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Number of Spanning Trees of Some of Pyramid Graphs Generated by a Wheel Graph

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Abstract: In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs. In this paper we define some classes of pyramid graphs and we derive simple formulas of the complexity, number of spanning trees, of these graphs, using linear algebra, Chebyshev polynomials and matrix analysis techniques.

Key Words: Number of spanning tree, Chebyshev Polynomial, pyramid graph.

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§1. Introduction

The study of the number of spanning trees in a graph has a long history and has been very active because computing this number is important: (1) in analyzing energy of masers in investigating the possible particle transitions; (2) in estimating the reliability of a network; (3) in designing electrical circuits; (4) in enumerating certain chemical isomers; (5) in counting the number of Eulerian circuits in a graph. See [1]-[7], [20],[22] and [24]. For a graph G , a spanning tree in G is a tree which has the same vertex set as G . The number of spanning trees of, also known as, the complexity of the graph, denoted by $\tau(G)$, this quantity is a well-studied quantity for long time. A classical result of Kirchhoff [19] can be used to determine the number of spanning trees for a graph G with p vertices. If $V = u_1, u_2, c \dots, u_p$, then the Kirchhoff matrix L defined as $p \times p$ characteristic matrix $L = D - A$ where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G ,$L = [X_{ij}]$ defined as follows:

(i) $X_{ij} = -1$ when U_i and U_j are adjacent and $i \neq j$;
(ii) X_{ij} equals the degree of vertex U_i if $i = j$, and
(iii) $X_{ij} = 0$ otherwise.

All of co-factors of L are equal to $\tau(G)$. There are other methods for calculating $\tau(G)$. Let $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_p$ denote the eignvalues of L matrix of a p point graph. It is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [18] have shown that $\tau(G) = \frac{1}{p} \prod_{j=1}^{p-1} \sigma_j$.

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For more results in this field, see [10]-[17].

Now, we introduce the following lemma.

Lemma 1.1([8]) $\tau(G) = \frac{1}{p^2} det(pI - \bar{D} + \bar{A})$ *where* $\bar{A}, \bar{D}$ *are the adjacency and degree matrices of* $\bar{G}$*, the complement of* G*, respectively, and* I *is the* $p \times p$ *unit matrix.*

The advantage of this formula is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

§2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, see Yuanping, et. al. [23].

Let $M_p(Z)$ be $p \times p$ matrix such that:

$$M_p(Z) = \begin{pmatrix} 2Z & -1 & 0 & \cdots & 0 \\ -1 & 2Z & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2Z & -1 \\ 0 & \cdots & 0 & -1 & 2Z \end{pmatrix},$$

Further, we recall that the Chebyshev polynomials of the first kind are defined by

$$T_p(Z) = \cos(p \arccos Z). \tag{1}$$

The Chebyshev polynomials of the second kind are defined by

$$U_{p-1}(Z) = \frac{1}{p}\frac{d}{dz}T_p(Z) = \frac{\sin(p \arccos Z)}{\cos(\arccos Z)}. \tag{2}$$

It is easily verified that

$$U_p(Z) - 2ZU_{p-1}(Z) + U_{p-2}(z) = 0. \tag{3}$$

It can then be shown from this recursion that by expanding one gets

$$U_p(Z) = det(M_p(z)), p \geqslant 1. \tag{4}$$

Furthermore, by using standard methods for solving the recursion (3), one obtains the explicit formula

$$U_p(Z) = \frac{1}{2\sqrt{Z^2-1}}\left[(Z+\sqrt{Z^2-1})^{p+1} - (Z-\sqrt{Z^2-1})^{p+1}\right], p \geqslant 1 \tag{5}$$

where the identity is true for all complex Z (except at $Z = \pm 1$, where the function can be taken as the limit). The definition of $U_p(Z)$ easily yields its zeros and it can therefore be verified that

$$U_{p-1}(Z) = 2^{p-1} \prod_{i=1}^{p-1} \left(Z - \cos \frac{i\pi}{p} \right). \tag{6}$$

One further note that

$$U_{p-1}(-Z) = (-1)^{p-1} U_{p-1}(Z). \tag{7}$$

These two results yield another formula for

$$U_{p-1}^2(Z) = 4^{p-1} \prod_{i=1}^{p-1} \left(Z^2 - \cos^2 \frac{i\pi}{p} \right). \tag{8}$$

Finally, a simple manipulation of the above formula yields the following formula (9), which is extremely useful to us latter:

$$U_{p-1}^2 (\sqrt{\frac{Z+2}{4}}) = \prod_{i=1}^{p-1} \left(Z - 2 \cos \frac{2i\pi}{p} \right). \tag{9}$$

Furthermore, one can show that

$$U_{p-1}^2(Z) = \frac{1}{2(1-z^2)} [1 - T_{2p}] = \frac{1}{2(1-z^2)} \left[1 - T_p \left(2Z^2 - 1 \right) \right], \tag{10}$$

$$T_p(Z) = \frac{1}{2} \left[(z + \sqrt{Z^2 - 1})^p + (z - \sqrt{z^2 -})^p \right]. \tag{11}$$

Now we introduce the following important two lemmas.

Lemma 2.1([8]) *Let $A_p(Z)$ be $p \times p$ circulant matrix such that:*

$$A_p(Z) = \begin{pmatrix} Z & 0 & 1 & \cdots & 1 & 0 \\ 0 & Z & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & Z & 1 \\ 0 & 1 & \cdots & 1 & 0 & Z \end{pmatrix},$$

Then for $p \geqslant 3$, $Z \geqslant 4$ we have:

$$\det(A_p(Z)) = \frac{2(Z+p-3)}{Z-3} \left[T_p(\frac{Z-1}{2}) - 1 \right].$$

Lemma 2.2([21]) *If $X \in F^{p \times p}$, $Y \in F^{p \times q}$, $Z \in F^{q \times p}$ andW $\in F^{q \times q}$. If X and W are* *nonsingular matrices, then*

$$\det\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \det(X - YW^{-1}Z)\det W = \det X \det(W - ZX^{-1}Y).$$

This Lemma give a sort of symmetry for some matrices which facilitates our calculations of the complexities of some special graphs.

§3. Main Results

Definition 3.1([9]) *The pyramid graph $P_p^{(q)}$ is the graph formed from the wheel graph W_{q+1} with vertices $U_0, U_1, U_2, \cdots, U_q$ and m sets of vertices, say, $V_1^1, V_2^1, \cdots, V_p^1, \cdots, V_1^q, V_2^q, \cdots, V_p^q$ such that for all $i = 1, 2, \cdots, p$ the vertex V_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, \cdots, q-1$ and v_i^q is adjacent to u_1 and u_q See Figure 1.*

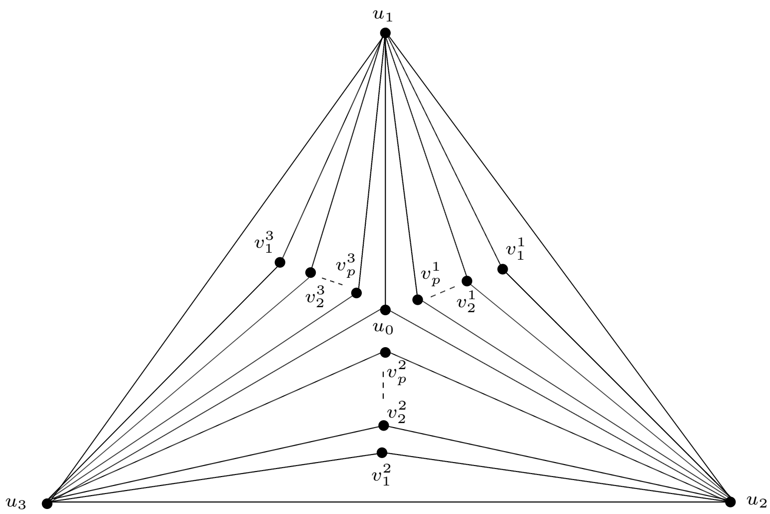

Figure 1 The pyramid graph $P_p^{(3)}$

Theorem 3.2([9]) *For $p \geqslant 0$, $q \geqslant 3$,*

$$\tau(P_p^{(q)}) = 2^{pq-q}\left[(p+3+\sqrt{2p+5})^q + \left[(p+3-\sqrt{2p+5})^q - 2(p+2)^q\right]\right].$$

Definition 3.3 *The pyramid graph $A_p^{(q)}$ is the graph formed from the wheel graph W_{q+1} with vertices $U_0, U_1, U_2, \cdots, U_q$ with double external edges and q sets of vertices, say*

$$V_1^1, V_2^1, \cdots, V_p^1, V_1^2, V_2^2, \cdots, V_p^2, \cdots, V_1^q, V_2^q, \cdots, V_p^q$$

such that for all $i = 1, 2, \cdots, p$ the vertex V_i^j is adjacent to u_j and u_{j+1} ,where $j = 1, 2, \cdots, q-1$

and v_i^q is adjacent to u_1 and u_q See Figure 2.

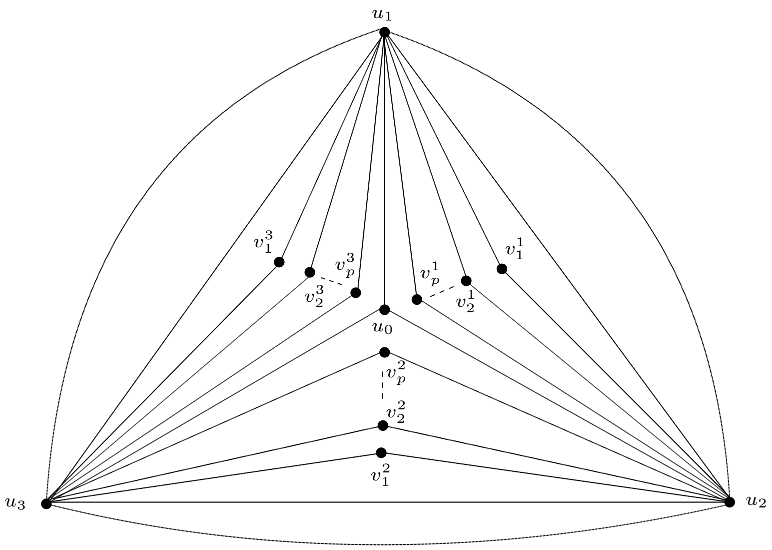

Figure 2 The pyramid graph $A_p^{(3)}$

Theorem 3.4 *For $p \geqslant 0$, $q \geqslant 3$,*

$$\tau(A_p^{(q)}) = 2^{pq-q}\left[(p+5+\sqrt{2p+9})^q + \left[(p+5-\sqrt{2p+9})^q - 2(p+4)^q\right]\right].$$

Proof Applying Lemma 1.1, We have

$$\begin{aligned} \tau(A_p^{(q)}) &= \frac{1}{(pq+q+1)^2} det((pq+p+1)I - \bar{D} + \bar{A}) \\ &= \frac{1}{(pq+q+1)^2} \times \det \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \end{aligned}$$

where,

$$A = \begin{pmatrix} q+1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 2(p+3) & -1 & 1 & \cdots & \cdots & \cdots & 1 & -1 \\ \cdots & -1 & 2(p+3)-1 & \cdots & \cdots & \cdots & \cdots & 1 & \\ \cdots & 1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ 0 & -1 & 1 & \cdots & \cdots & \cdots & 1 & -1 & 2(p+3) \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$C = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & 1 & \cdots & 1 & 1 \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad G = \begin{pmatrix} \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1 \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ H & I \end{pmatrix} = \begin{pmatrix} 3 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 1 \\ 1 & \cdots & 1 & 3 \end{pmatrix}$$

Let $j = (1 \cdots 1)$ be the $1 \times p$ matrix with all one, and J_p the $p \times p$ matrix with all one. Set $k = 2p + 4$ and $h = pq + q + 1$. Then we have

$\tau(A_p^{(q)})$

$$
= \frac{1}{h^2} \det \left(\begin{array}{ccccccccccccc}
(q+1) & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & J & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \cdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
0 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & \multicolumn{6}{c}{2I_{pq} + J_{pq}} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{array} \right)
$$

$$
= \frac{1}{h^2} \det \left(\begin{array}{ccccccccccccc}
h & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\
h & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & J & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \cdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
h & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
hj^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
hj^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & \multicolumn{6}{c}{2I_{pq} + J_{pq}} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\
hj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{array} \right)
$$

Hence, we know that

$$\tau(A_p^{(q)})$$

$$= \frac{1}{h}\det\left(\begin{array}{ccccccccccccc}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \cdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
1 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
1j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
1j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{array}\right)$$

$$= \frac{1}{h}\det\left(\begin{array}{ccccccccccccc}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & J & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \cdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
0 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
0 & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
0 & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\
0 & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{array}\right)$$

$$= \frac{1}{h} \det \begin{pmatrix} k & -1 & 1 & \cdots & 1 & -1 & -j & 0 & \cdots & \cdots & 0 & -j \\ -1 & k & -1 & \ddots & \ddots & 1 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & k & -1 & \vdots & \ddots & \ddots & \ddots & -j & 0 \\ -1 & 1 & \cdots & 1 & -1 & k & 0 & \cdots & \cdots & 0 & -j & -j \\ 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\ j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & 2I_{pq} & & & \\ j^t & \ddots & \ddots & \ddots & 0 & 0 & & & & & & \\ 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & & \end{pmatrix}$$

Using Lemma 2.2, yields

$$\tau(A_p^{(q)}) = \frac{1}{h} \begin{pmatrix} X & Y \\ Z & 2I_{pq} \end{pmatrix} = \frac{1}{h} det(X - Y \frac{1}{2I_{pq}} Z) 2^{pq}$$

$$= \frac{1}{h} 2^{pq} 2^{-q} \det \begin{pmatrix} 2k & p-2 & 2(p+1) & \cdots & 2(p+1) & p-2 \\ p-2 & 2k & p-2 & 2(p+1) & \cdots & 2(p+1) \\ 2(p+1) & p-2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 2(p+1) \\ 2(p+1) & \ddots & \ddots & \ddots & \ddots & p-2 \\ p-2 & 2(n+1) & \cdots & 2(p+1) & p-2 & 2k \end{pmatrix}$$

Straightforward induction using properties of determinants, we have

$$\tau(A_p^{(q)}) = \frac{1}{b} 2^{pq-q} \frac{2k + p(2q-4) + (2q-10)}{2k + p(q-4) + (4q-10)}$$

$$\times \det \begin{pmatrix} (2k-p+2) & 0 & 2(p+4) & \cdots & (p+4) & 0 \\ 0 & (2k-p+2) & 0 & (p+4) & \cdots & (p+4) \\ (p+4) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\ (p+4) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (p+1) & \cdots & (p+4) & 0 & (2k-p+2) \end{pmatrix}$$

$$= \frac{1}{h} 2^{pq-q} \frac{2h}{pq+4q+2} \det \begin{pmatrix} (2k-p+2) & 0 & 2(p+4) & \cdots & (p+4) & 0 \\ 0 & (2k-p+2) & 0 & (p+4) & \cdots & (p+4) \\ (p+4) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\ (p+4) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (p+1) & \cdots & (p+4) & 0 & (2k-p+2) \end{pmatrix}$$

$$= 2^{pq-q+1} \frac{(p+4)^q}{pq+4q+2} \det \begin{pmatrix} \frac{(2k-p+2)}{(p+4)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2k-p+2)}{(p+4)} & 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2k-p+2)}{(p+4)} \end{pmatrix}$$

Using Lemma 2.1, yields

$$\begin{aligned} \tau(A_p^{(q)}) &= 2^{pq-q+1} \times \frac{(p+4)^q}{pq+4q+2} \times \frac{2(\frac{2k-p+2}{p+4}+q-3)}{\frac{2k-p+2}{p+4}-3} \times \left[T_p(\frac{\frac{2k-p+2}{p+4}-1}{2}) - 1 \right] \\ &= 2^{pq-q+1} \times \frac{(p+4)^q}{pq+4q+2} \times (pq+4q+2) \times \left[T_q(\frac{2k-2p-2}{2(p+4)}) - 1 \right] \\ &= 2^{pq-p+1} \times (p+4)^q \times \left[T_q(\frac{p+5}{p+4}) - 1 \right]. \end{aligned}$$

Using Equation (11), yields the result. □

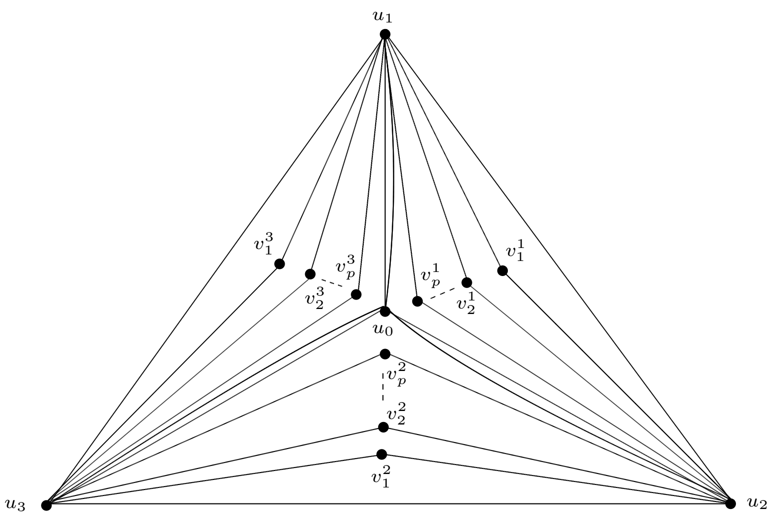

Figure 3 The pyramid graph $B_p^{(3)}$

Definition 3.5 *The pyramid graph $B_p^{(q)}$ is the graph formed from the wheel graph W_{q+1} with vertices $U_0, U_1, U_2, \cdots, U_q$ with double external edges and q sets of vertices, say*

$$V_1^1, V_2^1, \cdots, V_p^1, V_1^2, V_2^2, \cdots, V_p^2, \cdots, V_1^q, V_2^q, \cdots, V_p^q$$

such that for all $i = 1, 2, \cdots, p$ the vertex V_i^j is adjacent to u_j and u_{j+1}, where $j = 1, 2, \cdots, q-1$ and v_i^q is adjacent to u_1 and u_q. See Figure 3.

Theorem 3.6 *For $p \geqslant 0$, $q \geqslant 3$,*

$$\tau(B_p^{(q)}) = 2^{pq-q}\left[(p+4+2\sqrt{p+3})^q + \left[(p+4-2\sqrt{p+3})^q - 2(p+2)^q\right]\right].$$

Proof Applying Lemma 2.1, we get

$$\begin{aligned}\tau(B_p^{(q)}) &= \frac{1}{(pq+q+1)^2}\det((pq+p+1)I - \bar{D} + \bar{A}) \\ &= \frac{1}{(pq+q+1)^2} \times \det\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}\end{aligned}$$

where,

$$A = \begin{pmatrix} 2q+1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & 2p+5 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 \\ \cdots & 0 & 2p+5 & 0 & \ddots & \ddots & \ddots & \ddots & 1 \\ \cdots & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ -1 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 2p+5 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

$$C=\begin{pmatrix}
\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&1\\
\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&1&\cdots&1&0&\cdots&0\\
1&\cdots&1&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&1&\cdots&1\\
1&\cdots&1&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\
0&\cdots&0&1&\cdots&1&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\
\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\
\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots\\
1&\cdots&1&0&\cdots&0&0&\cdots&0&1&\cdots&1\\
\cdots&\cdots&\cdots&1&\cdots&1&0&\cdots&0&0&\cdots&0
\end{pmatrix}$$

$$D=\begin{pmatrix}
1&0&0&1&1&\cdots&\cdots&1&1\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
\cdots&0&0&1&1&\cdots&\cdots&1&1\\
\cdots&1&0&0&1&\cdots&\cdots&\cdots&\cdots\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
\cdots&1&0&0&1&\cdots&\cdots&\cdots&\cdots\\
\cdots&\cdots&1&0&0&\cdots&\cdots&\cdots&\cdots\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
\cdots&\cdots&1&0&0&\cdots&\cdots&\cdots&\cdots\\
\cdots&\cdots&\cdots&1&0&\cdots&\cdots&1&\cdots\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots
\end{pmatrix}$$

$$G=\begin{pmatrix}
\cdots&\cdots&\cdots&1&0&\cdots&\cdots&1&\cdots\\
\cdots&\cdots&\cdots&\cdots&1&\cdots&\cdots&0&1\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
\cdots&\cdots&\cdots&\cdots&1&\cdots&\cdots&0&1\\
\cdots&1&\cdots&\cdots&\cdots&\cdots&\cdots&0&0\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
\cdots&1&\cdots&\cdots&\cdots&\cdots&\cdots&0&0\\
\cdots&0&1&1&1&\cdots&\cdots&1&0\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\
1&0&1&1&1&\cdots&\cdots&1&0
\end{pmatrix}$$

$$\begin{pmatrix} E & F \\ H & I \end{pmatrix} = \begin{pmatrix} 3 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 1 \\ 1 & \cdots & 1 & 3 \end{pmatrix}$$

Let $j = (1 \cdots 1)$ be the $1 \times p$ matrix with all one, and J_p the $p \times p$ matrix with all one. Set $k = 2p + 5$ and $h = pq + q + 1$. Then we have

$$\tau(B_p^{(q)}) = \frac{1}{h^2} \det \begin{pmatrix}
2q+1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
-1 & k & 0 & 1 & \cdots & 1 & 0 & 0 & j & \ddots & \cdots & j & 0 \\
\vdots & 0 & k & 0 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & 0 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & k & 0 & \vdots & \ddots & \ddots & \ddots & 0 & j \\
-1 & 0 & 1 & \cdots & 1 & 0 & k & j & \cdots & \cdots & j & 0 & 0 \\
j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} + J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h^2} \det \begin{pmatrix}
h & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
h & k & 0 & 1 & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & 0 & k & 0 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
h & 0 & 1 & \cdots & 1 & 0 & k & j & \cdots & \cdots & j & 0 & 0 \\
hj^t & 0 & 0 & j^t & \ldots & j^t & j^t & & & & & & \\
hj^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} + J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
hj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

Therefore,

$$
\begin{aligned}
&\tau(B_p^{(q)})\\
&=\frac{1}{h}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & 0 & j & \cdots & \cdots & \cdots & \cdots & j\\
1 & k & 0 & 1 & \cdots & 1 & 0 & 0 & j & \cdots & \cdots & j & 0\\
\vdots & 0 & k & 0 & \ddots & \cdots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j\\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & j\\
1 & 0 & 1 & \cdots & 1 & 0 & k & j & \cdots & \cdots & j & 0 & 0\\
1j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
1j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}\\
&=\frac{1}{h}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j\\
0 & (k+1) & 1 & 2 & \cdots & 2 & 1 & -j & 0 & \cdots & \cdots & 0 & -j\\
\vdots & 1 & (k+1) & 1 & 2 & \cdots & 2 & -j & \ddots & \ddots & \ddots & \ddots & 0\\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & 1 & 2 & \cdots & 2 & 1 & (k+1) & 0 & \cdots & \cdots & 0 & -j & -j\\
0 & j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & & & & & & \\
0 & 2j^t & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & & & & & & \\
\vdots & 2j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
0 & j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & & & & & &
\end{pmatrix}
\end{aligned}
$$

$$
= \frac{1}{h}\det\begin{pmatrix}
(k+1) & 1 & 2 & \cdots & 2 & 1 & -j & 0 & \cdots & \cdots & 0 & -j \\
1 & (k+1) & 1 & 2 & \cdots & 2 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & 2 & \cdots & 2 & 1 & (k+1) & 0 & \cdots & \cdots & 0 & -j & -j \\
j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & & & & & & \\
2j^t & j^t & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & & & 2I_{pq} & & & \\
2j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & & & & & &
\end{pmatrix}
$$

Using Lemma 2.2 yields

$$
\tau(B_p^{(q)}) = \frac{1}{h}\begin{pmatrix} X & Y \\ Z & 2I_{pq} \end{pmatrix} = \frac{1}{b} det\left(X - Y\frac{1}{2I_{pq}}Z\right)2^{pq}
$$

$$
= \frac{1}{h}2^{pq}2^{-q}\det\begin{pmatrix}
(2k+2p+2) & (3p+2) & 4(p+1) & \cdots & 4(p+1) & (3p+2) \\
(3p+2) & (2k+2p+2) & (3p+2) & \ddots & \ddots & 4(p+1) \\
4(p+1) & (3p+2) & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(p+1) \\
4(p+1) & \ddots & \ddots & \ddots & \ddots & (3p+2) \\
(3p+2) & 4(p+1) & \cdots & 4(p+1) & (3p+2) & (2k+2p+2)
\end{pmatrix}
$$

Straightforward induction using properties of determinants, we have

$$
\begin{aligned}
\tau(B_p^{(q)}) \quad = \quad & \frac{1}{h}2^{pq-q}\frac{2k+p(4q-4)+(4q-6)}{2k+p(q-4)+(2q-6)} \\
& \times\det\begin{pmatrix}
(2k-p) & 0 & (p+2) & \cdots & (p+2) & 0 \\
0 & (2k-p) & 0 & (p+2) & \cdots & (p+2) \\
(p+2) & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & (p+2) \\
(p+2) & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & (p+2) & \cdots & (p+2) & 0 & (2k-p)
\end{pmatrix}
\end{aligned}
$$

$$= \frac{1}{h} 2^{pq-q} \frac{4h}{pq+2q+4} \det \begin{pmatrix} (2k-p) & 0 & (p+2) & \cdots & (p+2) & 0 \\ 0 & (2k-p) & 0 & (p+2) & \cdots & (p+2) \\ (p+2) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+2) \\ (p+2) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (p+2) & \cdots & (p+2) & 0 & (2k-p) \end{pmatrix}$$

$$= 2^{pq-q+2} \frac{(p+2)^q}{pq+2q+4} \det \begin{pmatrix} \frac{(2k-p)}{(p+2)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2k-p)}{(p+2)} & 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2k-p)}{(p+2)} \end{pmatrix}$$

Using Lemma 2.1 yields

$$\begin{aligned} \tau(B_p^{(q)}) &= 2^{pq-q+2} \times \frac{(p+2)^q}{pq+2q+4} \times \frac{2(\frac{2k-p}{p+2}+q-3)}{\frac{2k-p}{p+2}-3} \times \left[T_p(\frac{\frac{2k-p}{p+2}-1}{2}) - 1 \right] \\ &= 2^{pq-q+1} \times \frac{(p+2)^q}{pq+2q+4} \times (pq+2q+4) \times \left[T_q(\frac{2k-2p-2}{2(p+2)}) - 1 \right] \\ &= 2^{pq-p+1} \times (p+2)^q \times \left[T_q(\frac{p+4}{p+2}) - 1 \right]. \end{aligned}$$

Using Equation (11), yields the result. □

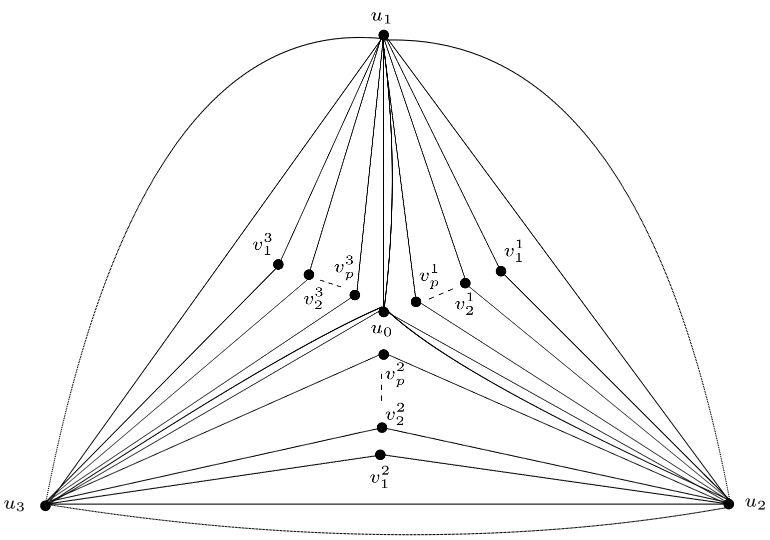

Figure 4 The pyramid graph $C_p^{(3)}$

Definition 3.7 *The pyramid graph $C_p^{(q)}$ is the graph formed from the wheel graph W_{q+1} with vertices $U_0, U_1, U_2, \cdots, U_q$ with double external edges and q sets of vertices, say,*

$$V_1^1, V_2^1, \cdots, V_p^1, V_1^2, V_2^2, \cdots, V_p^2, \cdots, V_1^q, V_2^q, \cdots, V_p^q$$

such that for all $i = 1, 2, \cdots, p$ the vertex V_i^j is adjacent to u_j and u_{j+1} ,where $j = 1, 2, \cdots, q-1$ and v_i^q is adjacent to u_1 and u_q. See Figure 4.

Theorem 3.8 *For $p \geqslant 0$, $q \geqslant 3$,*

$$\tau(C_p^{(q)}) = 2^{pq-q}\left[(p+6+2\sqrt{p+5})^q + \left[(p+6-2\sqrt{p+5})^q - 2(p+4)^q\right]\right].$$

Proof Applying Lemma 1.1, We have

$$\begin{aligned}\tau(C_p^{(q)}) &= \frac{1}{(pq+q+1)^2} det((pq+p+1)I - \bar{D} + \bar{A}) \\ &= \frac{1}{(pq+q+1)^2} \times \det\begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}\end{aligned}$$

where,

$$A = \begin{pmatrix} 2q+1 & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & 2p+7 & -1 & 1 & \cdots & \cdots & \cdots & 1 & -1 \\ \cdots & -1 & 2p+7 & -1 & \ddots & \ddots & \ddots & \ddots & 1 \\ \cdots & 1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ -1 & -1 & 1 & \cdots & \cdots & \cdots & 1 & -1 & 2p+7 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and

$$C=\begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0\\
1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 & \cdots & 1\\
1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 1\\
\cdots & \cdots & \cdots & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}$$

$$D=\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 1\\
\cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & 1 & 0 & 0 & 1 & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots\\
\cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}$$

$$G=\begin{pmatrix}
\cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 1 & \cdots\\
\cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & \cdots & \cdots & \cdots & 1 & \cdots & \cdots & 0 & 1\\
\cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
\cdots & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0\\
\cdots & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
1 & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 0
\end{pmatrix},\quad \begin{pmatrix} E & F\\ H & I\end{pmatrix}=\begin{pmatrix}
3 & 1 & \cdots & 1\\
\vdots & \vdots & \vdots & \vdots\\
\cdots & \cdots & \cdots & 1\\
1 & \cdots & 1 & 3
\end{pmatrix}$$

Let $j = (1 \cdots 1)$ be the $1 \times p$ matrix with all one, and J_p the $p \times p$ matrix with all one. Set $k = 2p + 7$ and $h = pq + q + 1$. Then we have

$$\tau(C_p^{(q)}) = \frac{1}{h^2} \det \begin{pmatrix}
2q+1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
-1 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \ddots & \cdots & j & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \ddots & \ddots & j \\
\vdots & 1 & -1 & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
-1 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} + J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h^2} \det \begin{pmatrix}
h & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
h & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & -1 & k & -1 & 1 & \cdots & 1 & 0 & 0 & \ddots & \cdots & \cdots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
h & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
hj^t & 0 & 0 & j^t & \ldots & j^t & j^t & & & & & & \\
hj^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} + J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
hj^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

Clearly,

$$\tau(C_p^{(q)}) = \frac{1}{h}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
1 & k & -1 & 1 & \cdots & 1 & -1 & 0 & j & \cdots & \cdots & j & 0 \\
\vdots & -1 & k & -1 & \ddots & \cdots & 1 & 0 & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & \vdots & j & \ddots & \ddots & \ddots & \ddots & j \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 1 & \ddots & \ddots & \ddots & \ddots & -1 & \vdots & \ddots & \ddots & \ddots & \ddots & j \\
1 & -1 & 1 & \cdots & 1 & -1 & k & j & \cdots & \cdots & j & 0 & 0 \\
1j^t & 0 & 0 & j^t & \cdots & j^t & j^t & & & & & & \\
1j^t & j^t & 0 & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq}+J_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
\vdots & j^t & \ddots & \ddots & \ddots & \ddots & 0 & & & & & & \\
1j^t & 0 & j^t & \cdots & \cdots & j^t & 0 & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h}\det\begin{pmatrix}
1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 & j & \cdots & \cdots & \cdots & \cdots & j \\
0 & (k+1) & 0 & 2 & \cdots & 2 & 0 & -j & 0 & \cdots & \cdots & 0 & -j \\
\vdots & 0 & (k+1) & 0 & 2 & \cdots & 2 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & 2 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 2 & \cdots & 2 & 0 & (k+1) & 0 & \cdots & \cdots & 0 & -j & -j \\
0 & j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & & & & & & \\
0 & 2j^t & j^t & \ddots & \ddots & \ddots & \vdots & & & 2I_{pq} & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & & & & & & \\
\vdots & 2j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
0 & j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & & & & & &
\end{pmatrix}$$

$$= \frac{1}{h}\det\begin{pmatrix}
(k+1) & 0 & 2 & \cdots & 2 & 0 & -j & 0 & \cdots & \cdots & 0 & -j \\
1 & (k+1) & 0 & 2 & \cdots & 2 & -j & \ddots & \ddots & \ddots & \ddots & 0 \\
2 & \ddots & \ddots & \ddots & \ddots & \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
2 & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & 2 & \cdots & 2 & 0 & (k+1) & 0 & \cdots & \cdots & 0 & -j & -j \\
j^t & j^t & 2j^t & \cdots & \cdots & 2j^t & & & & & & \\
2j^t & j^t & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots & & & & & & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 2j^t & & & 2I_{pq} & & & \\
2j^t & \ddots & \ddots & \ddots & \ddots & j^t & & & & & & \\
j^t & 2j^t & \cdots & \cdots & 2j^t & j^t & & & & & &
\end{pmatrix}$$

Using Lemma 2.2, yields

$$\begin{aligned}
\tau(C_p^{(q)}) &= \frac{1}{h}\begin{pmatrix} X & Y \\ Z & 2I_{pq} \end{pmatrix} \\
&= \frac{1}{h} det(X - Y\frac{1}{2I_{pq}}Z)2^{pq} = \frac{1}{h}2^{pq}2^{-q} \\
&\times \det\begin{pmatrix}
(2k+2p+2) & 3p & 4(p+1) & \cdots & 4(p+1) & 3p \\
3p & (2k+2p+2) & 3p & \ddots & \ddots & 4(p+1) \\
4(p+1) & 3p & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 4(p+1) \\
4(p+1) & \ddots & \ddots & \ddots & \ddots & 3p \\
3p & 4(p+1) & \cdots & 4(p+1) & 3n & (2k+2p+2)
\end{pmatrix}.
\end{aligned}$$

Straightforward induction using properties of determinants, we have

$$\begin{aligned}
\tau(C_p^{(q)}) &= \frac{1}{h}2^{pq-q}\frac{2k+p(4q-4)+(4q-10)}{2k+p(q-4)+(2q-10)} \\
&\times \det\begin{pmatrix}
(2k-p+2) & 0 & (p+4) & \cdots & (p+4) & 0 \\
0 & (2k-p+2) & 0 & (p+4) & \cdots & (p+4) \\
(p+4) & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\
(p+4) & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & (p+4) & \cdots & (p+4) & 0 & (2k-p+2)
\end{pmatrix}
\end{aligned}$$

$$= \frac{1}{h}2^{pq-q}\frac{4h}{pq+4q+4}$$

$$\times \det\begin{pmatrix} (2k-p+2) & 0 & (p+4) & \cdots & (p+4) & 0 \\ 0 & (2k-p+2) & 0 & (p+4) & \cdots & (p+4) \\ (p+4) & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & (p+4) \\ (p+4) & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & (p+4) & \cdots & (p+4) & 0 & (2k-p+2) \end{pmatrix}$$

$$= 2^{pq-q+2}\frac{(p+4)^q}{pq+4q+4}\det\begin{pmatrix} \frac{(2k-p+2)}{(p+4)} & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{(2k-p+2)}{(p+4)} & 0 & 1 & \cdots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{(2k-p+2)}{(p+4)} \end{pmatrix}$$

Using Lemma 2.1, yields

$$\begin{aligned} \tau(C_p^{(q)}) &= 2^{pq-q+2}\times\frac{(p+4)^q}{pq+4q+4}\times\frac{2(\frac{2k-p+2}{p+4}+q-3)}{\frac{2k-p+2}{p+4}-3}\times\left[T_p(\frac{\frac{2k-p+2}{p+4}-1}{2})-1\right] \\ &= 2^{pq-q+1}\times\frac{(p+4)^q}{pq+4q+4}\times(pq+4q+4)\times\left[T_q(\frac{2k-2p-2}{2(p+4)})-1\right] \\ &= 2^{pq-p+1}\times(p+4)^q\times\left[T_q(\frac{p+6}{p+4})-1\right]. \end{aligned}$$

By Equation (11), yields the result. □

§4. Numerical Results

The following tables Presents some number of spanning trees of studied pyramid graphs.

q	p	$\tau(P_p^{(q)})$	$\tau(A_p^{(q)})$	$\tau(B_p^{(q)})$	$\tau(C_p^{(q)})$
3	0	16	49	50	128
3	1	242	578	676	1444
3	2	3136	6400	8192	15488
3	3	36992	67712	92416	160000
3	4	409600	692224	991232	1605632
3	5	4333568	6889472	10240000	15745024

Table 1

q	p	$\tau(P_p^{(q)})$	$\tau(A_p^{(q)})$	$\tau(B_p^{(q)})$	$\tau(C_p^{(q)})$
4	0	45	225	192	720
4	1	1792	6336	6400	18816
4	2	57600	163072	184320	458752
4	3	1622016	3932160	4816896	10616832
4	4	41746432	90243072	117440512	235929600
4	5	1006632960	19992294400	2717908992	5075107840

Table 2

q	p	$\tau(P_p^{(q)})$	$\tau(A_p^{(q)})$	$\tau(B_p^{(q)})$	$\tau(C_p^{(q)})$
5	0	121	961	722	3872
5	1	12482	64082	58564	232324
5	2	984064	3810304	3964928	12781568
5	3	65619968	208406528	237899776	658640896
5	4	3901751296	10696523776	13088325632	32245809152
5	5	213408284672	522192945152	674448277504	1514986799104

Table 3

§5. Conclusion

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and Lemmas and their proofs.

References

[1] D. L. Applegate, R. E. V. Bixby, Chvtal, and W. J. Cook, *The Traveling Salesman Problem: A Computational Study*, Princeton University Press, (2006).

[2] T. Atajan and H. Inaba, Network reliability analysis by counting the number of spanning trees, ISCIT 2004, *IEEE International symposium on Communication and Information Technology*, 1 (2004), 601-604.

[3] F. T. Boesch, On unreliability polynomials and graph connectivity in reliable network synthesis, *J. Graph Theory*, 10 (1986), 339-352.

[4] F. T. Boesch and A. Salyanarayana in C.L. Suffel, A survey of some network reliability

analysis and synthesis results, *Networks*, 54 (2009), 99-107.

[5] T. J. N. Brown, R. B. Mallion, P. Pollak, and A. Roth, Some methods for counting the spanning trees in labelled molecular graphs, examined in relation to certain fullerenes, *Discrete Appl. Math.*, 67 (1996), 51-66.

[6] G. Chen, B. Wu, and Z. Zhang, Properties and applications of Laplacian spectra for Koch networks, *J. Phys. A: Math. Theor.*, 45 (2012), 025102.

[7] D. Cvetkovi, M. Doob and H. Sachs, *Spectra of Graphs: Theory and Applications*(Third Edition), Johann Ambrosius Barth, Heidelberg, (1995).

[8] S.N. Daoud, Chebyshev polynomials and spanning tree formulas, *International J. Math. Combin.*, Vol.4 (2013), 68-79.

[9] S.N.Daoud, On a class of some pyramid graphs and Chebyshev polynomials, *Journal of Mathematical Problems in Engineering*, Hindawi Publishing Corporation Vol. 2013, Article ID 820549, 11 pages.

[10] S.N. Daoud, Number of Spanning Trees of Different Products of Complete and Complete Bipartite Graphs, *Journal of Mathematical Problems in Engineering*, Hindawi pub. Cor.Vol. 2014 (2014) Article ID 965105, 20 pages.

[11] S.N. Daoud, The deletion-contraction method for counting the number of spanning trees of graphs, *Eur. Phys. J. Plus*, 130 (2015), 1- 14.

[12] S.N. Daoud, Complexity of graphs generated by wheel graph and their asymptotic limits, *Journal of the Egyptian Mathematical Society*, Volume 25, Issue 4, 424-433(2017).

[13] S.N. Daoud, Number of spanning trees in different products of complete and complete tripartite graphs, *Ars Cmbinatoria*, Vol. 139. 85-103 (2018).

[14] S.N. Daoud, Complexity of join and corona graphs and Chebyshev polynomials. *Journal of Taibah University for Sciences*,Vol. 12, NO. 5, (2018),557 -572.

[15] S.N. Daoud, Number of Spanning trees of cartesian and composition products of graphs and Chebyshev polynomials, *IEEE Access*, Vol.7(2019) 71142 -71157.

[16] S.N. Daoud, Number of spanning trees of some families of graphs generated by a triangle, *Journal of Taibah University for Sciences*, Vol.13, NO. 1, (2019)731-739.

[17] Jia-Bao Liu and S. N. Daoud, Number of spanning trees in the sequence of some graphs, *Complecity, Complexity*, Hindawi Publ. Corp. Volume 2019, Article ID 4271783, 22. pages (2019).

[18] A. K. Kelmans and V. M. Chelnokov, A certain polynomials of a graph and graphs with an extermal number of trees, *J. Comb. Theory* (B) 16, (1974), 197-214.

[19] G. G. Kirchhoff, Uber die Auflosung der Gleichungen, auf welche man be ider Untersuchung der Linearen Verteilung galvanischer Storme gefuhrt wird, *Ann. Phys. Chem.*, 72, (1847) 497-508.

[20] E. C. Kirby, D. J. Klein, R. B. Mallion, P. Pollak, and H. Sachs, A theorem for counting spanning trees in general chemical graphs and its particular application to toroidal fullerenes, *Croat. Chem. Acta*, 77 (2004), 263-278.

[21] M. Marcus, *A Survey of Matrix Theory and Matrix Inequalities*, Unvi. Allyn and Bacon. Inc. Boston (1964).

[22] F.Y. Wu, Number of spanning trees on a lattice, *J. Phys.* A 10, (1977), 113-115.

[23] Z. Yuanping, Y.Xuerong and J. Mordecai, Chebyshev polynomials and spanning trees formulas for circulant and related graphs, *Discrete Mathematics*, 298 (2005), 334-364.

[24] F. Zhang and X. Yong, Asymptotic enumeration theorems for the number of spanning trees and Eulerian trail in circulant digraphs & graphs, *Sci. China*, Ser. A43, 2(1999), 264-271.