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Laplacian and Signless Laplacian Degree Sum Distance Energy of Some Graphs

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Abstract: In this paper we define the Laplacian degree sum distance matrix and signless Laplacian degree sum distance matrix from the well known degree sum distance matrix and define Laplacian degree sum distance energy and signless Laplacian degree sum distance energy. We evaluate the Laplacian degree sum distance energy and signless Laplacian degree sum distance energy of some graphs. We also obtain some bounds for Laplacian degree sum distance energy.

Key Words: Laplacian degree sum distance energy, average degree, signless Laplacian degree sum distance energy.

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§1. Introduction

All the graphs considered here are finite simple, connected and undirected. Let G be such a graph of order n.The degree of a vertex v_i, $d(v_i)$ is the number of edges incident on it and we denote d_{ij} as the distance between vertex v_i and v_j.

Several results on Laplacian energy of graph G are reported in the literature [2, 3, 4, 5, 6]. The signless Laplacian energy is also studied in the literature rigorously [7, 8, 9]. We have discussed degree sum distance matrix in [1], as DSD(G).

We now define the Laplacian and the signless Laplacian degree sum distance matrix of a connected graph G as

$$L_{DSD}(G) = [ldsd_{ij}]$$

where,

$$\begin{aligned} ldsd_{ij} &= -d_{ij}(d(v_i) + d(v_j)) \text{ if } i \neq j \\ &= 2d_i \text{ if } i = j \end{aligned}$$

and $Q_{DSD}(G) = [qdsd_{ij}]$ where,

$$\begin{aligned} qdsd_{ij} &= d_{ij}(d(v_i) + d(v_j)) \text{ if } i \neq j \\ &= 2d_i \text{ if } i = j \end{aligned}$$

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The following are obvious for $L_{DSD}(G)$ and $Q_{DSD}(G)$.

(1) Both $L_{DSD}(G)$ and $Q_{DSD}(G)$ are real symmetric, hence their eigenvalues are real.

(2) If β_i and γ_i, i=$1, 2, 3 \cdots, n$ are eigenvalues of $L_{DSD}(G)$ and $Q_{DSD}(G)$ respectively then they can be arranged in non-increasing order as $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$ respectively.

Definition 1.1 *Let G be graph of order n and size m.If $2avd(G)$ denotes double average degree of a graph given by, $2\frac{\sum_1^n d_i(G)}{n}$, then analogous to usual Laplacian and signless Laplacian energy we define the Laplacian and signless Laplacian degree sum distance energy as,*

$$LE_{DSD}(G) = \sum_{i=1}^{n} |\beta_i - 2avd(G)| \ \ and \ \ QE_{DSD}(G) = \sum_{i=1}^{n} |\gamma_i - 2avd(G)|.$$

The double average degree is taken to be consistent with the degree sum entries defined in the matrix defined

Example 1.2 For the graph G given in Figure 1,

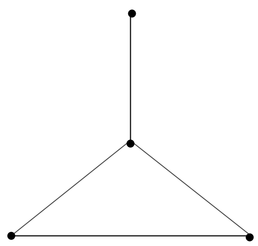

Figure 1

The double average degree of G is $2 \times \frac{4}{2} = 4$.

$L_{DSD}(G) = \begin{bmatrix} 2 & -4 & -6 & -6 \\ -4 & 6 & -5 & -5 \\ -6 & -5 & 4 & -4 \\ -6 & -5 & -4 & 4 \end{bmatrix}$	$Q_{DSD}(G) = \begin{bmatrix} 2 & 4 & 6 & 6 \\ 4 & 6 & 5 & 5 \\ 6 & 5 & 4 & 4 \\ 6 & 5 & 4 & 4 \end{bmatrix}$
Laplacian degree sum distance eigenvalues are $\beta_1 = 11.1686$, $\beta_2 = 8.065$, $\beta_3 = 8$,$\beta_4 = -11.2342$	signless Laplacian degree sum distance eigenvalues are $\gamma_1 = 19.0266$, $\gamma_2 = 1.0761$, $\gamma_3 = 0$,$\gamma_4 = -4.1027$
$LE_{DSD}(G) = \|11.1686 - 4\| + \|8.065 - 4\|$ $+\|8 - 4\| + \|11.2342 + 4\| = 30.4678.$	$QE_{DSD}(G) = \|19.0266 - 4\| + \|4 - 1.0761\|.$ $+\|0 + 4\| + \|4 + 4.1027\| = 30.0532.$

Table 1

§2. Bounds on Laplacian Degree Sum Distance Energy

We introduce the auxiliary Laplacian degree sum distance eigenvalue μ_i, defined as,

$$\mu_i = \beta_i - 2\frac{1}{n}\sum_{i=1}^{n} d_i.$$

Lemma 2.1 *Let G be a graph of order n, then we have*

$$\sum_{i=1}^{n} \mu_i = 0 \quad and \quad \sum_{i=1}^{n} \mu_i^2 = 2R,$$

where,

$$R = \frac{1}{2}\left(\frac{4}{n}\sum_{i=1}^{n} d_i^2 + 2\sum_{i=1,i<j}^{n}((d_i + d_j)d_{ij})^2\right).$$

The Laplacian degree sum distance energy of G can be another as

$$LE_{DSD}(G) = \sum_{i=1}^{n} |\mu_i| = \sum_{i=1}^{n}\left|\beta_i - 2\frac{1}{n}\sum_{i=1}^{n} d_i\right|.$$

Proposition 2.2 *Let G be a graph of order $n \geq 2$, then, $2\sqrt{R} \leq LE_{DSD}(G) \leq \sqrt{2nR}$, where R is defined above.*

Proof Consider the equation

$$\begin{aligned} N &= \sum_{i=1}^{n}\sum_{j=1}^{n}(|\mu_i| + |\mu_j|)^2 = 2n\sum_{i=1}^{n}|\mu_i|^2 - 2\left(\sum_{i=1}^{n}|\mu_i|\right)\left(\sum_{j=1}^{n}|\mu_j|\right) \\ &= 2n2R - 2(LE_{DSD}(G))^2 = 4nR - 2(LE_{DSD}(G))^2 \end{aligned}$$

from Lemma[2.1].

Note that, $N \geq 0$, i.e, $4nR - 2(LE_{DSD}(G))^2 \geq 0$ which implies, $LE_{DSD}(G) \leq \sqrt{2nR}$. Again from Lemma 2.1 we have $\left(\sum_{i=1}^{n}\mu_i\right)^2 = 0$, the fact that R≥0 and

$$\begin{aligned} \sum_{i=1}^{n}\mu_i^2 &= \left(\sum_{i=1}^{n}\mu_i\right)^2 - 2\sum_{1\leq i<j\leq n}\mu_i\mu_j \\ &\leq 2\sum_{1\leq i<j\leq n}|\mu_i\mu_j| \leq \sum_{1\leq i<j\leq n}|\mu_i|.|\mu_j|, \\ 2R &\leq 2\sum_{1\leq i<j\leq n}|\mu_i|.|\mu_j| \end{aligned}$$

so that

$$\begin{aligned} LE^2_{DSD}(G) &= \left(\sum_{i=1}^{n} |\mu_i|\right)^2 \\ &= \sum_{i=1}^{n} |\mu_i|^2 + 2\sum_{1\le i<j\le n} |\mu_i|.|\mu_j| \\ &= 2R + 2R = 4R. \end{aligned}$$

Then, $LE_{DSD}(G) \geq 2\sqrt{R}$. □

Lemma 2.3([10]) *Let $a_1, a_2, \cdots, a_n$ be non-negative numbers, then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - (\prod_{i=1}^{n} a_i)^{\frac{1}{n}}\right] \leq n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2 \leq n(n-1)\left[\frac{1}{n}\sum_{i=1}^{n} a_i - (\prod_{i=1}^{n} a_i)^{\frac{1}{n}}\right].$$

Proposition 2.4 *Let G be a graph with n vertices and m edges, then*

$$\sqrt{2R + n(n-1)\Delta^{\frac{2}{n}}} \leq LE_{DSD}(G) \leq \sqrt{2(n-1)R + n\Delta^{\frac{2}{n}}},$$

where,

$$\Delta = \left| det\left(L_{DSD}(G) - \left[\frac{2}{n}\sum_{i=1}^{n} d_i\right] I_n\right)\right|.$$

Proof We assume that $\Delta \neq 0$, by setting $a_i = \mu_i^2$ where $i = 1, 2, \cdots, n$, and

$$P = n\left[\frac{1}{n}\sum_{i=1}^{n}\mu_i^2 - \left(\prod_{i=1}^{n}\mu_i^2\right)^{\frac{1}{n}}\right] \geq 0$$

From Lemma 2.3, we have

$$P \leq n\sum_{i=1}^{n}\mu_i^2 - \left(\prod_{i=1}^{n}|\mu_i|^2\right) \leq (n-1)P,$$

which can be further expressed as $P \leq 2nR - (LE(G))^2 \leq (n-1)P$.

Hence,

$$P = n\left[\frac{1}{n}\sum_{i=1}^{n}\mu_i^2 - \left(\prod_{i=1}^{n}\mu_i^2\right)^{\frac{1}{n}}\right] = n\left[\frac{1}{n}2P - \Delta^{\frac{2}{n}}\right] = 2P - n\Delta^{\frac{2}{n}}$$

By substituting in the above inequality, we obtain desired result. □

Proposition 2.5 *If G is any graph of order n and Δ as defined above, then*

$$\sqrt{2R+n(n-1)\Delta^{\frac{2}{n}}} \leq LE_{DSD}(G) \leq \sqrt{2Rn}.$$

Proof For lower bound consider,

$$[LE_{DSD}(G)]^2 = \sum_{i=1}^{n}(|\mu_i|)^2 = \sum_{i=1}^{n}(\mu_i)^2 + 2\sum_{i<j}|\mu_i||\mu_j|$$

.

Since $AM \geq GM$, we have

$$\begin{aligned}\frac{1}{n(n-1)}\sum_{i\neq j}|\mu_i||\mu_j| &\geq (\prod_{i\neq j}|\mu_i||\mu_j|)^{\frac{1}{n(n-1)}}\\ &= \prod_{i=1}^{n}(|\mu_i|^{2n-2})^{\frac{1}{n(n-1)}}\\ &= (\prod_{i=1}^{n}|\mu_i|^{\frac{2}{n}}) = \Delta^{\frac{2}{n}}.\end{aligned}$$

Therefore,

$$\prod_{i\neq j}|\mu_i||\mu_j| \geq n(n-1)\Delta^{\frac{2}{n}}.$$

Hence,

$$[LE_{DSD}(G)]^2 \geq 2R + n(n-1)\Delta^{\frac{2}{n}},$$

i.e.,

$$LE_{DSD}(G) \geq \sqrt{2R+n(n-1)\Delta^{\frac{2}{n}}}. \quad (1)$$

For upper bound we define

$$\begin{aligned}X &= \sum_{i=1}^{n}\sum_{j=1}^{n}(|\mu_i|+|\mu_j|)^2\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(|\mu_i|^2+|\mu_j|^2\right) + 2\left(\sum_{i=1}^{n}|\mu_i||\mu_j|\right)\\ &= n\sum_{i=1}^{n}(\mu_i)^2 + n\sum_{i=1}^{n}(\mu_j)^2 - 2\left(\sum_{i=1}^{n}|\mu_i||\mu_j|\right)\\ &= 2nR + 2nR - 2[LE_{DSD}(G)]^2 = 4nR - 2[LE_{DSD}(G)]^2\end{aligned}$$

Since $X \geq 0$, we get that

$$LE_{DSD}(G) \leq \sqrt{2Rn}. \quad (2)$$

Combining (1) and (2) we obtain the desired result. □

§3. LE_{DSD} of Some Graphs

Lemma 3.1 *If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of any matrix P of order n, then the eigenvalues of the matrix $kI_n \pm P$ are $k \pm \lambda_1, k \pm \lambda_2, \cdots, k \pm \lambda_n$.*

Using Lemma[3.1] one can directly obtain the Laplacian and signless Laplacian degree sum distance eigenvalues from the degree sum distance eigenvalues for a regular graph G.The degree sum distance energy is already discussed by the present authors in [1].

In general the Laplacian degree sum distance energy and signless Laplacian degree sum distance energy are equal for a regular graph G.This is consistent with the equality of Laplacian and signless Laplacian energy for regular graph G. Hence we discuss graphs which are not regular.

Lemma 3.2 *Let a and b be two arbitrary constants, I is the identity matrix and J is $n \times n$ matrix whose all entries are $1's$. If $A = (a-b)I + bJ$ then the characteristic polynomial of A, is, $|\lambda I - A| = [\lambda - a + b]^{n-1}[\lambda - a - (n-1)b]$.*

Theorem 3.3 *The Laplacian degree sum distance energy of the complete bipartite graph $K_{m,n}$ is*

$$\begin{aligned} LE_{DSD}(K_{m,n}) &= \left|\frac{2n(m+3n)(m-1)}{m+n}\right| + \left|\frac{2m(n+3m)(n-1)}{m+n}\right| \\ &\quad + \left|\beta_1 + \frac{4mn}{m+n}\right| + \left|\frac{4mn}{m+n} - \beta_2\right|, \end{aligned}$$

where β_1 and β_2 are the roots of the equation

$$\beta^2 + 2(4mn - 3n - 3m)\beta + mn(14mn - 24m - 24n - m^2 - n^2 + 36) = 0.$$

Proof In $K_{m,n}$, m vertices have degree n and n vertices have degree m.The diameter being 2 the structure of the degree sum distance matrix is

$$L_{DSD}(K_{m,n}) = \begin{pmatrix} 2nI_m - 4mA(K_m) & -(m+n)J_{m\times n} \\ -(m+n)J_{n\times m} & 2mI_n - 4nA(K_n) \end{pmatrix}$$

where J is matrix of all 1's and A the adjacency matrix.The Laplacian degree sum distance polynomial is then given by

$$|\beta I - L_{DSD}(K_{m,n})| = \begin{vmatrix} (\beta - 2n)I_m + 4mA(K_m) & (m+n)J_{m\times n} \\ (m+n)J_{n\times m} & (\beta - 2m)I_n + 4nA(K_n) \end{vmatrix}$$

Using Lemma 3.2 with

$$a = \beta - 2m - \frac{(m+n)^2(n-1)}{\beta - 4m(n-1)} \quad \text{and} \quad b = -4n - \frac{(m+n)^2(n-1)}{\beta - 4m(n-1)},$$

we get that

$$\begin{aligned}|\beta I - L_{DSD}(K_{m,n})| &= [\beta - 6n]^{m-1}[\beta - 6m]^{n-1}[\beta^2 + 2(4mn - 3n - 3m)\beta \\ &\quad + mn(14mn - 24m - 24n - m^2 - n^2 + 36)].\end{aligned}$$

Using double average degree as, $doubleavd(K_{m,n}) = \dfrac{4mn}{m+n}$, we get desired result. □

Corollary 3.4 *If $m = 1$, we get star graph $K_{1,n}$ whose Laplacian degree sum distance energy is,*

$$LE_{DSD}(K_{1,n}) = \frac{(2n+6)(n-1)}{n+1} + |\frac{4n}{n+1} - \beta_1| + |\frac{4n}{n+1} - \beta_2|,$$

where β_1 and β_2 are roots of the equation,

$$[\beta^2 + 2(n-3)\beta - n(n^2 - 6n + 5)] = 0.$$

Let K_n+e and K_n-e denote the graph obtained by adding or deleting an edge e respectively to complete graph K_n, both are of diameter 2.

Theorem 3.5 *The LE_{DSD} of $K_n + e$ is*

$$\begin{aligned}LE_{DSD}(K_n + e) &= \left|\frac{2(n^2 - n - 2)}{n+1} - (2(n-1) - (2n-2))(2n-2)\right| + \left|\frac{2(n^2 - n - 2)}{n+1} + \beta_1\right| \\ &\quad + \left|\frac{2(n^2 - n - 2)}{n+1} - \beta_2\right| + \left|\frac{2(n^2 - n - 2)}{n+1} - \beta_3\right|,\end{aligned}$$

where β_1,β_2 and β_3 are roots of the equation,

$$\begin{aligned}&\beta^3 - ((2n+2) - 2(n-1)(n-3))\beta^2 - (4(n^2-1)(n-3) + 4n^2(n-1) + (2n-1)^2(n-1) \\ &+(n-1)^2)\beta + 8n^3(n-1) + 2(2n-1)^2(n-1) + 4n(n^2-1)(2n-1) - 2(n-1)^3(n-3) = 0.\end{aligned}$$

Proof In $K_n + e$ one vertex has degree n, one vertex has degree 1 remaining having degree $n - 1$, then we have

$$L_{DSD}(K_n + e) = \begin{bmatrix} 2n & -(n+1) & -(2n-1) & \ldots & -(2n-1) & \ldots & -(2n-1) \\ -(n+1) & 2 & -2n & \ldots & -2n & \ldots & -2n \\ -(2n-1) & -2n & 2(n-1) & \ddots & -(2n-2) & \ldots & -(2n-2) \\ -(2n-1) & -2n & -(2n-2) & \ldots & 2(n-1) & \ldots & -(2n-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(2n-1) & -2n & -(2n-2) & \ldots & -(2n-2) & \ldots & 2(n-1) \end{bmatrix}$$

The Laplacian degree sum distance polynomial is

$$|\beta I - L_{DSD}(K_n+e)| = \begin{bmatrix} \beta-2n & (n+1) & (2n-1) & \dots & (2n-1) & \dots & (2n-1) \\ (n+1) & \beta-2 & 2n & \dots & 2n & \dots & 2n \\ (2n-1) & 2(n-1) & \beta-2(n-1) & \ddots & (2n-2) & \dots & (2n-2) \\ (2n-1) & 2n & (2n-2) & \dots & \beta-2(n-1) & \dots & (2n-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (2n-1) & 2(n-1) & (2n-2) & \dots & (2n-2) & \dots & \beta-2(n-1) \end{bmatrix}$$

Hence,

$$\begin{aligned} |\beta I - L_{DSD}(K_n+e)| &= [\beta - 2(n-1) - (2n-2)]^{n-2}\left[\beta^3 - ((2n+2) - 2(n-1)(n-3))\beta^2 \right. \\ &\quad -(4(n^2-1)(n-3) + 4n^2(n-1) + (2n-1)^2(n-1) + (n-1)^2)\beta \\ &\quad +8n^3(n-1) + 2(2n-1)^2(n-1) + 4n(n^2-1)(2n-1) \\ &\quad \left. -2(n-1)^3(n-3)\right]. \end{aligned}$$

Using double average degree as

$$doubleavd(K_n+e) = \frac{2(n^2-n-2)}{n+1},$$

we get the desired result. □

Theorem 3.6 *The LE_{DSD} of $K_n - e$ is*

$$\begin{aligned} LE_{DSD}(K_n - e) &= \left|\frac{2(n-2)(n+1)}{n} - 4(n-1)\right|(n-3) + \left|\frac{2(n-2)(n+1)}{n} - 6(n-2)\right| \\ &\quad + \left|\frac{2(n-2)(n+1)}{n} - \beta_1\right| + \left|\frac{2(n-2)(n+1)}{n} + \beta_2\right|, \end{aligned}$$

where β_1 and β_2 are roots of the equation

$$[\beta^2 + (2n^2 - 8n + 4)\beta - 2(2n^3 - 6n^2 + 5n - 2)] = 0.$$

Proof In $K_n - e$ two vertices are of degree $n-2$ and remaining are of degree $n-1$. Proceeding in a way similar to Theorem 3.5, whose Laplacian degree sum distance polynomial of $K_n - e$ is

$$|\beta I - L_{DSD}(K_n - e)| = [\beta - 2(2n-2)]^{n-3}[\beta - 6(n-2)][\beta^2 + (2n^2 - 8n + 4)\beta - 2(2n^3 - 6n^2 + 5n - 2)].$$

Using double average degree as

$$doubleavd(K_n - e) = \frac{2(n-2)(n+1)}{n},$$

we get the desired result. □

Let K_n be a complete graph of order n then the vertex coalescence of K_n with K_n will be denoted by $K_nO_vK_n$ and the edge coalescence by $K_nO_eK_n$. $K_nO_vK_n$ has $2n-1$ vertices and $2\times(nC_2)$ edges whereas $K_nO_eK_n$ has $2n-2$ vertices and $2\times(nC_2-1)$ edges.

Theorem 3.7 *The LE_{DSD} of $K_nO_vK_n$ is*

$$\begin{aligned} LE_{DSD}(K_nO_vK_n) &= \left|\frac{4n(n-1)}{2n-1}-2(n-1)(n+1)\right|+\left|\frac{4n(n-1)}{2n-1}-4(n-1)\right|(2n-4) \\ &\quad +\left|\frac{4n(n-1)}{2n-1}-\beta_1\right|+\left|\beta_2-\frac{4n(n-1)}{2n-1}\right|, \end{aligned}$$

where β_1 and β_2 are the roots of the equation,

$$\beta^2+(6n^2-20n+14)\beta-2(n-1)(21n^2-50n+29)=0.$$

Proof The graph $K_nO_vK_n$ is of diameter 2 has two sets of vertices one at a distance 2 from each other and other at 1.There is one vertex of degree $(2n-2)$ and remaining $(2n-2)$ of degree $(n-1)$. With suitable labeling the L_{DSD} of $K_nO_vK_n$ takes the form

$L_{DSD}(K_nO_vK_n)=$

$$\begin{bmatrix} 2(2n-2) & -(3n-3) & -(3n-3) & \dots & -(3n-3) & -(3n-3) & -(3n-3) & \dots & -(3n-3) \\ -(3n-3) & 2(n-1) & -2(n-1) & \dots & -2(n-1) & -4(n-1) & -4(n-1) & \dots & -4(n-1) \\ -(3n-3) & -2(n-1) & 2(n-1) & \dots & -2(n-1) & -4(n-1) & -4(n-1) & \dots & -4(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -(3n-3) & -2(n-1) & -2(n-1) & \dots & 2(n-1) & -4(n-1) & -4(n-1) & \dots & -4(n-1) \\ -(3n-3) & -4(n-1) & -4(n-1) & \dots & -4(n-1) & 2(n-1) & -2(n-1) & \dots & -2(n-1) \\ -(3n-3) & -4(n-1) & -4(n-1) & \dots & -4(n-1) & -2(n-1) & 2(n-1) & \dots & -2(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -(3n-3) & -4(n-1) & -4(n-1) & \dots & -4(n-1) & -2(n-1) & -2(n-1) & \dots & 2(n-1) \end{bmatrix}$$

So that the Laplacian degree sum distance polynomial is

$|\beta I-L_{DSD}(K_nO_vK_n)|=$

$$\begin{bmatrix} \beta-2(2n-2) & (3n-3) & (3n-3) & \dots & (3n-3) & (3n-3) & (3n-3) & \dots & (3n-3) \\ (3n-3) & \beta-2(n-1) & 2(n-1) & \dots & 2(n-1) & 4(n-1) & 4(n-1) & \dots & 4(n-1) \\ (3n-3) & 2(n-1) & \beta-2(n-1) & \dots & 2(n-1) & 4(n-1) & 4(n-1) & \dots & 4(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (3n-3) & 2(n-1) & 2(n-1) & \dots & \beta-2(n-1) & 4(n-1) & 4(n-1) & \dots & 4(n-1) \\ (3n-3) & 4(n-1) & 4(n-1) & \dots & 4(n-1) & \beta-2(n-1) & 2(n-1) & \dots & 2(n-1) \\ (3n-3) & 4(n-1) & 4(n-1) & \dots & 4(n-1) & 2(n-1) & \beta-2(n-1) & \dots & 2(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (3n-3) & 4(n-1) & 4(n-1) & \dots & 4(n-1) & 2(n-1) & 2(n-1) & \dots & \beta-2(n-1) \end{bmatrix}$$

$$\begin{aligned} |\beta I-L_{DSD}(K_nO_vK_n)| &= [\beta-2(n-1)(n+1)][\beta-4(n-1)]^{2n-4} \\ &\quad \times[\beta^2+(6n^2-20n+14)\beta-2(n-1)(21n^2-50n+29)]. \end{aligned}$$

Using double average degree as

$$doubleavd(K_nO_vK_n) = \frac{4n(n-1)}{2n-1},$$

we get the desired result. □

On similar lines we get the Laplacian degree sum distance energy of $K_nO_eK_n$.

Theorem 3.8 *The LE_{DSD} of $K_nO_eK_n$ is,*

$$\begin{aligned} LE_{DSD}(K_nO_eK_n) &= \left|2\left(\frac{n^2-n-1}{n-1}\right) - 4(n-1)\right|(2n-4) \\ &+ \left|2\left(\frac{n^2-n-1}{n-1}\right) - 4(2n-3)\right| \\ &+ \left|2\left(\frac{n^2-n-1}{n-1}\right) - \beta_1\right| + \left|2\left(\frac{n^2-n-1}{n-1}\right) - \beta_2\right|, \end{aligned}$$

where β_1 and β_2 are roots of the equation, $\beta^2 + 2(n-1)(3n-8)\beta - 4(3n-4)^2(n-2) = 0$.

Proof The graph $K_nO_eK_n$ has two sets of vertices one at a distance 2 from each other and another at distance 1, being of diameter 2. There are two vertices of degree $(2n-3)$ and remaining $(2n-4)$ of degree $(n-1)$. With suitable labeling the L_{DSD} of $K_nO_eK_n$ takes the form

$$L_{DSD}(K_nO_eK_n) =$$

$$\begin{bmatrix} 2(2n-3) & -2(2n-3) & -(3n-4) & \dots & -(3n-4) & -(3n-4) & -(3n-4) & \dots & -(3n-4) \\ -2(2n-3) & 2(2n-3) & -(3n-4) & \dots & -(3n-4) & -(3n-4) & -(3n-4) & \dots & -(3n-4) \\ -(3n-4) & -(3n-4) & 2(n-1) & \dots & -2(n-1) & -4(n-1) & -4(n-1) & \dots & -4(n-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \\ -(3n-4) & -(3n-4) & -2(n-1) & \dots & 2(n-1) & -4(n-1) & -4(n-1) & \dots & -4(n-1) \\ -(3n-4) & -(3n-4) & -4(n-1) & \dots & -4(n-1) & 2(n-1) & -4(n-1) & \dots & -4(n-1) \\ -(3n-4) & -(3n-4) & -4(n-1) & \dots & -4(n-1) & -2(n-1) & 2(n-1) & \dots & -2(n-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \\ -(3n-4) & -(3n-4) & -4(n-1) & \dots & -4(n-1) & -2(n-1) & -2(n-1) & \dots & 2(n-1) \end{bmatrix}$$

So that the Laplacian degree sum distance polynomial is

$$|\beta - L_{DSD}(K_nO_eK_n)| = [\beta-4(n-1)]^{2n-6}[\beta-4(2n-3)][\beta^2+2(n-1)(3n-8)\beta-4(3n-4)^2(n-2)]$$

Using double average degree as

$$doubleavd(K_nO_eK_n) = 2\left(\frac{n^2-n-1}{n-1}\right),$$

we get the desired result. □

§4. QE_{DSD} of Some Graphs

We now state without proof results on QE_{DSD} which follow on similar lines like LE_{DSD} proved in previous section.

Theorem 4.1 *The signless Laplacian degree sum distance energy of the complete bipartite graph $K_{m,n}$ is*

$$\begin{aligned} QE_{DSD}(K_{m,n}) &= \left|\frac{2n(m-n)(m-1)}{m+n}\right| + \left|\frac{2m(m-n)(n-1)}{m+n}\right| \\ &+ \left|\frac{4mn}{m+n} - \gamma_1\right| + \left|\frac{4mn}{m+n} - \gamma_2\right|, \end{aligned}$$

where γ_1 and γ_2 are the roots of the equation

$$\gamma^2 + 2(n+m-4mn)\gamma + mn(14mn - m^2 - n^2 - 8m - 8n + 4) = 0.$$

Corollary 4.2 *If $m = 1$ we get star graph $K_{1,n}$ whose signless Laplacian degree sum distance energy is*

$$QE_{DSD}(K_{1,n}) = \left|\frac{4n}{n+1} - 2\right|(n-1) + \left|\gamma_1 - \frac{4n}{n+1}\right| + \left|\gamma_2 - \frac{4n}{n+1}\right|,$$

where γ_1 and γ_1 are roots of the equation, $\gamma^2 - 2(3n-1)\gamma - n(n^2 - 6n + 5) = 0$.

Theorem 4.3 *The QE_{DSD} of $K_n - e$ is*

$$\begin{aligned} QE_{DSD}(K_n - e) &= \left|\frac{2(n-2)(n+1)}{n}\right|(n-3) + \left|\frac{2(n-2)(n+1)}{n} + 2(n-1)\right| \\ &+ \left|\frac{2(n-2)(n+1)}{n} - \gamma_1\right| + \left|\frac{2(n-2)(n+1)}{n} - \gamma_2\right|, \end{aligned}$$

where γ_1 and γ_2 are roots of the equation,$\gamma^2 + (2n^2 - 8)\gamma + (4n^3 - 20n^2 + 30n - 12) = 0$.

Theorem 4.4 *The QE_{DSD} of $K_n + e$ is*

$$\begin{aligned} QE_{DSD}(K_n + e) &= \left|\frac{2(n^2-n+2)}{n+1} + 2(n-1) - (2n-2)\right|(n-2) \\ &+ \left|\frac{2(n^2-n+2)}{n+1} + 2(n-1) + (2n-2)(n-2)\right| \\ &+ \left|\frac{2(n^2-n+2)}{n+1} - \gamma_1\right| + \left|\frac{2(n^2-n+2)}{n+1} - \gamma_2\right| + \left|\frac{2(n^2-n+2)}{n+1} - \gamma_3\right|, \end{aligned}$$

where γ_1, γ_2 *and* γ_3 *are roots of the equation*

$$\begin{aligned}&\gamma^3-2(2n+(n-1)(n-2))\gamma^2+(4(n^2-1)(n-1)-4n^2(n-1)\\&-(2n-1)^2(n-1)-(n+1)^2+4n)\gamma+2((2n-1)^2(n-1)+4n^3(n-1)\\&+(n-1)^4-2n(n^2-1)(2n-1))=0.\end{aligned}$$

Theorem 4.5 *The* QE_{DSD} *of* $K_nO_vK_n$ *is*

$$\begin{aligned}QE_{DSD}(K_nO_vK_n) &= \left|\frac{4n(n-1)}{2n-1}\right|(2n-4)+\left|2(n-1)^2-\frac{4n(n-1)}{2n-1}\right|\\&+\left|\frac{4n(n-1)}{2n-1}-\gamma_1\right|+\left|\frac{4n(n-1)}{2n-1}-\gamma_2\right|,\end{aligned}$$

where γ_1 *and* γ_2 *are roots of the equation* $\gamma^2-2(n-1)(3n-1)\gamma+6(n-1)^3=0$.

Theorem 4.6 *The* QE_{DSD} *of* $K_nO_eK_n$ *is*

$$\begin{aligned}QE_{DSD}(K_nO_eK_n) &= \left|2\left(\frac{n^2-n-1}{n-1}\right)\right|(2n-5)+\left|2\left(\frac{n^2-n-1}{n-1}\right)-(2n-4)(n-1)\right|\\&+\left|\gamma_1-2\left(\frac{n^2-n-1}{n-1}\right)\right|+\left|\gamma_2-2\left(\frac{n^2-n-1}{n-1}\right)\right|,\end{aligned}$$

where γ_1 *and* γ_2 *are the roots of equation* $\gamma^2-2n(3n-5)\gamma+4(n-2)(3n^2-6n+2)=0$.

§5. Conclusion

We defined the L_{DSD} and Q_{DSD} of a graph, obtained expressions for energy of some graphs.

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