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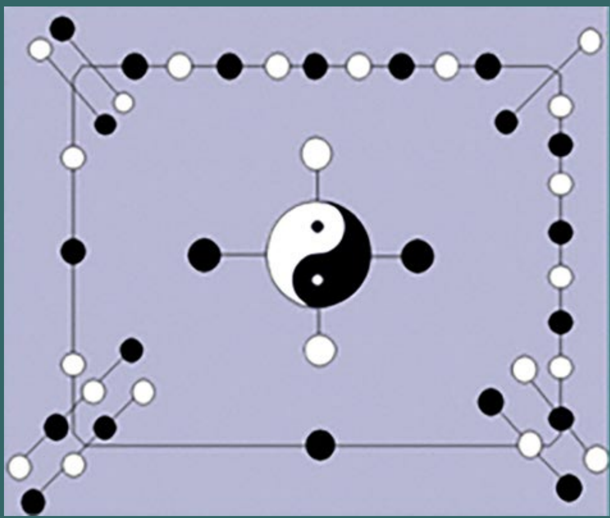

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The science of mathematics presents the most brilliant example of how pure reason may successfully enlarge its domain without the aid of experience.

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Conformal Yamabe Soliton and Conformal Gradient Yamabe Soliton on Para-Kenmotsu Manifold

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Abstract: The goal of this article is to study conformal Yamabe soliton and conformal gradient Yamabe soliton on the para-Kenmotsu manifold. Firstly, we have proved some results of para-Kenmotsu manifold when its admit conformal Yamabe soliton. Later, we have worked on conformal gradient Yamabe soliton on the para-Kenmotsu manifold.

Key Words: Yamabe soliton, conformal Yamabe soliton, gradient Yamabe soliton, conformal gradient Yamabe soliton, para-Kenmotsu manifold.

AMS(2010): 53C25, 53D15, 53C15.

§1. **Introduction**

The concept of Yamabe flow was introduced by Hamilton [9] in order to produce Yamabe metrics on compact Riemannian manifolds. The evaluation of the metric g_0 in time t to $g = g(t)$ using the equation is known as Yamabe flow. The equation of this is

$$\frac{\partial}{\partial t}g(t) = -r(t)g(t), \qquad g(0) = 0, t \geq 0,$$

where r is the scalar curvature of the Riemannian metric g. In dimension 2, the Yamabe flow is similar to the Ricci flow. However, the Yamabe flow and the Ricci flow exhibit distinct behaviors at higher dimensions. The Yamabe soliton [1] is a specific solution of the Yamabe flow that moves via a homothetic family of one-parameter difeomorphisms, much like the Ricci soliton [9]. The equation of the Yamabe soliton is

$$\frac{1}{2}\mathcal{L}_X g = (r - \lambda)g,$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X. Many researches have studied on Yamabe soliton such as [5, 6, 8, 11, 17] and many others.

In 2021, Roy, Dey and Bhattacharyya [13] generalized the notation of Yamabe solition and

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they introduced conformal Yamabe soliton which is

$$(\mathcal{L}_X g)(U,V) = \Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big] g(U,V), \tag{0.1}$$

where $\mathcal{L}_X$ denotes the Lie derivative along X, r is the scalar curvature, λ is a constant and p is the time dependent scalar field. $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ confirmed that conformal Yamabe soliton is expanding, steady and shrinking respectively.

When a smooth function f's gradient is represented by X, it can be substituted by Df to create the conformal gradient Yamabe soliton, for which the equation (1.1) takes on the following form

$$\nabla^2 f = \Big\{r - \lambda + (p + \frac{2}{n})\Big\}, \tag{1.1}$$

where, $\nabla^2 f$ is the Hessian of f and this defined as $Hess_f(U,V) = g(\nabla_U Df, V)$, D denotes the gradient [1] operator.

This paper is constructed as follows:

After a brief introduction, we have covered some necessary results of para-Kenmotsu manifold in section two. In section 3, we have worked on conformal Yamabe soliton on para-Kenmotsu manifold. Here we have proved that the scalar r curvature is dependent on p, the soliton vector field X and the Reeb vector field ξ are Killing, X is constant multiple of ξ, the soliton is shrinking, steady and expanding if $p > \frac{34}{3}$, $p = \frac{34}{3}$ and $p < \frac{34}{3}$ respectively and some other results are also proved. In section 4, we have worked on conformal gradient Yamabe soliton.

§2. **Preliminaries**

An $n-$ dimensional smooth manifold M^n is said to be an almost para-contact manifold ([3], [10], [12]) if it admits an $(1,1)$ tensor field ϕ, a unit vector field ξ, the smooth 1-form η and the pseudo-Riemannian metric g such that

$$\phi^2 U = U - \eta(U)\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta \circ \phi = 0, \tag{2.1}$$

$$g(U,\xi) = \eta(U), \tag{2.2}$$

$$g(\xi,\xi) = 1, \tag{2.3}$$

$$g(\phi U, \phi V) = -g(U,V) + \eta(U)\eta(V), \tag{2.4}$$

for $\forall U, V \in \chi(M)$, where $\chi(M)$ denotes Lie algebra of smooth vector fields on M.

$$d\eta(U,V) = g(U, \phi V), \tag{2.5}$$

for every $U, V \in \chi(M)$.

An almost para-contact metric manifold is said to be paraKenmotsu manifold if it satisfies

$$(\nabla_U \phi)V = g(\phi U, V) - \eta(V)\phi U, \tag{2.6}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian metric g.

Moreover, in a para-Kenmotsu manifold, we have the following relations [7]

$$\nabla_U \xi = U - \eta(U)\xi, \tag{2.7}$$

$$(\nabla_U \eta)V = g(U, V) - \eta(U)\eta(V), \tag{2.8}$$

$$R(U, V)\xi = \eta(U)V - \eta(V)U, \tag{2.9}$$

$$R(\xi, U)V = \eta(V)U - g(U, V)\xi, \tag{2.10}$$

$$R(\xi, U)\xi = U - \eta(U)\xi,$$

$$S(U, \xi) = -(n-1)\eta(U), \tag{2.11}$$

where Q and R denotes the Ricci operator and the Riemann curvature tensor respectively and $g(QU, V) = S(U, V)$.

It's known that the Ricci tensor of a $3-$dimensional para-Kenmotsu manifold is

$$S(U, V) = \frac{1}{2}\Big[(r+2)g(U, V) - (r-6)\eta(U)\eta(V)\Big]. \tag{2.12}$$

Several authors have studied on para-Kenmotsu manifold such as [2, 14, 15, 16] and many others.

§3. Conformal Yamabe Soliton

Theorem 3.1 *If a para-Kenmotsu manifold M^n admits conformal Yamabe soliton (g, ξ, λ, p), then the scalar curvature is dependent on p and the Reeb vector field ξ is Killing.*

Proof If ξ is the Reeb vector field then

$$(\mathcal{L}_\xi g)(U, V) = g(\nabla_U \xi, V) + g(U, \nabla_V \xi).$$

Using (2.7) in the above equation and then applying (2.2), we get

$$(\mathcal{L}_\xi g)(U, V) = 2[g(U, V) - \eta(U)\eta(V)]. \tag{3.1}$$

Again from equation (1.1), we have

$$(\mathcal{L}_\xi g)(U, V) = \Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big]g(U, V). \tag{3.2}$$

Equating (3.1) and (3.2), we get

$$\Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big]g(U,V) = 2[g(U,V) - \eta(U)\eta(V)]. \tag{3.3}$$

Substituting ξ in the place of V in the previous equation, we get

$$\Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big]\eta(U) = 0. \tag{3.4}$$

Since $\eta(U) \neq 0$, it gives

$$r = \lambda - \Big(\frac{p}{2} + \frac{1}{n}\Big), \tag{3.5}$$

where λ is a constant so, the scalar curvature r is dependent on p.

Using (3.5) in (3.2), we get $(\mathcal{L}_\xi g) = 0$. Hence, the Reeb vector field ξ is Killing. □

Theorem 3.2 *Let a* $3-$*dimensional para-Kenmotsu manifold* M^3 *admits conformal Yamabe soliton* (g, ξ, λ, p), ξ *being the Reeb vector field and if the manifold is Ricci symmetric, then*

$$6\lambda - 3p = -34.$$

Proof Using (3.5) in (2.12) for $3-$dimensional, we obtain

$$\begin{aligned} S(U,V) &= \frac{1}{2}\Big[\Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 2\Big\}g(U,V) \\ &\quad -\Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6\Big\}\eta(U)\eta(V)\Big]. \end{aligned} \tag{3.6}$$

Taking covariant derivative of the above equation along Z, we get

$$\begin{aligned} (\nabla_Z S)(U,V) &= -\frac{1}{2}\Big[\Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6\Big\} \\ &\quad \times\Big\{\eta(U)(\nabla_Z \eta)V + \eta(V)(\nabla_Z \eta)U\Big]. \end{aligned} \tag{3.7}$$

The manifold is Ricci symmetric i,e, $(\nabla_Z S)(U,V) = 0$, then from (3.7), we get

$$\Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6\Big\}\Big\{\eta(U)(\nabla_Z \eta)V + \eta(V)(\nabla_Z \eta)U\Big\} = 0. \tag{3.8}$$

Applying (2.8), in the foregoing equation (3.8), we obtain

$$\Big\{\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6\Big\}\Big\{g(\phi U, \phi V\Big\} = 0. \tag{3.9}$$

Since $g(\phi U, \phi V) \neq 0$, it yields

$$\lambda - \Big(\frac{p}{2} + \frac{1}{3}\Big) + 6 = 0,$$

Hence, from the above

$$6\lambda - 3p = -34. \tag{3.10}$$

This completes the proof. □

Corollary 3.3 *If a 3−dimensional para-Kenmotsu manifold M^3 admits conformal Yamabe soliton (g, ξ, λ, p) and if the manifold is Ricci symmetric, then the soliton is shrinking if $p > \frac{34}{3}$, steady if $p = \frac{34}{3}$ and expanding if $p < \frac{34}{3}$.*

Proof From equation (3.10), we get

$$6\lambda = 3p - 34.$$

The definition of shrinking, steady and expanding is that $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

So, from the above soliton is shrinking, steady and expanding if $p > \frac{34}{3}$, $p = \frac{34}{3}$ and $p < \frac{34}{3}$ respectively. □

Theorem 3.4 *Let a n−dimensional para-Kenmotsu manifold admits conformal Yamabe soliton (g, X, λ, p), such that the soliton vector field X is pointwise collinear with ξ, then X is a constant multiple of ξ and X is a Killing vector field.*

Proof Let $X = c\xi$, where c is a function and ξ is the Reeb vector field then

$$(\mathcal{L}_{c\xi}g)(U, V) = g(\nabla_U c\xi, V) + g(U, \nabla_V c\xi).$$

Using (2.7) in the above equation and then applying (2.2), we get

$$(\mathcal{L}_{c\xi}g)(U, V) = (Uc)\eta(V) + (Vc)\eta(U) + 2c\{g(U, V) - \eta(U)\eta(V)\}. \tag{3.11}$$

Again from equation (1.1), we have

$$(\mathcal{L}_{c\xi}g)(U, V) = \Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big]g(U, V). \tag{3.12}$$

Equating (3.11) and (3.12), we get

$$\begin{aligned}\Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big)\Big]g(U, V) \;&=\; (Uc)\eta(V) + (Vc)\eta(U) \\ &\quad +2c\{g(U, V) - \eta(U)\eta(V)\}.\end{aligned} \tag{3.13}$$

Putting $V = \xi$, in the previous equation, we obtain

$$(Uc) = \Big[2r - 2\lambda + \Big(p + \frac{2}{n}\Big) - \xi c\Big]\eta(U). \tag{3.14}$$

Again, Substituting $U = \xi$ in above equation, we get

$$(\xi c) = \Big[r - \lambda + \Big(\frac{p}{2} + \frac{1}{n}\Big)\Big]. \tag{3.15}$$

Using (3.15) in (3.14) becomes

$$(Uc) = \Big[r - \lambda + \Big(\frac{p}{2} + \frac{1}{n}\Big)\Big]\eta(U). \tag{3.16}$$

Now, taking exterior differentiation of (3.16), we get

$$\Big[r - \lambda + \Big(\frac{p}{2} + \frac{1}{n}\Big)\Big]d\eta = 0. \tag{3.17}$$

Since $d\eta \neq 0$, the above equation becomes

$$\Big[r - \lambda + \Big(\frac{p}{2} + \frac{1}{n}\Big)\Big] = 0. \tag{3.18}$$

Using (3.18) in (3.16) gets

$$Uc = 0,$$

which implies that c is constant.

If we are using (3.18) in (1.1) yields

$$(\mathcal{L}_X g)(U, V) = 0.$$

Hence, X is a Killing vector field. □

§4. **Conformal Gradiant Yamabe Soliton**

Theorem 4.1 *If a $n-$dimensional para-Kenmotsu manifold admits conformal gradient Yamabe soliton with potential function f, then if the scalar curvature is constant then the potential function f is also constant and conversely.*

Proof From equation (1.2), we gets

$$\nabla_U Df = \Big[r - \lambda + \Big(p + \frac{2}{n}\Big)\Big]U. \tag{4.1}$$

Taking covariant differentiation (4.1) along the vector field V, we get

$$\nabla_V \nabla_U Df = (Vr)U + \Big\{r - \lambda + \Big(p + \frac{2}{n}\Big)\Big\}\nabla_V U. \tag{4.2}$$

Interchanging U and V in the above equation, we get

$$\nabla_U \nabla_V Df = (Ur)V + \Big\{r - \lambda + \Big(p + \frac{2}{n}\Big)\Big\}\nabla_U V. \tag{4.3}$$

Again, from (4.1) we have

$$\nabla_{[U,V]}Df = \Big[r - \lambda + \Big(p + \frac{2}{n}\Big)\Big](\nabla_U V - \nabla_V U). \tag{4.4}$$

As is widely known that

$$R(U,V)Df = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]}Df,$$

Using (4.2),(4.3) and (4.4) in the previous equation, we get

$$R(U,V)Df = (Ur)V - (Vr)U. \tag{4.5}$$

Contracting (4.5) over U, we get

$$S(V,Df) = -(n-1)g(V,Dr). \tag{4.6}$$

Substituting, $V = \xi$ and using (2.11) in (4.6), we get $\xi f = \xi r$.
Putting $U = \xi$ in (4.5), we obtain

$$R(\xi,V)Df = (\xi r)V - (Vr)\xi. \tag{4.7}$$

Taking inner product with U, yields

$$g(R(\xi,V)Df,U) = (\xi r)g(U,V) - (Vr)\eta(U). \tag{4.8}$$

From (2.10)

$$g(R(\xi,V)Df,U) = [\eta(U)(Vf) - g(U,V)(\xi f)]. \tag{4.9}$$

As we know,

$$g(R(\xi,V)Df,U) = -g(R(\xi,V)U,Df).$$

So, from equation (4.8) and (4.9) we get,

$$(\xi r)g(V,U) - (Vr)\eta(U) = -[\eta(U)(Vf) - g(U,V)(\xi f)], \tag{4.10}$$

which gives the following after antisymmetrizing

$$(Ur)\eta(V) - (Vr)\eta(U) = (Uf)\eta(V) - (Vf)\eta(U). \tag{4.11}$$

Replacing V by ξ in the previous equation (4.11) and using $\xi f = \xi r$ implies that $Df = Dr$. So, if the scalar curvature is constant, then the potential function is also constant and conversely. This completes the proof. □

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References

[1] E. Barbosa, E. Ribeiro Jr, On conformal solutions of the Yamabe flow, *Archiv der Mathematik*, 101(1), 2013, 79-89.

[2] A. M. Blaga, η-Ricci solitons on para-Kenmotsu manifolds, arXiv preprint *arXiv:1402.0223. 2014*, Feb 2.

[3] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture note in Math, springer-Verlag, Berlin, New York, 1976, 509.

[4] H. D. Cao, X. Sun, Y. Zhang, On the structure of gradient Yamabe solitons, arXiv preprint *arXiv:1108.6316. 2011* Aug 31.

[5] G. Calvaruso, Homogeneous paracontact metric three-manifolds, *Ill. J. Math.* 55, 697-718 (2011)

[6] S. Deshmukh, B. Y. Chen, A note on Yamabe solitons, *Balkan J. Geom. Appl.* 23(1); 2018; 37-43.

[7] K. Dube, Study of curvatures on para-kenmotsu manifold, *The Nepali Math. Sci. Report*, 15, 1996, (1-2).

[8] A. Ghosh, Yamabe soliton and quasi Yamabe soliton on Kenmotsu manifold, *Mathematica Slovaca*, 70(1) 2020, 151-160.

[9] R. S. Hamilton, The Ricci flow on surfaces, mathematics and general relativity, *Contemp. Math.*, 71, 1988; 237-261.

[10] S. Kaneyuki and F. L. Williams, Almost paracontact and parahodge structures on manifolds, *Nagoya Math. J.*, 99, 173-187 (1985)

[11] L. Ma, V. Miquel; Remarks on scalar curvature of Yamabe solitons, *Annals of Global Analysis and Geometry*, 42, 2012, 195-205.

[12] A. Perrone, Some results on almost paracontact metric manifolds, *Mediterr. J. Math.* 13, 3311-3326 (2016)

[13] S. Roy, S. Dey, A. Bhattacharyya, Conformal Yamabe soliton and $*$-Yamabe soliton with torse forming potential vector field, arXiv preprint *arXiv:2105.13885. 2021* May 28.

[14] A. Sardar and U. C. De, η-Ricci solitons on para-Kenmotsu manifolds, *Differential Geometry - Dynamical Systems*, Vol.22, 2020, 218-228.

[15] S. Sarkar, S. Dey, X. Chen, Certain results of conformal and *-conformal Ricci soliton on para-cosymplectic and para-Kenmotsu manifolds. *Filomat*,35(15), 2021,5001-15.

[16] V. Venkatesha, H.A. Kumara, and D.M. Naik, Almost $*$-Ricci soliton on paraKenmotsu manifolds, it Arab. J. Math. 9, 715-726 (2020).

[17] Y. Wang, Yamabe solitons on three-dimensional Kenmotsu manifolds, *Bulletin of the Belgian Mathematical Society-Simon Stevin*, 23(3), 2016, 345-355.

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On the Klein Cubic Threefold in $PG(4,2)$

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Abstract: This paper investigates the structural properties of the Klein cubic threefold $\mathcal{F}$ in 4−dimensional projective space over the finite field $GF(2)$. We focus on the intersection properties of the lines and the planes with $\mathcal{F}$ in $PG(4,2)$. Notably, it is identified six spreads, each containing five lines in $\mathcal{F}$. Additionally, two distinct affine plane models are presented by using the tangent planes of $\mathcal{F}$. Furthermore, it is shown that Desargues' theorem does not hold in $\mathcal{F}$.

Key Words: Klein cubic threefold, projective spaces, spread, Galois field.

AMS(2010): 51E20, 51E30.

§1. Introduction

Cubic surfaces have been extensively studied in algebraic geometry and have applications in fields such as those of the computer graphics, physics and engineering. One notable early example is the non-singular Klein cubic threefold studied by Klein in 1879, [11]. The classification of non-singular cubic surfaces, particularly over finite fields, remains a significant area of research. For instance, it has been shown that a non-singular cubic surface over the field $GF(2)$ can have 15, 9, 5, 3, 2, 1, or 0 lines in [5,6,10]. In [13], Rosati further demonstrated that when q is odd, the number of lines must be one of 27, 15, 9, 7, 5, 3, 2, 1, or 0. In the 1960s, Hirschfeld initiated a program to classify cubic surfaces with 27 lines over finite fields, [8]. This work is a substantial contribution to this problem. Some examples of the nonsingular cubic surfaces were given in [9].

In order to classify projective spaces, tools from Veronesean embedding and quadric theory were used [1]-[4], [7], [12].

In this paper, we delve into the structural properties of the Klein cubic threefold $\mathcal{F}$ situated in 4−dimensional projective space over the finite field $GF(2)$. Our investigation primarily focuses on the intricate intersection properties of lines and planes with $\mathcal{F}$. Notably, every point on the Klein cubic threefold is identified as an Eckardt point, marking a fundamental characteristic of $\mathcal{F}$.

Utilizing Schläfli labeling, we systematically notate the 15 lines comprising $\mathcal{F}$. Through our analysis, we observe that each line intersects six others while remaining skew to eight additional

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lines. Moreover, $\mathcal{F}$ has six spreads, each consisting of five lines.

An important finding in our study is the determination of the point $(1, 1, 1, 1, 1)$, which is the nucleus of $\mathcal{F}$. We meticulously examine the tangent planes to $\mathcal{F}$ and present two distinct affine plane models based on their specific properties.

Furthermore, we rigorously demonstrate that Desargues' theorem, a cornerstone of projective geometry, does not hold in $\mathcal{F}$. This observation underscores the unique geometric and algebraic characteristics that distinguish $\mathcal{F}$.

Throughout this paper, we aim to provide a comprehensive exploration of these structural properties, offering insights into the rich interplay between algebraic geometry and combinatorial structure in the context of cubic threefold $PG(4, 2)$.

§2. Preliminaries

Let $GF(q)$ denote Galois field of order $q = p^k$ where p is a prime. If any $(n+1)-$dimensional vector space V, the $n-$dimensional projective space $PG(n, q)$ over $GF(q)$ is the set of all subspaces of V distinct from the trivial subspaces. $1-$dimensional subspaces are called the points of $PG(n, K)$, $2-$dimensional subspaces are called the (projective) lines and $3-$dimensional ones are called (projective) planes. We remark that by going from a vector space to the associated projective space, the dimension drops by one unit. Hence an $(n+1)-$dimensional vector space V gives rise to an $n-$dimensional projective space $PG(n, K)$ [3]-[7].

This is, the points in projective space $PG(n, q)$ are defined by equivalence classes of nonzero vectors in the vector space V.

For example, for $4-$dimensional vector space, the associated $3-$dimensional projective space would be where points are represented by equivalence classes of vectors (w, x, y, z), reducing the dimension by one unit.

The $4-$dimensional projective space $PG(4, q)$ over $GF(q)$ contains $q^4 + q^3 + q^2 + q + 1$ points and $PG(4, 2)$ has 31 points. The points of $PG(4, 2)$ are respectively listed as follows:

$P_1(0,0,0,0,1)$, $P_2(0,0,0,1,0)$, $P_3(0,0,1,0,0)$, $P_4(0,0,1,0,1)$, $P_5(0,0,1,1,1)$,
$P_6(0,1,0,0,0)$, $P_7(0,1,0,0,1)$, $P_8(0,1,0,1,0)$, $P_9(0,1,1,1,0)$, $P_{10}(1,0,0,0,0)$,
$P_{11}(1,0,0,1,0)$, $P_{12}(1,0,0,1,1)$, $P_{13}(1,0,1,0,0)$, $P_{14}(1,1,0,0,1)$, $P_{15}(1,1,1,0,0)$,
$P_{16}(0,0,0,1,1)$, $P_{17}(0,0,1,1,0)$, $P_{18}(0,1,1,0,0)$, $P_{19}(1,0,0,0,1)$, $P_{20}(1,1,0,0,0)$,
$P_{21}(0,1,0,1,1)$, $P_{22}(0,1,1,0,1)$, $P_{23}(1,0,1,0,1)$, $P_{24}(1,0,1,1,0)$, $P_{25}(1,1,0,1,0)$,
$P_{26}(0,1,1,1,1)$, $P_{27}(1,0,1,1,1)$, $P_{28}(1,1,0,1,1)$, $P_{29}(1,1,1,0,1)$, $P_{30}(1,1,1,1,0)$,
$P_{31}(1,1,1,1,1)$.

The Klein cubic threefold $\mathcal{F}$ is given by the equation

$$\mathcal{F} : x^2y + y^2z + z^2v + v^2w + w^2x = 0,$$

where x, y, z, v and w represent the coordinates of a point (v, w, x, y, z) in the projective space

$PG(4,q)$ over $GF(q)$. In algebraic geometry, a cubic threefold is a hypersurface of degree 3 in 4−dimensional projective space. This Klein cubic threefold $\mathcal{F}$ over the field $F(q)$ is the zero set of a homogeneous cubic equation in five variables over $GF(q)$.

2.1. Klein Cubic Threefold $\mathcal{F}$ with 15 Lines

We use the combinatorial definition where a line is considered as a subsets of points on $\mathcal{F}$.

Proposition 2.1 *Every Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$ contains 15 points and 15 lines. Every line has three points on it.*

Proof In $PG(4,2)$, a point is denoted by $P(a_0, a_1, a_{2,} a_3, a_4)$. A line through the points $P(a_0, a_1, a_2, a_3, a_4)$ and $P(b_0, b_1, b_{2,} b_3, b_4)$ is denoted by

$$l = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix}.$$

Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. $\mathcal{F}$ can be identified with a set of the points P_i satisfying the equation $x^2y + y^2z + z^2v + v^2w + w^2x = 0$ in $PG(4,2)$ such that $P_1(0,0,0,0,1)$, $P_2(0,0,0,1,0)$, $P_3(0,0,1,0,0)$, $P_4(0,0,1,0,1)$, $P_5(0,0,1,1,1)$, $P_6(0,1,0,0,0)$, $P_7(0,1,0,0,1)$, $P_8(0,1,0,1,0)$, $P_9(0,1,1,1,0)$, $P_{10}(1,0,0,0,0)$, $P_{11}(1,0,0,1,0)$, $P_{12}(1,0,0,1,1)$, $P_{13}(1,0,1,0,0)$, $P_{14}(1,1,0,0,1)$ and $P_{15}(1,1,1,0,0)$. The incidence relation on $\mathcal{F}$ over the field $GF(2)$ is given as the following table: We define the incidence matrix $A = [a_{ij}]$ of $\mathcal{F}$ such that $a_{ij} = 1$, where $i, j \in \{1, 2, \cdots, 15\}$ if and only if the line l_i is incident to the point P_j and 0 otherwise; here the rows represent the lines and the columns the points of $\mathcal{F}$. This matrix represents the incidence relation between the points P_j and the lines l_i of the Klein cubic threefold over the field $GF(2)$ in Table 1.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
l_1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
l_2	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0
l_3	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0
l_4	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0
l_5	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0
l_6	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0
l_7	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0
l_8	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0
l_9	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1
l_{10}	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0
l_{11}	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0
l_{12}	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1
l_{13}	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0
l_{14}	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0
l_{15}	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1

Table 1. Incidence relation between the points and the lines of $\mathcal{F}$

This completes the proof. □

Proposition 2.2 *Every point of Klein cubic threefold over the field $GF(2)$ is an Eckardt point, and the three Eckardt points lie on a line on the cubic threefold $\mathcal{F}$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$ of characteristic 2. Every point of Klein cubic threefold lies on three lines in $\mathcal{F}$. So, the number of Eckardt points of Klein cubic threefold over the field $GF(2)$ is 15. Eckardt points with their coordinates as the point of concurrency of three labeled lines can be seen in Table 1. Also, the three Eckardt points lie on a line on the cubic threefold $\mathcal{F}$. For example, P_1, P_3 and P_4 lie on the line l_1 of $\mathcal{F}$. □

Proposition 2.3 *Each line in $\mathcal{F}$ intersects exactly six others and is skew to the remaining eight lines in $\mathcal{F}$.*

Proof Consider the subset of 15 lines of the Klein cubic threefold over the field $GF(2)$ labeled according to the Schläfli notation. The intersection properties of these lines indicate that each line intersects exactly six others and is skew to the remaining eight. The intersection table for these 15 lines is provided in Table 2, where intersections are marked and non-intersecting lines are represented by 0.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}
l_1	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
l_2	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0
l_3	1	0	0	0	0	0	0	0	0	0	1	1	0	0	0
l_4	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0
l_5	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0
l_6	0	1	0	0	0	0	0	0	0	1	1	0	0	0	0
l_7	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0
l_8	0	0	1	0	0	0	0	0	0	1	0	0	1	0	0
l_9	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1
l_{10}	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0
l_{11}	0	0	0	0	1	0	0	0	0	0	0	1	1	0	0
l_{12}	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1
l_{13}	0	0	0	0	0	0	1	0	0	1	0	0	0	1	0
l_{14}	0	0	0	0	0	0	0	1	0	0	0	1	0	1	0
l_{15}	0	0	0	0	0	0	0	0	1	0	1	0	0	0	1

Table 2. Pairwise intersection table of the 15 lines of $\mathcal{F}$

This completes the proof. □

§3. Classifications the Lines of the Klein Cubic Threefold $\mathcal{F}$ Modulo 2

3.1. Line Spreads of $\mathcal{F}$

A (crisp) $k-$spread, or simply spread of the projective geometry $PG(n, K)$ is a partition of the point set of $PG(n, K)$ into $k-$spaces, for some k, $1 \leq k \leq n-1$. We now give line spreads of Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$.

Proposition 3.1 *Every Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$ has* 6 *spreads with five* *lines. Moreover, every line of the threefold $\mathcal{F}$ belongs to exactly two spreads.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. Let S_i be the set of lines spreads of Klein cubic threefold $\mathcal{F}$. $1-$spread S_i of $\mathcal{F}$ must be five lines because of a partition of the point set of $\mathcal{F}$. It is easily obtained that there are six line spreads of $\mathcal{F}$ from Table 1.

$l_1 = \{P_1, P_3, P_4\}$	$l_6 = \{P_2, P_{10}, P_{11}\}$	$l_{11} = \{P_5, P_{12}, P_{13}\}$
$l_2 = \{P_1, P_6, P_7\}$	$l_7 = \{P_3, P_8, P_9\}$	$l_{12} = \{P_6, P_{13}, P_{15}\}$
$l_3 = \{P_1, P_{11}, P_{12}\}$	$l_8 = \{P_3, P_{10}, P_{13}\}$	$l_{13} = \{P_7, P_{10}, P_{14}\}$
$l_4 = \{P_2, P_4, P_5\}$	$l_9 = \{P_4, P_{14}, P_{15}\}$	$l_{14} = \{P_8, P_{12}, P_{14}\}$
$l_5 = \{P_2, P_6, P_8\}$	$l_{10} = \{P_5, P_7, P_9\}$	$l_{15} = \{P_9, P_{11}, P_{15}\}$.

Table 3. Lines spreads of Klein cubic threefold $\mathcal{F}$

and lines spreads of Klein cubic threefold $\mathcal{F}$ are S_i, $i = 1, 2, \cdots, 6$, where

$$S_1 = \{l_1, l_5, l_{11}, l_{13}, l_{15}\}, \quad S_2 = \{l_1, l_6, l_{10}, l_{12}, l_{14}\}, \quad S_3 = \{l_2, l_4, l_8, l_{14}, l_{15}\},$$
$$S_4 = \{l_2, l_6, l_7, l_9, l_{11}\}, \quad S_5 = \{l_3, l_4, l_7, l_{12}, l_{13}\}, \quad S_6 = \{l_3, l_5, l_8, l_9, l_{10}\}.$$

From Table 3, two different spreads of $\mathcal{F}$ have a common line. □

3.2. Skew-Tangent-Secant Lines to $\mathcal{F}$

The study of lines in relation to the Klein cubic threefold $\mathcal{F}$ reveals geometric properties and relationships. The classification of lines as skew, tangent, or secant provides insight into the structure and interaction of $\mathcal{F}$ within the projective space $PG(4,2)$. The following theorem presents detailed characteristics of these lines in relation to $\mathcal{F}$.

Theorem 3.2 *Let $\mathcal{F}$ be a Klein cubic threefold with exactly 15 lines. Then,*

(1) *There are four tangent lines and eight secant lines passing through any point in $\mathcal{F}$;*

(2) *$PG(4,2)$ has exactly 20 lines not intersecting with $\mathcal{F}$, 60 tangent lines to $\mathcal{F}$, and 60 secant lines to $\mathcal{F}$;*

(3) *Four lines of the lines passing through any point in $PG(4,2)\backslash\mathcal{F}$ intersect with $\mathcal{F}$ at two points; seven lines of them intersect with $\mathcal{F}$ at one point; and four lines of them intersect with $\mathcal{F}$ at no point;*

(4) *Eight lines of the lines passing through any point on $\mathcal{F}$ intersect with $\mathcal{F}$ at two points; four lines of them intersect with $\mathcal{F}$ at one point; and three lines of them intersect with $\mathcal{F}$ at three points.*

Proof (1) Let $\mathcal{F}$ be the Klein cubic threefold with exactly 15 lines. Since the incidence relation between the points and the lines of $\mathcal{F}$, there are three points on any line and three lines passing through any point P_i, $i = 1, 2, \cdots, 15$ in $\mathcal{F}$. Also, three of the 15 lines in $PG(4,2)$ passing through any point of $\mathcal{F}$ belong to $\mathcal{F}$. There are seven points of $\mathcal{F}$ on these three lines, and there are eight points of $\mathcal{F}$ apart from these lines. Therefore, 8 lines passing through any point P_i, $i = 1, 2, \cdots, 15$ of $\mathcal{F}$ are formed secant lines. Thus, the remaining 4 lines passing through any point P_i of $\mathcal{F}$ are tangent lines.

(2) Since the number of tangent lines passing through each of the 15 points on $\mathcal{F}$ is 4, the total number of tangent lines to $\mathcal{F}$ in $PG(4,2)$ is 60. In addition, since the number of secant lines passing through a point on F is 8, the total number of secant lines is calculated $\frac{15.8}{2} = 60$.

(3) Let the points of $PG(4,2)$ be denoted by their indices, and the lines of $PG(4,2)$ denoted by the points on them. There exist 15 lines passing through any point in $PG(4,2)$. It is easily seen that four lines of them do not intersect with $\mathcal{F}$, four lines of them intersect with the $\mathcal{F}$ at two points, and seven lines of them intersect with $\mathcal{F}$ at one point. The lines in $PG(4,2)$ through the points not on $\mathcal{F}$ are listed according to the intersection points with $\mathcal{F}$ in the following tables.

16-20-28	17-19-27	18-19-29	19-26-30	21-24-29
16-18-26	17-20-30	18-24-25	20-22-23	22-24-28
16-23-24	17-21-22	18-27-28	20-26-27	22-25-27
16-29-30	17-28-29	19-21-25	21-23-30	23-25-26

Table 4. Lines not intersecting with $\mathcal{F}$

16- 4 -17	17- 12 -23	19- 5 -24	21- 3 -26	23- 8 -31	26- 10 -31
16- 6 -21	17- 15 -25	19- 7 -20	21- 13 -31	23- 9 -28	26- 11 -29
16- 9 -22	17- 14 -31	19- 8 -28	21- 15 -27	24- 1 -27	26- 13 -28
16- 11 -19	18- 1 -22	19- 9 -31	21- 10 -28	24- 6 -30	27- 6 -31
16- 13 -27	18- 5 -21	19- 15 -22	22- 2 -26	24- 7 -31	27- 7 -30
16- 14 -25	18- 11 -30	20- 2 -25	22- 11 -31	24- 14 -26	27- 8 -29
16- 15 -31	18- 12 -31	20- 4 -29	22- 10 -29	25- 1 -28	28- 3 -31
17- 7 -26	18- 13 -20	20- 5 -31	22- 12 -30	25- 3 -30	28- 4 -30
17- 8 -18	18- 14 -23	20- 9 -24	23- 2 -27	25- 4 -31	29- 2 -31
17- 10 -24	19- 3 -23	20- 12 -21	23- 6 -29	25- 5 -29	30- 1 -31

Table 5. Tangent lines to $\mathcal{F}$

16- 1 -2	18- 4 -7	21- 1 -8	23- 5 -11	26- 1 -9	28- 6 -12
16- 3 -5	18- 10 -15	21- 2 -7	23- 7 -15	26- 4 -8	28- 7 -11
16- 7 -8	19- 1 -10	21- 4 -9	24- 2 -13	26- 5 -6	29- 1 -15
16- 10 -12	19- 2 -12	21- 11 -14	24- 3 -11	26- 12 -15	29- 3 -14
17- 1 -5	19- 4 -13	22- 3 -7	24- 4 -12	27- 3 -12	29- 9 -12
17- 2 -3	19- 6 -14	22- 4 -6	24- 8 -15	27- 4 -11	29- 7 -13
17- 6 -9	20- 1 -14	22- 5 -8	25- 11 -6	27- 9 -14	30- 2 -15
17- 11 -13	20- 3 -15	22- 13 -14	25- 7 -12	27- 5 -10	30- 5 -14
18- 2 -9	20- 6 -10	23- 1 -13	25- 8 -10	28- 2 -14	30- 8 -13
18- 3 -6	20- 8 -11	23- 4 -10	25- 9 -13	28- 5 -15	30- 9 -10

Table 6. Secant lines to $\mathcal{F}$

(4) It is easily seen that eight lines of the lines passing through any point on $\mathcal{F}$ intersect with $\mathcal{F}$ at two points from Table 6; four lines of them intersect with $\mathcal{F}$ at one point from Table 5; and three lines of them intersect with $\mathcal{F}$ at three points from Table 3. □

Let B_n be a set of $q^{n-1} + q^{n-2} + \cdots + q + 1$ points, not all on a hyperplane in the $n-$dimensional projective space $PG(n,q)$ over the Galois field $GF(q)$, $n \geq 2$. A point not in B_n is called a nucleus of B_n if every line through it meets B_n (exactly once, of course). The set of all nuclei of B_n is denoted by $N(B_n)$.

Proposition 3.3 *The point $P_{31} = (1,1,1,1,1)$ not on $\mathcal{F}$ in $PG(4,2)$ is nucleus of $\mathcal{F}$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$ of characteristic 2. The projective space $PG(4,2)$ contains 31 points and 135 lines. There are 15 lines through every point. 15 points of these 31 points are on $\mathcal{F}$ and these points are labeled P_i, $i = 1, \cdots, 15$. Consider the point $P_{31} = (1,1,1,1,1)$ in $PG(4,2)$ not satisfying the equation

$$x^2y + y^2z + z^2v + v^2w + w^2x = 0$$

and every line passing through the point P_{31} is a tangent line of $\mathcal{F}$ from Table 6. So, P_{31} is a nucleus of $\mathcal{F}$. □

§4. Geometric Structures Associated with the Klein Cubic Threefold $\mathcal{F}$

In this section, we investigate well-known geometric structures such as the Fano plane, affine plane, and Desargues configuration associated with the Klein cubic threefold $\mathcal{F}$.

First of all, we show that F does not include any projective plane. Then, we determine the planes that are tangent to $\mathcal{F}$. We give two different affine plane models with these tangent planes. Finally, we show that the Desarg theorem is not valid in $\mathcal{F}$.

Proposition 4.1 *There is no any projective plane in Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. If there is a projective plane in Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$,then there are seven points and seven lines such that three points on any line and three lines passing through any point in this projective plane in $\mathcal{F}$. It is well known that seven points of the projective plane on three lines passing through any point. But the remaining 4 lines of the projective plane are secant lines from Table 7 in $PG(4,2)$. So, there is no any projective plane in Klein cubic threefold $\mathcal{F}$. □

Proposition 4.2 *Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$ in $PG(4,2)$. There is a single tangent projective plane at every point of the Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. Let the points of $\mathcal{F}$ be shown with their indices. Table 7 shows the tangent projective planes π_i at the points P_i,

$i = 1, 2, \cdots, 15$, to $\mathcal{F}$ over the field $GF(2)$.

$$\begin{array}{l}
\pi_1 = \{\{1,18,22\},\{1,24,27\},\{1,25,28\},\{18,27,28\},\{22,24,28\},\{22,25,27\},\{18,24,25\}\} \\
\pi_2 = \{\{2,20,25\},\{2,22,26\},\{2,23,27\},\{22,25,27\},\{20,26,27\},\{23,25,26\},\{20,22,23\}\} \\
\pi_3 = \{\{3,19,23\},\{3,21,26\},\{3,25,30\},\{23,25,26\},\{19,21,25\},\{19,26,30\},\{21,23,30\}\} \\
\pi_4 = \{\{4,16,17\},\{4,20,29\},\{4,20,28\},\{16,29,30\},\{17,20,30\},\{17,28,29\},\{16,20,28\}\} \\
\pi_5 = \{\{5,18,21\},\{5,19,24\},\{5,25,29\},\{18,24,25\},\{19,21,25\},\{21,24,29\},\{18,19,29\}\} \\
\pi_6 = \{\{6,16,21\},\{6,23,29\},\{6,24,30\},\{16,23,24\},\{16,29,30\},\{21,24,29\},\{21,23,30\}\} \\
\pi_7 = \{\{7,17,26\},\{7,19,20\},\{7,27,30\},\{17,20,30\},\{19,26,30\},\{20,26,27\},\{17,19,27\}\} \\
\pi_8 = \{\{8,17,18\},\{8,19,28\},\{8,27,29\},\{17,28,29\},\{18,19,29\},\{18,27,28\},\{17,19,27\}\} \\
\pi_9 = \{\{9,16,22\},\{9,20,24\},\{9,23,28\},\{16,23,24\},\{20,22,23\},\{22,24,28\},\{16,20,28\}\} \\
\pi_{10} = \{\{10,17,24\},\{10,21,28\},\{10,22,29\},\{17,28,29\},\{21,24,29\},\{22,24,28\},\{17,21,22\}\} \\
\pi_{11} = \{\{11,16,19\},\{11,18,30\},\{11,26,29\},\{16,29,30\},\{18,19,29\},\{19,26,30\},\{16,18,26\}\} \\
\pi_{12} = \{\{12,17,23\},\{12,20,21\},\{12,22,30\},\{17,21,22\},\{20,22,23\},\{21,23,30\},\{17,20,30\}\} \\
\pi_{13} = \{\{13,16,27\},\{13,18,20\},\{13,26,28\},\{16,20,28\},\{18,27,28\},\{16,18,26\},\{20,26,27\}\} \\
\pi_{14} = \{\{14,16,25\},\{14,18,23\},\{14,24,26\},\{16,23,24\},\{18,24,25\},\{23,25,26\},\{16,18,26\}\} \\
\pi_{15} = \{\{15,17,25\},\{15,21,27\},\{15,19,22\},\{17,19,27\},\{19,21,25\},\{22,25,27\},\{17,21,22\}\}
\end{array}$$

Table 7. Tangent projective planes to $\mathcal{F}$

This completes the proof. □

The following results are obtained from the Table 7.

Corollary 4.3 (i) *Each tangent plane of the surface $\mathcal{F}$ contains three tangent lines passing through the tangent point on $\mathcal{F}$ and four lines not intersecting the surface $\mathcal{F}$.*

(ii) *Three tangent planes of $\mathcal{F}$ intersect along a line not intersecting with $\mathcal{F}$.*

(iii) *The nucleus of $\mathcal{F}$ is not on any tangent planes of $\mathcal{F}$ in $PG(4,2)$.*

(iv) *There are four tangent lines any point in $\mathcal{F}$.*

Theorem 4.4 *Three of the planes passing through the nucleus of $\mathcal{F}$ intersect along a line with $\mathcal{F}$, and four of them intersect with $\mathcal{F}$ at two points.*

Proof The planes D_i, $i = 1, \cdots, 7$, passing through the nucleus of $\mathcal{F}$ can be listed as

$$\begin{array}{l}
D_1 = \{\{1,6,7\},\{1,24,27\},\{1,30,31\},\{6,24,30\},\{6,37,31\},\{7,27,30\},\{7,24,31\}\}, \\
D_2 = \{\{1,3,4\},\{1,25,28\},\{1,30,31\},\{3,25,30\},\{3,28,31\},\{4,28,30\},\{4,25,31\}\}, \\
D_3 = \{\{1,11,12\},\{1,18,22\},\{1,30,31\},\{11,18,30\},\{11,22,31\},\{12,22,30\},\{12,18,31\}\}, \\
D_4 = \{\{1,5,17\},\{1,14,20\},\{1,30,31\},\{5,14,30\},\{5,20,31\},\{14,17,31\},\{17,20,30\}\}, \\
D_5 = \{\{1,9,26\},\{1,10,19\},\{1,30,31\},\{9,10,30\},\{9,19,31\},\{10,26,31\},\{19,26,30\}\}, \\
D_6 = \{\{1,8,21\},\{1,13,23\},\{1,30,31\},\{8,13,30\},\{8,23,31\},\{13,21,31\},\{21,23,30\}\}, \\
D_7 = \{\{1,2,16\},\{1,15,29\},\{1,30,31\},\{2,15,30\},\{2,29,31\},\{15,16,31\},\{16,29,30\}\}.
\end{array}$$

It is easily seen that the planes D_1, D_2, and D_3 intersect along a line with $\mathcal{F}$, and the others intersect with $\mathcal{F}$ at two points. □

Theorem 4.5 *There are six-tangent planes of $\mathcal{F}$ passing through any point not on $\mathcal{F}$ in $PG(4,2)$. Moreover, these planes form four different plane bundles that intersect a line three by three, so that two different bundles have a common tangent plane.*

Proof It is seen from Table 7 that any point P_i, $P_i \in \{16, \cdots, 30\}$ not on $\mathcal{F}$ in $PG(4,2)$ is contained in six-tangent planes. For example, the point $(0,1,1,0,1)$ not on $\mathcal{F}$ is on the tangent planes π_1, π_2, π_9, π_{10}, π_{12},and π_{15}. The triplets of different tangent planes passing through a common line are $\{\pi_1, \pi_2, \pi_{10}\}$, $\{\pi_1, \pi_{12}, \pi_{15}\}$, $\{\pi_2, \pi_9, \pi_{12}\}$ and $\{\pi_{10}, \pi_{12}, \pi_{15}\}$. □

Theorem 4.6 *An affine plane in $PG(4,2)$ can be formed with six tangent planes that passes through a point outside the surface $\mathcal{F}$.*

(1) *Let each of the triplets of different tangent planes tangent to the surface from a point outside the surface in $PG(4,2)$ passing through a common line be called a point and each of the tangent planes be called a line. The incidence relation between a point and a line means that each point is on the three tangent planes (three lines) that form it;*

(2) *Let each of the triplets of different tangent planes that are tangent to the surface from a point outside the surface in $PG(4,2)$ not containing a common line be a point and each of the tangent planes be a line. The incidence relation means that every point is on the tangent planes that form it.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. An affine plane is a collection of points and lines in space that follow the following fairly sensible rules:

$(A1)$ *Given any two points, there is a unique line joining any two points.*

$(A2)$ *Given a point P and a line L not containing P , there is a unique line that contains P and does not intersect L.*

$(A3)$ *There are four points, no three of which are collinear.*

Since every tangent plane in only two triplets of different tangent planes, A1 is satisfied. Since a tangent plane is in only two triplets and not in two triplets of different tangent planes, A2 is satisfied. Since there are only four triplets, and any three of these have not common tangent plane, A3 is satisfied. □

Example 4.7 The six tangent planes passing through the point $(0,1,1,0,1)$, which is not on the surface $\mathcal{F}$ are π_1, π_2, π_9, π_{10}, π_{12} and π_{15}. The triplets of different tangent planes passing through a common line are

$$\{\pi_1, \pi_2, \pi_{10}\},\ \{\pi_1, \pi_{12}, \pi_{15}\},\ \{\pi_2, \pi_9, \pi_{12}\} \text{ and } \{\pi_{10}, \pi_{12}, \pi_{15}\}.$$

If we define respectively the points set, line set and the incidence relation as:

the point set

$$\mathcal{P} = \{\{\pi_1, \pi_2, \pi_{10}\}, \{\pi_1, \pi_{12}, \pi_{15}\}, \{\pi_2, \pi_9, \pi_{12}\}, \{\pi_{10}, \pi_{12}, \pi_{15}\}\},$$

the line set

$$\mathcal{L} = \{\pi_1, \pi_2, \pi_9, \pi_{10}, \pi_{12}, \pi_{15}\},$$

the incidence relation

$$I: \text{ the point } \{\pi_i, \pi_j, \pi_k\}\ I \text{ the lines } \pi_i,\ \pi_j,\ \text{and } \pi_k$$

then, the structure $(\mathcal{P}, \mathcal{L}, I)$ is an affine plane of order 2.

Example 4.8 The six tangent planes passing through the point 22, which is not on the surface $\mathcal{F}$ are π_1, π_2, π_9, π_{10}, π_{12} and π_{15}. The triplets of different tangent planes not containing a common line are

$$\{\pi_1, \pi_2, \pi_9\},\ \{\pi_1, \pi_{10}, \pi_{15}\},\ \{\pi_2, \pi_{12}, \pi_{15}\} \text{ and } \{\pi_9, \pi_{10}, \pi_{12}\}.$$

If we define respectively the points set, line set and the incidence relation as:

the point set

$$\mathcal{P}'= \{\{\pi_1, \pi_2, \pi_9\}, \{\pi_1, \pi_{10}, \pi_{15}\}, \{\pi_2, \pi_{12}, \pi_{15}\}, \{\pi_9, \pi_{10}, \pi_{12}\}\},$$

the line set

$$\mathcal{L}'= \{\pi_1, \pi_2, \pi_9, \pi_{10}, \pi_{12}, \pi_{15}\},$$

the incidence relation

$$I' : \text{ the point } \{\pi_i, \pi_j, \pi_k\}\ I' \text{ the lines } \pi_i, \pi_j, \text{and } \pi_k,$$

then, the structure $(\mathcal{P}', \mathcal{L}', I')$ is an affine plane of order 2.

Proposition 4.9 *If two triangles are in perspective centrally in Klein cubic threefold $\mathcal{F}$ over the field $GF(2)$, then they are not in perspective from an axis in $PG(4,2)\backslash\mathcal{F}$.*

Proof Let $\mathcal{F}$ be Klein cubic threefold over the field $GF(2)$. Denote the three vertices of one triangle by P_3, P_6 and P_{11} and those of the other by P_4, P_7 and P_{12}. Central perspectivity means that the three lines P_3P_4, P_6P_7, and $P_{11}P_{12}$ are concurrent at the point P_1 called the center of perspectivity. Axial perspectivity means that lines P_3P_6 and P_4, P_7 meet in the point P_{18}, lines P_3P_{11} and P_4P_{12} meet in a second point P_{24} and lines P_6P_{11} and P_7P_{12} meet in a third point P_{25} and that these three points all lie on a common line called the axis of perspectivity. This axis $\{P_{18}, P_{24}, P_{25}\}$ of perspectivity is not on $\mathcal{F}$. The line joining the three collinear points of intersection of the extensions of corresponding sides in perspective triangles is not intersecting with $\mathcal{F}$. □

§5. Conclusion

This study has delved into the structural properties of the cubic threefold $\mathcal{F}$ over the field $GF(2)$ in 4−dimensional projective space, focusing particularly on its intersection properties with lines and planes. The analysis has revealed the presence of six spreads, each composed of five lines, offering insights into the combinatorial structure of $\mathcal{F}$. Exploring tangent planes to $\mathcal{F}$ has led to the formulation of two distinct affine plane models, highlighting its geometric versatility. Additionally, the non-validity of Desargues' theorem within $\mathcal{F}$ underscores its departure from classical projective geometry norms. This research contributes to advancing our understanding of cubic threefolds over finite fields, pointing towards further investigations into their algebraic and geometric intricacies.

References

[1] Z. Akça, A. Bayar, S. Ekmekçi, R. Kaya, T. J A, and H. Van Maldeghem, Generalized Veronesean embeddings of projective spaces Part II the lax case, *Ars Combinatoria*, 103: 65–80, 2012.

[2] Z. Akça, A. Bayar, S. Ekmekçi, R. Kaya, H. Van Maldeghem, *Bazıgeometrilerin sonlu projektif uzaylara gömülmeleri üzerine*, Tübitak Proje No: 108T340, 2010.

[3] Z. Akça and A. Altıntaş, Fuzzy counterpart of Klein quadric, *International Electronic Journal of Geometry*, 16(2): 680–688,2023.

[4] A. Bayar, Z. Akça, and S. Ekmekçi, On embedding the projective plane PG(2,4) to the projective space PG(4,4), *New Trends in Mathematical Sciences*, 10(4): 142–150, Dec. 2022.

[5] A. Betten, F. Karaoğlu, Cubic surfaces over small finite fields, *Designs, Codes and Cryptography*, 87 (4): 931–953, 2019.

[6] L.E., Dickson, Projective classification of cubic surfaces modulo 2, *Annals of Mathematics*, 16: 139–157, 1915.

[7] S. Ekmekçi, A. Bayar, Z. Akça, On the projective planes in projective space PG(4,4), *Erciyes Üniversitesi Fen Bilimleri Enstitüsü Fen Bilimleri Dergisi*, Vol. 38, No. 3, pp. 519–524, Dec. 2022.

[8] Hirschfeld JWP, Classical configurations over finite fields. I. The double- six and the cubic surface with 27 lines, *Rendiconti di Matematica e delle sue Applicazioni*, 26 (5): 115–152, 1967.

[9] F., Karaoğlu, Nonsingular cubic surfaces over $\mathbb{F}_{2^k}$, *Turk. J. Math.*, 45: 2492 – 2510, 2021.

[10] F., Karaoglu, A., Smooth cubic surfaces with 15 lines, *Applicable Algebra in Engineering, Communication and Computing*, 33: 823–853, 2022.

[11] F. Klein, Uber die Ausflsung Gleichungen siebenten und achten Grades, *Math. Ann.*, 15: 251-282, 1879.

[12] H. J. Kroll, R. Vincenti, Partitions of the Klein quadric, *Electronic Notes in Discrete Mathematics*, 26(1): 151-157, 2006.

[13] L.A. Rosati, L'equazione delle 27 rette della superficie cubica generale in un corpo finito, *Bollettino dell'Unione Matematica Italiana*, 12(3): 612-626, 1957.

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A Coupled Fixed Point Theorem via Implicit Function in Partially Ordered Partial Metric Spaces and Application

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Abstract: The aim of this manuscript is to discuss the existence of coupled fixed points in the context of partially ordered partial metric spaces via implicit relations. Moreover, we provide some consequences of the established results. We also state an example to illustrate our work. Our main result extends and generalizes various results in the literature. Especially, our result extends and generalizes the corresponding result of Bhaskar and Lakshmikantham [9] from partially ordered complete metric spaces to partially ordered complete partial metric spaces. Finally, an application to the integral equation is included.

Key Words: Coupled fixed point, implicit function, partial metric space, partially ordered set.

AMS(2010): 47H10, 54H25.

§1. Introduction

The notion of coupled fixed point for a partially ordered set Ω was introduced by Bhashkar and Lakshmikantham [9]. Several other authors such as Ćirić and Lakshmikantham [10], Sabetghadam et al. [33] and Oleleru et al. [26] have established some coupled fixed point theorems in metric spaces. Afterwards, many researchers have obtained coupled fixed point results for mappings under various contractive conditions in the setting of metric spaces and generalized metric spaces (see [1], [6], [14], [19], [24], [37]).

In 1994, the notion of partial metric space was introduced by Matthews (see, [23]) as part of the study of denotational semantics of dataflow networks. It is well-known that partial metric spaces play an important role in the theory of computation (see, e.g., [15], [21], [32], [36]). The PMS is a generalization of usual metric spaces in which the self-distance need not be zero. Later, Matthews proved the partial metric version of Banach fixed point theorem [8].

Several famous mathematicians have contributed to the development of this research fields. Masiha et al. [22] proved some fixed point results for weakly contractive type mappings in partially ordered partial metric spaces. They applied their results to nonlinear fractional boundary value problem. Altun et al. [5] established some fixed point theorems for generalized contractive type mappings on partial metric spaces. They also proved a homotopy result. Aydi et

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al. [7] introduced the concept of partial Hausdorff metric and they initiated the study of fixed point theory for multi-valued mappings on partial metric spaces using the partial Hausdorff metric and proved an analogous to the well-known Nadler's fixed poin theorem. Heckmann [15] introduced the concept of weak partial metric function and established some fixed point results. Oltra ans Valero [27] generalized the Matthews results in the sense of O'Neil [28] in complete partial metric space (see, also [2], [3], [6], [12], [13], [17], [20], [25]).

The practice of improving contraction conditions in proving fixed point and common fixed point theorems is still in fashion. Recently, with a view to accommodate many contraction conditions, Popa [29] and Popa et al. [31] introduced implicit functions which are proving fruitful due to their unifying power besides admitting new contractive conditions.

In nonlinear analysis, especially in fixed point theory, implicit relations on metric spaces have been investigated highly in many articles (see, e.g., [4], [16], [30], [34] and references therein).

Inspired and motivated by the works of [6, 9, 29] and others, the purpose of this article is to examine the existence of a coupled fixed point theorem for mappings satisfying mixed monotone property in the context of partial metric spaces by using implicit relations. In addition, we provide some consequences of the established result. We also state an example to illustrate our result. Finally, an application to the integral equation is included. Our results extend, generalize and enrich several results in the existing literature.

§2. **Preliminaries**

In this section, we give some definitions and lemmas related to partial metric spaces which will be useful in the proof of our main results.

Definition 2.1([23]) *Let Ω be a nonempty set. A partial metric on Ω is a function $p\colon \Omega\times\Omega \to [0,+\infty)$ such that for all $v_1, v_2, u \in \Omega$ the followings are satisfied:*

($p1$) $v_1 = v_2 \Leftrightarrow p(v_1, v_1) = p(v_1, v_2) = p(v_2, v_2)$;
($p2$) $p(v_1, v_1) \le p(v_1, v_2)$;
($p3$) $p(v_1, v_2) = p(v_2, v_1)$;
($p4$) $p(v_1, v_2) \le p(v_1, u) + p(u, v_2) - p(u, u)$.

Then, p is called a partial metric on Ω and the pair (Ω, p) is called a partial metric space.

It is clear that if $p(v_1, v_2) = 0$, then from ($p1$), ($p2$), and ($p3$), $v_1 = v_2$. But if $v_1 = v_2$, $p(v_1, v_2)$ may not be 0.

If p is a partial metric on Ω, then the function $d^p\colon \Omega\times\Omega \to [0,+\infty)$ given by

$$d^p(v_1, v_2) = 2p(v_1, v_2) - p(v_1, v_1) - p(v_2, v_2), \tag{2.1}$$

is a usual metric on Ω.

Each partial metric p on Ω generates a T_0 topology τ_p on Ω with the family of open p-balls $\{B_p(y,\varepsilon) : y \in \Omega, \varepsilon > 0\}$ where $B_p(y,\varepsilon) = \{z \in \Omega : p(y, z) < p(y, y) + \varepsilon\}$ for all $y \in \Omega$ and

$\varepsilon > 0$. Similarly, closed p-ball is defined as $B_p[y,\varepsilon] = \{z \in \Omega : p(y,z) \leq p(y,y) + \varepsilon\}$ for all $y \in \Omega$ and $\varepsilon > 0$.

Example 2.2([7]) Let $\Omega = [0,+\infty)$ and $p \colon \Omega \times \Omega \to [0,+\infty)$ be given by $p(y,z) = \max\{y,z\}$ for all $y,z \in \Omega$. Then (Ω,p) is a partial metric space.

Example 2.3([7]) Let $\Omega = I$, where I denote the set of all intervals $[y_1,z_1]$ for any real numbers $y_1 \leq z_1$. Let $p \colon \Omega \times \Omega \to [0,\infty)$ be a function such that $p\big([y_1,z_1],[y_2,z_2]\big) = \max\{z_1,z_2\} - \min\{y_1,y_2\}$. Then, (Ω,p) is a partial metric space.

Example 2.4([11]) Let $\Omega = \mathbb{R}$ and $p \colon \Omega \times \Omega \to \mathbb{R}^+$ be given by $p(y,z) = e^{\max\{y,z\}}$ for all $y,z \in \Omega$. Then (Ω,p) is a partial metric space.

Definition 2.5([23]) *Let (Ω,p) be a partial metric space. Then,*

(A) A sequence $\{y_n\}$ converges to a point $y \in \Omega$ if and only if $\lim_{n\to\infty} p(y,y_n) = p(y,y)$;

(B) A sequence $\{y_n\}$ in Ω is called a Cauchy sequence if and only if $\lim_{m,n\to\infty}$ $p(y_m,y_n)$ exists (and finite);

(C) A partial metric space (Ω,p) is said to be complete if every Cauchy sequence $\{y_n\}$ in Ω converges, with respect to τ_p, to a point $y \in \Omega$, such that, $\lim_{m,n\to\infty} p(y_m,y_n) = p(y,y)$;

(D) A mapping $f \colon \Omega \to \Omega$ is said to be continuous at $y_0 \in \Omega$ if for every $\varepsilon > 0$, there exists $\eta > 0$ such that $f\big(B_p(y_0,\eta)\big) \subset B_p\big(f(y_0),\varepsilon\big)$.

Definition 2.6([23]) *A partial metric space (Ω,p) is said to be complete if every Cauchy sequence $\{y_n\}$ in Ω converges to a point $y \in \Omega$ with respect to τ_p. Furthermore,*

$$\lim_{m,n\to\infty} p(y_m,y_n) = \lim_{n\to\infty} p(y_n,y) = p(y,y).$$

Definition 2.7([9]) *Let $(\Omega,\leq)$ be a partially ordered set. The mapping $H \colon \Omega\times\Omega \to \Omega$ is said to have the mixed monotone property if $H(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is, for any $x,y \in \Omega$,*

$$x_1,x_2 \in \Omega,\;\; x_1 \leq x_2 \;\Rightarrow\; H(x_1,y) \leq H(x_2,y),$$

and

$$y_1,y_2 \in \Omega,\;\; y_1 \leq y_2 \;\Rightarrow\; H(x,y_1) \geq H(x,y_2).$$

Definition 2.8([9,10]) *An element $(x,y) \in \Omega \times \Omega$ is said to be a coupled fixed point of the mapping $H \colon \Omega \times \Omega \to \Omega$ if $H(x,y) = x$ and $H(y,x) = y$.*

Example 2.9 Let $\Omega = [0,+\infty)$ and $H \colon \Omega \times \Omega \to \Omega$ be defined by $H(x,y) = \frac{x+y}{3}$ for all $x,y \in \Omega$. Then one can easily see that H has a unique coupled fixed point $(0,0)$.

Example 2.10 Let $\Omega = [0,+\infty)$ and $H \colon \Omega \times \Omega \to \Omega$ be defined by $H(x,y) = \frac{x+y}{2}$ for all $x,y \in \Omega$. Then we see that H has two coupled fixed point $(0,0)$ and $(1,1)$, that is, the coupled fixed point is not unique.

Lemma 2.11([6, 23]) (1) *A sequence* $\{y_n\}$ *is Cauchy in a partial metric space* (Ω, p) *if and only if* $\{y_n\}$ *is Cauchy in a metric space* (Ω, d^p) *where*

$$d^p(y,z) = 2p(y,z) - p(y,y) - p(z,z) \ \text{for all} \ y, z \in \Omega.$$

(2) *A partial metric space* (Ω, p) *is complete if a metric space* (Ω, d^p) *is complete, i.e.,*

$$\lim_{n\to\infty} d^p(y_n, y) = 0 \Leftrightarrow p(y,y) = \lim_{n\to\infty} p(y_n, y) = \lim_{n,m\to\infty} p(y_n, y_m).$$

Lemma 2.12([17]) *Let* (Ω, p) *be a partial metric space.*

($1'$) *If* $y, z \in \Omega$, $p(y,z) = 0$, *then* $y = z$;
($2'$) *If* $y \neq z$, *then* $p(y,z) > 0$.

One of the characterization of continuity of mappings in partial metric spaces was given by Samet et al. [35] as follows.

Lemma 2.13([35]) *Let* (Ω, p) *be a partial metric space. The function* $F\colon \Omega \to \Omega$ *is continuous if given a sequence* $\{y_n\}_{n\in\mathbb{N}}$ *and* $y \in \Omega$ *such that* $p(y,y) = \lim_{n\to\infty} p(y, y_n)$, *then* $p(Fy, Fy) = \lim_{n\to\infty} p(Fy, Fy_n)$.

Example 2.14([35]) Let $\Omega = [0, +\infty)$ endowed with the partial metric $p\colon \Omega \times \Omega \to [0, +\infty)$ defined $p(y,z) = \max\{y, z\}$ for all $y, z \in \Omega$. Let $F\colon \Omega \to \Omega$ be a non-decreasing function. If F is continuous with respect to the standard metric $d(y,z) = |y - z|$ for all $y, z \in \Omega$, then F is continuous with respect to the partial metric p.

Lemma 2.15([11]) *Let* $y_n \to y$ *as* $n \to \infty$ *in a partial metric space* (Ω, p) *where* $p(y,y) = 0$. *Then* $\lim_{n\to\infty} p(y_n, u) = p(y, u)$ *for all* $u \in \Omega$.

A set of *implicit relations*, denoted by $\mathbb{V}$, is the collection of all continuous functions $V\colon (\mathbb{R}^+)^5 \to \mathbb{R}$ which satisfy:

($V1$) $V(t_1, t_2, t_3, t_4, t_5)$ is non-increasing in t_4 and t_5, and
($V2$) there exists a function $\psi \in \Psi$ such that

$$V(u, v, w, u+v, u+v) \leq 0 \text{ implies } u \leq v + \psi(w),$$

where Ψ denotes the set of all functions $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with the properties:

(i) ψ is continuous and non-decreasing;
(ii) $\psi(t) < t$ for each $t > 0$ and $\psi(0) = 0$.

Example 2.16 It is easy to check that the following functions are in $\mathbb{V}$.

($V_{1'}$) $V(t_1, t_2, t_3, t_4, t_5) = t_1 - a\,t_2 - b\,t_3 - c\,t_4 - d\,t_5$, where a, b, c, d are non-negative real numbers such that $a+b+2c+2d<1$;

$(V_{2'})$ $V(t_1,t_2,t_3,t_4,t_5)=t_1-a\max\left\{t_2,t_3,\frac{t_4}{2},\frac{t_5}{2}\right\}$, where $a\in(0,1)$;

$(V_{3'})$ $V(t_1,t_2,t_3,t_4,t_5)=t_1-\psi(\max\{t_2,t_3\})$, where $\psi\in\Psi$.

§3. Main Results

In this section, we shall prove a coupled fixed point theorem via implicit function in the framework of partially ordered partial metric spaces.

Theorem 3.1 *Let $(\Omega,p,\leq)$ be a partially ordered complete partial metric space. Suppose that $H\colon\Omega\times\Omega\to\Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V\in\mathbb{V}$ such that*

$$V\begin{pmatrix} p(H(u,v),H(y,z)),p(u,y),p(v,z),\\ p(H(u,v),u)+p(H(y,z),y),p(H(u,v),y)\end{pmatrix}\leq 0, \tag{3.1}$$

for all $u,v,y,z\in\Omega$ with $u\geq y$ and $v\leq z$. Suppose that either

(a) H is continuous or

(b) Ω has the following property

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u\in\Omega$, then $u_n\leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v\in\Omega$, then $v\leq v_n$ for all n.

If there exist two elements $u_0,v_0\in\Omega$ with $u_0\leq H(u_0,v_0)$ and $v_0\geq H(v_0,u_0)$, then H has a coupled fixed point in Ω.

Proof Let $u_0,v_0\in\Omega$ be such that $u_0\leq H(u_0,v_0)$ and $v_0\geq H(v_0,u_0)$. We construct the iterative sequences $\{u_n\}$ and $\{v_n\}$ in Ω as follows: let $u_1=H(u_0,v_0)$ and $v_1=H(v_0,u_0)$. Then $u_0\leq u_1$ and $v_0\geq v_1$. Again, let $u_2=H(u_1,v_1)$ and $v_2=H(v_1,u_1)$. Since H has the mixed monotone property on Ω, then we have $u_1\leq u_2$ and $v_1\geq v_2$. Continuing the same way as above, we get

$$u_{n+1}=H(u_n,v_n)\ \text{ and }\ v_{n+1}=H(v_n,u_n)\ \text{ for all } n\geq 0, \tag{3.2}$$

and

$$u_0\leq u_1\leq\cdots\leq u_n\leq u_{n+1}\leq\ldots,\quad v_0\geq v_1\geq\cdots\geq v_n\geq v_{n+1}\geq\ldots. \tag{3.3}$$

If there exists $n_0\in\mathbb{N}\cup\{0\}$ such that $u_{n_0}=u_{n_0+1}$ and $v_{n_0}=v_{n_0+1}$, then

$$u_{n_0}=u_{n_0+1}=H(u_{n_0},v_{n_0})\text{ and }v_{n_0}=v_{n_0+1}=H(v_{n_0},u_{n_0}),$$

which concludes that (u_{n_0},v_{n_0}) is a coupled fixed point of H. So, we assume that $u_{n_0}\neq u_{n_0+1}$ or $v_{n_0}\neq v_{n_0+1}$ for all n. By Lemma 2.12 $(2')$, we have $p(u_{n+1},u_n)>0$ and $p(v_{n+1},v_n)>0$ for all n.

Since $u_{n+1}\geq u_n$ and $v_{n+1}\leq v_n$, from equation (3.1) with $u=u_{n+1}$, $v=v_{n+1}$, $y=u_n$

and $z = v_n$, we have

$$V\begin{pmatrix} p(H(u_{n+1}, v_{n+1}), H(u_n, v_n)), p(u_{n+1}, u_n), p(v_{n+1}, v_n), \\ p(H(u_{n+1}, v_{n+1}), u_{n+1}) + p(H(u_n, v_n), u_n), p(H(u_{n+1}, v_{n+1}), u_n) \end{pmatrix} \leq 0,$$

or

$$V\begin{pmatrix} p(u_{n+2}, u_{n+1}), p(u_{n+1}, u_n), p(v_{n+1}, v_n), \\ p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n), p(u_{n+2}, u_n) \end{pmatrix} \leq 0. \tag{3.4}$$

By PMS condition ($p4$), we have

$$\begin{aligned} p(u_{n+2}, u_n) &\leq p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n) - p(u_{n+1}, u_{n+1}) \\ &\leq p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n). \end{aligned} \tag{3.5}$$

By the properties of V and equation (3.5), the inequality (3.4) reduces to

$$V\begin{pmatrix} p(u_{n+2}, u_{n+1}), p(u_{n+1}, u_n), p(v_{n+1}, v_n), \\ p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n), p(u_{n+2}, u_{n+1}) + p(u_{n+1}, u_n) \end{pmatrix} \leq 0, \tag{3.6}$$

which yields that

$$p(u_{n+2}, u_{n+1}) \leq p(u_{n+1}, u_n) + \psi(p(v_{n+1}, v_n)). \tag{3.7}$$

Similarly, we can show that

$$p(v_{n+2}, v_{n+1}) \leq p(v_{n+1}, v_n) + \psi(p(u_{n+1}, u_n)). \tag{3.8}$$

By adding equations (3.7)-(3.8) and using the properties of ψ, we have

$$S_n \leq S_{n-1} + \psi(S_{n-1}), \tag{3.9}$$

where $S_n = p(u_{n+2}, u_{n+1}) + p(v_{n+2}, v_{n+1})$.

If there exists $n_1 \in \mathbb{N} \cup \{0\}$ such that $p(u_{n_1+2}, u_{n_1+1}) = 0$, $p(v_{n_1+2}, v_{n_1+1}) = 0$, then $u_{n_1+1} = u_{n_1+2} = H(u_{n_1+1}, v_{n_1+1})$, $v_{n_1+1} = v_{n_1+2} = H(v_{n_1+1}, u_{n_1+1})$ and (u_{n_1+1}, v_{n_1+1}) is a coupled fixed point of H and thus the proof is finished. Suppose, on the contrary, that $p(u_{n_1+2}, u_{n_1+1}) \neq 0$, $p(v_{n_1+2}, v_{n_1+1}) \neq 0$ for all $n \in \mathbb{N}$. Then by the properties of function ψ, we have

$$S_n \leq S_{n-1} + \psi(S_{n-1}) \leq S_{n-1}, \tag{3.10}$$

where S_n is a non-negative sequence and hence convergent to a limit, say S^*. Taking the limit when $n \to \infty$ in equation (3.10), we get

$$S^* \leq S^* + \psi(S^*) \tag{3.11}$$

and consequently, we have $\psi(S^*) = 0$. By the property of function ψ, we obtain $S^* = 0$, that is, $\lim_{n\to\infty} S_n = 0$. Thus

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} p(u_{n+1}, u_n) + p(v_{n+1}, v_n) = 0$$

$$\Rightarrow \lim_{n\to\infty} p(u_{n+1}, u_n) = \lim_{n\to\infty} p(v_{n+1}, v_n) = 0. \tag{3.12}$$

Next, we prove that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{u_n\}$ or $\{v_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ for which we can find subsequences $\{u_{n(k)}\}$, $\{u_{m(k)}\}$ of $\{u_n\}$ and $\{v_{n(k)}\}$, $\{v_{m(k)}\}$ of $\{v_n\}$ with $n(k) > m(k) \geq k$ such that

$$p(u_{n(k)}, u_{m(k)}) \geq \varepsilon, \text{ for all } k = 1, 2, 3, \cdots. \tag{3.13}$$

Furthermore, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies equation (3.13). Then, we have

$$p(u_{n(k)-1}, u_{m(k)}) < \varepsilon. \tag{3.14}$$

Using the triangle inequality, we have

$$\begin{aligned} p(u_{n(k)}, u_{m(k)}) &\leq p(u_{n(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{m(k)}) \\ &\quad - p(u_{n(k)-1}, u_{n(k)-1}) \\ &\leq p(u_{n(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{m(k)}) \\ &< p(u_{n(k)}, u_{n(k)-1}) + \varepsilon. \end{aligned} \tag{3.15}$$

Similarly, we have

$$\begin{aligned} p(v_{n(k)}, v_{m(k)}) &\leq p(v_{n(k)}, v_{n(k)-1}) + p(v_{n(k)-1}, v_{m(k)}) \\ &\quad - p(v_{n(k)-1}, v_{n(k)-1}) \\ &\leq p(v_{n(k)}, v_{n(k)-1}) + p(v_{n(k)-1}, v_{m(k)}) \\ &< p(v_{n(k)}, v_{n(k)-1}) + \varepsilon. \end{aligned} \tag{3.16}$$

From equations (3.13) and (3.15), we have

$$\varepsilon \leq p(u_{n(k)}, u_{m(k)}) \leq p(u_{n(k)}, u_{n(k)-1}) + \varepsilon. \tag{3.17}$$

Letting $k \to \infty$ in equation (3.17) and using equation (3.12), we get

$$\lim_{k\to\infty} p(u_{n(k)}, u_{m(k)}) = \varepsilon. \tag{3.18}$$

Similarly, one can prove that

$$\lim_{k\to\infty} p(v_{n(k)}, v_{m(k)}) = \varepsilon. \tag{3.19}$$

By the triangle inequality, we have

$$\begin{aligned} p(u_{m(k)}, u_{n(k)}) &\leq p(u_{m(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{n(k)}) - p(u_{n(k)-1}, u_{n(k)-1}) \\ &\leq p(u_{m(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{n(k)}), \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} p(u_{m(k)}, u_{n(k)-1}) &\leq p(u_{m(k)}, u_{n(k)}) + p(u_{n(k)}, u_{n(k)-1}) - p(u_{n(k)}, u_{n(k)}) \\ &\leq p(u_{m(k)}, u_{n(k)}) + p(u_{n(k)}, u_{n(k)-1}). \end{aligned} \tag{3.21}$$

Taking the limit as $k \to \infty$ in equations (3.20), (3.21) and using equations (3.12), (3.18), we get

$$\lim_{k\to\infty} p(u_{n(k)-1}, u_{m(k)}) = \varepsilon. \tag{3.22}$$

Again, by triangle inequality, we have

$$\begin{aligned} p(u_{n(k)-1}, u_{m(k)}) &\leq p(u_{n(k)-1}, u_{m(k)-1}) + p(u_{m(k)-1}, u_{m(k)}) \\ &\quad - p(u_{m(k)-1}, u_{m(k)-1}) \\ &\leq p(u_{n(k)-1}, u_{m(k)-1}) + p(u_{m(k)-1}, u_{m(k)}), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} p(u_{n(k)-1}, u_{m(k)-1}) &\leq p(u_{n(k)-1}, u_{m(k)}) + p(u_{m(k)}, u_{m(k)-1}) \\ &\quad - p(u_{m(k)}, u_{m(k)}) \\ &\leq p(u_{n(k)-1}, u_{m(k)}) + p(u_{m(k)}, u_{m(k)-1}). \end{aligned} \tag{3.24}$$

Taking the limit as $k \to \infty$ in equations (3.23), (3.24) and using equations (3.12), (3.22), we get

$$\lim_{k\to\infty} p(u_{n(k)-1}, u_{m(k)-1}) = \varepsilon. \tag{3.25}$$

Once again using triangle inequality, we have

$$\begin{aligned} p(u_{n(k)-1}, u_{m(k)-1}) &\leq p(u_{n(k)-1}, u_{n(k)}) + p(u_{n(k)}, u_{m(k)-1}) \\ &\quad - p(u_{n(k)}, u_{n(k)}) \\ &\leq p(u_{n(k)-1}, u_{n(k)}) + p(u_{n(k)}, u_{m(k)-1}), \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} p(u_{n(k)}, u_{m(k)-1}) &\leq p(u_{n(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{m(k)-1}) \\ &\quad - p(u_{n(k)-1}, u_{n(k)-1}) \\ &\leq p(u_{n(k)}, u_{n(k)-1}) + p(u_{n(k)-1}, u_{m(k)-1}). \end{aligned} \tag{3.27}$$

Taking the limit as $k \to \infty$ in equations (3.26), (3.27) and using equations (3.12), (3.25), we get

$$\lim_{k\to\infty} p(u_{n(k)}, u_{m(k)-1}) = \varepsilon. \tag{3.28}$$

Since $n(k) > m(k)$, $u_{n(k)-1} \geq u_{m(k)-1}$ and $v_{n(k)-1} \leq v_{m(k)-1}$. From equation (3.1), we have

$$V\begin{pmatrix} p(H(u_{n(k)-1}, v_{n(k)-1}), H(u_{m(k)-1}, v_{m(k)-1})), \\ p(u_{n(k)-1}, u_{m(k)-1}), p(v_{n(k)-1}, v_{m(k)-1}), \\ p(H(u_{n(k)-1}, v_{n(k)-1}), u_{n(k)-1}) \\ +p(H(u_{m(k)-1}, v_{m(k)-1})), u_{m(k)-1}), \\ p(H(u_{n(k)-1}, v_{n(k)-1}), u_{m(k)-1}) \end{pmatrix} \leq 0,$$

or

$$V\begin{pmatrix} p(u_{n(k)}, u_{m(k)}), p(u_{n(k)-1}, u_{m(k)-1}), p(v_{n(k)-1}, v_{m(k)-1}), \\ p(u_{n(k)}, u_{n(k)-1}) + p(u_{m(k)}, u_{m(k)-1}), p(u_{n(k)}, u_{m(k)-1}) \end{pmatrix} \leq 0. \tag{3.29}$$

Letting $k \to \infty$ in equation (3.29), using equations (3.12), (3.18), (3.25) and (3.28), we obtain

$$V(\varepsilon, \varepsilon, \varepsilon, 0, \varepsilon) \leq 0. \tag{3.30}$$

Hence, we find

$$V(\varepsilon, \varepsilon, \varepsilon, 0 + \varepsilon, \varepsilon + 0) \leq V(\varepsilon, \varepsilon, \varepsilon, 0, \varepsilon) \leq 0,$$

which implies $\varepsilon \leq \varepsilon + \psi(\varepsilon)$. Thus, $\psi(\varepsilon) = 0$ and so $\varepsilon = 0$ by the property of ψ. Which is a contradiction. Thus $\{u_n\}$ is a Cauchy sequence. Using the same arguments as above, we can show that $\{v_n\}$ is also a Cauchy sequence. Since Ω is complete, there exist $u, v \in \Omega$ such that

$$\begin{aligned} \lim_{n,m\to\infty} p(u_n, u_m) &= \lim_{n\to\infty} p(u_n, u) = p(u, u), \\ \lim_{n,m\to\infty} p(v_n, v_m) &= \lim_{n\to\infty} p(v_n, v) = p(v, v). \end{aligned} \tag{3.31}$$

Now, we want to show that

$$p(u, u) = 0 = p(v, v).$$

Suppose, on the contrary, that

$$p(u, u) = \mu > 0 \text{ and } p(v, v) = \nu > 0. \tag{3.32}$$

Then, we see that

$$V\begin{pmatrix} p(H(u_{n-1}, v_{n-1}), H(u_{m-1}, v_{m-1})), \\ p(u_{n-1}, u_{m-1}), p(v_{n-1}, v_{m-1}), \\ p(H(u_{n-1}, v_{n-1}), u_{n-1}) \\ +p(H(u_{m-1}, v_{m-1})), u_{m-1}), \\ p(H(u_{n-1}, v_{n-1}), u_{m-1}) \end{pmatrix} \leq 0,$$

or

$$V\begin{pmatrix} p(u_n, u_m), p(u_{n-1}, u_{m-1}), p(v_{n-1}, v_{m-1}), \\ p(u_n, u_{n-1}) + p(u_m, u_{m-1}), p(u_n, u_{m-1}) \end{pmatrix} \leq 0.$$

By using the triangle inequality $(p4)$, we get

$$V\begin{pmatrix} p(u_n, u_m), p(u_{n-1}, u_{m-1}), p(v_{n-1}, v_{m-1}), \\ p(u_n, u_{n-1}) + p(u_m, u_{m-1}), \\ p(u_n, u_{n-1}) + p(u_{n-1}, u_{m-1}) \end{pmatrix} \leq 0.$$

Letting $n, m \to \infty$ and using equation (3.12), we obtain

$$V(\mu, \mu, 0, 0, \mu) \leq 0.$$

Hence, we get that

$$V(\mu, \mu, \mu, 0 + \mu, \mu + 0) \leq V(\mu, \mu, \mu, 0, \mu) \leq 0,$$

which implies that $\mu \leq \mu + \psi(\mu)$. Thus, $\psi(\mu) = 0$ and so $\mu = 0$ by the property of ψ. Hence $p(u, u) = 0$. By similar fashion, we can show that $p(v, v) = 0$.

Now, suppose that the assumption (a) holds. Then, we have

$$\begin{aligned} p(u, H(u_n, v_n)) &\leq p(u, u_{n+1}) + p(u_{n+1}, H(u_n, v_n)) \\ &\quad - p(u_{n+1}, u_{n+1}) \\ &\leq p(u, u_{n+1}) + p(u_{n+1}, H(u_n, v_n)) \\ &= p(u, H(u_n, v_n)) + p(H(u_n, v_n), H(u_n, v_n)). \end{aligned} \tag{3.33}$$

Taking the limit as $n \to \infty$ in equation (3.33), using equation (3.31) and continuity of H, we obtain

$$p(u, H(u, v)) = 0.$$

Similarly, we can show that

$$p(v, H(v, u)) = 0.$$

Therefore, $u = H(u, v)$ and $v = H(v, u)$. This shows that (u, v) is a coupled fixed point of H in Ω.

Finally, suppose that assumption (b) holds. Since $\{u_n\}$ is a non-decreasing sequence and $u_n \to u$ as $n \to \infty$ and $\{v_n\}$ is a non-increasing sequence and $v_n \to v$ as $n \to \infty$, by the assumption, we have $u_n \leq u$ and $v_n \geq v$ for all n. From equations (3.1) and (3.31), we have

$$\lim_{n\to\infty} p(u_n, u) = p(u, u) = \lim_{n\to\infty} p(H(u_n, v_n), u), \tag{3.34}$$

and

$$\lim_{n\to\infty} p(v_n, v) = p(v, v) = \lim_{n\to\infty} p(H(v_n, u_n), v). \tag{3.35}$$

We also have

$$V\begin{pmatrix} p(H(u_n,v_n),H(u,v)),p(u_n,u),p(v_n,v), \\ p(H(u_n,v_n),u_n)+p(H(u,v),u),p(H(u_n,v_n),u) \end{pmatrix} \le 0.$$

Letting $n \to \infty$ and using equations (3.34) and (3.35), we have

$$V(p(u,H(u,v)),0,0,p(u,H(u,v)),p(u,H(u,v))) \le 0,$$

which implies that $p(u,H(u,v)) \le 0+\psi(0) = 0$. Hence $u = H(u,v)$.

Similarly, one can show that $v = H(v,u)$. Thus in all the above cases, we proved that H has a coupled fixed point in Ω. This completes the proof. □

From Example 2.16 and Theorem 3.1, we obtain the following results.

Corollary 3.2 *Let $(\Omega,p,\le)$ be a partially ordered complete partial metric space. Suppose that $H\colon \Omega\times\Omega \to \Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V \in \mathbb{V}$ such that*

$$\begin{aligned} p(H(u,v),H(y,z)) \quad &\le \quad a_1\, p(u,y)+a_2\, p(v,z)+a_3\,[p(H(u,v),u) \\ &\qquad +p(H(y,z),y)]+a_4\, p(H(u,v),y) \end{aligned} \tag{3.36}$$

for all $u,v,y,z \in \Omega$ with $u \ge y$ and $v \le z$, where a_1,a_2,a_3,a_4 are non-negative reals such that $a_1+a_2+2a_3+2a_4 < 1$. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u \in \Omega$, then $u_n \le u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v \in \Omega$, then $v \le v_n$ for all n.

If there exist two elements $u_0,v_0 \in \Omega$ with $u_0 \le H(u_0,v_0)$ and $v_0 \ge H(v_0,u_0)$, then H has a coupled fixed point in Ω.

Corollary 3.3 *Let $(\Omega,p,\le)$ be a partially ordered complete partial metric space. Suppose that $H\colon \Omega\times\Omega \to \Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V \in \mathbb{V}$ such that*

$$\begin{aligned} p(H(u,v),H(y,z)) \quad &\le \quad a \max\Big\{p(u,y),p(v,z),\frac{1}{2}[p(H(u,v),u) \\ &\qquad +p(H(y,z),y)],\frac{1}{2}p(H(u,v),y)\Big\} \end{aligned} \tag{3.37}$$

for all $u,v,y,z \in \Omega$ with $u \ge y$ and $v \le z$, where $a \in (0,1)$ is a constant. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u \in \Omega$, then $u_n \leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v \in \Omega$, then $v \leq v_n$ for all n.

If there exist two elements $u_0, v_0 \in \Omega$ with $u_0 \leq H(u_0, v_0)$ and $v_0 \geq H(v_0, u_0)$, then H has a coupled fixed point in Ω.

Corollary 3.4 *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Suppose that $H: \Omega \times \Omega \to \Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V \in \mathbb{V}$ such that*

$$p(H(u,v), H(y,z)) \leq \psi\big(\max\{p(u,y), p(v,z)\}\big) \tag{3.38}$$

for all $u, v, y, z \in \Omega$ with $u \geq y$ and $v \leq z$, where $\psi \in \Psi$. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u \in \Omega$, then $u_n \leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v \in \Omega$, then $v \leq v_n$ for all n.

If there exist two elements $u_0, v_0 \in \Omega$ with $u_0 \leq H(u_0, v_0)$ and $v_0 \geq H(v_0, u_0)$, then H has a coupled fixed point in Ω.

If we take $a_1 = k$, $a_2 = l$ and $a_3 = a_4 = 0$ where $k, l \in (0,1)$ in Corollary 3.2, then we obtain the following result.

Corollary 3.5 *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Suppose that $H: \Omega \times \Omega \to \Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V \in \mathbb{V}$ such that*

$$p(H(u,v), H(y,z)) \leq k\, p(u,y) + l\, p(v,z) \tag{3.39}$$

for all $u, v, y, z \in \Omega$ with $u \geq y$ and $v \leq z$, where k, l are non-negative reals such that $k + l < 1$. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u \in \Omega$, then $u_n \leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v \in \Omega$, then $v \leq v_n$ for all n.

If there exist two elements $u_0, v_0 \in \Omega$ with $u_0 \leq H(u_0, v_0)$ and $v_0 \geq H(v_0, u_0)$, then H has a coupled fixed point in Ω.

If we take $k = l = m$ where $m \in (0, 1)$ in Corollary 3.5, then we obtain the following result.

Corollary 3.6 *Let $(\Omega, p, \leq)$ be a partially ordered complete partial metric space. Suppose that $H\colon \Omega\times\Omega \to \Omega$ be a mapping such that H has the mixed monotone property. Assume that there exists $V \in \mathbb{V}$ such that*

$$p(H(u,v), H(y,z)) \leq \frac{m}{2}\left[p(u,y) + p(v,z)\right] \tag{3.40}$$

for all $u, v, y, z \in \Omega$ with $u \geq y$ and $v \leq z$, where $m \in (0,1)$ is a constant. Suppose that either

(a) H is continuous or

(b) Ω has the following property:

(i) if a non-decreasing sequence $\{u_n\}$ in Ω converges to some point $u \in \Omega$, then $u_n \leq u$ for all n;

(ii) if a non-increasing sequence $\{v_n\}$ in Ω converges to some point $v \in \Omega$, then $v \leq v_n$ for all n.

If there exist two elements $u_0, v_0 \in \Omega$ with $u_0 \leq H(u_0, v_0)$ and $v_0 \geq H(v_0, u_0)$, then H has a coupled fixed point in Ω.

Remark 3.7 Corollary 3.6 extends and generalizes Theorems 2.1 and 2.2 of [9] from partially ordered complete metric spaces to partially ordered complete partial metric spaces.

Example 3.8([18]) Let $\Omega = [0, \infty)$ with usual order $\leq$. Then, $(\Omega, p, \leq)$ be a partially ordered partial metric space where $p(u, v) = \max\{u, v\}$. Suppose

$$H(u,v) = \begin{cases} \frac{u-v}{2}, & \text{if } u \geq v, \\ 0, & \text{otherwise,} \end{cases}$$

and $V(t_1, t_2, t_3, t_4, t_5) = t_1 - \frac{1}{2}\max\{t_2, t_3\}$. It is clear that all conditions of Theorem 3.1 are satisfied. Notice that $(0, 0)$ is the coupled fixed point of the operator H.

Now, note that if $(\Omega, \leq)$ is a partially ordered set, we endow the product space $\Omega\times\Omega$ with the partial order relation given by

$$(a,b) \leq (f,g) \quad \Leftrightarrow \quad f \geq a \text{ and } g \leq b.$$

We say that two pairs (p, q) and (r, s) are comparable, that is, every pair of elements has either a lower bound or an upper bound.

Theorem 3.9 *In addition to the hypotheses of Theorem* 3.1*, suppose that, for every $(a, b), (c, d) \in \Omega\times\Omega$, there exists a pair $(n, p) \in \Omega\times\Omega$ such that (n, p) is comparable to (a, b) and (c, d). Then H has a unique coupled fixed point. Moreover $p(t, t) = 0$.*

Proof Suppose that (x, y) and (s, t) are coupled fixed point of H, that is, $x = H(x, y)$, $y = H(y, x)$, $s = H(s, t)$ and $t = H(t, s)$.

Let (α, β) be an element of $\Omega \times \Omega$ comparable to both (x, y) and (s, t). Suppose that $(x, y) \geq (\alpha, \beta)$ (the proof is similar in other cases).

Assume that (x, y) and (s, t) are comparable, then from inequality (3.1), we have

$$V\begin{pmatrix} p(H(x,y), H(s,t)), p(x,s), p(y,t), \\ p(H(x,y), x) + p(H(s,t), s), p(H(x,y), s) \end{pmatrix} \leq 0,$$

$$V(p(x,s), p(x,s), p(y,t), p(x,x) + p(s,s), p(x,s)) \leq 0,$$

$$V(p(x,s), p(x,s), p(y,t), 0, p(x,s)) \leq 0,$$

or

$$\begin{aligned} & V(p(x,s), p(x,s), p(y,t), 0 + p(x,s), p(x,s) + 0) \\ & \leq V(p(x,s), p(x,s), p(y,t), 0, p(x,s)) \leq 0, \end{aligned}$$

which implies

$$p(x,s) \leq p(x,s) + \psi(p(y,t)). \tag{3.41}$$

By similar fashion, one can show that

$$p(y,t) \leq p(y,t) + \psi(p(x,s)). \tag{3.42}$$

From equations (3.41) and (3.42), we obtain

$$\begin{aligned} p(x,s) + p(y,t) & \leq p(x,s) + p(y,t) + \psi[p(x,s) + p(y,t)] \\ & \Rightarrow p(x,s) + p(y,t) = 0 \\ & \Rightarrow p(x,s) = p(y,t) = 0, \end{aligned}$$

and so, $x = s$ and $y = t$. Thus, $(x, y) = (s, t)$. This shows the uniqueness of coupled fixed point. This completes the proof. □

Theorem 3.10 *In addition to the hypotheses of Theorem* 3.1*, if* u_0*,* v_0 *are comparable, then the coupled fixed point* $(u, v) \in \Omega \times \Omega$ *satisfies* $u = v$*.*

Proof Assume that $u_0 \leq v_0$ (a similar argument applies for $v_0 \leq u_0$). Then, by using the mathematical induction

$$u_{n+1} = H(u_n, v_n) \leq H(v_n, u_n) = v_{n+1}.$$

Taking the limit as $n \to \infty$, we have

$$u = \lim_{n \to \infty} u_n \leq \lim_{n \to \infty} v_n = v.$$

From the contractive condition (3.1), we have

$$V\begin{pmatrix} p(H(u,v),H(v,u)),p(u,v),p(v,u), \\ p(H(u,v),u)+p(H(v,u),v),p(H(u,v),v)\end{pmatrix} \le 0,$$

$$V(p(u,v),p(u,v),p(v,u),p(u,u)+p(v,v),p(u,v)) \le 0,$$

$$V(p(u,v),p(u,v),p(u,v),p(u,v)+p(u,v),p(u,v)) \le 0 \quad \big(\text{by (p2),(p3)}\big)$$

or

$$V(p(u,v),p(u,v),p(u,v),p(u,v)+p(u,v),p(u,v)+p(u,v))$$

$$\le V(p(u,v),p(u,v),p(u,v),p(u,v)+p(u,v),p(u,v)) \le 0,$$

which implies

$$\begin{aligned} p(u,v) &\le p(u,v)+\psi(p(u,v)) \\ &\Rightarrow p(u,v)=0 \Rightarrow u=v, \end{aligned}$$

by the property of ψ. This completes the proof. □

Theorem 3.11 *Let $\Omega=[0,1]$. Then $(\Omega,\le)$ is a partially ordered set with a natural ordering of real numbers. Let $p\colon \Omega\times\Omega\to[0,1]$ be defined by $p(u,v)=|u-v|$ for all $u,v\in\Omega$. Consider the mapping $H\colon \Omega\times\Omega\to[0,1]$ defined by*

$$H(u,v)=\begin{cases} \frac{u^2-v^2+1}{3}, & \text{if } u\le v, \\ \frac{1}{3}, & \text{if } u>v, \end{cases}$$

for all $u,v\in\Omega$. Then,

(1) *(Ω,p) is a complete partial metric space since (Ω,d^p) is complete;*
(2) *H has the mixed monotone property;*
(3) *H is continuous;*
(4) *$0\le H(0,1)$ and $1\ge H(1,0)$;*
(5) *there exists a constant $0<m<1$ such that*

$$p\big(H(u,v),H(y,z)\big)\le \frac{m}{2}\big[p(u,y)+p(v,z)\big]$$

for all $u,v,y,z\in\Omega$ with $u\le y$ and $v\ge z$. Thus, by Corollary 3.6, H has a coupled fixed point. Moreover, $(\frac{1}{3},\frac{1}{3})$ is the unique coupled fixed point of H.

Proof The proofs of $(1)-(4)$ are obvious.

For any $u\le y$ and $v\ge z$, we have

$$p(u,y)=y-u,\quad p(v,z)=v-z.$$

The proof of (5) is divided into the following cases.

Case 1. If $y \leq z$. In this case, $u \leq y \leq z \leq v$, and so

$$H(u,v) = \frac{u^2 - v^2 + 1}{3}, \quad H(y,z) = \frac{y^2 - z^2 + 1}{3}.$$

Hence, we get

$$\begin{aligned} p\big(H(u,v), H(y,z)\big) &= p\Big(\frac{u^2 - v^2 + 1}{3}, \frac{y^2 - z^2 + 1}{3}\Big) \\ &= \frac{1}{3}(y^2 - z^2 - u^2 + v^2) = \frac{1}{3}\left[(y^2 - u^2) + (v^2 - z^2)\right] \\ &\leq \frac{1}{3}\left[(y - u) + (v - z)\right] = \frac{1}{3}\left[p(u,y) + p(v,z)\right] \\ &= \frac{m}{2}\left[p(u,y) + p(v,z)\right] \end{aligned}$$

with $m = \frac{2}{3} < 1$.

Case 2. If $y > z$. In this case, $u \leq y \leq v$, and so

$$H(u,v) = \frac{u^2 - v^2 + 1}{3}, \quad H(y,z) = \frac{1}{3}.$$

Hence, we get

$$\begin{aligned} p\big(H(u,v), H(y,z)\big) &= p\Big(\frac{u^2 - v^2 + 1}{3}, \frac{1}{3}\Big) = \frac{1}{3}(v^2 - u^2) \\ &\leq \frac{1}{3}(v^2 - u^2 + y^2 - z^2) = \frac{1}{3}\left[(y^2 - u^2) + (v^2 - z^2)\right] \\ &\leq \frac{1}{3}\left[(y - u) + (v - z)\right] = \frac{1}{3}\left[p(u,y) + p(v,z)\right] \\ &= \frac{m}{2}\left[p(u,y) + p(v,z)\right] \end{aligned}$$

with $m = \frac{2}{3} < 1$.

Case 3. If $u > v$. In this case, $y \leq z \leq v$, and so

$$H(u,v) = \frac{1}{3}, \quad H(y,z) = \frac{y^2 - z^2 + 1}{3}.$$

Hence, we get

$$\begin{aligned} p\big(H(u,v), H(y,z)\big) &= p\Big(\frac{1}{3}, \frac{y^2 - z^2 + 1}{3}\Big) = \frac{1}{3}(y^2 - z^2) \\ &\leq \frac{1}{3}(y^2 - z^2 + v^2 - u^2) = \frac{1}{3}\left[(y^2 - u^2) + (v^2 - z^2)\right] \\ &\leq \frac{1}{3}\left[(y - u) + (v - z)\right] = \frac{1}{3}\left[p(u,y) + p(v,z)\right] \\ &= \frac{m}{2}\left[p(u,y) + p(v,z)\right] \end{aligned}$$

with $m = \frac{2}{3} < 1$.

Thus, in all the above cases, the condition (5) is satisfied. Since $\Omega = [0, 1]$ is a totally ordered set, by Theorem 3.9, $(\frac{1}{3}, \frac{1}{3})$ is the unique coupled fixed point of H. $\square$

§4. An Application to the Integral Equation

In this section, we study the existence of solution of the nonlinear integral equations, as an application of the coupled fixed point theorem proved in the previous section.

Consider the following nonlinear integral equations

$$\begin{aligned} x(t) &= \mu(t) + \int_0^T R(t,p)g(p, x(p), y(p))dp, \\ y(t) &= \mu(t) + \int_0^T R(t,p)g(p, y(p), x(p))dp, \end{aligned} \tag{4.1}$$

where $t \in I = [0, T]$, with $T > 0$.

We consider the space $\Omega = C(I, \mathbb{R})$ of continuous functions defined in I. Define $p \colon \Omega \times \Omega \to [0, +\infty)$ by

$$p(x, y) = \max_{t \in I} |x(t) - y(t)| \tag{4.2}$$

for all $x, y \in \Omega$. Then (Ω, p) is a complete partial metric space.

Let $\Omega = C(I, \mathbb{R})$ with the natural partial order relation, that is, $x, y \in C(I, \mathbb{R})$,

$$x \le y \;\Leftrightarrow\; x(t) \le y(t),\ t \in I.$$

We consider the following assumptions:

(i) the mapping $g \colon I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\mu \colon I \to \mathbb{R}$ are continuous;

(ii) there exists a continuous $0 \le m < 1$ such that

$$|g(p, x, y) - g(p, u, v)| \le \frac{m}{2} (|x - u| + |y - v|) \tag{4.3}$$

for all $x, y, u, v \in \Omega$ and for all $p \in I$;

(iii) for all $t, p \in I$, there exists a continuous $R \colon I \times \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{t \in T} \int_0^T R(t,p)dp < 1; \tag{4.4}$$

(iv) there exist $x_0, y_0 \in \Omega$ such that

$$\begin{aligned} x_0(t) &\le \mu(t) + \int_0^T R(t,p)g(p, x_0(p), y_0(p))dp, \\ y_0(t) &\le \mu(t) + \int_0^T R(t,p)g(p, y_0(p), x_0(p))dp, \end{aligned} \tag{4.5}$$

where $t \in I$.

Theorem 4.1 *Consider the Corollary* 3.6 *and assume that conditions* (i) - (iv) *are satisfied. Then equation (*4.1*) has a unique solution in* Ω.

Proof Define the mapping $H\colon \Omega^2 \to \Omega$, $(x, y) \to H(x, y)$, where

$$H(x,y)(t) = \mu(t) + \int_0^T R(t,p)g(p, x(p), y(p))dp, \ t \in I \tag{4.6}$$

for all $x, y \in \Omega$ and $t \in I$.

Equation (4.1) can be stated as

$$x = H(x, y) \ \text{ and } \ y = H(y, x). \tag{4.7}$$

For $x, y, u, v \in \Omega$ be such that $x \leq u$ and $y \leq v$ and

$$\begin{aligned} H(x,y)(t) &= \mu(t) + \int_0^T R(t,p)g(p, x(p), y(p))dp \\ &\leq \mu(t) + \int_0^T R(t,p)g(p, u(p), v(p))dp \\ &= H(u,v)(t) \text{ for all } t \in I. \end{aligned} \tag{4.8}$$

From equations (4.2) and (4.3) for all $t \in I$, we have

$$\begin{aligned} p(H(x,y), H(u,v)) &= \max_{t \in I} |H(x,y)(t) - H(u,v)(t)| \\ &\leq \max_{t \in I} \int_0^T R(t,p)\Big|g(p, x(p), y(p)) - g(p, u(p), v(p))\Big|dp \\ &\leq \Big|g(p, x(p), y(p)) - g(p, u(p), v(p))\Big| \\ &\leq \frac{m}{2}(|x(p) - u(p)| + |y(p) - v(p)|) \\ &= \frac{m}{2}[p(x,u) + p(y,v)], \end{aligned}$$

where $0 \leq m < 1$.

So that

$$p(H(x,y), H(u,v)) \leq \frac{m}{2}[p(x,u) + p(y,v)],$$

which is the contractive condition in Corollary 3.6. Thus H has a coupled fixed point in Ω, that is, the system of nonlinear integral equation has a solution. Finally, let (p, q) be a coupled lower and upper solution of the integral equation (4.1), then by assumption (iv) of the Theorem 4.1, we have $p \leq H(p, q) \leq H(q, p) \leq q$. Corollary 3.6 gives us that H has a coupled fixed point, say $(m, n) \in \Omega \times \Omega$. Since $p \leq q$, Theorem 3.10 says us that $m = n$ and this implies $m = H(m, m)$ and m is the unique solution of the integral equation (4.1). □

The aforesaid application is illustrated by the following example.

Example 4.2 Let $\Omega = C([0,1],\mathbb{R})$, $g\colon I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\mu\colon I\to\mathbb{R}$. Now consider the following functional integral equation

$$\begin{aligned} x(t) &= \frac{t^2}{1+t^4} + \int_0^1 \frac{sin\,p\,3^{-p}e^{-p}}{9(t+3)}\Big(\frac{|x(p)|}{1+|x(p)|} + \frac{|y(p)|}{1+|y(p)|}\Big)dp \\ y(t) &= \frac{t^2}{1+t^4} + \int_0^1 \frac{sin\,p\,3^{-p}e^{-p}}{9(t+3)}\Big(\frac{|y(p)|}{1+|y(p)|} + \frac{|x(p)|}{1+|x(p)|}\Big)dp \end{aligned}$$

for all $x, y \in \Omega$ and $t \in I$. Observe that the above equation is a special case of equation (4.1) with

$$\begin{aligned} \mu(t) &= \frac{t^2}{1+t^4}. \\ R(t,p) &= \frac{3^{-p}e^{-p}}{t+3}. \\ g(p,x,y) &= \frac{sin\,p}{9}\Big(\frac{|x(p)|}{1+|x(p)|} + \frac{|y(p)|}{1+|y(p)|}\Big). \\ g(p,y,x) &= \frac{sin\,p}{9}\Big(\frac{|y(p)|}{1+|y(p)|} + \frac{|x(p)|}{1+|x(p)|}\Big). \end{aligned}$$

It is also easily seen that these functions are continuous.

For arbitrary $x, y, u, v \in \Omega$ and for all $p \in I$, we have

$$\begin{aligned} |g(p,x,y) - g(p,u,v)| &= \Big|\frac{sin\,p}{9}\Big(\frac{|x(p)|}{1+|x(p)|} + \frac{|y(p)|}{1+|y(p)|}\Big) \\ &\quad -\frac{sin\,p}{9}\Big(\frac{|u(p)|}{1+|u(p)|} + \frac{|v(p)|}{1+|v(p)|}\Big)\Big| \\ &\le \frac{1}{9}\big(|x-u| + |y-v|\big) = \frac{m}{2}\big(|x-u| + |y-v|\big). \end{aligned}$$

Therefore, the function g satisfies equation (4.3) with $m = \frac{2}{9} < 1$.

For all $t, p \in I$, there exists $R\colon I\times\mathbb{R}\to\mathbb{R}$ such that

$$\begin{aligned} \int_0^1 R(t,p)dp &= \int_0^1 \frac{3^{-p}e^{-p}}{t+3}dp = -\frac{1}{3}\Big(\frac{e^{-1}-3}{(ln\,3+1)(t+3)}\Big) \\ &= \Big(1-\frac{1}{3e}\Big)\frac{1}{(ln\,3+1)(t+3)} \le 1 - \frac{1}{3e} \le \frac{9}{10} < 1. \end{aligned}$$

We put $x_0(t) = \frac{5t^2}{7(1+t^4)}$ and obtain

$$\begin{aligned} x_0(t) &= \frac{5t^2}{7(1+t^4)} \le \frac{t^2}{1+t^4} \\ &\le \frac{t^2}{1+t^4} + \int_0^1 \frac{sin\,p}{9}\Big(\frac{|x(p)|}{1+|x(p)|} + \frac{|y(p)|}{1+|y(p)|}\Big)dp \\ &= \mu(t) + \int_0^T R(t,p)g(p, x_0(p), y_0(p))dp. \end{aligned}$$

Similarly, we have

$$y_0(t) \leq \mu(t) + \int_0^T R(t,p)g(p, y_0(p), x_0(p))dp.$$

This shows that equation (4.5) holds.

Hence, the integral equation (4.1) has a unique solution in Ω with $\Omega = C([0,1], \mathbb{R})$.

§5. **Conclusion**

In this paper, we prove some coupled fixed point theorems via implicit relations in the setting of partially ordered partial metric spaces. Furthermore, we give some consequences of the main result. We provide some illustrative examples to validate the established results. An application to the integral equation is also given. Our results extend and generalize various results in the literature.

References

[1] M. Abbas, M. Ali Khan and S. Radenović, Common coupled fixed point theorems in cone metric spaces for w-compatible mappings, *Appl. Math. Comput.*, 217(1) (2010), 195-202.

[2] M. Abbas, T. Nazir and S. Romaguera, Fixed point results for generalized cyclic contraction mappings in partial metric spaces, *Rev. R. Acad. Cienc. Exactas Fis. Nat.*, Ser. A Mat. RACSAM. 106(1) (2012), 287-297.

[3] T. Abdeljawad, E, Karapınar and K. Tas, Existence and uniqueness of a common fixed point on partial metric spaces, *Appl. Math. Lett.*, 24(11) (2011), 1894-1899.

[4] I. Altun and D. Turkoglu, Some fixed point theorems for weakly compatible multivalued mappings satisfying an implicit relation, *Filomat*, 22 (2008), 13-23.

[5] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, *Topology and its Appl.*, 157 (2010), 2778-2785.

[6] H. Aydi, Some coupled fixed point results on partial metric spaces, *International J. Math. Math. Sci.*, 2011, Article ID 647091, 11 pages.

[7] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, *Topology and Its Appl.*, 159 (2012), No. 14, 3234-3242.

[8] S. Banach, Surles operation dans les ensembles abstraits et leur application aux equation integrals, *Fund. Math.*, 3(1922), 133-181.

[9] T. G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Analysis: TMA*, 65(7) (2006), 1379-1393.

[10] L. Ciric and V. Lakshmikantham, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Analysis: TMA*, 70(12) (2009), 4341-4349.

[11] S. Chandok, D. Kumar and M. S. Khan, Some results in partial metric space using auxiliary functions, *Applied Math. E-Notes*, 15 (2015), 233-242.

[12] Pooja Dhawan and Jatinderdeep Kaur, Some common fixed point theorems in ordered

partial metric spaces via $\mathcal{F}$-generalized contractive type mappings, *Mathematics*, 2019, 7, 193; doi:10.3390/math7020193.

[13] I. Erhan, E. Karapinar and D. Turkoglu, Different types Meir-Keeler contractions on partial metric spaces, *J. Comput. Anal. Appl.*, 14(6) (2012), 1000-1005.

[14] R. O. Gregorio, Coupled fixed point theorems in partially ordered metric spaces, *Adv. Fixed Point Theory*, 9(1) (2019), 17-28.

[15] R. Heckmann, Approximation of metric spaces by partial metric spaces, *Appl. Categ. Structures*, 7, No. 1-2, (1999), 71-83.

[16] N. M. Hung, E. Karapinar and N. V. Luong, Coupled coincidence point theorem in partially ordered metric spaces via implicit relation, *Abstr. Appl. Anal.*, (2012), Art. ID 796964, 2012.

[17] E. Karapinar, W. Shatanawi and K. Tas, Fixed point theorems on partial metric spaces involving rational expressions, *Miskolc Math. Notes*, 14 (2013), 135-142.

[18] N. V. Luong and N. X. Thuan, Coupled fixed points in ordered generalized metric spaces and application to integro-differential equations, (Submitted).

[19] B. Khomdram, Y. Rohen and T. C. Singh, Coupled fixed point theorems in G_b-metric space satisfying some rational contractive conditions, *Springer Plus*, (2016) 5:1261. Doi:10.1186/s40064-016-2925-7.

[20] D. Kumar, S. Sadat, J. R. Lee and C. Park, Some theorems in partial metric spaces using auxiliary functions, *AIMS Maths.*, 6(7) (2021), 6734-6748.

[21] H. P. A. Künzi, H. Pajoohesh and M. P. Schellekens, Partial quasi-metrics, *Theoritical Comput. Sci.*, 365(3) (2006), 237-246.

[22] H. P. Masiha, F. Sabetghadam and N. Shahzad, Fixed point theorems in partial metric spaces with an application, *Filomat*, 27(4) (2013), 617-624.

[23] S. G. Matthews, Partial metric topology, Proceedings of the 8th summer conference on topology and its applications, *Annals of the New York Academy of Sciences*, 728 (1994), 183-197.

[24] H. K. Nashine and W. Shatanawi, Coupled common fixed point theorems for pair of commuting in partially ordered complete metric spaces, *Comput. Math. Appl.*, 62 (2011), 1984-1993.

[25] M. Nazam, Z. Hamid, H. Al Sulami and A. Hussain, Common fixed point theorems in the partial b-metric spaces and an application to the system of boundary value problems, *J. Function Spaces*, Vol. 2021, Art. ID 7777754, https://doi.org/10.1155/2021/7777754.

[26] J. O. Olaleru, G. A. Okeke and H. Akewe, Coupled fixed point theorems for generalized φ-mappings satisfying contractive condition of integral type on cone metric spaces, *Int. J. Math. Model. Comput.*, 2(2) (2012), 87-98.

[27] S. Oltra and O. Valero, Banach's fixed point theorem for partial metric spaces, *Rend. Ist. Mat. Univ. Trieste*, 36(1-2) (2004), 17-26.

[28] S. J. O'Neil, Partial metrics, valuations and domain theory, in: *Proc. 11th Summer Conference on General Topology and Application*, also in *Annals of the New York Academy of Sciences*, Vol.806 (1996), 304-315.

[29] V. Popa, On some fixed point theorems for compatible mappings satisfying an implicit

relation, *Demonstr. Math.*, 32(1) (1999), 157-163.
[30] V. Popa, A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation, *Demonstr. Math.*, 33 (2000), 159-164.
[31] V. Popa, M. Imdad and J. Ali, Using implicit relations to prove unified fixed point theorems in metric and 2-metric spaces, *Bull. Malays. Math. Sci. Soc.(2)*, 33(1) (2010), 105-120.
[32] S. Romaguera and M. P. Schellekens, Duality and quasi-normability for complexity spaces, *Appl. General Topology*, 3 (2002), 91-112.
[33] F. Sabetghadamt, H. P. Masiha and A. H. Sanatpour, Some coupled fixed point theorems in cone metric spaces, *Fixed Point Theory Appl.*, Article ID 125426, (2009), 8 pages.
[34] G. S. Saluja, Fixed point theorems on cone S-metric spaces using implicit relation, *Cubo, A Mathematical Journal*, 22(2) (2020), 273-288.
[35] B. Samet, M. Rajović, R. Lazović and R. Stoiljković, Common fixed point results for nonlinear contractions in ordered partial metric spaces, *Fixed Point Theory Appl.*, 2011, 2011:71.
[36] M. P. Schellekens, The correspondence between partial metrics and semivaluations, *Theoritical Comput. Sci.*, 315(1) (2004), 135-149.
[37] W. Shatanawi, B. Samet and M. Abbas, Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces, *Math. Comput. Modell.*, 55 (2012), Nos. 3-4, 680-687.

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Limits for the Some Topological Indices of Vertex Corona Graph

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Abstract: Topological indices are mathematical tools that numerically correlate chemical structures with various physical properties, chemical reactivity and biological activity.This paper presents lower and upper bounds for certain topological indices, such as inverse sum indeg index, third redefined Zagreb index, forgotten index, SK, SK_1 and SK_2 indices.

Key Words: Corona product, topological index, degree based topological indices, Smarandachely ξ-graph.

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§1. Introduction

The use of topological indices in chemistry began in 1947, when chemist Harnold Wiener developed the most widely known topological descriptor, the Wiener index and used it to determine the physical properties of alkanes known as paraffin. A topological descriptor was computed from a molecular graph representing the information about the corresponding chemical compound. These descriptors are commonly used in mathematical chemistry, particularly QSPR and QSAR research. Topological indices are useful for predicting the physico-chemical behavior of chemical compounds. Encouraged by Manjunath Muddalapuram [7], we obtained the upper and lower bounds for the additional topological indices of ξ-graph. Throughout this study, we consider $G = (V(G), E(G))$ as a simple and connected undirected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges.

Now we recall some well known topological indices. In 2010, Vukicevic [14] introduced inverse sum index and is defined as

$$I(G) = \sum_{uv \in E(G)} \frac{d(u)d(v)}{d(u) + d(v)}.$$

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In 2013, Ranjani et al. [10] introduced third redefined Zagreb index and is defined as

$$ReZG_3(G) = \sum_{uv \in E(G)} (d_u d_v)(d_u + d_v).$$

In 2015, Furtula and Gutman [2] introduced forgotten index and is defined as

$$F(G) = \sum_{uv \in E(G)} (d_u^2 + d_v^2).$$

In 2016, Shigehalli et al. [11] introduced the SK, SK_1 and SK_2 topological indices of a graph G and is defined as

$$SK(G) = \frac{1}{2} \sum_{(v_i, v_j) \in E(G)} [d(v_i) + d(v_j)],$$

$$SK_1(G) = \frac{1}{2} \sum_{(v_i, v_j) \in E(G)} [d(v_i) d(v_j)]$$

and

$$SK_2(G) = \frac{1}{4} \sum_{(v_i, v_j) \in E(G)} [d(v_i) + d(v_j)]^2.$$

Definition 1.1([6]) *The corona product of $G \odot H$ of these two graphs is obtained by taking one copy of G, n_1 copies of H and by joining each vertex of the i^{th} copy of H to the i^{th} vertex of G, where $1 \leq i \leq n_1$.*

In order to study bounds on ξ-graph, we divide the paper into few sections. The section one contains preliminaries, definitions of well known topological indices which are useful to prove our main results. Section two deals with new class of operator graph with their properties. In section three consisting of results related to bounds for defined class of graph using recalled topological indices. Paper conclude with the conclusion and references.

§2. Essential Prerequisite

Throughout the article, we utilized finite simple connected graphs with the following notions.

Let G and H be graphs with vertex sets $V(G)$, $V(H)$ and edge sets $E(G)$, $E(H)$ respectively. The degree of vertex v is the number of vertices adjacent to v.

Let $V(G) \bigcap V(H) = \emptyset, g \in V(G), h \in V(H)$. The number of vertices and number of edges in the graphs G and H are represented by n_1 , n_2 and m_1, m_2 respectively. we have

$$\Delta_G \geq deg_G(g), \qquad \delta_G \leq deg_G(g),$$
$$\Delta_H \geq deg_H(h), \qquad \delta_H \leq deg_H(h).$$

Now, the bounds for different topological indices are obtained by many researchers for graphs [4].

Definition 2.1([7]) *The $G \odot_R H = \xi$, is a graph obtained from one copy of graph G and m_1 copies of H and joining a vertex of $V(G)$, that is, on the i^{th} vertex in G is adjacent to every vertex of i^{th} copy of H.*

Furthermore, let $H_0 \prec H$ be a typical subgraph of H. If the i^{th} vertex in G is adjacent to every vertex of i^{th} copy of $H \setminus H_0$, such a ξ-graph is said to be a Smarandachely ξ-graph of G, H on H_0 and denoted by ξ_S. Certainly, if $H_0 = \emptyset$ then $\xi_S = \xi$.

§3. Limits on Various Topological Indices of ξ Graph

In this section, we formulate the limits on the ISI, $ReZG_3$, F, SK, SK_1 and SK_2 indices of ξ-graph.

Theorem 3.1 *Let G and H are two simple connected graphs, then the limits for the inverse sum indeg index of ξ-graph are given by*

$$ISI[\xi] \leq m_1\left[\frac{(2\Delta_G+n_2)}{2}+\frac{4(2\Delta_G+n_2)}{2\Delta_G+n_2+2}\right]+n_1m_2\left[\frac{(\Delta_H+1)^2}{2(\Delta_H+1)}\right]$$
$$+n_1n_2\left[\frac{(\Delta_H+1)(2\Delta_G+n_2)}{2\Delta_G+\Delta_H+n_2+1}\right]$$

and

$$ISI[\xi] \geq m_1\left[\frac{(2\delta_G+n_2)}{2}+\frac{4(2\delta_G+n_2)}{2\delta_G+n_2+2}\right]+n_1m_2\left[\frac{(\delta_H+1)^2}{2(\delta_H+1)}\right]$$
$$+n_1n_2\left[\frac{(\delta_H+1)(2\delta_G+n_2)}{2\delta_G+\delta_H+n_2+1}\right].$$

Proof Using definition of inverse sum indeg index, we get

$$ISI(G) = \sum_{uv\in E(G)}\frac{d(u)d(v)}{d(u)+d(v)},$$

$$ISI[\xi] = m_1\left[\frac{(2d_G+n_2)(2d_G+n_2)}{(2d_G+n_2)+(2d_G+n_2)}\right]+2m_1\left[\frac{2(2d_G+n_2)}{2d_G+n_2+2}\right]$$
$$+n_1m_2\left[\frac{(d_H+1)(d_H+1)}{(d_H+1)+(d_H+1)}\right]+n_1n_2\left[\frac{(d_H+1)(2d_G+n_2)}{(d_H+1)+(2d_G+n_2)}\right],$$

$$ISI[\xi] = m_1\left[\frac{(2d_G+n_2)}{2}+\frac{4(2d_G+n_2)}{2d_G+n_2+2}\right]+n_1m_2\left[\frac{(d_H+1)^2}{2(d_H+1)}\right]$$
$$+n_1n_2\left[\frac{(d_H+1)(2d_G+n_2)}{2d_G+d_H+n_2+1}\right],$$

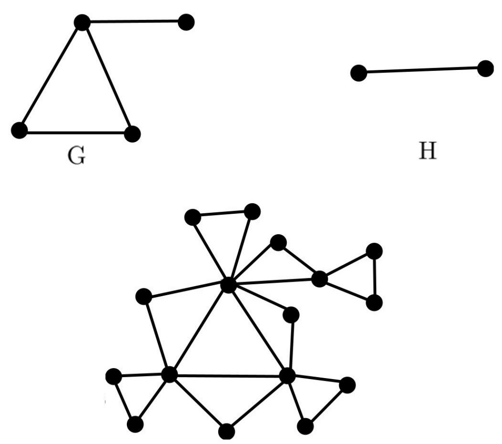
$$ISI[\xi] \leq m_1\left[\frac{(2\Delta_G+n_2)}{2}+\frac{4(2\Delta_G+n_2)}{2\Delta_G+n_2+2}\right]+n_1m_2\left[\frac{(\Delta_H+1)^2}{2(\Delta_H+1)}\right]$$
$$+n_1n_2\left[\frac{(\Delta_H+1)(2\Delta_G+n_2)}{2\Delta_G+\Delta_H+n_2+1}\right].$$

Similarly,

$$ISI[\xi] \geq m_1\left[\frac{(2\delta_G+n_2)}{2}+\frac{4(2\delta_G+n_2)}{2\delta_G+n_2+2}\right]+n_1m_2\left[\frac{(\delta_H+1)^2}{2(\delta_H+1)}\right]$$
$$+n_1n_2\left[\frac{(\delta_H+1)(2\delta_G+n_2)}{2\delta_G+\delta_H+n_2+1}\right].$$

This completes the proof. □

Illustration 3.1 Let the ξ-graph of G and H be shown in Figure 1.



ξ-graph of G and H

Figure 1

Then,

$$ISI(G)=\sum_{uv\in E(G)}\frac{d(u)d(v)}{d(u)+d(v)},$$
$$=\left(\frac{6\times 6}{6+6}\right)+8\left(\frac{6\times 2}{6+2}\right)+2\left(\frac{8\times 6}{8+6}\right)+5\left(\frac{8\times 2}{8+2}\right)$$
$$+\left(\frac{8\times 4}{8+4}\right)+3\left(\frac{4\times 2}{4+2}\right)+4\left(\frac{2\times 2}{2+2}\right)=40.37,$$

$$ISI[\xi] \leq m_1\left[\frac{(2\Delta_G+n_2)}{2}+\frac{4(2\Delta_G+n_2)}{2\Delta_G+n_2+2}\right]+n_1m_2\left[\frac{(\Delta_H+1)^2}{2(\Delta_H+1)}\right]$$
$$+n_1n_2\left[\frac{(\Delta_H+1)(2\Delta_G+n_2)}{2\Delta_G+\Delta_H+n_2+1}\right],$$

$$\leq 4\left[\frac{(2\times 3+2)}{2}+\frac{4(2\times 3+2)}{2\times 3+2+2}\right]+4\left[\frac{(1+1)^2}{2(1+1)}\right]+8\left[\frac{(1+1)(2\times 3+2)}{2\times 3+1+2+1}\right]$$
$$= 45.6.$$

$$ISI[\xi] \geq m_1\left[\frac{(2\delta_G+n_2)}{2}+\frac{4(2\delta_G+n_2)}{2\delta_G+n_2+2}\right]+n_1m_2\left[\frac{(\delta_H+1)^2}{2(\delta_H+1)}\right]$$
$$+n_1n_2\left[\frac{(\delta_H+1)(2\delta_G+n_2)}{2\delta_G+\delta_H+n_2+1}\right].$$
$$\geq 4\left[\frac{(2\times 1+2)}{2}+\frac{4(2\times 1+2)}{2\times 1+2+2}\right]+4\left[\frac{(1+1)^2}{2(1+1)}\right]+8\left[\frac{(1+1)(2\times 1+2)}{2\times 1+1+2+1}\right].$$
$$= 33.324$$

and therefore,

$$33.32 \;\leq\; 40.37 \;\leq\; 45.8.$$

Theorem 3.2 *Let G and H are two simple connected graphs, then the limits for the redefined third Zagreb index of ξ-graph are given by*

$$ReZG_3[\xi] \leq m_1\left\{2(2\Delta_G+n_2)\big[(2\Delta_G+n_2)^2+2(2\Delta_G+n_2+2)\big]\right\}+n_1m_2\Big[2(\Delta_H+1)^3\Big]$$
$$+n_1n_2\Big[(\Delta_H+1)(2\Delta_G+n_2)(\Delta_H+2\Delta_G+n_2+1)\Big].$$

and

$$ReZG_3[\xi] \geq m_1\left\{2(2\delta_G+n_2)\big[(2\delta_G+n_2)^2+2(2\delta_G+n_2+2)\big]\right\}+n_1m_2\Big[2(\delta_H+1)^3\Big]$$
$$+n_1n_2\Big[(\delta_H+1)(2\delta_G+n_2)(\delta_H+2\delta_G+n_2+1)\Big].$$

Proof Using definition of redefined third Zagreb index, we get

$$ReZG_3(G) = \sum_{uv\in E(G)}(d_ud_v)(d_u+d_v),$$

$$ReZG_3[\xi] = m_1\Big[(2d_G+n_2)(2d_G+n_2)(2d_G+n_2+2d_G+n_2)\Big]$$
$$+2m_1\Big[2(2d_G+n_2)(2d_G+n_2+2)\Big]+n_1m_2\Big[(d_H+1)(d_H+1)(d_H+1+d_H+1)\Big]$$
$$+n_1n_2\Big[(d_H+1)(2d_G+n_2)(d_H+2d_G+n_2+1)\Big],$$

$$ReZG_3[\xi] = m_1\left\{2(2d_G+n_2)\big[(2d_G+n_2)^2+2(2d_G+n_2+2)\big]\right\}+n_1m_2\Big[2(d_H+1)^3\Big]$$
$$+n_1n_2\Big[(d_H+1)(2d_G+n_2)(d_H+2d_G+n_2+1)\Big],$$

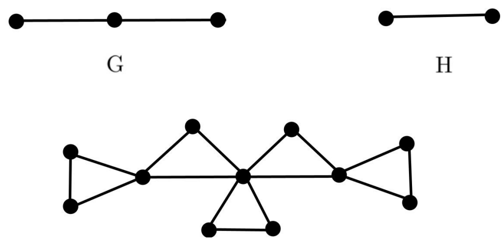
$$ReZG_3[\xi] \leq m_1\Big\{2(2\Delta_G+n_2)\big[(2\Delta_G+n_2)^2+2(2\Delta_G+n_2+2)\big]\Big\}+n_1m_2\Big[2(\Delta_H+1)^3\Big]$$
$$+n_1n_2\Big[(\Delta_H+1)(2\Delta_G+n_2)(\Delta_H+2\Delta_G+n_2+1)\Big].$$

Similarly,

$$ReZG_3[\xi] \geq m_1\Big\{2(2\delta_G+n_2)\big[(2\delta_G+n_2)^2+2(2\delta_G+n_2+2)\big]\Big\}+n_1m_2\Big[2(\delta_H+1)^3\Big]$$
$$+n_1n_2\Big[(\delta_H+1)(2\delta_G+n_2)(\delta_H+2\delta_G+n_2+1)\Big].$$

This completes the proof. □

Illustration 3.2 Let the ξ-graph of G, H be shown in Figure 2.



ξ**-graph of** G **and** H

Figure 2

Then,

$$ReZG_3(G) = \sum_{uv\in E(G)} (d_ud_v)(d_u+d_v),$$
$$= 3(4\times 4)+6(8\times 6)+2(24\times 10)+4(12\times 8),$$
$$= 1200,$$

$$ReZG_3[\xi] \leq m_1\Big\{2(2\Delta_G+n_2)\big[(2\Delta_G+n_2)^2+2(2\Delta_G+n_2+2)\big]\Big\}+n_1m_2\Big[2(\Delta_H+1)^3\Big]$$
$$+n_1n_2\Big[(\Delta_H+1)(2\Delta_G+n_2)(\Delta_H+2\Delta_G+n_2+1)\Big].$$
$$\leq 2\Big\{2(2\times 2+2)\big[(2\times 2+2)^2+2(2\times 2+2+2)\big]\Big\}+3\Big[2(1+1)^3\Big]$$
$$+6\Big[(1+1)(2\times 2+2)(1+2\times 2+2+1)\Big]$$
$$= 1872,$$

$$\begin{aligned} ReZG_3[\xi] &\geq m_1\Big\{2(2\delta_G+n_2)\big[(2\delta_G+n_2)^2+2(2\delta_G+n_2+2)\big]\Big\}+n_1m_2\Big[2(\delta_H+1)^3\Big] \\ &\quad +n_1n_2\Big[(\delta_H+1)(2\delta_G+n_2)(\delta_H+2\delta_G+n_2+1)\Big] \\ &= 2\Big\{2(2\times 1+2)\big[(2\times 1+2)^2+2(2\times 1+2+2)\big]\Big\}+3\Big[2(1+1)^3\Big] \\ &\quad +6\Big[(1+1)(2\times 1+2)(1+2\times 1+2+1)\Big] \\ &= 784 \end{aligned}$$

and therefore,

$$784 \;\leq\; 1200 \;\leq\; 1872.$$

Theorem 3.3 *Let G and H are two simple connected graphs, then the limits for the forgotten index of ξ-graph are given by*

$$F[\xi]\leq 4m_1\Big[(2\Delta_G+n_2)^2+2\Big]+2n_1m_2(\Delta_H+1)^2+n_1n_2\Big[(\Delta_H+1)^2+(2\Delta_G+n_2)^2\Big]$$

and

$$F[\xi]\geq 4m_1\Big[(2\delta_G+n_2)^2+2\Big]+2n_1m_2(\delta_H+1)^2+n_1n_2\Big[(\delta_H+1)^2+(2\delta_G+n_2)^2\Big].$$

Proof Using definition of forgotten index

$$F(G)=\sum_{uv\in E(G)}(d_u^2+d_v^2),$$

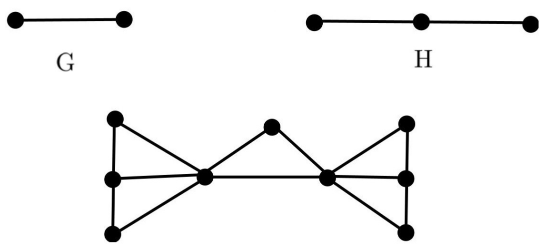
$$\begin{aligned} F[\xi] &= m_1\Big[(2d_G+n_2)^2+(2d_G+n_2)^2\Big]+2m_1\Big[(2d_G+n_2)^2+2^2\Big] \\ &\quad +n_1m_2\Big[(d_H+1)^2+(d_H+1)^2\Big]+n_1n_2\Big[(d_H+1)^2+(2d_G+n_2)^2\Big], \\ F[\xi] &= 4m_1\Big[(2d_G+n_2)^2+2\Big]+2n_1m_2(d_H+1)^2+n_1n_2\Big[(d_H+1)^2+(2d_G+n_2)^2\Big], \\ F[\xi] &\leq 4m_1\Big[(2\Delta_G+n_2)^2+2\Big]+2n_1m_2(\Delta_H+1)^2+n_1n_2\Big[(\Delta_H+1)^2+(2\Delta_G+n_2)^2\Big]. \end{aligned}$$

Similarly,

$$F[\xi]\geq 4m_1\Big[(2\delta_G+n_2)^2+2\Big]+2n_1m_2(\delta_H+1)^2+n_1n_2\Big[(\delta_H+1)^2+(2\delta_G+n_2)^2\Big].$$

This completes the proof. □

Illustration 3.3 Let the ξ-graph of G, H be shown in Figure 3.



ξ-graph of G and H

Figure 3

Then,

$$F(G) = \sum_{uv\in E(G)} (d_u^2 + d_v^2)$$
$$= 6(2^2 + 5^2) + 2(3^2 + 5^2) + 4(3^2 + 2^2) + (5^2 + 5^2)$$
$$= 344,$$

$$F[\xi] \leq 4m_1\Big[(2\Delta_G + n_2)^2 + 2\Big] + 2n_1m_2(\Delta_H + 1)^2$$
$$+ n_1n_2\Big[(\Delta_H + 1)^2 + (2\Delta_G + n_2)^2\Big]$$
$$= 4\Big[(2\times 1 + 3)^2 + 2\Big] + 8(2+1)^2 + 6\Big[(2+1)^2 + (2\times 1 + 3)^2\Big]$$
$$= 384,$$
$$F[\xi] \geq 4m_1\Big[(2\delta_G + n_2)^2 + 2\Big] + 2n_1m_2(\delta_H + 1)^2 + n_1n_2\Big[(\delta_H + 1)^2 + (2\delta_G + n_2)^2\Big]$$
$$= 4\Big[(2\times 1 + 3)^2 + 2\Big] + 8(1+1)^2 + 6\Big[(1+1)^2 + (2\times 1 + 3)^2\Big]$$
$$= 314$$

and therefore,

$$314 \ \leq\ 344 \ \leq\ 384.$$

Theorem 3.4 *Let G and H are two simple connected graphs, then the limits for the SK index of ξ-graph are given by*

$$SK(G) \leq \frac{1}{2}\Big[4m_1(2\Delta_G + n_2 + 1) + 2n_1m_2(\Delta_H + 1) + n_1n_2(\Delta_H + 1 + 2\Delta_G + n_2)\Big].$$

and

$$SK(G) \geq \frac{1}{2}\Big[4m_1(2\delta_G + n_2 + 1) + 2n_1m_2(\delta_H + 1) + n_1n_2(\delta_H + 1 + 2\delta_G + n_2)\Big].$$

Proof Using definition of SK index

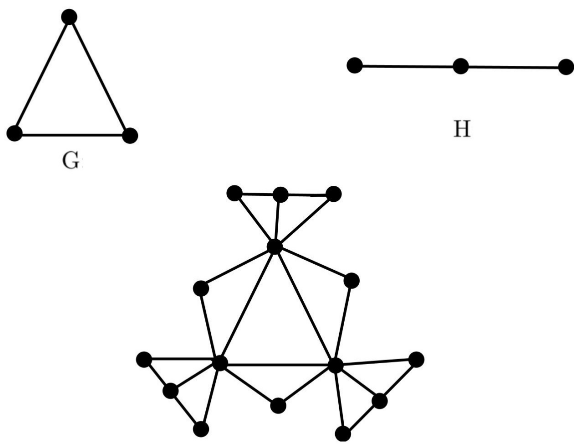
$$\begin{aligned} SK(G) &= \frac{1}{2}\sum_{(v_i,v_j)\in E(G)}[d(v_i) + d(v_j)], \\ &= \frac{1}{2}\Big[m_1(2d_G + n_2 + 2d_G + n_2) + 2m_1(2 + 2d_G + n_2) + n_1m_2(d_H + 1 + d_H + 1) \\ &\quad + n_1n_2(d_H + 1 + 2d_G + n_2)\Big], \\ &= \frac{1}{2}\Big[4m_1(2d_G + n_2 + 1) + 2n_1m_2(d_H + 1) + n_1n_2(d_H + 1 + 2d_G + n_2)\Big], \\ &\leq \frac{1}{2}\Big[4m_1(2\Delta_G + n_2 + 1) + 2n_1m_2(\Delta_H + 1) + n_1n_2(\Delta_H + 1 + 2\Delta_G + n_2)\Big]. \end{aligned}$$

Similarly,

$$SK(G) \geq \frac{1}{2}\Big[4m_1(2\delta_G + n_2 + 1) + 2n_1m_2(\delta_H + 1) + n_1n_2(\delta_H + 1 + 2\delta_G + n_2)\Big].$$

This completes the proof. □

Illustration 3.4 Let the ξ-graph of G, H be shown in Figure 4.



ξ-graph of G and H

Figure 4

Then,

$$SK(G) = \frac{1}{2}\sum_{(v_i,v_j)\in E(G)}[d(v_i)+d(v_j)]$$
$$= \frac{1}{2}\Big[6(3+2)+12(7+2)+3(7+3)+3(7+7)\Big] = 106.5,$$
$$SK(G) \le \frac{1}{2}\Big[4m_1(2\Delta_G+n_2+1)+2n_1m_2(\Delta_H+1)+n_1n_2(\Delta_H+1+2\Delta_G+n_2)\Big]$$
$$= \frac{1}{2}\Big[12(2\times 2+3+1)+12(2+1)+9(2+1+2\times 2+3)\Big] = 111,$$
$$SK(G) \ge \frac{1}{2}\Big[4m_1(2\delta_G+n_2+1)+2n_1m_2(\delta_H+1)+n_1n_2(\delta_H+1+2\delta_G+n_2)\Big]$$
$$= \frac{1}{2}\Big[12(2\times 2+3+1)+12(1+1)+9(1+1+2\times 2+3)\Big] = 105$$

and therefore,

$$105 \ \le\ 106.5 \ \le\ 111.$$

Theorem 3.5 *Let G and H are two simple connected graphs, then the limits for the SK_1 index of ξ-graph are given by*

$$SK_1(\xi) \le \frac{1}{2}\Big\{m_1(2\Delta_G+n_2)[2\Delta_G+n_2+4]+n_1m_2(2\Delta_H+1)^2$$
$$+\,n_1n_2[(\Delta_H+1)(2\Delta_G+n_2)]\Big\}$$

and

$$SK_1(\xi) \ge \frac{1}{2}\Big\{m_1(2\delta_G+n_2)[2\delta_G+n_2+4]+n_1m_2(2\delta_H+1)^2+n_1n_2[(\delta_H+1)(2\delta_G+n_2)]\Big\}.$$

Proof Using definition of SK_1 index, we get

$$SK_1(G) = \frac{1}{2}\sum_{(v_i,v_j)\in E(G)}[d(v_i)d(v_j)],$$
$$SK_1(\xi) = \frac{1}{2}\Big\{m_1[(2d_G+n_2)(2d_G+n_2)]+2m_1[2(2d_G+n_2)]+n_1m_2[(2d_H+1)(2d_H+1)]$$
$$+\,n_1n_2[(d_H+1)(2d_G+n_2)]\Big\}$$
$$= \frac{1}{2}\Big\{m_1(2d_G+n_2)[2d_G+n_2+4]+n_1m_2(2d_H+1)^2+n_1n_2[(d_H+1)(2d_G+n_2)]\Big\},$$

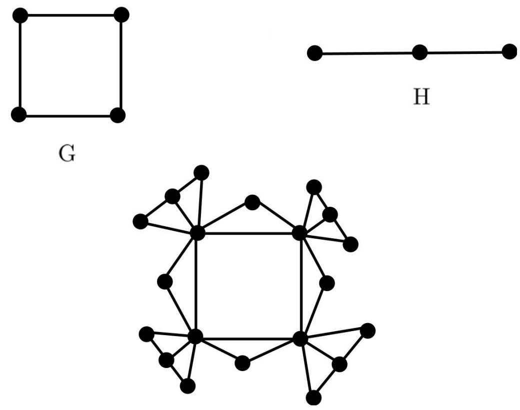
$$SK_1(\xi) \leq \frac{1}{2}\Big\{m_1(2\Delta_G + n_2)[2\Delta_G + n_2 + 4] + n_1 m_2(2\Delta_H + 1)^2$$
$$+ n_1 n_2[(\Delta_H + 1)(2\Delta_G + n_2)]\Big\}.$$

Similarly,

$$SK_1(\xi) \geq \frac{1}{2}\Big\{m_1(2\delta_G + n_2)[2\delta_G + n_2 + 4] + n_1 m_2(2\delta_H + 1)^2 + n_1 n_2[(\delta_H + 1)(2\delta_G + n_2)]\Big\}.$$

This completes the proof. □

Illustration 3.5 Let the ξ-graph of G, H be shown in Figure 5.



ξ**-graph of** G **and** H

Figure 5

Then,

$$SK_1(G) = \frac{1}{2}\sum_{(v_i,v_j)\in E(G)} [d(v_i)d(v_j)]$$
$$= \frac{1}{2}\Big[8(3\times 2) + 4(3\times 7) + 16(7\times 2) + 4(7\times 7)\Big] = 276,$$
$$SK_1(\xi) \leq \frac{1}{2}\Big\{m_1(2\Delta_G + n_2)[2\Delta_G + n_2 + 4] + n_1 m_2(2\Delta_H + 1)^2$$
$$+ n_1 n_2[(\Delta_H + 1)(2\Delta_G + n_2)]\Big\}$$
$$= \frac{1}{2}\Big\{4(2\times 2 + 3)(2\times 2 + 3 + 4) + 8(2\times 2 + 1)^2 + 12\big[(2+1)(2\times 2 + 3)\big]\Big\}$$
$$= 380,$$

$$SK_1(\xi) \geq \frac{1}{2}\Big\{m_1(2\delta_G + n_2)[2\delta_G + n_2 + 4] + n_1m_2(2\delta_H + 1)^2$$
$$+ n_1n_2[(\delta_H + 1)(2\delta_G + n_2)]\Big\}$$
$$= \frac{1}{2}\Big\{4(2\times 2 + 3)(2\times 2 + 3 + 4) + 8(2\times 1 + 1)^2 + 12\big[(1+1)(2\times 2 + 3)\big]\Big\}$$
$$= 274$$

and therefore,

$$274 \ \leq \ 276 \ \leq \ 380.$$

Theorem 3.6 *Let G and H are two simple connected graphs, then the limits for the Harmonic index of ξ-graph are given by*

$$SK_2[\xi] \leq \frac{1}{4}\Big\{2m_1\big[2(2\Delta_G + n_2)^2 + (2\Delta_G + n_2 + 2)^2\big] + 4n_1m_2(\Delta_H + 1)^2$$
$$+ n_1n_2(\Delta_H + 2\Delta_G + n_2 + 1)^2\Big\}$$

and

$$SK_2[\xi] \geq \frac{1}{4}\Big\{2m_1\big[2(2\delta_G + n_2)^2 + (2\delta_G + n_2 + 2)^2\big] + 4n_1m_2(\delta_H + 1)^2$$
$$+ n_1n_2(\delta_H + 2\delta_G + n_2 + 1)^2\Big\}.$$

Proof Using definition of Harmonic index, we get

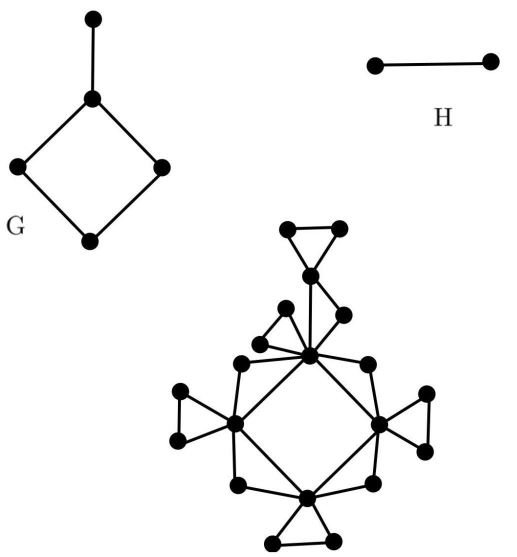
$$SK_2(G) = \frac{1}{4}\sum_{(v_i,v_j)\in E(G)}[d(v_i) + d(v_j)]^2,$$
$$SK_2[\xi] = \frac{1}{4}\Big\{m_1[(2d_G + n_2) + (2d_G + n_2)]^2 + 2m_1[(2d_G + n_2) + 2]^2$$
$$+ n_1m_2[(d_H + 1) + (d_H + 1)]^2 + n_1n_2[d_H + 1 + 2d_G + n_2]^2\Big\}$$
$$= \frac{1}{4}\Big\{2m_1\big[2(2d_G + n_2)^2 + (2d_G + n_2 + 2)^2\big] + 4n_1m_2(d_H + 1)^2$$
$$+ n_1n_2(d_H + 2d_G + n_2 + 1)^2\Big\},$$
$$SK_2[\xi] \leq \frac{1}{4}\Big\{2m_1\big[2(2\Delta_G + n_2)^2 + (2\Delta_G + n_2 + 2)^2\big] + 4n_1m_2(\Delta_H + 1)^2$$
$$+ n_1n_2(\Delta_H + 2\Delta_G + n_2 + 1)^2\Big\}.$$

Similarly,

$$SK_2[\xi] \geq \frac{1}{4}\Big\{2m_1\big[2(2\delta_G+n_2)^2+(2\delta_G+n_2+2)^2\big]+4n_1m_2(\delta_H+1)^2$$
$$+n_1n_2(\delta_H+2\delta_G+n_2+1)^2\Big\}.$$

This completes the proof. □

Illustration 3.6 Let the ξ-graph of G, H be shown in Figure 6.



ξ-graph of G and H

Figure 6

Then,

$$SK_2(G) = \frac{1}{4}\sum_{(v_i,v_j)\in E(G)}[d(v_i)+d(v_j)]^2$$
$$= \frac{1}{4}\Big[5(2+2)^2+3(4+2)^2+5(2+8)^2+12(2+6)^2+2(6+6)^2$$
$$+2(6+8)^2+(4+8)^2\Big] = 570,$$

$$SK_2[\xi] \leq \frac{1}{4}\Big\{2m_1\big[2(2\Delta_G+n_2)^2+(2\Delta_G+n_2+2)^2\big]+4n_1m_2(\Delta_H+1)^2$$
$$+n_1n_2(\Delta_H+2\Delta_G+n_2+1)^2\Big\}$$

$$= \frac{1}{4}\Big\{10\big[2(2\times 3+2)^2+(2\times 3+2+2)^2\big]+20(1+1)^2$$

$$+10(1+2\times 3+2+1)^2\Big\} = 850,$$

$$SK_2[\xi] \geq \frac{1}{4}\Big\{2m_1\big[2(2\delta_G+n_2)^2+(2\delta_G+n_2+2)^2\big]+4n_1m_2(\delta_H+1)^2$$

$$+n_1n_2(\delta_H+2\delta_G+n_2+1)^2\Big\}$$

$$= \frac{1}{4}\Big\{10\big[2(2\times 1+2)^2+(2\times 1+2+2)^2\big]+20(1+1)^2$$

$$+10(1+2\times 1+2+1)^2\Big\} = 280$$

and therefore,

$$280 \ \leq \ 570 \ \leq \ 840.$$

§4. Conclusion

This work focused on six significant topological indices and determined their limits while considering the ξ-graph. Similarly, researchers can consider various classes of topological indices and ascertain the limits that correspond to them for ξ-graphs.

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References

[1] E. Estrada, Atom-bond connectivity and the energetic of branched alkanes, *Chemical Physics Letters*, 463, (2008), 422-425.

[2] B. Furtula and I. Gutman, A forgotten topological index, *Journal of Mathematical Chemistry*, 53, (2015), 1184-1190.

[3] I. Gutman and N. Trinajstic, Graph theory and molecular obitals. Total π-electron energy of alternate hydrocarbons, *Chemical Physics Letters*, 17, (1972), 535-538.

[4] V. Lokesha, B. Shwetha Shetty, P. S. Ranjini, I. N. Cangul and A. S. Cevik, New bounds for Randic and GA indices, *Journal of Inequalities and Applications*, 1(180), (2013), 01-07.

[5] V. Lokesha, M. Manjunath, B. Chaluvaraju, K. M. Devendraiah, I. N. Cangul and A. S. Cevik, Computation of Adriatic indices of certain operators of regular and complete bipartite graphs, *Advanced Studies in Contemporary Mathematics*, 28(2), (2018), 231-244.

[6] M. Manjunath and V. Lokesha, S-Corona operations of standard graphs in terms of degree sequences, *Proceedings of the Jangjeon Mathematical Society*, 23(2), (2020), 149-157.

[7] Manjunath Muddulapuram, Deepika Togarichetu, Narendra Vaderahalli Halappa and Lokesha Veerabhadraiah, Bounds analysis on topological indices of semi total point graph of vertex corona graph, *AIP Conference Proceedings*, 2649, 030061, (2023), 1-7.

[8] A. Milicevic, S. Nikolic and N. Trinajstic, On reformulated Zagreb indices, *Molecular Divers-s*, 8, (2004), 393-399.

[9] Nirupadi K., Manjunath M. and V.Lokesha, Bounds for the Topological Indices of ℧-graph, *Palestine Journal of Mathematics*, 13(2), (2024), 210C220.

[10] P.S.Ranjini, V.Lokesha and A. Usha, Relation between phenylene and hexagonal squeeze using harmonic index, *International Journal of Graph Theory*, 1(4), (2013), 116-121.

[11] V. Shigehalli and R. Kanabur, Computing degree-based topological indices of Polyhex Nanotubes, *Journal of Mathematical Nanoscience*, 6(1-2), (2016), 47-55.

[12] D. Vukicevic and B. Furtula, Topological index based on the ratios of geometrical and arithmetical mean of end-vertex degrees of edges of edges, *Journal of Mathematical Chemistry*, 26, (2009), 1369-1376.

[13] D. Vukicevic and M. Gasperov, Bond additive modeling 1. Adriatic indices, *Croatica Chemical Acta*, (2010), 243-260.

[14] D. Vukicevic, Bond additive modeling 2. Mathematical properties of max-min Rodeg index, *Croatica Chemical Acta*, 83(3),(2010), 261-273.

[15] Winer. H, Structural determination of paraffin boiling points, *Journal of American Chemical Society*, 69, (1947), 17-20.

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α-Separation Axioms on Fuzzy Soft T_0 Spaces

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Abstract: The main objective of this article is to introduce four new inferences of fuzzy soft T_0 spaces by using the concept of fuzzy soft topological spaces. We present several new theories and some implications of such spaces. We also show that all these notions preserve some soft invariance properties such as *soft hereditary* and *soft topological* properties.

Key Words: Soft sets, fuzzy soft sets, soft topology, fuzzy soft topology, fuzzy soft open sets, fuzzy soft mapping, image of fuzzy soft mapping.

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§1. Introduction

It is an undeniable fact that the invention of fuzzy sets by Zadeh [1] in 1965 was a groundbreaking event. This type of set is used in control system engineering, image processing, industrial automation, robotics, consumer electronics, and other branches of applied sciences. Besides, it is connected to fuzzy logic giving the opportunity to model under conditions of uncertainty that are vague or not precisely defined, thus succeeding to mathematically solve problems whose statements are expressed in our natural language. Since then, a lot of research has been carried out for generalizing and extending the fuzzy set theory for the purpose of tackling more effectively the existing uncertainty in problems of science, technology, and everyday life.

Consequently, the Russian mathematician Dmtri Molodstov [2] proposed the soft sets in 1999 to overcome the existing difficulty of properly defining the membership function of a fuzzy set. After the introduction of the notion of soft sets, several researchers improved this concept. Maji et al. [3]-[5] presented an application of soft sets in decision-making problems based on the reduction of parameters to keep the optimal choice objects. Pei and Miao [6] showed that soft sets are a class of special information systems.

Topological structures of soft sets were also studied by Sabir and Naz [7]. They defined the

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soft topological spaces over an initial universe with a fixed set of parameters and studied the concepts of soft open sets, soft closed sets, soft closure, soft interior points, soft neighborhood of a point, soft separation axioms as well as their basic properties. Later, Roy and Samanta [8] gave the definition of fuzzy soft topology over the initial universe set. It was further extended by Varol and Aygun [9] and Cetin and Aygun [10].

Sabir Hussain and Bashir Ahmad [11] redefined and explored several properties of soft T_i, $i = 0, 1, 2$, soft regular, soft T_3, soft normal, and soft T_4 axioms using the soft points defined by Zorlutuna [12]. They also discussed some soft invariance properties, namely soft topological property and soft hereditary property. In this work, we newly define in four different ways the notions of Fuzzy Soft T_0 spaces, develop several theories, and discuss various properties, namely hereditary and topological properties.

Throughout this paper, X and Y will be non-empty sets, ϕ will denote the empty set, and E will be the set of all parameters. F_E will denote the soft set, f_A will denote the fuzzy soft set, $\bar{T}$ and τ will represent the soft topology and fuzzy soft topology, respectively.

The rest of this paper is organized as follows: Section 2 presents a brief review of the relevant definitions such as fuzzy sets, soft sets, fuzzy soft sets, soft topology, fuzzy soft topology, fuzzy soft mapping, and the image of fuzzy soft mapping. In Section 3, we develop four ideas of fuzzy soft T_0 spaces, show some implications among them, and introduce several new theories on fuzzy soft T_0 spaces. The concepts of, *good extension*, *hereditary property*, and related theorems are given in Section 4. Finally, Section 5 presents the conclusion and further discussion of this paper.

§2. **Preliminaries**

We recall some basic definitions and known results of soft sets, fuzzy soft sets, operations on fuzzy soft sets, soft topology, fuzzy soft topology, and fuzzy soft mapping.

Definition 2.1([1]) *Let X be a non-empty set and $I = [0, 1]$. A fuzzy set in X is a function $u : X \to I$ which assigns to each element $x \in X$ a degree of membership $u(x) \in I$.*

Definition 2.2([16]) *A pair (F, E) denoted by F_E is called a soft set over X, where F is a mapping given by $F : E \to P(X)$. We denote the family of all soft sets over X by $SS(X, E)$.*

Definition 2.3([16]) *A soft set (F, E) over X is called a null soft set and denoted by $\bar{\phi}$ if $F(e) = \phi$ for every $e \in E$.*

Definition 2.4([16]) *A soft set (F, E) over X is called an absolute soft set and denoted by $\bar{X}$ if $F(e) = X$ for every $e \in E$.*

Definition 2.5([16]) *Let X be an initial universal set, and $A \subseteq E$. Let $\bar{T}$ be a subfamily of the family of all soft sets $S(X)$. We say that the family $\bar{T}$ is a soft topology on X if the following axioms hold:*

(1) $\bar{\phi}_A, \bar{X}_A \in \bar{T}$;

(2) *If* $F_A, G_A \in \bar{T}$, *then* $F_A \cap G_A \in \bar{T}$;

(3) *If* $G_{iA} \in \bar{T}$ *for each* $i \in \Lambda$, *then* $\bigcup_{i\in\Lambda} G_{iA} \in \bar{T}$.

Then, the triple $(\bar{X}_A, \bar{T}, A)$ *is called a soft topological space (STS, for short) and the members of* $\bar{T}$ *are called soft open sets (SOS for short). A soft set* F_A *is called soft closed set (SCS, for short) if and only if its complement is a soft open set. That is,* $F_A^c \in \bar{T}$.

Definition 2.6([7]) *A soft topological space* $(F_A, \bar{T}, A)$ *is called soft* T_0 (ST_0) *space if for each* $x_1, x_2 \in X$ *with* $x_1 \neq x_2$, *there exists a SOS* $F_A \in \bar{T}$ *such that* $x_1 \in F_A, x_2 \notin F_A$ *or* $x_1 \notin F_A, x_2 \in F_A$.

Definition 2.7([9]) *A fuzzy soft set* f_A *on the universe* X *is a mapping from the parameter set* E *to* I^X, *i.e.,* $f_A : E \to I^X$, *where* $f_A(e) \neq 0_X$ *if* $e \in A \subseteq E$ *and* $f_A(e) = 0_X$ *if* $e \notin A$, *where* 0_X *is the empty fuzzy set on* X.

From now on, we will use $F(X, E)$ instead of the family of all fuzzy soft sets over X. A classical soft set F_A over a universe X can be seen as a fuzzy soft set by using the characteristic function of the set $F_A(e)$:

$$f_A(e)(a) = \chi_{F_A(e)}(a) = \begin{cases} 1, & \text{if } a \in F_A(e), \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.8([13]) *Two fuzzy soft sets* f_A *and* g_B *on* X, *we say that* f_A *is called a fuzzy soft subset of* g_B *and write* $f_A \subseteq g_B$ *if* $f_A(e) \leq g_B(e)$ *for every* $e \in E$.

Definition 2.9([13]) *Two fuzzy soft sets* f_A *and* g_B *on* X *are called equal if* $f_A \subseteq g_B$ *and* $g_B \subseteq f_A$.

Definition 2.10([13]) *Let* $f_A, g_B \in (X, E)$. *Then the union of* f_A *and* g_B *is also a fuzzy soft set* h_C, *defined by* $h_C(e) = f_A(e) \vee g_B(e)$ *for all* $e \in E$, *where* $C = A \cup B$. *Here we write* $h_C = f_A \cup g_B$.

Definition 2.11([13]) *Let* $f_A, g_B \in (X, E)$. *Then the intersection of* f_A *and* g_B *is also a fuzzy soft set* h_C, *defined by* $h_C(e) = f_A(e) \wedge g_B(e)$ *for all* $e \in E$, *where* $C = A \cap B$. *Here we write* $h_C = f_A \cap g_B$.

Definition 2.12([13]) *A fuzzy soft set* f_E *on* X *is called a null fuzzy soft set, denoted by* 0_E, *if* $f_E(E) = 0_X$ *for each* $e \in E$.

Definition 2.13([13]) *A fuzzy soft set* f_E *on* X *is called an absolute fuzzy soft set, denoted by* 1_E, *if* $f_E(E) = 1_X$ *for each* $e \in E$.

Definition 2.14([13]) *Let* $f_A \in (X, E)$. *Then the complement of* f_A *is denoted by* f_A^c *and is defined by* $f_A^c(e) = 1 - f_A(e)$ *for each* $e \in E$.

Definition 2.15([14]) *A fuzzy soft set* g_A *is said to be a fuzzy soft point, denoted by* e_{g_A}, *if for the element* $e \in E$, $g(e) \neq \tilde{\phi}$ *and* $g(e') = \tilde{\phi}$ *for all* $e' \in A - \{e\}$.

Definition 2.16([14]) *A fuzzy soft point e_{g_A} is said to be in a fuzzy soft set h_A, denoted by $e_{g_A} \in h_A$, if for the element $e \in A$, $g(e) \leq h(e)$.*

Definition 2.17([14]) *Let f_A be a fuzzy soft set, $\mathcal{FS}(f_A)$ be the set of all fuzzy soft subsets of f_A, and τ be a subfamily of $\mathcal{FS}(f_A)$. Then τ is called a fuzzy soft topology on f_A if the following conditions are satisfied:*

(1) $\tilde{\phi}_A, f_A \in \tau$;

(2) $f_{1A}, f_{2A} \in \tau \Rightarrow f_{1A} \cap f_{2A} \in \tau$;

(3) *For any index set I, if $f_{iA} \in \tau$ for any $i \in I$, then $\bigcup_{i\in I} f_{iA} \in \tau$.*

Then, the pair (f_A, τ) is called fuzzy soft topological space (FSTS, for short), and the members of τ are called fuzzy soft open sets (FSOS, for short). A fuzzy soft open set g_A is called a fuzzy soft closed set (FSCS, for short) if $g_A^c \in \tau$, where g_A^c is the complement of g_A.

Definition 2.18([14]) *A fuzzy soft topological space (f_A, τ) is said to be a fuzzy soft T_0 space if for every pair of distinct fuzzy soft points e_{h_A} and e_{g_B}, there exists a fuzzy soft open set containing one but not the other.*

Definition 2.19([9]) *Let (f_A, τ_1) and (g_B, τ_2) be two fuzzy soft topological spaces (FSTSs), on the two universal sets X and Y, respectively. Then a fuzzy soft mapping $(\varphi, \psi) : (f_A, \tau_1) \to (g_B, \tau_2)$ is called:*

(1) *Fuzzy soft continuous if $(\varphi, \psi)^{-1}(g_B) \in \tau_1$, for all $g_B \in \tau_2$;*

(2) *Fuzzy soft open if $(\varphi, \psi)(f_A) \in \tau_2$, for all $f_A \in \tau_1$;*

(3) *Fuzzy soft closed if $(\varphi, \psi)(f_A)$ is a fuzzy soft closed set of τ_2 for each fuzzy soft closed set f_A of τ_1;*

(4) *Fuzzy soft homeomorphism if (φ, ψ) is bijective, continuous, and open.*

Definition 2.20([13]) *Let $\varphi : X \to Y$ and $\psi : E \to F$ be two mappings, where E and F are parameter sets for the crisp sets X and Y, respectively. Then (φ, ψ) is called a fuzzy soft mapping from (X, E) into (Y, F) and denoted by $(\varphi, \psi) : (X, E) \to (Y, F)$.*

Definition 2.21([13]) *Let f_A and g_B be two fuzzy soft sets over X and Y, respectively, and (φ, ψ) be a fuzzy soft mapping from (X, E) into (Y, F).*

(1) *The image of f_A under the fuzzy soft mapping (φ, ψ), denoted by $(\varphi, \psi)(f_A)$, is defined as*

$$(\varphi, \psi)(f_A)k(y) = \begin{cases} \vee\varphi(x) = y \vee \psi(e) = k f_A(e)(x), & \text{if} \quad \varphi^{-1}(y) \neq \phi, \psi^{-1}(k) \neq \phi; \\ 0, & \quad\quad \text{otherwise,} \end{cases}$$

for all $k \in F$ and $y \in Y$.

(2) *The image of g_B under the fuzzy soft mapping (φ, ψ), denoted by $(\varphi, \psi)^{-1}(g_B)$, is defined as*

$$(\varphi, \psi)^{-1}(g_B)(e)(x) = g_B(\psi(e))(\varphi(x)) \text{ for all } e \in E \text{ and } x \in X.$$

Proposition 2.1([15]) *Let $(\bar{X}_A, \bar{T}, A)$ be a soft topological space over X. Then the collection $T_e = \{F(e) \mid (F, E) \in \bar{T}\}$ for each $e \in E$, defines a topology on X.*

§3. Definitions and Properties of Fuzzy Soft T_0 Spaces

Before we mentioned the definition of fuzzy soft T_0 space, and now in this section we introduce four new ideas of fuzzy soft T_0 spaces, establish some implications among them and develop several new theories on fuzzy soft T_0 spaces. We denote the grade of membership and the grade of non-membership of any point in fuzzy soft set is $\bar{1}$ and $\bar{0}$ respectively. Here $\bar{\alpha}$ means that the grade of membership of any point in fuzzy soft set lies between 0 and 1.

Definition 3.1 *A fuzzy soft topological space (FSTS) (f_A, τ) is called:*

(*a*) *$FST_0(i)$ if for any pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \bar{1}$, $f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}$, $f_{1A}(e)(x_2) = \bar{1}$;*

(*b*) *$FST_0(ii)$ if for any pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \bar{\alpha}$, $f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}$, $f_{1A}(e)(x_2) = \bar{\alpha}$ as $0 < \alpha < 1$;*

(*c*) *$FST_0(iii)$ if for any pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) > f_{1A}(e)(x_2)$, or $f_{1A}(e)(x_2) > f_{1A}(e)(x_1)$;*

(*d*) *$FST_0(iv)$ if for any pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) \neq f_1A(e)(x_2)$ or $f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1)$.*

Theorem 3.1 *Let (f_A, τ) be a fuzzy soft topological space. Then the above four notions of it form the following implications:*

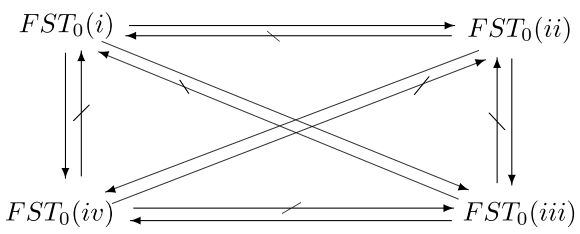

Figure 1 The implications of four notions are shown by a quadrilateral with two diagonals

Proof Let (f_A, τ) be a $FST_0(i)$. Then by definitions, for any pair of $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \bar{1}, f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}, f_{1A}(e)(x_2) = \bar{1}$.

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) = \bar{\alpha},\ f_{1A}(e)(x_2) = \bar{0}, \text{ or} \\ f_{1A}(e)(x_1) = \bar{0},\ f_{1A}(e)(x_2) = \bar{\alpha} \end{cases} \quad \text{as} \quad 0 < \alpha < 1 \tag{1}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) > f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) > f_{1A}(e)(x_1) \end{cases} \tag{2}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) \neq f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1). \end{cases} \tag{3}$$

Hence, from (1), (2), and (3), we see that $FST_0(i) \Rightarrow FST_0(ii) \Rightarrow FST_0(iii) \Rightarrow FST_0(iv)$.

Again, suppose that (f_A, τ) be a $FST_0(i)$. Then by definition, for any pair of $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \overline{1}, f_{1A}(e)(x_2) = \overline{0}$, or $f_{1A}(e)(x_1) = \overline{0}, f_{1A}(e)(x_2) = \overline{1}$.

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) > f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) > f_{1A}(e)(x_1) \end{cases} \tag{4}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) \neq f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1). \end{cases} \tag{5}$$

Hence, from (4) and (5), we see that $FST_0(i) \Rightarrow FST_0(iii)$, and $FST_0(i) \Rightarrow FST_0(iv)$.

Finally, let (f_A, τ) be a $FST_0(i)$. Then from (1), for any pair of $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) = \bar{\alpha}, f_{1A}(e)(x_2) = \overline{0}$, or $f_{1A}(e)(x_1) = \overline{0}, f_{1A}(e)(x_2) = \bar{\alpha}$, as $0 < \alpha < 1$.

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) \neq f_{1A}(e)(x_2), \text{ or} \\ f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1). \end{cases} \tag{6}$$

From (6), we see that $FST_0(ii) \Rightarrow FST_0(iv)$. □

None of the reverse implications is true in general, as can be seen in the following counter examples.

Example 3.1 Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ a set of parameters, $A = \{e_1, e_2\} \subset E$, and τ be a fuzzy soft topology on a universal set X generated by $\tau = \{\overline{0}, \overline{1}, f_{1A}\}$ where $f_{1A} = \{e_1 = \{0.6/x_1, 0.7/x_2\}, e_2 = \{0.7/x_1, 0.6/x_2\}\}$. Here $f_{1A}(e_1)(x_1) = 0.6, f_{1A}(e_1)(x_2) = 0.7$ and $f_{1A}(e_2)(x_1) = 0.7, f_{1A}(e_2)(x_2) = 0.6$.

Hence, we observe that (f_A, τ) is $FST_0(iv)$ but not $FST_0(i)$, $FST_0(ii)$, or $FST_0(iii)$. Therefore, $FST_0(iv) \nRightarrow FST_0(i)$, $FST_0(iv) \nRightarrow FST_0(ii)$, and $FST_0(iv) \nRightarrow FST_0(iii)$.

Example 3.2 Let $X = \{x_1, x_2\}$, $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters, $A = \{e_1, e_2\} \subset E$, and τ be a fuzzy soft topology on a universal set X generated by $\tau = \{\overline{0}, \overline{1}, f_{1A}\}$, where $f_{1A} = \{e_1 = \{0.7/x_1, 0.3/x_2\}, e_2 = \{0.3/x_1, 0.7/x_2\}\}$.

Here, $f_{1A}(e_1)(x_1) = 0.7$, $f_{1A}(e_1)(x_2) = 0.3$, and $f_{1A}(e_2)(x_1) = 0.3$, $f_{1A}(e_2)(x_2) = 0.7$.

$$f_{1A}(e_1)(x_1) = 0.7, \quad f_{1A}(e_1)(x_2) = 0.3$$

$$f_{1A}(e_2)(x_1) = 0.3, \quad f_{1A}(e_2)(x_2) = 0.7$$

Hence, we observe that (f_A, τ) is $FST_0(iii)$ but not $FST_0(i)$ and $FST_0(ii)$. Therefore,

$$FST_0(iii) \nRightarrow FST_0(i), \quad \text{and} \quad FST_0(iii) \nRightarrow FST_0(ii).$$

Finally, if we consider $f_{1A} = \{e_1 = \{\bar{\alpha}/x_1, 0/x_2\}, e_2 = \{0/x_1, \bar{\alpha}/x_2\}\}$, where $0 < \alpha < 1$, then we have

$$f_{1A}(e_1)(x_1) = \bar{\alpha}, \quad f_{1A}(e_1)(x_2) = \bar{0}$$

$$f_{1A}(e_2)(x_1) = \bar{0}, \quad f_{1A}(e_2)(x_2) = \bar{\alpha}$$

Thus, (f_A, τ) is $FST_0(ii)$ but not $FST_0(i)$.

Theorem 3.2 *If a fuzzy soft topological space (FSTS) (f_A, τ) is a fuzzy soft T_0 space, then the following statements are equivalent:*

(a) for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$, $\overline{f_{1A}}(e)(x_1) \wedge \overline{f_{1A}}(e)(x_2) \leq \bar{1}$;

(b) for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$ there exists an FSOS $f_{1A} \in \tau$ such that $f_{1A}(e)(x_1) > \bar{0}, f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}, f_{1A}(e)(x_2) > \bar{0}$.

Proof $(a) \Rightarrow (b)$: We have from (a) that,

$$\overline{f_{1A}}(e)(x_1) \wedge \overline{f_{1A}}(e)(x_2) \leq \bar{1} \Rightarrow \overline{f_{1A}}(e)(x_1) < \bar{1} \quad \text{or} \quad \overline{f_{1A}}(e)(x_2) < \bar{1}.$$

This implies

$$\bar{1} - \overline{f_{1A}}(e)(x_1) > \bar{0} \quad \text{or} \quad \bar{1} - \overline{f_{1A}}(e)(x_2) > \bar{0}.$$

Let $f_{1A} = \bar{1} - \overline{f_{1A}}$. Then we have

$$f_{1A}(e)(x_1) > \bar{0}, f_{1A}(e)(x_2) = \bar{0} \quad \text{or} \quad f_{1A}(e)(x_1) = \bar{0}, f_{1A}(e)(x_2) > \bar{0},$$

which is (b).

$(b) \Rightarrow (a)$: From (b) we have that, for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that

$$f_{1A}(e)(x_1) > \bar{0}, f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \quad f_{1A}(e)(x_1) = \bar{0}, f_{1A}(e)(x_2) > \bar{0}.$$

That implies

$$\bar{1} - f_{1A}(e)(x_1) < \bar{1}, \quad \bar{1} - f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \quad \bar{1} - f_{1A}(e)(x_2) < \bar{1}, \quad \bar{1} - f_{1A}(e)(x_1) = \bar{0}.$$

Since f_{1A} is a fuzzy soft open set (FSOS), therefore $\bar{1} - f_{1A}$ is a fuzzy soft closed set (FSCS). Hence, we have

$$\overline{f_{1A}}(e)(x_1) < \bar{1} \quad \text{or} \quad \overline{f_{1A}}(e)(x_2) < \bar{1} \;\Rightarrow\; \overline{f_{1A}}(e)(x_1) \wedge \overline{f_{1A}}(e)(x_2) \leq \bar{1},$$

which is (a). □

Theorem 3.3 *Let (f_A, τ, A) be a fuzzy soft topological space (FSTS) over a universal set X, and let $e_{x_1}, e_{x_2} \in f_A$ such that $e_{x_1} \neq e_{x_2}$ as $x_1 \neq x_2$ for every pair $x_1, x_2 \in X$. If there exist FSOSs (f_{1A}, A) and (f_{2A}, A) such that $e_{x_1} \in (f_{1A}, A)$ and $e_{x_2} \in (f_{1A}, A)^c$ or $e_{x_2} \in (f_{2A}, A)$ and $e_{x_1} \in (f_{2A}, A)^c$, then*

(*a*) *(f_A, τ, A) is an FST_0 space;*
(*b*) *(F_A, τ, A) is an ST_0 space;*
(*c*) *(X, τ_e) is a T_0 space.*

Proof Firstly we prove (a). It is clear that $e_{x_2} \in (f_{1A}, A)^c = (f^c_{1A}, A) \implies e_{x_2} \notin (f_{1A}, A)$, which implies that $f_{1A}(e)(x_2) = 0$. Similarly, $e_{x_1} \in (f_{2A}, A)^c = (f^c_{2A}, A) \implies e_{x_1} \notin (f_{2A}, A)$, which implies that $f_{2A}(e)(x_1) = 0$. Thus, we have $e_{x_1} \in (f_{1A}, A), e_{x_2} \notin (f_{1A}, A)$ or $e_{x_2} \in (f_{2A}, A), e_{x_1} \notin (f_{2A}, A)$. This proves that (f_A, τ, A) is an FST_0 space.

Secondly we prove (b), that is (F_A, τ, A) is an ST_0 space. To do this, we define a characteristic function 1_{F_A} such that

$$f_A(e)(x) = 1_{F_A}(e) = \begin{cases} 1 & \text{if } x \in F_A(e) \\ 0 & \text{otherwise.} \end{cases}$$

Let $f_A = (f_{1A}, f_{2A})$ and $F_A = (F_{1A}, F_{2A})$. Now, for any $x_1, x_2 \in X$ with $x_1 \neq x_2$ we have $f_{1A}(e)(x_1) = 1 \implies e_{x_1} \in (F_{1A}, A)$ and $f_{1A}(e)(x_2) = 0 \implies e_{x_2} \notin (F_{1A}, A)$. Therefore, $e_{x_1} \in (F_{1A}, A), e_{x_2} \notin (F_{1A}, A)$ or $e_{x_2} \in (F_{2A}, A), e_{x_1} \notin (F_{2A}, A)$. Thus, (F_A, τ, A) is an ST_0 space.

Finally to prove (c), for any $e \in A$, (X, T_e) is a topological space on X (see Proposition 2.1) and $e_{x_1} \in (F_{1A}, A), e_{x_2} \in (F_{1A}, A)^c$ or $e_{x_2} \in (F_{2A}, A), e_{x_1} \in (F_{2A}, A)^c$. So that $x_1 \in F_{1A}(e), x_2 \notin F_{1A}(e)$ or $x_2 \in F_{2A}(e), x_1 \notin F_{2A}(e)$. Thus, (X, τ_e) is a T_0 space. □

§4. Good Extension, Hereditary and Topological Property

In this section, we discuss some fuzzy soft invariance properties, namely *good extension, hereditary* and *soft topological properties.*

Definition 4.1 *Let $(F_A, \overline{T}, A)$ be a soft topological space and $\tau = \{1_{F_A} : F_A \in \overline{T}\}$, and $1_{F_A} = f_{1A}$. Then (f_A, τ) is the corresponding fuzzy soft topological space of $(F_A, \overline{T}, A)$. Let P be a property of soft topological spaces and FP be its fuzzy soft topological analogue. Then FP is called a Good extension of P if the statement $(F_A, \overline{T}, A)$ has P if and only if (f_A, τ) has FP holds true for every soft topological space $(F_A, \overline{T}, A)$.*

Theorem 4.1 *Let $(F_A, \bar{T}, A)$ be a soft T_0 space and (f_A, τ) be $FST_0(j)$ spaces, where $j = i, ii, iii, iv$. Then $(F_A, \bar{T}, A)$ will be $FST_0(j)$ spaces if and only if $FST_0(j)$ will also be a soft T_0 space.*

Proof Suppose that $(F_A, \bar{T}, A)$ is a soft T_0 (ST_0) space. We prove that $(F_A, \bar{T}, A)$ is $FST_0(j)$ spaces. Since $(F_A, \bar{T}, A)$ is a soft T_0 space, for each $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists a soft open set (SOS) $F_A \in \bar{T}$ such that $x_1 \in F_A$ and $x_2 \notin F_A$, or $x_1 \notin F_A$ and $x_2 \in F_A$. Then, by a characteristic function 1_{F_A}, we have

$$\Rightarrow \begin{cases} 1_{F_A}(e)(x_1) = \bar{1}, \quad 1_{F_A}(e)(x_2) = \bar{0}, \quad \text{or} \\ 1_{F_A}(e)(x_1) = \bar{0}, \quad 1_{F_A}(e)(x_2) = \bar{1}. \end{cases}$$

Let $1_{F_A} = f_{1A}$. Therefore,

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) = \bar{1}, \quad f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \\ f_{1A}(e)(x_1) = \bar{0}, \quad f_{1A}(e)(x_2) = \bar{1}, \end{cases} \tag{7}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) = \bar{\alpha}, \quad f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \\ f_{1A}(e)(x_1) = \bar{0}, \quad f_{1A}(e)(x_2) = \bar{\alpha}, \quad \text{as } 0 < \alpha < 1, \end{cases} \tag{8}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) > f_{1A}(e)(x_2) \quad \text{or} \\ f_{1A}(e)(x_2) > f_{1A}(e)(x_1), \end{cases} \tag{9}$$

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) \neq f_{1A}(e)(x_2) \quad \text{or} \\ f_{1A}(e)(x_2) \neq f_{1A}(e)(x_1). \end{cases} \tag{10}$$

Hence, from (7), (8), (9), and (10), we see that a soft T_0 space is $FST_0(i)$, $FST_0(ii)$, $FST_0(iii)$, and $FST_0(iv)$ spaces. This implies that a soft T_0 space is $FST_0(j)$ spaces, where $j = i, ii, iii, iv$.

Conversely, assume that (f_A, τ) is $FST_0(j)$ spaces. We will prove that (f_A, τ) is a soft T_0 space. To do this, we will first prove it for $j = i$. Since (f_A, τ) is $FST_0(i)$, by definition, for all $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau$ such that

$$\Rightarrow \begin{cases} f_{1A}(e)(x_1) = \bar{1}, \quad f_{1A}(e)(x_2) = \bar{0}, \quad \text{or} \\ f_{1A}(e)(x_1) = \bar{0}, \quad f_{1A}(e)(x_2) = \bar{1}. \end{cases}$$

Thus

$$\Rightarrow \begin{cases} f_{1A}^{-1}(e)(\bar{1}) = \{x_1\}, \quad f_{1A}^{-1}(e)(\bar{0}) = \{x_2\}, \quad \text{or} \\ f_{1A}^{-1}(e)(\bar{0}) = \{x_1\}, \quad f_{1A}^{-1}(e)(\bar{1}) = \{x_2\}. \end{cases}$$

Let $f_{1A}^{-1}(\bar{1}) = F_A$. Therefore, $F_A(e) = \{x_1\}$ or $F_A(e) = \{x_2\}$. Hence, for each $x_1, x_2 \in X$, with $x_1 \neq x_2$, and for all $e \in A$, there exists a soft open set (SOS) $F_A \in \bar{T}$ such that $x_1 \in F_A$ and $x_2 \notin F_A$, or $x_1 \notin F_A$ and $x_2 \in F_A$. Thus, $FST_0(i)$ is ST_0. Similarly, $FST_0(ii)$, $FST_0(iii)$, and $FST_0(iv)$ imply ST_0 space. □

Definition 4.2 *Let (f_A, τ) be a fuzzy soft topological space (FSTS) and $g_A \subset f_A$. Then the fuzzy soft topology $\tau_{g_A} = \{g_A \cap h_A \mid h_A \in \tau\}$ is called the fuzzy soft subspace topology, and (g_A, τ_{g_A}) is called the fuzzy soft subspace of (f_A, τ). A fuzzy soft topological property P is called hereditary if each subspace of a fuzzy soft topological space with property P also has property P.*

Theorem 4.2 *Let (f_A, τ) be a fuzzy soft topological space (FSTS) and $(g_A, \tau_{(g_A)})$ be a subspace of it. Then if (f_A, τ) is $FST_0(j)$, it implies that (g_A, τ_{g_A}) is also $FST_0(j)$, where $j = i, ii, iii, iv$.*

Proof We prove this theorem only for $j = i$. Suppose that (f_A, τ) is $FST_0(i)$. It will be shown that (g_A, τ_{g_A}) is $FST_0(i)$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, and for all $e \in A$ such that $g_A(e)(x_1) = \bar{1}$, $g_A(e)(x_2) = \bar{0}$. Since (f_A, τ) is $FST_0(i)$, by definition, for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $h_A \in \tau$ such that either $h_A(e)(x_1) = \bar{1}$, $h_A(e)(x_2) = \bar{0}$ or $h_A(e)(x_1) = \bar{0}$, $h_A(e)(x_2) = \bar{1}$. Since $g_A(e)(x_1) = \bar{1}$ and $h_A(e)(x_1) = \bar{1}$, we have $(g_A \cap h_A)(e)(x_1) = 1$ and similarly, $(g_A \cap h_A)(e)(x_2) = 0$ or $(g_A \cap h_A)(e)(x_1) = 0$, $(g_A \cap h_A)(e)(x_2) = 1$. Hence, (g_A, τ_{g_A}) is $FST_0(i)$. The cases for $j = ii, iii$, and iv can be proved in a similar way. □

Theorem 4.3 *Let (f_A, τ_1) and (g_B, τ_2) be two fuzzy soft topological spaces (FSTS's) on the two universal sets X and Y, respectively. Let $(\phi, \psi) : (f_A, \tau_1) \to (g_B, \tau_2)$ be a fuzzy soft one-to-one, onto, and continuous map. Then these spaces maintain the following features:*

(*a*) (f_A, τ_1) *is* $FST_0(i) \iff (g_B, \tau_2)$ *is* $FST_0(i)$;
(*b*) (f_A, τ_1) *is* $FST_0(ii) \iff (g_B, \tau_2)$ *is* $FST_0(ii)$;
(*c*) (f_A, τ_1) *is* $FST_0(iii) \iff (g_B, \tau_2)$ *is* $FST_0(iii)$;
(*d*) (f_A, τ_1) *is* $FST_0(iv) \iff (g_B, \tau_2)$ *is* $FST_0(iv)$.

Proof We prove only (a). Suppose (f_A, τ_1) is $FST_0(i)$. We will prove that (g_B, τ_2) is also $FST_0(i)$. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since (ϕ, ψ) is onto, there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$, such that $\phi(x_1) = y_1$, $\phi(x_2) = y_2$, and $\psi(e) = k$ for all parameters $e \in A$ and for all $k \in B$. Hence, $x_1 \neq x_2$ since ϕ is one-to-one. Since (f_A, τ_1) is $FST_0(i)$, we have that for all $x_1, x_2 \in X$, $x_1 \neq x_2$, and for all $e \in A$, there exists an FSOS $f_{1A} \in \tau_1$ such that either $f_{1A}(e)(x_1) = \bar{1}$ and $f_{1A}(e)(x_2) = \bar{0}$, or $f_{1A}(e)(x_1) = \bar{0}$ and $f_{1A}(e)(x_2) = \bar{1}$. Now, there exists an FSOS $(\phi, \psi)(f_{1A}) \in \tau_2$ such that $(\phi, \psi)(f_{1A})_k(y_1) = \bar{1}$ as $f_{1A}(e)(x_1) = \bar{1}$ and $(\phi, \psi)(f_{1A})_k(y_2) = \bar{0}$ as $f_{1A}(e)(x_2) = \bar{0}$. Similarly, $(\phi, \psi)(f_{1A})_k(y_1) = \bar{0}$ and $(\phi, \psi)(f_{1A})_k(y_2) = \bar{1}$. Hence, (g_B, τ_2) is $FST_0(i)$.

Conversely, suppose that (g_B, τ_2) is $FST_0(i)$. We prove that (f_A, τ_1) is $FST_0(i)$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. This implies that $\phi(x_1) \neq \phi(x_2)$ as ϕ is one-to-one. Put $\phi(x_1) = y_1$ and $\phi(x_2) = y_2$. Then $y_1 \neq y_2$. Since (g_B, τ_2) is $FST_0(i)$, there exists an FSOS $g_{1B} \in \tau_2$ such that either $g_{1B}(k)(y_1) = \bar{1}$ and $g_{1B}(k)(y_2) = \bar{0}$, or $g_{1B}(k)(y_1) = \bar{0}$ and $g_{1B}(k)(y_2) = \bar{1}$. Now, there exists an FSOS $(\phi, \psi)^{-1}(g_{1B}) \in \tau_1$ such that $(\phi, \psi)^{-1}(g_{1B})(e)(x_1) = g_{1B}(\psi(e))(\phi(x_1)) = g_{1B}(k)(y_1) = \bar{1}$ and $(\phi, \psi)^{-1}(g_{1B})(e)(x_2) = g_{1B}(\psi(e))(\phi(x_2)) = g_{1B}(k)(y_2) = \bar{0}$. Similarly, $(\phi, \psi)^{-1}(g_{1B})(e)(x_1) = \bar{0}$ and $(\phi, \psi)^{-1}(g_{1B})(e)(x_2) = \bar{1}$. Hence, (f_A, τ_1) is $FST_0(i)$. In the same way, (b), (c) and (d) can be proved. □

§5. Conclusion

This paper develops several new theories based on four new notions of $\alpha - separation$ axioms on fuzzy soft T_0 spaces. The *good extension, soft hereditary* and *soft topological properties* are widely discussed with examples and counter examples, which are extensively applicable in fuzzy logic and fuzzy topology. The similar concepts of such notions and relevant new theories will be investigated for lattice fuzzy soft T_0 spaces in our further study.

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References

[1] L. A. Zadeh, Fuzzy sets, *Information and Control*, 8, 1965, pp. 338-353.

[2] D. Molodtsov, Soft set theoryfirst results, *Comput. Math. Appl.*, 37, 1999, pp. 19-31.

[3] P. K. Maji and A. R. Roy, An application of soft set in decision making problem, *Computers Math. Appl.*, 44, 2002, pp. 1077-1083.

[4] P. K. Maji, R. Biswas, and A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.*, 9(3), 2001, pp. 589-602.

[5] P. K. Maji, R. Biswas, and A. R. Roy, Soft set theory, *Computers Math. Appl.*, 45, 2003, pp. 555-562.

[6] D. Pei and D. Miao, From soft sets to information systems, *Granular Computing*, 2, 2005, pp. 617-621.

[7] M. Shabir and M. Naz, On soft topological spaces, *Computers and Mathematics with Applications*, 61, 2011, pp. 1786-1799.

[8] S. Roy and T. K. Samanta, A note on fuzzy soft topological spaces, *Ann. Fuzzy Math. Inform.*, 3, 2011, pp. 305-311.

[9] B. P. Varol and H. Aygun, Fuzzy soft topology, *Hacettepe Journal of Mathematics and Statistics*, 41(3), 2012, pp. 407-419.

[10] V. Cetin and H. Aygun, A note on fuzzy soft topological spaces, *Proceeding of the 8th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2013)*, pp. 56-60.

[11] S. Hussain and B. Ahmad, Soft separation axioms in soft topological spaces, *Hacettepe Journal of Mathematics and Statistics*, 44, 2015, pp. 559-568.

[12] I. Zorlutana, N. Akdag, W. K. Min, and S. Atmaca, Remarks on soft topological spaces, *Annals of Fuzzy Mathematics and Informatics*, 3, 2012, pp. 171-185.

[13] G. Kalpana and C. Kalaivani, Fuzzy soft topology, *International Journal of Engineering Studies*, 9(1), 2017, pp. 45-56.

[14] J. Mahanta and P. K. Das, Results on fuzzy soft topological spaces, *arXiv:1203.0634v1* (2012).

[15] O. Gocur and A. Kopuzlu, On soft separation axioms, *Ann. Fuzzy Math. Inform.*, 9(5), 2015, pp. 817-822.

[16] O. Tantawy, S. A. El-Sheikh, S.Hamde, Remarks on soft topological spaces, *Separation Axioms on Fuzzy Soft Topological Space*, 8(2), 2018, pp. 1-15.

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Further Results on Level Matrix

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Abstract: A level index is a numerator of distance based Gini index which was introduced in 2017. Then, the level index is used to evaluate the balance of rooted trees. Level matrix and level characteristic polynomial concepts were introduced recently for rooted trees. The level characteristic polynomial of rooted binary caterpillars was computed in terms of distance characteristic polynomial of paths. In this paper, we compute the level index of rooted binary caterpillars as Octahedral numbers and show that level characteristic polynomials of the mentioned graphs can be computed by a recurrence relation in terms of Chebyshev polynomials of first- and second-kind. So, we can give an affirmative answer to a recent manuscript including a question about the computation of the level characteristic polynomial of a graph with recursive formula.

Key Words: Level index, level characteristic polynomial, rooted trees, binary caterpillar, Smarandachely rooted tree.

AMS(2010): 05C05, 05C50, 05C12, 05C35.

§1. Introduction

Corrado Gini [5] introduced in 1912 several economic summary statistics, among them what is now known as the Gini index, which it is a parameter that measures how equitably a resource is distributed throughout a population (for more details, see [4] and [7]. More recent, Balaji and Mahmoud [1] presented two distance-based molecular descriptors level index and Gini index. Clearly, the level index is a numerator of the distance-based Gini index which is used to evaluate the measure for a tree. Balaji and Mahmoud obtained a general phrase of the level index of trees. Level matrix and level characteristic polynomial concepts introduced recently [3]. Level characteristic polynomials of rooted stars, rooted double stars, and rooted binary caterpillars were obtained [3]. The level characteristic polynomials of rooted binary caterpillars were computed in terms of distance characteristic polynomials of paths. Moreover, distance characteristic polynomials of paths were obtained by Hosoya et al. in 1973 [6]. The spectrum of the level matrix was studied very recently in [2].

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In this paper, we compute the level characteristic polynomial of rooted binary caterpillar by a recurrence relation in terms of Chebyshev polynomials of the first-kind and second-kind. Therefore, we can present an affirmative answer to the question (Question 4) which appeares in [2] looking for a rooted tree that its level characteristic polynomial can be computed by a recursive formula.

§2. **Preliminaries**

We only consider simple, connected, and undirected graphs. A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$. The notation $d(u, v)$ is used to show the distance between two vertices u and v in a graph. Generally, a *Smarandachely rooted tree* T^S is a rooted tree T with s rooted vertices and the level l_u^S of $u \in V(T)$ is the minimum distance between u to rooted vertices. Particularly, if there are only one rooted vertex, such a T^S is nothing else but a rooted tree T. For a rooted tree T, the level of a vertex u is abbreviated to l_u, i.e., the distance between u and the rooted vertex, which is the objective in this paper. Certainly, the same question can be also considered on Smarandachely rooted tree T^S for a few of typical rooted vertices.

Definition 2.1([1]) *The level index of a rooted tree T, denoted by $LI(T)$, is given by*

$$LI(T) = \sum_{1 \le i < j \le n} |l_i(T) - l_j(T)|,$$

where $l_i(T)$ shows the level of the vertex v_i in T.

Definition 2.2([1]) *The level index of a rooted tree T of maximum level h is computed by the following equation such that N_i and N_{i+j} showing the number of vertices at level i and $i + j$ with the difference in depth of j*

$$LI(T) = \sum_{i=0}^{h} \sum_{j=0}^{h-i} j N_i N_{i+j}$$

Definition 2.3([3]) *Let T be a rooted tree and let its vertices be labeled as $v_1, v_2, \cdots, v_n$. The level of $v \in V(T)$ is the distance from the root of T to v. The level matrix of T is defined as the square matrix $L = L(T) = [l_{ij}]$ where l_{ij} is the absolute value of the levels' difference of vertices v_i and v_j in T.*

A rooted binary caterpillar T_k is obtained from a k vertex path by attaching a vertex to each vertex of the path which is illustrated in Figure 1.

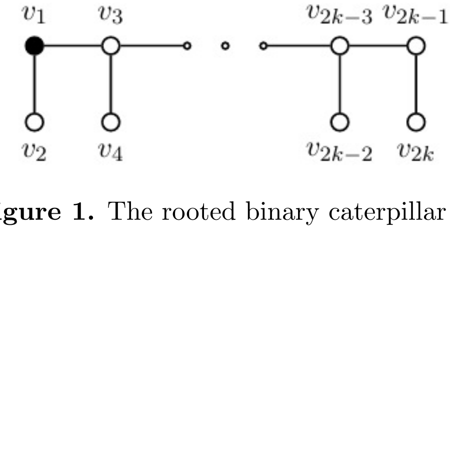

Figure 1. The rooted binary caterpillar T_k

Definition 2.4 *The level characteristic matrix of rooted binary caterpillar T_k is defined by Matrix M_k of order 2k:*

$$M_k = \begin{pmatrix} A_1 & A_2 & A_3 & \dots & A_k \\ A_{-2} & A_1 & A_2 & \ddots & \vdots \\ A_{-3} & A_{-2} & A_1 & \ddots & A_3 \\ \vdots & \ddots & \ddots & \ddots & A_2 \\ A_{-k} & \dots & A_{-3} & A_{-2} & A_1 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} x & -1 \\ -1 & x \end{pmatrix}, \quad \text{and} \quad A_k = \begin{pmatrix} -k+1 & -k \\ -k+2 & -k+1 \end{pmatrix}$$

for all integers $k \geq 2$, and $A_{-k} = A_k^t$ (transpose), for all $k \geq 1$.

For instance, the level characteristic matrix of T_5 is presented as follows

$$M_5 = \left(\begin{array}{cc|cc|cc|cc|cc} x & -1 & -1 & -2 & -2 & -3 & -3 & -4 & -4 & -5 \\ -1 & x & 0 & -1 & -1 & -2 & -2 & -3 & -3 & -4 \\ \hline -1 & 0 & x & -1 & -1 & -2 & -2 & -3 & -3 & -4 \\ -2 & -1 & -1 & x & 0 & -1 & -1 & -2 & -2 & -3 \\ \hline -2 & -1 & -1 & 0 & x & -1 & -1 & -2 & -2 & -3 \\ -3 & -2 & -2 & -1 & -1 & x & 0 & -1 & -1 & -2 \\ \hline -3 & -2 & -2 & -1 & -1 & 0 & x & -1 & -1 & -2 \\ -4 & -3 & -3 & -2 & -2 & -1 & -1 & x & 0 & -1 \\ \hline -4 & -3 & -3 & -2 & -2 & -1 & -1 & 0 & x & -1 \\ -5 & -4 & -4 & -3 & -3 & -2 & -2 & -1 & -1 & x \end{array}\right).$$

Theorem 2.5([6]) *The distance characteristic polynomial of a path P_n is given by*

$$\mathcal{C}_n(x) = x^n - \sum_{k=2}^{n} 2^{k-2}(k-1)\frac{n^2(n^2-1)(n^2-2^2)\dots(n^2-(k-1)^2)}{k^2(k^2-1)(k^2-2^2)\dots(k^2-(k-1)^2)}x^{n-k}.$$

Theorem 2.6([3]) *For $k > 2$, the characteristic polynomial of the rooted tree T_k is given by*

$$\varphi(\lambda) = (2\lambda)^{k-1}\big(\lambda \cdot \mathcal{C}_k(\lambda/2) + \mathcal{C}_{k+1}(\lambda/2)\big),$$

where $\mathcal{C}_n(x)$ is given in Theorem 2.5.

In order to obtain a general result, we can define a family Ŧ of trees such that $N_0 = 1$,$N_1 = N_2 = ... = N_{k-1} = 2$, and $N_k = 1$. Clearly $T_k \in$ Ŧ.

§3. Main Result

In this section, we first compute the level index of the rooted trees of the family Ŧ that the level indices of the members of Ŧ equal to Octahedral numbers. Moreover, we show that the determinant of the matrix M_k equals to level characteristic polynomial of only the members of Ŧ with level k. Finally, we obtain the determinant of M_k in terms of Chebyshev polynomials of the first-kind and second-kind.

Theorem 3.1 *If a tree $T \in$ Ŧ with level k, then the level index of T is given by*

$$LI(T) = \frac{k(2k^2+1)}{3}.$$

Proof By Definition 2.2, the level index of T can be computed by the following equation such that the numbers of the vertices at level i are ordered as follows $N_0 = 1$, $N_1 = N_2 = \cdots = N_{k-1} = 2$, and $N_k = 1$. Then

$$\begin{aligned} LI(T) &= \sum_{i=0}^{k}\sum_{j=0}^{k-i} jN_iN_{i+j} = k + 4\sum_{i=1}^{k-1} i(k-i) \\ &= k + 4k\sum_{i=1}^{k-1} i - 4\sum_{i=1}^{k-1} i^2 \\ &= k + 4k \times \frac{k(k-1)}{2} - 4 \times \frac{(k-1)k(2k-1)}{6} \\ &= \frac{4k^3+2k}{6} = \frac{k(2k^2+1)}{3}, \end{aligned}$$

as claimed. □

For instance, by Theorem 3.1, the initial terms of the level index of T_k are $LI(T_1) = 1$, $LI(T_2) = 6$, $LI(T_3) = 19$, $LI(T_4) = 44$, $LI(T_5) = 85$, and $LI(T_6) = 146$ (see Sequence A005900 in [8]).

Lemma 3.2 *Let T be a rooted tree. Then the level characteristic polynomial of T equals to determinant of M_k if and only if $T \epsilon$ Ŧ with level k.*

Proof The sufficient condition is clear. Then we will prove only the necessary condition. If $T \in$ Ŧ, then there are two vertices at each level except the last level k and there is one vertex at the last level k. It means that every two rows of the level matrix of T are equal because the vertices appeared on the same level. Since T is a rooted tree and there is one vertex at level k, the first row and the last row of the level matrix of T are equal in the reverse order.

Now assume that $T \notin$ Ŧ. It implies that there are at least three vertices at a level Then, at least three rows of the level matrix of T are the same is a contradiction. It is obtained that the level characteristic polynomial of T equals to determinant of M_k if and only if $T \in$ Ŧ. □

In order to find the determinant of the matrix M_k, we define the following matrices

- A_k (respectively, B_k, C_k) is the obtained matrix from M_k by removing the first row and first column (respectively, second column, $2k$th column).
- D_k is the obtained matrix from C_k by removing the first and last rows, and first and second columns.
- E_k is the obtained matrix from C_k by removing the first, second, and last rows, and the first three columns.

Then, by evaluating the determinant of the matrices $M_k, A_k, B_k, C_k, D_k, E_k$ according to the first row, we obtain the following recurrences

$$\begin{aligned}
\det(M_k) &= 2(x+1)\det(A_k) + x\det(B_k) + \det(C_k),\\
\det(A_k) &= 2x\det(M_{k-1}) - x^2\det(A_{k-1}),\\
\det(B_k) &= -x\det(M_{k-1}) + x\det(C_{k-1}),\\
\det(C_k) &= x^2\det(C_{k-1}) - 2x\det(D_{k-1}) + x^2\det(E_{k-1}),\\
\det(D_k) &= 2(x+1)\det(E_k) - x^2\det(D_{k-1}),\\
\det(E_k) &= 2x\det(D_{k-1}) - x^2\det(E_{k-1}).
\end{aligned}$$

By the recurrence of $\det(D_k)$ and $\det(E_k)$, we obtain

$$\det(D_k) = 2x(x+2)\det(D_{k-1}) - x^4\det(D_{k-2})$$

where $\det(D_1) = 2(1+x)$ and $\det(D_2) = 4x(1+x)(2+x)$. Hence, by induction on k, we have

$$\det(D_k) = 2(1+x)x^{2k-2}U_{k-1}(y),$$

where $y = 1 + 2/x$ and U_m is the mth Chebyshev polynomials of the second kind. Thus, by the recurrence of $\det(D_k)$, we have

$$\det(E_k) = x^{2k-2}(U_{k-1}(y) + U_{k-2}(y)).$$

By the recurrence of $\det(C_k)$, we have

$$\begin{aligned}
\det(C_k) &= -(1+x)x^{2k-2} + \sum_{j=0}^{k-2} x^{2j}(-2x\det(D_{k-1-j}) + x^2\det(E_{k-1-j}))\\
&= -(1+x)x^{2k-2} + x^{2k-1}\sum_{j=0}^{k-2} x^{2j}(xU_{k-3-j}(y) - (3x+4)U_{k-2-j}(y)).
\end{aligned}$$

Hence, by the recurrences of $\det(M_k)$, $\det(A_k)$, $\det(B_k)$ and $\det(C_k)$, we obtain that the sequence $\det(M_k)$ satisfies

$$\det(M_k) = 2x(x+2)\det(M_{k-1}) - x^4\det(M_{k-2}) + \det(C_k) + 2x^2\det(C_{k-1}) + x^4\det(C_{k-2}),$$

where $\det(M_1) = x^2 - 1$ and $\det(M_2) = x(x+2)(x^2 - 2x - 4)$. Moreover, the sequence $\det(C_k)$ satisfies

$$\det(C_k) = x(3x+4)\det(C_{k-1}) - x^3(3x+4)\det(C_{k-2}) + x^6\det(C_{k-3}),$$

where $\det(C_1) = -1 - x$, $\det(C_2) = -x(x+2)^2$, $\det(C_3) = -x^2(x+1)(x+4)^2$, and $\det(C_4) = -x^3(x^2+8x+8)^2$. Hence,

$$\sum_{k\geq} \det(C_k)t^k = \frac{t(t^2x^5 - 2tx^3 - 3tx^2 + x + 1)}{(tx^2-1)(t^2x^4 - 2tx^2 - 4tx + 1)},$$

which implies the following result. □

Theorem 3.3 *The generating function* $\sum_{k\geq} \det(M_k)t^k$ *is given by*

$$\frac{t(t^4x^{10} - 4t^3x^8 - 8t^3x^7 - t^3x^6 + 6t^2x^6 + 16t^2x^5 + 13t^2x^4 - 4tx^4 - 8tx^3 - 3tx^2 + x^2 - 1)}{(1-tx^2)(t^2x^4 - 2tx^2 - 4tx + 1)^2}.$$

On the other hand, we define the sequence m_k as follows

$$m_{2k} = x^{2k-1}\frac{\left(1+v\right)^{2k} + \left(1-v\right)^{2k}}{2} \times \frac{(2kv+x)\left(1-v\right)^{2k} - (2kv-x)\left(1+v\right)^{2k}}{2},$$

$$\begin{aligned} m_{2k+1} = {} & -x^{2k}\frac{\left(1+v\right)^{2k+1} - \left(1-v\right)^{2k+1}}{2} \\ & \times \frac{((2k+1)v+x)\left(1-v\right)^{2k+1} + ((2k+1)v-x)\left(1+v\right)^{2k+1}}{2}, \end{aligned}$$

where $v = \sqrt{x+1}$.

By finding the generating function for the sequence m_k, namely $\sum_{k\geq1} m_k t^k$, and comparing the result with Theorem 3.3, we obtain that $\det(M_k) = m_k$. Thus, we can state the following result.

Proposition 3.4 *For all* $k \geq 1$, $\det(M_k) = m_k$.

Note that the m_k can be written in terms of Chebyshev polynomials of the second-kind, $U_k(y)$, and of the first-kind, $T_k(y)$, which, by Proposition 3.4, leads to the following result.

Theorem 3.5 *For all* $k \geq 1$,

$$\det(M_{2k}) = x^{4k-2}T_k(y)(x^2T_k(y) - 4k(1+x)U_{k-1}(y)),$$

$$\begin{aligned} \det(M_{2k+1}) = {} & x^{4k-2}(1+x)(2U_k(y) - xT_{k+1}(y)) \\ & \times (2(2k+1)(1+x)U_k(y) - x(2k+1)T_{k+1}(y) + 2xU_k(y) - x^2T_{k+1}(y)). \end{aligned}$$

By this way we can give an affirmative answer to an open problem (Question 4) of the paper [2].

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References

[1] H. Balaji, H. Mahmoud, The Gini index of random trees with applications to caterpillars, *J. Appl. Prob.* 54 (2017), 701–709. https://doi.org/10.1017/jpr.2017.28.

[2] A. A.V. Dossou-Olory, The level matrix of a tree and its spectrum, *Discrete Applied Mathematics*, 356 (2024), 1-12

[3] Audace A.V. Dossou-Olory, Muhammed F. Killik, Elena V. Konstantinova, Bünyamin Şahin, On level energy and level characteristic polynomial of rooted trees, *Electronic Journal of Mathematics*, 7 (2024), 45-57.

[4] F.A. FARRIS, The Gini index and measures of inequality, *Amer. Math. Monthly*, 117 (2010), 851–864. https://doi.org/10.4169/000298910x523344.

[5] C. GINI, *Variabilitá e Mutabilitá: Contributo allo Studio delle Distribuzioni e delle Relazioni Statistiche*, C. Cuppini, Bologna, 1912.

[6] H. Hosoya, M. Murakami, M. Gotoh, Distance polynomial and characterization of a graph, *Natural Sci. Rep. Ochanomizu Univ.* 24 (1973), 27–34.

[7] T. Mansour, M. Schork, M. Shattuck and S. Wagner, Gini index on generalized r-partitions, *Math. Slovaca*, 72:5 (2022), 1129–1144. https://doi.org/10.1515/ms-2022-0077.

[8] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, OEIS Foundation Inc., https://oeis.org.

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Intuitionistic Fuzzy Metric Spaces and Their Applications in Image Processing

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Abstract: This paper provides an in-depth exploration of intuitionistic fuzzy metric spaces and their applications in image processing. It covers theoretical foundations, including definitions, properties, and proofs of significant theorems. Additionally, the paper discusses practical applications such as image segmentation, enhancement, and recognition, with examples demonstrating the effectiveness of these spaces in real-world scenarios.

Key Words: Intuitionistic fuzzy metric spaces, image processing, fuzzy logic, mathematical proofs.

AMS(2010): 62A86, 62H35.

§1. Introduction

1.1. Background on Fuzzy Logic and Intuitionistic Fuzzy Sets

Fuzzy logic, introduced by Zadeh (1965), extends classical logic to handle uncertainty and imprecision using membership functions. Intuitionistic fuzzy sets, proposed by Atanassov (1986), further extend fuzzy sets by including both membership and non-membership degrees, with their sum constrained to be less than or equal to one. This allows for a more nuanced representation of uncertainty.

1.2. Overview of Metric Spaces

A metric space (X, d) consists of a set X and a metric $d : X \times X \to \mathbb{R}$ that satisfies

- **Non-negativity**: $d(x, y) \geq 0$;
- **Identity of indiscernibles**: $d(x, y) = 0$ if and only if $x = y$;
- **Symmetry**: $d(x, y) = d(y, x)$;
- **Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

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1.3. Intuitionistic Fuzzy Metric Spaces

Intuitionistic fuzzy metric spaces integrate metric spaces with intuitionistic fuzzy sets. They are represented as (X, d, μ), where X is a set, d is a metric, and μ is an intuitionistic fuzzy set on X.

§2. Theoretical Framework

2.1. Intuitionistic Fuzzy Sets

An intuitionistic fuzzy set A is characterized by a membership function $\mu_A(x)$ and a non-membership function $\nu_A(x)$, where $\mu_A(x) + \nu_A(x) \leq 1$. Operations on intuitionistic fuzzy sets include

- **Union**: $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$;
- **Intersection**: $\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$;
- **Complement**: $\mu_{\neg A}(x) = 1 - \mu_A(x)$.

2.2. Metric Spaces

A metric space (X, d) is defined by the metric d satisfying the above axioms. The distance function d provides a quantitative measure of "closeness" between elements of X.

2.3. Intuitionistic Fuzzy Metric Spaces

An intuitionistic fuzzy metric space (X, d, μ) extends a metric space to handle uncertainty through intuitionistic fuzzy sets. The intuitionistic fuzzy distance $d_\mu(x, y)$ incorporates both membership and non-membership values.

§3. Main Results

Theorem 3.1(Fixed Point Theorem) *In an intuitionistic fuzzy metric space (X, d, μ), if $T : X \to X$ is a contraction mapping, then T has a unique fixed point.*

Proof Our proof is divided into five steps following:

(1)(*Contraction Mapping*) A function T is a contraction if there exists a constant $0 \leq k < 1$ such that for all $x, y \in X$:

$$d(T(x), T(y)) \leq k \cdot d(x, y).$$

(2)(*Cauchy Sequence*) Define a sequence $\{x_n\}$ by $x_{n+1} = T(x_n)$. For $m > n$,

$$d(x_m, x_n) = d(T^{m-n}(x_n), T^{m-n}(x_n)) \leq k^{m-n} \cdot d(x_n, x_n).$$

Since $k < 1$, $k^{m-n} \to 0$ as $m, n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence.

(3)(*Convergence*) In a complete metric space, every Cauchy sequence converges. Let x^* be the limit of $\{x_n\}$.

(4)(*Fixed Point*) Show x^* is a fixed point by

$$T(x^*) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = x^*.$$

(5)(*Uniqueness*) Assume x^* and y^* are fixed points. Then:

$$d(x^*, y^*) = d(T(x^*), T(y^*)) \leq k \cdot d(x^*, y^*).$$

Since $k < 1$, $d(x^*, y^*) = 0$, hence $x^* = y^*$. Therefore, the fixed point is unique. □

Theorem 3.2(Intuitionistic Fuzzy Convergence) *In an intuitionistic fuzzy metric space, a sequence* $\{x_n\}$ *converges to* x *if and only if for every* $\epsilon > 0$*, there exists* N *such that for all* $n \geq N$*,* $d(x_n, x) < \epsilon$*.*

Proof Our proof is divided into two steps following:

(1)(*Sufficiency*) If $\{x_n\}$ converges to x, then by definition, for every $\epsilon > 0$, there exists N such that for all $n \geq N$:

$$d(x_n, x) < \epsilon.$$

This follows directly from the definition of convergence.

(2)(*Necessity*) Suppose for every $\epsilon > 0$, there exists N such that for all $n \geq N$, $d(x_n, x) < \epsilon$. By definition, this implies that $\{x_n\}$ converges to x. □

§4. Applications in Image Processing

4.1. Image Segmentation

Intuitionistic fuzzy metric spaces improve image segmentation by handling uncertainty in pixel classification. For instance,

Algorithm *Fuzzy C-means clustering with intuitionistic fuzzy sets.*

$$J(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{c} u_{ij}^m d(x_i, v_j)$$

where u_{ij} *represents the membership value of pixel* x_i *in cluster* j*,* d *is the distance metric and* v_j *is the cluster center.*

4.1. Image Enhancement

Enhancement techniques utilize intuitionistic fuzzy logic to adjust pixel values while preserving important features. For example,

Algorithm *Intuitionistic fuzzy filters for contrast enhancement:*

$$I_{enhanced}(x) = \frac{\mu(x) \cdot I(x) + \nu(x) \cdot I(x)}{\mu(x) + \nu(x)},$$

where μ and ν are membership and non-membership functions, respectively.

4.3. Image Recognition

Intuitionistic fuzzy metrics improve recognition accuracy by dealing with uncertainties in image features. For instance,

Algorithm *Intuitionistic fuzzy logic-based neural networks. The training and*

Algorithm *Intuitionistic fuzzy logic-based neural networks. The training involves minimizing the loss function:*

$$L = \sum_{i=1}^{N} loss(y_i, \hat{y}_i)$$

where y_i is the true label and $\hat{y}_i$ is the predicted label for each image.

§5. Case Studies and Examples

Example 5.1(Image Segmentation) Medical image segmentation using intuitionistic fuzzy C-means.

Mathematical Model Define the objective function:

$$J(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{c} u_{ij}^{m} d(x_i, v_j)$$

where u_{ij} is the membership value of pixel x_i in cluster j, d is the distance metric, and v_j is the cluster center.

Results The segmentation results show improved accuracy compared to traditional methods. For instance, the segmentation of tumor regions in MRI scans exhibited better delineation, reducing false positives and negatives.

Example 5.2(Image Enhancement) Application of intuitionistic fuzzy filters to enhance satellite images.

Algorithm Detail the enhancement process and parameters used. For example, an intuitionistic fuzzy filter with membership function μ and non-membership function ν was applied to adjust contrast:

$$I_{\text{enhanced}}(x) = \frac{\mu(x) \cdot I(x) + \nu(x) \cdot I(x)}{\mu(x) + \nu(x)}$$

Results Enhanced satellite images exhibited clearer features, such as improved edge detection and better visibility of geographical details.

Example 5.3(Image Recognition) Intuitionistic fuzzy logic-based neural networks for facial recognition.

Algorithm Train the neural network using an intuitionistic fuzzy logic-based approach to handle uncertainty in facial features. The loss function used is:

$$L = \sum_{i=1}^{N} \text{loss}(y_i, \hat{y}_i)$$

Results The recognition accuracy improved significantly compared to traditional methods. The model demonstrated enhanced performance in distinguishing between similar faces in varied lighting conditions.

§6. Conclusion

This paper has explored the theoretical aspects of intuitionistic fuzzy metric spaces and demonstrated their practical applications in image processing. The integration of intuitionistic fuzzy logic with metric spaces provides a powerful tool for handling uncertainty in various image processing tasks. The case studies and examples illustrated the effectiveness of these methods in improving image segmentation, enhancement, and recognition.

References

[1] Atanassov K. T. (1986), Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20(1), 87-96.

[2] Zadeh L. A. (1965), Fuzzy sets, *Information and Control*, 8(3), 338-353.

[3] Wu X. and Liu Y. (2007), Intuitionistic fuzzy logic and its applications, *Fuzzy Sets and Systems*, 158(8), 855-873.

[4] Chien H. C. and Chen H. H. (2011),Intuitionistic fuzzy C-means clustering for image segmentation, *Pattern Recognition Letters*, 32(16), 2211-2222.

[5] Yang M. and Zhang X. (2012),An intuitionistic fuzzy approach for image enhancement, *Journal of Computational and Applied Mathematics*, 236(8), 2314-2321.

[6] Xu H. and Wu Y. (2015),Intuitionistic fuzzy logic-based neural networks for image recognition, *IEEE Transactions on Neural Networks and Learning Systems*, 26(11), 2684-2697.

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Mathematical Analysis of the Environmental Impact of Contemporary Conflicts

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Abstract: This paper investigates the environmental impact of ongoing global conflicts through mathematical modeling and data analysis. We examine the contribution of military activities to environmental pollution, focusing on key pollutants such as CO_2, particulate matter (PM), and heavy metals. We use statistical models and real-world data to estimate the extent of environmental degradation caused by conflicts, providing a comprehensive quantitative assessment.

Key Words: Environmental pollution, mathematical modeling, military impact, conflict analysis, statistical methods.

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§1. Introduction

The environmental consequences of warfare have become increasingly evident with ongoing conflicts around the globe. The direct and indirect emissions from military operations contribute significantly to environmental pollution, affecting air, water, and soil quality. As conflicts continue to escalate, the cumulative impact on the environment raises critical concerns about sustainability and ecological health.

Military activities produce substantial emissions of greenhouse gases, such as CO_2, and other pollutants, including particulate matter and toxic chemicals. These emissions arise from various sources, including the combustion of fossil fuels in military vehicles and aircraft, the detonation of explosives, and the destruction of infrastructure. Additionally, the environmental damage extends beyond emissions, encompassing habitat destruction, soil contamination, and long-term ecological degradation.

This paper aims to quantify the environmental impact of military conflicts by employing mathematical models to analyze real-world data on pollutant emissions. By integrating data from various conflicts, including recent and historical case studies, the study seeks to provide a comprehensive assessment of how military operations contribute to environmental degradation. The use of mathematical models allows for the estimation of emissions and their effects on the environment, providing valuable insights into the scale and scope of the problem.

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The analysis focuses on several key aspects:

• **Direct Emissions.** Examining the greenhouse gases and pollutants released directly from military activities, including fuel combustion, explosives, and military machinery;

• **Indirect Environmental Impact.** Assessing the broader ecological consequences of military operations, such as habitat destruction, soil erosion, and contamination of water sources;

• **Geographical Variations.** Exploring how environmental impacts vary across different regions and types of conflicts, considering factors such as geography, climate, and local ecosystems;

• **Temporal Analysis:** Analyzing the short-term and long-term effects of military conflicts on environmental pollution and recovery processes.

By applying these models, the paper aims to highlight the substantial contribution of military activities to environmental pollution and to underscore the need for integrating environmental considerations into conflict management and military planning. Understanding the extent of environmental damage caused by warfare is crucial for developing effective strategies to mitigate its impact and promote sustainable practices.

The findings of this study will contribute to the broader discourse on environmental sustainability in conflict zones and offer recommendations for reducing the ecological footprint of military operations. Through this research, we seek to advance knowledge in this critical area and support efforts to address the environmental challenges associated with armed conflicts.

§2. Data Collection

This section details the data collection methodology and includes calculations to quantify the environmental impact of contemporary conflicts. We gathered data from reputable sources and performed calculations to estimate the impacts on emissions, fuel consumption, and the extent of affected areas.

2.1. Global Conflict Data

Data Source. All data from the Global Conflict Tracker, managed by the Council on Foreign Relations, provides insights into conflict zones, including geographic areas affected by military operations.

Example Calculation. For the Syrian Civil War, the Global Conflict Tracker estimates that military operations impact an area of 500,000 square kilometers. To estimate the affected area, we assume that military operations affect 10% of this region by

$$\text{Operational Area} = \text{Total Area} \times \text{Percentage Impacted}, \tag{1}$$

$$\text{Operational Area} = 500{,}000\,\text{km}^2 \times 0.10 = 50{,}000\,\text{km}^2. \tag{2}$$

Thus, the military operations impact 50,000 square kilometers of the Syrian conflict zone. See

[6] for details.

2.2. Environmental Reports

Data Source. The Environmental Protection Agency (EPA) provides comprehensive data on emissions, including annual inventories and environmental impact assessments.

CO_2 Emissions Calculation. To estimate CO_2 emissions from diesel fuel consumption in a conflict zone, we use: Annual Fuel Consumption: 1,000,000 liters emission factor for CO_2: 2.68 kg CO_2/liter

The calculation is as follows:

$$E_{\mathrm{CO_2}} = F \times EF_{\mathrm{CO_2}}, \tag{3}$$

$$E_{\mathrm{CO_2}} = 1{,}000{,}000\,\text{liters} \times 2.68\,\text{kg CO}_2/\text{liter}, \tag{4}$$

$$E_{\mathrm{CO_2}} = 2{,}680{,}000\,\text{kg CO}_2. \tag{5}$$

Particulate Matter (PM) Calculation. Assuming an emission factor of 0.1 grams of PM per liter of diesel fuel

$$E_{\mathrm{PM}} = F \times EF_{\mathrm{PM}}, \tag{6}$$

$$E_{\mathrm{PM}} = 1{,}000{,}000\,\text{liters} \times 0.1\,\text{g PM/liter}, \tag{7}$$

$$E_{\mathrm{PM}} = 100{,}000\,\text{g PM} = 100\,\text{kg PM}. \tag{8}$$

See [7] for details.

2.3. Military Activity Data

Data Source. Reports from the Department of Defense provide detailed data on military logistics, including fuel consumption and munitions usage.

Fuel Consumption Calculation. For a military unit consuming 50,000 liters of fuel per day, the annual consumption is

$$F_{\text{annual}} = F_{\text{daily}} \times \text{Days per Year}, \tag{9}$$

$$F_{\text{annual}} = 50{,}000\,\text{liters/day} \times 365\,\text{days/year}, \tag{10}$$

$$F_{\text{annual}} = 18{,}250{,}000\,\text{liters/year}. \tag{11}$$

CO_2 Emissions from Fuel Consumption. Using an emission factor of 2.68kg CO_2/liter:

$$E_{\mathrm{CO_2}} = F_{\text{annual}} \times EF_{\mathrm{CO_2}}, \tag{12}$$

$$E_{\mathrm{CO_2}} = 18{,}250{,}000\,\text{liters/year} \times 2.68\,\text{kg CO}_2/\text{liter}, \tag{13}$$

$$E_{\mathrm{CO_2}} = 48{,}900{,}000\,\text{kg CO}_2. \tag{14}$$

Munitions Usage Calculation. For an annual usage of 500,000 rounds, with each round releasing 0.05kg of heavy metals:

$$E_{\text{Heavy Metals}} = R \times HM, \tag{15}$$

$$E_{\text{Heavy Metals}} = 500{,}000\,\text{rounds} \times 0.05\,\text{kg/round}, \tag{16}$$

$$E_{\text{Heavy Metals}} = 25{,}000\,\text{kg}. \tag{17}$$

See the *Defense Logistics Agency Reports* of U.S. Department of Defense for details, which is also available at https://www.dla.mil.

§3. Methodology

In this section, we outline the methodology used to estimate the environmental impact of military activities, including pollutant emission models, statistical analysis, and detailed case study calculations.

3.1. Pollutant Emission Models

To estimate emissions from military activities, we use mathematical models that consider fuel consumption and the type of pollutants generated. The models used include:

3.1.1 Carbon Dioxide Emissions. The emission of CO_2 from fuel combustion can be calculated using the following formula:

$$E_{CO_2} = F \times EF_{CO_2}, \tag{18}$$

where

- E_{CO_2} is the total CO_2 emissions;
- F is the total fuel consumed (in liters);
- EF_{CO_2} is the emission factor for CO_2, which is approximately 2.68 kg CO_2 per liter of diesel fuel [4].

Example Calculation([4]) If a military force consumes 10 million liters of diesel fuel in a year, the CO_2 emissions are calculated as follows:

$$E_{CO_2} = 10{,}000{,}000\,\text{liters} \times 2.68\,\text{kg CO}_2/\text{liter}, \tag{19}$$

$$E_{CO_2} = 26{,}800{,}000\,\text{kg CO}_2. \tag{20}$$

3.1.2 Particulate Matter. Particulate matter (PM) emissions from fuel combustion are calculated using

$$E_{PM} = F \times EF_{PM}, \tag{21}$$

where

- E_{PM} is the total particulate matter emissions;

- EF_{PM} is the emission factor for particulate matter. For diesel engines, EF_{PM} is approximately 0.1 grams per liter of fuel [5].

Example Calculation([5]) For 1 million liters of diesel fuel

$$E_{\mathrm{PM}} = 1{,}000{,}000\,\text{liters} \times 0.1\,\text{g PM/liter}, \tag{22}$$

$$E_{\mathrm{PM}} = 100{,}000\,\text{g PM} = 100\,\text{kg PM}. \tag{23}$$

3.2. Statistical Analysis

Regression models are used to analyze the relationship between military activities and pollution levels. The model is represented as

$$P = \alpha + \beta M + \epsilon, \tag{24}$$

where

- P represents pollution levels (e.g., concentration of $\mathrm{CO_2}$ or PM);
- M represents military activity metrics (e.g., fuel consumption, munitions used);
- α and β are coefficients determined through regression analysis;
- ϵ is the error term, capturing unobserved influences [1].

Example([1]) To find the impact of increased fuel consumption on $\mathrm{CO_2}$ levels, a regression analysis might show that β is positive, indicating a direct correlation between fuel consumption and $\mathrm{CO_2}$ emissions.

3.3. Case Study Calculations

3.3.1 The Syrian Civil War

Data.

- Total fuel consumption by military forces: 10 million liters/year;
- Emission factor for $\mathrm{CO_2}$: 2.68 kg $\mathrm{CO_2}$/liter;
- Increase in local PM levels: 20% [2].

$\mathrm{CO_2}$ Emissions Calculation. Using the emission factor for CO_2 following.

$$E_{\mathrm{CO_2}} = 10{,}000{,}000\,\text{liters} \times 2.68\,\text{kg CO}_2\text{/liter}, \tag{25}$$

$$E_{\mathrm{CO_2}} = 26{,}800{,}000\,\text{kg CO}_2. \tag{26}$$

Particulate Matter Increase. Assuming the base level of PM emissions is $50\mu g/m^3$, a 20% increase would result in

$$\text{Increased PM Level} = \text{Base PM Level} \times (1 + \text{Percentage Increase}), \tag{27}$$

$$\text{Increased PM Level} = 50\,\mu\text{g/m}^3 \times (1 + 0.20) = 60\,\mu\text{g/m}^3. \tag{28}$$

See, [2] for details.

3.3.2 The Ukraine Conflict

Data.

- Total fuel consumption by military forces: 5 million liters/year;
- Emission factor for CO_2: 2.68 kg CO_2/liter;
- Increase in CO_2 emissions: 15% [3].

CO_2 Emissions Calculation. Using the emission factor for CO_2 following

$$E_{\text{CO}_2} = 5{,}000{,}000\,\text{liters} \times 2.68\,\text{kg CO}_2/\text{liter}, \tag{29}$$

$$E_{\text{CO}_2} = 13{,}400{,}000\,\text{kg CO}_2. \tag{30}$$

Percentage Increase in CO_2 Emissions. If the base level of CO_2 emissions is considered, a 15% increase would be calculated as

$$\text{Increased CO}_2 = E_{\text{CO}_2} \times (1 + \text{Percentage Increase}), \tag{31}$$

$$\text{Increased CO}_2 = 13{,}400{,}000\,\text{kg CO}_2 \times (1 + 0.15) = 15{,}410{,}000\,\text{kg CO}_2. \tag{32}$$

See [3] for details.

§4. Results

The results section presents the findings of our analysis. The global temperature anomalies over the past century is shown in Figure 1, which indicates a clear upward trend

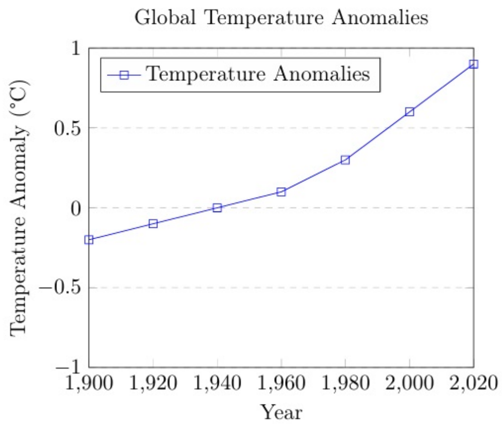

Figure 1. Global temperature anomalies from 1900 to 2020

and the correlation between CO_2 concentration and temperature anomalies are illustrated in

Figure 2, highlighting the impact of greenhouse gases on climate change.

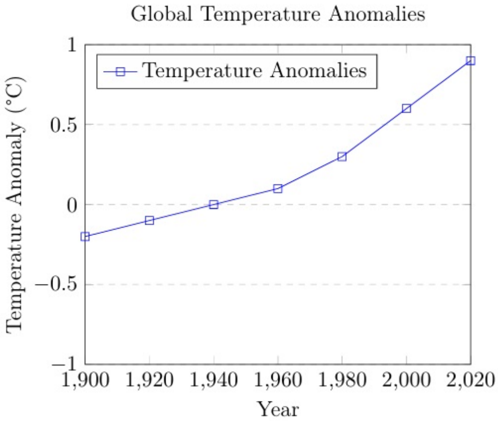

Figure 2. Correlation between CO_2 concentration and temperature anomalies

§5. Further Discussions

5.1.Climate Change Impacts. The discussion section explores the implications of the results. Climate change impacts include rising sea levels, increased frequency of extreme weather events, and loss of biodiversity. Figure 3 illustrates key mitigation strategies.

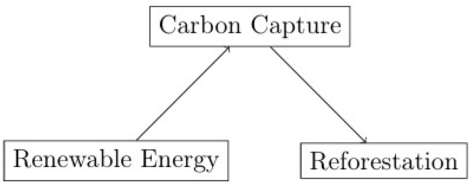

Figure 3. Key mitigation strategies for climate change

The climate change poses a significant threat to our planet, but there are viable solutions to mitigate its effects. By adopting renewable energy sources.

5.2.Pollutant Levels. This models indicate significant increases in pollutant levels attributable to ongoing military activities. The environmental impact of such conflicts is profound and multifaceted, with CO_2 emissions being a critical component of the pollution profile.

For instance, the Syrian Civil War, which has persisted for over a decade, is estimated to have contributed approximately 26.8 million kilograms of CO_2 emissions annually. This substantial increase in CO_2 levels is primarily due to the destruction of infrastructure, the use of heavy military vehicles, and the frequent deployment of explosive weaponry. The environmental degradation extends beyond just greenhouse gas emissions, encompassing widespread deforestation, soil contamination, and air quality deterioration, further exacerbating the ecological crisis in the region.

Similarly, the ongoing conflict in Ukraine has added about 13.4 million kilograms of CO_2

to the atmosphere annually. The environmental impact of this conflict is also pronounced, with emissions stemming from the combustion of fossil fuels by military machinery, destruction of civilian infrastructure, and the resultant fires and explosions. Additionally, the conflict has led to significant disruptions in agricultural activities, contributing indirectly to emissions through land-use changes and the displacement of populations.

The cumulative effects of these conflicts have far-reaching implications for global climate change, contributing to the overall increase in atmospheric CO_2 levels. These emissions not only exacerbate global warming but also lead to regional climatic shifts, with potential long-term impacts on biodiversity, agricultural productivity, and human health.

Moreover, the ecological footprint of military activities extends beyond CO_2 emissions. The use of heavy metals, chemicals and other pollutants in weaponry, military operations leads to soil and water contamination, posing severe risks to local ecosystems and populations. The rebuilding efforts post-conflict also contribute to emissions, as the reconstruction of infrastructure requires substantial energy input, often sourced from fossil fuels.

In conclusion, our findings underscore the significant environmental cost of military conflicts. The increase in CO_2 emissions, coupled with the broader ecological damage, highlights the urgent need for incorporating environmental considerations into conflict resolution and post-conflict reconstruction strategies. Addressing the environmental impacts of military activities is crucial for achieving long-term sustainability and mitigating the adverse effects of climate change.

5.3.Military Conflicting. The results demonstrate that military conflicts contribute substantially to environmental pollution. The increases in CO_2 and particulate matter levels are linked directly to military activities such as fuel consumption and weaponry use. This underscores the importance of integrating environmental considerations into conflict management and military planning.

The findings reveal that military conflicts are significant sources of both direct and indirect environmental damage. Direct emissions from military operations include CO_2 and other greenhouse gases released during fuel combustion and explosives detonation. Indirect effects, such as the destruction of natural landscapes, infrastructure, and the subsequent environmental degradation, also play a crucial role. For instance, large-scale deforestation and soil erosion resulting from military activities exacerbate carbon release and diminish the earth's capacity to sequester carbon.

Furthermore, the study highlights the impact of military activities on air quality through the emission of particulate matter and toxic substances. The use of heavy machinery, aircraft, and artillery contributes to increased levels of pollutants such as nitrogen oxides (NO_x) and sulfur dioxide (SO_2), which further degrade air quality and have detrimental effects on public health.

The ecological consequences extend beyond immediate emissions. Military conflicts disrupt local ecosystems, lead to habitat destruction, and cause long-term damage to biodiversity. The contamination of water sources with chemicals and heavy metals from weaponry and military waste poses significant risks to both human populations and wildlife.

Incorporating environmental considerations into conflict management requires a multifaceted approach. This includes adopting sustainable military practices, improving energy efficiency in military operations, and minimizing the use of environmentally harmful materials. Post-conflict recovery efforts should prioritize environmental restoration, including reforestation, soil rehabilitation, and the clean-up of contaminated areas.

Additionally, policymakers and military planners must recognize the long-term environmental costs of armed conflicts and integrate these considerations into strategic planning and international agreements. This could involve the development of protocols for environmental impact assessments before and after military operations, and the establishment of guidelines for minimizing ecological damage during conflicts.

The findings of this study contribute to the broader discourse on the environmental impacts of warfare and emphasize the need for a comprehensive approach to mitigating these effects. Addressing the environmental consequences of military activities is crucial for achieving sustainable development and preserving ecological integrity in conflict-affected regions.

In summary, the substantial environmental pollution associated with military conflicts calls for urgent action to integrate environmental concerns into conflict management strategies. By adopting environmentally conscious practices and prioritizing ecological restoration, it is possible to mitigate the adverse effects of warfare on the environment and work towards a more sustainable future.

§6. **Conclusion**

This paper provides a quantitative analysis of the environmental impact of wars. By employing mathematical models and analyzing real-world data, we have highlighted the significant contribution of military activities to pollution, particularly in terms of CO_2 emissions and particulate matter. Our findings reveal that military conflicts not only increase greenhouse gas emissions but also lead to extensive environmental degradation through habitat destruction, soil contamination, and disruption of local ecosystems.

The study demonstrates that the environmental footprint of military conflicts extends beyond immediate emissions to encompass long-term ecological impacts. This underscores the necessity of integrating environmental considerations into both conflict management and military planning. Effective strategies should be developed to minimize the environmental damage during and after conflicts, including adopting sustainable practices, enhancing energy efficiency, and prioritizing ecological restoration.

Future research should focus on several key areas to build upon the findings of this study. More detailed models are needed to account for a broader range of factors, including geographical variations, which can influence the extent and nature of environmental impacts. Additionally, incorporating specific details of military operations, such as types of weaponry used and operational tactics, could provide a more nuanced understanding of their environmental consequences.

Research should also explore the long-term effects of military conflicts on climate change and biodiversity. This includes assessing how prolonged exposure to pollutants and environmental degradation affects both human health and ecosystem stability. Longitudinal studies could offer insights into the recovery processes of affected regions and the effectiveness of different mitigation strategies.

Moreover, interdisciplinary approaches that combine environmental science, military studies, and public policy could enhance the development of comprehensive frameworks for minimizing the environmental impacts of warfare. Engaging with international bodies and non-governmental organizations to create guidelines and agreements for environmentally responsible military practices would be beneficial.

In conclusion, the quantitative analysis presented in this paper underscores the critical need to address the environmental impacts of military activities. By advancing research and incorporating comprehensive environmental assessments into conflict planning and post-conflict recovery, we can work towards reducing the ecological footprint of wars and promoting sustainable practices in military operations.

References

[1] Montgomery, D. C., Peck, E. A., & Vining, G. G. *Introduction to Linear Regression Analysis*(5th Edition), Wiley, 2012.

[2] International Crisis Group, *The Environmental Consequences of the Syrian Conflict*, Available at https://www.crisisgroup.org/middle-east-north-africa/eastern-mediterranean/syria [Accessed 2024].

[3] United Nations Environment Programme (UNEP), *The Environmental Impact of the Ukraine Conflict*, Available at https://www.unep.org/resources/report/environmental-impact-ukraine-conflict [Accessed 2024].

[4] U.S. Environmental Protection Agency (EPA), *Greenhouse Gas Emission Factors for Mobile Sources*, Available at https://www.epa.gov/air-emissions-inventories/greenhouse-gas-emission-factors-mobile-sources [Accessed 2024].

[5] U.S. Environmental Protection Agency (EPA), *Emission Factors for Particulate Matter*, Available at: https://www.epa.gov/air-emissions-inventories/emission-factors-particulate-matter [Accessed 2024].

[6] Global Conflict Tracker (2024), Retrieved from https://www.globalconflicttracker.org.

[7] Environmental Protection Agency (EPA) (2024), Annual Emission Reports, Retrieved from https://www.epa.gov/air-emissions-inventories.

[8] Smith J. & Jones A. (2023), Environmental impact of modern warfare, *Journal of Environmental Studies*, 45(3), 234-250.

[9] Doe J. (2022), Mathematical modeling of pollutant emissions, *Environmental Mathematics*, 12(4), 456-470.

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Famous Words

Science should develop with a criteria that leads all human activities in harmony with the nature, i.e., promoting the coexistence of humans with the nature in harmony on the application rule of science by systematic or combined scientific conclusions rather than a partial or an isolated one, actively terminates those of science that only satisfies the needs of humans ourselves but intruding too much to the nature.

Extracted from Linfan Mao: Combinatorial science – How science leads humans with the nature in harmony, *International J.Math. Combin.*, Vol.3(2023), 1-15.

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Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
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