some specific family of graphs are obtained. Also, its relationship with other graph theoretic parameters are investigated.

Key Words: Graph, domination number, neighborhood number, perfect neighborhood number, k - perfect neighborhood number.

AMS(2010): 05C69, 05C70.

§1. Introduction

Let $G = (V, E)$ be simple, undirected, and nontrivial graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Also $|V| = n$ and $|E| = m$ denote number of vertices and number of edges in G. The open neighborhood $N(v)$ of vertex v denotes number of vertices adjacent to v and its closed neighborhood $N[v] = N(v) \cup \{v\}$. The $\beta_1(G)$ is the minimum number of edges in a maximal independent set of edge of G. For notation and graph theoretic terminology, we generally follow [11].

A set $D \subseteq V$ is a dominating set if every vertex not in S is adjacent to one or more vertices in D. The cardinality of a smallest dominating set of G, denoted by $\gamma(G)$, is the domination number of G. For more details on domination theory, we refer to [12], [13], [14] and [19]. In 1985, E. Sampathkumar and P. S. Neeralagi [17], introduced an innovative concept of domination between the vertices and the edges, and vice-versa. They introduced a new parameter called the *neighborhood number* of a graph, as follows. A set $S \subseteq V$ is a neighborhood set of G, if $G = \bigcup_{v \in s} \langle N[v] \rangle$, where $\langle N[v] \rangle$ is the sub graph of G induced by v and all vertices adjacent to v. The neighborhood number $\eta(G)$ is the minimum cardinality of a neighborhood set of G. For more information on neighborhood number, we refer to [3], [15] and [16].

In 1993, Cockayne et al [7] introduced the concept of perfect domination following in this sense. A subset $D \subseteq V$ is a perfect dominating set of G if any vertex of G not in D is adjacent to exactly one vertex of D. In 2010, Chaluvaraju et al [4] and [5] was generalized perfect domination as follows. A vertex subset D of G is called a k - perfect dominating set of G, if any vertex v of V not in D is adjacent to exactly k - vertices of D. The minimum cardinality of a k - perfect dominating set of G is the k - perfect domination number $\gamma_{kp}(G)$.

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The following known results from [4] and [5] are used in the sequel.

Theorem 1.1 *Let T be a tree and G be a connected graph. Then,*

(*i*) $\frac{kn}{\Delta(G)+k} \leq \gamma_{kP}(G)$;
(*ii*) $\alpha(T) \leq \gamma_{kP}(T)$;
(*iii*) $\beta_1(T) \leq \gamma_{kP}(T)$;
(*iv*) $\lceil \frac{diam(G)}{2} \rceil \leq \gamma_{2P}(G)$.

Theorem 1.2 *Let $k = \Delta(G) - 1$. Then the graph G is a kPD - graph if and only if G satisfy one of the following conditions:*

(*i*) *there exists at least two adjacent vertices u and v in a graph G such that $deg(u) = deg(v) = \Delta(G)$;*
(*ii*) *there exists a vertex u such that $deg(u) = \Delta(G) - 1$.*

Several papers have been written on the subject of k - domination in graphs and when they exists, cf. [2], [6], [9] and [10]. Further, let k be a positive integer and G be a graph. A subset S of vertices in a graph G is a k - neighborhood set, if every vertex of $V - S$ is adjacent to at least k - vertices in S. The k - neighborhood number $\eta_k(G)$ is the minimum cardinality of a k - neighborhood set of a graph G. Hence for $k = 1$, 1 - neighborhood sets are the classical neighborhood set of a graph G, see [17].

Analogously, here we generalize the perfect neighborhood number as follows: A subset S of vertices in a graph G is a k - neighborhood set, if every vertex of $V - S$ is adjacent to exactly k - vertices in S. The k - perfect neighborhood number $\eta_{kp}(G)$ is the minimum cardinality of a k - perfect neighborhood set of a graph G. Hence for $k = 1$, 1 - perfect neighborhood sets are the usual perfect neighborhood sets. The concept of perfect neighborhood number was initiated by Sampatkumar and Neerlagi [18].

Note that every nontrivial graph G has a k - perfect neighborhood set, since the entire vertex set is such a set and there are graphs whose only k - perfect neighborhood set is $V(G)$. A graph G for which $\eta_{kp}(G) < n$ is called a k - perfect neighborhood graph, abbreviated kPN - graph and a tree T for which $\eta_{kp}(T) < n$ is called a kPN - tree.

§2. Specific Families of Graphs

Proposition 2.1 *For any complete graph K_n; $n \geq 2$ vertices with $1 \leq k \leq \Delta(G)$,*

$$\eta_{kP}(K_n) = k.$$

Proposition 2.2 *For any path P_n with $n \geq 3$ vertices,*

(*i*) $\eta_{1P}(P_n) = n - 2$;
(*ii*) $\eta_{2P}(P_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \text{ is odd} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even.} \end{cases}$

Proposition 2.3 *For any Cycle C_n with $n \geq 4$ vertices,*

(i) $\eta_{1P}(C_n)$ *does not exist;*

(ii) $\eta_{2P}(C_n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$

Proposition 2.4 *For any Fan graph $F_n = P_n + K_1$ with $n \geq 1$ vertices,*

(i) $\eta_{1P}(F_n) = 1$;

(ii) $\eta_{2P}(F_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{if } n = 3t + 1 \text{ for } t \geq 1 \\ \lceil \frac{n}{3} \rceil + 1, & \text{otherwise;} \end{cases}$

(iii) $\eta_{3P}(F_n) = \lceil \frac{n}{2} \rceil + 1$ *if* $n \geq 4$;

(iv) $\eta_{(n-1)P}(F_n) = n - 1$ *if* $n \geq 3$.

Proposition 2.5 *For any wheel graph $W_n = C_n + K_1$ with $n \geq 4$ vertices,*

(i) $\eta_{1P}(W_n) = 1$;

(ii) $\eta_{2P}(W_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor + 1, & if\ n = 3t,\ t \geq 2, \\ \lfloor \frac{n}{2} \rfloor, & \text{otherwise;} \end{cases}$

(iii) $\eta_{3P}(W_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd;} \end{cases}$

(iv) $\eta_{(n-1)P}(W_n) = n - 1$.

Proposition 2.6 *For any complete bipartite graph $K_{r,s}$ with $n = r + s$ vertices,*

(i) $\eta_{1P}(K_{1,s}) = 1$ *if r = 1 and s = n - 1;*

(ii) $\eta_{sP}(K_{1,s}) = s$ *if r = 1 and s = n - 1;*

(iii) $\eta_{rP}(K_{r,s}) = r$ *if* $2 \leq r \leq s$;

(iv) $\eta_{sP}(K_{r,s}) = s$ *if* $2 \leq r \leq s$.

§3. Properties and Bounds

Property 3.1 *For every graph G and positive integer k, every vertex with degree at most $k-1$ belongs to every k - perfect neighborhood set.*

Property 3.2 *Since $v \in V - S$ should be adjacent to k - vertices in S, the graph G is not a kPN - graph for $k \geq \Delta(G)$.*

Property 3.3 *Let v be a vertex wit $deg(v) = \Delta(G)$ and let $k = \Delta(G)$. Then $V - \{v\}$ is a Δ - perfect neighborhood set of G. Thus G is a kPN - graph for $k = \Delta(G)$.*

Property 3.4 *A graph will have two disjoint k - perfect neighborhood sets only if $k \leq \delta(G)$, since all the vertices with degree less than k belongs to every k - perfect neighborhood set.*

Property 3.5 *If S is a k - neighborhood set of a graph G, then S is a t - neighborhood set for* *every $t \leq k$. But this is not true in case of k - perfect neighborhood set.*

Property 3.6 *Every k - perfect neighborhood set is a k neighborhood set of a graph G and hence $\eta_k(G) \leq \eta_{kP}(G)$.*

Observation 3.1 An k- perfect neighborhood set is a k - perfect dominating set, and hence $\gamma_{kP}(G) \leq \eta_{kP}(G)$ for every graph G and positive integer k.

By above observation and Theorem 1.1, we have the following lower bounds.

Theorem 3.1 *Let T be a tree and G be a connected graph. Then,*

(i) $\frac{kn}{\Delta(G)+k} \leq \eta_{kP}(G)$;

(ii) $\alpha(T) \leq \eta_{kP}(T)$;

(iii) $\beta_1(T) \leq \eta_{kP}(T)$;

(iv) $\left\lceil \frac{diam(G)}{2} \right\rceil \leq \eta_{2P}(G)$.

Theorem 3.2 *Let G be a kPN - graph with $n \geq 2$ vertices. Then*

$$n - (m/k) \leq \eta_{kP}(G) \leq n - 1.$$

Proof Let S be a η_{kP} - set of a nontrivial graph G and $|V - S| = t$. Then there are t - times of k - edges from $V - S$ to S with $\eta_{kP}(G)= |S|$. Since $m > tk$, the lower bound follows. By the definition of kPN - graph, the upper bound follows. □

Theorem 3.3 *Let $\{x_1, x_2, \cdots, x_n\}$ be the degree sequence of a graph G with $degv_i = x_i$ for $i = 1, 2, \cdots, n$. If k is an integer such that $k \in \{x_1, x_2, \cdots, x_n\}$, Then G is a kPN - graph.*

Proof Let $k \in \{x_1, x_2, \ldots, x_n\}$. If $S = V - v$, where v is a vertex of degree k in a graph G, then S is a k - PNS of a graph G. Therefore G is a kPN - graph. □

Observation 3.2 The converse of above theorem is not true.

For example, we consider the following graph G_1.

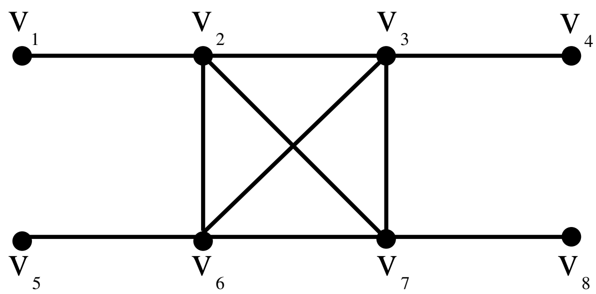

Figure 1. The graph G_1.

Here, the degree sequence of G_1 is $\{1, 4, 4, 1, 1, 4, 4, 1\}$, we have

(i) If $k = 2$, then η_{2P} - set S is $\{v_1, v_2, v_4, v_5, v_8\}$ and $V - S$ is $\{v_3, v_6, v_7\}$.

(ii) If $k = 3$, then η_{3P} - set S is $\{v_1, v_2, v_3, v_4, v_5, v_8\}$ and $V - S$ is $\{v_6, v_7\}$.

Clearly, these graphs are $2PN$ - graph and $3PN$ - graph. But $k = 2$ and 3 does not belong to the degree sequence of a graph G_1.

Theorem 3.4 *For any connected graph G,*

$$\frac{kn}{\Delta(G) + k} \leq \eta_{kP}(G).$$

Theorem 3.5 *Let G be a connected graph with $\eta_{kP}(G) = k$. Then*

$$\Delta(G) \geq Max.\{k, n - k\}.$$

Proof Let S be a η_{kP} - set of a graph G with $\eta_{kP}(G) = k$. Then we have the following cases.

Case 1. Suppose if $v \in V - S$, then the degree of v is greater than $|S| = k$. There fore $\Delta(G) \geq k$.

Case 2. Suppose if $v \notin V - S$, then the degree of $v \in S$ is greater than $|V - S| = n - k$. There fore $\Delta(G) \geq n - k$.

Thus, the result follows. □

§4. Concluding Remarks and Further Scope

Different graph theorists have defined wide varieties of neighborhood related graph parameters by imposing extra conditions on the neighborhood set S of a graph G, because the neighborhood number is closely related to the domination number of G. To stimulate further understanding or advancement in this generalized perfect neigbor based graph parameters, we pose the following open problems:

(i) *Obtain the complexity issues of $\eta_{kP}(G)$;*

(ii) *Characterize the class of graphs when $\gamma_{kP}(G) = \eta_{kP}(G)$?*

(iii) *Obtain some bound and characterization on $\eta_{kP}(G)$ in terms of other domination related parameters such as total domination, connected domination, independence domination and so on.,*

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Pair Difference Cordial Number of Some Degree Splitting Graph

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Abstract: In this paper we determine the pair difference cordial number of degree splitting graph of bistar, complete bipartite graph, ladder, wheel.

Key Words: Pair difference cordial labeling, Smarandachely pair difference cordial labeling, Smarandachely pair difference cordial graph, bipartite graph, bistar, degree splitting graph, ladder, wheel.

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§1. Introduction

We consider only finite, undirected and simple graphs. The notion of pair difference cordial labeling of a graph was introduced in [4]. Also we have investigated pair difference cordial labeling behavior of several graphs like path, cycle, star, wheel, some snake and butterfly graphs, graphs derived from ladder graph, degree splitting graph of some graphs have been investigated in [4,5,6,7,8,9,10]. Recently pair difference cordial number of a graph was introduced in [14]. In this paper we determine the pair difference cordial number of degree splitting graph of bistar, complete bipartite, ladder, wheel.

§2. Preliminaries

Definition 2.1([4]) *Let $G = (V, E)$ be a (p, q) graph. Define*

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm \rho\}$ called the set of labels. Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. Otherwise, if

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$\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \geq 2$, *such a labeling* f *is said to be a Smarandachely pair difference cordial labeling and a graph* G *for which there exists a pair difference cordial labeling or a Smarandache pair difference cordial labeling is called a pair difference cordial labeling graph or Smarandachely pair difference cordial graph.*

Definition 2.2([5]) *Let* $G = (V, E)$ *be a graph with* $V = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ *where each* S_i *is a set of vertices having at least two vertices and having the same degree and* $T = V - \bigcup_{i=1}^{t} S_i$*. The degree splitting graph degree splitting graph of* G *denoted by* $DS(G)$ *is obtained from* G *by adding vertices* $w_1, w_2 \ldots, w_t$ *and joining* w_i *to each vertex of* S_i $(1 \leq i \leq t)$.

Theorem 2.3([13]) $DS(K_{n,n})$ *is pair difference cordial if and only if* $n = 2$*, where* $K_{n,n}$ *is complete bipartite graph.*

Theorem 2.4([13]) $DS(L_n)$ *is pair difference cordial if and only if* $n \leq 3$*, where* L_n *is the ladder.*

Theorem 2.5([13]) $DS(B_{n,n})$ *is pair difference cordial if and only if* $n \leq 2$ *, where* $B_{n,n}$ *is the bistar.*

Theorem 2.6([13]) $DS(B_{1,n})$ *is pair difference cordial if and only if* $n \leq 4$.

Theorem 2.7([13]) $DS(W_n)$ *is not pair difference cordial for all* $n \geq 3$*, where* W_n *is the wheel.*

Theorem 2.8([4]) *The cycle* C_n *is pair difference cordial if and only if* $n > 3$.

§3. Pair Difference Cordial Number of a Graph

Definition 3.1 *Let* G *be a* (p, q) *graph. Pair difference cordial number of a graph* G *is the least positive integer* m *such that* $G \cup K_2$ *is pair difference cordial. It is denoted by* $PDC_\eta(G)$.

Remark 3.2 If G is pair difference cordial graph then $PDC_\eta(G) = 0$.

Theorem 3.3 *For any integer* $n \geq 1$,

$$PDC_\eta(B_{1,n}) = \begin{cases} 0 & \text{if } n \leq 4, \\ 2n-6 & \text{if } n \geq 5. \end{cases}$$

Proof Let $V(B_{1,n} \cup mK_2) = \{x, v, w, u, u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(B_{1,n} \cup mK_2) = \{vw, wu, xu_n\} \cup \{uu_i : 1 \leq i \leq n\} \cup \{v_iw_i : 1 \leq i \leq m\}$. Clearly $B_{1,n} \cup mK_2$ has $n + 2m + 4$ vertices and $2n + m + 3$ edges. There are two cases arises.

Case 1. $n \leq 4$.

The proof follows from Theorem 2.6.

Case 2. $n \geq 5$.

Subcase 2.1 n is odd.

Take $m = 2n-6$. Define $f : V(B_{1,n} \cup (2n-6)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{n+2m+3}{2} \rfloor\}$ as follows:

Assign the labels $1, 2, 3, 4, 4$ to the vertices v, w, u, u_1, u_2 and assign the labels $-1, -2, -3, -4$ to the vertices x, u_n, u_3, u_4. Next, assign the labels $5, 7, 9, \cdots, (n+2)$ to vertices $v_1, v_2, v_3, \cdots, v_{\frac{m}{2}}$ and assign the labels $6, 8, 10, \cdots, (n+3)$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{m}{2}}$. Next, assign the label $-5, -7, -9, \cdots, -(n+2)$ to the vertices $v_{\frac{m}{2}+1}, v_{\frac{m}{2}+1}, v_{\frac{m}{2}+1}, \cdots\ v_m$ and assign the label $-6, -8, -10, \cdots, -(n+3)$ to the vertices $w_{\frac{m}{2}+1}, w_{\frac{m}{2}+1}, w_{\frac{m}{2}+1}, \cdots\ w_m$. Finally assign the distinct labels to the vertices $u_5, u_6, u_7, \cdots, u_{n-1}$ from $\pm(n+1), \pm(n-1), \cdots, \pm(\frac{n+2m+3}{2})$.

Therefore, f is pair difference cordial labeling of $B_{1,n} \cup (2n-6)K_2$. The maximum number of edges with labels 1 from the $DS(B_{1,n})$ is, $\Delta_{f_1} = 5$, if n is odd.But the size of the $DS(B_{1,n})$ is $2n+3$. Hence $2n-6$ is the least integer such that $B_{1,n} \cup (2n-6)K_2$ is pair difference cordial.

Subcase 2.2 n is even.

Take $m = 2n-6$. Define $f : V(B_{1,n} \cup (2n-6)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{n+2m+4}{2} \rfloor\}$ as follows:

Assign the labels $1, 2, 3, 4$ to the vertices v, w, u, u_1 and assign the labels $-1, -2, -3, -4$ to the vertices x, u_n, u_2, u_3. Next assign the labels $5, 7, 9, \cdots, (n+2)$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{m}{2}}$ and assign the labels $6, 8, 10, \cdots, (n+3)$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{m}{2}}$. Next assign the label $-5, -7, -9, \cdots, -(n+2)$ to the vertices $v_{\frac{m}{2}+1}, v_{\frac{m}{2}+1}, v_{\frac{m}{2}+1}, \cdots\ v_m$ and assign the label $-6, -8, -10, \cdots, -(n+3)$ to the vertices $w_{\frac{m}{2}+1}, w_{\frac{m}{2}+1}, w_{\frac{m}{2}+1}, \cdots\ w_m$. Finally assign the distinct labels to the vertices $u_4, u_5, u_6, \cdots, u_{n-1}$ from $\pm(n+1), \pm(n-1), \cdots, \pm(\frac{n+2m+3}{2})$.

Therefore, f is pair difference cordial labeling of $B_{1,n} \cup (2n-6)K_2$. The maximum number of edges with labels 1 from the $DS(B_{1,n})$ is, $\Delta_{f_1} = 4$, if n is even.But the size of the $DS(B_{1,n})$ is $2n+3$. Hence $2n-6$ is the least integer such that $B_{1,n} \cup (2n-6)K_2$ is pair difference cordial. □

Theorem 3.4 *For any integer $n \geq 1$,*

$$PDC_\eta(DS(L_n)) = \begin{cases} 0 & \text{if } n \leq 3 \\ n-1 & \text{if } n \geq 5 \text{ and } n \text{ is odd,} \\ n-2 & \text{if } n \geq 4 \text{ and } n \text{ is even.} \end{cases}$$

Proof Let $V(DS(L_n) \cup mK_2) = \{x, x_i, y, y_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(DS(L_n) \cup mK_2) = \{xx_1, xx_n, xy_1, xy_n\} \cup \{yy_i, yx_i : 2 \leq i \leq n-1\} \cup \{v_i w_i : 1 \leq i \leq m\}$. Clearly $DS(L_n) \cup mK_2$ has $2n + 2m + 2$ vertices and $5n + m - 2$ edges. There are three cases arises.

Case 1. $n \leq 3$.

The proof follows from Theorem 2.4.

Case 2. $n \geq 5$ and n is odd.

Take $m = n-1$. Define $f : V(DS(L_n) \cup (n-1)K_2) \to \{\pm 1, \pm 2, \cdots, \pm n+m+1\}$ as follows:

Assign the labels $1, 2, 3, -1, -2, -3$ to the vertices x_1, x, x_n, y_1, y_2, y and assign the labels $4, 5, 6, \cdots, (n+1)$ to the vertices $x_{n-1}, x_{n-2}, x_{n-3}, \cdots, u_3, u_2$. Next assign the labels $-4, -5, -6, \cdots, -(n+1)$ to the vertices $y_3, y_4, y_5, \cdots, y_n$. Next assign the labels $(n+2)$, $(n+4)$, $(n+6), \cdots, (n+m)$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-1}{2}}$ and assign the labels $(n+3)$,$(n+5)$, $(n+7), \cdots, (n+m+1)$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{n-1}{2}}$. Now we assign the labels $-(n+2)$,$-(n+4)$, $-(n+6), \cdots, -(n+m)$ to the vertices $v_{\frac{n-1}{2}+1}, v_{\frac{n-1}{2}+2}, v_{\frac{n-1}{2}+3}, \cdots, v_{n-1}$ and assign the labels $-(n+3)$,$-(n+5)$,$-(n+7), \cdots$,$-(n+m+1)$ to the vertices $w_{\frac{n-1}{2}+1}, w_{\frac{n-1}{2}+2}, w_{\frac{n-1}{2}+3}, \cdots, w_{n-1}$.

Therefore, f is pair difference cordial labeling of $DS(L_n) \cup (n-1)K_2$. The maximum number of edges with labels 1 from the $DS(L_n)$ is, $\Delta_{f_1} = 2n$.But the size of the $DS(L_n)$ is $5n-2$. Hence $n-1$ is the least integer such that $DS(L_n) \cup (n-1)K_2$ is pair difference cordial.

Case 3. $n \geq 4$ and n is even.

Take $m = n-2$. Define $f : V(L_n \cup (n-2)K_2) \to \{\pm 1, \pm 2, \cdots, \pm n+m+1\}$ as follows:

Assign the labels $1, 2, 3, -1, -2, -3$ to the vertices x_1, x, x_n, y_1, y_2, y and assign the labels $4, 5, 6, \cdots, (n+1)$ to the vertices $x_{n-1}, x_{n-2}, x_{n-3}, \cdots, u_3, u_2$. Next assign the labels $-4, -5, -6, \cdots, -(n+1)$ to the vertices $y_3, y_4, y_5, \cdots, y_n$. Next assign the labels $(n+2)$,$(n+4)$,$(n+6), \cdots$,$(n+m)$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n-2}{2}}$ and assign the labels $(n+3)$,$(n+5)$,$(n+7), \cdots$,$(n+m+1)$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{n-2}{2}}$. Now we assign the labels $-(n+2)$,$-(n+4)$,$-(n+6), \cdots$,$-(n+m)$ to the vertices $v_{\frac{n-2}{2}+1}, v_{\frac{n-2}{2}+2}, v_{\frac{n-2}{2}+3}, \cdots, v_{n-2}$ and assign the labels $-(n+3)$,$-(n+5)$,$-(n+7), \cdots, -(n+m+1)$ to the vertices $w_{\frac{n-2}{2}+1}, w_{\frac{n-2}{2}+2}, w_{\frac{n-2}{2}+3}, \cdots, w_{n-2}$.

Therefore, f is pair difference cordial labeling of $DS(L_n) \cup (n-2)K_2$ and the maximum number of edges with labels 1 from the $DS(L_n)$ is $\Delta_{f_1} = 2n$. But the size of the $DS(L_n)$ is $5n-2$. Hence $n-2$ is the least integer such that $DS(L_n) \cup (n-2)K_2$ is a pair difference cordial graph. □

A pair difference cordial labeling on $DS(L_6) \cup 4K_2$ is shown in Figure 1.

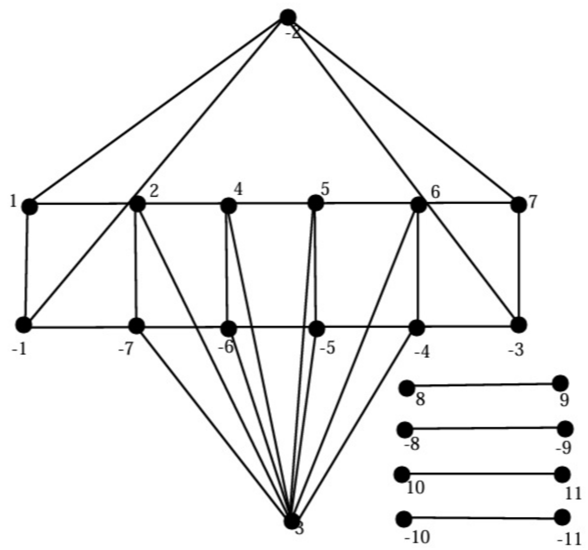

Figure 1

Theorem 3.5 *If $n \geq 3$, then*

$$PDC_\eta(DS(W_n)) = \begin{cases} n-1 & \text{if } n \text{ is odd,} \\ n-2 & \text{if } n \text{ is even.} \end{cases}$$

Proof Let $V(DS(W_n) \cup mK_2) = \{x, x_i, y : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(DS(W_n) \cup mK_2) = \{x_1x_n, x_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{xx_i, yx_i : 1 \leq i \leq n\} \cup \{v_iw_i : 1 \leq i \leq m\}$.

Clearly, $DS(W_n) \cup mK_2$ has $n + 2m + 2$ vertices and $3n + m$ edges. There are two cases arises.

Case 1. n is odd.

Take $m = n - 1$. Define

$$f : V(DS(W_n) \cup (n-1)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \frac{n + 2m + 2}{2}\}$$

as follows:

The maximum possible number of 1 occurs only when we assign the labels $3, 4, 5, \cdots, n+1$ to the vertices $x_3, x_4, x_5 \cdots, x_n$ and assign the labels $1, 2, n+2$ to the vertices x_1, x_2, x. Next assign the labels $-1, -3, -5, \cdots, -n$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n+1}{2}}$ and assign the labels $-2, -4, -6, \cdots, -(n+1)$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{n+1}{2}}$ and assign the labels $-(n+2), -(n+1)$ to the vertices $v_{\frac{n+3}{2}}, w_{\frac{n+3}{2}}$. Finally assign the labels $(n+3)$,$(n+4)$,$-(n+3)$,$-(n+4)$ to the vertices $v_{\frac{n+5}{2}}, w_{\frac{n+5}{2}}, v_{\frac{n+7}{2}}, w_{\frac{n+7}{2}}$ and assign the labels $(n+5)$,$(n+6)$,$-(n+5)$,$-(n+6)$ to the vertices $v_{\frac{n+9}{2}}, w_{\frac{n+9}{2}}, v_{\frac{n+11}{2}}, w_{\frac{n+11}{2}}$. Proceeding like this until we reach v_{n-1}, w_{n-1}.

Therefore, f is pair difference cordial labeling of $DS(W_n) \cup (n-1)K_2$. The maximum number of edges with labels 1 from the $DS(W_n)$ is, $\Delta_{f_1} = n + 1$.But the size of the $DS(W_n)$ is $3n$. Hence $n - 1$ is the least integer such that $DS(W_n) \cup (n-1)K_2$ is pair difference cordial.

Case 2. n is even.

Take $m = n-2$. Define $f : V(DS(W_n) \cup (n-2)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \frac{n+2m+2}{2}\}$ as follows:

The maximum possible number of 1 occurs only when we assign the labels $3, 4, 5, \cdots, n+1$ to the vertices $x_3, x_4, x_5 \cdots, x_n$ and assign the labels $1, 2, n+2$ to the vertices x_1, x_2, x. Next assign the labels $-1, -3, -5, \cdots, -(n+1)$ to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{n+2}{2}}$ and assign the labels $-2, -4, -6, \cdots, -(n+2)$ to the vertices $w_1, w_2, w_3, \cdots, w_{\frac{n+2}{2}}$. Finally assign the labels $(n+3)$,$(n+4)$,$-(n+3)$,$-(n+4)$ to the vertices $v_{\frac{n+4}{2}}, w_{\frac{n+4}{2}}, v_{\frac{n+6}{2}}, w_{\frac{n+6}{2}}$ and assign the labels $(n+5)$,$(n+6)$,$-(n+5)$,$-(n+6)$ to the vertices $v_{\frac{n+8}{2}}, w_{\frac{n+8}{2}}, v_{\frac{n+10}{2}}, w_{\frac{n+10}{2}}$. Proceeding like this until we reach v_{n-2}, w_{n-2}.

Therefore, f is pair difference cordial labeling of $DS(W_n) \cup (n-2)K_2$. The maximum number of edges with labels 1 from the $DS(W_n)$ is $\Delta_{f_1} = n + 1$. But the size of the $DS(W_n)$ is $3n$. Hence $n - 2$ is the least integer such that $DS(W_n) \cup (n-2)K_2$ is pair difference cordial. This completes the proof. □

Theorem 3.6 *For any integer $n \geq 1$,*

$$PDC_\eta(DS(K_{n,n})) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n = 2, \\ 5 & \text{if } n = 3, \\ n^2 - 2n - 2 & \text{if } n \geq 4. \end{cases}$$

Proof Let $V(DS(K_{n,n}) \cup mK_2) = \{x_i, y_i, x : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(DS(K_{n,n}) \cup mK_2) = E(K_{n,n}) \cup \{xx_i, xy_i : 1 \leq i \leq n\} \cup \{v_i w_i : 1 \leq i \leq m\}$. Clearly, $DS(K_{n,n}) \cup mK_2$ has $2n + 2m + 1$ vertices and $n^2 + 2n + m$ edges.

Case 1. $n = 1$.

In this case, $K_{n,n} \cong C_3$. Assign the labels $1, 1, 2, -1, -2$ to the vertices x, x_1, y_1, v_1, w_1.

Case 2. $n = 2$.

The proof follows from Theorem 2.3.

Case 3. $n = 3$.

In this case, A pair difference cordial labeling on $DS(K_{3,3}) \cup 5K_2$ is shown in Figure 2.

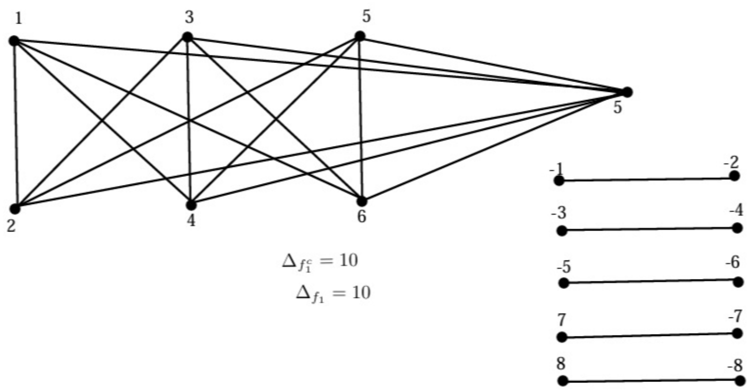

Figure 2

Case 4. $n \geq 4$.

Take $m = n^2 - 2n - 2$. Define $f : V(DS(K_{n,n}) \cup (n^2 - 2n - 2)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{n+2m+4}{2} \rfloor\}$ as follows:

Assign the labels $1, 3, 5, \cdots, 2n - 1$ to the vertices $x_1, x_2, x_3, \cdots, x_n$ and assign the labels $2, 4, 6, \cdots, 2n$ to the vertices $y_1, y_2, y_3, \cdots, y_n$. Next assign the labels $-1, -3, -5, \cdots, -(2n - 1)$ to the vertices $v_1, v_2, v_3, \cdots, v_n$ and assign the labels $-2, -4, -6, \cdots, -2n$ to the vertices $w_1, w_2, w_3, \cdots, w_m$. Next assign the labels $(2n+1)$,$(2n+2)$,$-(2n+1)$,$-(2n+2)$ to the vertices v_{n+1},w_{n+1},v_{n+2},w_{n+2} and assign the labels $(2n+3)$,$(2n+4)$,$-(2n+3)$,$-(2n+4)$ to the vertices v_{n+3},w_{n+3},v_{n+4},w_{n+4}. Proceeding like this until we reach $v_{n^2-2n-2}, w_{n^2-2n-2}$.

Therefore, f is pair difference cordial labeling of $K_{n,n} \cup (n^2 - 2n - 2)K_2$. The maximum number of edges with labels 1 from the $DS(K_{n,n})$ is, $\Delta_{f_1} = 2n+1$.But the size of the $DS(K_{n,n})$ is $n^2 + 2n$. Hence $n^2 - 2n - 2$ is the least integer such that $K_{n,n} \cup (n^2 - 2n - 2)K_2$ is pair difference cordial. □

Theorem 3.7 *For any integer* $n \geq 1$,

$$PDC_\eta(DS(K_{n,n+1})) = \begin{cases} 0 & \text{if } n = 1,2, \\ n^2 - n - 2 & \text{if } n \geq 3. \end{cases}$$

Proof Let $V(DS(K_{n,n+1}) \cup mK_2) = \{x_i, y_i, x, y : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq m\}$ and $E(DS(K_{n,n+1}) \cup mK_2) = E(K_{n,n+1}) \cup \{xx_i, yy_i : 1 \leq i \leq n\} \cup \{v_iw_i : 1 \leq i \leq m\}$. Clearly $DS(K_{n,n+1}) \cup mK_2$ has $2n + 3$ vertices and $n^2 + 3n + 1$ edges.

Case 1. $n = 1$.

In this case, $K_{n,n} \cong C_4$. The proof follows from Theorem 2.8.

Case 2. $n = 2$.

In this case, a pair difference cordial labeling on $DS(K_{2,3})$ is shown in Figure 3.

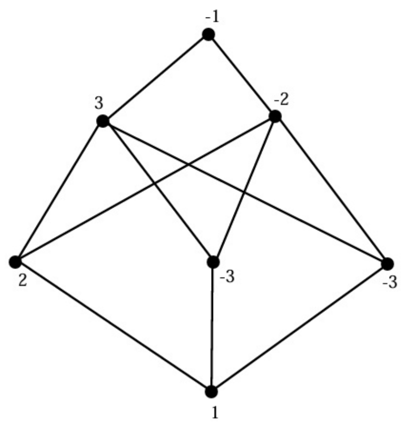

Figure 3

Case 3. $n \geq 3$.

Subcase 3.1 n is even.

Take $m = n^2-n-2$. Define $f : V(DS(K_{n,n+1}) \cup (n^2-n-2)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{2n+3}{2} \rfloor\}$ as follows:

Assign the labels $3, 5, 7, \cdots, n + 1$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n}{2}}$ and assign the labels $2, 4, 6, \cdots, n$ to the vertices $x_1, x_2, x_3, \cdots, x_{\frac{n}{2}}$. Next assign the labels $-3, -5, -7, \cdots, -(n+1)$ to the vertices $x_{\frac{n}{2}+1}, x_{\frac{n}{2}+2}, x_{\frac{n}{2}+3}, \cdots, x_n$ and assign the labels $-2, -4, -6, \cdots, -n$ to the vertices $y_{\frac{n}{2}+1}, y_{\frac{n}{2}+2}, y_{\frac{n}{2}+3}, \cdots, y_n$. Now assign the labels $1, -1, -(n-1)$ to the vertices x, y, x_{n+1}. Next assign the labels $(n+2)$,$(n+3)$,$-(n+2)$,$-(n+3)$ to the vertices v_1,w_1,v_2,w_2 and assign the labels $(n+4)$,$(n+5)$,$-(n+4)$,$-(n+5)$ to the vertices v_{n+3},w_3,v_4,w_4. Proceeding like this until we reach v_{n^2-n-2}, w_{n^2-n-2}.

Therefore, f is pair difference cordial labeling of $K_{n,n+1} \cup (n^2-2n-2)K_2$. The maximum number of edges with labels 1 from the $DS(K_{n,n+1})$ is, $\Delta_{f_1} = 2n+1$.But the size of the $DS(K_{n,n+1})$ is n^2+3n+1. Hence n^2-n-2 is the least integer such that $DS(K_{n,n+1}) \cup (n^2-n-2)K_2$ is pair difference cordial.

Subcase 3.2 n is odd.

Take $m = n^2-n-2$. Define $f : V(DS(K_{n,n+1}) \cup (n^2-n-2)K_2) \to \{\pm 1, \pm 2, \cdots, \pm \lfloor \frac{2n+3}{2} \rfloor\}$ as follows:

Assign the labels $3, 5, 7, \cdots, n$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n-1}{2}}$ and assign the labels $2, 4, 6, \cdots, n+1$ to the vertices $x_1, x_2, x_3, \cdots, x_{\frac{n+1}{2}}$. Next assign the labels $-3, -5, -7, \cdots, -n$ to the vertices $x_{\frac{n+1}{2}+1}, x_{\frac{n+1}{2}+2}, x_{\frac{n+1}{2}+3}, \cdots, x_n$ and assign the labels $-2, -4, -6, \cdots, -(n+1)$ to the vertices $y_{\frac{n-1}{2}+1}, y_{\frac{n-1}{2}+2}, y_{\frac{n-1}{2}+3}, \cdots, y_n$. Now assign the labels $1, -1, -n$ to the vertices x, y, x_{n+1}. Next assign the labels $(n+2)$,$(n+3)$,$-(n+2)$,$-(n+3)$ to the vertices v_1,w_1,v_2,w_2 and assign the labels $(n+4)$,$(n+5)$,$-(n+4)$,$-(n+5)$ to the vertices v_{n+3},w_3,v_4,w_4. Proceeding like this until we reach v_{n^2-n-2}, w_{n^2-n-2}.

Therefore, f is pair difference cordial labeling of $K_{n,n+1} \cup (n^2-2n-2)K_2$. The maximum number of edges with labels 1 from the $DS(K_{n,n+1})$ is, $\Delta_{f_1} = 2n+1$.But the size of the $DS(K_{n,n+1})$ is n^2+3n+1. Hence n^2-n-2 is the least integer such that $DS(K_{n,n+1}) \cup (n^2-n-2)K_2$ is pair difference cordial. □

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A Counterexample to a Theorem about Orthogonal Latin Squares

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Abstract: We give a counterexample to a theorem of Vadiraja and Shankar about orthogonality of Latin squares induced by bivariate polynomials in $(\mathbb{Z}/n\mathbb{Z})[X, Y]$.

Key Words: Bivariate polynomials, Latin squares, orthogonal Latin squares.

AMS(2010): 08B15.

The topic of orthogonal Latin squares has a rich history dating back to Euler. The main result of a paper by Vadiraja and Shankar asserts that certain Latin squares are orthogonal to one another. In this note we give a counterexample to this result. We need some preliminaries in order to state the result.

Let n be a positive integer, write $R := \mathbb{Z}/n\mathbb{Z}$, and pick any polynomials $f(X, Y), g(X, Y) \in R[X, Y]$. Let S_f be the n-by-n matrix with rows and columns indexed by $0, 1, 2, \cdots, n-1$ and whose entry in row i and column j is $f(i, j)$. The matrix S_f is called a *Latin square* if, for each $c \in R$, each of the polynomials $f(X, c)$ and $f(c, Y)$ permutes R. If both S_f and S_g are Latin squares then these Latin squares are *orthogonal* if, for each choice of $u, v \in R$, there exist unique $i, j \in R$ for which $f(i, j) = u$ and $g(i, j) = v$. If S_f is a Latin square then we define its "mirror image" to be $S_{\widehat{f}}$ where $\widehat{f}(X, Y) := f(X, -1-Y)$. Note that $S_{\widehat{f}}$ is the matrix obtained from S_f by reversing the order of the entries in each row. It is clear that if S_f is a Latin square then also $S_{\widehat{f}}$ is a Latin square. In light of this, it is natural to ask when S_f and $S_{\widehat{f}}$ are orthogonal. It is easy to see that this never occurs when n is even [1, Theorem 2.3]. Theorem 2.9 of [1] and Theorem 6.2 of [2] each assert that it always occurs when n is odd.

Theorem A (Vadiraja–Shankar) *If n is odd and S_f is a Latin square then S_f and $S_{\widehat{f}}$ are orthogonal.*

However, Theorem A is not true in general. One counterexample to this conclusion is $f(X, Y) = -X^3Y^2 - X^2Y^3 - X^2Y + XY^2 + X + Y$ with $n = 5$. For, we have

$$\begin{aligned} f(X,0) &= X, & f(0,Y) &= Y, \\ f(X,1) &= -(X-1)^3, & f(1,Y) &= -Y^3+1, \\ f(X,2) &= X^3+2, & f(2,Y) &= (Y-2)^3, \end{aligned}$$

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$$f(X,3) = X^3 - 2, \qquad f(3,Y) = (Y+2)^3,$$
$$f(X,4) = -(X+1)^3, \qquad f(4,Y) = -Y^3 - 1.$$

Since X^3 permutes $Z/5\mathbb{Z}$, we see that S_f is a Latin square. But $f(0,0) = 0 = f(-1,-1)$ and

$$\widehat{f}(0,0) = f(0,-1) = -1 = f(-1,0) = \widehat{f}(-1,-1),$$

so that each of the pairs $(i,j) = (0,0)$ and $(i,j) = (-1,-1)$ satisfies $f(i,j) = 0$ and $\widehat{f}(i,j) = -1$. It follows that S_f and $S_{\widehat{f}}$ are not orthogonal. This concludes the proof that Theorem A is false.

In light of this counterexample, it is natural to reexamine the published proofs of Theorem A. The proof of Theorem 2.9 in [1] consists of restating the orthogonality condition (incorrectly) as pairwise distinctness of the pairs $\big(f(i,j), f(-1-i,j)\big)$ with $i,j \in R$, and then asserting without further justification that this distinctness follows from S_f being a Latin square.

The proof of Theorem 6.2 in [2] notes that there are n^2 distinct triples $\big(i,j,f(i,j)\big)$ with $i,j \in R$, and also n^2 distinct triples $\big(-1-i,j,f(i,j)\big)$ with $i,j \in R$, and then asserts orthogonality without further justification. Thus, the mistake in the proofs of both [1] and [2] is that the conclusion of Theorem A was claimed to follow at once from the hypothesis after an immediate reformulation, when in fact the hypothesis does not imply the conclusion.

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Corrigendum: Variations of Orthogonality of Latin Squares

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In the proof of Theorem 2.9 ([1]), we clearly mentioned that the result applies to bivariate linear polynomials. Unfortunately, the word *linear* was omitted from the theorem's statement and in the conclusion due to a typographical error. The corrected statement of the theorem should be

Theorem 2.9 *For odd n, Latin square over $\mathbb{Z}_n$ formed by a bivariate linear permutation polynomial $P(x, y)$ is orthogonal with its mirror image.*

The reason of corrected conclusion is as follows:

Identifying a pair of bivariate polynomials modulo n which represent a pair of orthogonal Latin squares is not obvious. But for odd n, a Latin square formed by a bivariate linear polynomial is orthogonal to its mirror image. Moreover, no two bivariate polynomials over $\mathbb{Z}_n$, when n is even can form orthogonal Latin squares.

And so, all words "*In the general case, $\cdots$. Hence they are orthogonal*" in the Proof of Theorem 2.9 should be deleted.

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Famous Words

Gravity explains the motions of the planets, but it can not explain who sets the planets in motion.

By *Isaac Newton*, a British physicist, mathematician and philosopher.

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