International J.Math. Combin. Vol.2 (2011), 98-103

New Characterizations for Bertrand Curves in Minkowski 3-Space

Bahaddin Bukcu

Gaziosmanpasa University, Faculty of Sciences and Arts
Department of Mathematics, Tasliciftlik Campus, 60250, Tokat-Turkey)

Murat Kemal Karacan

Usak University, Faculty of Sciences and Arts
Department of Mathematics, 1 Eylul Campus, 64200, Usak-Turkey

Nural Yuksel

Erciyes University, Faculty of Art and Sciences, Department of Mathematics, Kayseri -Turkey

E-mail: bbukcu@yahoo.com, murat.karacan@usak.edu.tr, yukseln@erciyes.edu.tr

Abstract: Bertrand curves have been investigated in Lorentzian and Minkowski spaces and some characterizations have been given in [1,2,6]. In this paper, we have investigated the relations between Frenet vector fields and curvatures and torsions of Bertrand curves at the corresponding points in Minkowski 3-space.

Key Words: Bertrand curves, constant curvature and torsion, Minkowski 3- Space.

AMS(2010): 53A04,53A35,53B30

§1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are the very interesting and important problem. The well-known Bertrand curve is characterized as a kind of such corresponding relation between the two curves. J. Bertrand studied curves in Euclidean 3-space whose principal normals are the principal normals of another curve. Such a curve is nowadays called a Bertrand curve. Bertrand curves have a characteristic property that curvature and torsion are in linear relation.In the recent work [2], the authors studied spacelike and timelike Bertrand curves in Minkowski 3-space. (See also independently obtained results by [6]).

In this paper, we have investigated the relations between Frenet vector fields and curvatures and torsions of Bertrand curves at the corresponding points in Minkowski 3-space.

---

[1] Received October 29, 2010. Accepted May 30, 2011.

§2. Preliminaries

The Minkowski 3-space E_1^3 is the Euclidean 3-space E^3 provided with the standard flat metric given by

$$\langle , \rangle = -dx_1 + dx_3 + dx_3$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since $\langle , \rangle$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$ and null (lightlike) if $\langle v, v\rangle = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null (lightlike).

Minkowski space is originally from the relativity in Physics. In fact, a timelike curve corresponds to the path of an observer moving at less than the speed of light. Denote by $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ in the space E_1^3. For an arbitrary curve $\alpha(s)$ in the space E_1^3, the following Frenet formulae are given. If α is timelike curve, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1.1}$$

where $\langle T, T\rangle_L = -1, \langle N, N\rangle_L = 1, \langle B, B\rangle_L = 1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0$. If α is a spacelike curve with a spacelike principal normal, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1.2}$$

where $\langle T, T\rangle_L = \langle N, N\rangle_L = 1, \langle B, B\rangle_L = -1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0$.If α is a spacelike curve with a spacelike binormal, then the Frenet formulae read

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{1.3}$$

where $\langle T, T\rangle_L = \langle B, B\rangle_L = 1, \langle N, N\rangle_L = -1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0$ [4,7,11].

§3. Bertrand Curves in Minkowski 3-Space

Definition 3.1([1,2,6]) *Let β_1 and β_2 be two unit speed regular curves in E_1^3, and $\{T_1, N_1, B_1\}$ and $\{T_2, N_2, B_2\}$also be Frenet Frames on these curves, respectively. β_1 and β_2 are called Bertrand curves if N_1 and N_2 are linearly dependent. We say that β_2 is a Bertrand mate for β_1 and β_2 are Bertrand curves. And (β_1, β_2) is called a Bertrand couple and we can write*

$$\beta_2(s) = \beta_1(s) + rN_1(s).$$

Theorem 3.1 *If there exists a one-to-one correspondence between the points of the spacelike curves C_1 and C_2 with timelike principal normal, such that at corresponding points P_1 on C_1 and P_2 on C_2,then the following statements hold:*

(1) *The curvature κ_1 of C_1 is a constant;*
(2) *The torsion τ_2 of C_2 is constant;*
(3) *The unit tangent vector T_1 of C_1 is parallel to the unit tangent vector T_2 of C_2.*

Then the curve C generated by P that divides the segment P_1P_2 in ratio $m:1$ is a spacelike Bertrand curve with timelike principal normal.

Proof We shall use the subscripts 1, 2 to designate the geometric quantites corresponding to the curves C_1, C_2 while the same letters without subscripts will refer to the spacelike curve C with timelike principal normal.

Let $\alpha(s), \alpha_1(s), \alpha_2(s)$ be the coordinat vectors at the points P, P_1, P_2 on the curves C,C_1,C_2 respectively. Then the from convex combination of points P_1 and P_2 , the equation of point P is

$$\alpha(s) = m\alpha_1(s) + (1-m)\alpha_2(s),\ m \in R, \tag{2.1}$$

while by hypothesis,

$$\|T_1\| = \|T_2\| = 1, \quad T_1 = T_2. \tag{2.2}$$

On differentiating Eq.(2.1) we have

$$Tds = mT_1ds_1 + (1-m)T_2ds_2 = (mds_1 + (1-m)ds_2)\,T_1 \tag{2.3}$$

which shows that T is parallel to T_1 and T_2 and always can be chosen so that

$$T = T_1 = T_2, \tag{2.4}$$

and

$$ds = mds_1 + (1-m)ds_2. \tag{2.5}$$

Differentiating of Eq.(2.4) gives

$$\kappa Nds = \kappa_1N_1ds_1 = \kappa_2N_2ds_2, \tag{2.6}$$

and if we assume that $\kappa, \kappa_1, \kappa_2$ are positive, then

$$N = N_1 = N_2, \tag{2.7}$$

and

$$\kappa ds = \kappa_1ds_1 = \kappa_2ds_2. \tag{2.8}$$

From Eq.(2.4) and Eq.(2.7)

$$B = B_1 = B_2, \tag{2.9}$$

and differentiating

$$\tau Nds = \tau_1N_1ds_1 = \tau_2N_2ds_2. \tag{2.10}$$

Elimination of ds, ds_1, ds_2 gives

$$\left(\frac{m}{\kappa_1}\right)\kappa + \left(\frac{1-m}{\tau_2}\right)\tau = 1; \kappa_1 \neq 0, \tau_2 \neq 0,$$

which is the desired result, since m, κ_1, τ_2 are constant. If instead of $T_1 = T_2$ were given the condition $B_1 = B_2$, the same result would follow in the same manner. □

Theorem 3.2 *If condition (c) of Theorem 3.1 is modifed so that at corresponding points P_1 and P_2, the binormals B_1 and B_2 are parallel, then the curve C is a spacelike Bertrand curve with timelike principal normal.*

Proof Since $B_1 = B_2$ then

$$\tau_1 N_1 = \tau_2 N_2 \frac{ds_2}{ds_1}. \tag{2.11}$$

where N_1 and N_2 are the unit normal vectors of α_1 and α_2 at the points $P_1$and P_2 with arc-length parametrization. Hence $N_1 = N_2$ and $T_1 = N_1 \times B_1 = N_2 \times B_2$, we know T_1 parallel to T_2. By Theorem 2.1, C is a spacelike Bertrand curve with timelike principal normal. □

Theorem 3.3 *If condition (c) of Theorem 3.1 is modified so that at corresponding points P_1 and P_2 the tangent at $P_1$is parallel to the binormal B_2 at P_2, then the curve C is a spacelike Bertrand curve with timelike principal normal.*

Proof Since $T_1 = B_2$, it follow that

$$\kappa_1 N_1 = \tau_2 N_2 \frac{ds_2}{ds_1}. \tag{2.12}$$

Hence N_1 is parallel to N_2 and since N_1 and N_2 are unit vectors,

$$N_1 = N_2 \tag{2.13}$$

and

$$\frac{ds_2}{ds_1} = \frac{\kappa_1}{\tau_2}, \tag{2.14}$$

since $B_1 = T_1 \times N_1 = B_2 \times N_2 = -T_2$,we have

$$B_1 = -T_2. \tag{2.15}$$

Let $\alpha, \alpha_1, \alpha_2$ be the coordinate vectors at the points P, P_1, P_2 on the curves C,C_1,C_2, respectively. Then

$$\alpha = m\alpha_1 + (1-m)\alpha_2 \tag{2.16}$$

$$\begin{aligned}
\frac{d\alpha}{ds} &= m\frac{d\alpha_1}{ds} + (1-m)\frac{d\alpha_2}{ds} \\
\frac{d\alpha}{ds} &= m\frac{d\alpha_1}{ds_1}\frac{ds_1}{ds} + (1-m)\frac{d\alpha_2}{ds_2}\frac{ds_2}{ds} \\
T &= \left(mT_1 + (1-m)\frac{ds_2}{ds_1}T_2\right)\frac{ds_1}{ds} \\
&= \left(mT_1 + \frac{\kappa_1}{\tau_2}(1-m)T_2\right)\frac{ds_1}{ds}
\end{aligned}$$

$$T = m_1 T_1 + m_2 T_2 \ , \tag{2.17}$$

where

$$\begin{aligned} m_1 &= m\frac{ds_1}{ds} = \frac{m\ \tau_2}{\sqrt{(m\ \tau_2)^2 + [\kappa_1(1-m)]^2}},\ m_1 = const. \\ m_2 &= (1-m)\frac{ds_2}{ds} = \frac{(1-m)\ \kappa_1}{\sqrt{(m\ \tau_2)^2 + [\kappa_1(1-m)]^2}}, m_2 = const. \end{aligned}$$

Differentiating Eq.(2.17), one gets

$$\kappa N = \kappa_1 m_1 N_1 \frac{ds_1}{ds} = \kappa_2 m_2 N_2 \frac{ds_2}{ds}. \tag{2.18}$$

Hence

$$N = N_1 = N_2 \tag{2.19}$$

and

$$\kappa = \kappa_1 m_1 \frac{ds_1}{ds} + \kappa_2 m_2 \frac{ds_2}{ds}. \tag{2.20}$$

Using Eq.(2.7) and Eq.(2.9), one finds that

$$B = m_1 B_1 + m_2 B_2. \tag{2.21}$$

Differentiating Eq.(2.11), one gets

$$\tau N = \tau_1 m_1 \frac{ds_1}{ds} N_1 + \tau_2 m_2 \frac{ds_2}{ds} N_2. \tag{2.22}$$

Hence

$$\tau = \tau_1 m_1 \frac{ds_1}{ds} + \tau_2 m_2 \frac{ds_2}{ds}. \tag{2.23}$$

Using Eq.(2.14) and Eq.(2.15), one gets

$$\frac{ds_2}{ds_1} = -\frac{\tau_1}{\kappa_2} = \frac{\kappa_1}{\tau_2}. \tag{2.24}$$

and

$$\frac{\tau_1}{\kappa_1} = -\frac{\kappa_2}{\tau_2}.$$

Let

$$M_1 = m_1 \frac{ds_1}{ds}, \qquad M_2 = m_2 \frac{ds_2}{ds}.$$

Then using Eq.(2.20) and Eq.(2.23), one gets

$$\begin{aligned} \frac{\kappa}{M_2\tau_2} + \frac{\tau}{M_1\kappa_1} &= \frac{\kappa_1}{\tau_2}\frac{M_1}{M_2} + \left(\frac{\kappa_2 M_2}{\tau_2 M_2} + \frac{\tau_1 M_1}{\kappa_1 M_1}\right) + \frac{\tau_2}{\kappa_1}\frac{M_2}{M_1} \\ &= \frac{\kappa_1}{\tau_2}\frac{M_1}{M_2} + \frac{\tau_2}{\kappa_1}\frac{M_2}{M_1} = \text{constant}, \quad \frac{\kappa_1}{\tau_2},\ \frac{M_1}{M_2} = \text{constant}. \end{aligned}$$

and this is the intrinsic equation of a spacelike Bertrand curve. □

References

[1] Balgetir H., Bektas M., Inoguchi J, Null bertrand curves in Minkowski 3-space and their characterizations, *Note di Matematica*, 23, No:1, 713, 2004.

[2] Balgetir H., Bektas,M., Ergut,M., Bertrand curves for nonnull curves in 3-dimensional Lorentzian space, *Hadronic Journal*, 27, 229236, 2004.

[3] Bioche C., Sur les courbes de M. Bertrand, *Bull. Soc.Math. France*, 17, pp. 109-112, 1889.

[4] Bukcu B., Karacan M.K., On the involute and evolute curves of the spacelike curve with a spacelike binormal in Minkowski 3-Space, *Int. J. Contemp. Math. Sciences*, Vol. 2, no. 5, 221 - 232, 2007.

[5] Burke J.F., Bertrands curves associated with a pair of curves, *Mathematics Magazine*, Vol. 34, No. 1. pp.60-62, (1960).

[6] Ekmekci, N. ve Ilarslan K., On Bertrand curves and their characterization, *Differential Geometry Dynamical Systems*, 3, No.2, 17-24,201.

[7] Karacan M.K.,Yayli Y., On the geodesics of tubular surfaces in Minkowski 3-space, *Bulletin of the Malaysian Mathematical Sciences Society*, 31 (1). pp. 1-10, 2008.

[8] Mellish A.P., Notes on differential geometry, *Annals of Mathematics*, Second Series, Vol. 32, No. 1, pp. 181-190, Jan. 1931.

[9] Oprea J., *Differential Geometry and its Applications*, Prentica Hall, Inc, 1997.

[10] Tajima J., On Bertrand curves, *Tohoku Math. J.*, 18, pp.128-133,1920.

[11] Walrave J., *Curves and Surfaces in Minkowski Space*, PhD Thesis,1995.