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Metric on L-Fuzzy Real Line

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Abstract: In this study, the concept of L-fuzzy real numbers which is given in [14] is extended by presenting the definition from both-sided. For each side, different functions are defined and it is proved that these functions are metrics. For that, it is shown that for a complete lattice L, given conditions in [14] for an equivalence relation $\sim$ on $md_{\mathbb{R}}(L)$ are equivalent. So condition is weakened in our work. A metric which is consistent with the Euclidean metric is defined by using two-sided metrics. Also, an example is given for L-Fuzzy metric.

Key Words: L-fuzzy real line, metric, chain.

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§1. Introduction

Firstly, we give the concepts of L-fuzzy real number and equivalence relation.

Definition 1.1([11]) *Let $(L, \leq)$ be a complete lattice. Then, $\lambda \in L^{\mathbb{R}}$ is called L-fuzzy real number $\Leftrightarrow$*

(*i*) *$\exists x_0 \in \mathbb{R}$ such that $\lambda(x_0) = 1$;*

(*ii*) *For all $a \in L$, $\lambda_{[a]}$ level subset is closed interval.*

Definition 1.2([9]) *Let $(L, \leq)$ be a complete lattice and*

$$md_{\mathbb{R}}(L) = \left\{ \lambda \in L^{\mathbb{R}} : \bigvee_{t\in\mathbb{R}} \lambda(t) = 1, \quad \bigwedge_{t\in\mathbb{R}} \lambda(t) = 0,\ \lambda \text{ monotonous decreasing} \right\}.$$

For $\forall \lambda \in md_{\mathbb{R}}(L)$ and all $t \in \mathbb{R}$, let

$$\lambda(t-) := \bigwedge_{s<t} \lambda(s) \quad \text{and} \quad \lambda(t+) := \bigvee_{s>t} \lambda(s).$$

Then, an equivalence relation "$\sim$" on $md_{\mathbb{R}}(L)$ is defined as follow:

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for $\lambda, \mu \in md_{\mathbb{R}}(L)$,

$$\lambda \sim \mu :\Leftrightarrow \forall t \in \mathbb{R},\ \lambda(t-) = \mu(t-)\ \ , \lambda(t+) = \mu(t+).$$

The set of equivalence classes containing λ is denoted by

$$[\lambda] := \{\mu \in md_{\mathbb{R}}(L) : \mu \sim \lambda\}.$$

Let $\mathbb{R}[L]_{right} := \left\{[\lambda] : \lambda \in md_{\mathbb{R}}(L)_{right}\right\}$ be set of all equivalence classes with respect to " $\sim$ " equivalence relation on $md_{\mathbb{R}}(L)$. $\mathbb{R}[L]_{right}$ is called L-fuzzy real line.

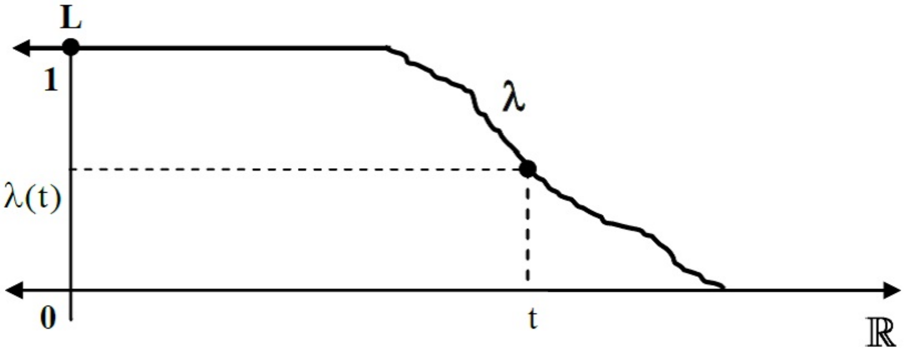

Figure 1. $[\lambda] \in \mathbb{R}[L]_{right}$

The concepts of L-fuzzy unit interval and L-fuzzy real line have an important place in L-fuzzy topoligical spaces. The metrics are very essential tools in various fields of sciences that measure the distance or difference between two points. Firstly, the concept of the L-fuzzy unit interval was given in 1975 by Hutton [7]. In 1982, Rodabaugh defined the fuzzy addition process on the fuzzy real line taking L-complete lattice instead of $[0,1]$ [11].

In 1983, Lowen examined the algebraic structure of the L-fuzzy real line [10]. Wang gave the necessary and sufficient condition on the convergence of infinite sums by giving infinite additive concepts on the fuzzy real line [12]. S. Göhler ve Werner Göhler examined the topological properties of the fuzzy real line by defining two special fuzzy metrics on the fuzzy real line [4]. Diamond [2] defined a metric for the triangular fuzzy numbers. Kaufmann et al. considered a distance of two fuzzy numbers combined by the interval of a-cuts of fuzzy numbers [8]. Heilpern proposed three definitions of the distance between two fuzzy numbers [5].

In 2007, Han-Liang Huang and Fu-Gui Shi gave the concepts of L-fuzzy numbers and L-fuzzy convex sets on L completely distributive lattice [6]. Recently, Jian-zhong Xiao and Xing-hua Zhu have studied the metric structure of the fuzzy real line by giving the semi-metric concept on the fuzzy real towards the L completely distributive lattice [13]. Allahviranloo et al. gave a metric based on modified Euclidean metric on interval numbers, for $L-R$ fuzzy numbers with fixed $L(.)$ and $R(.)$ is introduced [1]. García, J.G. and Kubiak showed how the Hutton's concept evolved [3].

§2. Main Results for Metric on the L-Fuzzy Real Line

In this section, some new concepts and some theorems and results related to these concepts are given as parallel to the concepts given in Definition 1.2. In addition, using theorems and the results in this section, a metric on the L-fuzzy real line was created for complete lattice $(L, \leq)$.

Theorem 2.1 *Let $(L, \leq)$ be a complete lattice, $\lambda, \mu \in md_{\mathbb{R}}(L)$ and for $t_0 \in \mathbb{R}$, $\lambda(t_0-) \neq \mu(t_0-)$. Then,*

(i) For $\forall \varepsilon > 0,\ \exists s_0 \in (t_0 - \varepsilon, t_0) : \lambda(t_0-) \nleq \mu(s_0)$ or $\mu(t_0-) \nleq \lambda(s_0)$;

(ii) For $\forall \varepsilon > 0,\ \exists s_0 \in (t_0 - \varepsilon, t_0) : \lambda(s_0+) \neq \mu(s_0+)$.

Proof (i) Let $\lambda(t_0-) \neq \mu(t_0-)$ for $t_0 \in \mathbb{R}$. Let's assume that

$$\text{For } \exists \varepsilon > 0 : \forall s \in (t_0 - \varepsilon, t_0), \quad \lambda(t_0-) \leq \mu(s) \text{ and } \mu(t_0-) \leq \lambda(s).$$

Then,

$$\lambda(t_0-) \leq \bigwedge_{t_0-\varepsilon<s<t_0} \mu(s) \text{ and } \mu(t_0-) \leq \bigwedge_{t_0-\varepsilon<s<t_0} \lambda(s).$$

Since λ and μ are decreasing,

$$\lambda(t_0-) \leq \bigwedge_{t_0-\varepsilon<s<t_0} \mu(s) = \bigwedge_{s<t_0} \mu(s) = \mu(t_0-) \Rightarrow \lambda(t_0-) \leq \mu(t_0-) \tag{2.1}$$

and

$$\mu(t_0-) \leq \bigwedge_{t_0-\varepsilon<s<t_0} \lambda(s) = \bigwedge_{s<t_0} \lambda(s) = \lambda(t_0-) \Rightarrow \mu(t_0-) \leq \lambda(t_0-). \tag{2.2}$$

From (2.1) and (2.2), $\lambda(t_0-) = \mu(t_0-)$. This contradicts the hypothesis of the theorem. So,

$$\forall \varepsilon > 0,\ \exists s_0 \in (t_0 - \varepsilon, t_0) : \lambda(t_0-) \nleq \mu(s_0) \text{ or } \mu(t_0-) \nleq \lambda(s_0).$$

(ii) Let $\lambda(t_0-) \neq \mu(t_0-)$ for $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. From (i),

$$\exists s_0 \in (t_0 - \varepsilon, t_0) : \lambda(t_0-) \nleq \mu(s_0) \text{ or } \mu(t_0-) \nleq \lambda(s_0).$$

Without loss of generality, let $\lambda(t_0-) \nleq \mu(s_0)$. Since μ is decreasing, $\mu(s_0+) \leq \mu(s_0)$. Hence,

$$\lambda(t_0-) \nleq \mu(s_0+). \tag{2.3}$$

On the other hand,

$$\{\lambda(s) : s_0 < s < t_0\} \subset \{\lambda(s) : s < t_0\}.$$

Therefore,

$$\bigvee_{s_0<s<t_0} \lambda(s) \geq \bigwedge_{s<t_0} \lambda(s) = \lambda(t_0-)$$

is obtained. So,

$$\lambda(t_0-) \leq \bigvee_{s_0<s<t_0} \lambda(s).$$

Since λ is decreasing,

$$\lambda(t_0-) \leq \bigvee_{s_0<s<t_0} \lambda(s) = \bigvee_{s_0<s} \lambda(s) = \lambda(s_0+).$$

From (2.3), since $\lambda(t_0-) \nleq \mu(s_0+)$, $\lambda(s_0+) \nleq \mu(s_0+)$. So, $\lambda(s_0+) \neq \mu(s_0+)$. □

Theorem 2.2 *Let* $(L, \leq)$ *be a complete lattice,* $\lambda, \mu \in md_{\mathbb{R}}(L)$ *and* $\lambda(t_0+) \neq \mu(t_0+)$ *for* $t_0 \in \mathbb{R}$*. Then,*

(i) *For* $\forall \varepsilon > 0,\ \exists s_0 \in (t_0, t_0+\varepsilon) : \mu(s_0) \nleq \lambda(t_0+)$ *or* $\lambda(s_0) \nleq \mu(t_0+)$;

(ii) *For* $\forall \varepsilon > 0,\ \exists s_0 \in (t_0, t_0+\varepsilon) : \lambda(s_0-) \neq \mu(s_0-)$.

Proof (i) Let $\lambda(t_0+) \neq \mu(t_0+)$ for $t_0 \in \mathbb{R}$. Let's assume that

$$\text{For } \exists \varepsilon > 0 : \forall s \in (t_0, t_0+\varepsilon),\quad \mu(s) \leq \lambda(t_0+) \text{ and } \lambda(s) \leq \mu(t_0+).$$

Then,

$$\bigvee_{t_0<s<t_0+\varepsilon} \mu(s) \leq \lambda(t_0+) \quad \text{and} \quad \bigvee_{t_0<s<t_0+\varepsilon} \lambda(s) \leq \mu(t_0+).$$

Since λ and μ are decreasing,

$$\lambda(t_0+) = \bigvee_{t_0<s} \lambda(s) = \bigvee_{t_0<s<t_0+\varepsilon} \lambda(s) \leq \mu(t_0+) \Rightarrow \lambda(t_0+) \leq \mu(t_0+) \tag{2.4}$$

and

$$\mu(t_0+) = \bigvee_{t_0<s} \mu(s) = \bigvee_{t_0<s<t_0+\varepsilon} \mu(s) \leq \lambda(t_0+) \Rightarrow \mu(t_0+) \leq \lambda(t_0+). \tag{2.5}$$

From (2.4) and (2.5), $\lambda(t_0+) = \mu(t_0+)$ is obtained. This contradicts the hypothesis of the theorem. So,

$$\forall \varepsilon > 0,\ \exists s_0 \in (t_0, t_0+\varepsilon) : \mu(s_0) \nleq \lambda(t_0+) \text{ or } \lambda(s_0) \nleq \mu(t_0+).$$

(ii) Let $\lambda(t_0+) \neq \mu(t_0+)$ for $t_0 \in \mathbb{R}$ and $\varepsilon > 0$. From (i),

$$\exists s_0 \in (t_0, t_0+\varepsilon) : \mu(s_0) \nleq \lambda(t_0+) \text{ or } \lambda(s_0) \nleq \mu(t_0+).$$

Without loss of generality, let $\lambda(s_0) \nleq \mu(t_0+)$. Since λ is decreasing, $\lambda(s_0) \leq \lambda(s_0-)$. Hence,

$$\lambda(s_0-) \nleq \mu(t_0+). \tag{2.6}$$

On the other hand,

$$\{\mu(s) : t_0 < s < s_0\} \subset \{\mu(s) : t_0 < s\}.$$

Therefore,

$$\bigwedge_{t_0<s<s_0} \mu(s) \leq \bigvee_{t_0<s} \mu(s) = \mu(t_0+)$$

is obtained. So, $\bigwedge_{t_0<s<s_0} \mu(s) \leq \mu(t_0+)$. Since μ is decreasing,

$$\mu(s_0-) = \bigwedge_{s<s_0} \mu(s) = \bigwedge_{t_0<s<s_0} \mu(s) \leq \mu(t_0+).$$

Hence $\mu(s_0-) \leq \mu(t_0+)$.

From (2.6), since $\lambda(s_0-) \nleq \mu(t_0+)$, $\lambda(s_0-) \nleq \mu(s_0-)$. So, $\lambda(s_0-) \neq \mu(s_0-)$. □

The following result is obtained from Theorem 2.1(ii) and Theorem 2.2(ii).

Corollary 2.1 *Let* $(L, \leq)$ *be a complete lattice,* $\lambda, \mu \in md_{\mathbb{R}}(L)$. *Then*

$$\lambda(t_0-) \neq \mu(t_0-) \quad \text{for a } t_0 \in \mathbb{R} \ \Leftrightarrow\ \lambda(s_0+) \neq \mu(s_0+) \quad \text{for a } s_0 \in \mathbb{R}.$$

Theorem 2.3 *Defined as maping* $d_{right} : \mathbb{R}[L]_{right} \times \mathbb{R}[L]_{right} \to [0, +\infty)$ *is a metric on set* $\mathbb{R}[L]_{right}$.

$$d_{right}([\lambda], [\mu]) := \sup\left\{ \left| \bigvee\{t : \lambda(t) \geq k\} - \bigvee\{t : \mu(t) \geq k\} \right| : k \in L \setminus \{0\} \right\}.$$

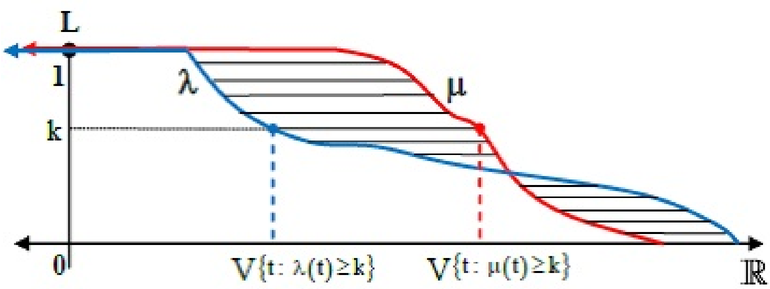

Figure 2. $\mathbb{R}[L]_{right}; d_{right}$

Proof (i) Let's show that $[\lambda] = [\mu] \Rightarrow d_{right}([\lambda], [\mu]) = 0$. Let $k \in L \setminus \{0\}$ be an arbitrary and constant. Let's define as

$$t_1 := \bigvee_{\lambda(t)\geq k} t, \quad t_2 := \bigvee_{\mu(t)\geq k} t$$

and

$$A := \{t : \lambda(t) \geq k\}, \qquad B := \{t : \mu(t) \geq k\}.$$

Then, $\exists t' \in B : t_2 - \varepsilon < t'$ for arbitrary and constant $\varepsilon > 0$. So, $\mu(t') \geq k$ and $t_2 - \varepsilon < t'$. Since $[\lambda] = [\mu]$, $\lambda(t'-) = \mu(t'-)$.

On the other hand, since μ is decreasing, $\mu(t') \leq \mu(t'-)$. Hence,

$$k \leq \mu(t') \leq \mu(t'-) = \lambda(t'-).$$

That is, $k \leq \lambda(t'-)$. On the other hand,

$$\exists t^* \in \mathbb{R} : t_2 - \varepsilon < t^* < t' \quad \text{for} \quad \varepsilon > 0.$$

Since $k \leq \lambda(t'-)$ and λ is decreasing, $k \leq \lambda(t'-) = \bigwedge_{s<t'} \lambda(s) \leq \lambda(t^*)$. So, $t_2 - \varepsilon < t^*$ and $t^* \in A$ are obtained. Since $\varepsilon > 0$ is arbitrary, $t_2 \leq \bigvee A = t_1$. Hence,

$$t_2 \leq t_1. \tag{2.7}$$

Now, let's show that $t_1 \leq t_2$ and $\exists t' \in A : t_1 - \varepsilon < t'$ for arbitrary and constant $\varepsilon > 0$. So $\lambda(t') \geq k$ and $t_1 - \varepsilon < t'$. Since $[\lambda] = [\mu]$, $\lambda(t'-) = \mu(t'-)$. On the other hand, since λ is decreasing, $\lambda(t') \leq \lambda(t'-)$. So $k \leq \lambda(t') \leq \lambda(t'-) = \mu(t'-)$. That is, $k \leq \mu(t'-)$.

On the other hand,

$$\exists t^* \in \mathbb{R} : t_1 - \varepsilon < t^* < t' \quad \text{for} \quad \varepsilon > 0.$$

Since, $k \leq \mu(t'-)$ and μ is decreasing $k \leq \mu(t'-) = \bigwedge_{s<t'} \mu(s) \leq \mu(t^*)$ is obtained. Hence, $t_1 - \varepsilon < t^*$ and $t^* \in B$ is obtained. Since $\varepsilon > 0$ is arbitrary, $t_1 \leq \bigvee B = t_2$. So

$$t_1 \leq t_2. \tag{2.8}$$

From (2.7) and (2.8), $t_1 = t_2$ is obtained. Since $k \in L\backslash\{0\}$ is arbitrary,

$$d_{right}([\lambda],[\mu]) = 0.$$

Conversely, let's show that $d_{right}([\lambda],[\mu]) = 0 \quad \Rightarrow \quad [\lambda] = [\mu]$. In fact, let's assume that $[\lambda] \neq [\mu]$. In this case,

$$\exists t_0 \in \mathbb{R} : \lambda(t_0-) \neq \mu(t_0-) \text{ or } \lambda(t_0+) \neq \mu(t_0+).$$

From Corollary 2.1, $\lambda(t_0-) \neq \mu(t_0-)$. Then, for $a := \mu(t_0-) = \bigwedge_{s<t_0} \mu(s)$,

$$a \nleq \bigwedge_{s<t_0} \lambda(s) \text{ or } a \ngeq \bigwedge_{s<t_0} \lambda(s)$$

is written. Without loss of generality, let's assume that $a \nleq \bigwedge_{s<t_0} \lambda(s)$. Then,

$$\exists s_0 \in \mathbb{R} : s_0 < t_0 \text{ and } a \nleq \lambda(s_0). \tag{2.9}$$

Let $A := \{t : \lambda(t) \geq a\}$ and $B := \{t : \mu(t) \geq a\}$. Then, there is the assertion $\bigvee B \geq t_0$. Notice that $\exists s' \in \mathbb{R} : t_0 - \varepsilon < s' < t_0$ for all $\varepsilon > 0$, $\mu(s') \geq \bigwedge_{s<t_0} \mu(s) = a$. Hence $\mu(s') \geq a$. So $s' \in B$. Since $\varepsilon > 0$ is arbitrary, $t_0 \leq \bigvee B$.

According to hypothesis, since

$$d_{right}([\lambda], [\mu]) = \sup\left\{\left|\bigvee\{t : \lambda(t) \geq k\} - \bigvee\{t : \mu(t) \geq k\}\right| : k \in L\setminus\{0\}\right\} = 0,$$

and $\bigvee_{\mu(t)\geq a} t = \bigvee_{\lambda(t)\geq a} t$. Hence,

$$t_0 \leq \bigvee B = \bigvee_{\mu(t)\geq a} t = \bigvee A \Rightarrow t_0 \leq \bigvee A.$$

So, $\exists t' \in A : t_0 - \varepsilon < t'$ for $\varepsilon := t_0 - s_0 > 0$. From here $a \leq \lambda(t')$ and $s_0 < t'$. Since λ is decreasing, $a \leq \lambda(t')$ and $\lambda(t') \leq \lambda(s_0)$. So $a \leq \lambda(s_0)$. This contradicts (2.9). So $[\lambda] = [\mu]$ is obtained.

(ii) It is clearly that $d_{right}([\lambda], [\mu]) = d_{right}([\mu], [\lambda])$.

(iii) Let's show that $d_{right}([\lambda], [\eta]) \leq d_{right}([\lambda], [\mu]) + d_{right}([\mu], [\eta])$ as follows:

Let's define

$$t_\lambda(k_0) := \bigvee_{\lambda(t)\geq k_0} t, \quad t_\mu(k_0) := \bigvee_{\mu(t)\geq k_0} t, \quad t_\eta(k_0) := \bigvee_{\eta(t)\geq k_0} t$$

for arbitrary $k_0 \in L\,(k_0 \neq 0)$. Then,

$$|t_\lambda(k_0) - t_\eta(k_0)| \leq |t_\lambda(k_0) - t_\mu(k_0)| + |t_\mu(k_0) - t_\eta(k_0)|.$$

Here,

$$\sup_{k\in L, k>0} |t_\lambda(k) - t_\eta(k)| \leq \sup_{k\in L, k>0} |t_\lambda(k) - t_\mu(k)| + \sup_{k\in L, k>0} |t_\mu(k) - t_\eta(k)|.$$

As a result,

$$d_{right}([\lambda], [\eta]) \leq d_{right}([\lambda], [\mu]) + d_{right}([\mu], [\eta])$$

is obtained. □

Definition 2.3 *Let* $(L, \leq)$ *be a complete lattice,* $\lambda \in L^{\mathbb{R}}$ *and*

$$mi_{\mathbb{R}}(L) := \left\{ \lambda \in L^{\mathbb{R}} : \bigvee_{t \in \mathbb{R}} \lambda(t) = 1, \bigwedge_{t \in \mathbb{R}} \lambda(t) = 0, \ \lambda \text{ is monotonous increasing} \right\}.$$

For all $\lambda \in mi_{\mathbb{R}}(L)$ *and all* $\forall t \in \mathbb{R}$ *let,*

$$\lambda(t-) := \bigvee_{s<t} \lambda(s) \quad \text{and} \quad \lambda(t+) := \bigwedge_{s>t} \lambda(s),$$

Then, an equivalence relation " $\sim$ " *on* $mi_{\mathbb{R}}(L)$ *is defined as following:*

for $\lambda, \mu \in mi_{\mathbb{R}}(L)$,

$$\lambda \sim \mu :\Leftrightarrow \forall t \in \mathbb{R}, \lambda(t-) = \mu(t-), \lambda(t+) = \mu(t+)$$

and the set of equivalence classes containing λ *is defined as*

$$[\lambda] := \{\mu \in mi_{\mathbb{R}}(L) : \mu \sim \lambda\},$$

the set of all equivalence classes with respect to " $\sim$ " *equivalence relation on* $mi_{\mathbb{R}}(L)$ *is defined as*

$$\mathbb{R}[L]_{left} := \{[\lambda] : \ \lambda \in mi_{\mathbb{R}}(L)\}.$$

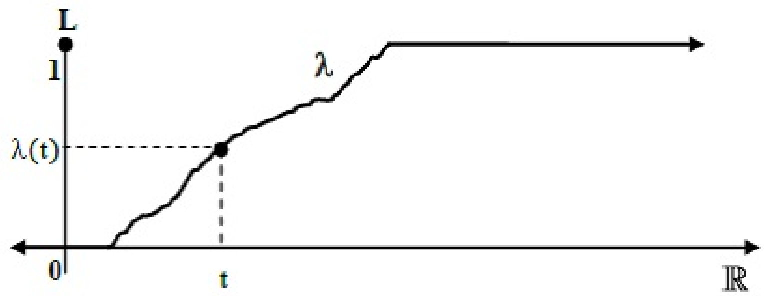

Figure 3. $[\lambda] \in \mathbb{R}[L]_{left}$

Theorem 2.4 *Let* $(L, \leq)$ *be a complete lattice,* $\lambda, \mu \in mi_{\mathbb{R}}(L)$ *and for* $t_0 \in \mathbb{R}$, $\lambda(t_0+) \neq \mu(t_0+)$. *Then,*

(*i*) *For* $\forall \varepsilon > 0$, $\exists s_0 \in (t_0, t_0 + \varepsilon) : \lambda(t_0+) \nleq \mu(s_0)$ *or* $\mu(t_0+) \nleq \lambda(s_0)$*;*
(*ii*) *For* $\forall \varepsilon > 0$, $\exists s_0 \in (t_0, t_0 + \varepsilon) : \lambda(s_0-) \neq \mu(s_0-)$.

Proof The proof of this theorem is similar to Theorem 2.1. □

Theorem 2.5 *Let* $(L, \leq)$ *be a complete lattice,* $\lambda, \mu \in mi_{\mathbb{R}}(L)_{left}$ *and for* $t_0 \in \mathbb{R}$, $\lambda(t_0-) \neq \mu(t_0-)$. *Then,*

(*i*) *For* $\forall \varepsilon > 0$, $\exists s_0 \in (t_0 - \varepsilon, t_0) : \mu(s_0) \nleq \lambda(t_0-)$ *or* $\lambda(s_0) \nleq \mu(t_0-)$*;*

(*ii*) *For* $\forall\varepsilon > 0,\ \exists s_0 \in (t_0 - \varepsilon, t_0) : \mu(s_0+) \neq \lambda(s_0+)$.

Proof The proof is similar to Theorem 2.2. □

The following result can be obtained from the Theorem 2.4(*ii*) and Theorem 2.5(*ii*).

Corollary 2.2 *Let* $(L, \leq)$ *be a complete lattice,* $\lambda, \mu \in mi_{\mathbb{R}}(L)$. *Then*

$$\lambda(t_0-) \neq \mu(t_0-) \quad \textit{for a } t_0 \in \mathbb{R} \Leftrightarrow \lambda(s_0+) \neq \mu(s_0+) \quad \textit{for a } s_0 \in \mathbb{R}.$$

Theorem 2.6 *Defined as the mapping* $d_{left} : \mathbb{R}[L]_{left} \times \mathbb{R}[L]_{left} \to [0, +\infty)$ *is a metric on the set* $\mathbb{R}[L]_{left}$.

$$d_{left}([\lambda], [\mu]) := \sup\left\{\left|\bigwedge\{t : \lambda(t) \geq k\} - \bigwedge\{t : \mu(t) \geq k\}\right| : k \in L \setminus \{0\}\right\}.$$

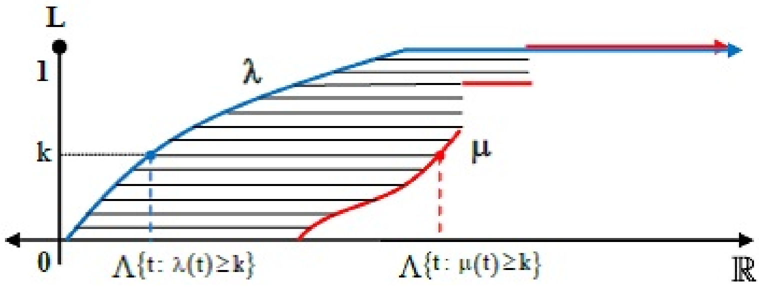

Figure 4. $\left(\mathbb{R}[L]_{left}, d_{left}\right)$

Definition 2.4 $\lambda \in L^{\mathbb{R}}$ *is called L-fuzzy number if the following conditions satisfy:*

(1) $\exists x_0 \in \mathbb{R} : \lambda(x_0) = 1$;
(2) $\forall s < s' \leq x_0,\ \lambda(s) \leq \lambda(s')$ *and* $\bigwedge_{t \leq x_0} \lambda(t) = 0$;
(3) $\forall x_0 \leq s < s',\ \lambda(s) \geq \lambda(s')$ *and* $\bigwedge_{x_0 \leq t} \lambda(t) = 0$.

The set of L-fuzzy numbers given this definition is denoted by $F\mathbb{R}[L]$.

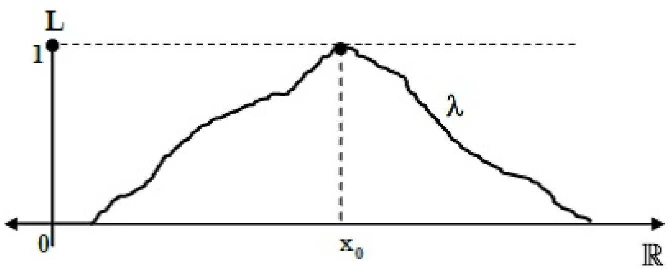

Figure 5. $[\lambda] \in F\mathbb{R}[L]$

Example 2.1 A real number $r_0 \in \mathbb{R}$ given in the classical sense is expressed as follows on the set $F\mathbb{R}[L]$:

$$\lambda(x) = \begin{cases} 1, & x = r_0 \\ 0, & x \neq r_0 \end{cases} .$$

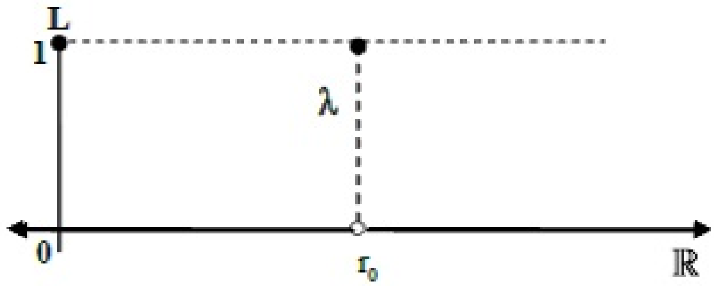

Figure 6. $r_0 \in F\mathbb{R}[L]$

After that, for ease of typing, we shall use λ instead of $[\lambda]$.

Definition 2.5 *Let $\lambda \in F\mathbb{R}[L]$. Then, mappings $\lambda_-, \lambda^- : \mathbb{R} \to L$ is defined as follows:*

$$\lambda_-(x) := \begin{cases} 1, & x < x_0 \\ \lambda(x), & x_0 \leq x \end{cases}, \quad \lambda^-(x) := \begin{cases} \lambda(x), & x \leq x_0 \\ 1, & x_0 < x \end{cases}.$$

Theorem 2.7 *Let $\lambda, \mu \in F\mathbb{R}[L]$. Then*

$$\lambda = \mu \Leftrightarrow \lambda_- = \mu_- \text{ and } \lambda^- = \mu^-.$$

Proof The "$\Rightarrow$" part is evident.

The "$\Leftarrow$" part should be $\exists x_1, x_2 \in \mathbb{R}$ such that $\lambda(x_1) = 1$ and $\mu(x_2) = 1$. Now, let

$$\lambda_-(x) = \begin{cases} 1, & x < x_1 \\ \lambda(x), & x_1 \leq x \end{cases}, \quad \lambda^-(x) = \begin{cases} \lambda(x), & x \leq x_1 \\ 1, & x_1 < x \end{cases}$$

$$\mu_-(x) = \begin{cases} 1, & x < x_2 \\ \mu(x), & x_2 \leq x \end{cases}, \quad \mu^-(x) = \begin{cases} \mu(x), & x \leq x_2 \\ 1, & x_2 < x \end{cases}.$$

Then

$$\lambda_-(x) = \mu_-(x) \Rightarrow \lambda(x) = \mu(x) \text{ for all } x \in \mathbb{R} \text{ satisfied } x_1 < x, \tag{2.10}$$

$$\lambda^-(x) = \mu^-(x) \Rightarrow \lambda(x) = \mu(x) \text{ for all } x \in \mathbb{R} \text{ satisfied } x < x_1. \tag{2.11}$$

This completes the proof. □

Theorem 2.8 *Let* $\lambda, \mu \in F\mathbb{R}[L]$ *and*

$$d_{right}(\lambda_-, \mu_-) = \sup\left\{\left|\bigvee\{t : \lambda_-(t) \geq k\} - \bigvee\{t : \mu_-(t) \geq k\}\right| : k \in L\setminus\{0\}\right\},$$
$$d_{left}\left(\lambda^-, \mu^-\right) = \sup\left\{\left|\bigwedge\left\{t : \lambda^-(t) \geq k\right\} - \bigwedge\left\{t : \mu^-(t) \geq k\right\}\right| : k \in L\setminus\{0\}\right\},$$

where d_{right} *and* d_{left} *are the metrics on the sets* $\mathbb{R}[L]_{right}$ *and* $\mathbb{R}[L]_{left}$ *respectively and* d *defined by*

$$d(\lambda, \mu) := \max\left\{d_{right}(\lambda_-, \mu_-), d_{left}\left(\lambda^-, \mu^-\right)\right\},$$

is a metric on $F\mathbb{R}[L]$.

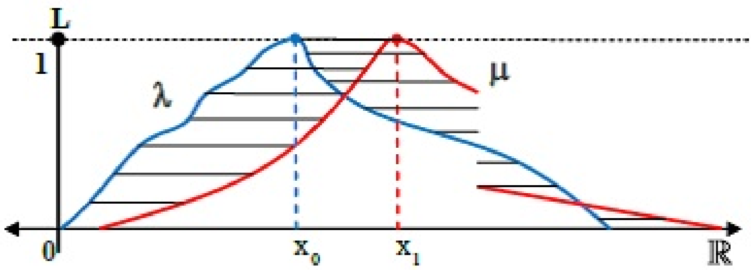

Figure 7. $(F\mathbb{R}[L], d)$

Proof (*i*) Let's show that $d(\lambda, \mu) = 0 \Leftrightarrow \lambda = \mu$.
First, the assertion "$\Rightarrow$". Let $d(\lambda, \mu) = 0$. From definition,

$$d_{right}(\lambda_-, \mu_-) = 0 \text{ and } d_{left}\left(\lambda^-, \mu^-\right) = 0.$$

Since d_{right} and d_{left} are metrics, $\lambda_- = \mu_-$ and $\lambda^- = \mu^-$. From Theorem 2.7, $\lambda = \mu$.

Second, the assertion "$\Leftarrow$". Let $\lambda = \mu$. From Theorem 2.7, $\lambda_- = \mu_-$ and $\lambda^- = \mu^-$. Since d_{right} and d_{left} are the metrics

$$d_{right}(\lambda_-, \mu_-) = 0 \text{ and } d_{left}\left(\lambda^-, \mu^-\right) = 0.$$

Hence,

$$d(\lambda, \mu) = \max\left\{d_{right}(\lambda_-, \mu_-), d_{left}\left(\lambda^-, \mu^-\right)\right\}.$$

That is,

$$d(\lambda, \mu) = 0.$$

(*ii*) $d(\lambda, \mu) = d(\mu, \lambda)$.
(*iii*) Let's show that $d(\lambda, \eta) \leq d(\lambda, \mu) + d(\mu, \eta)$. Let

$$d(\lambda, \eta) = \max\left\{d_{right}(\lambda_-, \eta_-), d_{left}\left(\lambda^-, \eta^-\right)\right\}.$$

Without loss of generality, we can take

$$d_{right}(\lambda_-, \eta_-) \geq d_{left}\left(\lambda^-, \eta^-\right).$$

Since d_{right} is a metric and

$$d(\lambda,\eta) = d_{right}(\lambda_-,\eta_-) \leq d_{right}(\lambda_-,\mu_-) + d_{right}(\mu_-,\eta_-) \leq d(\lambda,\mu) + d(\mu,\eta),$$

following inequality is obtained:

$$d(\lambda,\eta) \leq d(\lambda,\mu) + d(\mu,\eta).$$

This completes the proof. □

Example 2.2 The real numbers $3, 7 \in \mathbb{R}$ given in the classical sense is expressed as follows on the set $F\mathbb{R}[L]$:

$$\lambda_3(x) := \begin{cases} 1, & x = 3 \\ 0, & x \neq 3 \end{cases} \quad \text{and} \quad \lambda_7(x) := \begin{cases} 1, & x = 7 \\ 0, & x \neq 7 \end{cases}$$

$\lambda_{3,}\lambda_7 \in F\mathbb{R}[L]$.

$$d(\lambda_3,\lambda_7) = \max\left\{d_{right}\left((\lambda_-)_3,(\lambda_-)_7\right), d_{left}\left(\left(\lambda^-\right)_3,\left(\lambda^-\right)_7\right)\right\} = \max\{4,4\} = 4,$$

where

$$(\lambda_-)_3(x) := \begin{cases} 1, & x \leq 3 \\ 0, & x > 3 \end{cases}, \qquad \left(\lambda^-\right)_3(x) := \begin{cases} 1, & x \geq 3 \\ 0, & x < 3 \end{cases}$$

and

$$(\lambda_-)_7(x) := \begin{cases} 1, & x \leq 7 \\ 0, & x > 7 \end{cases} \qquad \left(\lambda^-\right)_7(x) := \begin{cases} 1, & x \geq 7 \\ 0, & x < 7 \end{cases}$$

are defined. Specially, since $I = [0,1]$ is complete, $I = [0,1]$ can be taken instead of L-complete lattice. In this case, d_{right} defined in Theorem 2.3, d_{left} defined in Theorem 2.6 and d defined in Theorem 2.8 satisfy the metric conditions.

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