Independent Complementary Distance Pattern Uniform Graphs

Germina K.A. and Beena Koshy

PG and Research Department of Mathematics of Kannur University
Mary Matha Arts & Science College, Vemom P.O., Mananthavady - 670645, India

E-mail: srgerminaka@gmail.com, beenakoshy1@yahoo.co.in

Abstract: A graph \( G = (V, E) \) is called to be Smarandachely uniform \( k \)-graph for an integer \( k \geq 1 \) if there exists \( M_1, M_2, \cdots, M_k \subset V(G) \) such that \( f_{M_i}(u) = \{d(u, v) : v \in M_i\} \) for \( \forall u \in V(G) - M_i \) is independent of the choice of \( u \in V(G) - M_i \) and integer \( i, 1 \leq i \leq k \). Each such set \( M_i, 1 \leq i \leq k \) is called a CDPU set [6, 7]. Particularly, for \( k = 1 \), a Smarandachely uniform 1-graph is abbreviated to a complementary distance pattern uniform graph, i.e., CDPU graphs. This paper studies independent CDPU graphs.

Key Words: Smarandachely uniform \( k \)-graph, complementary distance pattern uniform, independent CDPU.

AMS(2000): 05C22.

§1. Introduction

For all terminology and notation in graph theory, not defined specifically in this paper, we refer the reader to Harary [4]. Unless mentioned otherwise, all the graphs considered in this paper are simple, self-loop-free and finite.

Let \( G = (V, E) \) represent the structure of a chemical molecule. Often, a topological index (TI), derived as an invariant of \( G \), is used to represent a chemical property of the molecule. There are a number of TIs based on distance concepts in graphs [5] and some of them could be designed using distance patterns of vertices in a graph. There are strong indications in the literature cited above that the notion of CDPU sets in \( G \) could be used to design a class of TIs that represent certain stereochemical properties of the molecule.

Definition 1.1([6]) Let \( G = (V, E) \) be a \( (p, q) \) graph and \( M \) be any non-empty subset of \( V(G) \). Each vertex \( u \) in \( G \) is associated with the set \( f_M(u) = \{d(u, v) : v \in M\} \), where \( d(u, v) \) denotes the usual distance between \( u \) and \( v \) in \( G \), called the \( M \)-distance pattern of \( u \).

\(^1\text{Received Oct.9, 2009. Accepted Nov. 24, 2009.}\)
A graph $G = (V, E)$ is called to be Smarandachely uniform $k$-graph for an integer $k \geq 1$ if there exists $M_1, M_2, \ldots, M_k \subset V(G)$ such that $f_{M_i}(u) = \{d(u, v) : v \in M_i\}$ for $\forall u \in V(G) - M_i$ is independent of the choice of $u \in V(G) - M_i$ and integer $i$, $1 \leq i \leq k$. Each such set $M_i$, $1 \leq i \leq k$ is called a CDPU set. Particularly, for $k = 1$, a Smarandachely uniform 1-graph is abbreviated to a complementary distance pattern uniform graph, i.e., CDPU graphs. The least cardinality of the CDPU set is called the CDPU number denoted by $\sigma(G)$.

The following are some of the results used in this paper.

**Theorem 1.2([7])** Every connected graph has a CDPU set.

**Definition 1.3([7])** The least cardinality of CDPU set in $G$ is called the CDPU number of $G$, denoted $\sigma(G)$.

**Remark 1.4([7])** Let $G$ be a connected graph of order $p$ and let $(e_1, e_2, \ldots, e_k)$ be the non decreasing sequence of eccentricities of its vertices. Let $M$ consists of the vertices with eccentricities $e_1, e_2, \ldots, e_k - 1$ and let $|V - M| = p - m$ where $|M| = m$. Then $\sigma(G) \leq m$, since all the vertices in $V - M$ have $f_M(v) = \{1, 2, \ldots, e_{k - 1}\}$.

**Theorem 1.5([7])** A graph $G$ has $\sigma(G) = 1$ if and only if $G$ has at least one vertex of full degree.

**Corollary 1.6([7])** For any positive integer $n$, $\sigma(G + K_m) = 1$.

**Theorem 1.7([7])** For any integer $n$, $\sigma(P_n) = n - 2$.

**Theorem 1.8([7])** For all integers $a_1 \geq a_2 \geq \cdots \geq a_n \geq 2$, $\sigma(K_{a_1, a_2, \ldots, a_n}) = n$.

**Theorem 1.9([7])** $\sigma(C_n) = n - 2$, if $n$ is odd and $\sigma(C_n) = n/2$, if $n \geq 8$ is even. Also $\sigma(C_4) = \sigma(C_6) = 2$.

**Theorem 1.10([7])** If $\sigma(G_1) = k_1$ and $\sigma(G_2) = k_2$, then $\sigma(G_1 + G_2) = \min(k_1, k_2)$.

**Theorem 1.11([7])** Let $T$ be a CDPU tree. Then $\sigma(T) = 1$ if and only if $T$ is isomorphic to $P_2, P_3$ or $K_{1,n}$.

**Theorem 1.12([7])** The central subgraph of a maximal outerplanar graph has CDPU number 1 or 3.

**Remark 1.13([7])** For a graph $G$ which is not self centered, $\max f_M(v) = \text{diam}(G) - 1$.

**Theorem 1.14([7])** The shadow graph of a complete graph $K_n$ has exactly two $\sigma(K_n)$ disjoint CDPU sets.

The following were the problems identified by B. D. Acharya [6, 7].

**Problem 1.15** Characterize graphs $G$ in which every minimal CDPU-set is independent.

**Problem 1.16** What is the maximum cardinality of a minimal CDPU set in $G$. 
Problem 1.17 Determine whether every graph has an independent CDPU-set.

Problem 1.18 Characterize minimal CDPU-set.

Fig. 1 following depicts an independent CDPU graph.

![Independent CDPU graph](image)

Fig. 1: An independent CDPU graph with $M = \{v_2, v_4\}$

§2. Main Results

Definition 2.1 A graph $G$ is called an Independent CDPU graph if there exists an independent CDPU set for $G$.

Following two observations are immediate.

Observations 2.2 Complete graphs are independent CDPU.

Observations 2.3 Star graph $K_{1,n}$ is an Independent CDPU graph.

Proposition 2.4 $C_n$ with $n$ even is an Independent CDPU graph.

Proof Let $C_n$ be a cycle on $n$ vertices and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, where $n$ is even. Choose $M$ as the set of alternate vertices on $C_n$, say, $\{v_2, v_4, \ldots, v_n\}$. Then, $f_M(v_i) = \{1, 3, 5, \ldots, m - 1\}$ for $i = 1, 3, \ldots, n - 1$, if $C_n = 2m$ and $m$ is even and $f_M(v_i) = \{1, 3, 5, \ldots, m\}$, for $i = 1, 3, \ldots, n - 1$ if $C_n = 2m$ and $m$ odd. Therefore, $f_M(v_i)$ is identical depending on whether $m$ is odd or even. Hence, the alternate vertices $\{v_2, v_4, \ldots, v_n\}$ forms a CDPU set $M$. Also all the vertices in $M$ are non-adjacent. Hence $C_n, n$ even is an independent CDPU graph. \qed

Theorem 2.5 A cycle $C_n$ is an independent CDPU graph if and only if $n$ is even.

Proof Let $C_n$ be a cycle on $n$ vertices. Suppose $n$ is even. Then from Proposition 2.4, $C_n$ is an independent CDPU graph.

Conversely, suppose that $C_n$ is an independent CDPU graph. That is, there exist vertices in $M$ such that every pair of vertices are non adjacent. We have to prove that $n$ is even. Suppose $n$ is odd. Then from Theorem 1.9, $\sigma(C_n) = n - 2$, which implies that $|M| \geq n - 2$.\hfill \qed
But from \( n \) vertices, we cannot have \( n - 2 \) (or more) vertices which are non-adjacent. \( \square \)

**Theorem 2.6** A graph \( G \) which contains a full degree vertex is an independent CDPU.

**Proof** Let \( G \) be a graph which contains a full degree vertex \( v \). Then, from Theorem 1.5, \( G \) is CDPU with CDPU set \( M = \{ v \} \). Also \( M \) is independent. Therefore, \( G \) is an independent CDPU. \( \square \)

**Remark 2.7** If the CDPU number of a graph \( G \) is 1, then clearly \( G \) is independent CDPU.

**Theorem 2.8** A complete \( n \)-partite graph \( G \) is an independent CDPU graph for any \( n \).

**Proof** Let \( G = K_{a_1, a_2, \ldots, a_n} \) be a complete \( n \)-partite graph. Then, \( V(G) \) can be partitioned into \( n \) subsets \( V_1, V_2, \ldots, V_n \) where \( |V_1| = a_1, |V_2| = a_2, \ldots, |V_n| = a_n \). Take all the vertices from the partite set, say, \( V_i \) of \( K_{a_1, a_2, \ldots, a_n} \) to constitute the set \( M \). Since each element of a partite set is non-adjacent to the other vertices in it and is adjacent to all other partite sets, we get, \( f_M(u) = \{ 1 \}, \forall u \in V(K_{a_1, a_2, \ldots, a_n}) - M \). Hence, the complete \( n \)-partite graph \( G \) is an independent CDPU graph for any \( n \). \( \square \)

**Corollary 2.9** Complete \( n \)-partite graphs have \( n \) distinct independent CDPU sets.

**Proof** Let \( G = K_{a_1, a_2, \ldots, a_n} \) be a complete \( n \)-partite graph. Then, \( V(G) \) can be partitioned into \( n \) subsets \( V_1, V_2, \ldots, V_n \) where \( |V_1| = a_1, |V_2| = a_2, \ldots, |V_n| = a_n \). Take \( M_1 \) as the vertices corresponding to the partite set \( V_1 \), \( M_2 \) as the vertices corresponding to the partite set \( V_2, \ldots, M_i \) corresponds to the vertices of the partite set \( V_i, \ldots, M_n \) corresponds to the vertices of the partite set \( V_n \). Then from Theorem 2.8, each \( M_i, 1 \leq i \leq n \) form a CDPU set. Hence there are \( n \) distinct CDPU sets. \( \square \)

**Theorem 2.10** A path \( P_n \) is an independent CDPU graph if and only if \( n = 2, 3, 4, 5 \).

**Proof** Let \( P_n \) be a path on \( n \) vertices and \( V(P_n) = \{ v_1, v_2, \ldots, v_n \} \). When \( n = 2 \) and \( 3 \), \( P_2 \) and \( P_3 \) contains a vertex of full degree and hence from Theorem 2.6, \( P_2 \) and \( P_3 \) are independent.
CDPU. When \( n = 4 \), take \( M = \{ v_1, v_4 \} \). Then \( f_M(v_2) = f_M(v_3) = \{ 1, 2 \} \), whence \( M \) is independent CDPU. When \( n = 5 \), let \( V(G) = \{ v_1, v_2, \ldots, v_5 \} \) and choose \( M = \{ v_1, v_3, v_5 \} \). Then, \( f_M(v_2) = f_M(v_4) = \{ 1, 3 \} \). Hence, \( P_5 \) is an independent CDPU graph.

Conversely, suppose that \( P_n \) is an independent CDPU graph. That is, there exists a CDPU set \( M \) such that no two of the vertices are adjacent. From \( n \) vertices, we can have at most \( \frac{n}{2} \) or \( \frac{n+1}{2} \) vertices which are non-adjacent. From Theorem 1.7, \( \sigma(P_n) = n - 2, n \geq 3 \). When \( n \geq 6 \), we cannot choose a CDPU set \( M \) such that \( n - 2 \) vertices are non-adjacent. Hence \( P_n \) is independent CDPU only for \( n = 2, 3, 4 \) and \( 5 \).

\[ \square \]

Theorem 2.11 \( n \)-cube \( Q_n \) is an independent CDPU graph with \( |M| = 2^{n-1} \).

\[ \text{Proof} \] We have \( Q_n = K_2 \times Q_{n-1} \) and has \( 2^n \) vertices which may be labeled \( a_1a_2\ldots a_n \), where each \( a_i \) is either 0 or 1. Also two points in \( Q_n \) are adjacent if their binary representations differ at exactly one place. Take \( M \) as the set of all vertices whose binary representation differ at two places. Clearly the vertices in \( M \) are non-adjacent and also maximal. We have to check whether \( M \) is CDPU. For let \( M = \{ v_1, v_3, \ldots, v_{2^n-1} \} \). Consider a vertex \( v_i \) which does not belong to \( M \). Clearly \( v_i \) is adjacent to a vertex \( v_j \) in \( M \). Hence \( 1 \in f_M(v_i) \). Then, since \( v_j \) is in \( M, v_j \) is adjacent to a vertex \( v_k \) not in \( M \). Hence 2 does not belong to \( f_M(v_i) \). Since \( v_k \) is not an element of \( M \) and \( v_k \) is adjacent to a vertex \( v_l \) in \( M \), \( 3 \in f_M(v_i) \). Proceeding in the same manner, we get \( f_M(v_i) = \{ 1, 3, \ldots, n-1 \} \). Hence \( Q_n \) is independent CDPU with \( |M| = 2^n \). \( \square \)

Theorem 2.12 Ladder \( P_n \times K_2 \) is an independent CDPU graph if and only if \( n \leq 4 \).

\[ \text{Proof} \] First we have to prove that \( P_n \times K_2 \) is an independent CDPU graph for \( n \leq 4 \). When \( n = 2 \), take \( M = \{v_2,v_4\} \), so that \( f_M(v_i) = \{1\} \) for \( i = 1, 3 \).
When $n = 3$, take $M = \{v_1, v_4\}$, so that $f_M(v_i) = \{1, 2\}$, for $i = 2, 4, 6$.

When $n = 4$, take $M = \{v_1, v_3, v_5, v_7\}$, so that $f_M(v_i) = \{1, 3\}$ for $i = 2, 4, 6, 8$. Therefore, $P_n \times K_2$ is an independent CDPU graph for $n \leq 4$.

Conversely, suppose that $P_n \times K_2$ is an independent CDPU graph. We have to prove that $n \leq 4$. If possible, suppose $n = k \geq 5$. In $P_n \times K_2$, since the number of vertices is even, and the vertices in $P_n \times K_2$ form a Hamiltonian cycle, then the only possibility of $M$ to be an independent CDPU set is to choose $M$ as the set of all alternate vertices of the Hamiltonian cycle. Clearly, in this case $M$ is a maximal independent set. Denote $M_1 = \{v_1, v_3, \ldots, v_{2n-1}\}$ and $M_2 = \{v_2, v_4, \ldots, v_{2n}\}$. Consider $M_1 = \{v_2, v_4, \ldots, v_1, \ldots, v_{2n}\}$.

**Case 1** $n$ is odd.

In this case, $f_{M_1}(v_1) = \{1, 3, \ldots, n\}$. Since $n$ is odd we have two central vertices, say, $v_i$ and $v_j$ in $P_n \times K_2$. Since $v_i$ and $v_j$ are of the same eccentricity and $M_1$ is a maximal independent set, $v_j$ does not belong to $M_1$. Then, $f_{M_1}(v_j) = \{1, 3, \ldots, \frac{n+1}{2}\}$.

Thus, $f_{M_1}(v_1) \neq f_{M_1}(v_j)$. Hence $M_1$ is not a CDPU.

**Case 2** $n$ is even.

In this case, $f_{M_1}(v_1) = \{1, 3, \ldots, n-1\}$. Since $n$ is even, there are four central vertices $v_i, v_j, v_k, v_l$ in $P_n \times K_2$. Clearly the graph induced by $T = \{v_i, v_j, v_k, v_l\}$ is a cycle on four vertices. Since $M_1$ is maximal and consists of the alternate vertices of $P_n \times K_n$, $v_j, v_l$ should necessarily be outside $M_1$. Thus, $f_{M_1}(v_j) = \{1, 3, \ldots, \frac{n}{2}\}$.

Thus, $f_{M_1}(v_1) \neq f_{M_1}(v_j)$. Hence $M_1$ is not a CDPU.

Therefore $P_n \times K_2$ is not independent CDPU for $n \geq 5$. Hence the theorem. □

**Theorem 2.13** If $G_1$ and $G_2$ are independent CDPU graphs, then $G_1 + G_2$ is also an independent CDPU graph.
Proof Since $G_1$ and $G_2$ are independent CDPU graphs, there exist $M_1 \subset V(G_1)$ and $M_2 \subset V(G_2)$ such that no two vertices in $M_1$ (and in $M_2$) are adjacent. Now, in $G_1 + G_2$, every vertex of $G_1$ is adjacent to every vertices of $G_2$. Then clearly, independent CDPU set $M_1$ of $G_1$ (or $M_2$ of $G_2$) is an independent CDPU set for $G_1 + G_2$. Hence the theorem. □

Remark 2.14 If $G_1$ and $G_2$ are independent CDPU graphs, then the cartesian product $G_1 \times G_2$ need not have an independent CDPU set. But $G_i \times G_i$ is independent CDPU for $i = 1, 2$ as illustrated in Fig.5.

![Fig.5](image)

Definition 2.15 An independent set that is not a proper subset of any independent set of $G$ is called maximal independent set of $G$. The number of vertices in the largest independent set of $G$ is called the independence number of $G$ and is denoted by $\beta(G)$.

§3. Independence CDPU Number

The least cardinality of the independent cdpu set in $G$ is called the independent CDPU number of $G$, denoted by $\sigma_i(G)$. In general, for an independent CDPU graph, $\sigma_i(G) \leq \beta(G)$, where $\beta(G)$ is the independence number of $G$.

Theorem 3.1 If $G$ is an independent CDPU graph with $n$ vertices, then $r(G) \leq \sigma_i(G) \leq \lceil \frac{n}{2} \rceil$, where $r(G)$ is the radius of $G$.

Proof We have, $\beta(G) \leq \lceil \frac{n}{2} \rceil$ and hence $\sigma_i(G) \leq \lceil \frac{n}{2} \rceil$. Now we prove that $r(G) \leq \sigma_i(G)$. Suppose $r(G) = k$. Then, there are vertices with eccentricities $k, k + 1, k + 2, \ldots, d$, where $d$ is the diameter of $G$. Let $v$ be the central vertex of $G$ and $e = vw$. Since the central vertex $v$ of a graph on $n(\geq 3)$ vertices cannot be a pendant vertex, there exists a vertex $w$ which is adjacent to $v$. Hence, $w$ is of eccentricity $k + 1$. Also $u$ is of eccentricity $k + 1$. By a similar
argument there exists at least two vertices each of eccentricity \( k + 1, k + 2, \ldots, d \). Hence, the CDPU set should necessarily consists of all vertices with eccentricity \( k, k + 1, k + 2, \ldots, d - 1 \). Thus, \( \sigma(G) \geq 1 + \{2 + 2 + \ldots (d - 1 - k) \text{times} \} \geq k \). Whence, \( \sigma_i(G) \geq r(G) \). Therefore, \( r(G) \leq \sigma_i(G) \leq \lceil \frac{n}{2} \rceil \). □

**Theorem 3.2** A graph \( G \) has \( \sigma_i(G) = 1 \) if and only if \( G \) has at least one vertex of full degree.

![Peterson Graph](image)

**Fig.6:** A graph with \( \sigma_i(G) = 1 \)

**Proof** Suppose that \( G \) has one vertex \( v_i \) with full degree. Take \( M = \{v_i\} \). Then \( f_M(u) = \{1\} \), for every \( u \in V - M \). Also \( M \) is independent. Hence \( \sigma_i(G) = 1 \).

Conversely, suppose that \( G \) is a graph with \( \sigma_i(G) = 1 \). That is, there exists an independent set \( M \) which contains only one vertex \( v_i \) which is a CDPU set of \( G \). Also, \( \sigma_i(G) = 1 \) implies, \( v_i \) is adjacent to all other vertices. Hence \( v_i \) is a vertex with full degree. □

**Corollary 3.3** The independent CDPU number of a complete graph is 1.

**Corollary 3.4** If \( M \) is the maximal independent set of a graph \( G \) with \( |M| = 1 \), then \( G \) is an independent CDPU.

**Proof** The result follows since \( M \) is a maximal independent set and \( |M| = 1 \), there is a vertex \( v \) of full degree. □

**Theorem 3.5** Peterson Graph is an independent CDPU graph with \( \sigma_i(G) = 4 \).

**Proof** Let \( G \) be a Peterson Graph with \( V(G) = \{v_1, v_2, \ldots, v_{10}\} \). Let \( M \) be such that \( M \) contains two non adjacent vertices from the outer cycle and two non-adjacent vertices from the inner cycle. Let it be \( \{v_3, v_5, v_6, v_7\} \). Clearly, \( M \) is a maximal independent set of \( G \). Also \( f_M(v_i) = \{1, 2\} \), for every \( i = 1, 2, 4, 8, 9, 10 \). Thus, \( M \) is a CDPU set of \( G \). Hence, \( G \) is an independent CDPU graph with \( \sigma_i(G) \leq 4 \). To prove that \( \sigma_i(G) = 4 \), it is enough to prove that the deletion of any vertex from \( M \) does not form a CDPU set. For, let \( M_1 = \{v_3, v_5, v_7\} \). Then, \( f_M(v_i) = \{1, 2\} \), for \( i = 1, 2, 4, 8, 9, 10 \) and \( f_M(v_6) = \{2\} \). Hence \( M_1 \) cannot be a CDPU set for \( G \). Thus \( \sigma_i(G) = 4 \). □
Theorem 3.6  Shadow graphs of $K_n$ are independent CDPU with $|M| = n$.

Proof  Let $v_1, v_2, \ldots, v_n$ be the vertices of $K_n$ and $v'_1, v'_2, \ldots, v'_n$ be the corresponding shadow vertices. Clearly, $M = \{v'_1, v'_2, \ldots, v'_n\}$ is a maximal independent set of $S(K_n)$. Also, from Theorem 1.14, $M$ forms a CDPU set. Hence $|M| = n$. □

Definition 3.7  A set of points which covers all the lines of a graph $G$ is called a point cover for $G$. The smallest number of points in any point cover for $G$ is called its point covering number and is denoted by $\alpha_0(G)$.

It is natural to rise the following question by definition:

Does there exist any connection between the point covering for a graph and independent CDPU set?

Proposition 3.8  If $\alpha_0(G) = 1$, then $\sigma_i(G) = 1$

Proof  Since $\alpha_0(G) = 1$, we have to cover every edges by a single vertex. This implies that there exists a vertex of full degree. Hence from Theorem 3.2, $\sigma_i(G) = 1$. □

Remark 3.9  The converse of Proposition 3.8 need not be true. Note that in Figure 6, $\sigma_i(G) = 1$, but $\alpha_0(G) = 6$.

Theorem 3.10  The central subgraph $\langle C(G) \rangle$ of a maximal outerplanar graph $G$ is an independent CDPU graph with $\sigma_i(G) = 1, 2$ or 3.

Proof  Fig. 8 depicts all the central subgraphs of maximal outerplanar graph [3]. Since $G_1, G_2, G_3, G_4, G_5$ have a full degree vertex, those graphs are independent CDPU and $\sigma_i(G_j) = 1$, for $j = 1, 2, 3, 4, 5$.

In $G_6$, let $M = \{v_1, v_4\}$. Then, $f_M(v_i) = \{1, 2\}$, for every $v_i \in V - M$. Since $M$ is independent, $G_6$ is independent CDPU and $\sigma_i(G_6) = 2$. 

Fig. 7
Germina K.A. and Beena Koshy

In $G_7$, let $M = \{v_1, v_3, v_5\}$. Then, $f_M(v_i) = \{1, 2\}$ for every $v_i \in V - M$. Hence, $G_7$ is independent CDPU with $\sigma_i(G_7) = 3$. □

![Central subgraphs of a maximal outerplanar graph](image)

**Fig. 8:** Central subgraphs of a maximal outerplanar graph

**Theorem 3.11** The independent CDPU number of an even cycle $C_n$, $n \geq 8$ is $\frac{n}{2}$.

*Proof* From Proposition 2.4, the alternate vertices of the even cycle constitute the independent CDPU set. As already proved, removal of any vertex from $M$ does not give a cdpu set. Hence, $\sigma_i(C_n) = \frac{n}{2}$. □

**Remark 3.12** $\sigma_i(C_6) = 2$.

**Theorem 3.13** For all integers $a_1 \geq a_2 \geq \cdots \geq a_n \geq 2$, $\sigma_i(K_{a_1, a_2, \ldots, a_n}) = \min\{a_1, a_2, \ldots, a_n\}$.

*Proof* From Theorem 2.8 and Corollary 2.9, all the $n$ partite sets form an independent CDPU set. Hence the independent CDPU number is the minimum of all $a_i$’s. □

**Theorem 3.14** If $\sigma_i(G_1) = k_1$ and $\sigma_i(G_2) = k_2$, then $\sigma_i(G_1 + G_2) = \min\{k_1, k_2\}$.

*Proof* From Theorem 2.13, either $M_1$ or $M_2$ is an independent cdpu set for $G_1 + G_2$. Also $\sigma_i(G_1 + G_2)$ is the minimum among $M_1$ and $M_2$. □

**Theorem 3.15** If $G_1$ and $G_2$ are independent CDPU cycles with $n, m(\geq 4)$ vertices respectively, then $G_1 \times G_2$ is independent CDPU with $|M| = \frac{mn}{2}$.

*Proof* Since $G_1$ has $n$ vertices and $G_2$ has $m$ vertices, then $G_1 \times G_2$ has $mn$ vertices. Without loss of generality, assume that $m > n$. In the construction of $G_1 \times G_2$, $G_2$ is drawn $n$ times and then the corresponding adjacency is given according as the adjacency in $G_1$. Since $G_2$ is an independent CDPU cycle, from Theorem 3.11, $\sigma_i(G_2) = \frac{m}{2}$. Therefore in $G_1 \times G_2$ there are $\frac{mn}{2}$ vertices in the CDPU set. □
Remark 3.16 In Theorem 3.15, if any one of $G_1$ or $G_2$ is $C_3$, then $|M| = n$, since $\sigma_i(C_3) = 1$.

Fig. 9: Graphs whose subdivision graphs are bipartite complementary

Theorem 3.17 The connected graphs, whose subdivision graphs are bipartite complementary are independent CDPU.

Proof Fig. 9 depicts the seven graphs whose subdivision graphs are bipartite self-complementary [2]. In $G_4$, $M_1 = \{v_1, v_2\}$ gives $f_{M_1}(v_3) = f_{M_1}(v_4) = \{1, 2\}$.

In $G_5$, $M_2 = \{v_1, v_4\}$ gives $f_{M_2}(v_3) = f_{M_2}(v_2) = \{1\}$.

In $G_6$, $M_3 = \{v_2, v_3\}$ gives $f_{M_3}(v_1) = f_{M_3}(v_4) = \{1\}$.

In $G_7$, $M_4 = \{v_1\}$ gives $f_{M_4}(v_2) = f_{M_4}(v_3) = f_{M_4}(v_4) = \{1\}$. Hence $M_1, M_2, M_3, M_4$ are independent CDPU sets. Thus the connected graphs $G_4, G_5, G_6$ and $G_7$ are independent CDPU.

§4. Conclusion and Scope

As already stated in the introduction, the concept under study has important applications in the field of Chemistry. The study is interesting due to its applications in Computer Networks and Engineering, especially in Control System. In a closed loop control system, signal flow graph representation is used for gain analysis. So in certain control systems specified by certain characteristics, we can find out $M$, a set consisting of two vertices such that one vertex will be the take off point and other vertex will be the summing point.

Following are some problems that are under investigation:

1. Characterize independent CDPU trees.
2. Characterize unicyclic graphs which are independent CDPU.
3. What is the independent CDPU number for a generalized Peterson graph.
4. What are those classes of graphs with $r(G) = \sigma_i(G)$, where $r(G)$ is the radius of $G$. 
Acknowledgements

The authors are deeply indebted to Professor Dr. B.D. Acharya, SRC-IIIDMS, University of Mysore, Mysore-560 005, India for sparing his valuable time in sharing his many incisive thoughts to propel our vigorous discussion on the content of this paper. They are thankful to the Department of Science & Technology, Government of India for supporting this research under the project No. SR/S4/MS:277/06.

References


