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Neighborhoods of Certain Class of Analytic Functions Using Modified Sigmoid Function

Gehan Abo Elyazyd, Hanan Darwish and Awatif Shahin

(Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt)

E-mail: gehan.aldesoky@yahoo.com, Darwish333@yahoo.com

Abstract: By using an operator involving modified Sigmoid function we prove the neighborhoods problem.of more class for function $f_\gamma(z) \in T_\gamma$ in the unit disc $\mathfrak{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$.

Key Words: Modified sigmoid function, $(m, \delta)-$neighborhood, analytic function, Al-Oboudi operator, Sălăgean Operator.

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§1. Introduction

A sigmoid function is a mathematical function having an "S" shape (sigmoid curve). The sigmoid function, also called the sigmoidal curve or logistic function is the function of the form $\gamma(s) = \frac{1}{1+e^{-s}}$; $S \in \mathbb{R}$. A sigmoid function is a bounded differentiable real function that is defined for all real input values and has a positive derivative at each point. It is useful in compressing, or squashing outputs. It is a monotone function The sigmoid function is the most popular of the three activation function in the hardware implementation of artificial neural network. The Sigmoid function is defined as

$$G(s) = \frac{1}{1+e^{-s}} = \frac{1}{2} + \frac{1}{4}s - \frac{1}{48}s^3 + \frac{1}{480}s^5 - \frac{17}{80640}s^7 + \cdots.$$

Let $\gamma(s)$ be a modified Sigmoid function, that is

$$\gamma(s) = \frac{2}{1+e^{-s}} = 1 + \frac{1}{2}s - \frac{1}{24}s^3 + \frac{1}{240}s^5 - \frac{17}{40320}s^7 + \cdots \quad s \geq 0 \tag{1.1}$$

with $\gamma(s) = 1$ for $s = 0$ be a modified Sigmoid function. For detail information on Sigmoid function see ([4], [3], [8], [7]).

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.2}$$

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which are analytic in the open unit disk $\mathfrak{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

The Hadmard product of two functions $f(z) \in \mathcal{S}$ and $g(z) \in \mathcal{S}$ which denoted by $(f * g)(z)$, that is, if $f(z)$ is given by (**??**) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (z \in \mathfrak{U}),$$

then

$$(f * g)(z) = (g * f)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in \mathfrak{U}). \tag{1.3}$$

Also, we denote by T the subclass of $\mathcal{S}$ consisting of functions $f(z) \in \mathcal{S}$ which are analytic and univalent in $\mathfrak{U}$ and of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.4}$$

A function $f_\gamma(z) \in T_\gamma$ defined as

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \quad a_k \geq 0 \tag{1.5}$$

where $\gamma(s)$ defined by (1.1). Furthermore, we define identity function for T_γ as $e_\gamma(z) = z$.

§2. Differential Operators

2.1. Sălăgean Differential Operator

Definition 2.1([10]) *For $f(z) \in \mathcal{S}$ and $n \in \mathbb{N}_0$, the Sălăgean differential operator $D^n : \mathcal{S} \to \mathcal{S}$ is defined by*

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= z f'(z) \\ &\vdots \\ D^{n+1} f(z) &= z(D^n f(z))'. \end{aligned}$$

Remark 2.1 If $f(z)$ is given by (1.2), then

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad z \in \mathfrak{U}. \tag{2.1}$$

2.2. Al-Oboudi Differential Operator

Definition 2.2 *For $f(z) \in \mathcal{S}$ and $n \in \mathbb{N}_0$, Al-Oboudi differential operator $D_\delta^n f(z)$ is defined as* (see [1])

$$\begin{aligned} D^0 f(z) &= f(z) \\ D_\delta f(z) &= D^1 f(z) = (1-\delta) f(z) + \delta z f^{'}(z),\ \ \delta \geq 0 \\ &\vdots \\ D_\delta^n f(z) &= D_\delta(D^{n-1} f(z)). \end{aligned}$$

Remark 2.2 D_δ^n is a linear operator for $f(z) \in \mathcal{S}$, we get

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} \{1 + (k-1)\delta\}^n a_k z^k,\ \ z \in \mathfrak{U}, \delta \geq 0. \tag{2.2}$$

For $\delta = 1$, (2.2) becomes (2.1).

In [2], Darus and Ibrahim introduced a generalized differential operator

$$\begin{aligned} D^0 f(z) &= f(z) \\ D_{\alpha,\lambda}^1 f(z) &= (\alpha - \lambda) f(z) + (\lambda - \alpha + 1) z f^{'}(z) \\ D_{\alpha,\lambda}^n f(z) &= D_{\alpha,\lambda}^1 (D^{n-1} f(z)). \end{aligned}$$

Thus

$$D_{\alpha,\lambda}^n f(z) = z + \sum_{k=2}^{\infty} \{(k-1)(\lambda - \alpha) + k\}^n a_k z^k. \tag{2.3}$$

2.3. Differential Operator Involving Modified Sigmoid Function

In [4], Fadipe-Joseph et al. introduced Sălăgean differential operator involving modified sigmoid function which is defined as follows.

Let $f_\gamma(z) \in T_\gamma$, the Sălăgean differential operator $D^n f_\gamma(z)$ is defined by

$$\begin{aligned} D^0 f_\gamma(z) &= f_\gamma(z) \\ D^1 f(z) &= z\gamma(s) f_\gamma^{'}(z) \\ &\vdots \\ D^n f_\gamma(z) &= D(D^{n-1} f_\gamma(z)) = z\gamma(s)(D^{n-1} f_\gamma(z))^{'}. \end{aligned}$$

Hence

$$D^n f_\gamma(z) = \gamma^n(s) z + \sum_{k=2}^{\infty} \gamma^{n+1}(s) k^n a_k z^k. \tag{2.4}$$

2.4. Ruscheweyh Operator Involving Modified Sigmoid Function

Ruscheweyh differential operator involving the modified sigmoid function with $R^n : T_\gamma \to T_\gamma$ is defined as

$$R^n f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) B_k(n) a_k z^k, \ a_k \geq 0, n \in \mathbb{N}_0, \tag{2.5}$$

where

$$\begin{aligned} B_k(n) &= B(n,k) = \begin{pmatrix} n+k-1 \\ n \end{pmatrix} \\ &= \frac{(n+1)(n+2)\ldots(n+k-1)}{(k-1)!} = \frac{(n+1)_{k-1}}{(1)_{k-1}}. \end{aligned}$$

Hence

$$B(0,k) = \begin{pmatrix} k-1 \\ 0 \end{pmatrix} = \frac{(1)_{k-1}}{(1)_{k-1}} = 1.$$

See [4] and [9] for detail.

2.5. New Differential Operator Involving Modified Sigmoid Function

Definition 2.3 *Let $f_\gamma(z) \in T_\gamma$, then from (*2.3*) and (*2.4*) we obtain a generalized differential operator involving modified sigmoid function as follows*

$$D^n_{\lambda,\omega} f_\gamma(z) = \gamma^n(s) z - \sum_{k=2}^{\infty} \gamma^{n+1}(s) \left\{(k-1)(\lambda-\omega)+k\right\}^n a_k z^k, \tag{2.6}$$

for $\lambda, \omega \geq 0$ (see [4] and [2])*.*

2.6. Linear Combination of Generalized Sălăgean Differential Operator and Ruscheweyh Operator Involving Modified Sigmoid Function

We combine the generalised Sălăgean differential operator involving modified sigmoid function and the Ruscheweyh operator involving modified Sigmoid function to obtain a certain operator defined as

$$\begin{aligned} \Phi^n_{\lambda,\omega} f_\gamma(z) &= \mu D^n_{\lambda,\omega} f_\gamma(z) + (1-\mu) R^n f_\gamma(z) \\ &= \left[\mu\gamma^n(s) - \mu + 1\right] z \\ &\quad - \sum_{k=2}^{\infty} \gamma(s) \left\{\mu\left[\gamma^n(s)(k-1)(\lambda-\omega)+k\right]^n + (1-\mu) B_k(n)\right\} a_k z^k \end{aligned} \tag{2.7}$$

for $0 \leq \lambda, \mu \leq 1$ with special cases following:

(*i*) If $n = 0$ in (2.7), we get $f_\gamma(z)$ defined in (1.5);
(*ii*) If $\mu = 1$ in (2.7), then $\Phi^n_{\lambda,\omega} f_\gamma(z) = D^n_{\lambda,\omega} f_\gamma(z)$ defined in (2.6);
(*iii*) If $\mu = 0$ in (2.7), then $\Phi^n_{\lambda,\omega} f_\gamma(z) = R^n f_\gamma(z)$ defined in (2.5);
(*iv*) If $s = 0, \mu = 1$ and $\omega = 1$ in (2.7), then $\Phi^n_{\lambda,1} f_\gamma(z) = D^n_\delta f(z)$ defined in (2.2);
(*v*) If $s = 0, \mu = 1$ and $\lambda = \omega = 0$ in (2.7), then $\Phi^n_{0,0} f_\gamma(z) = D^n f(z)$ defined in (2.1).

Definition 2.4([8]) *For a function $f_\gamma(z) \in T_\gamma$ defined by (1.5) is in the class $T_\gamma \chi_\mu^n(\lambda, \omega, \alpha, p, \zeta, \eta)$ if*

$$\left| \frac{\left[\Phi_{\lambda,\omega}^n f_\gamma(z)\right]' - [\mu\gamma^n(s) - \mu + 1]}{p\zeta\left[\left(\Phi_{\lambda,\omega}^n f_\gamma(z)\right)' - \alpha\right] - \left[\left(\Phi_{\lambda,\omega}^n f_\gamma(z)\right)' - [\mu\gamma^n(s) - \mu + 1]\right]} \right| < \eta, \tag{2.8}$$

where $0 \le \alpha < \frac{1}{2}\zeta,\ 0 \le \mu \le 1,\ \lambda, \omega \ge 0,\ p \ge 2,\ \frac{1}{2} \le \zeta \le 1,\ 0 < \eta \le 1$ *and* $n \in \mathbb{N}_0$.

Note that:

(1) If $\alpha = 0,\ \mu = 1,\ p = 2$ and $\zeta = 1$, then

$$T_\gamma \chi^n(\lambda, \omega, 0, 2, 1, \eta) = S\chi_\gamma^*(n, \eta, \lambda, \omega) = \left| \frac{\left[D_{\lambda,\omega}^n f_\gamma(z)\right]' - \gamma^n(s)}{\left[D_{\lambda,\omega}^n f_\gamma(z)\right]' + \gamma^n(s)} \right| < \eta.$$

(2) If $\alpha = 0,\ \mu = 1,\ p = 2$ and $\zeta = \frac{1}{2}$, then

$$T_\gamma \chi^n(\lambda, \omega, 0, 2, \frac{1}{2}, \eta) = \chi_\gamma^*(n, \eta, \lambda, \omega) = \left| \frac{1}{\gamma^n(s)} \left[D_{\lambda,\omega}^n f_\gamma(z)\right]' - 1 \right| < \eta.$$

(3) If $\mu = 0,\ p = 2$ and $\zeta = 1$, then

$$T_\gamma \chi_0^n(0, 0, \alpha, 2, 1, \eta) = \chi_\gamma^*(n, \eta, \alpha) = \left| \frac{[R^n f_\gamma(z)]' - 1}{[R^n f_\gamma(z)]' - 2\alpha + 1} \right| < \eta.$$

(4) If $\alpha = 0,\ \mu = 0,\ p = 2$ and $\zeta = 1$, then

$$T_\gamma \chi_0^n(0, 0, 0, 2, 1, \eta) = S\chi_\gamma^*(n, \eta) = \left| \frac{[R^n f_\gamma(z)]' - 1}{[R^n f_\gamma(z)]' + 1} \right| < \eta.$$

(5) If $\alpha = 0,\ \mu = 0,\ p = 2$ and $\zeta = \frac{1}{2}$, then

$$T_\gamma \chi_0^n(0, 0, 0, 2, \frac{1}{2}, \eta) = \chi_\gamma^*(n, \eta) = \left| [R^n f_\gamma(z)]' - 1 \right| < \eta.$$

(6) If $\eta = 1,\ \mu = 0,\ p = 2,\ n = 0$ and $\zeta = 1$, then

$$T_\gamma \chi_0^0(0, 0, \alpha, 2, 1, 1) = \chi_\gamma^*(\alpha) = \left| \frac{f_\gamma'(z) - 1}{f_\gamma'(z) - 2\alpha + 1} \right| < 1.$$

(7) If $\eta = 1,\ \alpha = 0,\ \mu = 0,\ p = 2,\ n = 0$ and $\zeta = 1$, then

$$T_\gamma \chi_0^0(0, 0, 0, 2, 1, 1) = \chi^*(\gamma) = \left| \frac{f_\gamma'(z) - 1}{f_\gamma'(z) + 1} \right| < 1.$$

(8) If $\alpha = 0,\ \mu = 0,\ p = 2,\ n = 0$ and $\zeta = \frac{1}{2}$, then

$$T_\gamma \chi_0^0(0, 0, 0, 2, \frac{1}{2}, \eta) = \chi_\gamma^*(\eta) = \left| f_\gamma'(z) - 1 \right| < \eta.$$

(9) If $\mu = 0,\ p = 2,\ s = 0$ and $\zeta = 1$, then

$$T\chi_0^n(0,0,\alpha,2,1,\eta) = \chi^*(n,\eta,\alpha) = \left|\frac{[R^n f(z)]' - 1}{[R^n f(z)]' - 2\alpha + 1}\right| < \eta.$$

(10) If $\alpha = 0,\ \mu = 0,\ p = 2,\ s = 0$ and $\zeta = 1$, then

$$T\chi_0^n(0,0,0,2,1,\eta) = S\chi^*(n,\eta) = \left|\frac{[R^n f(z)]' - 1}{[R^n f(z)]' + 1}\right| < \eta.$$

(11) If $\alpha = 0,\ \mu = 1,\ p = 2,\ s = 0$ and $\zeta = \frac{1}{2}$, then

$$T\chi^n(\lambda,\omega,0,2,\frac{1}{2},\eta) = \chi^*(n,\eta,\lambda,\omega) = \left|\left[D_{\lambda,\omega}^n f(z)\right]' - 1\right| < \eta.$$

(12) If $\alpha = 0,\ s = 0,\ \mu = 0,\ p = 2$ and $\zeta = \frac{1}{2}$, then

$$T\chi_0^n(0,0,0,2,\frac{1}{2},\eta) = \chi^*(n,\eta) = \left|[R^n f(z)]' - 1\right| < \eta.$$

(13) If $\mu = 0,\ n = 0,\ p = 2,\ s = 0$ and $\zeta = 1$, then

$$T\chi_0^0(0,0,\alpha,2,1,\eta) = \chi^*(\eta,\alpha) = \left|\frac{f'(z) - 1}{f'(z) - 2\alpha + 1}\right| < \eta \quad \text{(Juneja and Mogra [?])}.$$

(14) If $\alpha = 0,\ \mu = 0,\ n = 0,\ p = 2,\ s = 0$ and $\zeta = 1$, then

$$T\chi_0^0(0,0,0,2,1,\eta) = S\chi^*(\eta) = \left|\frac{f'(z) - 1}{f'(z) + 1}\right| < \eta, \quad \text{(Kim and Lee [?])}.$$

(15) If $\alpha = 0,\ n = 0,\ s = 0,\ \mu = 0,\ p = 2$ and $\zeta = \frac{1}{2}$, then

$$T\chi_0^0(0,0,0,2,\frac{1}{2},\eta) = \chi^*(\eta) = |f(z)' - 1| < \eta, \quad \text{(Kim and Lee [?])}.$$

We begin by proving the necessary and sufficient condition for a function belongs to the class $T_\gamma\chi_\mu^n(\lambda,\omega,\alpha,p,\zeta,\eta)$.

2.7. Coefficient Estimates

Theorem 2.1 *If a function $f_\gamma(z)$ belongs to the class $T_\gamma\chi_\mu^n(\lambda,\omega,\alpha,p,\zeta,\eta)$, then*

$$\sum_{k=2}^{\infty} k\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^n(s)\left[(k-1)(\lambda-\omega)+k\right]^n + (1-\mu)B_k(n)\right\}a_k$$
$$\le \eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right]. \tag{2.9}$$

Proof Suppose $f_\gamma(z)$ belongs to the class $T_\gamma\chi_\mu^n(\lambda,\omega,\alpha,p,\zeta,\eta)$, by equation (2.7) and Definition 2.4, we have that

$$\left|-\sum_{k=2}^{\infty}k\gamma(s)\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_k(n)\right\}a_k z^{k-1}\right\}\Bigg|$$
$$\leq \eta\Bigg|p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]-\sum_{k=2}^{\infty}(1-p\zeta)k\gamma(s)\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}\right.$$
$$\left.+(1-\mu)B_k(n)\right\}a_k z^{k-1}\Big|,$$

$|z|\leq r$ and $r\to 1^{+}$, then

$$\sum_{k=2}^{\infty}k\gamma(s)\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_k(n)\right\}a_k$$
$$\leq \eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]$$
$$+\sum_{k=2}^{\infty}\eta(1-p\zeta)k\gamma(s)\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_k(n)\right\}a_k$$

or

$$\sum_{k=2}^{\infty}k\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_k(n)\right\}a_k$$
$$\leq \eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right].$$

Hence,

$$\sum_{k=2}^{\infty}a_k\leq\frac{\eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]}{k\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_k(n)\right\}}. \tag{2.10}$$

The result is sharp for

$$f(z)=z-\frac{\eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]}{k\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_k(n)\right\}}z^{k}$$

and $k\geq 2$. □

2.8. Neighborhoods for the Class $T_\gamma\chi_\mu^n(\lambda,\omega,\alpha,p,\zeta,\eta)$

Definition 2.5 *The* $(m,\delta)-$*neighborhood of the function* $f_\gamma(z)$ *belongs to the class* T_γ *by*

$$N_{m,\delta}(f_\gamma)=\left\{g_\gamma: g_\gamma\in T_\gamma,\ g_\gamma=z-\sum_{k=m+1}^{\infty}\gamma(s)b_k z^k\ \ and\ \sum_{k=m+1}^{\infty}k\gamma(s)\left|a_k-b_k\right|\leq\delta\right\}. \tag{2.11}$$

In particular, if identity function

$$e_\gamma(z)=z,$$

we immediately have

$$N_{m,\delta}(e_\gamma) = \left\{ g_\gamma : g_\gamma \in T_\gamma,\ g_\gamma = z - \sum_{k=m+1}^{\infty} \gamma(s) b_k z^k \ \textit{and} \sum_{k=m+1}^{\infty} k\gamma(s)\,|b_k| \leq \delta \right\}. \tag{2.12}$$

Lemma 2.1 *Let the function $f_\gamma(z) \in T_\gamma$ be defined by*

$$f_\gamma(z) = z - \sum_{k=m+1}^{\infty} \gamma(s) a_k z^k, \quad a_k \geq 0. \tag{2.13}$$

Then, $f_\gamma(z)$ is in the class $T_\gamma \chi_\mu^n(\lambda, \omega, \alpha, p, \zeta, \eta)$ if and only if

$$\sum_{k=m+1}^{\infty} k\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[(k-1)(\lambda - \omega) + k\right]^n + (1-\mu)B_k(n)\right\} a_k$$
$$\leq \eta p\zeta \left[\mu\gamma^n(s) - \mu + 1 - \alpha\right]. \tag{2.14}$$

Theorem 2.2 *Let*

$$\delta = \frac{\eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right]}{\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda - \omega) + m + 1\right]^n + (1-\mu)B_{m+1}(n)\right\}}. \tag{2.15}$$

If $\delta < 1$, then $T_\gamma \chi_\mu^n(\lambda, \omega, \alpha, p, \zeta, \eta) \subset N_{m,\delta}(e_\gamma)$.

Proof Let the function $f_\gamma(z)$ is in the class $T_\gamma \chi_\mu^n(\lambda, \omega, \alpha, p, \zeta, \eta)$, we have

$$(m+1)\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda - \omega) + m + 1\right]^n + (1-\mu)B_{m+1}(n)\right\}$$
$$\sum_{k=m+1}^{\infty} a_k \leq \eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right], \tag{2.16}$$

which readily yields

$$\sum_{k=m+1}^{\infty} a_k \leq \frac{\eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right]}{(m+1)\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda - \omega) + m + 1\right]^n + (1-\mu)B_{m+1}(n)\right\}}. \tag{2.17}$$

Applying (2.14) again, in conjunction with (2.17), we get

$$\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda - \omega) + m + 1\right]^n + (1-\mu)B_{m+1}(n)\right\}$$
$$\sum_{k=m+1}^{\infty} k a_k \leq \eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right].$$

So that

$$\begin{aligned}\sum_{k=m+1}^{\infty} ka_k &\leq \frac{\eta p\zeta\,[\mu\gamma^n(s)-\mu+1-\alpha]}{\gamma(s)\,[1+\eta(p\zeta-1)]\,\{\mu\gamma^n(s)\,[m(\lambda-\omega)+m+1]^n+(1-\mu)B_{m+1}(n)\}}\\ &= \delta. \end{aligned}\tag{2.18}$$

The proof is completed. □

Corollary 2.1 *Let*

$$\delta = 1 - \frac{[\gamma(s)\,(1+\eta)\,[m(\lambda-\omega)+m+1]^n - 2\eta]}{\gamma(s)\,(1+\eta)\,[m(\lambda-\omega)+m+1]^n}.$$

Then, $S\chi^*_\gamma(n,\eta,\lambda,\omega) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.2 *Let*

$$\delta = 1 - \frac{[\gamma(s)\,[m(\lambda-\omega)+m+1]^n - \eta]}{\gamma(s)\,[m(\lambda-\omega)+m+1]^n}.$$

Then, $\chi^*_\gamma(n,\eta,\lambda,\omega) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.3 *Let*

$$\delta = 1 - \frac{[\gamma(s)\,(1+\eta)\,B_{m+1}(n) - 2\eta(1-\alpha)]}{\gamma(s)\,(1+\eta)\,B_{m+1}(n)}.$$

Then, $\chi^*_\gamma(n,\eta,\alpha) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.4 *Let*

$$\delta = 1 - \frac{[\gamma(s)\,(1+\eta)\,B_{m+1}(n) - 2\eta]}{\gamma(s)\,(1+\eta)\,B_{m+1}(n)}.$$

Then, $S\chi^*_\gamma(n,\eta) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.5 *Let*

$$\delta = 1 - \frac{[\gamma(s)B_{m+1}(n) - \eta]}{\gamma(s)B_{m+1}(n)}.$$

Then, $\chi^*_\gamma(n,\eta) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.6 *Let*

$$\delta = 1 - \frac{[\gamma(s) - (1-\alpha)]}{\gamma(s)}.$$

Then, $\chi^*_\gamma(\alpha) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.7 *Let*

$$\delta = 1 - \frac{[\gamma(s) - 1]}{\gamma(s)}.$$

Then, $\chi^*(\gamma) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.8 *Let*

$$\delta = 1 - \frac{[\gamma(s) - \eta]}{\gamma(s)}.$$

Then, $\chi^*_\gamma(\eta) \subset N_{m,\delta}(e_\gamma)$.

Definition 2.6 *A function* $f_\gamma(z) \in T_\gamma$ *is said to be in the class* $T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$ *if there exist a function* $h_\gamma(z) \in T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$ *such that*

$$\left|\frac{f_\gamma(z)}{h_\gamma(z)} - 1\right| < 1-\rho, \quad (z \in \mathfrak{U}, 0 \le \rho < 1). \tag{2.19}$$

Theorem 2.3 *If* $h_\gamma(z) \in T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$ *and*

$$\rho = 1 - \frac{A(\lambda,\omega,\alpha,p,\zeta,\eta)}{B(\lambda,\omega,\alpha,p,\zeta,\eta)}$$

where

$$\begin{aligned} A(\lambda,\omega,\alpha,p,\zeta,\eta) &= \delta(m+1)\gamma(s)\left[1+\eta(p\zeta-1)\right]\{\mu\gamma^n(s)\left[m(\lambda-\omega)+m+1\right]^n \\ &\quad +(1-\mu)B_{m+1}(n)\}, \\ B(\lambda,\omega,\alpha,p,\zeta,\eta) &= (m+1)^2\gamma(s)\left[1+\eta(p\zeta-1)\right]\{\mu\gamma^n(s)\left[m(\lambda-\omega)+m+1\right]^n \\ &\quad +(1-\mu)B_{m+1}(n)\} - (m+1)\eta p\zeta\left[\mu\gamma^n(s)-\mu+1-\alpha\right], \end{aligned}$$

then $N_{m,\delta}(h_\gamma) \subset T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$.

Proof Let $f_\gamma(z) \in N_{m,\delta}(h_\gamma)$. We find from (2.11) that

$$\sum_{k=m+1}^{\infty} k\gamma(s)\left|a_k - b_k\right| \le \delta, \tag{2.20}$$

which ready implies that

$$\sum_{k=m+1}^{\infty} \gamma(s)\left|a_k - b_k\right| \le \frac{\delta}{m+1}. \tag{2.21}$$

Next, since $h_\gamma(z) \in T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$, we have

$$\sum_{k=m+1}^{\infty} \gamma(s)b_k \le \frac{\eta p\zeta\left[\mu\gamma^n(s)-\mu+1-\alpha\right]}{(m+1)\left[1+\eta(p\zeta-1)\right]\{\mu\gamma^n(s)\left[m(\lambda-\omega)+m+1\right]^n +(1-\mu)B_{m+1}(n)\}},$$

so that

$$\left|\frac{f_\gamma(z)}{h_\gamma(z)} - 1\right| \le \frac{\sum\limits_{k=m+1}^{\infty} \gamma(s)\left|a_k - b_k\right|}{1 - \sum\limits_{k=m+1}^{\infty} \gamma(s)\left|b_k\right|}$$

$$\leq \frac{\delta}{m+1}\frac{\begin{array}{c}(m+1)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[m(\lambda-\omega)+m+1\right]^{n}\right.\\ \left.+(1-\mu)B_{m+1}(n)\right\}\end{array}}{\begin{array}{c}(m+1)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[m(\lambda-\omega)+m+1\right]^{n}\right.\\ \left.+(1-\mu)B_{m+1}(n)\right\}-\eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]\end{array}}$$

$$= \frac{A(\lambda,\omega,\alpha,p,\zeta,\eta)}{B(\lambda,\omega,\alpha,p,\zeta,\eta)} = 1-\rho.$$

This completes the proof. □

Corollary 2.9 *If* $h_\gamma(z) \in S\chi^*_\gamma(n,\eta,\lambda,\omega)$ *and*

$$\rho = 1 - \frac{\delta\gamma^{n+1}(s)\,(1+\eta)\left[m(\lambda-\omega)+m+1\right]^{n}}{(m+1)\gamma(s)\,(1+\eta)\left\{\gamma^{n}(s)\left[m(\lambda-\omega)+m+1\right]^{n}\right\}-2\eta\left(\gamma^{n}(s)-\alpha\right)}$$

then $N_{m,\delta}(h_\gamma) \subset S\chi^*_\gamma(n,\eta,\lambda,\omega)$.

Corollary 2.10 *If* $h_\gamma(z) \in \chi^*_\gamma(n,\eta,\lambda,\omega)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)\left[m(\lambda-\omega)+m+1\right]^{n}}{(m+1)\gamma(s)\left[m(\lambda-\omega)+m+1\right]^{n}-\eta}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*_\gamma(n,\eta,\lambda,\omega)$.

Corollary 2.11 *If* $h_\gamma(z) \in \chi^*_\gamma(n,\eta,\alpha)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)(1+\eta)B_{m+1}(n)}{(m+1)\gamma(s)(1+\eta)B_{m+1}(n)-2\eta(1-\alpha)}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*_\gamma(n,\eta,\alpha)$.

Corollary 2.12 *If* $h_\gamma(z) \in S\chi^*_\gamma(n,\eta)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)(1+\eta)B_{m+1}(n)}{(m+1)\gamma(s)(1+\eta)B_{m+1}(n)-2\eta}$$

then $N_{m,\delta}(h_\gamma) \subset S\chi^*_\gamma(n,\eta)$.

Corollary 2.13 *If* $h_\gamma(z) \in \chi^*_\gamma(n,\eta)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)B_{m+1}(n)}{(m+1)\gamma(s)B_{m+1}(n)-\eta}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*_\gamma(n,\eta)$.

Corollary 2.14 *If* $h_\gamma(z) \in \chi^*_\gamma(\alpha)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)}{(m+1)\gamma(s)-2(1-\alpha)}$$

then $N_{m,\delta}(h_\gamma) \subset \chi_\gamma^*(\alpha)$.

Corollary 2.15 *If* $h_\gamma(z) \in \chi^*(\gamma)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)}{(m+1)\gamma(s) - 2}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*(\gamma)$.

Corollary 2.16 *If* $h_\gamma(z) \in \chi_\gamma^*(\eta)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)}{(m+1)\gamma(s) - \eta}$$

then $N_{m,\delta}(h_\gamma) \subset \chi_\gamma^*(\eta)$.

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