

Multiplicative A -Metric Spaces

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Abstract: In this paper, we introduce multiplicative A -metric spaces (MA-spaces), a multi-point distance structure that unifies A -metric and multiplicative metric frameworks. The distance is defined on ordered n -tuples via a positive multiplicative function satisfying an A -type triangle inequality in product form. This setting includes multiplicative metric spaces ($n = 2$) and multiplicative S -metric spaces ($n = 3$) as special cases. We develop the basic topology of MA-spaces through multiplicative open balls, convergence, Cauchy sequences, and completeness. Furthermore, we establish fixed point results for multiplicative contractive and expansive mappings, extending classical theorems of Banach, Kannan, and Wang to a general multiplicative multi-point framework.

Key Words: Multiplicative metric, S -metric, A -metric, multiplicative S -metric, convergence, fixed point.

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§1. Introduction

Metric spaces have been a cornerstone of modern mathematics since their formal introduction by Fréchet in the early twentieth century. In a classical metric space (X, d) , each pair of points $\alpha, \beta \in X$ is assigned a nonnegative real number $d(\alpha, \beta)$ satisfying nonnegativity, identity of indiscernibles, symmetry, and the triangle inequality. However, the increasing complexity of applications in nonlinear analysis, optimization, and computer science has motivated the development of generalized distance structures, including multi-point distances and multiplicative (product-type) geometries.

One notable line of development began with Gähler's 2-metric space [6] in 1963 and Dhage's D -metric space [7] in 1984. Mustafa and Sims [3] addressed several structural issues by introducing the G -metric space, obtaining a well-behaved topology and fixed point theory.

Sedghi et al. [1] proposed the concept of an S -metric space, assigning a nonnegative real number $S(\alpha, \beta, \gamma)$ to each triple. Abbas et al. [2] extended S -metrics to an n -tuple structure called an A -metric space.

Parallel to these developments, Bashirov et al. [5] and later works such as [4] studied multiplicative metric spaces, where the usual additive triangle inequality is replaced by a multiplicative inequality and the neutral element is 1. A standard bridge between additive and

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multiplicative settings is provided by logarithmic/exponential transforms.

Fixed point theory has been central since Banach's contraction principle [9] and many contractive conditions exist, including Kannan-type mappings [10] and comparisons by Rhoades [11]. Expansive mappings also play an essential role (e.g. [12]-[21]).

The aim of the present paper is to unify the multi-point flexibility of A -metrics with multiplicative distance geometry into a single framework, called a *multiplicative A -metric space* (MA-space), and to develop both contractive and expansive fixed point principles in this setting.

§2. Preliminaries

In this section, we recall several generalized distance structures that are essential for the development of multiplicative A -metric spaces. These concepts extend the classical notion of metric distance by allowing multi-variable distance functions and, in some cases, multiplicative distance aggregation. Such generalizations play an important role in nonlinear analysis, fixed point theory, and generalized topology.

Throughout this paper, X denotes a nonempty set unless otherwise specified.

Definition 2.1(S -metric space, [1]) *Let X be a nonempty set. A mapping*

$$S : X^3 \rightarrow [0, \infty)$$

is called an S -metric on X if for all $\alpha, \beta, \gamma, \mu \in X$ the following conditions hold:

(1) **Nonnegativity and identity property.**

$$S(\alpha, \beta, \gamma) \geq 0, \quad S(\alpha, \beta, \gamma) = 0 \iff \alpha = \beta = \gamma.$$

(2) **Permutation invariance.** *The value of $S(\alpha, \beta, \gamma)$ remains unchanged under any permutation of its arguments.*

(3) **S -triangle inequality.**

$$S(\alpha, \beta, \gamma) \leq S(\alpha, \alpha, \mu) + S(\beta, \beta, \mu) + S(\gamma, \gamma, \mu).$$

In this case, the pair (X, S) is called an S -metric space.

Remark 2.2 Every classical metric space (X, d) generates an S -metric by defining

$$S(\alpha, \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma), \quad \alpha, \beta, \gamma \in X.$$

Thus, ordinary metric spaces are naturally embedded in the class of S -metric spaces.

Definition 2.3(Multiplicative S -metric, [8]) *Let X be a nonempty set. A mapping*

$$S^* : X^3 \rightarrow [1, \infty)$$

is called a multiplicative S-metric on X if for all $\alpha, \beta, \gamma, \mu \in X$:

(1) **Positivity and identity condition.**

$$S^*(\alpha, \beta, \gamma) \geq 1, \quad S^*(\alpha, \beta, \gamma) = 1 \iff \alpha = \beta = \gamma.$$

(2) **Permutation invariance.** $S^*(\alpha, \beta, \gamma)$ is invariant under any permutation of its arguments.

(3) **Multiplicative triangle inequality.**

$$S^*(\alpha, \beta, \gamma) \leq S^*(\alpha, \alpha, \mu) S^*(\beta, \beta, \mu) S^*(\gamma, \gamma, \mu).$$

Definition 2.4(A-metric space, [2]) Let $n \geq 2$ be fixed. A mapping

$$A : X^n \rightarrow [0, \infty)$$

is called an A-metric on X if for all $\alpha_1, \dots, \alpha_n, \mu \in X$ the following properties hold:

(1) **Nonnegativity and identity condition.**

$$A(\alpha_1, \dots, \alpha_n) \geq 0, \quad A(\alpha_1, \dots, \alpha_n) = 0 \iff \alpha_1 = \dots = \alpha_n.$$

(2) **Permutation symmetry.** The value of $A(\alpha_1, \dots, \alpha_n)$ is invariant under any permutation of its arguments.

(3) **A-type triangle inequality.**

$$A(\alpha_1, \dots, \alpha_n) \leq \sum_{i=1}^n A(\underbrace{\alpha_i, \dots, \alpha_i}_{n-1}, \mu).$$

In this case, (X, A) is called an A-metric space.

§3. Multiplicative A-Metric Spaces

In this section, we introduce the concept of multiplicative A-metric spaces, which can be viewed as a multiplicative multi-point extension of classical A-metric spaces. This structure combines the multi-variable distance mechanism of A-metrics with the product-based geometry of multiplicative metrics. The resulting framework allows distances to be measured simultaneously among multiple points while preserving multiplicative consistency. Throughout this section, X denotes a nonempty set and $n \geq 3$ is a fixed integer.

Definition 3.1(Multiplicative A-metric space) Let $n \geq 3$ be fixed. A mapping

$$\mathcal{A}^* : X^n \rightarrow [1, \infty)$$

is called a multiplicative A-metric on X if for all $\alpha_1, \dots, \alpha_n, \mu \in X$ the following conditions

hold:

(MA1) **Positivity and identity condition.**

$$\mathcal{A}^*(\alpha_1, \dots, \alpha_n) \geq 1,$$

and

$$\mathcal{A}^*(\alpha_1, \dots, \alpha_n) = 1 \quad \text{if and only if} \quad \alpha_1 = \dots = \alpha_n.$$

(MA2) **Permutation invariance.** *The value of $\mathcal{A}^*(\alpha_1, \dots, \alpha_n)$ remains unchanged under any permutation of its arguments.*

(MA3) **Multiplicative A-triangle inequality.**

$$\mathcal{A}^*(\alpha_1, \dots, \alpha_n) \leq \prod_{i=1}^n \mathcal{A}^*(\underbrace{\alpha_i, \dots, \alpha_i}_{n-1}, \mu).$$

The pair (X, \mathcal{A}^*) is called a multiplicative A-metric space.

Remark 3.2 In multiplicative distance settings, the neutral element is 1, which plays the role analogous to zero in additive metric spaces. Consequently, closeness between points is measured by values approaching 1, and convergence of sequences is characterized by multiplicative distances tending to 1.

Definition 3.3(Induced two-point distance) *Let (X, \mathcal{A}^*) be a multiplicative A-metric space. Define a two-point distance function*

$$D : X \times X \rightarrow [1, \infty)$$

by

$$D(\alpha, \beta) := \mathcal{A}^*(\underbrace{\alpha, \dots, \alpha}_{n-1}, \beta), \quad \alpha, \beta \in X.$$

Lemma 3.4 *For all $\alpha, \beta \in X$,*

- (1) $D(\alpha, \beta) \geq 1$, and $D(\alpha, \beta) = 1$ if and only if $\alpha = \beta$;
- (2) $D(\alpha, \beta) = D(\beta, \alpha)$.

Proof From Definition 3.3,

$$D(\alpha, \beta) = \mathcal{A}^*(\alpha, \dots, \alpha, \beta).$$

Applying condition (MA1) to this n -tuple gives $D(\alpha, \beta) \geq 1$, and equality holds only when all entries are equal, which is equivalent to $\alpha = \beta$. The symmetry property follows immediately from (MA2), since permutation of arguments does not change the value of \mathcal{A}^* . \square

Lemma 3.5(Two-point multiplicative inequality) *For all $\alpha, \beta, \gamma \in X$,*

$$D(\alpha, \gamma) \leq D(\alpha, \beta)^{n-1} D(\beta, \gamma).$$

Proof Applying (MA3) to the n -tuple $(\underbrace{\alpha, \dots, \alpha}_{n-1}, \gamma)$ with $\mu = \beta$ yields

$$\mathcal{A}^*(\underbrace{\alpha, \dots, \alpha}_{n-1}, \gamma) \leq \prod_{i=1}^n \mathcal{A}^*(\underbrace{\alpha_i, \dots, \alpha_i}_{n-1}, \beta).$$

The first $n - 1$ factors equal $\mathcal{A}^*(\underbrace{\alpha, \dots, \alpha}_{n-1}, \beta) = D(\alpha, \beta)$, while the last factor equals $\mathcal{A}^*(\underbrace{\gamma, \dots, \gamma}_{n-1}, \beta)$, which by permutation invariance equals $\mathcal{A}^*(\underbrace{\beta, \dots, \beta}_{n-1}, \gamma) = D(\beta, \gamma)$. Substituting these identities gives the desired inequality. \square

Definition 3.6(Multiplicative open and closed balls) *Let $\alpha \in X$ and $r > 1$. Define*

$$B(\alpha, r) := \{\beta \in X : D(\alpha, \beta) < r\}, \quad \overline{B}(\alpha, r) := \{\beta \in X : D(\alpha, \beta) \leq r\}.$$

Lemma 3.7 *Let $\alpha \in X$, $r > 1$, and $\xi \in B(\alpha, r)$. Then there exists $\varepsilon > 1$ such that*

$$B(\xi, \varepsilon) \subset B(\alpha, r).$$

Consequently, the family $\{B(\alpha, r) : \alpha \in X, r > 1\}$ forms a base for a topology on X .

Proof Let $C := D(\alpha, \xi)$ so that $1 \leq C < r$. From Lemma 3.5, for any $\gamma \in X$ we have

$$D(\alpha, \gamma) \leq C^{n-1} D(\xi, \gamma).$$

Choose $\varepsilon = r/C^{n-1} > 1$. If $\gamma \in B(\xi, \varepsilon)$ then $D(\xi, \gamma) < \varepsilon$, which implies

$$D(\alpha, \gamma) < C^{n-1} \varepsilon = r,$$

hence $\gamma \in B(\alpha, r)$. Therefore $B(\xi, \varepsilon) \subset B(\alpha, r)$, proving the base property. \square

Remark 3.8 The balls $B(\alpha, r)$ generate a topology $\tau_{\mathcal{A}^*}$ on X . A sequence $\{\alpha_k\}$ converges to α if and only if $D(\alpha, \alpha_k) \rightarrow 1$.

Definition 3.9(Convergence) *A sequence $\{\alpha_k\} \subset X$ converges to $\alpha \in X$ if*

$$\lim_{k \rightarrow \infty} D(\alpha, \alpha_k) = 1.$$

Definition 3.10(Multiplicative A -Cauchy sequence) *A sequence $\{\alpha_k\} \subset X$ is called multiplicative A -Cauchy if for every $\varepsilon > 1$ there exists $N \in \mathbb{N}$ such that*

$$D(\alpha_k, \alpha_m) < \varepsilon \quad \text{for all } k, m \geq N.$$

Definition 3.11(Completeness) *A multiplicative A -metric space (X, \mathcal{A}^*) is said to be complete if every multiplicative A -Cauchy sequence converges to a point in X in the sense of Definition 3.9.*

Proposition 3.12 *Let (X, d) be a metric space and let $n \geq 3$. Define*

$$\mathcal{A}^*(\alpha_1, \dots, \alpha_n) := \exp\left(\sum_{1 \leq i < j \leq n} d(\alpha_i, \alpha_j)\right).$$

Then \mathcal{A}^ is a multiplicative A -metric on X .*

Proof Since $d(\alpha_i, \alpha_j) \geq 0$, the sum $\sum_{i < j} d(\alpha_i, \alpha_j)$ is nonnegative, and therefore $\mathcal{A}^*(\alpha_1, \dots, \alpha_n) \geq 1$. Moreover, the sum equals zero if and only if $d(\alpha_i, \alpha_j) = 0$ for all i, j , which is equivalent to $\alpha_1 = \dots = \alpha_n$ by the metric identity property. Hence condition (MA1) holds.

The definition of \mathcal{A}^* depends only on pairwise distances and the summation is taken over all unordered pairs, hence invariance under permutation of arguments is immediate, so (MA2) holds.

Let $\mu \in X$ be arbitrary. By the triangle inequality,

$$d(\alpha_i, \alpha_j) \leq d(\alpha_i, \mu) + d(\alpha_j, \mu) \quad \text{for all } i < j.$$

Summing over all pairs $1 \leq i < j \leq n$ yields

$$\sum_{i < j} d(\alpha_i, \alpha_j) \leq \sum_{i < j} (d(\alpha_i, \mu) + d(\alpha_j, \mu)).$$

Each term $d(\alpha_k, \mu)$ appears exactly $(n-1)$ times in the right-hand double sum, hence

$$\sum_{i < j} d(\alpha_i, \alpha_j) \leq (n-1) \sum_{k=1}^n d(\alpha_k, \mu).$$

Now observe that

$$\mathcal{A}^*(\underbrace{\alpha_i, \dots, \alpha_i}_{n-1}, \mu) = \exp((n-1)d(\alpha_i, \mu)).$$

Therefore

$$\prod_{i=1}^n \mathcal{A}^*(\underbrace{\alpha_i, \dots, \alpha_i}_{n-1}, \mu) = \exp\left((n-1) \sum_{i=1}^n d(\alpha_i, \mu)\right).$$

Exponentiating the earlier inequality gives (MA3). Hence, \mathcal{A}^* is nothing else but a multi-

plicative A-metric. □

Proposition 3.13 *Let (X, d) be a metric space and let $n \geq 3$. Define*

$$\mathcal{A}^*(\alpha_1, \dots, \alpha_n) := \prod_{1 \leq i < j \leq n} (1 + d(\alpha_i, \alpha_j)).$$

Then \mathcal{A}^ is a multiplicative A-metric on X .*

Proof Since each factor satisfies $1 + d(\alpha_i, \alpha_j) \geq 1$, their product is at least 1. The product equals 1 if and only if each factor equals 1, which holds precisely when $d(\alpha_i, \alpha_j) = 0$ for all i, j , implying $\alpha_1 = \dots = \alpha_n$. Hence (MA1) holds.

Because the definition involves a product over all unordered pairs, permutation of indices leaves the value unchanged, so (MA2) holds.

Fix $\mu \in X$. Using the triangle inequality,

$$d(\alpha_i, \alpha_j) \leq d(\alpha_i, \mu) + d(\alpha_j, \mu),$$

and the elementary inequality $1 + a + b \leq (1 + a)(1 + b)$ for $a, b \geq 0$, we obtain

$$1 + d(\alpha_i, \alpha_j) \leq (1 + d(\alpha_i, \mu))(1 + d(\alpha_j, \mu)).$$

Taking products over all pairs $i < j$ gives

$$\prod_{i < j} (1 + d(\alpha_i, \alpha_j)) \leq \prod_{i < j} (1 + d(\alpha_i, \mu))(1 + d(\alpha_j, \mu)).$$

Each factor $(1 + d(\alpha_k, \mu))$ appears exactly $(n - 1)$ times in the right-hand product, hence

$$\prod_{i < j} (1 + d(\alpha_i, \alpha_j)) \leq \prod_{k=1}^n (1 + d(\alpha_k, \mu))^{n-1}.$$

Since

$$\mathcal{A}^*(\underbrace{\alpha_i, \dots, \alpha_i}_{n-1}, \mu) = (1 + d(\alpha_i, \mu))^{n-1},$$

condition (MA3) follows. □

Proposition 3.14 *Let (X, d) be a metric space and let $n \geq 3$. Define*

$$\mathcal{A}^*(\alpha_1, \dots, \alpha_n) := \exp\left(\max_{1 \leq i < j \leq n} d(\alpha_i, \alpha_j)\right).$$

Then \mathcal{A}^ is a multiplicative A-metric on X .*

Proof Since the maximum of nonnegative numbers is nonnegative, $\mathcal{A}^* \geq 1$. Equality holds only when all pairwise distances vanish, which is equivalent to equality of all arguments. Hence (MA1) holds. Symmetry follows from symmetry of the maximum over unordered pairs.

Let $\mu \in X$. For any pair $i < j$, the triangle inequality gives

$$d(\alpha_i, \alpha_j) \leq d(\alpha_i, \mu) + d(\alpha_j, \mu) \leq \sum_{k=1}^n d(\alpha_k, \mu).$$

Therefore

$$\max_{i < j} d(\alpha_i, \alpha_j) \leq \sum_{k=1}^n d(\alpha_k, \mu).$$

Also,

$$\mathcal{A}^*(\underbrace{\alpha_i, \dots, \alpha_i}_{n-1}, \mu) = \exp(d(\alpha_i, \mu)).$$

Hence

$$\prod_{i=1}^n \mathcal{A}^*(\underbrace{\alpha_i, \dots, \alpha_i}_{n-1}, \mu) = \exp\left(\sum_{k=1}^n d(\alpha_k, \mu)\right),$$

which implies (MA3). □

Proposition 3.15 *Let (X, d) be a metric space, $n \geq 3$, and $q \in (0, 1]$. Define*

$$\mathcal{A}^*(\alpha_1, \dots, \alpha_n) := \exp\left(\sum_{1 \leq i < j \leq n} d(\alpha_i, \alpha_j)^q\right).$$

Then \mathcal{A}^ is a multiplicative A -metric on X .*

Proof Since $d(\alpha_i, \alpha_j)^q \geq 0$, the exponential is at least 1. Equality holds only when all distances are zero, which implies equality of all points, hence (MA1) holds. Symmetry follows from symmetry of the pairwise sum.

Fix $\mu \in X$. Since $q \in (0, 1]$, the function $t \mapsto t^q$ is subadditive on $[0, \infty)$, so for all $a, b \geq 0$, $(a + b)^q \leq a^q + b^q$. Using triangle inequality,

$$d(\alpha_i, \alpha_j)^q \leq (d(\alpha_i, \mu) + d(\alpha_j, \mu))^q \leq d(\alpha_i, \mu)^q + d(\alpha_j, \mu)^q.$$

Summing over all pairs gives

$$\sum_{i < j} d(\alpha_i, \alpha_j)^q \leq (n-1) \sum_{i=1}^n d(\alpha_i, \mu)^q \quad \text{and also} \quad \mathcal{A}^*(\underbrace{\alpha_i, \dots, \alpha_i}_{n-1}, \mu) = \exp((n-1)d(\alpha_i, \mu)^q).$$

Exponentiation yields (MA3), completing the proof. □

§4. Topological Properties

In this section, we investigate several fundamental topological characteristics of multiplicative A -metric spaces. In particular, we study convergence behavior, uniqueness of limits, and separation properties of the topology generated by multiplicative open balls. These results

confirm that multiplicative A -metric spaces retain many essential structural features analogous to classical metric spaces, while reflecting the multiplicative nature of the underlying distance function.

Lemma 4.1 *A sequence $\{\alpha_k\} \subset X$ converges to $\alpha \in X$ (that is, $D(\alpha, \alpha_k) \rightarrow 1$) if and only if for every $\lambda > 1$ there exists $N \in \mathbb{N}$ such that $\alpha_k \in B(\alpha, \lambda)$ for all $k \geq N$.*

Proof Suppose first that $D(\alpha, \alpha_k) \rightarrow 1$. Let $\lambda > 1$ be arbitrary. Since $D(\alpha, \alpha_k)$ converges to 1, by definition of limit there exists $N \in \mathbb{N}$ such that

$$|D(\alpha, \alpha_k) - 1| < \lambda - 1 \quad \text{for all } k \geq N.$$

Because $D(\alpha, \alpha_k) \geq 1$, this implies $D(\alpha, \alpha_k) < \lambda$ for all $k \geq N$. Hence $\alpha_k \in B(\alpha, \lambda)$ for all $k \geq N$.

Conversely, suppose that for every $\lambda > 1$ there exists N such that $\alpha_k \in B(\alpha, \lambda)$ whenever $k \geq N$. Then for every $\lambda > 1$ there exists N such that $D(\alpha, \alpha_k) < \lambda$ for all $k \geq N$. Since $D(\alpha, \alpha_k) \geq 1$ for all k , it follows that $D(\alpha, \alpha_k)$ is eventually contained in the interval $[1, \lambda)$. Because this holds for every $\lambda > 1$, the only possible limit value is 1. Hence $D(\alpha, \alpha_k) \rightarrow 1$. \square

Lemma 4.2 *Limits are unique in a multiplicative A -metric space (X, \mathcal{A}^*) .*

Proof Suppose $\{\alpha_k\}$ converges to both α and β . Then

$$D(\alpha, \alpha_k) \rightarrow 1 \quad \text{and} \quad D(\beta, \alpha_k) \rightarrow 1.$$

From Lemma 3.5, for every k ,

$$D(\alpha, \beta) \leq D(\alpha, \alpha_k)^{n-1} D(\alpha_k, \beta).$$

Since $D(\alpha, \alpha_k) \rightarrow 1$ and $D(\alpha_k, \beta) = D(\beta, \alpha_k) \rightarrow 1$, the right-hand side converges to 1. Hence

$$D(\alpha, \beta) \leq 1.$$

Because $D(\alpha, \beta) \geq 1$ always holds, we obtain $D(\alpha, \beta) = 1$, and therefore $\alpha = \beta$. \square

Lemma 4.3 *The topology generated by the multiplicative open balls $\{B(\alpha, r) : \alpha \in X, r > 1\}$ is Hausdorff.*

Proof Let $\alpha, \beta \in X$ with $\alpha \neq \beta$. Then $D(\alpha, \beta) > 1$. Choose $\lambda > 1$ such that

$$\lambda^n < D(\alpha, \beta).$$

Such a choice is possible because the function $\lambda \mapsto \lambda^n$ is continuous and strictly increasing on $(1, \infty)$.

Assume, for contradiction, that there exists $\gamma \in B(\alpha, \lambda) \cap B(\beta, \lambda)$. Then

$$D(\alpha, \gamma) < \lambda \quad \text{and} \quad D(\beta, \gamma) < \lambda.$$

Using Lemma 3.5 with (α, γ, β) , we obtain

$$D(\alpha, \beta) \leq D(\alpha, \gamma)^{n-1} D(\gamma, \beta).$$

Hence

$$D(\alpha, \beta) < \lambda^{n-1} \lambda = \lambda^n,$$

which contradicts the choice of λ . Therefore

$$B(\alpha, \lambda) \cap B(\beta, \lambda) = \emptyset,$$

and the topology is Hausdorff. □

Remark 4.4 In contrast to classical metric spaces, closed multiplicative balls need not be closed in the induced topology when $n \geq 3$. Indeed, let $\{\beta_k\}$ be a sequence such that $\beta_k \rightarrow \beta$ and suppose $D(\alpha, \beta_k) \leq \lambda$ for all k . From Lemma 3.5, we obtain

$$D(\alpha, \beta) \leq D(\alpha, \beta_k)^{n-1} D(\beta_k, \beta).$$

Taking limit as $k \rightarrow \infty$ yields

$$D(\alpha, \beta) \leq \lambda^{n-1}.$$

If $n \geq 3$, then $\lambda^{n-1} > \lambda$ whenever $\lambda > 1$, so this estimate does not imply $D(\alpha, \beta) \leq \lambda$. Thus the limit point β need not belong to $\overline{B}(\alpha, \lambda)$, showing that closed balls are not necessarily topologically closed without additional structural assumptions.

§5. Fixed Points for Contractive Mappings

In this section, we establish fixed point results for contractive-type self-mappings defined on multiplicative A -metric spaces.

Throughout this section, let (X, \mathcal{A}^*) be a multiplicative A -metric space and let D denote the associated two-point distance.

Lemma 5.1 *Let $\{\alpha_k\}$ be a finite sequence in X and let $m > k$. Then*

$$D(\alpha_m, \alpha_k) \leq \prod_{j=k}^{m-1} D(\alpha_{j+1}, \alpha_j)^{n-1}.$$

Proof Let $m > k$ be fixed. The proof is based on repeated application of the two-point

multiplicative inequality stated in Lemma 3.5, which asserts that for all $\alpha, \beta, \gamma \in X$,

$$D(\alpha, \gamma) \leq D(\alpha, \beta)^{n-1} D(\beta, \gamma).$$

Applying this inequality with $(\alpha, \beta, \gamma) = (\alpha_m, \alpha_{m-1}, \alpha_k)$ yields

$$D(\alpha_m, \alpha_k) \leq D(\alpha_m, \alpha_{m-1})^{n-1} D(\alpha_{m-1}, \alpha_k).$$

The term $D(\alpha_{m-1}, \alpha_k)$ can again be estimated using the same inequality, this time applied with $(\alpha, \beta, \gamma) = (\alpha_{m-1}, \alpha_{m-2}, \alpha_k)$, giving

$$D(\alpha_{m-1}, \alpha_k) \leq D(\alpha_{m-1}, \alpha_{m-2})^{n-1} D(\alpha_{m-2}, \alpha_k).$$

Substituting this inequality into the previous bound produces

$$D(\alpha_m, \alpha_k) \leq D(\alpha_m, \alpha_{m-1})^{n-1} D(\alpha_{m-1}, \alpha_{m-2})^{n-1} D(\alpha_{m-2}, \alpha_k).$$

Continuing this substitution procedure repeatedly reduces the remaining distance term by shifting the intermediate index downward one position at each stage, while simultaneously introducing an additional multiplicative factor of the form $D(\alpha_{r+1}, \alpha_r)^{n-1}$ for decreasing indices r .

Since the sequence is finite and $m > k$, after finitely many such substitutions the remaining distance term becomes $D(\alpha_{k+1}, \alpha_k)$.

Consequently, after completing all substitutions, the inequality takes the form

$$D(\alpha_m, \alpha_k) \leq D(\alpha_m, \alpha_{m-1})^{n-1} D(\alpha_{m-1}, \alpha_{m-2})^{n-1} \cdots D(\alpha_{k+1}, \alpha_k)^{n-1}.$$

Rewriting this product using index notation yields

$$D(\alpha_m, \alpha_k) \leq \prod_{j=k}^{m-1} D(\alpha_{j+1}, \alpha_j)^{n-1}.$$

This establishes the desired multiplicative chain estimate. □

Theorem 5.2(Multiplicative Banach type) *Let (X, \mathcal{A}^*) be a complete multiplicative A-metric space and let D be the induced two-point distance. Assume that there exists a constant $q \in (0, 1)$ such that*

$$D(\Phi\alpha, \Phi\beta) \leq D(\alpha, \beta)^q \quad \text{for all } \alpha, \beta \in X.$$

Then Φ has a unique fixed point $\xi \in X$. Moreover, for every $\alpha_0 \in X$, the Picard iteration

$$\alpha_{k+1} = \Phi(\alpha_k), \quad k \geq 0,$$

converges to ξ in the multiplicative sense, i.e. $D(\alpha_k, \xi) \rightarrow 1$.

Proof Fix $\alpha_0 \in X$ and define the Picard sequence $\alpha_{k+1} = \Phi(\alpha_k)$ for $k \geq 0$. Set

$$\delta_k := D(\alpha_{k+1}, \alpha_k) \quad (k \geq 0).$$

By Lemma 3.4, each $\delta_k \geq 1$. If $\delta_0 = 1$, then $\alpha_1 = \alpha_0$ and hence α_0 is a fixed point, so the conclusion holds. Assume therefore that $\delta_0 > 1$.

Applying the contractive hypothesis to (α_k, α_{k-1}) gives, for every $k \geq 1$,

$$\delta_k = D(\Phi\alpha_k, \Phi\alpha_{k-1}) \leq D(\alpha_k, \alpha_{k-1})^q = \delta_{k-1}^q.$$

Iterating yields

$$\delta_k \leq \delta_0^{q^k} \quad (k \geq 0).$$

Since $q^k \rightarrow 0$ and $\delta_0 > 0$, we have $\delta_0^{q^k} = \exp(q^k \ln \delta_0) \rightarrow \exp(0) = 1$, hence $\delta_k \rightarrow 1$.

To prove that $\{\alpha_k\}$ is multiplicative A -Cauchy, fix integers $m > k$. Lemma 5.1 implies

$$D(\alpha_m, \alpha_k) \leq \prod_{j=k}^{m-1} \delta_j^{n-1}.$$

Taking logarithms (valid since each factor is ≥ 1) yields

$$\ln D(\alpha_m, \alpha_k) \leq (n-1) \sum_{j=k}^{m-1} \ln \delta_j.$$

Using $\delta_j \leq \delta_0^{q^j}$ and monotonicity of \ln on $(0, \infty)$ gives

$$\ln \delta_j \leq \ln(\delta_0^{q^j}) = q^j \ln \delta_0.$$

Therefore,

$$\ln D(\alpha_m, \alpha_k) \leq (n-1) \ln \delta_0 \sum_{j=k}^{m-1} q^j \leq (n-1) \ln \delta_0 \sum_{j=k}^{\infty} q^j = \frac{(n-1) \ln \delta_0}{1-q} q^k.$$

The right-hand side tends to 0 as $k \rightarrow \infty$, uniformly in $m > k$. Hence

$$\lim_{k \rightarrow \infty} \sup_{m > k} \ln D(\alpha_m, \alpha_k) = 0 \quad \text{equivalently} \quad \lim_{k \rightarrow \infty} \sup_{m > k} D(\alpha_m, \alpha_k) = 1.$$

This is precisely the multiplicative A -Cauchy property (Definition 3.10). By completeness, there exists $\xi \in X$ such that

$$D(\alpha_k, \xi) \rightarrow 1.$$

To show that ξ is a fixed point, apply the contractive hypothesis to (α_k, ξ) :

$$D(\alpha_{k+1}, \Phi\xi) = D(\Phi\alpha_k, \Phi\xi) \leq D(\alpha_k, \xi)^q \rightarrow 1.$$

Thus $D(\alpha_{k+1}, \Phi\xi) \rightarrow 1$ and also $D(\alpha_{k+1}, \xi) \rightarrow 1$. By uniqueness of limits in the induced topology (Lemma 4.2), we must have $\Phi\xi = \xi$, so ξ is a fixed point.

Finally, let η be any fixed point of Φ . Then

$$D(\xi, \eta) = D(\Phi\xi, \Phi\eta) \leq D(\xi, \eta)^q.$$

If $D(\xi, \eta) > 1$, then raising to the power $q \in (0, 1)$ strictly decreases the value, so $D(\xi, \eta) \leq D(\xi, \eta)^q$ is impossible. Hence $D(\xi, \eta) = 1$, and Lemma 3.4 yields $\xi = \eta$. Therefore the fixed point is unique. \square

Theorem 5.3(Multiplicative Kannan type) *Let (X, \mathcal{A}^*) be a complete multiplicative A -metric space and let D be the induced two-point distance. Assume that there exists $q \in (0, \frac{1}{2})$ such that*

$$D(\Phi\alpha, \Phi\beta) \leq (D(\alpha, \Phi\alpha) D(\beta, \Phi\beta))^q \quad \text{for all } \alpha, \beta \in X.$$

Then Φ has a unique fixed point $\xi \in X$. Moreover, for every $\alpha_0 \in X$, the Picard iteration $\alpha_{k+1} = \Phi(\alpha_k)$ converges multiplicatively to ξ , i.e. $D(\alpha_k, \xi) \rightarrow 1$.

Proof Fix $\alpha_0 \in X$ and define the Picard sequence $\alpha_{k+1} = \Phi(\alpha_k)$ for $k \geq 0$. Set

$$\delta_k := D(\alpha_{k+1}, \alpha_k) \quad (k \geq 0).$$

By Lemma 3.4, $\delta_k \geq 1$ for every k . If $\delta_0 = 1$, then $\alpha_1 = \alpha_0$ and hence α_0 is a fixed point; uniqueness (proved below) then yields the conclusion. Assume henceforth that $\delta_0 > 1$.

Applying the Kannan-type condition to (α_k, α_{k-1}) gives, for every $k \geq 1$,

$$\delta_k = D(\Phi\alpha_k, \Phi\alpha_{k-1}) \leq (D(\alpha_k, \Phi\alpha_k) D(\alpha_{k-1}, \Phi\alpha_{k-1}))^q = (\delta_k \delta_{k-1})^q.$$

Since $\delta_k \geq 1$, we may divide by δ_k^q to obtain

$$\delta_k^{1-q} \leq \delta_{k-1}^q.$$

Taking logarithms (all terms are ≥ 1) yields

$$(1-q) \ln \delta_k \leq q \ln \delta_{k-1}, \quad \text{equivalently} \quad \ln \delta_k \leq c \ln \delta_{k-1}, \quad c := \frac{q}{1-q} \in (0, 1),$$

because $q \in (0, \frac{1}{2})$ implies $0 < c < 1$. Iterating the last inequality gives

$$\ln \delta_k \leq c^k \ln \delta_0 \quad (k \geq 0),$$

hence $\ln \delta_k \rightarrow 0$ and therefore $\delta_k \rightarrow 1$.

To show that $\{\alpha_k\}$ is multiplicative A -Cauchy, let $m > k$. Lemma 5.1 yields

$$D(\alpha_m, \alpha_k) \leq \prod_{j=k}^{m-1} \delta_j^{n-1}.$$

Taking logarithms gives

$$\ln D(\alpha_m, \alpha_k) \leq (n-1) \sum_{j=k}^{m-1} \ln \delta_j.$$

Using $\ln \delta_j \leq c^j \ln \delta_0$ and summing the geometric series,

$$\sum_{j=k}^{\infty} \ln \delta_j \leq \ln \delta_0 \sum_{j=k}^{\infty} c^j = \frac{c^k}{1-c} \ln \delta_0,$$

and thus

$$\ln D(\alpha_m, \alpha_k) \leq (n-1) \frac{c^k}{1-c} \ln \delta_0.$$

The right-hand side tends to 0 as $k \rightarrow \infty$, uniformly in $m > k$. Hence

$$\limsup_{k \rightarrow \infty} \ln D(\alpha_m, \alpha_k) = 0, \quad \text{equivalently} \quad \limsup_{k \rightarrow \infty} D(\alpha_m, \alpha_k) = 1,$$

so $\{\alpha_k\}$ is multiplicative A -Cauchy. Completeness provides $\xi \in X$ such that

$$D(\alpha_k, \xi) \rightarrow 1.$$

To verify that ξ is a fixed point, apply the Kannan inequality to (α_k, ξ) :

$$D(\alpha_{k+1}, \Phi\xi) = D(\Phi\alpha_k, \Phi\xi) \leq (D(\alpha_k, \Phi\alpha_k) D(\xi, \Phi\xi))^q = (\delta_k D(\xi, \Phi\xi))^q.$$

Letting $k \rightarrow \infty$ and using $\delta_k \rightarrow 1$ gives

$$\limsup_{k \rightarrow \infty} D(\alpha_{k+1}, \Phi\xi) \leq D(\xi, \Phi\xi)^q.$$

On the other hand, since $D(\alpha_{k+1}, \xi) \rightarrow 1$, Lemma 3.5 implies

$$D(\xi, \Phi\xi) \leq D(\xi, \alpha_{k+1})^{n-1} D(\alpha_{k+1}, \Phi\xi),$$

and letting $k \rightarrow \infty$ yields

$$D(\xi, \Phi\xi) \leq \liminf_{k \rightarrow \infty} D(\alpha_{k+1}, \Phi\xi).$$

Combining the last two estimates gives

$$D(\xi, \Phi\xi) \leq D(\xi, \Phi\xi)^q.$$

Since $D(\xi, \Phi\xi) \geq 1$ and $q \in (0, 1)$, the inequality forces $D(\xi, \Phi\xi) = 1$, hence $\Phi\xi = \xi$.

Finally, to prove uniqueness, let η be another fixed point. Then the Kannan condition gives

$$D(\xi, \eta) = D(\Phi\xi, \Phi\eta) \leq (D(\xi, \Phi\xi) D(\eta, \Phi\eta))^q = (1 \cdot 1)^q = 1.$$

Because $D(\xi, \eta) \geq 1$, we obtain $D(\xi, \eta) = 1$, and Lemma 3.4 implies $\xi = \eta$. \square

Example 5.4 Let $X = \mathbb{R}$ and fix an integer $n \geq 3$. Consider the standard metric $d(x, y) = |x - y|$ on \mathbb{R} and define \mathcal{A}^* by the construction in Proposition 3.12, namely

$$\mathcal{A}^*(\alpha_1, \dots, \alpha_n) = \exp\left(\sum_{1 \leq i < j \leq n} |\alpha_i - \alpha_j|\right).$$

Then $(\mathbb{R}, \mathcal{A}^*)$ is a multiplicative A -metric space. The induced two-point distance $D(\alpha, \beta) = \mathcal{A}^*(\underbrace{\alpha, \dots, \alpha}_{n-1}, \beta)$ is explicitly given by

$$D(\alpha, \beta) = \exp((n-1)|\alpha - \beta|).$$

Indeed, for the n -tuple $(\alpha, \dots, \alpha, \beta)$ the only nonzero pairwise distances are those involving β , and there are exactly $(n-1)$ such pairs, each equal to $|\alpha - \beta|$, hence the sum of all pairwise distances equals $(n-1)|\alpha - \beta|$, which yields the above formula.

Define $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ by $\Phi(\alpha) = c\alpha$ where $c \in (0, 1)$. For any $\alpha, \beta \in \mathbb{R}$,

$$|\Phi(\alpha) - \Phi(\beta)| = |c\alpha - c\beta| = c|\alpha - \beta|.$$

Substituting into the expression for D gives

$$\begin{aligned} D(\Phi\alpha, \Phi\beta) &= \exp((n-1)|\Phi(\alpha) - \Phi(\beta)|) = \exp((n-1)c|\alpha - \beta|) \\ &= \left(\exp((n-1)|\alpha - \beta|)\right)^c = (D(\alpha, \beta))^c. \end{aligned}$$

Thus the multiplicative Banach-type inequality

$$D(\Phi\alpha, \Phi\beta) \leq D(\alpha, \beta)^q$$

holds with $q = c \in (0, 1)$. Since (\mathbb{R}, D) is complete (because $D(\alpha, \beta) = \exp((n-1)|\alpha - \beta|)$ is equivalent to the usual metric via the logarithmic transform), Theorem 5.2 applies and yields a unique fixed point.

Finally, the fixed point equation $\Phi(\xi) = \xi$ reduces to $c\xi = \xi$, i.e. $(c-1)\xi = 0$. Since $c \neq 1$, it follows that $\xi = 0$. Therefore 0 is the unique fixed point, and for every $\alpha_0 \in \mathbb{R}$ the Picard iteration $\alpha_{k+1} = c\alpha_k$ converges multiplicatively to 0, equivalently $D(\alpha_k, 0) \rightarrow 1$.

Example 5.5 Let $X = [0, \infty)$, fix $n \geq 3$, and equip X with the usual metric $d(x, y) = |x - y|$. Define \mathcal{A}^* as in Proposition 3.12 by

$$\mathcal{A}^*(\alpha_1, \dots, \alpha_n) := \exp\left(\sum_{1 \leq i < j \leq n} |\alpha_i - \alpha_j|\right).$$

Then (X, \mathcal{A}^*) is a multiplicative A -metric space and the induced two-point distance is

$$D(\alpha, \beta) = \mathcal{A}^*(\underbrace{\alpha, \dots, \alpha}_{n-1}, \beta) = \exp((n-1)|\alpha - \beta|), \quad \alpha, \beta \in X.$$

Fix constants $c \in (0, 1)$ and $b \geq 0$, and define $\Phi : X \rightarrow X$ by

$$\Phi(\alpha) = c\alpha + b, \quad \alpha \in X.$$

Then $\Phi(X) \subset X$ because $c\alpha + b \geq 0$ for all $\alpha \geq 0$.

For any $\alpha, \beta \in X$, the translation term cancels and

$$|\Phi(\alpha) - \Phi(\beta)| = |c\alpha + b - (c\beta + b)| = c|\alpha - \beta|.$$

Substituting this into the explicit formula for D gives

$$\begin{aligned} D(\Phi\alpha, \Phi\beta) &= \exp((n-1)|\Phi(\alpha) - \Phi(\beta)|) = \exp((n-1)c|\alpha - \beta|) \\ &= (\exp((n-1)|\alpha - \beta|))^c = (D(\alpha, \beta))^c. \end{aligned}$$

Hence Φ satisfies the multiplicative Banach-type inequality

$$D(\Phi\alpha, \Phi\beta) \leq D(\alpha, \beta)^q \quad \text{for all } \alpha, \beta \in X$$

with $q = c \in (0, 1)$. Moreover, (X, D) is complete because $D(\alpha, \beta) = \exp((n-1)|\alpha - \beta|)$ is equivalent to the usual metric on $[0, \infty)$ via the logarithmic transform.

Therefore Theorem 5.2 applies and yields existence and uniqueness of a fixed point $\xi \in X$ and multiplicative convergence of the Picard iteration for every initial value.

Finally, the fixed point equation $\Phi(\xi) = \xi$ gives

$$\xi = c\xi + b, \quad \text{hence} \quad (1-c)\xi = b,$$

so

$$\xi = \frac{b}{1-c} \in [0, \infty).$$

Consequently, Φ has the unique fixed point $\xi = \frac{b}{1-c}$, and for every $\alpha_0 \in X$ the iteration $\alpha_{k+1} = c\alpha_k + b$ converges multiplicatively to ξ , i.e. $D(\alpha_k, \xi) \rightarrow 1$.

§6. Fixed Points for Expansive Mappings

Throughout this section let (X, \mathcal{A}^*) be a multiplicative \mathcal{A} -metric space with parameter $n \geq 3$ and let D denote the induced two-point multiplicative distance.

Definition 6.1 (k -expansive mapping) *Let $k > 1$. A mapping $\Phi : X \rightarrow X$ is called k -expansive if*

$$D(\Phi(\alpha), \Phi(\beta)) \geq (D(\alpha, \beta))^k \quad \forall \alpha, \beta \in X.$$

Lemma 6.2 *If Φ is k -expansive with $k > 1$, then Φ has at most one fixed point.*

Proof Assume ξ, η are fixed points. Then

$$D(\xi, \eta) = D(\Phi\xi, \Phi\eta) \geq D(\xi, \eta)^k.$$

Since $D(\xi, \eta) \geq 1$ and $k > 1$, if $D(\xi, \eta) > 1$ then

$$D(\xi, \eta)^k > D(\xi, \eta),$$

which contradicts the above inequality. Hence $D(\xi, \eta) = 1$, and therefore $\xi = \eta$. \square

Definition 6.3(Multiplicative diameter)

$$\text{diam}(X) := \sup_{\alpha, \beta \in X} D(\alpha, \beta) \in [1, \infty].$$

Theorem 6.4 *If $\text{diam}(X) < \infty$ and there exists a k -expansive mapping $\Phi : X \rightarrow X$ with $k > 1$, then X is a singleton.*

Proof Fix arbitrary $\alpha, \beta \in X$. Repeated application of expansiveness gives

$$D(\Phi^m(\alpha), \Phi^m(\beta)) \geq D(\alpha, \beta)^{k^m} \quad (m \geq 0).$$

Since $\Phi^m(\alpha), \Phi^m(\beta) \in X$, we have

$$D(\Phi^m(\alpha), \Phi^m(\beta)) \leq \text{diam}(X).$$

Hence

$$D(\alpha, \beta)^{k^m} \leq \text{diam}(X), \quad \forall m.$$

If $D(\alpha, \beta) > 1$, then since $k^m \rightarrow \infty$, the left-hand side diverges to infinity, which contradicts bounded diameter. Thus $D(\alpha, \beta) = 1$, hence $\alpha = \beta$. Since α, β were arbitrary, X is a singleton, completing the proof. \square

Definition 6.5(D -bounded set) *A subset $B \subset X$ is called D -bounded if*

$$\sup_{u, v \in B} D(u, v) < \infty.$$

Theorem 6.6 *Let Φ be k -expansive with $k > 1$. If there exists a nonempty D -bounded set B such that $\Phi(B) \subset B$, then every point of B is a fixed point. Consequently, B is a singleton.*

Proof Fix $\gamma \in B$ and set $\delta = \Phi(\gamma)$. Define $\gamma_m = \Phi^m(\gamma)$. Since $\Phi(B) \subset B$, it follows that $\gamma_m \in B$ for all m .

Let

$$M := \sup_{u, v \in B} D(u, v) < \infty.$$

By repeated expansiveness,

$$D(\gamma_m, \gamma_{m+1}) = D(\Phi^m \gamma, \Phi^m \delta) \geq D(\gamma, \delta)^{k^m}.$$

Since $\gamma_m, \gamma_{m+1} \in B$, we must have

$$D(\gamma_m, \gamma_{m+1}) \leq M.$$

If $D(\gamma, \delta) > 1$, then $D(\gamma, \delta)^{k^m} \rightarrow \infty$, contradicting boundedness. Hence $D(\gamma, \delta) = 1$ and therefore

$$\Phi(\gamma) = \gamma.$$

Since γ was arbitrary in B , every point of B is a fixed point. By Lemma 6.2, there is at most one fixed point, hence B is a singleton. \square

Theorem 6.7 *Assume D is continuous with respect to the multiplicative topology. If X is compact and $\Phi : X \rightarrow X$ is continuous and k -expansive for some $k > 1$, then X is a singleton.*

Proof Since X is compact, the product space $X \times X$ is compact. By continuity of D , the image

$$D(X \times X) \subset [1, \infty)$$

is compact and hence bounded. Therefore $\text{diam}(X) < \infty$. The conclusion now follows directly from Theorem 6.4. \square

Example 6.8 Let $X = \mathbb{R}$ and let \mathcal{A}^* be defined as in Proposition 3.12 with $d(x, y) = |x - y|$. Then

$$D(\alpha, \beta) = \exp((n-1)|\alpha - \beta|).$$

Let $\Phi(\alpha) = c\alpha$ with $|c| \geq k > 1$. Then

$$\begin{aligned} D(\Phi\alpha, \Phi\beta) &= \exp((n-1)|c||\alpha - \beta|) \\ &\geq \exp((n-1)k|\alpha - \beta|) = D(\alpha, \beta)^k, \end{aligned}$$

so Φ is k -expansive. The fixed point equation $c\xi = \xi$ gives $(c-1)\xi = 0$, hence the unique fixed point is $\xi = 0$.

Proposition 6.9 *For $\Phi(\alpha) = c\alpha$ with $|c| > 1$, the forward orbit $\{\Phi^m(\alpha)\} = \{c^m\alpha\}$ is D -bounded if and only if $\alpha = 0$.*

Proof If $\alpha = 0$, the orbit is constant and therefore bounded. If $\alpha \neq 0$, then $|c^m\alpha| \rightarrow \infty$ and hence

$$D(0, c^m\alpha) = \exp((n-1)|c^m\alpha|) \rightarrow \infty.$$

So the orbit is not D -bounded. \square

§7. Applications

Throughout this section let

$$X := C([0, 1]), \quad \|u\|_\infty := \sup_{t \in [0, 1]} |u(t)|.$$

Then $(X, \|\cdot\|_\infty)$ is a Banach space.

Fix $n \geq 3$ and define the multiplicative A-metric

$$\mathcal{A}^*(u_1, \dots, u_n) := \exp\left(\sum_{1 \leq i < j \leq n} \|u_i - u_j\|_\infty\right). \quad (1)$$

The induced two-point multiplicative distance is therefore

$$D(u, v) = \exp((n-1)\|u - v\|_\infty). \quad (2)$$

Lemma 7.1 *The multiplicative A-metric space $(C([0, 1]), \mathcal{A}^*)$ is complete. Equivalently, (X, D) is multiplicatively complete.*

Proof Let $\{u_m\} \subset X$ be multiplicative A-Cauchy. By definition, for every $\varepsilon > 1$ there exists N such that

$$D(u_m, u_\ell) < \varepsilon \quad \text{for all } m, \ell \geq N.$$

Using (2), this is equivalent to

$$\exp((n-1)\|u_m - u_\ell\|_\infty) < \varepsilon,$$

which implies

$$\|u_m - u_\ell\|_\infty < \frac{\ln \varepsilon}{n-1}.$$

Hence $\{u_m\}$ is Cauchy in the Banach space $(X, \|\cdot\|_\infty)$, and therefore there exists $u \in X$ such that $u_m \rightarrow u$ uniformly. Passing back to the multiplicative distance,

$$D(u_m, u) = \exp((n-1)\|u_m - u\|_\infty) \rightarrow 1,$$

which shows multiplicative convergence. Hence (X, \mathcal{A}^*) is complete. \square

7.1 Nonlinear Volterra Integral Equation

Define the operator $\Phi : X \rightarrow X$ by

$$(\Phi u)(t) := g(t) + \int_0^t K(t, s, u(s)) ds, \quad t \in [0, 1], \quad (3)$$

where $g \in C([0, 1])$ and K is continuous and satisfies the Lipschitz condition

$$|K(t, s, x) - K(t, s, y)| \leq L|x - y|, \quad L \in [0, 1]. \quad (4)$$

Theorem 7.2 *Under condition (4) with $L \in [0, 1]$, the nonlinear Volterra equation*

$$u(t) = g(t) + \int_0^t K(t, s, u(s)) ds$$

admits a unique solution $\xi \in C([0, 1])$. Moreover, for every $u_0 \in C([0, 1])$, the Picard sequence $u_{m+1} = \Phi(u_m)$ converges to ξ in the multiplicative A -sense and hence uniformly.

Proof Let $u, v \in X$. Using (3) and (4),

$$|(\Phi u)(t) - (\Phi v)(t)| \leq \int_0^t L|u(s) - v(s)| ds \leq L\|u - v\|_\infty.$$

Taking supremum over $t \in [0, 1]$ yields

$$\|\Phi u - \Phi v\|_\infty \leq L\|u - v\|_\infty.$$

Using (2),

$$D(\Phi u, \Phi v) = \exp((n-1)\|\Phi u - \Phi v\|_\infty) \leq \exp((n-1)L\|u - v\|_\infty) = D(u, v)^L.$$

Since $L \in (0, 1)$, Φ is multiplicative Banach-type contractive. Completeness follows from Lemma 7.1, hence Theorem 5.2 yields existence, uniqueness, and multiplicative convergence. \square

7.2 Second-order Boundary Value Problem via Green Operator

Consider the boundary value problem

$$u''(t) = F(t, u(t)), \quad u(0) = \alpha, \quad u(1) = \beta,$$

where F is continuous and satisfies

$$|F(t, x) - F(t, y)| \leq L|x - y|.$$

Let the Green kernel be

$$G(t, s) = \begin{cases} s(1-t), & s \leq t, \\ t(1-s), & t \leq s, \end{cases}$$

which satisfies

$$\sup_{t \in [0, 1]} \int_0^1 G(t, s) ds = \frac{1}{8}.$$

Define

$$(\Phi u)(t) = \alpha(1-t) + \beta t + \int_0^1 G(t,s)F(s,u(s))ds.$$

Theorem 7.3 *If $L < 8$, then the boundary value problem admits a unique solution $\xi \in C^2([0, 1])$. Moreover, Picard iteration converges multiplicatively to ξ .*

Proof For $u, v \in X$,

$$\|\Phi u - \Phi v\|_\infty \leq \sup_t \int_0^1 G(t,s)L|u(s) - v(s)|ds \leq \frac{L}{8}\|u - v\|_\infty.$$

Hence

$$D(\Phi u, \Phi v) \leq D(u, v)^{L/8}.$$

Since $L/8 < 1$, Φ is multiplicative Banach-type contractive. Existence and uniqueness follow. Standard elliptic regularity of Green operators implies $\xi \in C^2([0, 1])$. \square

7.3 Expansive Linear Operator and Bounded Orbits

Let $(E, \|\cdot\|)$ be a Banach space and define

$$D(x, y) = \exp((n-1)\|x - y\|).$$

Assume $\Phi : E \rightarrow E$ is linear and satisfies

$$\|\Phi x - \Phi y\| \geq c\|x - y\|, \quad c > 1.$$

Theorem 7.4 *Φ is k -expansive with $k = c$. The only point having a D -bounded forward orbit is 0, and 0 is the unique fixed point.*

Proof Using monotonicity of the exponential function,

$$D(\Phi x, \Phi y) = \exp((n-1)\|\Phi x - \Phi y\|) \geq \exp((n-1)c\|x - y\|) = D(x, y)^c.$$

Hence Φ is k -expansive.

Taking $y = 0$ gives $\|\Phi x\| \geq c\|x\|$. Iterating yields $\|\Phi^m x\| \geq c^m\|x\|$. If $x \neq 0$, then $\|\Phi^m x\| \rightarrow \infty$ and therefore $D(0, \Phi^m x) \rightarrow \infty$. Thus only $x = 0$ has bounded orbit. Fixed point uniqueness follows from expansiveness. \square

§8. Conclusion

In this work, we introduced the framework of multiplicative A -metric spaces (MA-spaces), which provides a natural synthesis of the multi-point structure of A -metrics and the product-type geometry inherent in multiplicative distance models. This construction allows distances to be measured simultaneously across multiple points while preserving the multiplicative structure that frequently arises in nonlinear growth processes, stability theory, and exponential-type

transformations.

Within this setting, we developed the fundamental topological structure induced by the multiplicative distance, including multiplicative open balls, neighborhood systems, and the associated multiplicative topology. We established basic sequential properties of MA-spaces, including characterization of convergence via the induced two-point multiplicative distance, uniqueness of limits, and Hausdorff separation. These results confirm that MA-spaces provide a robust analytical environment comparable to classical metric frameworks while offering additional structural flexibility.

On the analytical side, we established fixed point results for multiplicative contractive mappings, including Banach-type and Kannan-type principles. These theorems extend classical fixed point theory to a multi-point multiplicative setting and demonstrate that iterative schemes converge under natural multiplicative contractive conditions. In contrast, for strictly expansive mappings, we identified strong rigidity phenomena, showing that the presence of bounded invariant sets or finite multiplicative diameter forces structural triviality of the space. These results highlight a fundamental dichotomy between multiplicative contraction and expansion behaviors in MA-spaces.

To demonstrate applicability, we constructed a natural exponential embedding of MA-geometry into function spaces, particularly $C([0, 1])$ equipped with the supremum norm. Using this embedding, we established existence and uniqueness results for nonlinear Volterra integral equations and second-order boundary value problems via Green operator methods. These applications show that MA-spaces provide an effective analytical framework for studying nonlinear functional equations in infinite-dimensional settings.

Overall, multiplicative A -metric spaces offer a unified platform connecting multi-point distance geometry, multiplicative analysis, and nonlinear fixed point theory. The framework opens several directions for further investigation, including extensions to fractional operators, stochastic functional equations, and generalized multi-point multiplicative structures.

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