Laplacian Energy of Certain Graphs

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Abstract: Let $G$ be a graph with $n$ vertices and $m$ edges. Let $\mu_1, \mu_2, \cdots, \mu_n$ be the eigenvalues of the Laplacian matrix of $G$. The Laplacian energy $LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$. In this paper, we calculate the exact Laplacian energy of complete graph, complete bipartite graph, path, cycle and friendship graph.

Key Words: Complete graph, complete bipartite graph, path, cycle, friendship graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected, simple graph $G$ with $n$ vertices and $m$ edges. Let $d_i$ be the degree of the $i^{th}$ vertex of $G$, $i = 1, 2, \cdots, n$.

Definition 1.1([3]) Let $A(G) = [a_{ij}]$ be the $(0, 1)$ adjacency matrix, $D(G) = \text{diag}(d_1, d_2, \cdots, d_n)$, the diagonal matrix with vertex degrees $d_1, d_2, \cdots, d_n$ of its vertices $v_1, v_2, \cdots, v_n$ of a graph $G$. Then $L(G) = D(G) - A(G)$ is called the Laplacian matrix of the graph $G$.

It is symmetric, singular and positive semi-definite. All its eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ are real and nonnegative and form the Laplacian spectrum. It is well known that one of the eigenvalues is zero.

Definition 1.2([3]) If $G$ is a graph with $n$ vertices and $m$ edges, and its Laplacian eigen values are $\mu_1, \mu_2, \cdots, \mu_n$ then the Laplacian energy of $G$, denoted by $LE(G)$, is $\sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|$. i.e.,

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$ 

This quantity has a long known chemical application for details see the surveys [1,4,5]. If the graph $G$ has one vertex then the Laplacian energy is zero.

Property 1.3([3])

(1) $LE(G) \leq \sqrt{2Mn}$;

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(2) \[ LE(G) \leq \frac{2m}{n} + \sqrt{(n-1) \left[ 2M - \left( \frac{2m}{n} \right)^2 \right]}; \]

(3) \[ 2\sqrt{M} \leq LE(G) \leq 2M, \text{ where } M = m + \frac{1}{2} \sum_{i=1}^{n} \left( d_i - \frac{2m}{n} \right)^2. \]

§2. The Laplacian Energy of Complete Graphs

\textbf{Definition 2.1} ([2]) A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph.

\textbf{Theorem 2.2} The Laplacian energy of the complete graph \( K_n \) on \( n \) vertices is \( 2(n-1) \).

\textit{Proof} The eigenvalues of the Laplacian matrix of the complete graph \( K_n \) on \( n \) vertices and \( \frac{n(n-1)}{2} \) edges are \( \mu_1 = 0 \) and multiplicity of the eigen values \( n \) as \( n-1 \), i.e., \( \mu_1 = 0, \mu_2 = \mu_3 = \cdots = \mu_n = n \). Thus

\[ LE(K_n) = \sum_{i=1}^{n} |(\mu_i - (n-1)| = |0-(n-1)| + (n-1)|n-(n-1)| = 2(n-1). \] \[ \square \]

§3. The Laplacian Energy of Complete Bipartite Graphs

\textbf{Definition 3.1} ([2]) A bipartite graph is one whose vertex set can be partitioned into two subsets \( X \) and \( Y \), so that each edge has one end in \( X \) and one end in \( Y \); such a partition \((X,Y)\) is called a bipartition of the graph.

\textbf{Definition 3.2} ([2]) A complete bipartite graph is a simple bipartite graph with bipartition \((X,Y)\) in which each vertex of \( X \) is joined to each vertex of \( Y \); if \( |X| = m \) and \( |Y| = n \), such a graph is denoted by \( K_{m,n} \).

\textbf{Definition 3.3} ([6]) The Star graph \( K_{1,n} \) is a tree on \( n+1 \) vertices with one vertex having degree \( n \) and the other \( n \) vertices having degree 1.

\textbf{Theorem 3.4} The Laplacian energy of the complete bipartite graph \( K_{m,n} \) with \( m + n \) vertices and \( mn \) edges is

\[ \frac{(m+n)^2 + |m-n| (2mn - (m+n))}{(m+n)}. \]

\textit{Proof} In this graph, the Laplacian spectrum is \( \mu_1 = 0 \), the multiplicity of the eigen values \( m \) as \( n-1 \), the multiplicity of the eigen values \( n \) as \( m-1 \) and \( \mu_{m+n} = m+n \).
The Laplacian energy

\[ LE(K_{m,n}) = \sum_{i=1}^{n+m} \left| \mu_i - \frac{2mn}{m+n} \right| \]

\[ = |0 - \frac{2mn}{m+n}| + (n-1)|m - \frac{2mn}{m+n}| + (m-1)|n - \frac{2mn}{m+n}| + (m+n) - \frac{2mn}{m+n} \]

\[ = \frac{2mn}{m+n} + \frac{m(n-1)}{m+n}|m-n| + \frac{n(m-1)}{m+n}|n-m| \]

\[ = \frac{(m+n)^2 + |m-n|(2mn - (m+n))}{m+n} \].

\[ \square \]

**Corollary 3.5** The Laplacian energy of a star graph \( K_{1,n} \) is \( \frac{2(n^2+1)}{n+1} \).

**Proof** Let \( m \) be replaced by one in Theorem 3.4. We get the following

\[ LE(K_{1,n}) = \frac{(1+n)^2 + |1-n|(2n-(1+n))}{1+n} = \frac{2(n^2+1)}{n+1} \].

\[ \square \]

§4. The Laplacian Energy of Paths \( P_n \) and Cycles \( C_n \)

**Definition 4.1** A path \( P_n \) with \( n \) vertices has \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \) for its vertex set and \( E(P_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n\} \) is its edge set. This path \( P_n \) is said to have length \( n-1 \).

**Definition 4.2** A cycle \( C_n \) with \( n \) points is a graph with vertex set \( V(C_n) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\} \).

**Theorem 4.3** The Laplacian energy of the path \( P_n \) with \( n \) vertices is

\[ \sum_{i=0}^{n-1} \left| 2 \left( \frac{1}{n} - \cos \left( \frac{\pi i}{n} \right) \right) \right| . \]

**Proof** The eigen values of the Laplacian matrix of \( P_n \) are \( 2 \left[ 1 - \cos \left( \frac{\pi i}{n} \right) \right] \), \( i = 0, 1, \ldots, n-1 \). Then,

\[ LE(P_n) = \sum_{i=0}^{n-1} \left| 2 \left( 1 - \cos \left( \frac{\pi i}{n} \right) \right) \right| - \frac{2(n-1)}{n} = \sum_{i=0}^{n-1} \left| 2 \left( \frac{1}{n} - \cos \left( \frac{\pi i}{n} \right) \right) \right| . \]

\[ \square \]

**Theorem 4.4** The Laplacian energy of the cycle \( C_n \) with \( n \) vertices is \( 2 \sum_{i=0}^{n-1} \left| \cos \left( \frac{2\pi i}{n} \right) \right| . \)

**Proof** The Laplacian spectrum of the cycle \( C_n \) is \( 2 \left[ 1 - \cos \left( \frac{2\pi i}{n} \right) \right] \), \( i = 0, 1, \ldots, (n-1) \). Then

\[ LE(C_n) = \sum_{i=0}^{n-1} \left| 2 \left[ 1 - \cos \left( \frac{2\pi i}{n} \right) \right] - 2 \right| = 2 \sum_{i=0}^{n-1} \left| \cos \left( \frac{2\pi i}{n} \right) \right| . \]

\[ \square \]
§5. The Laplacian Energy of Friendship Graphs

**Definition 5.1**\([6]\) The friendship graph \(F_r (r \geq 1)\) consists of \(r\) triangles with a common vertex.

**Illustration.** The Friendship graph \(F_4\) consists of 4 triangles with a common vertex is as shown in Fig.1.

![Fig.1 Friendship graph \(F_4\)](image)

The Laplacian matrix of \(F_2\) is

\[
\begin{bmatrix}
  4 & -1 & -1 & -1 & -1 \\
  -1 & 2 & -1 & 0 & 0 \\
  -1 & -1 & 2 & 0 & 0 \\
  -1 & 0 & 0 & 2 & -1 \\
  -1 & 0 & 0 & -1 & 2 \\
\end{bmatrix}
\]

**Theorem 5.2** The Laplacian energy of the friendship graph \(F_r\) is \(\frac{8r^2 + 2r + 2}{2r + 1}\), where \(r \geq 1\).

**Proof** The friendship graph \(F_r\) has \(2r + 1\) vertices and \(3r\) edges. Its Laplacian matrix has \(2r + 1\) eigen values. These eigen values are \(\mu_1 = 2r + 1\), the multiplicity of the eigen value 3 as \(r\), the multiplicity of the eigen value 1 as \(r - 1\) and \(\mu_{2r+1} = 0\).

By definition, the Laplacian energy

\[
LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.
\]

Thus,

\[
LE(F_r) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| = \sum_{i=1}^{n} \left| \mu_i - \frac{6r}{2r + 1} \right|
\]

\[
= \left| \frac{2r + 1 - 6r}{2r + 1} \right| + r \left| 3 - \frac{6r}{2r + 1} \right| + (r - 1) \left| 1 - \frac{6r}{2r + 1} \right| + 0 - \frac{6r}{2r + 1}
\]

\[
= \left| \frac{4r^2 - 2r + 1}{2r + 1} \right| + r \frac{3}{2r + 1} + (r - 1) \frac{4r - 1}{2r + 1} + \frac{6r}{2r + 1} = \frac{8r^2 + 2r + 2}{2r + 1}
\]
since \(4r^2 + 1 > 2r\) and \(1 - 4r < 0\).

**Corollary 5.1** If \(G\) is the friendship graph of \(n\) vertices then \(LE(G) = \frac{2n^2 - 3n + 3}{n}\).

**Proof** Replacing \(r\) by \(\frac{n-1}{2}\) in Theorem 5.2, we get the result.

**Corollary 5.2** If \(G\) is the friendship graph of \(m\) edges then \(LE(G) = \frac{2}{3} \left[ \frac{4m^2 + 3m + 9}{2m + 3} \right]\).

**Proof** Let \(r\) be replaced by \(\frac{m}{3}\) in Theorem 5.2, we get the result. From [3], \(M = m + \frac{1}{2} \sum_{i=1}^{n} (d_i - \frac{2m}{n})^2\). In a friendship graph \(M = \frac{r}{2r+1} (4r^2 - 2r + 7)\). Therefore, \(2Mn = 2r (4r^2 - 2r + 7)\). Hence, using Property 1.3, we get the following

\[
2\sqrt{\frac{r}{2r+1} (4r^2 - 2r + 7)} \leq LE(G) \leq \frac{2r}{2r+1} (4r^2 - 2r + 7).
\]

**References**


