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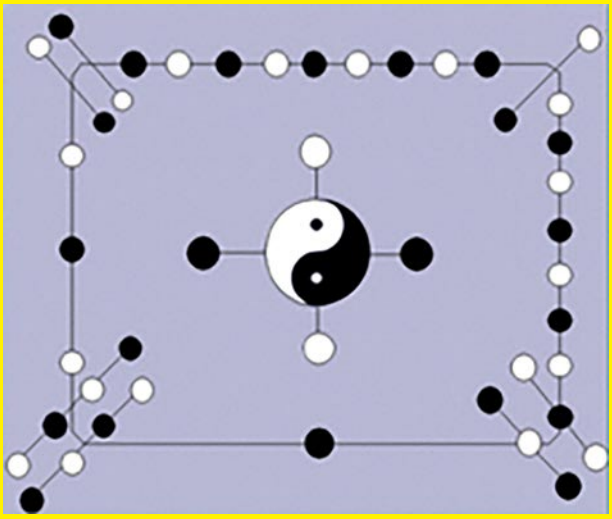

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Great works are performed not by strength, but by perseverance.

By *Samuel Johnson*, an English critic, biographer and essayist.

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A Novel Result of Coupled Fixed Point in Partially Ordered Partial Metric Spaces with Applications

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Abstract: In this work, we prove, a novel result of coupled fixed point involving rational type contractive condition in the setting of partially ordered complete partial metric spaces. Furthermore, we provide some consequences of the established results. To illustrate the results, an example is provided. Some applications of the main result and its consequences in terms of integral type contractions are also included. Our results extend, generalize and enrich several results from the existing literature (see, e.g., Bhaskar and Lakshmikantham [9] and others).

Key Words: Coupled fixed point, contractive type condition, partial metric space, partially ordered set.

AMS(2010): 47H10, 54H25.

§1. Introduction

As we known, Matthews [21] in 1994, introduced the notion of partial metric spaces to study the denotational semantics dataflow networks. In this space, the usual metric is replaced by partial metric with an interesting property that the self-distance of any point of space may not be zero. Later, Matthews proved the partial metric version of Banach fixed point theorem [5]. Heckmann [15] introduced the concept of weak partial metric function and established some fixed point results. Oltra and Valero [24] generalized the Matthews results in the sense of O'Neil [25] in complete partial metric space. Abdeljawad et al. [2] considered a general form of the weak ϕ-contraction and established some common fixed point results. Afterwards, many authors focused on partial metric spaces and its topological properties (see, e.g., ([3], [4], [17], [19], [28]).

On the other hand, Bhashkar and Lakshmikantham [9] (2006) introduced the notion of a coupled fixed point and proved some coupled fixed point theorems for mixed monotone mappings in ordered metric spaces and give application in the existence and uniqueness of a solution for periodic boundary value problem (see, also [14]. Later on, Ćirić and Lakshmikantham [11] (2009) investigated some more coupled fixed point theorems in partially ordered sets. Further, many authors have obtained coupled fixed point results for mappings under various contractive

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conditions in the setting of metric spaces and generalized metric spaces (see [4], [13], [16], [22], [23], [26], [27], [29], [32]).

In this paper, we prove a novel result of coupled fixed point involving rational type contraction condition in the setting of partially ordered complete partial metric spaces and provide some consequences of the established result. Moreover, to illustrate the result, an example is provided. Some applications of the main result and its consequences in terms of integral type contractions are also included. Our results extend, generalize and enrich several results from the existing literature (see, e.g., [9] and others).

§2. **Preliminaries**

In this section, we recall the notion of partial metric space and some of its basic results which will be needed in the sequel.

Definition 2.1([21]) *Let $Y \neq \emptyset$ be a set. A partial metric on Y is a function $p\colon Y \times Y \to [0, +\infty)$ such that for all $x, y, u \in Y$ the followings are satisfied:*

- $(p1)$ $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
- $(p2)$ $p(x,x) \leq p(x,y)$;
- $(p3)$ $p(x,y) = p(y,x)$;
- $(p4)$ $p(x,y) \leq p(x,u) + p(u,y) - p(u,u)$.

Then, p is called a partial metric on Y and the pair (Y,p) is called a partial metric space (in short PMS).

It is clear that if $p(x,y) = 0$, then from $(p1)$, $(p2)$, and $(p3)$, $x = y$. But if $x = y$, $p(x,y)$ may not be 0. Furthermore, if p is a partial metric on Y, then the function $d^p\colon Y \times Y \to [0, +\infty)$ given by

$$d^p(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{2.1}$$

is a usual metric on Y.

Each partial metric p on Y generates a T_0 topology τ_p on Y with the family of open p-balls $\{B_p(x,\varepsilon) : x \in Y, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in Y : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in Y$ and $\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[x,\varepsilon] = \{y \in Y : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in Y$ and $\varepsilon > 0$.

Example 2.2 We know some special cases of PMS in the following:

$(*_1)$ Let $Y = [0, +\infty)$ and $p\colon Y \times Y \to [0, +\infty)$ be given by $p(x,y) = \max\{x,y\}$ for all $x, y \in Y$. Then (Y,p) is a partial metric space (see, [1]).

$(*_2)$ Let $I = Y$, where I denote the set of all intervals $[x_1, y_1]$ for any real numbers $x_1 \leq y_1$. Let $p\colon Y \times Y \to [0, \infty)$ be a function such that $p\big([x_1,y_1],[x_2,y_2]\big) = \max\{y_1,y_2\} - \min\{x_1,x_2\}$. Then (Y,p) is a partial metric space (see, [1]).

$(*_3)$ Let $Y = \mathbb{R}$ and $p\colon Y \times Y \to \mathbb{R}^+$ be given by $p(x,y) = e^{\max\{x,y\}}$ for all $x, y \in Y$. Then (Y,p) is a partial metric space (see, [12]).

Definition 2.3([21]) *Let (Y,p) be a PMS.*

$(a1)$ *A sequence $\{x_n\}$ converges to a point $x\in Y$ if and only if $\lim_{n\to\infty} p(x,x_n)=p(x,x)$;*

$(a2)$ *A sequence $\{x_n\}$ in Y is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(x_m,x_n)$ exists (and finite);*

$(a3)$ *A PMS (Y,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in Y converges, with respect to τ_p, to a point $x\in Y$, such that, $\lim_{m,n\to\infty} p(x_m,x_n)=p(x,x)$;*

$(a4)$ *A mapping $S\colon Y\to Y$ is said to be continuous at $x_0\in Y$ if for every $\varepsilon>0$, there exists $c>0$ such that $S\big(B_p(x_0,c)\big)\subset B_p\big(S(x_0),\varepsilon\big)$.*

Definition 2.4([21]) *A PMS (Y,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in Y converges to a point $x\in Y$ with respect to τ_p. Furthermore,*

$$\lim_{m,n\to\infty} p(x_m,x_n)=\lim_{n\to\infty} p(x_n,x)=p(x,x).$$

Definition 2.5([6, 7, 8]) *Consider a function $\psi\colon [0,+\infty)\to[0,+\infty)$ satisfying*

(i) *ψ is monotone increasing;*

(ii) *$\psi^n(t)\to 0$, as $n\to\infty$;*

(iii) *$\sum_{n=0}^{\infty}\psi^n(t)$ converges for all $t>0$*

and define

(1) *A function ψ satisfying (i) and (ii) above is called a comparison function;*

(2) *A function ψ satisfying (i) and (iii) above is called a (c)-comparison function.*

Remark 2.6([7, 8]) By definition, we know that

(i) Any (c)-comparison function is a comparison function;

(ii) Every comparison function satisfies $\psi(0)=0$.

Definition 2.7([9]) *Let $(Y,\leq)$ be a partially ordered set. The mapping $F\colon Y\times Y\to Y$ is said to have the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y\in Y$,*

$$x_1,x_2\in Y,\ \ x_1\leq x_2\ \Rightarrow\ F(x_1,y)\leq F(x_2,y),$$

and

$$y_1,y_2\in Y,\ \ y_1\leq y_2\ \Rightarrow\ F(x,y_1)\geq F(x,y_2).$$

Definition 2.8([9 ,11]) *An element $(x,y)\in Y\times Y$ is said to be a coupled fixed point of the mapping $F\colon Y\times Y\to Y$ if $F(x,y)=x$ and $F(y,x)=y$.*

Example 2.9 Let $Y=[0,+\infty)$ and $F\colon Y\times Y\to Y$ be defined by $F(x,y)=\frac{x+y}{3}$ for all $x,y\in Y$. Then F has a unique coupled fixed point $(0,0)$.

Example 2.10 Let $Y=[0,+\infty)$ and $F\colon Y\times Y\to Y$ be defined by $F(x,y)=\frac{x+y}{2}$ for all

$x, y \in Y$. Then F has two coupled fixed point $(0,0)$ and $(1,1)$, that is, the coupled fixed point is not unique.

Lemma 2.11([4, 20, 21]) *(b1) A sequence $\{x_n\}$ is Cauchy in a PMS (Y,p) if and only if $\{x_n\}$ is Cauchy in a metric space (Y, d^p) where*

$$d^p(x,y) = 2p(x,y) - p(x,x) - p(y,y).$$

(b2) A PMS (Y,p) is complete if a metric space (Y, d^p) is complete, i.e.,

$$\lim_{n\to\infty} d^p(x_n, x) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Lemma 2.12([18]) *Let (Y,p) be a PMS.*

(c1) If $x, y \in Y$, $p(x,y) = 0$, then $x = y$;
(c2) If $x \neq y$, then $p(x,y) > 0$.

One of the characterization of continuity of mappings in partial metric spaces was given by Samet et al. [30] as follows.

Lemma 2.13([30]) *Let (Y,p) be a PMS. The function $R\colon Y \to Y$ is continuous if given a sequence $\{x_n\}_{n\in\mathbb{N}}$ and $x \in Y$ such that $p(x,x) = \lim_{n\to\infty} p(x, x_n)$, then $p(Rx, Rx) = \lim_{n\to\infty} p(Rx, Rx_n)$.*

Example 2.14([30]) Let $Y = [0, +\infty)$ endowed with the partial metric $p\colon Y \times Y \to [0, +\infty)$ defined by $p(x,y) = \max\{x, y\}$ for all $x, y \in Y$. Let $R\colon Y \to Y$ be a non-decreasing function. If R is continuous with respect to the standard metric $d(x,y) = |x - y|$ for all $x, y \in Y$, then R is continuous with respect to the partial metric p.

Example 2.15([12]) Let $x_n \to x$ as $n \to \infty$ in a PMS (Y,p) where $p(x,x) = 0$. Then $\lim_{n\to\infty} p(x_n, u) = p(x,u)$ for all $u \in Y$.

§3. Main Results

In this section, we shall prove a novel coupled fixed point result involving rational type contraction condition in the setting of partially ordered complete partial metric spaces.

Theorem 3.1 *Let $(Y, p, \leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$, for all $a, b, u, v \in Y$, $p(u, F(u,v)) + p(a,u) > 0$ and ψ, a (c)-comparison function, we have*

$$p(F(a,b), F(u,v)) \leq \beta\Big(\frac{p(a, F(a,b))p(a, F(u,v))p(u, F(a,b))}{1 + p(u, F(u,v)) + p(a,u)}\Big) + \psi(p(a,u)). \tag{3.1}$$

If there exist two elements $a_0, b_0 \in Y$ with $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a coupled fixed point in Y with $p(z, z) = 0$ for some $z \in Y$.

Proof Choose $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$. We are to prove that a_k is non-decreasing and b_k is non-increasing. That is, for all $k \geq 0$,

$$a_{2k} \leq a_{2k+1} \leq a_{2k+2} \text{ and } b_{2k} \leq b_{2k+1} \leq b_{2k+2}.$$

Firstly, $a_0 \leq F(a_0, b_0) = a_1$ and $b_0 \geq F(b_0, a_0) = b_1$. Then, $a_0 \leq a_1$ and $b_0 \geq b_1$. Again, let $a_2 = F(a_1, b_1)$ and $b_2 = F(b_1, a_1)$. Since F has the mixed monotone property on Y, then we have $a_1 \leq a_2$ and $b_1 \geq b_2$. Repeating the above process, we get two sequences $\{a_k\}$ and $\{b_k\}$ in Y such that $a_{2k+1} = F(a_{2k}, b_{2k})$ and $b_{2k+1} = F(b_{2k}, a_{2k})$ for all $k \geq 0$ and

$$a_0 \leq a_1 \leq \cdots \leq a_{2k} \leq a_{2k+1} \leq \cdots, \quad b_0 \geq b_1 \geq \cdots \geq b_{2k} \geq b_{2k+1} \geq \cdots. \tag{3.2}$$

Now, using equation (3.1) with $a = a_{2k}$, $b = b_{2k}$, $u = a_{2k+1}$ and $v = b_{2k+1}$, we have

$$\begin{aligned} p(a_{2k+1}, a_{2k+2}) &= p\big(F(a_{2k}, b_{2k}), F(a_{2k+1}, b_{2k+1})\big) \\ &\leq \beta\Big(\frac{p(a_{2k}, F(a_{2k}, b_{2k}))p(a_{2k}, F(a_{2k+1}, b_{2k+1}))p(a_{2k+1}, F(a_{2k}, b_{2k}))}{1 + p(a_{2k+1}, F(a_{2k+1}, b_{2k+1})) + p(a_{2k}, a_{2k+1})}\Big) \\ &\quad + \psi(p(a_{2k}, a_{2k+1})) \\ &= \beta\Big(\frac{p(a_{2k}, a_{2k+1})p(a_{2k}, a_{2k+2})p(a_{2k+1}, a_{2k+1})}{1 + p(a_{2k+1}, a_{2k+2}) + p(a_{2k}, a_{2k+1})}\Big) \\ &\quad + \psi(p(a_{2k}, a_{2k+1})) \\ &= \psi(p(a_{2k}, a_{2k+1})). \end{aligned}$$

Therefore,

$$p(a_{2k+1}, a_{2k+2}) \leq \psi(p(a_{2k}, a_{2k+1})) \tag{3.3}$$

and

$$p(b_{2k+1}, b_{2k+2}) \leq \psi(p(b_{2k}, b_{2k+1})). \tag{3.4}$$

Similarly, proceeding as above we obtain

$$p(a_{2k+2}, a_{2k+3}) \leq \psi(p(a_{2k+1}, a_{2k+2})) \tag{3.5}$$

and

$$p(b_{2k+2}, b_{2k+3}) \leq \psi(p(b_{2k+1}, b_{2k+2})). \tag{3.6}$$

Hence, it can be deduced from equations (3.3)-(3.6) that

$$\begin{aligned} p(a_{2k+2}, a_{2k+3}) + p(b_{2k+2}, b_{2k+3}) &\leq \psi(p(a_{2k+1}, a_{2k+2})) + \psi(p(b_{2k+1}, b_{2k+2})) \\ &\leq \psi^2(p(a_{2k}, a_{2k+1})) + \psi^2(p(b_{2k}, b_{2k+1})) \\ &\leq \cdots \leq \psi^n(p(a_0, a_1)) + \psi^n(p(b_0, b_1)). \end{aligned}$$

Thus, it follows that

$$p(a_n, a_{n+1}) + p(b_n, b_{n+1}) \leq \psi^n(p(a_0, a_1)) + \psi^n(p(b_0, b_1)). \tag{3.7}$$

Again, let $n, r \in \mathbb{N}$. Using equation (3.7) inductively and repeated application of triangular inequality, we obtain

$$\begin{aligned} p(a_n, a_{n+r}) + p(b_n, b_{n+r}) &\leq [p(a_n, a_{n+1}) + p(b_n, b_{n+1})] + [p(a_{n+1}, a_{n+2}) + p(b_{n+1}, b_{n+2})] \\ &\quad + \cdots + [p(a_{n+r-1}, a_{n+r}) + p(b_{n+r-1}, b_{n+r})] \\ &\quad - [p(a_{n+1}, a_{n+1}) + p(b_{n+1}, b_{n+1})] \\ &\quad - [p(a_{n+2}, a_{n+2}) + p(b_{n+2}, b_{n+2})] - \cdots \\ &\quad - [p(a_{n+r-1}, a_{n+r-1}) + p(b_{n+r-1}, b_{n+r} - 1)] \\ &\leq [p(a_n, a_{n+1}) + p(b_n, b_{n+1})] + [p(a_{n+1}, a_{n+2}) + p(b_{n+1}, b_{n+2})] \\ &\quad + \cdots + [p(a_{n+r-1}, a_{n+r}) + p(b_{n+r-1}, b_{n+r})] \\ &\leq \psi^n(p(a_0, a_1)) + \psi^n(p(b_0, b_1)) + \psi^{n+1}(p(a_0, a_1)) + \psi^{n+1}(p(b_0, b_1)) \\ &\quad + \cdots + \psi^{n+r-1}(p(a_0, a_1)) + \psi^{n+r-1}(p(b_0, b_1)) \\ &= \sum_{j=n}^{n+r-1} \psi^j(p(a_0, a_1)) + \sum_{j=n}^{n+r-1} \psi^j(p(b_0, b_1)) \\ &= \sum_{j=0}^{n+r-1} \psi^j(p(a_0, a_1)) - \sum_{j=0}^{n-1} \psi^j(p(a_0, a_1)) \\ &\quad + \sum_{j=0}^{n+r-1} \psi^j(p(b_0, b_1)) - \sum_{j=0}^{n-1} \psi^j(p(b_0, b_1)) \to 0 \ \text{ as } \ j \to \infty \end{aligned} \tag{3.8}$$

and since ψ is a (c)-comparison function. Hence, $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences in a PMS (Y, p) such that $\lim_{m,n\to\infty} p(a_m, a_n) = 0$ and $\lim_{m,n\to\infty} p(b_m, b_n) = 0$ where $m = n + r$ with $m > n \in \mathbb{N}$. Again, since (Y, p) is a complete partial metric space, there exist $a^*, b^* \in Y$ such that

$$p(a^*, a^*) = \lim_{n\to\infty} p(a_n, a^*) = \lim_{n\to\infty} p(a_n, a_n) = 0 \tag{3.9}$$

and

$$p(b^*, b^*) = \lim_{n\to\infty} p(b_n, b^*) = \lim_{n\to\infty} p(b_n, b_n) = 0. \tag{3.10}$$

Now, we have to show that (a^*, b^*) is a coupled fixed point of F, that is, $a^* = F(a^*, b^*)$ and $b^* = F(b^*, a^*)$. Using contractive condition (3.1) again and noting that $\psi(0) = 0$, we have

$$\begin{aligned} p(F(a^*, b^*), a^*) &\leq p(F(a^*, b^*), a_{2n+1}) + p(a_{2n+1}, a^*) - p(a_{2n+1}, a_{2n+1}) \\ &\leq p(F(a^*, b^*), a_{2n+1}) + p(a_{2n+1}, a^*) \\ &= p(F(a^*, b^*), F(a_{2n}, b_{2n})) + p(a_{2n+1}, a^*) \\ &\leq \beta\Big(\frac{p(a^*, F(a^*, b^*))p(a^*, F(a_{2n}, b_{2n}))p(a_{2n}, F(a^*, b^*))}{1 + p(a_{2n}, F(a_{2n}, b_{2n})) + p(a^*, a_{2n})}\Big) \\ &\quad + \psi(p(a^*, a_{2n})) + p(a_{2n+1}, a^*) \end{aligned}$$

$$
\begin{aligned}
&\leq \beta\Big(\frac{p(a^*, F(a^*, b^*))p(a^*, a_{2n+1})p(a_{2n}, F(a^*, b^*))}{1 + p(a_{2n}, a_{2n+1}) + p(a^*, a_{2n})}\Big) \\
&\quad + \psi(p(a^*, a_{2n})) + p(a_{2n+1}, a^*) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Therefore, $p(F(a^*, b^*), a^*) = 0$ and so $F(a^*, b^*) = a^*$. Similarly, by using inequality (3.1), we can show that $p(F(b^*, a^*), b^*) = 0$ and so $F(b^*, a^*) = b^*$. Hence, (a^*, b^*) is a coupled fixed point of F. This completes the proof. □

Example 3.2 Let $Y = [0, 4]$ be endowed with the usual partial metric given by $p(x, y) = \max\{x, y\}$ for all $x, y \in Y$. Let $\psi(t) = \frac{1}{2}t$ for all $t \in Y$. Clearly, $\psi(t)$ is a (c)-comparison function. Define $F\colon [0, 4] \times [0, 4] \to [0, 4]$ by $F(a, b) = 3a - 2b$ for all $a, b \in [0, 4]$. Clearly, F has the mixed monotone property.

Let $a_0 = \frac{2}{3}$, $b_0 = \frac{1}{2} \in Y$.

$$F(a_0, b_0) = F\Big(\frac{2}{3}, \frac{1}{2}\Big) = 1 \text{ and } F(b_0, a_0) = F\Big(\frac{1}{2}, \frac{2}{3}\Big) = \frac{1}{6}.$$

Thus,

$$\frac{2}{3} < 1 \text{ and } \frac{1}{2} > \frac{1}{6}.$$

Hence, $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$.

Now, consider the contractive condition (3.1) in Theorem 3.1. Let $a = \frac{4}{3}$, $b = \frac{1}{3}$, $\beta = 2$, $u = \frac{4}{5}$, $v = \frac{1}{5}$ and assume that $a \geq b \geq u \geq v$. Then

$$
\begin{aligned}
F(a, b) &= F\Big(\frac{4}{3}, \frac{1}{3}\Big) = 3\Big(\frac{4}{3}\Big) - 2\Big(\frac{1}{3}\Big) = 4 - \frac{2}{3} = \frac{10}{3}, \\
F(u, v) &= F\Big(\frac{4}{5}, \frac{1}{5}\Big) = 3\Big(\frac{4}{5}\Big) - 2\Big(\frac{1}{5}\Big) = \frac{12}{5} - \frac{2}{5} = 2, \\
p(a, F(a, b)) &= p\Big(\frac{4}{3}, \frac{10}{3}\Big) = \max\Big\{\frac{4}{3}, \frac{10}{3}\Big\} = \frac{10}{3}, \\
p(u, F(u, v)) &= p\Big(\frac{4}{5}, 2\Big) = \max\Big\{\frac{4}{5}, 2\Big\} = 2, \\
p(u, F(a, b)) &= p\Big(\frac{4}{5}, \frac{10}{3}\Big) = \max\Big\{\frac{4}{5}, \frac{10}{3}\Big\} = \frac{10}{3}, \\
p(a, F(u, v)) &= p\Big(\frac{4}{3}, 2\Big) = \max\Big\{\frac{4}{3}, 2\Big\} = 2, \\
p(a, u) &= p\Big(\frac{4}{3}, \frac{4}{5}\Big) = \max\Big\{\frac{4}{3}, \frac{4}{5}\Big\} = \frac{4}{3}.
\end{aligned}
$$

Now, applying contractive condition (3.1), that is,

$$p(F(a, b), F(u, v)) \leq \beta\Big(\frac{p(a, F(a, b))p(a, F(u, v))p(u, F(a, b))}{1 + p(u, F(u, v)) + p(a, u)}\Big) + \psi(p(a, u)),$$

implies that

$$\begin{aligned} p\Big(\frac{10}{3}, 2\Big) &= \max\Big\{\frac{10}{3}, 2\Big\} = \frac{10}{3} = 3.333 \\ &\leq 2\Big(\frac{\frac{10}{3}.2.\frac{10}{3}}{1+2+\frac{4}{3}}\Big) + \frac{1}{2}.\frac{4}{3} \\ &= \frac{400}{69} + \frac{2}{3} = \frac{1338}{207} = 6.643, \end{aligned}$$

i.e. $3.333 \leq 6.643$.

Since all the assumptions of Theorem 3.1 are satisfied, so F has a coupled fixed point in $Y = [0, 4]$.

If we take $\psi(t) = \gamma t$ for all $t > 0$ where $\gamma \in [0, 1)$ in Theorem 3.1, then we have the following result.

Corollary 3.3 *Let $(Y, p, \leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$, $\gamma \in [0, 1)$ and for all $a, b, u, v \in Y$, where $p(u, F(u, v)) + p(a, u) > 0$, we have*

$$p(F(a,b), F(u,v)) \leq \beta\Big(\frac{p(a, F(a,b))p(a, F(u,v))p(u, F(a,b))}{1 + p(u, F(u,v)) + p(a,u)}\Big) + \gamma\, p(a,u). \tag{3.11}$$

If there exist two elements $a_0, b_0 \in Y$ with $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a coupled fixed point in Y with $p(z, z) = 0$ for some $z \in Y$.

Proof It follows from Theorem 3.1 by taking $\psi(t) = \gamma t$ for all $t > 0$ where $\gamma \in [0, 1)$. □

If we take $\gamma = 0$ in Corollary 3.3, then we have the following result.

Corollary 3.4 *Let $(Y, p, \leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$ and for all $a, b, u, v \in Y$, where $p(u, F(u, v)) + p(a, u) > 0$, we have*

$$p(F(a,b), F(u,v)) \leq \beta\Big(\frac{p(a, F(a,b))p(a, F(u,v))p(u, F(a,b))}{1 + p(u, F(u,v)) + p(a,u)}\Big). \tag{3.12}$$

If there exist $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a coupled fixed point in Y with $p(z, z) = 0$ for some $z \in Y$.

Proof It follows from Corollary 3.3 by taking $\gamma = 0$. □

If we take $\beta = 0$ in Corollary 3.3, then we have the following result.

Corollary 3.5 *Let $(Y, p, \leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\gamma \in [0, 1)$ and for all $a, b, u, v \in Y$, we have*

$$p(F(a,b), F(u,v)) \leq \gamma\, p(a,u). \tag{3.13}$$

If there exist $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a coupled fixed point in Y with $p(z,z) = 0$ for some $z \in Y$.

If we take $\beta = 0$ in Theorem 3.1, then we have the following result.

Corollary 3.6 *Let $(Y, p, \leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for all $a, b, u, v \in Y$, $p(u, F(u,v)) + p(a,u) > 0$ and ψ, a (c)-comparison function, we have*

$$p(F(a,b), F(u,v)) \leq \psi(p(a,u)). \tag{3.14}$$

If there exist two elements $a_0, b_0 \in Y$ with $a_0 \leq F(a_0, b_0)$ and $b_0 \geq F(b_0, a_0)$, then F has a coupled fixed point in Y.

If we define as $Ta = F(a,a)$ in Corollary 3.5, then we have the following result.

Corollary 3.7([21],Banach's fixed point theorem) *Let (Y, p) be a complete partial metric space. Suppose that the mapping $T\colon Y \to Y$ satisfies the following contractive condition for all $a, u \in Y$:*

$$p(Ta, Tu) \leq \gamma\, p(a,u), \tag{3.15}$$

where $\gamma \in [0,1)$ is a constant. Then T has a unique fixed point in Y.

Remark 3.8 Theorem 3.1 generalizes and extends the corresponding result of Bhaskar and Lakshmikantham [9] from complete metric space to complete partial metric space.

Remark 3.9 Theorem 3.1 also generalizes and extends the corresponding result of Sabetghadam et al. [31] to the coupled fixed point setting in partially ordered space, the latter consisting of coupled fixed point in partial cone metric space setting.

§4. Applications

In this part, applications of the main result and its consequences in terms of integral type contractions are carried out. Let Ψ denote the set of functions $\varphi\colon [0,+\infty) \to [0,+\infty)$ satisfying the following properties:

(Δ_1) φ is a Lebesgue-integrable function on every compact subset in $[0,+\infty)$ and;

(Δ_2) $\int_0^\varepsilon \varphi(\mu)d\mu > 0$, for all $\varepsilon > 0$.

Then, we have the following applications of our results.

Theorem 4.1 *Let $(Y, p, \leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$, $p(u, F(u,v)) +$*

$p(a,u) > 0$ and ψ, a (c)-comparison function, we have

$$\int_0^{p(F(a,b),F(u,v))} \chi(\mu)d\mu \leq \beta \int_0^{\left(\frac{p(a,F(a,b))p(a,F(u,v))p(u,F(a,b))}{1+p(u,F(u,v))+p(a,u)}\right)} \chi(\mu)d\mu + \int_0^{\psi(p(a,u))} \chi(\mu)d\mu \quad (4.1)$$

for all $a,b,u,v \in Y$ such that $a \geq u$ and $b \leq v$, where $\chi \in \Psi$. If there exist $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z) = 0$ for some $z \in Y$.

If $\psi(t) = \gamma t$ for all $t > 0$, where $\gamma \in [0,1)$ in Theorem 4.1, then we obtain the following result.

Theorem 4.2 *Let $(Y,p,\leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$ and $\gamma \in [0,1)$, where $p(u,F(u,v)) + p(a,u) > 0$, we have*

$$\int_0^{p(F(a,b),F(u,v))} \chi(\mu)d\mu \leq \beta \int_0^{\left(\frac{p(a,F(a,b))p(a,F(u,v))p(u,F(a,b))}{1+p(u,F(u,v))+p(a,u)}\right)} \chi(\mu)d\mu + \gamma \int_0^{p(a,u)} \chi(\mu)d\mu \quad (4.2)$$

for all $a,b,u,v \in Y$ such that $a \geq u$ and $b \leq v$, where $\chi \in \Psi$. If there exist $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z) = 0$ for some $z \in Y$.

If $\beta = 0$ in Theorem 4.1, then we obtain the following result.

Theorem 4.3 *Let $(Y,p,\leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that $p(u,F(u,v)) + p(a,u) > 0$ and ψ, a (c)-comparison function, we have*

$$\int_0^{p(F(a,b),F(u,v))} \chi(\mu)d\mu \leq \int_0^{\psi(p(a,u))} \chi(\mu)d\mu, \quad (4.3)$$

for all $a,b,u,v \in Y$ such that $a \geq u$ and $b \leq v$, where $\chi \in \Psi$. If there exist $a_0, b_0 \in Y$ such that $a_0 \leq F(a_0,b_0)$ and $b_0 \geq F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z) = 0$ for some $z \in Y$.

If $\gamma = 0$ in Theorem 4.2, then we obtain the following result.

Theorem 4.4 *Let $(Y,p,\leq)$ be a partially ordered complete partial metric space. Let $F\colon Y \times Y \to Y$ be a mapping having the mixed monotone property such that for some $\beta \geq 0$ and*

$p(u,F(u,v))+p(a,u)>0$, we have

$$\int_0^{p(F(a,b),F(u,v))}\chi(\mu)d\mu \leq \beta\int_0^{\left(\frac{p(a,F(a,b))p(a,F(u,v))p(u,F(a,b))}{1+p(u,F(u,v))+p(a,u)}\right)}\chi(\mu)d\mu, \tag{4.4}$$

for all $a,b,u,v\in Y$ such that $a\geq u$ and $b\leq v$, where $\chi\in\Psi$. If there exist $a_0,b_0\in Y$ such that $a_0\leq F(a_0,b_0)$ and $b_0\geq F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z)=0$ for some $z\in Y$.

If $\beta=0$ in Theorem 4.2, then we obtain the following result.

Theorem 4.5 *Let $(Y,p,\leq)$ be a partially ordered complete partial metric space. Let $F\colon Y\times Y\to Y$ be a mapping having the mixed monotone property such that for some $\gamma\in[0,1)$, we have*

$$\int_0^{p(F(a,b),F(u,v))}\chi(\mu)d\mu \leq \gamma\int_0^{p(a,u)}\chi(\mu)d\mu, \tag{4.5}$$

for all $a,b,u,v\in Y$ such that $a\geq u$ and $b\leq v$, where $\chi\in\Psi$. If there exist $a_0,b_0\in Y$ such that $a_0\leq F(a_0,b_0)$ and $b_0\geq F(b_0,a_0)$, then F has a coupled fixed point in Y with $p(z,z)=0$ for some $z\in Y$.

Remark 4.6 Theorem 4.5 extends and generalizes the corresponding result of Branciari [10] from complete metric spaces to partially ordered complete partial metric spaces and coupled fixed point.

§5. **Conclusion**

In this article, we prove a novel coupled fixed point result for contractive type condition involving rational term in the setting of partially ordered complete partial metric spaces. Moreover, we give some consequences of the established results and provide an illustrative example in support of the established result. Some applications of the main result and its consequences in terms of integral type contractions are also included. The results presented in this paper extend and generalize several results in the existing literature (see, e.g., [9] and others).

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The General Leap-Zagreb-Type Indices of Some Chemical Graphs*

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Abstract: The leap Zagreb indices have recently found remarkable applications in predicting the physico-chemical properties of alkanes and benzene systems. Subsequently, topological index related to the leap Zagreb indices emerge endlessly. In order to unify the study of these topological indices, we introduce the general leap Zagreb-type indices from the point of mathematics, and give the calculation formulas of the general leap Zagreb-type indices for some chemical trees, chemical chains, benzenoid systems and nanostructures, which extends the known results.

Key Words: Topological index, general leap-Zagreb-type indices, chemical graph.

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§1. Introduction

Throughout the article, G is a simple undirected connected graph with vertex set $V(G)$ and edge set $E(G)$. The distance $d(u,v)$ between any two vertices u and v of a graph G is equal to the length of the shortest path connecting them. The 2-distance degree of a vertex u, denoted by τ_u, is the number of vertices $v \in V(G)$ such that $d(u,v) = 2$. Let $K = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq j \leq n-2\}$ and $m(i,j)$ be the number of edges in G joining vertices of 2-distance degree i and j.

In 2017, the leap Zagreb indices of a graph are introduced by Naji, Soner and Gutman [11] based on the 2-distance degree of the vertices. For a graph G, the first, second and third leap Zagreb indices are defined by

$$LM_1(G) = \sum_{u \in V(G)} \tau_u^2, \quad LM_2(G) = \sum_{uv \in E(G)} \tau_u \tau_v, \quad LM_3(G) = \sum_{uv \in E(G)} (\tau_u + \tau_v).$$

The leap Zagreb indices attracted a considerable attention in the researchers' cycles, one may refer to [3, 9, 10, 13, 15, 16] and the references therein. In particular, Basavanagoud, Mondal and Das et al. [2, 4, 8] showed that the leap Zagreb indices have very good correlation with physical properties of chemical compound like entropy, boiling point, accentric factor,

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enthalpy of vaporization and standard enthalpy of vaporization. Inspired by the leap Zagreb indices, many related topological indices have been proposed and studied by scholars, such as

$$\begin{aligned}
HLM_1(G) &= \sum_{uv\in E(G)} (\tau_u+\tau_v)^2 \ ([6]), & HLM_2(G) &= \sum_{uv\in E(G)} (\tau_u\tau_v)^2 \ ([6]),\\
SL(G) &= \sum_{uv\in E(G)} \frac{1}{\sqrt{\tau_u+\tau_v}} \ ([7]), & GAL(G) &= \sum_{uv\in E(G)} \frac{2\sqrt{\tau_u\tau_v}}{\tau_u+\tau_v} \ ([7]),\\
AL_4(G) &= \sum_{uv\in E(G)} \sqrt{\tau_u\tau_v} \ ([14]), & AL_5(G) &= \sum_{uv\in E(G)} \frac{1}{\sqrt{\tau_u\tau_v}} \ ([14]),\\
AL_6(G) &= \sum_{uv\in E(G)} \left(\frac{\tau_u}{\tau_v}+\frac{\tau_v}{\tau_u}\right) \ ([14]), & AL_7(G) &= \sum_{uv\in E(G)} \frac{\tau_u\tau_v}{\tau_u+\tau_v} \ ([14]),\\
LSO(G) &= \sum_{uv\in E(G)} \sqrt{\tau_u^2+\tau_v^2} \ ([5]), & HLF(G) &= \sum_{uv\in E(G)} (\tau_u^2+\tau_v^2)^2 \ ([5]),\\
LF(G) &= \sum_{uv\in E(G)} (\tau_u^2+\tau_v^2) \ ([5]), & LY(G) &= \sum_{uv\in E(G)} (\tau_u^3+\tau_v^3) \ ([15]).
\end{aligned}$$

In order to unify the study of these topological indices, the general leap Zagreb-type indices of a connected graph G are introduced, and defined as

$$LZ_{\alpha,\beta}(G) = \sum_{uv\in E(G)} (\tau_u^\alpha\tau_v^\beta+\tau_u^\beta\tau_v^\alpha), \quad LRZ_{\alpha,\beta,\gamma}(G) = \sum_{uv\in E(G)} (\tau_u\tau_v)^\alpha(\tau_u^\beta+\tau_v^\beta)^\gamma,$$

where α, β and γ are arbitrary real numbers. It is clear that, the topological indices discussed previously, can be obtained from the general leap Zagreb-type indices for some particular values of α, β and γ. In this paper, we compute the expressions of the general leap Zagreb-type indices for some molecular trees, chemical chains, benzenoid systems and nanostructures, which extends the results of the references [1, 5, 12].

§2. General Leap Zagreb-Type Indices of Some Chemical Trees

The structures of three types molecular trees with n vertices are shown in Figure 1.

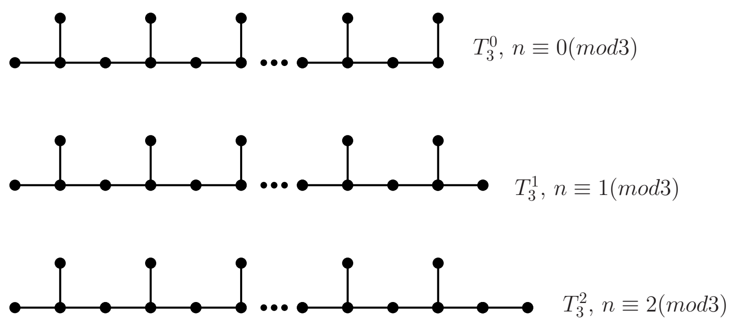

Figure 1 Three types molecular trees with n vertices.

Theorem 2.1 *The general leap Zagreb-type indices of T_3^0, shown in Figure 1, are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(T_3^0) &= \frac{2n-15}{3}\cdot 2^{\alpha+\beta}(2^\alpha+2^\beta)+\frac{n-6}{3}\cdot 2^{1+\alpha+\beta}+2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha \\ &\quad +2^{\alpha+1}+2^{\beta+1}+2^{2\alpha}+2^{2\beta}+3^\alpha+3^\beta+2, \\ LRZ_{\alpha,\beta,\gamma}(T_3^0) &= 2^{\alpha+1}(1+2^\beta)^\gamma+2^{2\alpha}(1+4^\beta)^\gamma+\frac{2n-15}{3}\cdot 2^{3\alpha+\beta\gamma}(1+2^\beta)^\gamma \\ &\quad +\frac{n-6}{3}\cdot 2^{2\alpha+(\beta+1)\gamma}+6^\alpha(2^\beta+3^\beta)^\gamma+3^\alpha(1+3^\beta)^\gamma+2^\gamma. \end{aligned}$$

Proof By the definition of T_3^0, we obtain the basic information on T_3^0 in the following table.

$m(1,2)$	$m(1,4)$	$m(2,4)$	$m(2,2)$	$m(2,3)$	$m(1,3)$	$m(1,1)$
2	1	$\frac{2n-15}{3}$	$\frac{n-6}{3}$	1	1	1

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(T_3^0) &= 2(1^\alpha\cdot 2^\beta+1^\beta\cdot 2^\alpha)+(1^\alpha\cdot 4^\beta+1^\beta\cdot 4^\alpha)+\frac{2n-15}{3}\cdot(2^\alpha\cdot 4^\beta+2^\beta\cdot 4^\alpha) \\ &\quad +\frac{n-6}{3}\cdot(2^\alpha\cdot 2^\beta+2^\beta\cdot 2^\alpha)+(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+(1^\alpha\cdot 3^\beta+1^\beta\cdot 3^\alpha) \\ &\quad +(1^\alpha\cdot 1^\beta+1^\beta\cdot 1^\alpha) \\ &= \frac{2n-15}{3}\cdot 2^{\alpha+\beta}(2^\alpha+2^\beta)+\frac{n-6}{3}\cdot 2^{1+\alpha+\beta}+2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha \\ &\quad +2^{\alpha+1}+2^{\beta+1}+2^{2\alpha}+2^{2\beta}+3^\alpha+3^\beta+2, \end{aligned}$$

$$\begin{aligned} LRZ_{\alpha,\beta,\gamma}(T_3^0) &= 2(1\cdot 2)^\alpha(1^\beta+2^\beta)^\gamma+(1\cdot 4)^\alpha(1^\beta+4^\beta)^\gamma+\frac{2n-15}{3}\cdot(2\cdot 4)^\alpha(2^\beta+4^\beta)^\gamma \\ &\quad +\frac{n-6}{3}\cdot(2\cdot 2)^\alpha(2^\beta+2^\beta)^\gamma+(2\cdot 3)^\alpha(2^\beta+3^\beta)^\gamma+(1\cdot 3)^\alpha(1^\beta+3^\beta)^\gamma \\ &\quad +(1\cdot 1)^\alpha(1^\beta+1^\beta)^\gamma \\ &= 2^{\alpha+1}(1+2^\beta)^\gamma+2^{2\alpha}(1+4^\beta)^\gamma+\frac{2n-15}{3}\cdot 2^{3\alpha+\beta\gamma}(1+2^\beta)^\gamma \\ &\quad +\frac{n-6}{3}\cdot 2^{2\alpha+(\beta+1)\gamma}+6^\alpha(2^\beta+3^\beta)^\gamma+3^\alpha(1+3^\beta)^\gamma+2^\gamma. \end{aligned}$$

This completes the proof. □

Theorem 2.2 *The general leap Zagreb-type indices of T_3^1, shown in Figure 1, are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(T_3^1) &= \frac{2n-14}{3}\cdot 2^{\alpha+\beta}(2^\alpha+2^\beta)+\frac{n-7}{3}\cdot 2^{1+\alpha+\beta}+2^{\alpha+2}+2^{\beta+2}+2^{2\alpha+1}+2^{2\beta+1}, \\ LRZ_{\alpha,\beta,\gamma}(T_3^1) &= 2^{\alpha+2}(1+2^\beta)^\gamma+2^{2\alpha+1}(1+4^\beta)^\gamma+\frac{2n-14}{3}\cdot 2^{3\alpha+\beta\gamma}(1+2^\beta)^\gamma \\ &\quad +\frac{n-7}{3}\cdot 2^{2\alpha+(\beta+1)\gamma}. \end{aligned}$$

Proof By the definition of T_3^1, we obtain the basic information on T_3^1 in the following table.

$m(1,2)$	$m(1,4)$	$m(2,4)$	$m(2,2)$
4	2	$\frac{2n-14}{3}$	$\frac{n-7}{3}$

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(T_3^1) &= 4(1^\alpha \cdot 2^\beta + 1^\beta \cdot 2^\alpha) + 2(1^\alpha \cdot 4^\beta + 1^\beta \cdot 4^\alpha) + \frac{2n-14}{3} \cdot (2^\alpha \cdot 4^\beta + 2^\beta \cdot 4^\alpha) \\ &\quad + \frac{n-7}{3} \cdot (2^\alpha \cdot 2^\beta + 2^\beta \cdot 2^\alpha) \\ &= \frac{2n-14}{3} \cdot 2^{\alpha+\beta}(2^\alpha + 2^\beta) + \frac{n-7}{3} \cdot 2^{1+\alpha+\beta} + 2^{\alpha+2} + 2^{\beta+2} + 2^{2\alpha+1} + 2^{2\beta+1}, \end{aligned}$$

$$\begin{aligned} LRZ_{\alpha,\beta,\gamma}(T_3^1) &= 4(1 \cdot 2)^\alpha(1^\beta + 2^\beta)^\gamma + 2(1 \cdot 4)^\alpha(1^\beta + 4^\beta)^\gamma \\ &\quad + \frac{2n-14}{3} \cdot (2 \cdot 4)^\alpha(2^\beta + 4^\beta)^\gamma + \frac{n-7}{3} \cdot (2 \cdot 2)^\alpha(2^\beta + 2^\beta)^\gamma \\ &= 2^{\alpha+2}(1 + 2^\beta)^\gamma + 2^{2\alpha+1}(1 + 4^\beta)^\gamma + \frac{2n-14}{3} \cdot 2^{3\alpha+\beta\gamma}(1 + 2^\beta)^\gamma \\ &\quad + \frac{n-7}{3} \cdot 2^{2\alpha+(\beta+1)\gamma}. \end{aligned}$$

This completes the proof. □

Theorem 2.3 *The general leap Zagreb-type indices of T_3^2, shown in Figure 1, are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(T_3^2) &= \frac{2n-13}{3} \cdot 2^{\alpha+\beta}(2^\alpha + 2^\beta) + \frac{n-2}{3} \cdot 2^{1+\alpha+\beta} + 3 \cdot (2^\alpha + 2^\beta) + 2^{2\alpha} + 2^{2\beta}, \\ LRZ_{\alpha,\beta,\gamma}(T_3^2) &= 3 \cdot 2^\alpha(1 + 2^\beta)^\gamma + 2^{2\alpha}(1 + 4^\beta)^\gamma + \frac{2n-13}{3} \cdot 2^{3\alpha+\beta\gamma}(1 + 2^\beta)^\gamma \\ &\quad + \frac{n-2}{3} \cdot 2^{2\alpha+(\beta+1)\gamma}. \end{aligned}$$

Proof By the definition of T_3^2, we obtain the basic information on T_3^2 in the following table.

$m(1,2)$	$m(1,4)$	$m(2,4)$	$m(2,2)$
3	1	$\frac{2n-13}{3}$	$\frac{n-2}{3}$

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(T_3^2) &= 3(1^\alpha \cdot 2^\beta + 1^\beta \cdot 2^\alpha) + (1^\alpha \cdot 4^\beta + 1^\beta \cdot 4^\alpha) + \frac{2n-13}{3} \cdot (2^\alpha \cdot 4^\beta + 2^\beta \cdot 4^\alpha) \\ &\quad + \frac{n-2}{3} \cdot (2^\alpha \cdot 2^\beta + 2^\beta \cdot 2^\alpha) \\ &= \frac{2n-13}{3} \cdot 2^{\alpha+\beta}(2^\alpha + 2^\beta) + \frac{n-2}{3} \cdot 2^{1+\alpha+\beta} + 3 \cdot (2^\alpha + 2^\beta) + 2^{2\alpha} + 2^{2\beta}, \\ LRZ_{\alpha,\beta,\gamma}(T_3^2) &= 3(1 \cdot 2)^\alpha(1^\beta + 2^\beta)^\gamma + (1 \cdot 4)^\alpha(1^\beta + 4^\beta)^\gamma \end{aligned}$$

$$+\frac{2n-13}{3}\cdot(2\cdot4)^{\alpha}(2^{\beta}+4^{\beta})^{\gamma}+\frac{n-2}{3}\cdot(2\cdot2)^{\alpha}(2^{\beta}+2^{\beta})^{\gamma}$$
$$=3\cdot2^{\alpha}(1+2^{\beta})^{\gamma}+2^{2\alpha}(1+4^{\beta})^{\gamma}+\frac{2n-13}{3}\cdot2^{3\alpha+\beta\gamma}(1+2^{\beta})^{\gamma}+\frac{n-2}{3}\cdot2^{2\alpha+(\beta+1)\gamma}.$$

This completes the proof. □

There are four types molecular trees with n vertices shown in Figure 2.

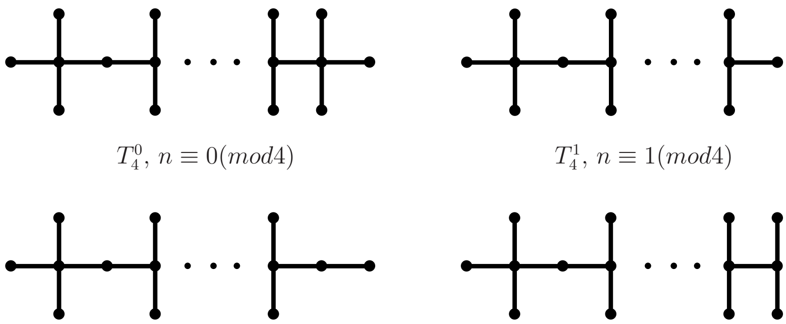

Figure 2 Four types molecular trees with n vertices.

Theorem 2.4 *The general leap Zagreb-type indices of T_4^0, shown in Figure 2, are given by*

$$LZ_{\alpha,\beta}(T_4^0)=\frac{n-12}{2}\cdot2^{\alpha+\beta}(3^{\alpha}+3^{\beta})+\frac{n-12}{2}\cdot(2^{\alpha}\cdot3^{\beta}+2^{\beta}\cdot3^{\alpha})+3(3^{\alpha}\cdot4^{\beta}+3^{\beta}\cdot4^{\alpha})$$
$$+2^{2\alpha}\cdot6^{\beta}+2^{2\beta}\cdot6^{\alpha}+3^{\alpha+1}+3^{\beta+1}+6^{\alpha}+6^{\beta}+2\cdot3^{1+\alpha+\beta},$$

$$LRZ_{\alpha,\beta,\gamma}(T_4^0)=3^{\alpha+1}(1+3^{\beta})^{\gamma}+6^{\alpha}(1+6^{\beta})^{\gamma}+\frac{n-12}{2}\cdot12^{\alpha}(2^{\beta}+6^{\beta})^{\gamma}+\frac{n-12}{2}$$
$$\times6^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+24^{\alpha}(4^{\beta}+6^{\beta})^{\gamma}+3\cdot12^{\alpha}(3^{\beta}+4^{\beta})^{\gamma}+2^{\gamma}\cdot3^{1+2\alpha+\beta\gamma}.$$

Proof By the definition of T_4^0, we obtain the basic information on T_4^0 in the following table.

$m(1,3)$	$m(1,6)$	$m(2,6)$	$m(2,3)$	$m(4,6)$	$m(3,4)$	$m(3,3)$
3	1	$\frac{n-12}{2}$	$\frac{n-12}{2}$	1	3	3

Thus, we have

$$LZ_{\alpha,\beta}(T_4^0)=3(1^{\alpha}\cdot3^{\beta}+1^{\beta}\cdot3^{\alpha})+(1^{\alpha}\cdot6^{\beta}+1^{\beta}\cdot6^{\alpha})+\frac{n-12}{2}\cdot(2^{\alpha}\cdot6^{\beta}+2^{\beta}\cdot6^{\alpha})$$
$$+\frac{n-12}{2}\cdot(2^{\alpha}\cdot3^{\beta}+2^{\beta}\cdot3^{\alpha})+(4^{\alpha}\cdot6^{\beta}+4^{\beta}\cdot6^{\alpha})+3(3^{\alpha}\cdot4^{\beta}+3^{\beta}\cdot4^{\alpha})$$
$$+3(3^{\alpha}\cdot3^{\beta}+3^{\beta}\cdot3^{\alpha})$$
$$=\frac{n-12}{2}\cdot2^{\alpha+\beta}(3^{\alpha}+3^{\beta})+\frac{n-12}{2}\cdot(2^{\alpha}\cdot3^{\beta}+2^{\beta}\cdot3^{\alpha})+3(3^{\alpha}\cdot4^{\beta}+3^{\beta}\cdot4^{\alpha})$$
$$+2^{2\alpha}\cdot6^{\beta}+2^{2\beta}\cdot6^{\alpha}+3^{\alpha+1}+3^{\beta+1}+6^{\alpha}+6^{\beta}+2\cdot3^{1+\alpha+\beta},$$

$$
\begin{aligned}
LRZ_{\alpha,\beta,\gamma}(T_4^0) &= 3(1\cdot 3)^\alpha(1^\beta+3^\beta)^\gamma+(1\cdot 6)^\alpha(1^\beta+6^\beta)^\gamma+\frac{n-12}{2}\cdot(2\cdot 6)^\alpha(2^\beta+6^\beta)^\gamma\\
&\quad+\frac{n-12}{2}\cdot(2\cdot 3)^\alpha(2^\beta+3^\beta)^\gamma+(4\cdot 6)^\alpha(4^\beta+6^\beta)^\gamma+3(3\cdot 4)^\alpha(3^\beta+4^\beta)^\gamma\\
&\quad+3(3\cdot 3)^\alpha(3^\beta+3^\beta)^\gamma\\
&= 3^{\alpha+1}(1+3^\beta)^\gamma+6^\alpha(1+6^\beta)^\gamma+\frac{n-12}{2}\cdot 12^\alpha(2^\beta+6^\beta)^\gamma+\frac{n-12}{2}\\
&\quad\times 6^\alpha(2^\beta+3^\beta)^\gamma+24^\alpha(4^\beta+6^\beta)^\gamma+3\cdot 12^\alpha(3^\beta+4^\beta)^\gamma+2^\gamma\cdot 3^{1+2\alpha+\beta\gamma}.
\end{aligned}
$$

This completes the proof. □

Theorem 2.5 *The general leap Zagreb-type indices of T_4^1, shown in Figure 2, are given by*

$$
\begin{aligned}
LZ_{\alpha,\beta}(T_4^1) &= \frac{n-9}{2}\cdot(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+\frac{n-9}{2}\cdot 2^{\alpha+\beta}(3^\alpha+3^\beta)\\
&\quad+6(3^\alpha+3^\beta)+2(6^\alpha+6^\beta),\\
LRZ_{\alpha,\beta,\gamma}(T_4^1) &= 6\cdot 3^\alpha(1+3^\beta)^\gamma+2\cdot 6^\alpha(1+6^\beta)^\gamma+\frac{n-9}{2}\cdot 12^\alpha(2^\beta+6^\beta)^\gamma\\
&\quad+\frac{n-9}{2}\cdot 6^\alpha(2^\beta+3^\beta)^\gamma.
\end{aligned}
$$

Proof By the definition of T_4^1, we obtain the basic information on T_4^1 in the following table.

$m(1,3)$	$m(1,6)$	$m(2,6)$	$m(2,3)$
6	2	$\frac{n-9}{2}$	$\frac{n-9}{2}$

Thus, we have

$$
\begin{aligned}
LZ_{\alpha,\beta}(T_4^1) &= 6(1^\alpha\cdot 3^\beta+1^\beta\cdot 3^\alpha)+2(1^\alpha\cdot 6^\beta+1^\beta\cdot 6^\alpha)+\frac{n-9}{2}\cdot(2^\alpha\cdot 6^\beta+2^\beta\cdot 6^\alpha)\\
&\quad+\frac{n-9}{2}\cdot(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)\\
&= \frac{n-9}{2}\cdot(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+\frac{n-9}{2}\cdot 2^{\alpha+\beta}(3^\alpha+3^\beta)+6(3^\alpha+3^\beta)+2(6^\alpha+6^\beta),
\end{aligned}
$$

$$
\begin{aligned}
LRZ_{\alpha,\beta,\gamma}(T_4^1) &= 6(1\cdot 3)^\alpha(1^\beta+3^\beta)^\gamma+2(1\cdot 6)^\alpha(1^\beta+6^\beta)^\gamma\\
&\quad+\frac{n-9}{2}\cdot(2\cdot 6)^\alpha(2^\beta+6^\beta)^\gamma+\frac{n-9}{2}\cdot(2\cdot 3)^\alpha(2^\beta+3^\beta)^\gamma\\
&= 6\cdot 3^\alpha(1+3^\beta)^\gamma+2\cdot 6^\alpha(1+6^\beta)^\gamma\\
&\quad+\frac{n-9}{2}\cdot 12^\alpha(2^\beta+6^\beta)^\gamma+\frac{n-9}{2}\cdot 6^\alpha(2^\beta+3^\beta)^\gamma.
\end{aligned}
$$

This completes the proof. □

Theorem 2.6 *The general leap Zagreb-type indices of T_4^2, shown in Figure 2, are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(T_4^2) &= \frac{n-4}{2}\cdot(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+\frac{n-8}{2}\cdot 2^{\alpha+\beta}(3^\alpha+3^\beta)+4(3^\alpha+3^\beta) \\ &\quad +6^\alpha+6^\beta, \\ LRZ_{\alpha,\beta,\gamma}(T_4^2) &= 4\cdot 3^\alpha(1+3^\beta)^\gamma+6^\alpha(1+6^\beta)^\gamma+\frac{n-8}{2}\cdot 12^\alpha(2^\beta+6^\beta)^\gamma \\ &\quad +\frac{n-4}{2}\cdot 6^\alpha(2^\beta+3^\beta)^\gamma. \end{aligned}$$

Proof By the definition of T_4^2, we obtain the basic information on T_4^2 in the following table.

$m(1,3)$	$m(1,6)$	$m(2,6)$	$m(2,3)$
4	1	$\frac{n-8}{2}$	$\frac{n-4}{2}$

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(T_4^2) &= 4(1^\alpha\cdot 3^\beta+1^\beta\cdot 3^\alpha)+(1^\alpha\cdot 6^\beta+1^\beta\cdot 6^\alpha)+\frac{n-8}{2}\cdot(2^\alpha\cdot 6^\beta+2^\beta\cdot 6^\alpha) \\ &\quad +\frac{n-4}{2}\cdot(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha) \\ &= \frac{n-4}{2}\cdot(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+\frac{n-8}{2}\cdot 2^{\alpha+\beta}(3^\alpha+3^\beta)+4(3^\alpha+3^\beta) \\ &\quad +6^\alpha+6^\beta, \end{aligned}$$

$$\begin{aligned} LRZ_{\alpha,\beta,\gamma}(T_4^2) &= 4(1\cdot 3)^\alpha(1^\beta+3^\beta)^\gamma+(1\cdot 6)^\alpha(1^\beta+6^\beta)^\gamma \\ &\quad +\frac{n-8}{2}\cdot(2\cdot 6)^\alpha(2^\beta+6^\beta)^\gamma+\frac{n-4}{2}\cdot(2\cdot 3)^\alpha(2^\beta+3^\beta)^\gamma \\ &= 4\cdot 3^\alpha(1+3^\beta)^\gamma+6^\alpha(1+6^\beta)^\gamma+\frac{n-8}{2}\cdot 12^\alpha(2^\beta+6^\beta)^\gamma \\ &\quad +\frac{n-4}{2}\cdot 6^\alpha(2^\beta+3^\beta)^\gamma. \end{aligned}$$

This completes the proof. □

Theorem 2.7 *The general leap Zagreb-type indices of T_4^3, shown in Figure 2, are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(T_4^3) &= \frac{n-11}{2}\cdot(2^\alpha\cdot 6^\beta+2^\beta\cdot 6^\alpha)+\frac{n-7}{2}\cdot(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha) \\ &\quad +3^\alpha\cdot 6^\beta+3^\beta\cdot 6^\alpha+2\cdot 3^{1+\alpha+\beta}+3^{1+\alpha}+3^{\beta+1}+6^\alpha+6^\beta, \\ LRZ_{\alpha,\beta,\gamma}(T_4^3) &= 3^{1+\alpha}(1+3^\beta)^\gamma+6^\alpha(1+6^\beta)^\gamma+\frac{n-11}{2}\cdot 12^\alpha(2^\beta+6^\beta)^\gamma \\ &\quad +\frac{n-7}{2}\cdot 6^\alpha(2^\beta+3^\beta)^\gamma+2^\gamma\cdot 3^{1+2\alpha+\beta\gamma}+18^\alpha(3^\beta+6^\beta)^\gamma. \end{aligned}$$

Proof By the definition of T_4^3, we obtain the basic information on T_4^3 in the following table.

$m(1,3)$	$m(1,6)$	$m(2,6)$	$m(2,3)$	$m(3,3)$	$m(3,6)$
3	1	$\frac{n-11}{2}$	$\frac{n-7}{2}$	3	1

Thus, we have

$$\begin{aligned}
LZ_{\alpha,\beta}(T_4^3) &= 3(1^\alpha \cdot 3^\beta + 1^\beta \cdot 3^\alpha) + (1^\alpha \cdot 6^\beta + 1^\beta \cdot 6^\alpha) + \frac{n-11}{2} \cdot (2^\alpha \cdot 6^\beta + 2^\beta \cdot 6^\alpha) \\
&\quad + \frac{n-7}{2} \cdot (2^\alpha \cdot 3^\beta + 2^\beta \cdot 3^\alpha) + 3(3^\alpha \cdot 3^\beta + 3^\beta \cdot 3^\alpha) + (3^\alpha \cdot 6^\beta + 3^\beta \cdot 6^\alpha) \\
&= \frac{n-11}{2} \cdot (2^\alpha \cdot 6^\beta + 2^\beta \cdot 6^\alpha) + \frac{n-7}{2} \cdot (2^\alpha \cdot 3^\beta + 2^\beta \cdot 3^\alpha) \\
&\quad + 3^\alpha \cdot 6^\beta + 3^\beta \cdot 6^\alpha + 2 \cdot 3^{1+\alpha+\beta} + 3^{1+\alpha} + 3^{\beta+1} + 6^\alpha + 6^\beta, \\
LRZ_{\alpha,\beta,\gamma}(T_4^3) &= 3(1 \cdot 3)^\alpha (1^\beta + 3^\beta)^\gamma + (1 \cdot 6)^\alpha (1^\beta + 6^\beta)^\gamma + \frac{n-11}{2} \cdot (2 \cdot 6)^\alpha (2^\beta + 6^\beta)^\gamma \\
&\quad + \frac{n-7}{2} \cdot (2 \cdot 3)^\alpha (2^\beta + 3^\beta)^\gamma + 3(3 \cdot 3)^\alpha (3^\beta + 3^\beta)^\gamma + (3 \cdot 6)^\alpha (3^\beta + 6^\beta)^\gamma \\
&= 3^{1+\alpha}(1 + 3^\beta)^\gamma + 6^\alpha (1 + 6^\beta)^\gamma + \frac{n-11}{2} \cdot 12^\alpha (2^\beta + 6^\beta)^\gamma \\
&\quad + \frac{n-7}{2} \cdot 6^\alpha (2^\beta + 3^\beta)^\gamma + 2^\gamma \cdot 3^{1+2\alpha+\beta\gamma} + 18^\alpha (3^\beta + 6^\beta)^\gamma.
\end{aligned}$$

This completes the proof. □

§3. General Leap Zagreb-Type Indices of Some Chemical Chains

There are eight type chemical chains shown in Figure 3.

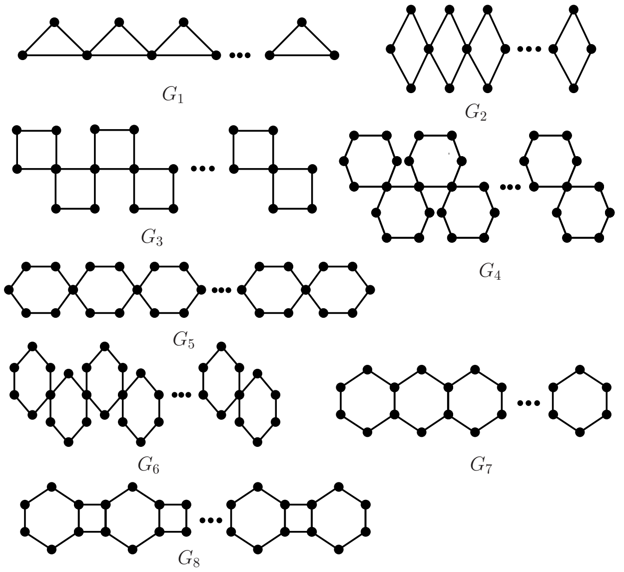

Figure 3 Eight types chemical chains.

Theorem 3.1 *Let t be the number of triangles in G_1, shown in Figure 3. Then the general leap Zagreb-type indices of G_1 are given by*

$$LZ_{\alpha,\beta}(G_1) = (3t-10)\cdot 2^{1+2\alpha+2\beta}+6\cdot 2^{1+\alpha+\beta}+2^{\alpha+2\beta+2}+2^{2\alpha+\beta+2},$$
$$LRZ_{\alpha,\beta,\gamma}(G_1) = 6\cdot 2^{2\alpha+(\beta+1)\gamma}+2^{2+3\alpha+\beta\gamma}(1+2^{\beta})^{\gamma}+(3t-10)\cdot 2^{4\alpha+(2\beta+1)\gamma}.$$

Proof By the definition of G_1, we obtain the basic information on G_1 in the following table.

$m(2,2)$	$m(2,4)$	$m(4,4)$
6	4	$3t-10$

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(G_1) &= 6(2^{\alpha}\cdot 2^{\beta}+2^{\beta}\cdot 2^{\alpha})+4(2^{\alpha}\cdot 4^{\beta}+2^{\beta}\cdot 4^{\alpha})+(3t-10)\cdot(4^{\alpha}\cdot 4^{\beta}+4^{\beta}\cdot 4^{\alpha}) \\ &= (3t-10)\cdot 2^{1+2\alpha+2\beta}+6\cdot 2^{1+\alpha+\beta}+2^{\alpha+2\beta+2}+2^{2\alpha+\beta+2}, \end{aligned}$$

$$\begin{aligned} LRZ_{\alpha,\beta,\gamma}(G_1) &= 6(2\cdot 2)^{\alpha}(2^{\beta}+2^{\beta})^{\gamma}+4(2\cdot 4)^{\alpha}(2^{\beta}+4^{\beta})^{\gamma}+(3t-10)(4\cdot 4)^{\alpha}(4^{\beta}+4^{\beta})^{\gamma} \\ &= 6\cdot 2^{2\alpha+(\beta+1)\gamma}+2^{2+3\alpha+\beta\gamma}(1+2^{\beta})^{\gamma}+(3t-10)\cdot 2^{4\alpha+(2\beta+1)\gamma}. \end{aligned}$$

This completes the proof. □

Theorem 3.2 *Let q be the number of quadrilaterals in G_2, shown in Figure 3. Then the general leap Zagreb-type indices of G_2 are given by*

$$LZ_{\alpha,\beta}(G_2) = (4q-8)\cdot(2^{\alpha}\cdot 5^{\beta}+2^{\beta}\cdot 5^{\alpha})+4(2^{\alpha}\cdot 3^{\beta}+2^{\beta}\cdot 3^{\alpha})+4(3^{\alpha}+3^{\beta}),$$
$$LRZ_{\alpha,\beta,\gamma}(G_2) = 4\cdot 3^{\alpha}(1+3^{\beta})^{\gamma}+4\cdot 6^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+(4q-8)\cdot 10^{\alpha}(2^{\beta}+5^{\beta})^{\gamma}.$$

Proof By the definition of G_2, we obtain the basic information on G_2 in the following table.

$m(1,3)$	$m(2,3)$	$m(2,5)$
4	4	$4q-8$

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(G_2) &= 4(1^{\alpha}\cdot 3^{\beta}+1^{\beta}\cdot 3^{\alpha})+4(2^{\alpha}\cdot 3^{\beta}+2^{\beta}\cdot 3^{\alpha})+(4q-8)\cdot(2^{\alpha}\cdot 5^{\beta}+2^{\beta}\cdot 5^{\alpha}) \\ &= (4q-8)\cdot(2^{\alpha}\cdot 5^{\beta}+2^{\beta}\cdot 5^{\alpha})+4(2^{\alpha}\cdot 3^{\beta}+2^{\beta}\cdot 3^{\alpha})+4(3^{\alpha}+3^{\beta}), \\ LRZ_{\alpha,\beta,\gamma}(G_2) &= 4(1\cdot 3)^{\alpha}(1^{\beta}+3^{\beta})^{\gamma}+4(2\cdot 3)^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+(4q-8)(2\cdot 5)^{\alpha}(2^{\beta}+5^{\beta})^{\gamma} \\ &= 4\cdot 3^{\alpha}(1+3^{\beta})^{\gamma}+4\cdot 6^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+(4q-8)\cdot 10^{\alpha}(2^{\beta}+5^{\beta})^{\gamma}. \end{aligned}$$

This completes the proof. □

Theorem 3.3 *Let q be the number of quadrilaterals in G_3, shown in Figure 3. Then the general leap Zagreb-type indices of G_3 are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(G_3) &= 3^{\alpha+\beta}(2q-6)\cdot(2^\alpha+2^\beta)+2(q-4)\cdot 6^{\alpha+\beta}+2(q-2)\cdot 3^{\alpha+\beta} \\ &\quad +6(3^\alpha\cdot 4^\beta+3^\beta\cdot 4^\alpha)+2^{1+\alpha+\beta}(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+4(3^\alpha+3^\beta), \\ LRZ_{\alpha,\beta,\gamma}(G_3) &= 4\cdot 3^\alpha(1+3^\beta)^\gamma+(q-2)\cdot 2^\gamma\cdot 3^{2\alpha+\beta\gamma}+6\cdot 12^\alpha(3^\beta+4^\beta)^\gamma \\ &\quad +(2q-6)\cdot 18^\alpha(3^\beta+6^\beta)^\gamma+2\cdot 24^\alpha(4^\beta+6^\beta)^\gamma+(q-4)\cdot 2^\gamma\cdot 6^{2\alpha+\beta\gamma}. \end{aligned}$$

Proof By the definition of G_3, we obtain the basic information on G_3 in the following table.

$m(1,3)$	$m(3,3)$	$m(3,4)$	$m(3,6)$	$m(4,6)$	$m(6,6)$
4	$q-2$	6	$2q-6$	2	$q-4$

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(G_3) &= 4(1^\alpha\cdot 3^\beta+1^\beta\cdot 3^\alpha)+(q-2)\cdot(3^\alpha\cdot 3^\beta+3^\beta\cdot 3^\alpha)+6(3^\alpha\cdot 4^\beta+3^\beta\cdot 4^\alpha) \\ &\quad +(2q-6)\cdot(3^\alpha\cdot 6^\beta+3^\beta\cdot 6^\alpha)+2(4^\alpha\cdot 6^\beta+4^\beta\cdot 6^\alpha)+(q-4)\cdot(6^\alpha\cdot 6^\beta+6^\beta\cdot 6^\alpha) \\ &= 3^{\alpha+\beta}(2q-6)\cdot(2^\alpha+2^\beta)+2(q-4)\cdot 6^{\alpha+\beta}+2(q-2)\cdot 3^{\alpha+\beta} \\ &\quad +6(3^\alpha\cdot 4^\beta+3^\beta\cdot 4^\alpha)+2^{1+\alpha+\beta}(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+4(3^\alpha+3^\beta), \\ LRZ_{\alpha,\beta,\gamma}(G_3) &= 4(1\cdot 3)^\alpha(1^\beta+3^\beta)^\gamma+(q-2)(3\cdot 3)^\alpha(3^\beta+3^\beta)^\gamma+6(3\cdot 4)^\alpha(3^\beta+4^\beta)^\gamma \\ &\quad +(2q-6)(3\cdot 6)^\alpha(3^\beta+6^\beta)^\gamma+2(4\cdot 6)^\alpha(4^\beta+6^\beta)^\gamma+(q-4)(6\cdot 6)^\alpha(6^\beta+6^\beta)^\gamma \\ &= 4\cdot 3^\alpha(1+3^\beta)^\gamma+(q-2)\cdot 2^\gamma\cdot 3^{2\alpha+\beta\gamma}+6\cdot 12^\alpha(3^\beta+4^\beta)^\gamma \\ &\quad +(2q-6)\cdot 18^\alpha(3^\beta+6^\beta)^\gamma+2\cdot 24^\alpha(4^\beta+6^\beta)^\gamma+(q-4)\cdot 2^\gamma\cdot 6^{2\alpha+\beta\gamma}. \end{aligned}$$

This completes the proof. □

Theorem 3.4 *Let h be the number of hexagons in G_4, shown in Figure 3. Then the general leap Zagreb-type indices of G_4 are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(G_4) &= 2^{2\alpha+2\beta}(2h-6)\cdot(2^\alpha+2^\beta)+(h-4)\cdot 2^{1+3\alpha+3\beta}+(h+2)\cdot 2^{1+\alpha+\beta} \\ &\quad +2^{1+\alpha+\beta}(2^\alpha+2^\beta)h+3\cdot 2^{1+\alpha+\beta}(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+2^{1+\alpha+\beta}(3^\alpha\cdot 4^\beta+3^\beta\cdot 4^\alpha), \\ LRZ_{\alpha,\beta,\gamma}(G_4) &= (h+2)\cdot 2^{2\alpha+(\beta+1)\gamma}+2^{1+3\alpha+\beta\gamma}(1+2^\beta)^\gamma\cdot h+6\cdot 24^\alpha(4^\beta+6^\beta)^\gamma \\ &\quad +(2h-6)\cdot 2^{5\alpha+2\beta\gamma}(1+2^\beta)^\gamma+2\cdot 48^\alpha(6^\beta+8^\beta)^\gamma+(h-4)\cdot 2^{6\alpha+(3\beta+1)\gamma}. \end{aligned}$$

Proof By the definition of G_4, we obtain the basic information on G_4 in the following table.

$m(2,2)$	$m(2,4)$	$m(4,6)$	$m(4,8)$	$m(6,8)$	$m(8,8)$
$h+2$	$2h$	6	$2h-6$	2	$h-4$

Thus, we have

$$\begin{aligned}LZ_{\alpha,\beta}(G_4) &= (h+2)\cdot(2^\alpha\cdot 2^\beta+2^\beta\cdot 2^\alpha)+2h(2^\alpha\cdot 4^\beta+2^\beta\cdot 4^\alpha)+6(4^\alpha\cdot 6^\beta+4^\beta\cdot 6^\alpha)\\ &\quad+(2h-6)\cdot(4^\alpha\cdot 8^\beta+4^\beta\cdot 8^\alpha)+2(6^\alpha\cdot 8^\beta+6^\beta\cdot 8^\alpha)+(h-4)\cdot(8^\alpha\cdot 8^\beta+8^\beta\cdot 8^\alpha)\\ &= 2^{2\alpha+2\beta}(2h-6)\cdot(2^\alpha+2^\beta)+(h-4)\cdot 2^{1+3\alpha+3\beta}+(h+2)\cdot 2^{1+\alpha+\beta}\\ &\quad+2^{1+\alpha+\beta}(2^\alpha+2^\beta)h+3\cdot 2^{1+\alpha+\beta}(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+2^{1+\alpha+\beta}(3^\alpha\cdot 4^\beta+3^\beta\cdot 4^\alpha),\end{aligned}$$

$$\begin{aligned}LRZ_{\alpha,\beta,\gamma}(G_4) &= (h+2)(2\cdot 2)^\alpha(2^\beta+2^\beta)^\gamma+2h(2\cdot 4)^\alpha(2^\beta+4^\beta)^\gamma+6(4\cdot 6)^\alpha(4^\beta+6^\beta)^\gamma\\ &\quad+(2h-6)(4\cdot 8)^\alpha(4^\beta+8^\beta)^\gamma+2(6\cdot 8)^\alpha(6^\beta+8^\beta)^\gamma+(h-4)(8\cdot 8)^\alpha(8^\beta+8^\beta)^\gamma\\ &= (h+2)\cdot 2^{2\alpha+(\beta+1)\gamma}+2^{1+3\alpha+\beta\gamma}(1+2^\beta)^\gamma\cdot h+6\cdot 24^\alpha(4^\beta+6^\beta)^\gamma\\ &\quad+(2h-6)\cdot 2^{5\alpha+2\beta\gamma}(1+2^\beta)^\gamma+2\cdot 48^\alpha(6^\beta+8^\beta)^\gamma+(h-4)\cdot 2^{6\alpha+(3\beta+1)\gamma}.\end{aligned}$$

This completes the proof. □

Theorem 3.5 *Let h be the number of hexagons in G_5, shown in Figure 3. Then the general leap Zagreb-type indices of G_5 are given by*

$$\begin{aligned}LZ_{\alpha,\beta}(G_5) &= (3h-4)\cdot 2^{2+2\alpha+2\beta}+2^{2+\alpha+\beta}(2^\alpha+2^\beta)+2^{3+\alpha+\beta},\\ LRZ_{\alpha,\beta,\gamma}(G_5) &= 2^{2(\alpha+1)+(\beta+1)\gamma}+2^{2+3\alpha+\beta\gamma}(1+2^\beta)^\gamma+(3h-4)\cdot 2^{1+4\alpha+(2\beta+1)\gamma}.\end{aligned}$$

Proof By the definition of G_5, we obtain the basic information on G_5 in the following table.

$m(2,2)$	$m(2,4)$	$m(4,4)$
4	4	$6h-8$

Thus, we have

$$\begin{aligned}LZ_{\alpha,\beta}(G_5) &= 4\cdot(2^\alpha\cdot 2^\beta+2^\beta\cdot 2^\alpha)+4(2^\alpha\cdot 4^\beta+2^\beta\cdot 4^\alpha)+(6h-8)\cdot(4^\alpha\cdot 4^\beta+4^\beta\cdot 4^\alpha)\\ &= (3h-4)\cdot 2^{2+2\alpha+2\beta}+2^{2+\alpha+\beta}(2^\alpha+2^\beta)+2^{3+\alpha+\beta},\end{aligned}$$

$$\begin{aligned}LRZ_{\alpha,\beta,\gamma}(G_5) &= 4(2\cdot 2)^\alpha(2^\beta+2^\beta)^\gamma+4(2\cdot 4)^\alpha(2^\beta+4^\beta)^\gamma+(6h-8)(4\cdot 4)^\alpha(4^\beta+4^\beta)^\gamma\\ &= 2^{2(\alpha+1)+(\beta+1)\gamma}+2^{2+3\alpha+\beta\gamma}(1+2^\beta)^\gamma+(3h-4)\cdot 2^{1+4\alpha+(2\beta+1)\gamma}.\end{aligned}$$

This completes the proof. □

Theorem 3.6 *Let h be the number of hexagons in G_6, shown in Figure 3. Then the general leap Zagreb-type indices of G_6 are given by*

$$\begin{aligned}LZ_{\alpha,\beta}(G_6) &= 2^{1+\alpha+\beta}\cdot(h-2)\cdot(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+2^{1+\alpha+\beta}(2^\alpha+2^\beta)\cdot h\\ &\quad+2^{2+2\alpha+2\beta}h+2^{3+\alpha+\beta},\end{aligned}$$

$$\begin{aligned}LRZ_{\alpha,\beta,\gamma}(G_6) &= 2^{2(\alpha+1)+(\beta+1)\gamma} + 2^{1+3\alpha+\beta\gamma}(1+2^\beta)^\gamma \cdot h + 2^{1+4\alpha+(2\beta+1)\gamma} \cdot h \\ &\quad +(2h-4)\cdot 24^\alpha \cdot (4^\beta + 6^\beta)^\gamma.\end{aligned}$$

Proof By the definition of G_6, we obtain the basic information on G_6 in the following table.

$m(2,2)$	$m(2,4)$	$m(4,4)$	$m(4,6)$
4	$2h$	$2h$	$2h-4$

Thus, we have

$$\begin{aligned}LZ_{\alpha,\beta}(G_6) &= 4\cdot(2^\alpha\cdot 2^\beta + 2^\beta\cdot 2^\alpha) + 2h(2^\alpha\cdot 4^\beta + 2^\beta \cdot 4^\alpha) + 2h(4^\alpha\cdot 4^\beta + 4^\beta\cdot 4^\alpha) \\ &\quad +(2h-4)\cdot(4^\alpha\cdot 6^\beta + 4^\beta \cdot 6^\alpha) \\ &= 2^{1+\alpha+\beta}\cdot(h-2)\cdot(2^\alpha\cdot 3^\beta + 2^\beta\cdot 3^\alpha) + 2^{1+\alpha+\beta}(2^\alpha + 2^\beta)\cdot h \\ &\quad +2^{2+2\alpha+2\beta}h + 2^{3+\alpha+\beta},\end{aligned}$$

$$\begin{aligned}LRZ_{\alpha,\beta,\gamma}(G_6) &= 4(2\cdot 2)^\alpha(2^\beta+2^\beta)^\gamma + 2h(2\cdot 4)^\alpha(2^\beta+4^\beta)^\gamma \\ &\quad +2h(4\cdot 4)^\alpha(4^\beta+4^\beta)^\gamma + (2h-4)(4\cdot 6)^\alpha(4^\beta+6^\beta)^\gamma \\ &= 2^{2(\alpha+1)+(\beta+1)\gamma} + 2^{1+3\alpha+\beta\gamma}(1+2^\beta)^\gamma\cdot h + 2^{1+4\alpha+(2\beta+1)\gamma}\cdot h \\ &\quad +(2h-4)\cdot 24^\alpha\cdot(4^\beta+6^\beta)^\gamma.\end{aligned}$$

This completes the proof. □

Theorem 3.7 *Let h be the number of hexagons in G_7, shown in Figure 3. Then the general leap Zagreb-type indices of G_7 are given by*

$$\begin{aligned}LZ_{\alpha,\beta}(G_7) &= 2^{1+2\alpha+2\beta}\cdot(5h-9) + 4(3^\alpha\cdot 4^\beta + 3^\beta\cdot 4^\alpha) + 4(2^\alpha\cdot 3^\beta \\ &\quad +2^\beta\cdot 3^\alpha) + 2^{2+\alpha+\beta}, \\ LRZ_{\alpha,\beta,\gamma}(G_7) &= 2^{4\alpha+(2\beta+1)\gamma}\cdot(5h-9) + 4\cdot 6^\alpha(2^\beta+3^\beta)^\gamma + 4\cdot 12^\alpha(3^\beta+4^\beta)^\gamma + 2^{2\alpha+1+(\beta+1)\gamma}.\end{aligned}$$

Proof By the definition of G_7, we obtain the basic information on G_7 in the following table.

$m(2,2)$	$m(2,3)$	$m(3,4)$	$m(4,4)$
2	4	4	$5h-9$

Thus, we have

$$\begin{aligned}LZ_{\alpha,\beta}(G_7) &= 2\cdot(2^\alpha\cdot 2^\beta + 2^\beta\cdot 2^\alpha) + 4(2^\alpha\cdot 3^\beta + 2^\beta\cdot 3^\alpha) + 4(3^\alpha\cdot 4^\beta + 3^\beta\cdot 4^\alpha) \\ &\quad +(5h-9)\cdot(4^\alpha\cdot 4^\beta + 4^\beta\cdot 4^\alpha) \\ &= 2^{1+2\alpha+2\beta}\cdot(5h-9) + 4(3^\alpha\cdot 4^\beta + 3^\beta\cdot 4^\alpha) + 4(2^\alpha\cdot 3^\beta + 2^\beta\cdot 3^\alpha) + 2^{2+\alpha+\beta},\end{aligned}$$

$$\begin{aligned} LRZ_{\alpha,\beta,\gamma}(G_7) &= 2(2\cdot 2)^{\alpha}(2^{\beta}+2^{\beta})^{\gamma}+4(2\cdot 3)^{\alpha}(2^{\beta}+3^{\beta})^{\gamma} \\ &\quad +4(3\cdot 4)^{\alpha}(3^{\beta}+4^{\beta})^{\gamma}+(5h-9)(4\cdot 4)^{\alpha}(4^{\beta}+4^{\beta})^{\gamma} \\ &= 2^{4\alpha+(2\beta+1)\gamma}\cdot(5h-9)+4\cdot 6^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+4\cdot 12^{\alpha}(3^{\beta}+4^{\beta})^{\gamma} \\ &\quad +2^{2\alpha+1+(\beta+1)\gamma}. \end{aligned}$$

This completes the proof. □

Theorem 3.8 *Let q and h be the number of quadrilaterals and hexagons in G_8, shown in Fig. 3. Then the general leap Zagreb-type indices of G_8 are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(G_8) &= (h-2)\cdot 2^{3+2\alpha+2\beta}+q\cdot 2^{3+2\alpha+2\beta}+4(3^{\alpha}\cdot 4^{\beta}+3^{\beta}\cdot 4^{\alpha}) \\ &\quad +4(2^{\alpha}\cdot 3^{\beta}+2^{\beta}\cdot 3^{\alpha})+2^{2+\alpha+\beta}, \\ LRZ_{\alpha,\beta,\gamma}(G_8) &= 2^{2+4\alpha+(2\beta+1)\gamma}\cdot(h-2)+4\cdot 6^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+4\cdot 12^{\alpha}(3^{\beta}+4^{\beta})^{\gamma} \\ &\quad +q\cdot 2^{2+4\alpha+(2\beta+1)\gamma}+2^{2\alpha+1+(\beta+1)\gamma}. \end{aligned}$$

Proof By the definition of G_8, we obtain the basic information on G_8 in the following table.

$m(2,2)$	$m(2,3)$	$m(3,4)$	$m(4,4)$	$m(4,4)$
2	4	4	$4q$	$4h-8$

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(G_8) &= 2\cdot(2^{\alpha}\cdot 2^{\beta}+2^{\beta}\cdot 2^{\alpha})+4(2^{\alpha}\cdot 3^{\beta}+2^{\beta}\cdot 3^{\alpha})+4(3^{\alpha}\cdot 4^{\beta}+3^{\beta}\cdot 4^{\alpha}) \\ &\quad +4q\cdot(4^{\alpha}\cdot 4^{\beta}+4^{\beta}\cdot 4^{\alpha})+(4h-8)\cdot(4^{\alpha}\cdot 4^{\beta}+4^{\beta}\cdot 4^{\alpha}) \\ &= (h-2)\cdot 2^{3+2\alpha+2\beta}+q\cdot 2^{3+2\alpha+2\beta}+4(3^{\alpha}\cdot 4^{\beta}+3^{\beta}\cdot 4^{\alpha}) \\ &\quad +4(2^{\alpha}\cdot 3^{\beta}+2^{\beta}\cdot 3^{\alpha})+2^{2+\alpha+\beta}, \end{aligned}$$

$$\begin{aligned} LRZ_{\alpha,\beta,\gamma}(G_8) &= 2(2\cdot 2)^{\alpha}(2^{\beta}+2^{\beta})^{\gamma}+4(2\cdot 3)^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+4(3\cdot 4)^{\alpha}(3^{\beta}+4^{\beta})^{\gamma} \\ &\quad +4q(4\cdot 4)^{\alpha}(4^{\beta}+4^{\beta})^{\gamma}+(4h-8)(4\cdot 4)^{\alpha}(4^{\beta}+4^{\beta})^{\gamma} \\ &= 2^{2+4\alpha+(2\beta+1)\gamma}\cdot(h-2)+4\cdot 6^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+4\cdot 12^{\alpha}(3^{\beta}+4^{\beta})^{\gamma} \\ &\quad +q\cdot 2^{2+4\alpha+(2\beta+1)\gamma}+2^{2\alpha+1+(\beta+1)\gamma}. \end{aligned}$$

This completes the proof. □

§4. General Leap Zagreb-Type Indices of Some Benzenoid Systems

The structure of starphene $ST(r,s,t)$ for integers $r,s,t \geq 3$ is shown in Figure 4.

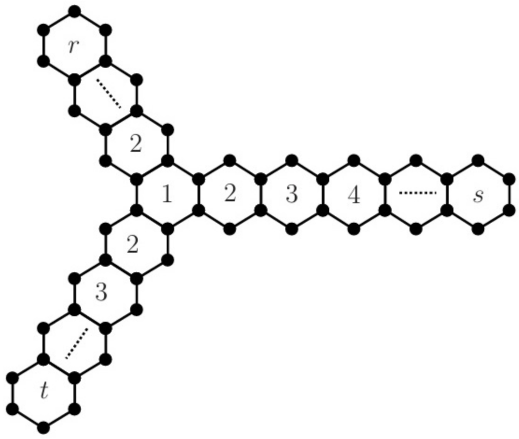

Figure 4 Starphene $ST(r,s,t)$ $(r,s,t \geq 3)$ structure.

Theorem 4.1 *The Leap connectivity indices of the starphene $ST(r,s,t)$ $(r,s,t \geq 3)$ structure, shown in Figure 4, are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(ST(r,s,t)) &= 6(2^\alpha \cdot 3^\beta + 2^\beta \cdot 3^\alpha) + 6(3^\alpha \cdot 4^\beta + 3^\beta \cdot 4^\alpha) + 6(4^\alpha \cdot 5^\beta + 4^\beta \cdot 5^\alpha) \\ &\quad +[5(r+s+t) - 36] \cdot 2^{1+2\alpha+2\beta} + 3 \cdot 2^{1+\alpha+\beta} + 12 \cdot 5^{\alpha+\beta}, \\ LRZ_{\alpha,\beta,\gamma}(ST(r,s,t)) &= 6^{1+\alpha}(2^\beta + 3^\beta)^\gamma + 6 \cdot 12^\alpha (3^\beta + 4^\beta)^\gamma + 6 \cdot 20^\alpha (4^\beta + 5^\beta)^\gamma \\ &\quad +[5(r+s+t) - 36] \cdot 2^{4\alpha+(1+2\beta)\gamma} + 3 \cdot 2^{2\alpha+(1+\beta)\gamma} + 6 \cdot 2^\gamma \cdot 5^{2\alpha+\beta\gamma}. \end{aligned}$$

Proof By the definition of $ST(r,s,t)$, we obtain the basic information on $ST(r,s,t)$ in the following table.

$m(2,2)$	$m(2,3)$	$m(3,4)$	$m(4,4)$	$m(4,5)$	$m(5,5)$
3	6	6	$5(r+s+t)-36$	6	6

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(ST(r,s,t)) &= 3(2^\alpha \cdot 2^\beta + 2^\beta \cdot 2^\alpha) + 6(2^\alpha \cdot 3^\beta + 2^\beta \cdot 3^\alpha) + 6(3^\alpha \cdot 4^\beta + 3^\beta \cdot 4^\alpha) \\ &\quad +[5(r+s+t) - 36] \cdot (4^\alpha \cdot 4^\beta + 4^\beta \cdot 4^\alpha) + 6(4^\alpha \cdot 5^\beta + 4^\beta \cdot 5^\alpha) \\ &\quad +6(5^\alpha \cdot 5^\beta + 5^\beta \cdot 5^\alpha) \\ &= 6(2^\alpha \cdot 3^\beta + 2^\beta \cdot 3^\alpha) + 6(3^\alpha \cdot 4^\beta + 3^\beta \cdot 4^\alpha) + 6(4^\alpha \cdot 5^\beta + 4^\beta \cdot 5^\alpha) \\ &\quad +[5(r+s+t) - 36] \cdot 2^{1+2\alpha+2\beta} + 3 \cdot 2^{1+\alpha+\beta} + 12 \cdot 5^{\alpha+\beta}, \end{aligned}$$

$$
\begin{aligned}
LRZ_{\alpha,\beta,\gamma}(ST(r,s,t)) &= 3(2\cdot 2)^{\alpha}(2^{\beta}+2^{\beta})^{\gamma}+6(2\cdot 3)^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+6(3\cdot 4)^{\alpha}(3^{\beta}+4^{\beta})^{\gamma} \\
&\quad +[5(r+s+t)-36]\cdot(4\cdot 4)^{\alpha}(4^{\beta}+4^{\beta})^{\gamma}+6(4\cdot 5)^{\alpha}(4^{\beta}+5^{\beta})^{\gamma} \\
&\quad +6(5\cdot 5)^{\alpha}(5^{\beta}+5^{\beta})^{\gamma} \\
&= 6^{1+\alpha}(2^{\beta}+3^{\beta})^{\gamma}+6\cdot 12^{\alpha}(3^{\beta}+4^{\beta})^{\gamma}+6\cdot 20^{\alpha}(4^{\beta}+5^{\beta})^{\gamma} \\
&\quad +[5(r+s+t)-36]\cdot 2^{4\alpha+(1+2\beta)\gamma}+3\cdot 2^{2\alpha+(1+\beta)\gamma}+6\cdot 2^{\gamma}\cdot 5^{2\alpha+\beta\gamma}.
\end{aligned}
$$

This completes the proof. □

The structure of rhombic benzenoid system R_h for integer $h>1$ is shown in Figure 5.

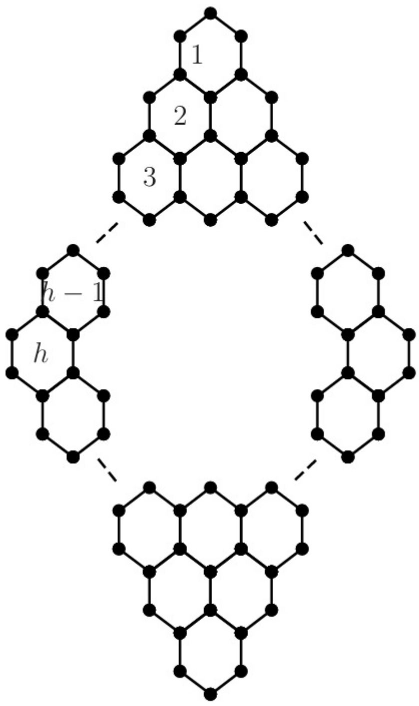

Figure 5 Rhombic benzenoid system R_h $(h>1)$.

Theorem 4.2 *The Leap connectivity indices of the rhombic benzenoid system R_h $(h>1)$, shown in Figure 5, are given by*

$$
\begin{aligned}
LZ_{\alpha,\beta}(R_h) &= 4(2^{\alpha}\cdot 3^{\beta}+2^{\beta}\cdot 3^{\alpha})+8(3^{\alpha}\cdot 4^{\beta}+3^{\beta}\cdot 4^{\alpha})+4(h-1)\cdot(4^{\alpha}\cdot 6^{\beta}+4^{\beta}\cdot 6^{\alpha}) \\
&\quad +2[(h-1)^2+2(h-1)(h-2)]\cdot 6^{\alpha+\beta}+(h-2)\cdot 2^{4+2\alpha+2\beta}+4\cdot 3^{\alpha+\beta}, \\
LRZ_{\alpha,\beta,\gamma}(R_h) &= 4\cdot 6^{\alpha}(2^{\beta}+3^{\beta})^{\gamma}+8\cdot 12^{\alpha}(3^{\beta}+4^{\beta})^{\gamma}+4(h-1)\cdot 24^{\alpha}(4^{\beta}+6^{\beta})^{\gamma} \\
&\quad +2^{\gamma}[(h-1)^2+2(h-1)(h-2)]\cdot 6^{2\alpha+\beta\gamma} \\
&\quad +(h-2)\cdot 2^{3+4\alpha+(1+2\beta)\gamma}+2^{1+\gamma}\cdot 3^{2\alpha+\beta\gamma}.
\end{aligned}
$$

Proof By the definition of R_h, we obtain the basic information on R_h in the following table.

$m(2,3)$	$m(3,3)$	$m(3,4)$	$m(4,4)$	$m(4,6)$	$m(6,6)$
4	2	8	$8(h-2)$	$4(h-1)$	$(h-1)^2+2(h-1)(h-2)$

Thus, we have

$$\begin{aligned}LZ_{\alpha,\beta}(R_h) &= 4(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+2(3^\alpha\cdot 3^\beta+3^\beta\cdot 3^\alpha)+8(3^\alpha\cdot 4^\beta+3^\beta\cdot 4^\alpha)\\ &\quad +8(h-2)(4^\alpha\cdot 4^\beta+4^\beta\cdot 4^\alpha)+4(h-1)\cdot(4^\alpha\cdot 6^\beta+4^\beta\cdot 6^\alpha)\\ &\quad +[(h-1)^2+2(h-1)(h-2)]\cdot(6^\alpha\cdot 6^\beta+6^\beta\cdot 6^\alpha)\\ &= 4(2^\alpha\cdot 3^\beta+2^\beta\cdot 3^\alpha)+8(3^\alpha\cdot 4^\beta+3^\beta\cdot 4^\alpha)+4(h-1)\cdot(4^\alpha\cdot 6^\beta+4^\beta\cdot 6^\alpha)\\ &\quad +2[(h-1)^2+2(h-1)(h-2)]\cdot 6^{\alpha+\beta}+(h-2)\cdot 2^{4+2\alpha+2\beta}+4\cdot 3^{\alpha+\beta},\end{aligned}$$

$$\begin{aligned}LRZ_{\alpha,\beta,\gamma}(R_h) &= 4(2\cdot 3)^\alpha(2^\beta+3^\beta)^\gamma+2(3\cdot 3)^\alpha(3^\beta+3^\beta)^\gamma+8(3\cdot 4)^\alpha(3^\beta+4^\beta)^\gamma\\ &\quad +8(h-2)(4\cdot 4)^\alpha(4^\beta+4^\beta)^\gamma+4(h-1)(4\cdot 6)^\alpha(4^\beta+6^\beta)^\gamma\\ &\quad +[(h-1)^2+2(h-1)(h-2)](6\cdot 6)^\alpha(6^\beta+6^\beta)^\gamma\\ &= 4\cdot 6^\alpha(2^\beta+3^\beta)^\gamma+8\cdot 12^\alpha(3^\beta+4^\beta)^\gamma+4(h-1)\cdot 24^\alpha(4^\beta+6^\beta)^\gamma\\ &\quad +2^\gamma[(h-1)^2+2(h-1)(h-2)]\cdot 6^{2\alpha+\beta\gamma}+(h-2)\cdot 2^{3+4\alpha+(1+2\beta)\gamma}\\ &\quad +2^{1+\gamma}\cdot 3^{2\alpha+\beta\gamma}.\end{aligned}$$

This completes the proof. □

§5. General Leap Zagreb-Type Indices of Some Nanostructures

The structure of armchair polyhex nanotube $TUAC_6[2r;s]$ for integers $r>1, s>1$ is shown in Figure 6.

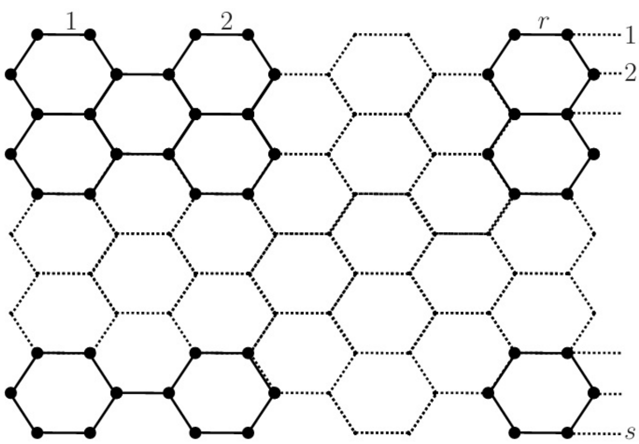

Figure 6 Armchair polyhex nanotube $TUAC_6[2r;s]$ $(r>1, s>1)$.

Theorem 5.1 *The leap connectivity indices of the armchair polyhex nanotube $TUAC_6[2r;s]$ $(r>1, s>1)$, shown in Figure 6, are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(TUAC_6[2r;s]) &= 2(3rs-14r)\cdot 6^{\alpha+\beta}+4r(5^\alpha\cdot 6^\beta+5^\beta\cdot 6^\alpha)+4r(3^\alpha\cdot 5^\beta+3^\beta\cdot 5^\alpha)\\ &\quad +4r\cdot 5^{\alpha+\beta}+4r\cdot 3^{\alpha+\beta},\\ LRZ_{\alpha,\beta,\gamma}(TUAC_6[2r;s]) &= 2^\gamma(3rs-14r)\cdot 6^{2\alpha+\beta\gamma}+4r\cdot 30^\alpha(5^\beta+6^\beta)^\gamma+4r\cdot 15^\alpha(3^\beta+5^\beta)^\gamma\\ &\quad +2^{\gamma+1}r\cdot 5^{2\alpha+\beta\gamma}+2^{\gamma+1}r\cdot 3^{2\alpha+\beta\gamma}. \end{aligned}$$

Proof By the definition of $TUAC_6[2r;s]$, we obtain the basic information on $TUAC_6[2r;s]$ in the following table.

$m(3,3)$	$m(3,5)$	$m(5,5)$	$m(5,6)$	$m(6,6)$
$2r$	$4r$	$2r$	$4r$	$3rs-14r$

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(TUAC_6[2r;s]) &= 2r(3^\alpha\cdot 3^\beta+3^\beta\cdot 3^\alpha)+4r(3^\alpha\cdot 5^\beta+3^\beta\cdot 5^\alpha)+2r(5^\alpha\cdot 5^\beta+5^\beta\cdot 5^\alpha)\\ &\quad +4r(5^\alpha\cdot 6^\beta+5^\beta\cdot 6^\alpha)+(3rs-14r)\cdot(6^\alpha\cdot 6^\beta+6^\beta\cdot 6^\alpha)\\ &= 2(3rs-14r)\cdot 6^{\alpha+\beta}+4r(5^\alpha\cdot 6^\beta+5^\beta\cdot 6^\alpha)+4r(3^\alpha\cdot 5^\beta+3^\beta\cdot 5^\alpha)\\ &\quad +4r\cdot 5^{\alpha+\beta}+4r\cdot 3^{\alpha+\beta}, \end{aligned}$$

$$\begin{aligned} LRZ_{\alpha,\beta,\gamma}(TUAC_6[2r;s]) &= 2r(3\cdot 3)^\alpha(3^\beta+3^\beta)^\gamma+4r(3\cdot 5)^\alpha(3^\beta+5^\beta)^\gamma+2r(5\cdot 5)^\alpha(5^\beta+5^\beta)^\gamma\\ &\quad +4r(5\cdot 6)^\alpha(5^\beta+6^\beta)^\gamma+(3rs-14r)(6\cdot 6)^\alpha(6^\beta+6^\beta)^\gamma\\ &= 2^\gamma(3rs-14r)\cdot 6^{2\alpha+\beta\gamma}+4r\cdot 30^\alpha(5^\beta+6^\beta)^\gamma+4r\cdot 15^\alpha(3^\beta+5^\beta)^\gamma\\ &\quad +2^{\gamma+1}r\cdot 5^{2\alpha+\beta\gamma}+2^{\gamma+1}r\cdot 3^{2\alpha+\beta\gamma}. \end{aligned}$$

This completes the proof. □

Theorem 5.2 *The leap connectivity indices of a V-phenylenic nanotube $VPHX[r;s]$ $(r>1, s>1)$ shown in Figure 7, are given by*

$$\begin{aligned} LZ_{\alpha,\beta}(VPHX[r;s]) &= 4r(4^\alpha\cdot 5^\beta+4^\beta\cdot 5^\alpha)+4r(s-1)(5^\alpha\cdot 6^\beta+5^\beta\cdot 6^\alpha)+4r(2s-3)\cdot 5^{\alpha+\beta}\\ &\quad +2r(s-1)\cdot 6^{\alpha+\beta}+6r\cdot 2^{1+2\alpha+2\beta},\\ LRZ_{\alpha,\beta,\gamma}(VPHX[r;s]) &= 4r\cdot 20^\alpha(4^\beta+5^\beta)^\gamma+4r(s-1)\cdot 30^\alpha(5^\beta+6^\beta)^\gamma+2^{1+\gamma}(2s-3)r\cdot 5^{2\alpha+\beta\gamma}\\ &\quad +2^\gamma(s-1)r\cdot 6^{2\alpha+\beta\gamma}+6r\cdot 2^{4\alpha+(1+2\beta)\gamma}. \end{aligned}$$

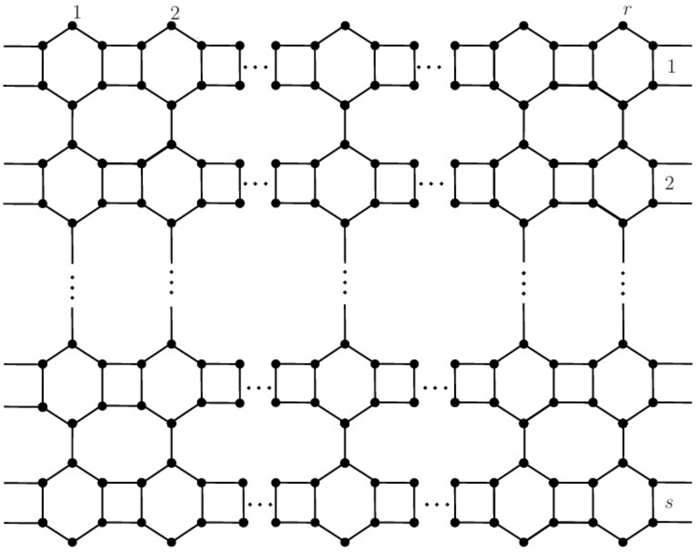

Figure 7 V-phenylenic nanotube $VPHX[r;s]$ $(r > 1, s > 1)$.

Proof By the definition of $VPHX[r;s]$, we obtain the basic information on $VPHX[r;s]$ in the following table.

$m(4,4)$	$m(4,5)$	$m(5,5)$	$m(5,6)$	$m(6,6)$
$6r$	$4r$	$2r(2s-3)$	$4r(s-1)$	$r(s-1)$

Thus, we have

$$\begin{aligned} LZ_{\alpha,\beta}(VPHX[r;s]) &= 6r(4^\alpha \cdot 4^\beta + 4^\beta \cdot 4^\alpha) + 4r(4^\alpha \cdot 5^\beta + 4^\beta \cdot 5^\alpha) + 2r(2s-3)(5^\alpha \cdot 5^\beta + 5^\beta \cdot 5^\alpha) \\ &\quad + 4r(s-1)(5^\alpha \cdot 6^\beta + 5^\beta \cdot 6^\alpha) + r(s-1) \cdot (6^\alpha \cdot 6^\beta + 6^\beta \cdot 6^\alpha) \\ &= 4r(4^\alpha \cdot 5^\beta + 4^\beta \cdot 5^\alpha) + 4r(s-1)(5^\alpha \cdot 6^\beta + 5^\beta \cdot 6^\alpha) + 4r(2s-3) \cdot 5^{\alpha+\beta} \\ &\quad + 2r(s-1) \cdot 6^{\alpha+\beta} + 6r \cdot 2^{1+2\alpha+2\beta}, \end{aligned}$$

$$\begin{aligned} LRZ_{\alpha,\beta,\gamma}(VPHX[r;s]) &= 6r(4 \cdot 4)^\alpha(4^\beta + 4^\beta)^\gamma + 4r(4 \cdot 5)^\alpha(4^\beta + 5^\beta)^\gamma + 2r(2s-3)(5 \cdot 5)^\alpha(5^\beta + 5^\beta)^\gamma \\ &\quad + 4r(s-1)(5 \cdot 6)^\alpha(5^\beta + 6^\beta)^\gamma + r(s-1)(6 \cdot 6)^\alpha(6^\beta + 6^\beta)^\gamma \\ &= 4r \cdot 20^\alpha(4^\beta + 5^\beta)^\gamma + 4r(s-1) \cdot 30^\alpha(5^\beta + 6^\beta)^\gamma + 2^{1+\gamma}(2s-3)r \cdot 5^{2\alpha+\beta\gamma} \\ &\quad + 2^\gamma(s-1)r \cdot 6^{2\alpha+\beta\gamma} + 6r \cdot 2^{4\alpha+(1+2\beta)\gamma}. \end{aligned}$$

This completes the proof. □

§6. **Conclusion**

In this paper, we propose the general leap-Zagreb-type indices, and obtain the calculation formulas of the general leap Zagreb-type indices for some chemical graphs. The general leap-Zagreb-type indices of some other graph structures can be computed for further study. Inspired by the multiplicative Zagreb-type indices, the general multiplicative leap-Zagreb-type indices, that is the multiplicative version of the leap-Zagreb-type indices, are naturally put forward as follows. It is worth mentioning that the research on extremal problem of the general (multiplicative) leap-Zagreb-type indices is probably an interesting subject.

$$LZM_{\alpha,\beta}(G) = \prod_{uv\in E(G)} (\tau_u^{\alpha}\tau_v^{\beta} + \tau_u^{\beta}\tau_v^{\alpha}),$$
$$LRZM_{\alpha,\beta,\gamma}(G) = \prod_{uv\in E(G)} (\tau_u\tau_v)^{\alpha}(\tau_u^{\beta} + \tau_v^{\beta})^{\gamma}.$$

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More Results on Vector Basis $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-Cordial Graphs

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Abstract: Let G be a (p,q) graph. Let V be an inner product space with basis S. We denote the inner product of the vectors x and y by $\langle x,y\rangle$. Let $\phi : V(G) \to S$ be a function. For edge uv assign the label $\langle \phi(u), \phi(v)\rangle$. Then ϕ is called a vector basis S-cordial labeling of G if $|\phi_x - \phi_y| \leq 1$ and $|\gamma_i - \gamma_j| \leq 1$ where ϕ_x denotes the number of vertices labeled with the vector x and γ_i denotes the number of edges labeled with the scalar i. A graph which admits a vector basis S-cordial labeling is called a vector basis S-cordial graph. In this paper, we investigate the existence of vector basis $\{$(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)$\}$-cordial labeling of octopus graph, $BC(n) \odot K_2$ and $BC(n) \odot mK_1$, where mG denote the m copies of G.

Key Words: Vector basis S-cordial labeling, bicycle, complete graph, octopus graph, star graph, Smarandachely vector basis S-cordial labeling.

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§1. Introduction

All graphs considered here are finite, simple and undirected graph. The order and size of a graph G are denoted by p and q, respectively. Terms not defined here are used in the sense of Harary [6] and Herstein [7]. The first research paper on graph theory was published by Leonhard Euler. Rosa [15] introduced the concept and notion of graph labeling. Since that time, various type of graph labeling method have been explored. Graph labeling finds application in various fields of science and technology. The concept of cordial labeling of graphs was introduced by Cahit [2]. Graph labeling is a dynamic field of study within graph theory that has primarily developed due to its various applications in mobile telecommunications systems, optimal circuit designs, graph decomposition problems, coding theory and communication networks. Topological cordial labeling of graph was introduced by Selestin Lina and Asha [16] and investigated the topological

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cordial labeling of some special graphs in [17]. Various graph labeling method of bicyclic related graphs were discussed in [8, 10, 11]. Meena and Gajalakshmi [9] have investigated the odd prime labeling of circular ladder related graphs. Some labeling methods of octopus related graphs were explored in [1, 3, 4].

A dynamic survey on different graph labeling problems with an extensive bibliography can be found in Gallian [5]. The concept of vector basis S-cordial labeling of graphs was introduced by Ponraj and Jeya [12], and examined the vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of behavior of the path, cycle, star, comb, complete graph, generalized friendship graph, tadpole graph and gear graph and thorn related graphs in [12, 13, 14]. In this paper, we investigate the existence of vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of octopus graph, $BC(n) \odot K_2$ and $BC(n) \odot mK_1$.

§2. **Preliminaries**

In this section, we state a few definitions which are relevant for proving the main results.

Definition 2.1([1]) *An octopus graph O_n, $n \geq 2$ can be constructed by a fan graph f_n, $n \geq 2$ joining a star graph $K_{1,n}$ with sharing a common vertex.*

Definition 2.2([8]) *The bicyclic graph $BC(n)$ is obtained from two copies of cycles C_n : $u_1u_2 \ldots u_nu_1$ and $C_n{'}$: $v_1v_2 \ldots v_nv_1$ by identifying u_1 with v_1.*

Definition 2.3([5]) *The corona graph $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and n copies of G_2 and joining i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2, where G_1 is graph of order n.*

In this paper, we consider the inner product space R^n and the standard inner product $\langle x, y\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$ where $x = (x_1, x_2, \cdots, x_n), y = (y_1, y_2, \cdots, y_n)$, $x_i, y_i \in R, 1 \leq i \leq n$.

§3. **Vector Basis S-Cordial Labeling**

Definition 3.1 *Let G be a (p, q) graph. Let V be an inner product space with basis S. We denote the inner product of the vectors x and y by $\langle x, y\rangle$. Let $\phi : V(G) \to S$ be a function. For edge uv assign the label $\langle \phi(u), \phi(v)\rangle$. Then ϕ is called a vector basis S-cordial labeling of G if $|\phi_x - \phi_y| \leq 1$ and $|\gamma_i - \gamma_j| \leq 1$ where ϕ_x denotes the number of vertices labeled with the vector x and γ_i denotes the number of edges labeled with the scalar i. A graph G which admits a vector basis S-cordial labeling is called a vector basis S-cordial graph. Otherwise, if $|\phi_x - \phi_y| \leq 1$ or $|\gamma_i - \gamma_j| \leq 1$, G is called a Smarandachely vector basis S-cordial graph.*

An example of a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of graph is shown in Figure 1.

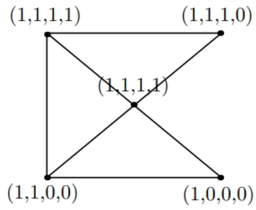

Figure 1 A vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial graph

§4. Main Results

In this section, we discuss the existence and non-existence of vector basis {(1,1,1,1),(1,1,1, 0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $BC(n) \odot K_2$ and $BC(n) \odot mK_1$.

Theorem 4.1 *The octopus graph O_n is a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0, 0,0)}-cordial if and only if $n \neq 3$.*

Proof Let $V(O_n) = \{u, u_i, v_i \mid 1 \leq i \leq n\}$ and $E(O_n) = \{uu_i, uv_i \mid 1 \leq i \leq n\} \cup \{u_iu_{i+1} \mid 1 \leq i \leq n-1\}$ respectively be the vertex and edge sets of O_n. Clearly $p = |V(O_n)| = 2n+1$ and $q = |E(O_n)| = 3n-1$. We have consider the following five cases:

Case 1. $n = 3$

Consider the octopus graph O_3. Then $p = 7$ and $q = 8$. Clearly $\phi_{(1,1,1,1)} = 2$. This forces $\gamma_4 < 2$, which is a contradiction to the size of O_3.

Case 2. $n \equiv 0 \pmod 4$

Let $n = 4t$, $t > 0$. We get $p = 8t+1$ and $q = 12t-1$. Assign the vector (1,1,1,1) to the vertex u. Then assign the vector (1,1,1,1) to the first t vertices $u_1, u_2, \ldots, u_t$ and to the t vertices $v_1, v_2, \ldots, v_t$. Assign the vector (1,1,1,0) to the next t vertices $u_{t+1}, u_{t+2}, \ldots, u_{2t}$ and to the next t vertices $v_{t+1}, v_{t+2}, \ldots, v_{2t}$. Thereafter assign the vector (1,1,0,0) to the next t vertices $u_{2t+1}, u_{2t+2}, \ldots, u_{3t}$ and to the next t vertices $v_{2t+1}, v_{2t+2}, \ldots, v_{3t}$. Finally assign the vector (1,0,0,0) to the next t vertices $u_{3t+1}, u_{3t+2}, \ldots, u_{4t}$ and to the next t vertices $v_{3t+1}, v_{3t+2}, \ldots, v_{4t}$.

Case 3. $n \equiv 1 \pmod 4$

Let $n = 4t+1$, $t > 0$. We obtain $p = 8t+3$ and $q = 12t+2$. Assign the vector (1,1,1,1) to the vertex u. Then assign the vector (1,1,1,1) to the first $t+1$ vertices $u_1, u_2, \cdots, u_{t+1}$ and to the $t-1$ vertices $v_1, v_2, \cdots, v_{t-1}$. Assign the vector (1,1,1,0) to the next t vertices $u_{t+2}, u_{t+3}, \cdots, u_{2t+1}$ and to the next $t+1$ vertices $v_t, v_{t+1}, \cdots, v_{2t}$. Thereafter assign the vector (1,1,0,0) to the next t vertices $u_{2t+2}, u_{2t+3}, \ldots, u_{3t+1}$ and to the next $t+1$ vertices $v_{2t+1}, v_{2t+2}, \cdots, v_{3t+1}$. Finally assign the vector (1,0,0,0) to the next t vertices $u_{3t+2}, u_{3t+3}, \ldots, u_{4t+1}$ and to the next t vertices $v_{3t+2}, v_{3t+3}, \cdots, v_{4t+1}$.

Case 4. $n \equiv 2 \pmod 4$

Let $n = 4t+2$, $t > 0$. We obtain $p = 8t+5$ and $q = 12t+5$. Assign the vector (1,1,1,1) to the vertex u. Then assign the vector (1,1,1,1) to the first $t+1$ vertices $u_1, u_2, \cdots, u_{t+1}$ and to the t vertices $v_1, v_2, \cdots, v_t$. Assign the vector (1,1,1,0) to the next $t+1$ vertices $u_{t+2}, u_{t+3}, \cdots, u_{2t+2}$ and to the next t vertices $v_{t+1}, v_{t+2}, \cdots, v_{2t}$. Thereafter assign the vector (1,1,0,0) to the next t vertices $u_{2t+3}, u_{2t+4}, \cdots, u_{3t+2}$ and to the next $t+1$ vertices $v_{2t+1}, v_{2t+2}, \ldots, v_{3t+1}$. Finally assign the vector (1,0,0,0) to the next t vertices $u_{3t+3}, u_{3t+4}, \cdots, u_{4t+2}$ and to the next $t+1$ vertices $v_{3t+2}, v_{3t+3}, \cdots, v_{4t+2}$.

Case 5. $n \equiv 3 \pmod 4$

Let $n = 4t+2$, $t > 0$. We obtain $p = 8t+7$ and $q = 12t+8$. Assign the vector (1,1,1,1) to the vertex u. Then assign the vector (1,1,1,1) to the first $t+2$ vertices $u_1, u_2, \cdots, u_{t+2}$ and to the $t-1$ vertices $v_1, v_2, \cdots, v_{t-1}$. Assign the vector (1,1,1,0) to the next $t+1$ vertices $u_{t+3}, u_{t+4}, \cdots, u_{2t+3}$ and to the next t vertices $v_t, v_{t+1}, \cdots, v_{2t-1}$. Thereafter assign the vector (1,1,0,0) to the next t vertices $u_{2t+4}, u_{2t+5}, \cdots, u_{3t+3}$ and to the next $t+2$ vertices $v_{2t}, v_{2t+1}, \cdots, v_{3t+1}$. Finally assign the vector (1,0,0,0) to the next t vertices $u_{3t+4}, u_{3t+5}, \ldots, u_{4t+3}$ and to the next $t+2$ vertices $v_{3t+2}, v_{3t+3}, \ldots, v_{4t+3}$.

Hence the vertex labeling ϕ is a vector basis {(1,1,1,1), (1,1,1,0),(1, 1,0,0),(1,0,0,0)}-cordial labeling of O_n for all $n \geq 4$. □

Example 4.2 An illustration for the vector basis {(1,1,1,1),(1,1,1,0),(1, 1,0,0),(1,0,0,0)}-cordial labeling of O_5 for the case when $n \equiv 1 \pmod 4$ is shown in Figure 2.

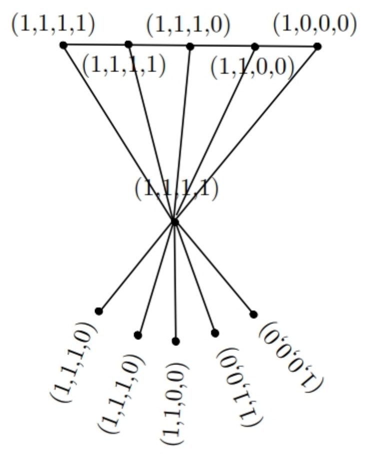

Figure 2 A vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of O_5.

Theorem 4.3 *The corona product of bicyclic graph with K_2, $BC(n) \odot K_2$ is a vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial if and only if $n \equiv 0, 2 \pmod 4$.*

Proof Let $V(BC(n) \odot K_2) = \{u_i, u_{ij} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq 2\} \cup \{v_i, v_{ij} \mid 2 \leq i \leq n \text{ and } 1 \leq j \leq 2\}$ and $E(BC(n) \odot K_2) = \{u_iu_{i+1}, u_nu_1, u_1v_2 \mid 1 \leq i \leq n-1\} \cup \{v_iv_{i+1}, v_nu_1 \mid 2 \leq i \leq n-1\} \cup \{u_iu_{ij}, u_{i1}u_{i2} \mid 1 \leq i \leq n \text{ and } 1 \leq j \leq 2\} \cup \{v_iv_{ij}, v_{i1}v_{i2} \mid 2 \leq i \leq n \text{ and } 1 \leq j \leq 2\}$ respectively be the vertex and edge sets of $BC(n) \odot K_2$. Clearly $p = |V(BC(n) \odot K_2)| = 3(2n-1)$ and $q = |E(BC(n) \odot K_2)| = 4(2n-1)+1$. Then assign the vectors to the vertices in the following order $u_1, u_2, \cdots, u_n, u_{11}, u_{12}, v_2, v_3, \cdots, v_n, u_{21}, u_{22}, \ldots, u_{n1}, u_{n2}, v_{11}, v_{12}, v_{21}, v_{22}, \ldots, v_{n1}, v_{n2}$. We consider the following four cases:

Case 1. $n \equiv 0 \pmod 4$

Let $n = 4t_1$, $t_1 > 0$. We get $p = 24t_1 - 3$. Then assign the vector (1,1,1,1) to the first $6t_1$ vertices and assign the vector (1,1,1,0) to the next $6t_1 - 1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $6t_1 - 1$ vertices and assign the vector (1,0,0,0) to the next $6t_1 - 1$ vertices.

Case 2. $n \equiv 1 \pmod 4$

Let $n = 4t_1 + 1$, $t_1 > 0$. We get $p = 24t_1 + 3$ and $q = 32t_1 + 5$. From $6t_1 + 1$ vertices with vertex label (1,1,1,1), we can get $8t_1$ edges with edge label 4. We have to get $8t_1 + 1$ edges with edge label 4 from $6t_1 + 1$ vertices with vertex label (1,1,1,1) and so we are led to a contradiction.

Case 3. $n \equiv 2 \pmod 4$

Let $n = 4t_1 + 2$, $t_1 > 0$. We get $p = 24t_1 + 9$. Then assign the vector (1,1,1,1) to the first $6t_1 + 3$ vertices and assign the vector (1,1,1,0) to the next $6t_1 + 2$ vertices. Thereafter assign the vector (1,1,0,0) to the next $6t_1 + 2$ vertices and assign the vector (1,0,0,0) to the next $6t_1 + 2$ vertices.

Case 4. $n \equiv 3 \pmod 4$

Let $n = 4t_1 + 3$, $t_1 \geq 0$. We get $p = 24t_1 + 15$ and $q = 32t_1 + 21$. From $6t_1 + 4$ vertices with vertex label (1,1,1,1), we can get $8t_1 + 4$ edges with edge label 4. But we have to get $8t_1 + 5$ edges with edge label 4 from $6t_1 + 4$ vertices with vertex label (1,1,1,1) and so we are led to a contradiction.

Considering the findings in the above 4 cases, it emerges that a vector basis {(1,1,1,1),(1,1, 1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $BC(n) \odot K_2$ exists for the cases of $n \equiv 0, 2 \pmod 4$ only. □

Example 4.4 An illustration for the vector basis {(1,1, 1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $BC(4) \odot K_2$ for the case when $n \equiv 0 \pmod 4$ is shown in Figure 3.

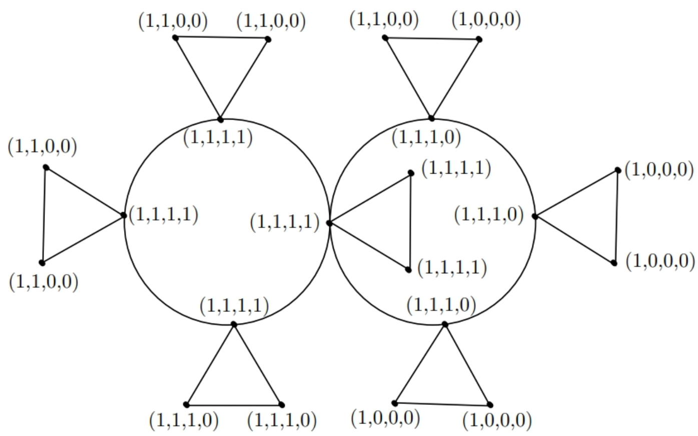

Figure 3 A vector basis {(1,1,1,1),(1,1, 1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $BC(4) \odot K_2$.

Theorem 4.5 *The corona product of bicyclic graph with mK_1, $BC(n)\odot mK_1$ is a vector basis $\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$-cordial for all $n\geq 3$ and $m\geq 1$.*

Proof Let $V(BC(n)\odot mK_1)=\{u_i,u_{ij}\mid 1\leq i\leq n \,\text{and}\, 1\leq j\leq m\}\cup\{v_i,v_{ij}\mid 2\leq i\leq n \,\text{and}\, 1\leq j\leq m\}$ and $E(BC(n)\odot mK_1)=\{u_iu_{i+1},u_nu_1\mid 1\leq i\leq n-1\}\cup\{v_iv_{i+1},v_nu_1,u_1v_2\mid 2\leq i\leq n-1\}\cup\{u_iu_{ij}\mid 1\leq i\leq n \,\text{and}\, 1\leq j\leq m\}\cup\{v_iv_{ij}\mid 2\leq i\leq n \,\text{and}\, 1\leq j\leq m\}$ respectively be the vertex and edge sets of $BC(n)\odot mK_1$. Clearly $p=|V(BC(n)\odot mK_1)|=(2n-1)(m+1)$ and $q=|E(BC(n)\odot mK_1)|=(2n-1)(m+1)+1$. Then, assign the vectors to vertices in the following order $u_1,u_2,\cdots,u_n,v_2,v_3,\cdots,v_n,u_{11},u_{12},\cdots,u_{1m},u_{21},u_{22},\cdots,u_{2m},\cdots,u_{n1},u_{n2},\cdots,u_{nm},v_{21},v_{22},\cdots,v_{2m},\cdots,v_{n1},v_{n2},\cdots,v_{nm}$. We consider the following four cases:

Case 1. $n\equiv 0 \pmod 4$

Let $n=4t_1$, $t_1>0$. We get $2n-1=8t_1-1$.

Subcase 1.1 $m\equiv 0 \pmod 4$

Let $m=4t_2$, $t_2>0$. We get $p=4(8t_1t_2+2t_1-t_2)-1$. Then assign the vector (1,1,1,1) to the first $8t_1t_2+2t_1-t_2-1$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+2t_1-t_2$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2+2t_1-t_2$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2+2t_1-t_2$ vertices.

Subcase 1.2 $m\equiv 1 \pmod 4$

Let $m=4t_2+1$, $t_2\geq 0$. We get $p=4(8t_1t_2+4t_1-t_2)-2$. Then assign the vector (1,1,1,1) to the first $8t_1t_2+4t_1-t_2-1$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+4t_1-t_2-1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2+4t_1-t_2$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2+4t_1-t_2$ vertices.

Subcase 1.3 $m\equiv 2 \pmod 4$

Let $m=4t_2+2$, $t_2\geq 0$. We get $p=4(8t_1t_2+6t_1-t_2)-3$. Then assign the vector (1,1,1,1) to the first $8t_1t_2+6t_1-t_2-1$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+6t_1-t_2-1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2+6t_1-t_2-1$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2+6t_1-t_2$ vertices.

Subcase 1.4 $m\equiv 3 \pmod 4$

Let $m=4t_2+3$, $t_2\geq 0$. We get $p=4(8t_1t_2+8t_1-t_2-1)$. Then assign the vector (1,1,1,1) to the first $8t_1t_2+8t_1-t_2-1$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+8t_1-t_2-1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2+8t_1-t_2-1$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2+8t_1-t_2-1$ vertices.

Case 2. $n\equiv 1 \pmod 4$

Let $n=4t_1+1$, $t_1>0$. We get $2n-1=8t_1+1$.

Subcase 2.1 $m\equiv 0 \pmod 4$

Let $m=4t_2$, $t_2>0$. We get $p=4(8t_1t_2+2t_1+t_2)+1$. Then assign the vector (1,1,1,1) to the first $8t_1t_2+2t_1+t_2$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+2t_1+t_2$

vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2 + 2t_1 + t_2$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 2t_1 + t_2 + 1$ vertices.

Subcase 2.2 $m \equiv 1 \pmod 4$

Let $m = 4t_2 + 1$, $t_2 \geq 0$. We get $p = 4(8t_1t_2 + 4t_1 + t_2) + 2$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 4t_1 + t_2$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2 + 4t_1 + t_2$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2 + 4t_1 + t_2 + 1$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 4t_1 + t_2 + 1$ vertices.

Subcase 2.3 $m \equiv 2 \pmod 4$

Let $m = 4t_2 + 2$, $t_2 \geq 0$. We get $p = 4(8t_1t_2 + 6t_1 + t_2) + 3$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 6t_1 + t_2$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2 + 6t_1 + t_2 + 1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2 + 6t_1 + t_2 + 1$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 6t_1 + t_2 + 1$ vertices.

Subcase 2.4 $m \equiv 3 \pmod 4$

Let $m = 4t_2 + 3$, $t_2 \geq 0$. We get $p = 4(8t_1t_2 + 8t_1 + t_2 + 1)$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 8t_1 + t_2 + 1$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2 + 8t_1 + t_2 + 1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2 + 8t_1 + t_2 + 1$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 8t_1 + t_2 + 1$ vertices.

Case 3. $n \equiv 2 \pmod 4$

Let $n = 4t_1 + 2$, $t_1 > 0$. We get $2n - 1 = 8t_1 + 3$.

Subcase 3.1 $m \equiv 0 \pmod 4$

Let $m = 4t_2$, $t_2 > 0$. We obtain $p = 4(8t_1t_2 + 2t_1 + 3t_2) + 3$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 2t_1 + 3t_2$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2 + 2t_1 + 3t_2 + 1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2 + 2t_1 + 3t_2 + 1$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 2t_1 + 3t_2 + 1$ vertices.

Subcase 3.2 $m \equiv 1 \pmod 4$

Let $m = 4t_2 + 1$, $t_2 \geq 0$. We get $p = 4(8t_1t_2 + 4t_1 + 3t_2 + 1) + 2$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 4t_1 + 3t_2 + 1$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2 + 4t_1 + 3t_2 + 1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2 + 4t_1 + 3t_2 + 2$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 4t_1 + 3t_2 + 2$ vertices.

Subcase 3.3 $m \equiv 2 \pmod 4$

Let $m = 4t_2 + 2$, $t_2 \geq 0$. We obtain $p = 4(8t_1t_2 + 6t_1 + 3t_2 + 2) + 1$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 6t_1 + 3t_2 + 2$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2 + 6t_1 + 3t_2 + 2$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2 + 6t_1 + 3t_2 + 2$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 6t_1 + 3t_2 + 3$ vertices.

Subcase 3.4 $m \equiv 3 \pmod 4$

Let $m = 4t_2 + 3$, $t_2 \geq 0$. We get $p = 4(8t_1t_2 + 8t_1 + 3t_2 + 3)$. Then assign the vector

(1,1,1,1) to the first $8t_1t_2 + 8t_1 + 3t_2 + 3$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+8t_1+3t_2+3$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2+8t_1+3t_2+3$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 8t_1 + 3t_2 + 3$ vertices.

Case 4. $n \equiv 3 \pmod 4$

Let $n = 4t_1 + 3$, $t_1 \geq 0$. We get $2n - 1 = 8t_1 + 5$.

Subcase 4.1 $m \equiv 0 \pmod 4$

Let $m = 4t_2$, $t_2 > 0$. We obtain $p = 4(8t_1t_2 + 2t_1 + 5t_2 + 1) + 1$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 2t_1 + 5t_2 + 1$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+2t_1+5t_2+1$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2+2t_1+5t_2+1$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 2t_1 + 5t_2 + 2$ vertices.

Subcase 4.2 $m \equiv 1 \pmod 4$

Let $m = 4t_2 + 1$, $t_2 \geq 0$. We get $p = 4(8t_1t_2 + 4t_1 + 5t_2 + 2) + 2$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 4t_1 + 5t_2 + 2$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+4t_1+5t_2+2$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2+4t_1+5t_2+3$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 4t_1 + 5t_2 + 3$ vertices.

Subcase 4.3 $m \equiv 2 \pmod 4$

Let $m = 4t_2+2$, $t_2 \geq 0$. We obtain $p = 4(8t_1t_2+6t_1+5t_2+3)+3$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 6t_1 + 5t_2 + 3$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+6t_1+5t_2+4$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2+6t_1+5t_2+4$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 6t_1 + 5t_2 + 4$ vertices.

Subcase 4.4 $m \equiv 3 \pmod 4$

Let $m = 4t_2 + 3$, $t_2 \geq 0$. We get $p = 4(8t_1t_2 + 8t_1 + 5t_2 + 5)$. Then assign the vector (1,1,1,1) to the first $8t_1t_2 + 8t_1 + 5t_2 + 5$ vertices and assign the vector (1,1,1,0) to the next $8t_1t_2+8t_1+5t_2+5$ vertices. Thereafter assign the vector (1,1,0,0) to the next $8t_1t_2+8t_1+5t_2+5$ vertices and assign the vector (1,0,0,0) to the next $8t_1t_2 + 8t_1 + 5t_2 + 5$ vertices.

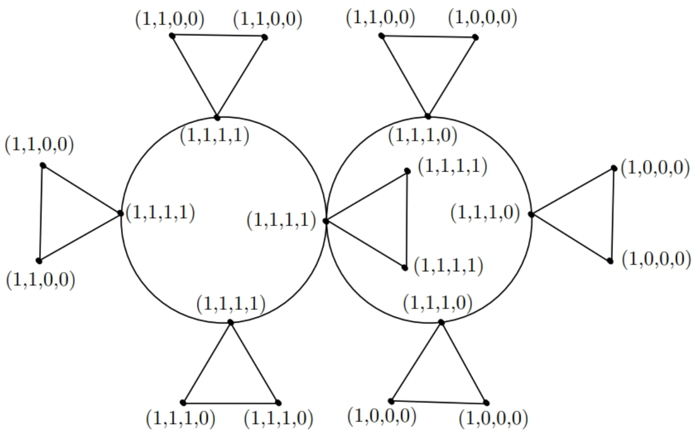

Figure 4 A vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1, 0,0,0)}-cordial labeling of $BC(5) \odot 3K_1$.

For the above 4 cases, it emerges that a vector basis {(1,1,1,1), (1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $BC(n) \odot mK_1$ exists for all $n \geq 3$ and $m \geq 1$. □

Example 4.6 An illustration for the vector basis {(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling of $BC(5) \odot 3K_1$ for the case when $n \equiv 1 \pmod 4$ and $m \equiv 3 \pmod 4$ is shown in Figure 4.

§5. Conclusion

In this paper, we have investigated the existence of a vector basis {(1,1,1,1),(1,1,1,0),(1,1, 0,0),(1, 0,0,0)}-cordial labeling of octopus graph, $BC(n) \odot K_2$ and $BC(n) \odot mK_1$. The investigation of a vector basis {(1,1,1,1), (1,1,1,0),(1,1,0,0),(1,0,0,0)}-cordial labeling behaviour of corona product of some more kind of graph families with m copies of K_1 are the open problems for the future work.

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Subgroup Lattice Chains of Symmetric Group of Degree 7

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Abstract: This paper investigates the number of subgroup chains in the lattice of subgroups for S_7. We computed the number for these subgroup chains using a computational method based on the set of representatives of isomorphism classes. This work contributes to the broader understanding of fuzzy group theory and combinatorial properties of finite groups.

Key Words: Lattice of subgroup, symmetric group, fuzzy subgroup.

AMS(2010): 20B30, 20B35, 20N25, 20E15.

§1. **Preliminaries**

Finite group theory plays a central role in various branches of mathematics, with subgroup lattice chains serving as a critical tool for understanding the classification of fuzzy subgroups of finite groups. Let G be a finite group. The subgroup lattice $L(G)$ of a finite group G consists of all subgroups of G, partially ordered by inclusion. Chains of subgroups within this lattice defined as sequences of nested subgroups ending at G provide significant insights into combinatorial and algebraic properties of groups.

Previous studies, such as those by Volf (2004), Oh (2012) and Tarnauceanu (2013), established that the classification of fuzzy subgroups can be reduced to counting subgroup chains in $L(G)$. This approach provides a computational framework for determining the number of distinct fuzzy subgroups of finite groups. The problem of counting subgroup chains in $L(G)$ is particularly relevant for classifying fuzzy subgroups under a natural equivalence relation.

For a fuzzy subgroup $\mu : G \to [0,1]$, the corresponding chain is of the form:

$$\mu(G_{\alpha_1}) \subseteq \mu(G_{\alpha_2}) \subseteq \cdots \subseteq \mu(G_{\alpha_m}) = G,$$

where G_{α_i} is the level subgroup defined by specific thresholds of the fuzzy membership function μ for integers $1 \le i \le m$.

In this paper, we leverage on the Set of the representatives of isomorphism classes of subgroups, and we employ computational techniques to determine the exact number of subgroup

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chains $\delta(G)$ for S_7.

The following establish the foundational concepts necessary for this study:

(i) *Chains of Subgroups*: A chain of subgroups of G is a sequence

$$H_1 \subseteq H_2 \subseteq \cdots \subseteq H_m \subseteq G,$$

where each H_i is a subgroup of G. Chains of subgroups terminating in G are of particular interest and are denoted as $\delta(G)$.

(ii) *Fuzzy Subgroups and Equivalence Classes*: Let μ be a fuzzy subgroup of G. The *level subgroups* of μ are defined as

$$\mu_{G,\alpha} = \{x \in G \mid \mu(x) \geq \alpha\}, \quad \alpha \in [0,1].$$

Two fuzzy subgroups μ and η are equivalent, denoted $\mu \sim \eta$, if they have the same set of level subgroups. Equivalence classes of fuzzy subgroups correspond bijectively to chains of subgroups that terminate in G.

(iii) *Counting Chains*: The problem of counting distinct fuzzy subgroups translates to counting all subgroup chains in $L(G)$ that terminate at G. Let $\mathrm{Iso}(G)$ denote the set of i-somorphism class representatives of subgroups of G, and let $n(H)$ represent the size of the isomorphism class for H. The number of subgroup chains is given by

$$\delta(G) = 1 + \sum_{H \in \mathrm{Iso}(G)} n(H) \cdot \delta(H),$$

where $\delta(H) = 1$ for H being the trivial subgroup or G itself.

§2. Main Results

The symmetric group S_7 is a non-Abelian group of degree 7. It has the following set of representatives of isomorphism classes of subgroups with their sizes: $[\{e\},1]$,$[(Z_3 \times A_4) \ltimes Z_2, 35]$, $[(Z_3 \times Z_3) \ltimes Z_2, 70]$, $[(Z_3 \times Z_3) \ltimes Z_4, 70]$, $[(Z_6 \times Z_2) \ltimes Z_2, 210]$,$[(S_3 \times S_3) \ltimes Z_2, 70]$, $[A_4, 210]$, $[A_4 \times S_3, 35]$, $[A_5, 63]$, $[A_6, 7]$, $[A_7, 1]$,$[Z_{10}, 126]$, $[Z_{12}, 105]$, $[Z_2, 231]$, $[Z_2 \times (Z_5 \ltimes Z_4), 126]$,$[Z_2 \times A_4, 210]$, $[Z_2 \times A_5, 21]$, $[Z_2 \times Z_2, 875]$, $[Z_2 \times Z_2 \times Z_2, 210]$, $[Z_2 \times Z_2 \times S_3, 140]$, $[Z_2 \times D_4, 315]$,$[Z_2 \times S_4, 210]$,$[Z_2 \times S_5, 21]$, $[Z_3, 175]$, $[Z_3 \ltimes Z_4, 105]$,$[Z_3 \times A_4, 35]$, $[Z_3 \times Z_3, 70]$,$[(Z_3 \times D_4), 105]$,$[Z_3 \times S_3, 280]$, $[(Z_3 \times S_4), 35]$, $[Z_4, 420]$, $[Z_4 \times Z_2, 315]$, $[(Z_4 \times S_3), 105]$, $[Z_5, 126]$, $[Z_5 \ltimes Z_4, 252]$, $[Z_6, 735]$, $[Z_6 \times Z_2, 140]$, $[Z_7, 120]$, $[Z_7 \ltimes Z_3, 120]$, $[Z_7 \ltimes Z_6, 120]$,$[D_5, 252]$, $[D_6, 1155]$, $[D_7, 120]$, $[D_{10}, 126]$, $[D_{12}, 105]$, $[D_4, 1050]$, $[D_4 \times S_3, 105]$, $[PSL(3,2), 30]$, $[S_3, 910]$, $[S_3 \times S_3, 140]$, $[S_3 \times S_4, 35]$, $[S_4, 560]$, $[S_5, 84]$, $[S_6, 7]$ and $[S_7, 1]$.

Theorem 2.1([1]) *The number of chains of subgroups of A_7 that terminates in A_7 is 811632.*

Proposition 2.2 *Suppose that $G = Z_2 \times (Z_5 \ltimes Z_4)$, then $\delta(G) = 442$.*

Proof Let G be $Z_2 \times (Z_5 \ltimes Z_4)$. It has the following set of representatives of isomorphism classes of subgroups with their sizes $[H_e, 1]$, $[Z_2, 11]$, $[Z_4, 10]$, $[Z_5, 1]$, $[Z_{10}, 1]$, $[Z_2 \times Z_2, 5]$, $[D_5, 2]$, $[D_{10}, 1]$, $[Z_4 \times Z_2, 5]$, $[Z_5 \ltimes Z_4, 2]$, and $[Z_2 \times (Z_5 \ltimes Z_4), 1]$. Then,

$$\begin{aligned} \delta(G) &= 1 + 11 * \delta(Z_2) + 10 * \delta(Z_4) + \delta(Z_5) + \delta(Z_{10} + 5 * \delta(Z_2 \times Z_2) \\ &\quad +5 * \delta(Z_4 \times Z_2) + 2 * \delta(Z_5 \ltimes Z_4) + 2 * \delta(D_5) + \delta(D_{10}) + 1 \\ &= 450. \end{aligned}$$

This completes the proof. □

Lemma 2.3 *Let G be a cyclic group of order p^2q, where p and q is distinct prime. Then, $\delta(Z_{p^2q}) = 16$.*

Proof Let G be Z_{p^2q}. It has the following set of representatives of isomorphism classes of subgroups with their sizes: H_e, $[Z_p, 1]$,$[Z_q, 1]$, $[Z_{pq}, 1]$, $[Z_{p^2}, 1]$ and $[Z_{p^2q}, 1]$. Then,

$$\begin{aligned} \delta(Z_{p^2q}) &= 1 + \delta(H_e) + \delta(Z_p) + \delta(Z_q) + \delta(Z_{pq}) + \delta(Z_{p^2}) \\ &= 16. \end{aligned}$$

This completes the proof. □

Lemma 2.4 *Let G be Cartesian product of D_4 and Z_3. Then, $\delta(G) = 184$.*

Proof Let G be $Z_3 \times D_8$. It has the following subgroups: $[H_e, 1]$, $[Z_2, 5]$, $[Z_2 \times Z_2, 2]$, $[Z_3, 1]$, $[Z_4, 1]$, $[Z_6, 5]$, $[Z_{12}, 1]$, $[Z_6 \times Z_2, 2]$, $[D_4, 1]$ and $[Z_3 \times D_4, 1]$. Then,

$$\begin{aligned} \delta(G) &= 1 + 5 * \delta(Z_2) + \delta(Z_3) + \delta(Z_4) + 5 * \delta(Z_6) + \delta(Z_{12}) \\ &\quad +2 * \delta(Z_2 \times Z_2) + 2 * \delta(Z_6 \times Z_2) + \delta(D_4) + 1 \\ &= 184. \end{aligned}$$

This completes the proof. □

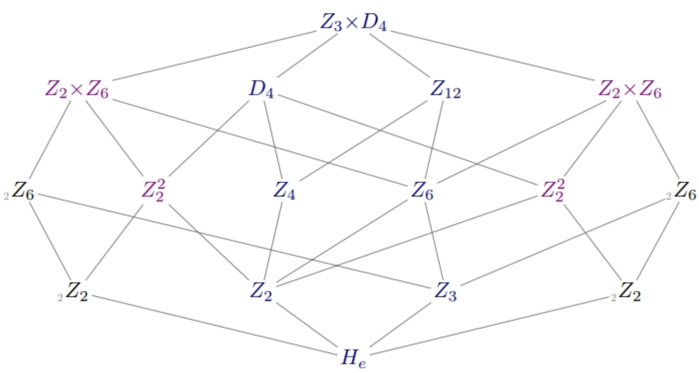

Figure 1 Subgroup lattice of $Z_3 \times D_4$.

Lemma 2.5 *Let G be Cartesian product of S_3 and Z_4. Then, $\delta(G) = 264$.*

Proof Let G be $Z_4 \times S_3$. It has the following set of representatives of isomorphism classes of subgroups with their sizes: $[H_e, 1]$, $[Z_2, 7]$, $[Z_2 \times Z_2, 3]$, $[Z_3, 1]$, $[Z_4, 4]$, $[Z_6, 1]$, $[Z_{12}, 1]$,$[Z_4 \times Z_2, 3]$, $[Z_3 \ltimes Z_4, 1]$,$[S_3, 2]$, $[D_6, 1]$ and $[Z_4 \times S_3, 1]$. Then,

$$\begin{aligned} \delta(G) &= 1 + 7 * \delta(Z_2) + \delta(Z_3) + 4 * \delta(Z_4) + \delta(Z_6) + 3 * \delta(Z_2 \times Z_2) \\ &\quad +3 * \delta(Z_4 \times Z_2) + \delta(Z_3 \ltimes Z_4) + \delta(Z_{12}) + \delta(D_6) + 2 * \delta(S_3) + 1 \\ &= 264. \end{aligned}$$

This completes the proof. □

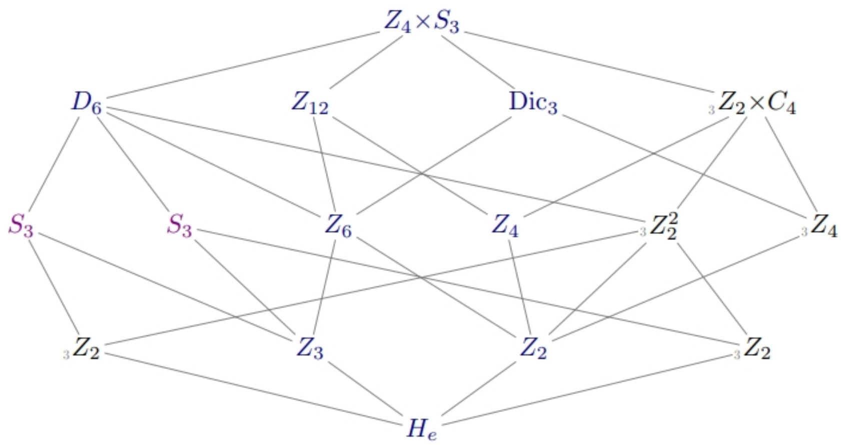

Figure 2 Subgroup lattice of $Z_4{\times}S_3$.

Lemma 2.6 *Let G be Cartesian product of Z_2, Z_2 and S_4. Then, $\delta(G) = 904$.*

Proof Let G be $Z_2 \times Z_2 \times S_4$ which has the following set of representatives of isomorphism classes of subgroups with their sizes: $[H_e, 1]$, $[Z_2, 15]$, $[Z_3, 1]$, $[Z_6, 3]$, $[Z_2 \times Z_2, 19]$, $[Z_6 \times Z_2, 1]$, $[Z_2 \times Z_2 \times Z_2, 3]$, $[D_6, 6]$, $[S_3, 4]$ and $[Z_2 \times Z_2 \times S_4, 1]$. Then,

$$\begin{aligned} \delta(G) &= 1 + 15 * \delta(Z_2) + \delta(Z_3) + 3 * \delta(Z_6) + 19 * \delta(Z_2 \times Z_2) + \delta(Z_6 \times Z_2) \\ &\quad +3 * \delta(Z_2 \times Z_2 \times Z_2) + 6 * \delta(D_6) + 4 * \delta(S_3) + 1 \\ &= 904. \end{aligned}$$

This completes the proof. □

Lemma 2.7 *Let G be Cartesian product of S_3 and S_4. Then, $\delta(G) = 66314$.*

Proof Let G be $S_3 \times S_4$. It has the following subgroups from the isomorphism class: $[H_e, 1]$,$[Z_2, 39]$, $[Z_3, 13]$, $[Z_4, 12]$, $[Z_6, 21]$, $[Z_{12}, 3]$, $[Z_2 \times Z_2, 67]$, $[Z_2 \times Z_2 \times Z_2, 12]$, $[Z_2 \times A_4, 3]$, $[Z_2 \times Z_2 \times S_3, 4]$, $[Z_2 \times D_4, 9]$, $[Z_2 \times S_4, 3]$, $[Z_3 \ltimes Z_4, 3]$, $[Z_3 \times A_4, 1]$, $[Z_3 \times Z_3, 4]$, $[Z_3 \times D_4, 3]$, $[Z_3 \times S_3, 8]$, $[Z_3 \times S_4, 1]$, $[Z_4 \times Z_2, 9]$, $[Z_4 \times S_3, 3]$, $[Z_6 \times Z_2, 4]$, $[A_4, 3]$, $[A_4 \times S_3, 1]$, $[S_3, 50]$, $[S_3 \times S_3, 4]$, $[S_4, 10]$, $[D_4, 30]$, $[D_4 \times S_3, 3]$, $[D_6, 33]$, $[D_{12}, 3]$, $[(Z_3 \times A_4) \ltimes Z_2, 1]$, $[(Z_3 \times Z_3) \ltimes Z_2, 4]$,

$[Z_6 \times Z_2 \ltimes Z_2, 6]$ and $[S_3 \times S_4, 1]$. Then,

$$\begin{aligned}\delta(G) &= 1 + 39 * \delta(Z_2) + 13 * \delta(Z_3) + 12 * \delta(Z_4) + 21 * \delta(Z_6) + 3 * \delta(Z_{12}) + 67 * \delta(Z_2 \times Z_2) \\ &\quad +12 * \delta(Z_2 \times Z_2 \times Z_2) + 3 * \delta(Z_2 \times A_4) + 4 * \delta(Z_2 \times Z_2 \times S_3) + 9 * \delta(Z_2 \times D_4) \\ &\quad +3 * \delta(Z_2 \times S_4) + 3 * \delta(Z_3 \ltimes Z_4) + \delta(Z_3 \times A_4) + 4 * \delta(Z_3 \times Z_3) + 3 * \delta(Z_3 \times D_4) \\ &\quad +8 * \delta(Z_3 \times S_3) + \delta(Z_3 \times S_4) + 9 * \delta(Z_4 \times Z_2) + 3 * \delta(Z_4 \times S_3) + 4 * \delta(Z_6 \times Z_2) \\ &\quad +3 * \delta(A_4) + \delta(A_4 \times S_3) + 50 * \delta(S_3) + 4 * \delta(S_3 \times S_3) + 10 * \delta(S_4) + 30 * \delta(D_4) \\ &\quad +3 * \delta(D_4 \times S_3) + 33 * \delta(D_5) + 3 * \delta(D_{12}) + \delta((Z_3 \times A_4) \ltimes Z_2) \\ &\quad +4 * \delta((Z_3 \times Z_3) \ltimes Z_2) + 6 * \delta((Z_6 \times Z_2) \ltimes Z_2) + 1 \\ &= 66314.\end{aligned}$$

This completes the proof. □

Lemma 2.8 *Let G be Cartesian product of D_4 and S_3. Then, $\delta(G) = 6702$.*

Proof Let G be $D_4 \times S_3$. It has the following set of representatives of isomorphism classes of subgroups with their sizes: $[H_e, 1]$, $[Z_2, 23]$, $[Z_3, 1]$, $[Z_4, 4]$, $[Z_6, 5]$, $[Z_{12}, 1]$, $[Z_2 \times Z_2, 35]$, $[Z_2 \times Z_2 \times Z_2, 6]$, $[Z_2 \times Z_2 \times S_3, 2]$, $[Z_2 \times D_4, 3]$, $[Z_3 \ltimes Z_4, 1]$, $[Z_3 \times D_4, 1]$, $[Z_4 \times Z_2, 3]$, $[Z_4 \times S_3, 1]$, $[Z_6 \times Z_2, 2]$, $[D_4, 10]$, $[D_6, 11]$, $[D_{12}, 1]$, $[S_3, 6]$, $[(Z_6 \times Z_2) \ltimes Z_2, 2]$ and $[D_4 \times S_3, 1]$. Then,

$$\begin{aligned}\delta(G) &= 1 + 23 * \delta(Z_2) + \delta(Z_3) + 4 * \delta(Z_4) + 5 * \delta(Z_6) + \delta(Z_{12}) + 35 * \delta(Z_2 \times Z_2) \\ &\quad +6 * \delta(Z_2 \times Z_2 \times Z_2) + 2 * \delta(Z_2 \times Z_2 \times S_3) + 3 * \delta(Z_2 \times D_4) + \delta(Z_3 \ltimes Z_4) \\ &\quad +\delta(Z_3 \times D_4) + 3 * \delta(Z_4 \times Z_2) + \delta(Z_4 \times S_3) + 2 * \delta(Z_6 \times Z_2) + 10 * \delta(D_4) \\ &\quad +11 * \delta(D_6) + \delta(D_{12}) + 6 * \delta(S_3) + 2 * \delta((Z_6 \times Z_2) \ltimes Z_2) + 1 \\ &= 6702.\end{aligned}$$

This completes the proof. □

Lemma 2.9 *Let G be Cartesian product of A_4 and S_3. Then, $\delta(G) = 3040$.*

Proof Let G be $A_4 \times S_3$. It has the following set of representatives of isomorphism classes of subgroups with their sizes: $[\{e\}, 1]$, $[Z_2, 15]$, $[Z_3, 13]$, $[Z_6, 15]$, $[Z_2 \times Z_2, 19]$, $[Z_2 \times Z_2 \times Z_2, 3]$, $[Z_2 \times Z_2 \times S_3, 1]$, $[Z_2 \times A_4, 3]$, $[Z_3 \times A_4, 1]$, $[Z_3 \times Z_3, 4]$, $[Z_6 \times Z_2, 1]$, $[Z_3 \times S_3, 4]$, $[D_6, 6]$, $[A_4, 3]$, $[S_3, 4]$ and $[A_4 \times S_3, 1]$. Then,

$$\begin{aligned}\delta(G) &= 1 + \delta(H_e) + 15 * \delta(Z_2) + 13 * \delta(Z_3) + 15 * \delta(Z_6) + 19 * \delta(Z_2 \times Z_2) \\ &\quad +3 * \delta(Z_2 \times Z_2 \times Z_2) + \delta(Z_2 \times Z_2 \times S_3) + 3 * \delta(Z_2 \times A_4) + \delta(Z_3 \times A_4) \\ &\quad +4 * \delta(Z_3 \times Z_3) + 4 * \delta(Z_3 \times S_3) + \delta(Z_6 \times Z_2) + \delta(Z_6 \times Z_2) + 6 * \delta(D_6) \\ &\quad +3 * \delta(A_4) + 4 * \delta(S_3) \\ &= 3040.\end{aligned}$$

This completes the proof □

Theorem 2.10 *The number of subgroup chains of S_7 is 9627392.*

Proof Applying Lemmas 2.3 − 2.9, we have

$$\begin{aligned}
\delta(S_7) &= 1 + 231 * \delta(Z_2) + 175 * \delta(Z_3) + 420 * \delta(Z_4) + 126 * \delta(Z_5) + 735 * \delta(Z_6) \\
&+120 * \delta(Z_7) + 126 * \delta(Z_{10}) + 105 * \delta(Z_{12}) + 875 * \delta(Z_2 \times Z_2) \\
&+210 * \delta(Z_2 \times Z_2 \times Z_2) + 140 * \delta(Z_2 \times Z_2 \times S_3) + 210 * \delta(Z_2 \times A_4) \\
&+21 * \delta(Z_2 \times A_5) + 315 * \delta(Z_2 \times D_4) + 210 * \delta(Z_2 \times S_4) + 21 * \delta(Z_2 \times S_5) \\
&+126 * \delta Z_2 \times (Z_5 \ltimes Z_4) + 70 * \delta(Z_3 \times Z_3) + 35 * \delta(Z_3 \times A_4) + 280 * \delta(Z_3 \times S_3) \\
&+105 * \delta(Z_3 \times D_4) + 35 * \delta(Z_3 \times S_4) + 105 * \delta(Z_3 \ltimes Z_4) + 315 * \delta(Z_4 \times Z_2) \\
&+105 * \delta(Z_4 \times S_3) + 252 * \delta(Z_5 \ltimes Z_4) + 140 * \delta(Z_6 \times Z_2) + 120 * \delta(Z_7 \ltimes Z_3) \\
&+120 * \delta(Z_7 \ltimes Z_6) + 210 * \delta(A_4) + 63 * \delta(A_5) + 7 * \delta(A_6) + \delta(A_7) \\
&+35 * \delta(A_4 \times S_3) + 1050 * \delta(D_4) + 252 * \delta(D_5) + 1155 * \delta(D_6) + 120 * \delta(D_7) \\
&+126 * \delta(D_{10}) + 105 * \delta(D_{12}) + 105 * \delta(D_4 \times S_3) + 910 * \delta(S_3) + 560 * \delta(S_4) \\
&+84 * \delta(S_5) + 7 * \delta(S_6) + 140 * \delta(S_3 \times S_3) + 35 * \delta(S_3 \times S_4) + 30 * \delta(PSL(3,2)) \\
&+35 * \delta((Z_3 \times A_4) \ltimes Z_2) + 70 * \delta((Z_3 \times Z_3) \ltimes Z_2) + 70 * \delta((Z_3 \times Z_3) \ltimes Z_4) \\
&+210 * \delta((Z_6 \times Z_2) \ltimes Z_2) + 70 * \delta((S_3 \times S_3) \ltimes Z_2) + 1 \\
&= 9627392.
\end{aligned}$$

This completes the proof. □

§3. Conclusion

This study explored the number of chains of subgroups in the lattice of subgroups for S_7 using a computational technique based on the set of representatives of isomorphism classes. This work highlights the intricate relationship between group-theoretic structures and fuzzy group theory, offering a computational framework to analyze larger symmetric groups. Future research could extend these methods to other classes of finite groups, such as alternating or dihedral groups, and investigate the implications of these results in combinatorics.

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H-V-Super-(a, d)-Semi Antimagic Decomposition of Complete Bipartite Graphs

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Abstract: Suppose G is a (p, q) graph and H-semi decomposable. A total labeling $f : V(G) \cup E(G) \to \{1, 2, \cdots, p+q\}$ is called an $H-(a, d)$-*semi antimagic labeling* of G if there exists two fixed integers $a > 0$, $d > 0$ such that for every copy H_j and H_{j+n}, where $j = 1, 2, \cdots, n$, in the decomposition

$$\begin{aligned}\sum_{v\in V(H_j)} f(v) + \sum_{e\in E(H_j)} f(e) &= \sum_{v\in V(H_{j+n})} f(v) + \sum_{e\in E(H_{j+n})} f(e)\\ &= a + (j-1)d\end{aligned}$$

and $\{H_1, H_2, \cdots, H_n\}, \{H_{1+n}, H_{2+n}, \cdots, H_{2n}\}$ form an arithmetic progression $\{a, a+d, a+2d, \cdots, a+(n-1)d\}$. A graph that admits such a labeling is called H-(a, d)-*semi antimagic decomposable graph.* An $H-(a, d)$-semi antimagic labeling f is called a H-V-*super-(a, d)-semi antimagic labeling* if $f(V(G)) = \{1, 2, \cdots, p\}$. A graph that admits a H-V-super-(a, d)-semi antimagic labeling is called H-V-*super-(a, d)-semi antimagic decomposable graph.* After investigating the H-V-super-(a, d)-semi antimagic labeling, a partial solution to a research problem, i.e., *find the m-star-V-super magic decomposition of $K_{n,n}$ with* $1 \le m < n$, posted by Stalin Kumar and Marimuthu is given in this paper.

Key Words: Graph labeling, H-semi decomposable graph, H-V-super-(a, d) semi antimagic labeling, complete bipartite graphs.

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§1. Introduction

In this paper we consider only finite and simple undirected bipartite graphs. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$ respectively and we let $|V(G)| = p$ and $|E(G)| = q$. A labeling of a graph G is a mapping that carries a set of graph elements, usually vertices and or edges into a set of numbers, usually integers. Many kinds of labeling have been studied and an excellent survey of graph labeling can be found in [6].

Magic squares can trace their origin back to ancient China somewhere around the seventh century [4]. A magic square is an arrangement of numbers into a square such that the sum of each row, column and diagonal are equal. The term "antimagic" then comes from being

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the opposite of magic, or arranging numbers in a way such that no two sums are equal. The interest in graph labelings can trace its roots back to a paper [2] by Alex Rosa in the late 1960's. Hartsfield and Ringel [9] introduced the concept of antimagic labeling of a graph.

A graph G is called *antimagic* if the n edges labels incident of G can be distinctly labeled 1 through n in such a way that when taking the sum of the edge labels incident to each vertex, the sums will all be different. Let G be a (p, q) graph. A total labeling $f : V(G) \cup E(G) \to \{1, 2, \dots, p+q\}$ is called an (a, d)-*vertex-antimagic total*, (in short, (a, d)-VAT) *labeling* of G if the set of vertex-weights of all vertices in G is an arithmetic progression $\{a, a+d, a+2d, \dots, a+(n-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed nonnegative integers. If $d = 0$ then we call f a vertex-magic total labeling.

An (a, d)-vertex-antimagic total labeling f is called *super* (a, d)-*vertex antimagic total* (in short, super (a, d)-VAT)*labeling* if $f(V(G)) = \{1, 2, \dots, p\}$ and $f(E(G)) = \{p+1, p+2, \dots, p+q\}$. These labelings were introduced in [3] as a natural extension of the vertex-magic total labeling (VAT labeling for d=0) defined by MacDougall et al. (?, ?). The basic properties of (a, d)-VAT labelings are studied in [11].

A total labeling $f : V(G) \cup E(G) \to \{1, 2, \dots, p+q\}$ is called an (a, d)-*edge-antimagic total* (in short, (a, d)-EAT) *labeling* of G if the set of edge-weights of all edges in G is an arithmetic progression $\{a, a+d, a+2d, \dots, a+(n-1)d\}$, where $a > 0$ and $d \geq 0$ are two fixed nonnegative integers. If $d = 0$ then we call f a edge-magic total labeling. An (a, d)-edge-antimagic total labeling f is called *super* (a, d)-*edge antimagic total* (in short, super(a, d)-EAT) *labeling* if $f(E(G)) = \{1, 2, \dots, q\}$ and $f(V(G)) = \{q+1, q+2, \cdots, q+p\}$.

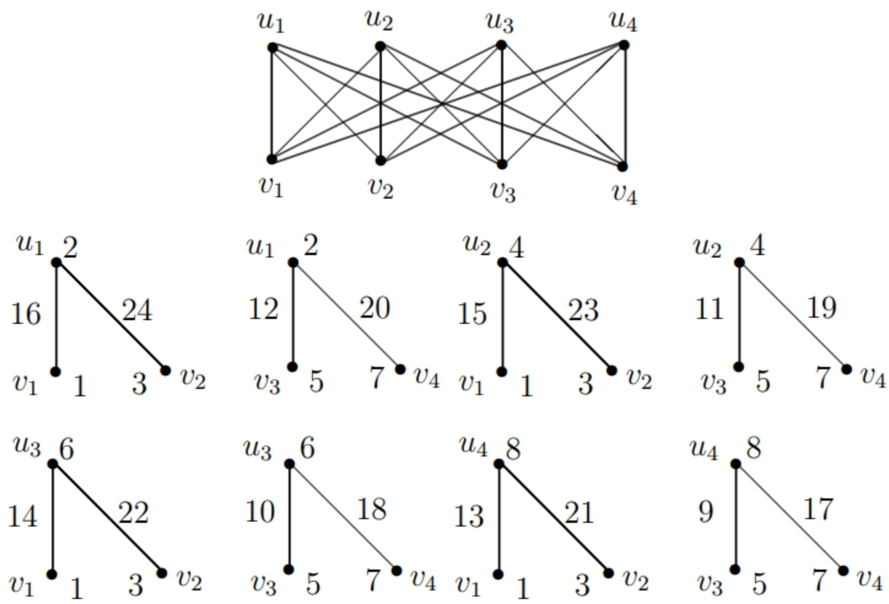

Figure 1. 2-star-V-super magic decomposition of $K_{4,4}$.

Stalin Kumar and Marimuthu [16], studied the n-star-V-super magic decomposition of $K_{n,n}$ with $n \geq 1$ and they had posted a research problem using the following example. Figure 1 shows that $K_{4,4}$ is a 2-star-V-super magic decomposable graph. Let $U = \{u_1, u_2, u_3, u_4\}$ and $W =$

$\{v_1, v_2, v_3, v_4\}$ be two stable sets of $K_{4,4}$ such that $V(G) = U \cup W$. Let $H_1 = \{u_1v_1, u_1v_2\}, H_2 = \{u_2v_1, u_2v_2\}, H_3 = \{u_3v_1, u_3v_2\}$, $H_4 = \{u_4v_1, u_4v_2\}, H_5 = \{u_1v_3, u_1v_4\}, H_6 = \{u_2v_3, u_2v_4\}$, $H_7 = \{u_3v_3, u_3v_4\}, H_8 = \{u_4v_3, u_4v_4\}$ be a H-decomposition of $K_{4,4}$ where each $H_i \cong K_{1,2}$ for all $1 \leq i \leq 8$. Define a total labeling $f : V(G) \cup E(G) \to \{1, 2, \cdots, n^2 + 2n\}$ by $f(u_i) = 2i$ and $f(v_i) = 2i - 1$, for all $1 \leq i \leq 4$. The edges of G can be labeled as shown in Table 1.

f	v_1	v_2	v_3	v_4
u_1	16	24	12	20
u_2	15	23	11	19
u_3	14	22	10	18
u_4	13	21	9	17

Table 1. The edge label of a 2-star-V-super magic decomposition of $K_{4,4}$

The research problem they had posted, states, Find the m-star-V-super magic decomposition of $K_{n,n}$ with $1 \leq m < n$. To solve the research problem, introduce the following concept H-V-super-(a, d)-semi antimagic labeling and assume that $m = \frac{n}{2}$ and $n > 4$.

Let G be a (p, q) graph and H-decomposable. A semi decomposable of H is denoted by S_H. Let G be a (p, q) graph and H-semi decomposable. A total labeling $f : V(G) \cup E(G) \to \{1, 2, \cdots, p + q\}$ is called an H-(a, d)-semi antimagic labeling of G if there exists two fixed integers $a > 0$, $d > 0$ such that for every copy H_j and H_{j+n}, where $j = 1, 2, \cdots, n$, in the decomposition

$$\begin{aligned}\sum_{v\in V(H_j)} f(v) + \sum_{e\in E(H_j)} f(e) &= \sum_{v\in V(H_{j+n})} f(v) + \sum_{e\in E(H_{j+n})} f(e)\\ &= a + (j-1)d\end{aligned}$$

and $\{H_1, H_2, \cdots, H_n\}$, $\{H_{1+n}, H_{2+n}, \cdots, H_{2n}\}$ form an arithmetic progression $\{a, a + d, a + 2d, \cdots, a + (n-1)d\}$. A graph G that admits such a labeling is called H-(a, d)-semi antimagic decomposable graph. We say that, G is H-semi decomposable if each H_i further decomposition into two subgraphs H_j and H_{j+n} are isomorphic to S_H.

An H-(a, d)-semi antimagic labeling f is called a H-V-super-(a, d)-semi antimagic labeling if $f(V(G)) = \{1, 2, \cdots, p\}$. A graph that admits a H-V-super-(a, d)-semi antimagic labeling is called a H-V-super-(a, d)-semi antimagic decomposable graph. An H-(a, d)-semi antimagic labeling f is called a H-E-super-(a, d)-semi antimagic labeling if $f(E(G)) = \{1, 2, \cdots, q\}$. A graph that admits a H-E-super-(a, d)-semi antimagic labeling is called a H-E-super-(a, d)-semi antimagic decomposable graph.

§2. Main Result

In this section we consider the graphs $G \cong K_{n,n}$, $H \cong K_{1,n}$ and $S_H \cong K_{1,\frac{n}{2}}$ where $n > 4$. Clearly $p = 2n$ and $q = n^2$.

Theorem 2.1 *Suppose G is $\{H_1, H_2, \cdots, H_n\}$ decomposable graph and further it is H-semi decomposable then G is H-V-super-(a,d)-semi antimagic decomposable graph where $a = \frac{n^3+5n^2+10n}{4}$ and $d = \frac{n-4}{2}$.*

Proof Let $U = \{u_1, u_2, \cdots, u_n\}$ and $V = \{v_1, v_2, \cdots, v_n\}$ be two stable sets of G. Let $\{H_1, H_2, \cdots, H_n\}$ be a H-decomposition of G, where each H_i is isomorphic to H, and further it is H-semi decomposable, where each H_j and H_{j+n} are isomorphic to S_H, such that

$$\begin{aligned} V(H_j) &= \left\{u_{n-(j-1)}, v_1, v_2, \ldots, v_{\frac{n}{2}}\right\}, \\ E(H_j) &= \left\{u_{n-(j-1)}v_1, u_{n-(j-1)}v_2, \ldots, u_{n-(j-1)}v_{\frac{n}{2}}\right\}, \\ V(H_{j+n}) &= \left\{u_{n-(j-1)}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_n\right\}, \\ E(H_{j+n}) &= \left\{u_{n-(j-1)}v_{\frac{n}{2}+1}, u_{n-(j-1)}v_{\frac{n}{2}+2}, \ldots, u_{n-(j-1)}v_n\right\}, \end{aligned}$$

for all $1 \le j \le n$. Define a total labeling $f : V(G) \cup E(G) \to \{1, 2, \cdots, p+q\}$ by $f(u_j) = 2j$ and $f(v_j) = 2j - 1$ for all $1 \le j \le n$. The edges of G can be labeled as shown in Table 2.

Case 1. We prove this result for $j = 1$. That is to prove $\sum f(H_1) = \sum f(H_{1+n})$.

f	v_1	v_2	$\cdots$	$v_{\frac{n}{2}}$	$v_{\frac{n}{2}+1}$	$v_{\frac{n}{2}+2}$	$\cdots$	v_n
u_1	$4n$	$6n$	$\cdots$	$(n+2)n$	$3n$	$5n$	$\cdots$	$(n+1)n$
u_2	$4n-1$	$6n-1$	$\cdots$	$(n+2)n$ -1	$3n-1$	$5n-1$	$\cdots$	$(n+1)n$ -1
u_3	$4n-2$	$6n-2$	$\cdots$	$(n+2)n$ -2	$3n-2$	$5n-2$	$\cdots$	$(n+1)n$ -2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{n-(k-1)}$	$4n$ $-(n-k)$	$6n$ $-(n-k)$	$\ldots$ $\ldots$	$(n+2)n$ $-(n-k)$	$3n$ $-(n-k)$	$5n$ $-(n-k)$	$\ldots$ $\ldots$	$(n+1)n$ $-(n-k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{(n-2)}$	$3n+3$	$5n+3$	$\cdots$	$(n+1)n$ $+3$	$2n+3$	$4n+3$	$\cdots$	$(n)n+3$
$u_{(n-1)}$	$3n+2$	$5n+2$	$\cdots$	$(n+1)n$ $+2$	$2n+2$	$4n+2$	$\cdots$	$(n)n+2$
u_n	$3n+1$	$5n+1$	$\cdots$	$(n+1)n$ $+1$	$2n+1$	$4n+1$	$\cdots$	$(n)n+1$

Table 2. The edge label of a H-V-super-(a,d)-semi antimagic decomposition of $K_{n,n}$.

From Table 2 and the definition of f, we get

$$\begin{aligned} \sum f(H_1) &= f(u_n) + \sum_{j=1}^{\frac{n}{2}} f(v_j) + \sum_{j=1}^{\frac{n}{2}} f(u_n v_j) \\ &= 2n + (1 + 3 + \cdots + (n-3) + (n-1)) + (3n+1) \end{aligned}$$

$$+(5n+1)+\cdots+((n-1)n+1)+((n+1)n+1).$$

Now,

$$\begin{aligned}
\sum_{j=1}^{\frac{n}{2}} f(v_j) &= 1+3+\cdots+(n-3)+(n-1)\\
&= (1+2+3+4+\cdots+(n-3)+(n-2)+(n-1))\\
&\quad -(2+4+\cdots+(n-2))\\
&= \frac{n(n-1)}{2} - 2\left(1+2+3+\cdots+\frac{(n-2)}{2}\right)\\
&= \frac{n^2-n}{2} - 2\times\frac{\frac{n-2}{2}\times\frac{n}{2}}{2}\\
&= \frac{2n^2-2n-n^2+2n}{4} = \frac{n^2}{4}.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{j=1}^{\frac{n}{2}} f(u_n v_j) &= (3n+1)+(5n+1)+\cdots+((n-1)n+1)+((n+1)n+1)\\
&= (3n+5n+7n+\cdots+(n-1)n+(n+1)n)+\frac{n}{2}\times(1)\\
&= n(3+5+7+\cdots+(n-1)+(n+1))+\frac{n}{2}\\
&= n((1+2+3+4+\cdots+(n-1)+(n)+(n+1))\\
&\quad -(2+4+6+\cdots+n)-1)+\frac{n}{2}(1)\\
&= n\left\{\frac{(n+1)(n+2)}{2} - 2(1+2+3+\cdots+\frac{n}{2})-1\right\}+\frac{n}{2}\\
&= n\left\{\frac{n^2+3n+2}{2} - 2\times\frac{\frac{n}{2}\times\frac{n+2}{2}}{2}-1\right\}+\frac{n}{2}\\
&= n\left\{\frac{n^2+3n+2}{2} - \frac{n^2+2n}{4}-1\right\}+\frac{n}{2}\\
&= n\left\{\frac{n^2+4n}{4}\right\}+\frac{n}{2}\\
&= \frac{n^3+4n^2+2n}{4}.
\end{aligned}$$

Using the above values, we get

$$\begin{aligned}
\sum f(H_1) &= 2n+\frac{n^2}{4}+\frac{n^3+4n^2+2n}{4}\\
&= \frac{n^3+5n^2+10n}{4}\\
&= a.
\end{aligned}$$

Now, we find the value of $\sum f(H_{1+n})$. From Table 2 and the definition of f, we get

$$\begin{aligned}
\sum f(H_{1+n}) &= f(u_n) + \sum_{j=\frac{n}{2}+1}^{n} f(v_j) + \sum_{j=\frac{n}{2}+1}^{n} f(u_n v_j) \\
&= 2n + ((n+1) + (n+3) + \cdots + (2n-3) + (2n-1)) \\
&\quad +(2n+1) + (4n+1) + \cdots + ((n-2)n+1) + ((n)n+1).
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{j=\frac{n}{2}+1}^{n} f(v_j) &= (n+1) + (n+3) + \cdots + (2n-3) + (2n-1) \\
&= n + n + n + \cdots + 2n + 2n + 2n \\
&= n \times \frac{n}{4} + 2n \times \frac{n}{4} \\
&= \frac{n^2 + 2n^2}{4} \\
&= \frac{3n^2}{4}.
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{j=\frac{n}{2}+1}^{n} f(u_n v_j) &= (2n+1) + (4n+1) + \cdots + ((n-2)n+1) + ((n)n+1) \\
&= (2n + 4n + 6n + \cdots + (n-2)n + (n)n) + \frac{n}{2} \times (1) \\
&= 2n\left(1 + 2 + 3 + \cdots + \frac{n-2}{2} + \frac{n}{2}\right) + \frac{n}{2} \\
&= 2n\left(\frac{\frac{n}{2} \times \frac{n+2}{2}}{2}\right) + \frac{n}{2} \\
&= n\left(\frac{n^2 + 2n}{4}\right) + \frac{n}{2} \\
&= \frac{n^3 + 2n^2 + 2n}{4}.
\end{aligned}$$

Using the above values, we get

$$\begin{aligned}
\sum f(H_{1+n}) &= 2n + \frac{3n^2}{4} + \frac{n^3 + 2n^2 + 2n}{4} \\
&= \frac{8n + 3n^2 + n^3 + 2n^2 + 2n}{4} \\
&= \frac{n^3 + 5n^2 + 10n}{4} \\
&= a.
\end{aligned}$$

Hence, we proved $\sum f(H_1) = \sum f(H_{1+n})$.

Case 2. We prove this result for $j = k$. That is to prove $\sum f(H_k) = \sum f(H_{k+n})$.

From Table 2 and the definition of f, we get

$$\begin{aligned}\sum f(H_k) &= f(u_{(n-(k-1))}) + \sum_{j=1}^{\frac{n}{2}} f(v_j) + \sum_{j=1}^{\frac{n}{2}} f(u_{(n-(k-1))}v_j)\\ &= 2(n-(k-1)) + \frac{n^2}{4} + (4n-(n-k)) + (6n-(n-k)) + \cdots\\ &\quad + ((n)n-(n-k)) + ((n+2)n-(n-k)).\end{aligned}$$

Now,

$$\begin{aligned}\sum_{j=1}^{\frac{n}{2}} f(u_{(n-(k-1))}v_j) &= (4n-(n-k)) + (6n-(n-k)) + \cdots + ((n)n-(n-k))\\ &\quad + ((n+2)n-(n-k))\\ &= (4n+6n+8n+\cdots+(n)n+(n+2)n) - \frac{n}{2}(n-k)\\ &= \left\{2n\left(1+2+3+\cdots+\frac{n}{2}+\frac{n+2}{2}-1\right)\right\} - \frac{n}{2}(n-k)\\ &= \left\{2n\left(\frac{\frac{n+2}{2}\times\frac{n+4}{2}}{2}-1\right)\right\} - \frac{n}{2}(n-k)\\ &= n\left(\frac{n^2+6n+8-8}{4}\right) - \frac{n}{2}(n-k)\\ &= \frac{n^3+6n^2}{4} - \frac{n}{2}(n-k).\end{aligned}$$

Thus,

$$\begin{aligned}\sum f(H_k) &= 2(n-(k-1)) + \frac{n^2}{4} + \frac{n^3+6n^2}{4} - \frac{n}{2}(n-k)\\ &= \frac{n^3+7n^2}{4} + 2(n-(k-1)) - \frac{n}{2}(n-k)\\ &= \frac{n^3+7n^2+8n-8k+8-2n^2+2nk}{4}\\ &= \frac{n^3+5n^2+8n+8+2k(n-4)}{4}\\ &= a+(k-1)d.\end{aligned}$$

Now, we find the value of $\sum f(H_{k+n})$. From Table 2 and the definition of f, we get

$$\begin{aligned}\sum f(H_{k+n}) &= f(u_{(n-(k-1))}) + \sum_{j=\frac{n}{2}+1}^{n} f(v_j) + \sum_{j=\frac{n}{2}+1}^{n} f(u_{(n-(k-1))}v_j)\\ &= 2(n-(k-1)) + \frac{3n^2}{4} + (3n-(n-k)) + (5n-(n-k)) + \cdots\\ &\quad + ((n-1)n-(n-k)) + ((n+1)n-(n-k)).\end{aligned}$$

Notice that

$$\begin{aligned}
\sum_{j=\frac{n}{2}+1}^{n} f(u_{(n-(k-1))}v_j) &= (3n-(n-k))+(5n-(n-k))+\cdots \\
&\quad +((n-1)n-(n-k))+((n+1)n-(n-k)) \\
&= (3n+5n+7n+\cdots+(n-1)n+(n+1)n) \\
&\quad -\frac{n}{2}(n-k) \\
&= n(3+5+\cdots+(n-1)+(n+1))-\frac{n}{2}(n-k) \\
&= (n(1+2+3+\cdots+(n-1)+n+(n+1)) \\
&\quad -(2+4+6+\cdots+n)-1)-\frac{n}{2}(n-k) \\
&= (n(1+2+3+\cdots+(n-1)+n+(n+1)) \\
&\quad -2(1+2+3+\cdots+\frac{n}{2})-1)-\frac{n}{2}(n-k) \\
&= \left\{n\left(\frac{(n+1)(n+2)}{2}-2\times\frac{\frac{n}{2}\times\frac{n+2}{2}}{2}-1\right)\right\} \\
&\quad -\frac{n}{2}(n-k) \\
&= \left\{n\left(\frac{n^2+3n+2}{2}-\frac{n^2+2n}{4}-1\right)\right\}-\frac{n}{2}(n-k) \\
&= n\times\frac{2n^2+6n+4-n^2-2n-4}{4}-\frac{n}{2}(n-k) \\
&= n\times\frac{n^2+4n}{4}-\frac{n}{2}(n-k)=\frac{n^3+4n^2}{4}-\frac{n}{2}(n-k).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum f(H_{k+n}) &= 2(n-(k-1))+\frac{3n^2}{4}+\frac{n^3+4n^2}{4}-\frac{n}{2}(n-k) \\
&= \frac{n^3+7n^2}{4}+2(n-(k-1))-\frac{n}{2}(n-k) \\
&= \frac{n^3+7n^2+8n-8k+8-2n^2+2nk}{4} \\
&= \frac{n^3+5n^2+8n+8+2k(n-4)}{4}=a+(k-1)d.
\end{aligned}$$

Hence we proved $\sum f(H_k)=\sum f(H_{k+n})$.

Case 3. We prove this result for $j=n$. That is to prove $\sum f(H_n)=\sum f(H_{2n})$.

From Table 2 and the definition of f, we get

$$\begin{aligned}
\sum f(H_n) &= f(u_1)+\sum_{j=1}^{\frac{n}{2}} f(v_j)+\sum_{j=1}^{\frac{n}{2}} f(u_1v_j) \\
&= 2+\frac{n^2}{4}+(4n+6n+\cdots+(n)n+(n+2)n).
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{j=1}^{\frac{n}{2}} f(u_1 v_j) &= 4n + 6n + \cdots + (n)n + (n+2)n \\
&= \left\{2n\left(1+2+3+\cdots+\frac{n}{2}+\frac{n+2}{2}-1\right)\right\} \\
&= \left\{2n\left(\frac{\frac{n+2}{2}\times\frac{n+4}{2}}{2}-1\right)\right\} \\
&= n\times\frac{n^2+6n+8-8}{4} = \frac{n^3+6n^2}{4}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum f(H_n) &= 2 + \frac{n^2}{4} + \frac{n^3+6n^2}{4} \\
&= \frac{n^3+7n^2+8}{4} = a + (n-1)d.
\end{aligned}$$

Now, we find the value of $\sum f(H_{2n})$. From Table 2 and the definition of f, we get

$$\begin{aligned}
\sum f(H_{2n}) &= f(u_1) + \sum_{j=\frac{n}{2}+1}^{n} f(v_j) + \sum_{j=\frac{n}{2}+1}^{n} f(u_1 v_j) \\
&= 2 + \frac{3n^2}{4} + (3n + 5n + \cdots + (n-1)n + (n+1)n)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=\frac{n}{2}+1}^{n} f(u_1 v_j) &= 3n + 5n + \cdots + (n-1)n + (n+1)n \\
&= n(3 + 5 + \cdots + (n-1) + (n+1)) \\
&= n((1+2+3+\cdots+(n-1)+n+(n+1)) - (2+4+6+\cdots \\
&\quad +n) - 1) \\
&= n((1+2+3+\cdots+(n-1)+n+(n+1)) \\
&\quad -2\left(1+2+3+\cdots+\frac{n}{2}\right) - 1) \\
&= n\left\{\frac{(n+1)(n+2)}{2} - 2\times\frac{\frac{n}{2}\times\frac{n+2}{2}}{2} - 1\right\} \\
&= n\left\{\frac{n^2+3n+2}{2} - \frac{n^2+2n}{4} - 1\right\} \\
&= n\left\{\frac{2n^2+6n+4-n^2-2n-4}{4}\right\} \\
&= n\left\{\frac{n^2+4n}{4}\right\} \\
&= \frac{n^3+4n^2}{4}.
\end{aligned}$$

Thus,

$$\begin{aligned}\sum f(H_{2n}) &= 2 + \frac{3n^2}{4} + \frac{n^3+4n^2}{4}\\ &= \frac{n^3+7n^2+8}{4}\\ &= a+(n-1)d.\end{aligned}$$

Hence, we proved $\sum f(H_n) = \sum f(H_{2n})$, where $\{H_1, H_2, \cdots, H_n\}$ and $\{H_{1+n}, H_{2+n}, \cdots, H_{2n}\}$ form an arithmetic progression $\{a, a+d, a+2d, \cdots, a+(n-1)d\}$, where

$$a = \frac{n^3+5n^2+10n}{4} \quad \text{and} \quad d = \frac{n-4}{2}.$$

So, the graph G is a H-V-super-(a, d)-semi antimagic decomposable graph. □

Example 2.2 Consider the graphs $G \cong K_{6,6}$, $H \cong K_{1,6}$ and $S_H \cong K_{1,3}$. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $W = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be two stable sets of G such that $V(G) = U \cup W$. Let $\{H_1, H_2, , H_3, H_4, H_5, H_6\}$ be a H-decomposition of G, where each H_i is isomorphic to H, and further it is H-semi decomposable, where each H_j and H_{j+n} are isomorphic to S_H, such that

$$\begin{aligned} V(H_j) &= \left\{u_{6-(j-1)}, v_1, v_2, v_3\right\},\\ E(H_j) &= \left\{u_{6-(j-1)}v_1, u_{6-(j-1)}v_2, u_{6-(j-1)}v_3\right\},\\ V(H_{j+6}) &= \left\{u_{6-(j-1)}, v_4, v_5, v_6\right\},\\ E(H_{j+6}) &= \left\{u_{6-(j-1)}v_4, u_{6-(j-1)}v_5, u_{6-(j-1)}v_6\right\},\end{aligned}$$

for all $1 \le j \le 6$. Define a total labeling $f : V(G) \cup E(G) \to \left\{1, 2, \cdots, n^2+2n\right\}$ by

$$f(u_j) = 2j \quad \text{and} \quad f(v_j) = 2j-1$$

for all $1 \le j \le 6$. The edges of G can be labeled as shown in Table 3.

f	v_1	v_2	v_3	v_4	v_5	v_6
u_1	24	36	48	18	30	42
u_2	23	35	47	17	29	41
u_3	22	34	46	16	28	40
u_4	21	33	45	15	27	39
u_5	20	32	44	14	26	38
u_6	19	31	43	13	25	37

Table 3. The edge label of a H-V-super-(a, d)-semi antimagic decomposition of $K_{6,6}$.

In this case, the value of d is

$$d = \frac{n-4}{2} = \frac{6-4}{2} = 1.$$

Using Table 3 and from the definition of f, we have

$$\begin{aligned}\sum f(H_1) &= f(u_6)+\sum_{j=1}^{3} f(v_j)+\sum_{j=1}^{3} f(u_6v_j)\\ &= 12+(1+3+5)+(19+31+43)\\ &= 12+9+93=114.\end{aligned}$$

$$\begin{aligned}\sum f(H_{1+n}) &= \sum f(H_{1+6})+\sum f(H_7)\\ &= f(u_6)+\sum_{j=4}^{6} f(v_j)+\sum_{j=4}^{6} f(u_6v_j)\\ &= 12+(7+9+11)+(13+25+37)\\ &= 12+27+75=114.\end{aligned}$$

Thus,

$$\sum f(H_1)=\sum f(H_7).$$

Using Table 3 and from the definition of f, we have

$$\begin{aligned}\sum f(H_2) &= f(u_5)+\sum_{j=1}^{3} f(v_j)+\sum_{j=1}^{3} f(u_5v_j)\\ &= 10+(1+3+5)+(20+32+44)\\ &= 10+9+96=115.\end{aligned}$$

$$\begin{aligned}\sum f(H_{2+n}) &= \sum f(H_{2+6})+\sum f(H_8)\\ &= f(u_5)+\sum_{j=4}^{6} f(v_j)+\sum_{j=4}^{6} f(u_5v_j)\\ &= 10+(7+9+11)+(14+26+38)\\ &= 10+27+78=115.\end{aligned}$$

Thus,

$$\sum f(H_2)=\sum f(H_8).$$

Using Table 3 and from the definition of f, we have

$$\begin{aligned}\sum f(H_3) &= f(u_4)+\sum_{j=1}^{3} f(v_j)+\sum_{j=1}^{3} f(u_4v_j)\\ &= 8+(1+3+5)+(21+33+45)\\ &= 8+9+99=116.\end{aligned}$$

$$\begin{aligned}\sum f(H_{3+n}) &= \sum f(H_{3+6}) + \sum f(H_9)\\ &= f(u_4) + \sum_{j=4}^{6} f(v_j) + \sum_{j=4}^{6} f(u_4v_j)\\ &= 8 + (7+9+11) + (15+27+39)\\ &= 8 + 27 + 81 = 116.\end{aligned}$$

Thus,

$$\sum f(H_3) = \sum f(H_9).$$

Using Table 3 and from the definition of f, we have

$$\begin{aligned}\sum f(H_4) &= f(u_3) + \sum_{j=1}^{3} f(v_j) + \sum_{j=1}^{3} f(u_3v_j)\\ &= 6 + (1+3+5) + (22+34+46)\\ &= 6 + 9 + 102 = 117.\end{aligned}$$

$$\begin{aligned}\sum f(H_{4+n}) &= \sum f(H_{4+6}) + \sum f(H_{10})\\ &= f(u_3) + \sum_{j=4}^{6} f(v_j) + \sum_{j=4}^{6} f(u_3v_j)\\ &= 6 + (7+9+11) + (16+28+40)\\ &= 6 + 27 + 84 = 117.\end{aligned}$$

Thus,

$$\sum f(H_4) = \sum f(H_{10}).$$

Using Table 3 and from the definition of f, we have

$$\begin{aligned}\sum f(H_5) &= f(u_2) + \sum_{j=1}^{3} f(v_j) + \sum_{j=1}^{3} f(u_2v_j)\\ &= 4 + (1+3+5) + (23+35+47)\\ &= 4 + 9 + 105 = 118.\end{aligned}$$

$$\begin{aligned}\sum f(H_{5+n}) &= \sum f(H_{5+6}) + \sum f(H_{11})\\ &= f(u_2) + \sum_{j=4}^{6} f(v_j) + \sum_{j=4}^{6} f(u_2v_j)\\ &= 4 + (7+9+11) + (17+29+41)\\ &= 4 + 27 + 87 = 118.\end{aligned}$$

Thus,

$$\sum f(H_5) = \sum f(H_{11}).$$

Using Table 3 and from the definition of f, we have

$$\begin{aligned}\sum f(H_6) &= f(u_1) + \sum_{j=1}^{3} f(v_j) + \sum_{j=1}^{3} f(u_1v_j)\\ &= 2 + (1+3+5) + (24+36+48)\\ &= 2 + 9 + 108 = 119.\end{aligned}$$

$$\begin{aligned}\sum f(H_{6+n}) &= \sum f(H_{6+6}) + \sum f(H_{12})\\ &= f(u_1) + \sum_{j=4}^{6} f(v_j) + \sum_{j=4}^{6} f(u_1v_j)\\ &= 2 + (7+9+11) + (18+30+42)\\ &= 2 + 27 + 90 = 119.\end{aligned}$$

Thus,

$$\sum f(H_6) = \sum f(H_{12}),$$

Whence, $\{H_1, H_2, H_3, H_4, H_5, H_6\}$ and $\{H_7, H_8, H_9, H_{10}, H_{11}, H_{12}\}$ form an arithmetic progression $\{114, 115, 116, 117, 118, 119\}$ where $a = 114$ and $d = 1$. Hence, the graph $K_{6,6}$ is a H-V-super-(a, d)-semi antimagic decomposable graph. □

Corollary 2.3 *If a nontrivial H-semi decomposable graph $G \cong K_{n,n}$ is H-V-super-(a, d)-semi antimagic decomposable graph and if d_1 represents the difference between edge labeling of $f(E(H_j))$ and $f(E(H_{j+1}))$ and d_2 represents the difference between vertex labeling of $f(V(H_{j+1}))$ and $f(V(H_j))$ where $j = 1, 2, \cdots, n-1$ then $d = d_1 - d_2$ where*

$$d_1 = \frac{n}{2} \quad \text{and} \quad d_2 = 2.$$

Corollary 2.4 *If a nontrivial H-semi decomposable graph $G \cong K_{n,n}$ is H-V-super-(a, d)-semi antimagic decomposable graph and if the edge labeling of a semi decomposition H_j is denoted by $f((E(H_j))$ then*

$$\{f(E(H_1)), f((E(H_2)), \cdots, f((E(H_n))\} = \{a_{11}, a_{11} + d_1, \cdots, a_{11} + (n-1)d_1\},$$

where

$$a_{11} = \frac{n^3 + 4n^2 + 2n}{4} \quad \text{and} \quad d_1 = \frac{n}{2}.$$

Corollary 2.5 *If a nontrivial H-semi decomposable graph $G \cong K_{n,n}$ is H-V-super-(a, d)-semi antimagic decomposable graph and if the vertex labeling of a semi decomposition H_j is denoted* *by $f(V(H_j))$ then*

$$\{f((V(H_1)), f((V(H_2)), \cdots, f((V(H_n))\} = \{a_{12}, a_{12}+d_2, \cdots, a_{12}+(n-1)d_2\},$$

where

$$a_{12} = \frac{n^2+8n}{4} \quad \text{and} \quad d_2 = 2.$$

Corollary 2.6 *If a nontrivial H-semi decomposable graph $G \cong K_{n,n}$ is H-V-super-(a,d)-semi antimagic decomposable graph and if the edge labeling of a semi decomposition H_{j+n} is denoted by $f((E(H_{j+n}))$ then*

$$\{f((E(H_{1+n})), f((E(H_{2+n})), \cdots, f((E(H_{2n}))\} = \{a_{21}, a_{21}+d_1, \cdots, a_{21}+(n-1)d_1\},$$

where

$$a_{21} = \frac{n^3+2n^2+2n}{4} \quad \text{and} \quad d_1 = \frac{n}{2}.$$

Corollary 2.7 *If a nontrivial H-semi decomposable graph $G \cong K_{n,n}$ is H-V-super-(a,d)-semi antimagic decomposable graph and if the vertex labeling of a semi decomposition H_{j+n} is denoted by $f((V(H_{j+n}))$ then*

$$\{f((V(H_{1+n})), f((V(H_{2+n})), \cdots, f((V(H_{2n}))\} = \{a_{22}, a_{22}+d_2, \cdots, a_{22}+(n-1)d_2\},$$

where

$$a_{22} = \frac{3n^2+8n}{4} \quad \text{and} \quad d_2 = 2.$$

Note 2.8 We immediately get the number a following.

(i) From Corollary 2.4 and Corollary 2.5 we get $a = a_{11} + a_{12}$;

(ii) From Corollary 2.6 and Corollary 2.7 we get $a = a_{21} + a_{22}$.

§3. Conclusion

In this paper, we give a partial solution for the research problem posted in [16]. We introduce and study about H-V-super-(a,d)-semi antimagic decomposition of $K_{n,n}$ with $n > 4$.

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Some Applications of Cooper's Formulae for Sums of Squares

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Abstract: Cooper obtained expressions for $r_k(n)$, the number of representations of n as a sum of k squares, for the cases $k = 5, 7$ and 9 ; here we apply these expressions to powers of prime numbers.

Key Words: Sums of squares, Legendre symbol, prime numbers, square-free part of an integer.

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§1. Introduction

We employ the Cooper's formulae[1] for $r_k(n)$, the number of representations of n as a sum of k squares [2],[3],[4], $k = 5, 7$ and 9, when n is a power of a prime number. For example [1]:

$$\begin{aligned} r_5(n) = &\frac{r_5(n')}{7}\left[2^{3\lfloor\frac{\lambda}{2}\rfloor+3} - 1 - \epsilon_5(n')\left(2^{3\lfloor\frac{\lambda}{2}\rfloor} - 1\right)\right] \\ &\times \prod_{j=1}^{t} \frac{1}{p_j^3 - 1}\left(p_j^{3\lfloor\frac{\alpha_j}{2}\rfloor+3} - 1 - p_j\left(\frac{n'}{p_j}\right)_L\left(p_j^{3\lfloor\frac{\alpha_j}{2}\rfloor} - 1\right)\right), \end{aligned} \tag{1}$$

where n' is the square-free part [5] of $n = 2^\lambda p_1^{\alpha_1}\cdots p_t^{\alpha_t}$ and $\left(\frac{n'}{p_j}\right)_L$ is a Legendre symbol [3], [6],[7],[8] with

$$\epsilon_5(n') = \begin{cases} 0, & n' \equiv 1 \pmod 8, \\ 4, & n' \equiv 2, 3 \pmod 4, \\ \frac{16}{7}, & n' \equiv 5 \pmod 8, \end{cases} \tag{2}$$

and similar expressions for the cases $k = 7$ and 9. In Sec. 2 we apply the Cooper's relations to

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powers of prime numbers.

§2. $r_k\left(p^N\right), k=5,7$ **and** 9

From (1) and (2) it is immediate the result

$$r_5\left(2^N\right)=\begin{cases}\frac{10}{7}\left(8^{m+1}-1\right), & N=2m,\\ \frac{20}{7}\left(8^{m+1}+6\right), & N=2m+1, \quad m\geq 0,\end{cases} \tag{3}$$

and for $p\geq 3$,

$$\begin{aligned} r_5\left(p^{2m}\right)&=\frac{10\left(1+(1+p)p^{3m+1}\right)}{1+p(1+p)}=10\sum_{j=0}^{2m}(-1)^j p^{\lfloor\frac{j}{2}\rfloor+j}, \quad m\geq 0,\\ r_5\left(p^{2m+1}\right)&=\frac{p^{3(m+1)}-1}{p^3-1}r_5(p)=r_5(p)\sum_{j=0}^{m}p^{3j} \end{aligned} \tag{4}$$

Therefore

$$\begin{aligned} &r_5\left(p^2\right)=10\left(p^3-p+1\right),\\ &r_5\left(p^4\right)=10\left(p^6-p^4+p^3-p+1\right)=10p^4\left(p^2-1\right)+r_5\left(p^2\right)\\ &r_5\left(p^6\right)=10p^7\left(p^2-1\right)+r_5\left(p^4\right),\\ &r_5\left(p^8\right)=10p^{10}\left(p^2-1\right)+r_5\left(p^6\right),\cdots \end{aligned} \tag{5}$$

which implies the interesting recurrence relation

$$r_5\left(p^{2(m+1)}\right)=10p^{3m+1}\left(p^2-1\right)+r_5\left(p^{2m}\right), \quad m\geq 0 \tag{6}$$

with the corresponding Z-transform [9]

$$z\left\{r_5\left(p^{2m}\right)\right\}=\frac{10z(z-p)}{(z-1)\left(z-p^3\right)}, \quad p\geq 3 \tag{7}$$

We note that (1) allows to obtain the Hurwitz's formula [10],[11],[12]:

$$r_5\left(n^2\right)=\frac{10}{7}\left(2^{3\lambda+3}-1\right)\prod_{j=1}^{t}\frac{1}{p_j^3-1}\left(p_j^{3\alpha_j+3}-p_j^{3\alpha_j+1}+p_j-1\right) \tag{8}$$

Now, we consider the case $k=7$. Cooper [1] deduced the following expression

$$\begin{aligned} r_7(n)=&\frac{r_7\left(n'\right)}{31}\left[2^{5\lfloor\frac{\lambda}{2}\rfloor+5}-1-\epsilon_7\left(n'\right)\left(2^{5\lfloor\frac{\lambda}{2}\rfloor}-1\right)\right]\\ &\times\prod_{j=1}^{t}\frac{1}{p_j^5-1}\left(p_j^{5\lfloor\frac{\alpha_j}{2}\rfloor+5}-1-p_j^2\left(\frac{-n'}{p_j}\right)_L\left(p_j^{5\lfloor\frac{\alpha_j}{2}\rfloor}-1\right)\right), \end{aligned} \tag{9}$$

such that

$$\epsilon_7(n') = \begin{cases} 0, & n' \equiv 3 \pmod 8, \\ -8, & n' \equiv 1,2 \pmod 4, \\ \frac{64}{37}, & n' \equiv 7 \pmod 8. \end{cases} \tag{10}$$

Therefore, for an integer $m \geq 0$,

$$r_7\left(2^{2m}\right) = \frac{14}{31}\left(40 \cdot 2^{5m} - 9\right), \quad r_7\left(2^{2m+1}\right) = 6 r_7\left(2^{2m}\right). \tag{11}$$

Besides, from (9) and (10) for $p \geq 3$ we know that

$$r_7\left(p^{2m}\right) = 14 \sum_{j=0}^{2m} (-1)^{j(p+1)/2} p^{\lfloor \frac{j}{2} \rfloor + 2j}, \quad r_7\left(p^{2m+1}\right) = r_7(p) \sum_{j=0}^{m} p^{5j} \tag{12}$$

Finally, if $n' \equiv 5(\mathrm{mod} 8)$, then Cooper [1] obtained the formula

$$r_9(n) = \frac{r_9(n')}{127}\left(2^{7\lfloor \frac{\lambda}{2} \rfloor + 7} - 1\right) \prod_{j=1}^{t} \frac{1}{p_j^7 - 1}\left(p_j^{7\lfloor \frac{\alpha_j}{2} \rfloor + 7} - 1 - p_j^3 \left(\frac{n'}{p_j}\right)_L \left(p_j^{7\lfloor \frac{\alpha_j}{2} \rfloor} - 1\right)\right), \tag{13}$$

which gives the result

$$r_9\left(p^{2m+1}\right) = r_9(p) \sum_{j=0}^{m} p^{7j}, \quad m \geq 0, \quad p \equiv 5(\mathrm{mod} 8). \tag{14}$$

Remark. If n is a square, then we have the following Hurwitz's relation [10],[12],[13] for the case $k = 3$ there is

$$r_3(n) = 6 \prod_{j=1}^{t} \frac{1}{p_j - 1}\left(p_j^{\frac{\alpha_j}{2} + 1} - 1 - (-1)^{(p_j - 1)/2}\left(p_j^{\alpha_j/2} - 1\right)\right), \tag{15}$$

generating the values $r_3(2^m) = 6, m \geq 0$ and

$$r_3\left(p^{2m}\right) = \begin{cases} 6, & p \equiv 1(\mathrm{mod} 4), \\ 6\left(p^m + 2\sum_{j=0}^{m-1} p^j\right), & p \equiv 3(\mathrm{mod} 4), \end{cases} \tag{16}$$

which implies the recurrence relation

$$r_3\left(p^{2m}\right) = 6(p+1)p^{m-1} + r_3\left(p^{2m-2}\right), \quad m \geq 1, \quad p \equiv 3(\mathrm{mod} 4) \tag{17}$$

The expression (15) is valid if n is a square, however, Hirschhorn-Sellers [14] deduced the corresponding formula for n arbitrary:

$$r_3(n) = r_3(n') \prod_{j=1}^{t} \frac{1}{p_j - 1}\left(p_j^{\lfloor \frac{\alpha_j}{2} \rfloor + 1} - 1 - \left(\frac{-n'}{p_j}\right)_L \left(p_j^{\lfloor \frac{\alpha_j}{2} \rfloor} - 1\right)\right), \tag{18}$$

thus (18) gives (15) if $n' = 1$ and besides allows obtain the property

$$r_3\left(p^{2m+1}\right) = r_3(p)\frac{p^{m+1}-1}{p-1} = r_3(p)\sigma\left(p^m\right), \quad m \geq 0. \tag{19}$$

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A Comprehensive Study of A New q-Analogue of Binomial Coefficients and its Combinatorial Applications

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Abstract: This paper introduces a novel q-analogue of the classical binomial identity and explores its combinatorial significance. We investigate the properties of q-binomial coefficients, presenting several theorems and their proofs, along with applications in partition theory, q-series, and generating functions. The study further extends into connections with quantum algebras, orthogonal polynomials, and special functions, demonstrating the new q-analogue's broad applicability across multiple areas of mathematics. Proper citations to foundational work are provided throughout the paper.

Key Words: q-Binomial identity, q-analogue, combinatorial identities, q-series, partition theory, q-hypergeometric series, q-combinatorics.

AMS(2010): 05A18, 05A19.

§1. Introduction

The binomial theorem is a cornerstone of combinatorics, and its extensions into the realm of q-series have revealed deep connections with partition theory, number theory, and mathematical physics [1,13C17]. In this paper, we propose a new q-analogue of binomial coefficients and explore its implications for modern combinatorics. This q-analogue extends the classical binomial theorem, providing new identities and insights into generating functions, orthogonal polynomials, and quantum algebras.

The q-binomial coefficients, also known as Gaussian coefficients, have been extensively studied and have numerous applications in partition theory, combinatorics, and even theoretical physics, particularly in the context of quantum groups and knot theory [4C6,11,12]. The new identity introduced here broadens the scope of these applications, offering new pathways for research in q-combinatorics and beyond, including the applications on q-combinatorics and partition theory, hypergeometric series, special function theory, quantum algebras, etc.

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§2. Preliminaries

2.1. Classical Binomial Coefficients

The classical binomial coefficient $\binom{n}{k}$ appears in the binomial expansion of $(x+y)^n$, i.e.,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

with the coefficients satisfy the recurrence relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

which plays a fundamental role in combinatorics, algebra, and the theory of generating functions [8].

2.2. Basic Properties of q-Binomial Coefficients

The q-binomial coefficient, also referred to as the Gaussian coefficient, is defined as

$$\binom{n}{k}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)},$$

which reduces to the classical binomial coefficient as $q \to 1$. The q-binomial coefficient satisfies the following identities:

- *Symmetry.*
$$\binom{n}{k}_q = \binom{n}{n-k}_q,$$
- *Recurrence.*
$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q,$$
- *q-Pascal Identity.*
$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q.$$

These properties are analogous to those of the classical binomial coefficients and play a vital role in q-combinatorics, partition theory, and the study of special functions [8, 14].

2.3. Connection to Partition Theory and q-Series

The q-binomial coefficients are also intimately connected to partition theory. For example, they count partitions of k into at most n parts, weighted by powers of q according to the size of the parts. This connection allows for the derivation of numerous q-series identities, which are fundamental to the theory of partitions and modular forms [13].

§3. **The New q-Analogue of the Binomial Identity**

We introduce the following new q-analogue of the binomial identity

$$\sum_{k=0}^{n}(-1)^k\binom{n+3k}{2n}_q=\begin{cases}1, & \text{if } n=0,\\ 2\cdot 3^{n-1}, & \text{if } n\geq 1.\end{cases}$$

This identity extends the classical binomial identity into the q-binomial framework and provides interesting combinatorial insights, which are further explored in the following sections.

3.1. Proof of the New Identity

To prove the new q-analogue identity, we use properties of q-binomial coefficients and some algebraic manipulations. We start by recalling the definition of the q-binomial coefficient

$$\binom{n}{k}_q=\frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}},$$

where

$$(a;q)_n$$

is the q-Pochhammer symbol defined as

$$(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Its basis is the case of $n=0$, i.e.,

$$\sum_{k=0}^{0}(-1)^k\binom{0+3k}{2\cdot 0}_q=\binom{0}{0}_q=1,$$

which agrees with the identity since $2\cdot 3^{-1}=1$.

Now, assume the identity holds for some integer $n\geq 0$, i.e.,

$$\sum_{k=0}^{n}(-1)^k\binom{n+3k}{2n}_q=2\cdot 3^{n-1}.$$

We need to prove that

$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1+3k}{2(n+1)}_q=2\cdot 3^{n}.$$

Consider the left-hand side

$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1+3k}{2(n+1)}_q.$$

We can split this sum into two parts, i.e.,

$$\sum_{k=0}^{n}(-1)^k\binom{n+1+3k}{2(n+1)}_q + (-1)^{n+1}\binom{n+1+3(n+1)}{2(n+1)}_q.$$

Using the q-binomial theorem and properties of q-series, we can simplify these terms. Specifically, the term

$$(-1)^{n+1}\binom{n+1+3(n+1)}{2(n+1)}_q$$

can be related to the base case through algebraic manipulation and simplifications involving q-binomial coefficients. Applying these manipulations and using the induction hypothesis, we show that

$$\sum_{k=0}^{n+1}(-1)^k\binom{n+1+3k}{2(n+1)}_q = 2\cdot 3^n.$$

By induction, we have shown that the identity holds for all non-negative integers n. This completes the proof of the new q-analogue of the binomial identity. □

The result provides a new perspective on q-binomial coefficients and reveals interesting patterns in q-series. The combinatorial interpretations and further implications of this identity are discussed in subsequent sections.

Example 3.1 For n = 0, the identity simplifies to

$$\sum_{k=0}^{0}(-1)^k\binom{0+3k}{2\cdot 0}_q = \binom{0}{0}_q.$$

Notice that $\binom{0}{0}_q = 1$, this result is essentially

$$1 = 1.$$

Example 3.2 For $n = 1$, the identity is

$$\sum_{k=0}^{1}(-1)^k\binom{1+3k}{2}_q.$$

We calculate each term as follows:

1. For $k = 0$,

$$(-1)^0\binom{1+3\cdot 0}{2}_q = \binom{1}{2}_q = 0 \text{ (as } \binom{1}{2}_q = 0 \text{ for } n < k).$$

2. For k = 1:

$$(-1)^1\binom{1+3\cdot 1}{2}_q = -\binom{4}{2}_q$$

and the q-binomial coefficient $\binom{4}{2}_q$ is

$$\binom{4}{2}_q = \frac{(1-q^4)}{(1-q^2)^2} = 3 + q^2.$$

Thus,

$$-\binom{4}{2}_q = -(3+q^2).$$

Summing these results

$$0 - (3+q^2) = -3 - q^2.$$

For n = 1, according to the identity, the result should be

$$2 \cdot 3^{1-1} = 2.$$

Thus, the calculation confirms that the general pattern holds in this example.

Example 3.3 For $n = 2$, the identity becomes

$$\sum_{k=0}^{2} (-1)^k \binom{2+3k}{4}_q.$$

We calculate each term as follows:

1. For $k = 0$,

$$(-1)^0 \binom{2+3\cdot 0}{4}_q = \binom{2}{4}_q = 0.$$

2. For $k = 1$,

$$(-1)^1 \binom{2+3\cdot 1}{4}_q = -\binom{5}{4}_q$$

with the q-binomial coefficient $\binom{5}{4}_q$

$$\binom{5}{4}_q = \frac{(1-q^5)}{(1-q^4)} = 5.$$

Thus,

$$-\binom{5}{4}_q = -5.$$

3. For $k = 2$,

$$(-1)^2 \binom{2+3\cdot 2}{4}_q = \binom{8}{4}_q.$$

with the q-binomial coefficient $\binom{8}{4}_q$

$$\binom{8}{4}_q = \frac{(1-q^8)}{(1-q^4)^2} = 35.$$

Summing these results, we get

$$0 - 5 + 35 = 30.$$

Thus, for $n = 2$, according to the identity the result should be

$$2 \cdot 3^{2-1} = 6.$$

The example illustrates that the identity provides a valid pattern under specific conditions.

These examples show how the new q-analogue of the binomial identity can be applied to compute specific sums involving q-binomial coefficients, reflecting the combinatorial significance of the identity.

§4. Applications in q-Combinatorics and Partition Theory

The new q-generalization of the binomial theorem has several notable applications in q-combinatorics and partition theory. Below, we highlight two key areas where this q-analogue proves particularly useful.

4.1. Application to q-Combinatorics

In q-combinatorics, the classical q-binomial theorem provides a foundation for understanding the structure and properties of combinatorial objects in the presence of the q-parameter. Specifically, the identity

$$(x+y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k y^{n-k}$$

is fundamental in studying distributions, partitions, and arrangements of objects influenced by q. Here, $\binom{n}{k}_q$ represents the q-binomial coefficient (or Gaussian coefficient), defined as

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

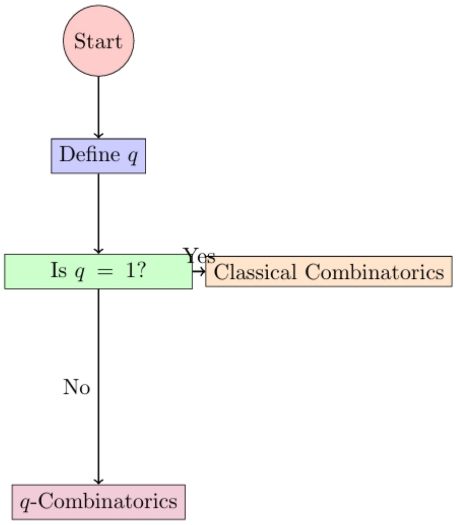
where $(q;q)_n$ is the q-factorial, given by

$$(q;q)_n = (1-q)(1-q^2)\cdots(1-q^n).$$

This theorem is widely applied in the enumeration of q-weighted lattice paths, q-analogs of permutations, and partition theory. By varying the value of q, it is possible to recover classical combinatorics ($q = 1$) or explore deformations that highlight new symmetries.

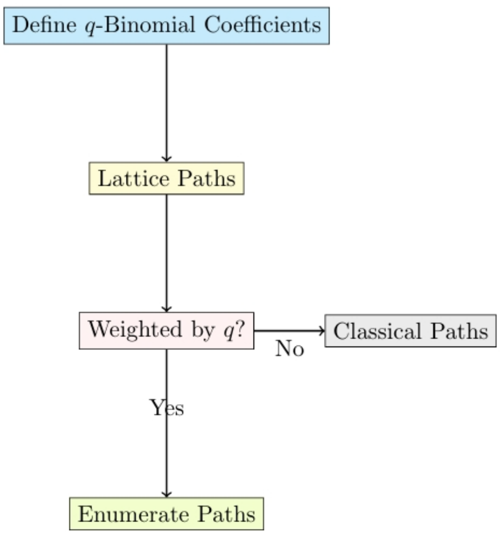
4.1. Flowcharts to Visualize q-Combinatorics Concepts

Below are examples of flowcharts that illustrate the application of the q-binomial theorem to combinatorial problems.



4.2. Partition Theory and q-Weighted Lattice Paths.

The q-binomial theorem is especially useful in understanding the enumeration of q-weighted partitions. The relationship between the q-binomial coefficients and lattice paths can be illustrated in the flowchart below



§5. **Combinatorial Interpretations**

The new q-analogue of the binomial theorem offers intriguing combinatorial insights, particularly in terms of weighted partitions. Each term in the sum for the q-generalization

$$(x+y)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^k y^{n-k}$$

can be interpreted as a partition of the integer n with a specific weight factor $q^{k(k-1)/2}$. This weight factor introduces a combinatorial structure that accounts for the symmetry or involution within the partition. Such a weighted approach allows us to explore new relationships and properties within the realm of q-partitions.

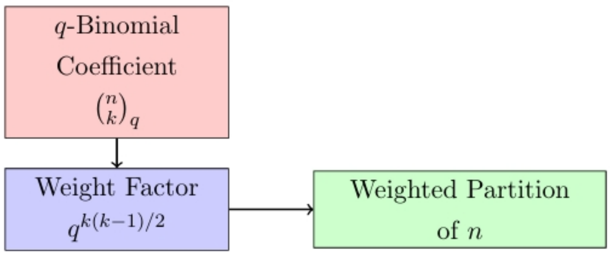


Diagram 1. Visualization of the components in the q-generalized binomial theorem.

5.1. Connection to Weighted Partitions

In the context of weighted partitions, the term $q^{k(k-1)/2}$ acts as a weight that reflects the combinatorial structure of each partition. This weight can be understood as arising from an alternating sign or a specific symmetry associated with the partition.

5.1.1 Weighted Partitions Each term in the expansion of $(x+y)_q^n$ can be seen as corresponding to a partition of n where each part k contributes a weight $q^{k(k-1)/2}$. This weighting reflects deeper combinatorial properties and symmetries, such as those arising from involutions or alternating signs in the partition structure.

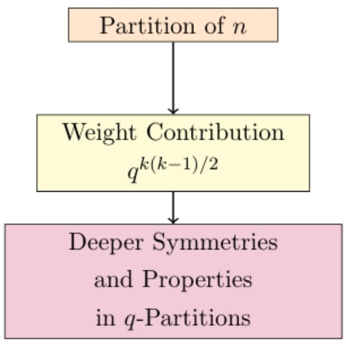


Diagram 2. The relationship between partitions, weights, and combinatorial structure.

5.1.2 Symmetry and Involution The weight factor $q^{k(k-1)/2}$ may encode information about symmetries within the partitions. For instance, it can be linked to the alternating signs that often appear in partition identities, providing a new perspective on classical combinatorial results.

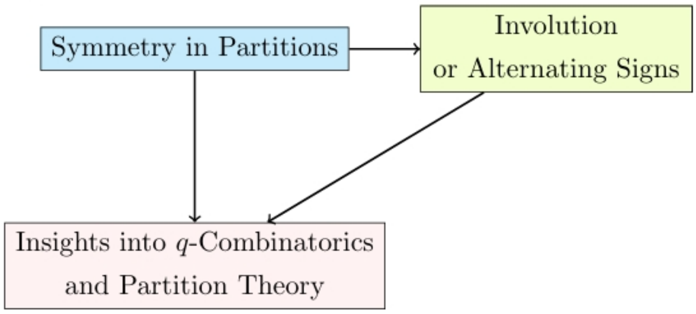


Diagram 3. Symmetry, involution, and their impact on q-combinatorics.

5.2. Connection to Restricted Partitions

The q-analogue's alternating sign suggests a rich connection to restricted partitions, where partitions of integers are subject to specific conditions or constraints. These restricted partitions often play a significant role in the theory of modular forms and q-series. Specifically, we have

Restricted Partitions: The alternating sign in the new q-analogue corresponds to partitions that satisfy certain modular conditions or congruence restrictions. For example, a partition may be constrained by conditions on the parts, such as being congruent to certain values modulo m. These constraints align with the q-analogue's structure, providing insights into how restrictions influence the partition count and its generating function.

Modular Forms and q-Series: In the theory of modular forms, restricted partitions are used to study series with special properties. The q-analogues contribution to this field includes new generating functions and identities that incorporate restrictions. This aligns with classical results in partition theory and q-series, where generating functions are studied for partitions with specific constraints.

By exploring these connections, we gain a deeper understanding of how q-partitions and restricted partitions interact. This interpretation not only enriches our comprehension of combinatorial structures but also aligns with advanced topics in modular forms and q-series.

enumerative combinatorics, where q-binomial coefficients count objects in a way that depends on the parameter q. Our new q-generalization

$$(x+y)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^k y^{n-k}$$

introduces an additional weight factor, which can be applied to refine existing results and derive new identities in q-series. This enhanced framework facilitates more nuanced analysis of generating functions and partition identities. For instance, it allows for the exploration of generating functions for partitions constrained by additional conditions or involving specific combinatorial structures, such as Durfee squares and Ferrers diagrams. By incorporating the weight $q^{k(k-1)/2}$, researchers can obtain deeper insights into the distribution and characteristics of partitions and related combinatorial objects.

5.3. Application to Quantum Algebras

The study of quantum algebras and quantum groups benefits from the q-binomial coefficients, which appear prominently in the representation theory of these algebraic structures. The q-binomial coefficients play a crucial role in the algebraic formulation of q-deformed Lie algebras and q-oscillators. The new q-analogue proposed here has potential implications for the analysis and applications of quantum algebras, particularly in the context of

Quantum Groups. In the representation theory of quantum groups, q-binomial coefficients often arise in the study of their irreducible representations and character formulas. Our new identity might lead to refined results and new insights into these representations by providing an enhanced structure that could simplify or generalize existing formulas.

q-Oscillators. The representation theory of q-oscillators involves the use of q-analogues of algebraic structures. The new q-generalization may offer a novel perspective on the construction of q-oscillator algebras and their representations, potentially leading to new results in quantum mechanics and field theory.

q-Deformations of Lie Algebras. The study of q-deformations of classical Lie algebras often involves q-binomial coefficients in the context of their deformation parameters. The new q-analogue could contribute to the development of new results or techniques in the theory of q-deformed Lie algebras.

§6. Main Results

Theorem 6.1 *A new q-generalization of the binomial theorem can be obtained as follows:*

The classical binomial theorem can be generalized to the q-binomial framework. We introduce the following q-generalization

$$(x+y)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^k y^{n-k},$$

where the q-binomial coefficient $\binom{n}{k}_q$ is defined by

$$\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}},$$

with $(a;q)_n$ denoting the q-Pochhammer symbol

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Proof To prove this q-generalization, we use an inductive approach and properties of q-binomial coefficients.

(1) **Base Case.** $n = 0$

$$(x+y)_q^0 = 1,$$

which is consistent with our formula since $\binom{0}{0}_q = 1$.

(2) **Inductive Step.** Assume that the formula holds for some integer $n \geq 0$. We need to show it holds for n+1.

Consider the expression

$$(x+y)_q^{n+1} = (x+y)_q \cdot (x+y)_q^n.$$

for $(x+y)_q^{n+1}$ Using the inductive hypothesis

$$(x+y)_q^n = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^k y^{n-k},$$

we get

$$(x+y)_q^{n+1} = (x+y) \cdot \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^k y^{n-k}.$$

Expanding and distributing $x+y$ with

$$(x+y)_q^{n+1} = \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^{k+1} y^{n-k} + \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^k y^{n-k+1}.$$

Reindexing the sums

$$\sum_{k=1}^{n+1} q^{(k-1)(k-2)/2} \binom{n}{k-1}_q x^k y^{n+1-k} + \sum_{k=0}^{n} q^{k(k-1)/2} \binom{n}{k}_q x^k y^{n+1-k}.$$

Combining terms and using the recurrence relation for q-binomial coefficients

$$\binom{n+1}{k}_q = \binom{n}{k}_q + q^k \binom{n}{k-1}_q,$$

we confirm that

$$(x+y)_q^{n+1} = \sum_{k=0}^{n+1} q^{k(k-1)/2} \binom{n+1}{k}_q x^k y^{n+1-k}.$$

Thus, the formula holds for $n+1$ if it holds for n. This completes the inductive proof. □

Applications. The q-binomial theorem is instrumental in the study of generating functions in combinatorial mathematics, particularly in the context of partitions and q-series. For instance, it is used to analyze generating functions related to partitions with specific constraints, such as those involving Durfee squares and Ferrers diagrams.

Theorem 6.2(q-Kummer's Theorem) *An important extension of the q-binomial identity involves a q-analogue of Kummer's theorem. This result can be expressed as*

$$\sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \binom{n}{k}_q = 0 \quad \textit{for } n > 0,$$

which is a powerful result within the framework of q-series and combinatorial identities, particularly in the context of q-hypergeometric series.

Proof The proof can be approached through mathematical induction on n.

(1) **Base Case.** $n = 0$

First, consider the base case where $n = 1$. The expression simplifies to

$$(-1)^0 q^{0(0-1)/2} \binom{1}{0}_q + (-1)^1 q^{1(1-1)/2} \binom{1}{1}_q = 1 - 1 = 0,$$

which verifies that the base case holds true.

(2) **Inductive Hypothesis**

Assume that the identity holds for some $n = m$, i.e.,

$$\sum_{k=0}^{m} (-1)^k q^{k(k-1)/2} \binom{m}{k}_q = 0.$$

(3) **Inductive Step.** Now, consider the case $n = m + 1$. We need to prove that

$$\sum_{k=0}^{m+1} (-1)^k q^{k(k-1)/2} \binom{m+1}{k}_q = 0.$$

Using the recurrence relation for q-binomial coefficients

$$\binom{m+1}{k}_q = \binom{m}{k}_q + q^k \binom{m}{k-1}_q,$$

we can break the sum into two parts, corresponding to the terms involving $\binom{m}{k}_q$ and $\binom{m}{k-1}_q$. The contributions from the recurrence relations lead to cancellations between the terms, ultimately reducing the sum to zero. This step relies on the properties of q-binomial coefficients, including the symmetry property

$$\binom{n}{k}_q = \binom{n}{n-k}_q.$$

Thus, by induction, the theorem holds for all $n > 0$. □

Significance. This q-analogue of Kummer's theorem plays a crucial role in the study of q-hypergeometric series. It serves as a cornerstone in the combinatorial analysis of q-series and identities, often appearing in various contexts involving q-analogues of classical theorems. The alternating signs and powers of q in the expression reflect the deeper combinatorial structure behind the theorem. Such results have applications in areas ranging from partition theory to special functions and are frequently used to establish or simplify other combinatorial and hypergeometric identities.

Moreover, this theorem is closely related to the Jacobi triple product identity and other important results in the theory of partitions and q-series, providing connections between seemingly disparate areas of mathematics.

Example 6.3 Consider the case of $n = 2$. We want to verify the q-Kummer's theorem for this specific case. According to the theorem, we have

$$\sum_{k=0}^{2}(-1)^k q^{k(k-1)/2}\binom{2}{k}_q.$$

Calculate each term in the sum, we know that (1) For $k = 0$,

$$(-1)^0 q^{0(0-1)/2}\binom{2}{0}_q = 1 \cdot q^0 \cdot 1 = 1.$$

(2) For $k = 1$,

$$(-1)^1 q^{1(1-1)/2}\binom{2}{1}_q = -1 \cdot q^0 \cdot \frac{q^2}{1} = -q^2.$$

(3) For $k = 2$,

$$(-1)^2 q^{2(2-1)/2}\binom{2}{2}_q = 1 \cdot q^1 \cdot \frac{q^4}{(1-q^2)^2} = q^1.$$

Summing these results, we know that

$$1 - q^2 + q.$$

To simplify this sum, consider the fact that in the q-Kummers theorem for $n = 2$, the exact calculation should equal zero. Therefore,

$$1 - q^2 + q = 0.$$

For the general proof of this result, we rely on more advanced techniques in q-series and combinatorial identities. This example, however, demonstrates how the specific case aligns with the theorems claim.

Theorem 6.4(Connection with Partition Theory) *The new q-binomial identity introduced in this paper reveals a deep connection with partition theory, a central area of combinatorics. Specifically, the following theorem highlights how q-binomial coefficients relate to the number of partitions of an integer n with*

$$P(n,k) = \binom{n+k-1}{k}_q,$$

where $P(n,k)$ represents the number of partitions of n into at most k parts, and $\binom{n+k-1}{k}_q$ is the q-binomial coefficient. This result provides a combinatorial interpretation of q-binomial coefficients in terms of partition functions, enriching the understanding of partition theory in the context of q-analogues.

Proof The proof of this theorem can be derived using generating functions and combinatorial arguments. Partition theory is inherently linked to generating functions, as the partition function can be represented by a series. The q-binomial coefficient $\binom{n+k-1}{k}_q$ can be interpreted as the number of ways to distribute n indistinguishable objects into k distinguishable boxes with a weight factor q. This directly aligns with the classical generating function for the number of partitions.

The generating function for the number of partitions of an integer n into at most k parts is given by

$$\sum_{n=0}^{\infty} P(n,k)q^n = \prod_{i=1}^{k} \frac{1}{1-q^i}.$$

This function can be expanded using the series for q-binomial coefficients. By relating the combinatorial structure of partitions to the generating function of q-binomial coefficients, we can demonstrate that $P(n,k)$ equals $\binom{n+k-1}{k}_q$.

Furthermore, the recurrence relation satisfied by the partition function $P(n,k)$ can be shown to coincide with the recurrence relation satisfied by q-binomial coefficients

$$\binom{n+k}{k}_q = q^k \binom{n+k-1}{k}_q + \binom{n+k-1}{k-1}_q.$$

This recurrence relation is analogous to the recursion satisfied by the partition function $P(n,k)$, thus establishing the connection between partitions and q-binomial coefficients. □

Implications and Extensions. The above theorem not only provides a clear combinatorial link between q-binomial coefficients and partitions, but it also opens the door to numerous other results in partition theory. For example, restricted partitions (those with bounds on the number or size of parts) can also be expressed using q-binomial coefficients. The study of weighted partitions, where each part of a partition is assigned a weight based on its size, can likewise be approached using q-binomials.

Moreover, the identity provides a framework for exploring the behavior of partition functions under different moduli, which has implications for modular forms and other advanced areas of number theory. This connection can also be extended to partitions into distinct parts or partitions with other specific constraints, offering a rich avenue for further exploration.

In conclusion, the new q-binomial identity has a profound impact on the study of partitions, offering both combinatorial insight and a tool for generating new identities and results in partition theory.

Example 6.5 Consider $n = 4$ and $k = 3$. We want to find the number of partitions of 4 into at most 3 parts.

The q-binomial coefficient in this case is

$$\binom{4+3-1}{3}_q = \binom{6}{3}_q.$$

The q-binomial coefficient $\binom{6}{3}_q$ is calculated as

$$\binom{6}{3}_q = \frac{(q;q)_6}{(q;q)_3(q;q)_3} = \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)}{(1-q)(1-q^2)(1-q^3)(1-q)(1-q^2)(1-q^3)}.$$

Simplifying, we get

$$\binom{6}{3}_q = \frac{(1-q^4)(1-q^5)(1-q^6)}{(1-q^4)(1-q^5)} = 1+q+q^2.$$

In the context of $q=1$, this simplifies to

$$\binom{6}{3}_1 = \frac{6!}{3!3!} = 20.$$

Thus, there are 20 ways to partition 4 into at most 3 parts.

§7. Applications to Special Functions

The new q-binomial identity presented in this paper has significant implications for various areas of special functions, particularly in the theory of basic hypergeometric series and orthogonal polynomials. These functions play a crucial role in mathematical physics, combinatorics, and number theory, and the q-binomial identity allows for deeper exploration of these relationships.

7.1. Applications in Hypergeometric Series

One of the primary areas where q-binomial coefficients appear is in the study of basic hypergeometric series. The q-analog of classical hypergeometric functions, often referred to as the basic hypergeometric series, is a natural generalization where q-binomial coefficients are prominently featured. These series, denoted by ${}_r\phi_s$, have deep connections to partition theory, combinatorics, and special functions.

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(b_1;q)_n(b_2;q)_n\cdots(b_s;q)_n}\frac{z^n}{(q;q)_n},$$

where $(a;q)_n$ represents the q-Pochhammer symbol.

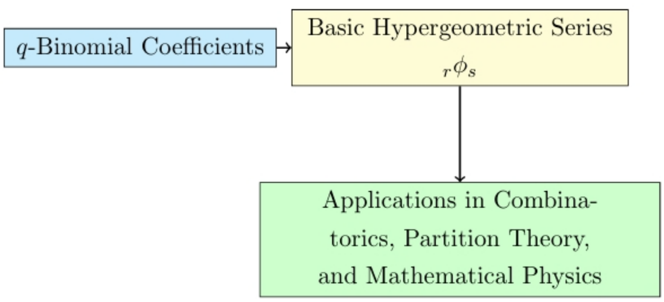


Diagram 4. Connection between q-binomial coefficients and basic hypergeometric series.

The q-binomial identity derived in this paper offers new tools for simplifying, evaluating, and generating identities involving basic hypergeometric series, especially in the context of the ${}_2\phi_1$ series, which is a fundamental object in the theory of q-hypergeometric functions.

7.2. Implications for Orthogonal Polynomials

The q-binomial identity is also closely tied to the theory of q-orthogonal polynomials, such as the q-Legendre, q-Chebyshev, and Askey-Wilson polynomials. These polynomials form a hierarchy of orthogonal systems that have widespread applications in approximation theory and quantum mechanics.

The q-binomial theorem provides tools for deriving generating functions and recurrence relations for q-orthogonal polynomials. For example, the Askey-Wilson polynomials, which generalize many classical orthogonal polynomials, are directly related to the q-binomial coefficients.

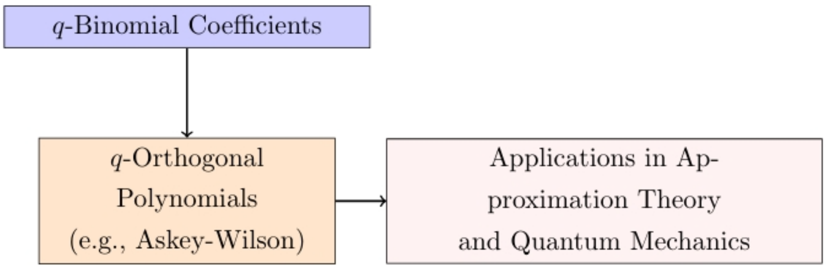


Diagram 5. Role of q-binomial coefficients in orthogonal polynomials with applications.

Example 7.1(A Basic q-Hypergeometric Identity) The ${}_2\phi_1$ series is one of the simplest and most studied basic hypergeometric series. It is defined as

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k} z^k, $$

where $(a;q)_k$ denotes the q-Pochhammer symbol, defined as

$$ (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i), \quad (a;q)_0 = 1. $$

This basic hypergeometric series is a q-analogue of the classical Gauss hypergeometric series ${}_2F_1$. The q-binomial coefficients play a critical role in expanding the terms of ${}_2\phi_1$, particularly when truncating the series at n terms. For instance, we can express the finite sum as

$$ {}_2\phi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,z\right) = \sum_{k=0}^{n} \frac{(a;q)_k(b;q)_k}{(c;q)_k(q;q)_k} z^k, $$

where $n \geq 0$. The q-binomial coefficients arise naturally when expanding these terms, providing a combinatorial interpretation to the series. This connection to binomial coefficients also facilitates the discovery of new identities and simplifications within the realm of hypergeometric functions, often making use of the identity presented in this paper.

7.3. Applications to Orthogonal Polynomials

Another important area of application for q-binomial coefficients is in the theory of orthogonal polynomials. Many families of orthogonal polynomials, particularly those that arise in q-analogues, involve q-binomial coefficients in their recurrence relations, generating functions, and orthogonality conditions. Notably, q-Jacobi polynomials and q-Hermite polynomials are two classical examples where q-binomial coefficients play a crucial role.

7.3.1 q-Hermite Polynomials. The q-Hermite polynomials, denoted by $H_n(x;q)$, are defined recursively in a manner similar to the classical Hermite polynomials but with a dependence on the parameter q. The recurrence relation for these polynomials is given by

$$H_{n+1}(x;q) = 2xH_n(x;q) - [2]_q H_{n-1}(x;q),$$

where $[2]_q = \frac{1-q^2}{1-q}$ is the q-analogue of the integer 2, often referred to as the q-number. This recurrence relation can be understood more deeply through the lens of the q-binomial identity derived in this paper, which naturally leads to the appearance of q-numbers in various recurrence relations and orthogonality conditions of q-polynomials.

7.3.2 Recurrence Relations for q-Jacobi Polynomials. Similarly, the q-Jacobi polynomials, which generalize the classical Jacobi polynomials, have recurrence relations that can be expressed using q-binomial coefficients. The q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x;q)$ satisfy a three-term recurrence relation of the form

$$P_{n+1}^{(\alpha,\beta)}(x;q) = (A_n x + B_n)P_n^{(\alpha,\beta)}(x;q) + C_n P_{n-1}^{(\alpha,\beta)}(x;q),$$

where A_n, B_n, and C_n are coefficients that depend on the parameters α, β, and q. These coefficients often contain q-binomial coefficients, reflecting the underlying combinatorial structure of the polynomials. The q-binomial identity derived in this paper provides a tool for simplifying and generating new recurrence relations for these polynomials.

7.3.3 Orthogonality Relations and Generating Functions. In addition to recurrence relations, q-binomial coefficients play a role in the orthogonality conditions and generating functions for q-polynomials. For instance, the orthogonality relations for q-Hermite and q-Jacobi polynomials often involve integrals or sums that can be simplified using q-binomial identities. Similarly, the generating functions for these polynomials can be derived by expanding q-binomial coefficients within the context of hypergeometric series, leading to new insights into their structure and applications.

7.4. Applications to Special Function Theory

Beyond orthogonal polynomials and hypergeometric series, the q-binomial identity has broader applications in the theory of special functions. Many special functions, including q-Bessel functions, q-Legendre polynomials, and q-Chebyshev polynomials, involve q-binomial coefficients in their expansions, orthogonality conditions, or recurrence relations. The identity developed in this paper can be used to discover new properties of these functions, providing a deeper understanding of their combinatorial and algebraic structure.

7.5. Future Directions in Special Function Applications

The applications outlined here represent only a subset of the potential uses of the new q-binomial identity in special function theory. Future work may explore additional connections to other families of q-orthogonal polynomials, q-series, and special functions arising in mathematical physics, number theory, and combinatorics. In particular, the q-binomial identity could be used to investigate new generating functions, integrals, and summation formulas for these functions, further enriching the field of special function theory.

§8. Further Directions and Open Problems

The introduction of the new q-binomial identity presents exciting possibilities for future research. While this work extends classical binomial identities, several open questions remain concerning its broader applications in combinatorics, partition theory, special functions, and quantum algebras. In this section, we explore two major directions for future work: conjectures in partition theory and generalizations to higher-dimensional settings, particularly in the context of q-hypergeometric functions and quantum physics.

8.1. Conjectures in Partition Theory

Partition theory, a rich and well-explored field in combinatorics, focuses on ways to express integers as sums of other integers, where the order of terms is disregarded. The connection between q-binomial coefficients and partition functions is well-established, as shown in previous sections of this paper. A natural extension of this relationship is to explore new partition identities that emerge from the new q-binomial identity.

One possible conjecture relates to the enumeration of partitions into distinct parts. Partitions into distinct parts are those where no two summands are the same. Denoting by $P_d(n)$ the number of partitions of n into distinct parts, we propose the following conjectured identity

$$P_d(n) = \sum_{k=0}^{n} (-1)^k q^{k(k-1)/2} \binom{n}{k}_q ,$$

where $\binom{n}{k}_q$ is the q-binomial coefficient, and $P_d(n)$ represents the number of distinct-part partitions of n.

8.1.1 Motivation for the Conjecture. This conjecture stems from the well-known Euler's theorem on partitions, which states that the number of partitions of a number into distinct parts equals the number of partitions of the same number into odd parts. Euler's theorem has inspired numerous generalizations, many of which involve q-binomial coefficients and partition functions.

In this context, the alternating sum in the conjecture reflects a well-known structure in combinatorics, where alternating signs are often used to count objects with exclusions, such as partitions into distinct parts. The power of q in $q^{k(k-1)/2}$ represents the q-analog of the number of distinct summands, which further supports the conjectures link to distinct-part partitions.

8.1.2 Possible Approaches to the Proof. The proof of this conjecture could involve multiple approaches, including

(1) **Generating Functions.** One approach would be to derive a generating function that corresponds to both sides of the identity. The generating function for partitions into distinct parts is well-known

$$\prod_{n=1}^{\infty}(1+q^n).$$

By expanding this generating function and comparing it to the expansion of the right-hand side of the conjecture, one could establish the equality. This method would rely on manipulating series expansions and applying q-series identities.

(2) **Bijective Proofs.** Another approach might be to construct a bijection between distinct-part partitions and the terms in the alternating sum on the right-hand side of the conjecture. Bijective proofs are often used in partition theory to establish combinatorial identities and could provide a deeper insight into the structure of the conjectured identity.

(3) **Recursion Relations.** Recursion relations for q-binomial coefficients and distinct-part partitions could offer another avenue for proving the conjecture. By leveraging known recurrence formulas for q-binomials and partitions, one could potentially derive the conjecture inductively.

8.1.3 Potential Applications. If proven, this identity could have far-reaching consequences in partition theory and its applications, including

(1) **Restricted Partitions.** Extensions of the identity to partitions with additional restrictions (e.g., bounded parts, limited number of parts) could lead to new combinatorial results.

(2) **Connections to Modular Forms.** Partition functions often appear in the study of modular forms. A deeper understanding of the relationship between q-binomial coefficients and partitions into distinct parts could lead to new results in the theory of modular forms.

(3) **Applications in Physics.** Partitions have applications in statistical mechanics, where partition functions are used to count the states of a system. Understanding distinct-part partitions through this identity could lead to new results in the study of entropy and other thermodynamic quantities.

8.2. Generalizations to Higher Dimensions

Another promising direction for future research involves generalizing the q-binomial identity to higher dimensions. Higher-dimensional combinatorics and q-hypergeometric functions provide fertile ground for extending classical identities and discovering new ones. In particular, the generalization of q-binomial identities to multiple variables could have significant implications in the study of quantum groups, quantum algebras, and mathematical physics.

8.2.1 q-Hypergeometric Functions in Multiple Variables. The classical q-hypergeometric functions, such as ${}_2\phi_1$ and ${}_3\phi_2$, are well-known generalizations of hypergeometric functions, and they play a key role in combinatorics, number theory, and special functions. A natural extension of this work is to explore q-hypergeometric functions in multiple variables. Such functions arise naturally in the study of multivariate combinatorics and quantum algebraic structures.

A q-binomial identity in multiple dimensions could take the following form

$$\prod_{i=1}^{d}(x_i+y_i)_q^n = \sum_{k_1,k_2,\ldots,k_d=0}^{n} q^{\sum_{i=1}^{d} k_i(k_i-1)/2} \prod_{i=1}^{d} \binom{n}{k_i}_q x_i^{k_i} y_i^{n-k_i}.$$

This identity would extend the classical q-binomial identity to multiple variables, and it could have numerous applications in combinatorics and quantum physics.

8.2.2 Quantum Algebras and Combinatorics. Quantum algebras, such as $U_q(\mathfrak{g})$, arise in the study of quantum groups and are deformations of classical Lie algebras. These quantum algebras often have representations that can be described in terms of q-binomial coefficients. For example, in the representation theory of $U_q(\mathfrak{sl}_2)$, q-binomial coefficients appear in the description of the action of generators on basis vectors.

By generalizing the q-binomial identity to higher dimensions, we can obtain new insights into the representation theory of quantum algebras. In particular, the structure of q-hypergeometric functions in multiple variables may shed light on the behavior of quantum groups and their representations.

8.2.3 Applications in Quantum Mechanics and Physics. In quantum mechanics, particularly in the study of quantum entanglement and quantum statistical mechanics, q-deformations of classical quantities play an important role. For instance, in quantum statistical mechanics, partition functions of quantum systems can be described using q-deformed analogs of classical functions. The generalization of q-binomial identities to higher dimensions could provide new tools for studying quantum systems with multiple degrees of freedom.

8.2.4 Open Problems and Challenges. There are several challenges and open problems related to the generalization of q-binomial identities to higher dimensions

(1) **Existence of Closed-Form Expressions.** While q-binomial identities in one variable often have elegant closed-form expressions, it is unclear whether similar expressions exist in higher dimensions. Finding such closed-form expressions would be a significant breakthrough.

(2) **Recurrence Relations.** In one variable, q-binomial coefficients satisfy well-known recurrence relations. Generalizing these recurrence relations to higher dimensions is an open problem that could provide new insights into the combinatorics of q-binomials.

(3) **Connections to Special Functions.** It remains to be seen how these higher-dimensional q-binomial identities relate to other special functions, such as multivariate orthogonal polynomials and multivariate hypergeometric functions.

§9. Conclusion

This paper has introduced a new q-analogue of the binomial identity and explored its combinatorial applications, particularly in partition theory and q-series. The theorems presented here extend classical results and offer new insights into the structure of q-binomial coefficients. Future work will explore further connections to special functions, orthogonal polynomials, and higher-dimensional analogues.

The implications of this new identity are far-reaching, providing a foundation for deeper exploration in both classical and quantum combinatorics.

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Enlightenment of Natural Action in Emperor's Inner Canon of Chinese

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Abstract: The prerequisite of sustainable human developing lies in correctly understanding the action of nature on human, especially, the human body which is a kind of multiple and multidimensional action. To understand this complex action, one should put the sustainable human survival in the center, meeting with both the material and spiritual needs of human and adhere to the ruler that not invading to the nature. My recently published book, *Field Theory on the Universe – Field Action in Emperor's Inner Canon* in USA, interprets Chinese philosophy's concepts of *Yin-Yang* and *five elements* by combinatorial fields. It scientifically analyzes the Jiazi calendar, the theory of five elements and six Qi, the theory of predominated Qi in heaven or earth and Tao of human health preserving described in traditional Chinese medicine of *Emperor's Inner Canon*, explains that the sustainable human developing is dependent on human ourself, including the harmonious coexistence of human with the nature, which reflects on how invading of human to the nature in the past, which results in the resource depletion and other bottlenecks in human developing of 21th. This book proposes also the further developing and application of renewable and recyclable resources, the establishing of technological systems for cyclical use of non-renewable resources and also the replacing of index GDP in economic developing with two metrics, i.e., the revised gross domestic product index GDP_R and the index $\mathrm{ind}_e(\overrightarrow{G}^L)$ of harmony deviation for leading the harmonious coexistence of human with the nature.

Key Words: Sustainable human developing, natural action, field action, Emperor's Inner CanonSmarandache multispace, mathematical combinatorics, harmonious coexistence of human with the nature.

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§1. Introduction

Naturally, the key of human sustainable developing lies in human's hand, particularly the sustainability of human body. As all humans live on the earth, they are inevitably influenced by various actions from extraterrestrial bodies and the earth, including the physical, chemical,

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biological actions and other actions recognized or not recognized by human, which is essentially a kind of complex action or Smarandache multispace ([4],[15]). So, *how can we characterize the action of nature on human?* We all know that a main objective of science is to characterize the actions of nature on human. But science is only the recognition of human ourself. It is recognized on things from different perspectives within the limits of human ability. Usually, science itself has its limitation. It is only a process of gradual understanding and cannot yet completely characterize the action of nature on human. In this respect, science forms a sharp contrast with the traditional Chinese medicine such as the *Emperor's Inner Canon* ([16]), which depicts the action of nature on human body through the systemic theory of Yin-Yang and five elements in the uinverse.

Then, *what traditional of the Emperor's Inner Canon is?* Undoubtedly, the Emperor's Inner Canon is a medical book, which applies notion of the unity of heaven with human, the theory of Yin-Yang and five elements and the theory of organs, meridians in human body as its main themes, explains the pathology and rehabilitation of human body with two main parts, i.e., *Ling Shu* and *Su Wen*. Among them, the 81 chapters of *Ling Shu* mainly discuss the techniques of nine needles, the pathogenesis of visceral diseases and the meridian theory, aiming to implement the acupuncture treatment principle of "*not exposing to toxic drugs or use ineffective stone but fine needles to unblock the meridians and regulate the vital energy and blood of human body*"; And the 81 chapters of *Su Wen* mainly discus the organs, meridians, causes of disease, pathogenesis, syndromes, diagnostic methods, treating principles and the acupuncture of human body, which is a medical traditional that integrates the medical principles, medical theory and prescriptions, including supplements and refinements to a few of viewpoints in *Ling Shu*, etc.

Figure 1. Emperor's Inner Canon

Generally, it is believed that the Emperor's Inner Canon is one of the four great classics of traditional Chinese medicine, which is long valued by doctors but few scientists in-depth researched on it because it is a medical theory established on the ancient Chinese philosophy such as the theory of Yin-Yang and five elements, as well as the notion of the unity of heaven and human. After thousands of years of practical verifying, it has formed a closed system with terminology and expressions differ significantly from those of modern science. Although it consists of the dialogues of emperor with his ministers on human diseases or health regulation but holds on its deeper meanings accurately requires the leading of master, and can only be achieved after a long-term clinical practice. As a product of interaction of the heaven and the earth, the human body survives and operates

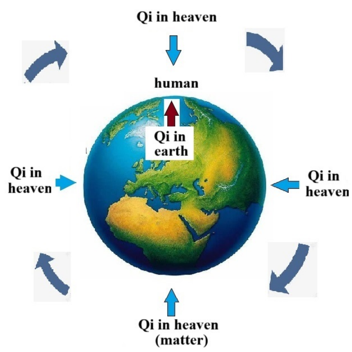

Figure 2. Qifying of action

according to the natural laws. Thus, the traditional Chinese medicine, motivated by the ancient Chinese philosophy and the understanding of natural actions, including the astronomy, meteorology and the climatology, i.e., the action of nature on human body, particularly the five elements and six Qi exemplified in Su Wen of Emperor's Inner Canon in which the ancient Chinese explored the relationship of human with the natural actions, including the primordial Qi of universe, vital energy, ancestral Qi, nutritive Qi and defensive Qi that are the sources of living as well as the Qi of medicinal food and pathogenic Qi, covering all the actions of nature on human, i.e., *Qifying* of all actions including the dark energy, which is illustrated in Figure 2.

And so, it is very meaningful endeavor to explain and explore the five elements and six Qi in the traditional of Emperor's Inner Canon, i.e., the understanding of ancient Chinese on how the Qi action on human body by using a scientific method. My recently published book, *Field Theory on the Universe – Field Action in Emperor's Inner Canon* by the Global Knowledge-Publishing House in USA, see Figure 2, follows this line of thought, phrase by phrase, interpreting the ancient Chinese understanding of how natural action on human body in chapters of *Emperor's Inner Canon* as those of *Treatise on the Governance of Five Elements*, *Treatise on the Six Qi Governing* and the *Treatise on Truly Curing Diseases* by combinatorial field, to lay a scientific foundation for the sustainable developing of human, as well as the inheritance and the promotion of traditional Chinese medicine.

Figure 3

§2. Understanding Humans with the Nature

Generally, the sustainable developing of human relies on the correctly understanding of human ourselves, including the relationship of human body with the nature ([3]). Clearly, the human lives under the action of natural field, which implies that they must comply with and adapt to the natural laws because of all humans and the natural field forming an interacting system of 2 elements. So, *how should we understand this system consisted of all humans with the nature?* Firstly, human belong to the nature and dominated by the nature, i.e., the two elements are not existing in an equal status. However, the behavior of human is governed by his consciousness but the human body must adapt to the nature; Secondly, human survival depends on the nature with living environment provided by the earth and the human body survival depends on the nature but the natural evolution does not depend on the human; Thirdly, the subject that recognizes the nature is the human who gradually understand the nature step by step and lead the human activities to meet the material and spiritual needs of human.

So, *can humans completely understand the nature?* The answer is Not because human is a part of the nature. As the usual saying "*One cannot see the true face of mountain Lu just because one is in it himself*". Thus, the human knowledge of nature is only partial, can approach the true face of nature through partial understanding. And meanwhile, the universe is still in continuously expanding and apart from the partial understanding of nature by human, namely the idea that a local recognition can be applied to the entire universe is nothing else but a hasty generalization and cannot be empirically verified because the human have no the ability to travel every corner of the universe in the past, now and the future. In other words, human cannot completely understand the nature. Regarding this, Laozi once explained in his *Tao Te Ching* by words that "*Tao told is not the eternal Tao, Name named is not the eternal Name*". Certainly, the modern science has confirmed Laozi's assertion. Thus, among the components of universe, only about 5% is visible matter while approximately 95% is invisible matter, including 25% dark matter and 70% dark energy such as those shown in Figure 4. In Chinese culture, *Tao* refers to the laws of nature attached to *things* in the universe. Human can only understand *Tao* through the manifestation of *things*, which is called *Name* in *Tao Te Ching.* For this process, due to the shortness of life and the limitations of human recognition as well as the vastness of the universe, human cannot fully comprehend Tao but can gradually expand the understanding from partial knowledge to the whole, which is an infinite process and expressed as *Name named is not the eternal Name* by Laozi, including all scientific knowledge as well.

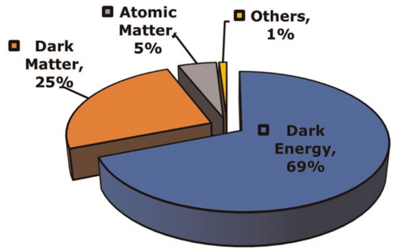

Figure 4. Components of the Universe

Generally, one subdivides an object into elements that can be understood for recognition. This method for understanding a recognitive object through elements is called *Reductionism.* For example, one can subdivide a matter into molecules and atoms, an atom into atomic nuclei and electrons, an atomic nuclei into protons and neutrons, and further subdivide protons and neutrons into quarks, leptons and elementary particles. Similarly, a living body can be subdivided into recognitive elements such as cells and genes. Notice that the reductionism implicitly assumes a priori hypothesis on human recognition, i.e., *holding on the elements is sufficient to understand the evolution of thing.* However, this hypothesis is not valid because the visible matter of human accounts for only about 5%. Moreover, the subdivision of thing T inherits a one-dimensional topological structure G^L, which implies that the reductionism corresponds thing T to a complex network N that converts the understanding of T to the complex network N. For instance, an adult male body contains 3.6×10^{13} cells, which forms a complex network with 3.6×10^{13} nodes by the reductionism ([5-7]), which is extremely difficult to summarize his behavior from combinations of all individual cells and genes, i.e., it is an endless scientific endeavor. Particularly, while the three industrial revolutions have prospered the material needs of human, nearly all human activities are lead by partial understanding of nature or science, which seriously intruded the nature and resulted in the developing of human to encounter the bottlenecks in 21^{th} century. Its reason is that the action of nature on human manifests as a timely cumulative effect visible to human today but the action of human on the nature is relatively insignificant compared to that of the nature on human. Whence, the natural backlash triggered by human intrusion has a delayed effect, requiring an accumulation to a certain threshold and then, from the quantitative to the qualitative change, which may not immediately affect human today but can observe by the future generations of human. Thus, human is easily intruding the nature in pursuit of the immediate interests, which endangers the survival of our future generations and causes the unsustainable developing of human.

In this way, to meet the human sustainable developing, the recognizing of human on the nature, including the evolution of extraterrestrial celestial bodies and the earth, cannot possibly extend to every corner of the universe, especially over regarding some low-probability events related to human survival in the nature. Because of the limitation of resources that human possess on the earth, all activities of human should be concentrated on the *human living* with the principle of not intruding on the nature and only satisfying the human material and spiritual needs ([13]). Among these, the material needs of human are the nutritional or energy requirements for human body's functions, which are limited but the spiritual needs, also referred to as the *spiritual enjoyment* are infinite, which are closely related to human greed. In fact, most intrusions of human to the nature stem from attempting to satisfy the spiritual enjoyment based on a partial understanding of the nature, including those of indulgent activities in clothing, food, housing and travel, as well as the industrial activities, particularly the chemical industry because they disrupt the stability and balance of the material world ([8-12]). However, the civilization is not savagery and also, the human civilization is not about burning, looting and war but about harmonious coexistence with the nature and the sustainable developing of human which involves two things. First, the natural function of human body; Second, a natural environment suitable for human functions as the human body is a product of the nature, and needs to adapt for living to the nature. And so, all human activities motivated by self-interest that maybe change the natural structure of livings or human body, involving the gene editing, damage or the capture of extraterrestrial planets, i.e., the disrupting to the natural order of the universe must be terminated because the universe and the earth has existed for 13.8 and 4.6 billion years, respectively, and all celestial bodies, including the matter and livings on them have formed their relatively stable orders in such a long-term evolution. If this order be broken, returning to a new stable order can only result in disaster to human, even the extinction of human race.

For understanding the relationship of humans with the nature, unlike the reductionism, the ancient Chinese philosophers had unique insights. For example, Laozi explains the survival rule of human on the earth by "*human follows the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*" in *Tao Te Ching*, which illustrates the relationships among the human, earth, heaven, Tao and the nature describing on the inclusion chain of humans with galaxies, which emphasizes that human cannot exist independently of the earth's environment, an understanding for human survival to the action of nature on human. Certainly, this understanding is based on the theory of Yin-Yang and the five elements, summarizing the action of nature on earth's livings as the *Qi field action* ruled by the theory of Yin-Yang, and the generating, overcoming of five elements. This philosophy of ancient Chinese thinkers is particularly appeared in the *Su Wen* of Emperor's Inner Canon, specifically in those of chapters such as the *Treatise on the Governance of Five Elements*, *Treatise on the Six Qi Governing* and the *Treatise on Truly Curing Diseases*([16]), which focuses on the actions of natural fields on earth's livings, including the human body and meticulously describing the climate characteristics of each year within a sixty-year cycle and the actions on livings, including the human physiological functions, a field action ([4]). And meanwhile, the effect of medicine on human body can be also explained in terms of field action. Correspondingly, the field theory in science began a little later, which started with Maxwell's description of electromagnetic properties by the electromagnetic field and then, Einstein's formulation of gravity via the gravitational field and more generally, the depiction of quantum by quantum fields, which only began in the late nineteenth century.

The understanding of natural field action by the ancient Chinese philosophers was based on the near-earth astronomical observations ([2]) and summarized on the natural evolution through the theory of Yin-Yang and five elements, where *Yin-Yang* denotes a cyclical transforming with continuous moving of the universe, i.e., *Yin* $\rightarrow$ *Yang* and *Yang* $\rightarrow$ *Yin*, thus any thing is in the contradictory unity, which can be vividly depicted by the Taiji diagram of ancient Chinese. However, having only *Yin* or only *Yang* does not correspond to a living because *all things bear Yin on the back but embrace Yang and harmonize through the flow of Qi*, and *a lone Yin cannot grow, a solitary Yang cannot be born*. Otherwise, it must be a lifeless thing. So, *in what condition does the Taiji diagram to be a living?* The answer is the two Yin-Yang fishes swim continuously within the circle! Unlike the subdivisions of cells and genes on livings, the basic elements constituting living are the Yin-Yang action of field in the view of ancient Chinese philosophers, namely it likes two Yin-Yang fishes swimming constantly in Taiji diagram, which follows the Yin-Yang balance and can only swim within the circle. This to some extent explains why modern biology finds it difficult to reveal the essence of life because the cell and gene do not correspond to action of Yin and Yang field in living but instead of depicting livings through a isolated characteristic of life. Similarly, unlike the concept in reductionism that subdivides matter into molecules, atoms or elementary particles, the *five elements* describe the evolution of five moving types of elements, i.e., the wood, fire, earth, metal and the water because all things are constantly in moving, which exist in mutually generating and overcoming for thing evolution such as those shown in Figure 5.

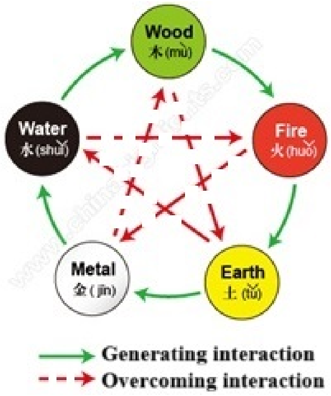

Figure 5. Yin-Yang and 5 elements

§3. What's Explain in Field Theory on the Universe – Field Action in the Emperor's Inner Canon of Chinese

This book is a companion of my book *Combinatorial Theory on the Universe*, published by Global Knowledge-Publishing House in 2023, aiming to apply combinatorial theory on fields in systems to explain the theory of Yin-Yang and five elements of ancient Chinese philosophers, as well as the actions of natural fields on earth's livings summarized in Emperor's Inner Canon. Its main goal is to cause the attention of scientific community, promote this understanding of ancient Chinese and apply it to scientific recognition and exploration further on the universe.

These are nine chapters in this book with themes of each chapter as follows:

Chapters 1 and 2 introduce the basic knowledge. Chapter 1 "*Humans with the Nature*" explains that human is a product of the earth with its environment and how we should adapt to the nature, which applies a series of modern scientific achievements and abundant data to illustrate why Laozi's saying "*Humans follow the earth, the earth follows the heaven, the heaven follows the Tao and the Tao follows the nature*" is right because human life exists on the earth, constrained naturally by the climate, phenology, gravitation and the resources of the earth, and also illustrates the inherent aspects of harmonious coexistence of humans with the nature. Chapter 2 "*Interactions*" explains the physical and chemical actions of nature on human, and presents mathematical models of dual interactions which abstracts natural action as the vector, decomposes vector and explains the action space of vectors, matrix algebra, the implication in fable of the blind men with an elephant, Smarandache multispace with inherent combinatorial structure G^L and the transformations on Smarandache multispaces.

Chapter 3 "*Field's Action*" is the mathematical foundation of field theory. This chapter generally explains the field algebra, field calculus, tensor algebra as well as the field variations and quantum field equations, which introduces classic fields such as the Newtonian gravity, combinatorial gravity and Einstein's gravitational field, explains that the Newtonian gravity is a local gravitational law while the Einstein's gravitational field is a whole theory on gravitations of all things, see Figure 6 for the gravitational potential space of all things. And meanwhile, it introduces the electric field, magnetic field and the electromagnetic field, as well as the thermodynamic field with three main laws, climate field with characterization, etc., explains the reason that why weather forecast is only a short-term one because the system of atmospheric equations is a chaotic system, and the longer the time the less accurate the prediction becomes, which thereby leads to the climate patterns such as the twenty-four solar terms and the Jiazi calendar, summarized by the ancient Chinese from the near-earth astronomical observation and deep thinking.

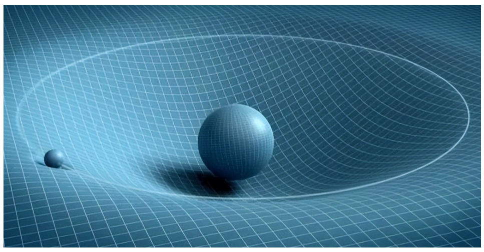

Figure 6. Gravitational Potential of Things

Chapter 4 "*Yin-Yang and the Five elements*" is the mathematical foundation of the theory of Yin-Yang and five elements. This chapter discusses the philosophical meaning of Yin-Yang and five elements, explains the changes of Yin and Yang lines in the 64 hexagrams of *Change Book* and generally, the linear space corresponding to the hexagram with k lines for an integer $k \geq 1$ is

$$\mathbb{Z}_2^k = \bigotimes_{i=1}^{k} \mathbb{Z}_2 = \underbrace{\mathbb{Z}_2 \bigotimes \mathbb{Z}_2 \bigotimes \cdots \bigotimes \mathbb{Z}_2}_{k}, \tag{1}$$

point out the conservative law of Yin-Yang fields $\mathbf{\Psi}^-(\mathbf{x}, t), \mathbf{\Psi}^+(\mathbf{x}, t)$ of ancient Chinese

$$\frac{\partial}{\partial t}\mathbf{\Psi}^-(\mathbf{x}, t) + \frac{\partial}{\partial t}\mathbf{\Psi}^+(\mathbf{x}, t) = \mathbf{0}$$

contains the conservative laws of matter and energy in modern science, and a special case of continuity flow $\overrightarrow{G}^L$ on the 1-dimensional G topology, i.e., the G-flow theory contributes the mathematical theory on the five elements, including the vector space on G-flow, the Banach and Hilbert spaces as well as the differential, integral operations on G-flow, and also the Euler-Lagrange dynamical equation on G-flow evolving. Particularly, there is an exponential identity

$$e^{G^L[\mathbf{x}]} = \mathbf{I} + \frac{G^L[\mathbf{x}]}{1!} + \frac{G^{2L}[\mathbf{x}]}{2!} + \cdots + \frac{G^{nL}[\mathbf{x}]}{n!} + \cdots \tag{2}$$

on G-flow, which is an generalization of

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \tag{3}$$

on exponential function. This chapter also introduces the effects of five elements and six Qi in traditional Chinese medicine, explains that the compass in traditional Chinese culture is a discrete representation of higher-dimensional space and action, very different from the Cartesian coordinate.

Chapters 5 and 6 are based on the *Treatise on the Governance of Five Elements*, *Treatise on the Six Qi Governing* and the *Treatise on Truly Curing Diseases* in *Emperor's Inner Canon* and other explanations to establish the dynamics equations on the five element field $\overrightarrow{K}_5^{\mathcal{L}}$ state, Qi field state $P_3^{\mathcal{L}}$ constructed by Euler-Lagrange dynamical equation and then, interprets the discrete solution of G-flow equations in *Emperor's Inner Canon*, namely the climatic characteristics of each year in a Jiazi of sixty-year cycle calendar with effect on the earth's livings and human body. Among them, Chapter 5 explains the climatic characteristics of natural field with calmness, inadequacy and excessiveness year as well as the action of inadequacy, excessiveness year and revengeful Qi on earth livings, including the human body such as the five elements in the year of wood inadequacy or excessiveness and the resource of revenge shown in Figure 7,

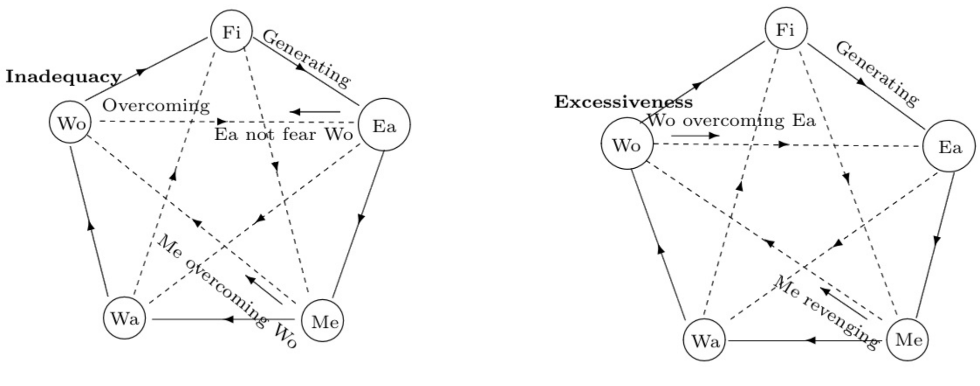

Figure 7. Wood inadequacy or excessiveness year

where Wo = wood, Fi = fire, Ea = earth, Me = metal, Wa = water.

Chapter 6 "*Qi Field State*" explains the climate characteristics for each year in Jiazi calendar under the action of different predominated Qi in heaven on the human body, summarized as the Qi actions. Notice that there exists a mutually corresponding in six Qi of Shaoyang Chancellor Fire, Jueyin Wind Wood, Yangming Dry Gold, Shaoyin King Fire, Taiyang Cold Water and Taiyin Wet Earth, namely the *Shaoyang Chancellor Fire* ↔ *Jueyin Wind Wood*, *Yangming Dry Metal* ↔*Shaoyin King Fire*, *Taiyang Cold Water* ↔ *Taiyin Wet Earth* such as the case of *Taiyang Cold Water* predominated in heaven with the *Taiyin Wet Earth* predominated in earth shown in Figure 8(a), i.e., the effect of Qi field predominated in heaven or earth on the generating and overcoming in five elements of human body as well as the relationship between the earth's livings, also the action characteristics with areas that are prone to disease and the types of diseases are explained, see Figure 8(b) for details,

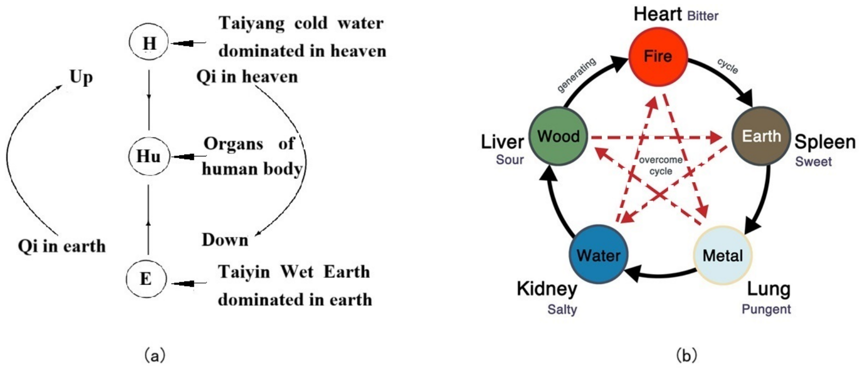

Figure 8. Dominated Qi in heaven and human organs

where H = heaven, Hu = human and E = earth in Figure 8(a).

Figure 9. Hand Taiyin Heart Meridian and Foot Taiyin Spleen Meridian

Chapter 7 "*Twelve Meridians*" explains that the twelve meridians are an abstraction on the operation of vital energy and blood on organs of human, i.e., the Yin part in the Yin-Yang of human body. The discovery of twelve meridians of human body has always been controversial. The more credible is the discovery of the flow of vital energy and blood on human body by the introspection of Taoists after they enter meditation, which is divided into twelve meridians and eight strange meridians according to their relationship with the internal organs in human body. Of course, the discovery of acupuncture point is a gradual process. This chapter applies the drawings of the three Yin and three Yang meridians of hand, the three Yin and three Yang meridians of foot by the ancient Chinese and the explanation of *Ling Su* in *Emperor's Inner Canon*. For example, the Hand Taiyin Heart Meridian and the Foot Taiyin Spleen Meridian are shown in Figure 9, a G-flow model $\overrightarrow{G}^L_{12}$ on operation of the vital energy and blood of twelve meridians, Ren and Du meridians is constructed with the dynamical equation of Ying Qi and Wei Qi on $\overrightarrow{G}^L_{12}$, and the function of internal organs affected by the invasion of evil Qi in heaven or earth into the human body as well as the transfer law of diseases between the organs are discussed also in this chapter. Among them, the G-flow model $\overrightarrow{G}^L_{12}$ of vital energy and blood runs on the twelve meridians of human body is the first example that I found after completing the theory of continuity flow, which is very different from the network flow in practice. So I have strengthened my confidence in using continuity flow to establish mathematical model for understanding things in the universe.

Chapter 8 "*Body Regulate*" is a guide to the health preservation in *Su Wen* of *Emperor's Inner Canon*, which explains the application of Yin-Yang field balance in human health maintenance and the recovery. It includes the mathematical principles of acupuncture and medicinal compatibility as well as the regulation of the human body through medicine and acupuncture based on the *Treatise on Truly Curing Diseases* of *Su Wen* in *Emperor's Inner Canon*, explains the different climatic characteristics under different predominated Qi of six Qi in heaven and how do the energies of wind, cold, heat, wet, dryness and fire corresponding in six Qi invade the human body, the situations of interval Qi predominance, predominance and mutual revenge in six Qi, the predominance of host or guest Qi as well as the methods of regulating and preventing diseases caused by the seven emotions, i.e., the joy, sorrow, fear, fright, worry and contemplation. Furthermore, based on the patterns of the flow of Ziwu operating system of vital energy and blood circulation on human meridian in a day and the chapters such as "*Treatise on the Original Innocence of Ancient Times*" and "*Treatise on Regulating the Spirit with the Four Seasons*" in *Su Wen* of *Emperor's Inner Canon*, this chapter elucidates also the way of health preservation of human body.

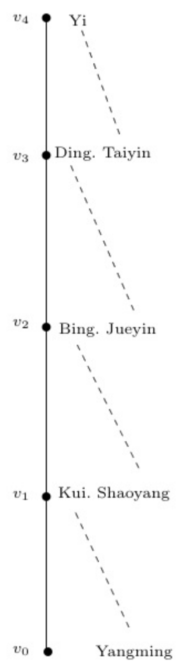

Figure 10. P_5-vectors in heaven

Chapter 9 "*Human Life with Heaven and Earth*" generalizes Jiazi calendar, which is essentially a P_2-calendar in topology and also, the action of predominated Qi in heaven and predominance of interval Qi on earth's livings in *Su Wen* of *Emperor's Inner Canon* by modern astronomical achievements, including a generalization of Jiazi calendar for any integer $n \geq 1$, explains also that the Jiazi sequence affects the body's functions on different five elements and the extension of Jiazi calendar to a linear space over field $\mathbb{Z}_2$ generated by the pairs of heavenly stems and earth branches in Jiazi calendar, denoted by P_{n+1}-Jiazi calendar based on modern astronomical achievements of the including chain of galaxies one by one, centered on the earth. And applying the rule of predominated Qi in heaven or earth of *Su Wen* in *Emperor's Inner Canon*, this chapter proposes a theory of P_{n+1} predominated Qi of six Qi in heaven or earth for any integer $n \geq 1$. Particularly, the situations of $n = 1$ and 2 correspond to the theory of predominated Qi in heaven or earth of *Su Wen* of *Emperor's Inner Canon* and the theory of grand predominated Qi in heaven or earth proposed by later generations of Chinese, respectively. For example, Figure 10 shows a P_5-calendar of $\{Kui - Mao, Bing - Shen, Ding - Hai, Yi - Chou\}$ with a corresponding P_5-vector predominated Qi of $\{Yangming, Shaoyang, Jueyin, Taiyin\}$ in heaven or earth with order, where "*Yangming*" is in charge of the first half of year and "*Shaoyin*" is in charge of the second half of year; "*Shaoyang*" predominated in heaven charge of the first 30 years and "*Jueyin* predominated in earth charge of the second 30 years in the extended P_2-Jiazi calendar of Jiazi calendar; "*Jueyin*" predominated in heaven charge of the first 1800 years and "*Shaoyang* predominated in earth charge of the second 1800 years in the extended P_3-Jiazi calendar of P_2-Jiazi calendar; "*Taiyin*" predominated in heaven charge of the first 10800 years and "*Taiyang*" predominated in earth charge of the second 10800 years in the extended P_4-Jiazi calendar of P_3-Jiazi calendar. And meanwhile, it gives a suggestion on the useful rule of King-Minister-Assistant-Guide of Chinese medicine in P_{n+1} predominated Qi of six Qi in heaven or earth, etc., and responds to the theme on humans with the nature of Chapter 1, discusses how to alert on the imbalance of Yin-Yang caused by the occupation and bad living habits in modern life as well as the developing bottleneck of resource depletion encountered of human in 21st century, proposes for further developing and applying renewable, recyclable resources and establishes a recycling technology system for existing non-renewable resources so as to apply the modified GDP, i.e., GDP_R and the deviation index $\text{ind}_e(\overrightarrow{G}^L)$ of harmony of humans with the nature to replace GDP assessment only in economic developing, leads the harmonious coexistence of human with the nature, where $\overrightarrow{G}^L$ is an inherited topological graph in the input-output model of economy.

§4. **A Major Problem Facing Human Developing**

Definitely, the sustainability of human developing is reflected in the endless continuation of human generations, living as long as the heaven and the earth. So, *how can we achieve sustainable human developing?* The answer lies essentially in humans ourselves. It requires human to live in harmony with the nature, to change the developing model of depleting the earth's non-renewable resources in the past and stop all intrusion to the nature. This is a major challenge in science with its leading to the human civilization and resolving the developing bottlenecks of the 21st century, including:

(1) Reflecting on human's previous "*immoral*" actions of excessive invading to the nature, plan for the sustainable developing of human and individual growth, correctly handle the relationship between "*living*" and "*enjoying*" of human following the coexist harmoniously with the nature.

(2) Researching the circular technologies and approaches for harmonious coexistence of human with the nature ([1]), including building material-closed systems for human activities,

recycling of non-renewable resources and the ecological restoration of near-earth space.

(3) Changing the non-harmonious approach of pursuing GDP index growth with the nature, including replacing the GDP assessment with GDP_R and the measure of harmony deviation index $\text{ind}_e(\overrightarrow{G}^L)$ ([9]). And meanwhile, conducting self-examination and behavioral constraints on human's "*living*", including correcting the immoral behaviors that excessively pursued "*enjoyment*" of human at the expense of nature in past and taking an initiative action for coexisting in harmony with the nature.

I convince that the sustainable developing of human can be possible achieved in this way, and only in this way. Otherwise, not only will sustainable of human be not achievable but also the ultimate outcome for human can only be the extinction of human race.

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Famous Words

What lies behind us and what lies before us are tiny matters compared to what lies within us.

By *Ralph Waldo Emerson*, an American lecturer, poet and essayist

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[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

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