

Number of Spanning Trees of Some of the Families of Sequence Graphs Generated by Triangle Graph

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Abstract: The number of spanning trees is an important quantity characterizing the reliability of a network. In this paper, we find the explicit formulas of the number of spanning trees of some of the families of sequence graphs generated by triangle graph with its special feature in iteration. Using the electrically equivalent transformations, we obtain the weights of corresponding equivalent graphs and we further derive relationships for spanning trees between these graphs and transformed graphs. Finally, we compare the entropy of our graphs together and with other studied graphs of average degree.

Key Words: Number of spanning trees, electrically equivalent transformations, entropy, sequence graphs.

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§1. Introduction

Calculating number of spanning trees in a graph is one of the well studied problems in Graph Theory. A spanning tree of a connected graph G with n vertices is a connected $(n - 1)$ -edge subgraph of G . The number of spanning trees of a graph G denoted by $\tau(G)$, also called the complexity of G [1], is a well-studied quantity in graph theory, and appears in a number of applications. Most memorable application fields are network reliability [18], recounting certain chemical isomers [2], and counting the number of Eulerian circuits in a graph [1]. In particular, counting spanning trees is an essential step in many methods for computing, bounding, and approximating network reliability [4]. In a network modeled by a graph, intercommunication between all nodes of the network implies that the graph must contain a spanning tree and thus maximizing the number of spanning trees is a way of maximizing reliability.

In 1847, a classical result of Kirchoff [16] can be used to determine the number of spanning trees for a connected graph $G = (V, E)$ with n vertices $\{v_1, v_2, \dots, v_n\}$, and the Kirchoff matrix L is defined as $n \times n$ characteristic matrix $L = D - A$, where D is the diagonal matrix of the

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degrees of G and A is the adjacency matrix of G , $L = [a_{ij}]$, where

$$a_{ij} = \begin{cases} \text{deg}(v_j), & \text{if } i = j \\ -1, & \text{if } (v_i, v_j) \in E(G) \\ 0, & \text{if } (v_i, v_j) \notin E(G) \end{cases}$$

All of co-factors of L are equal to the number of spanning trees of the graph G .

Another method for calculating $\tau(G)$ is described as follows:

Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n = 0$ denote the eigenvalues of L matrix of a graph G with n vertices. In 1974, Kelmans and Chelnokov [15] has shown that

$$\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i. \tag{1}$$

One common method for finding the number of spanning trees, $\tau(G)$, is the deletion-contraction method. This method is a dependable method which allows to calculate the number of spanning trees of a multigraph G . This method uses the fact that

$$\tau(G) = \tau(G - e) + \tau(G/e), \tag{2}$$

where $G - e$ denotes the graph obtained by deleting an arbitrary edge e and G/e denotes the graph obtained by contracting an arbitrary edge e [1,5]. For more methods and other techniques see [6]-[22].

§2. Electrically Equivalent Transformations

An electrical network is an interconnection of electrical components (eg. inductors, capacitors, batteries, resistors, switches, etc.).

In this section, we provide the relationships between electrical networks and spanning trees. Let G be an edge weighted graph, G' be the corresponding electrically equivalent graph and $\tau(G)$ denote the weighted number of spanning trees of G . Using the results in [20,21], we have the following transformation rules:

Parallel edges: If two parallel edges with conductances a and b in G are merged into a single edge with conductances $a + b$ in G' , then $\tau(G') = \tau(G)$.

Serial edges: If two serial edges with conductances a and b in G are merged into a single edge with conductance $\frac{ab}{a+b}$ in G' , then $\tau(G') = \frac{1}{a+b} \tau(G)$.

Delta-Wye transformation: If a triangle with conductances a , b and c in G is changed into an electrically equivalent star graph with conductances $r = \frac{ab+bc+ac}{a}$, $s = \frac{ab+bc+ac}{b}$ and $t = \frac{ab+bc+ac}{c}$ in G' , then $\tau(G') = \frac{(ab+bc+ac)^2}{abc} \tau(G)$.

Wye-Delta transformation: If a star graph with conductances a , b and c in G is changed into an electrically equivalent triangle with conductances $r = \frac{bc}{a+b+c}$, $s = \frac{ac}{a+b+c}$ and $t = \frac{ab}{a+b+c}$

in G' , then $\tau(G') = \frac{1}{a+b+c}\tau(G)$.

§3. Main Results

In Mathematics, one always tries to get new structure from given once. This also applies to the realm of graphs, where one can generate many new graphs from a given set of graphs.

Consider the sequence of graphs G_1, G_2, \dots, G_n constructed as shown in Fig.1. According to the construction, the number of total vertices $|V(G_n)|$ and edges $|E(G_n)|$ are $|V(G_n)| = 6n - 3$ and $|E(G_n)| = 15n - 12$, $n = 1, 2, \dots$. It is clear that the average degree is approximately 5 for a large n .

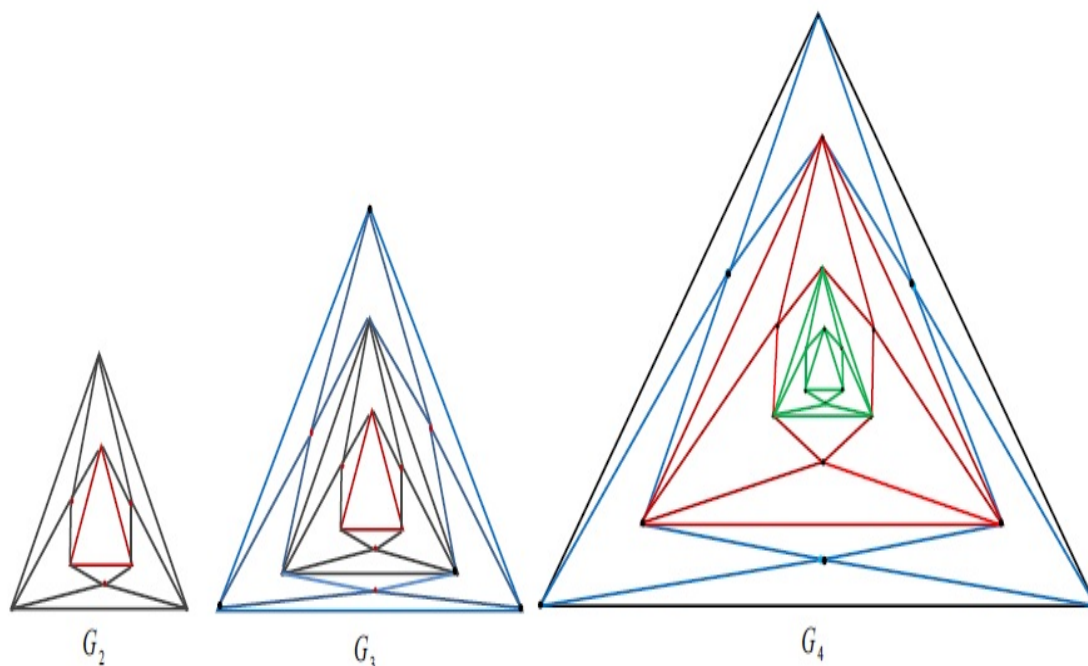


Figure 1

Theorem 3.1 For $n \geq 1$, the number of spanning trees in the sequence of the graph G_n is given by $\frac{A}{B}$, where

$$A = 4^{n-4}((98 - 15\sqrt{42})(13 + 2\sqrt{42})^n + (13 - 2\sqrt{42})^n(98 + 15\sqrt{42}))^2 \\ \times ((135 + 23\sqrt{42})(337 + 52\sqrt{42})^n - 11(1173 + 181\sqrt{42}))^2$$

$$B = 147(3707 + 572\sqrt{42} + (17 + 2\sqrt{42})(337 + 52\sqrt{42})^n)^2.$$

Proof We use the electrically equivalent transformation to transform G_i to G_{i-1} . Fig.2 illustrates the transformation process from G_2 to G_1 .

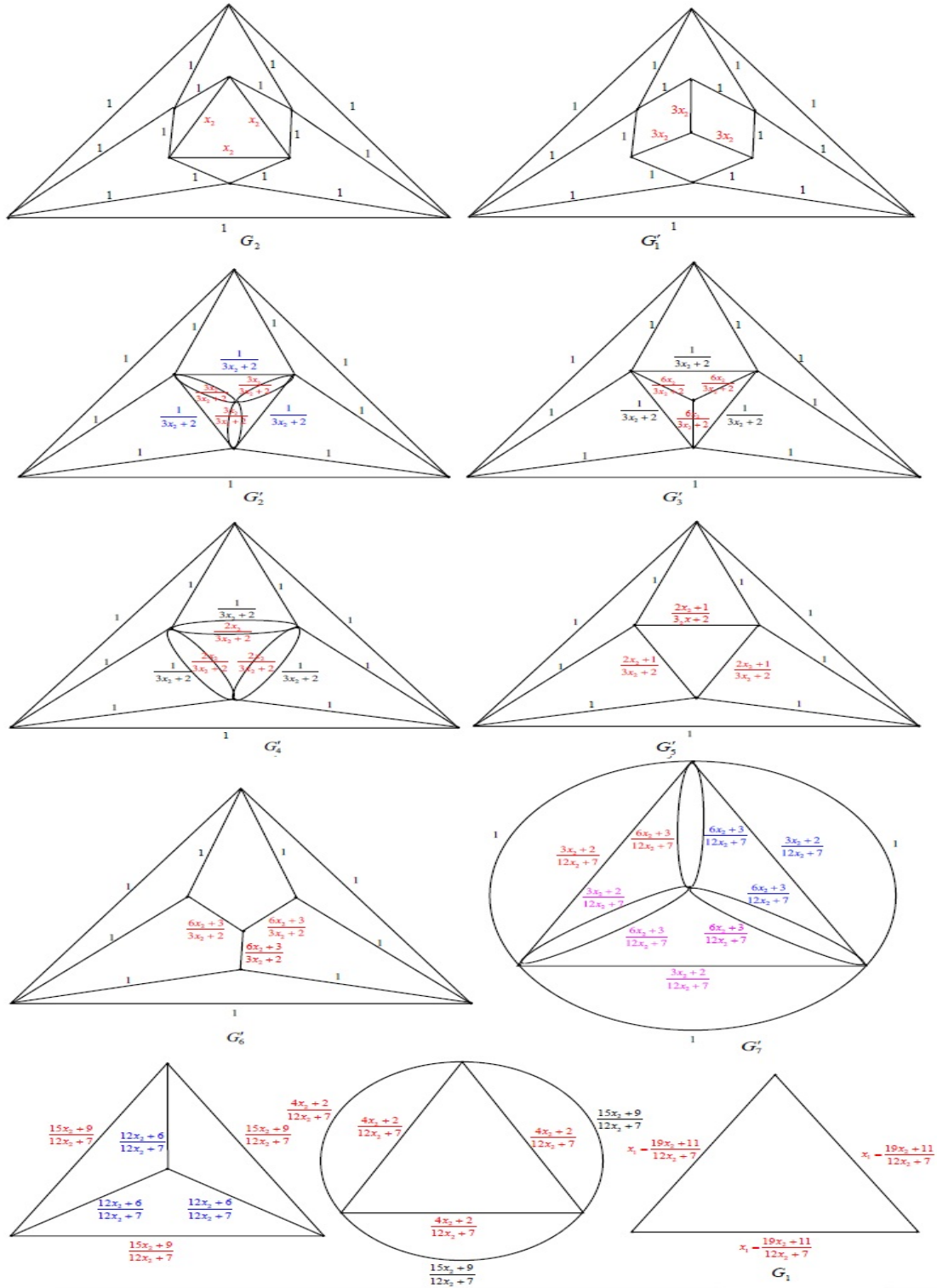


Figure 2

Using the properties given in section 2, we have the following transformations

$$\begin{aligned}
\tau(G'_1) &= 9x_2\tau(G_2), & \tau(G'_2) &= \frac{1}{(3x_2+2)^3}\tau(G'_1), \\
\tau(G'_3) &= \tau(G'_2), & \tau(G'_4) &= \frac{3x_2+2}{18x_2}\tau(G'_3), \\
\tau(G'_5) &= \tau(G'_4), & \tau(G'_6) &= 9\left(\frac{2x_2+1}{3x_2+2}\right)\tau(G'_5), \\
\tau(G'_7) &= \left(\frac{3x_2+2}{12x_2+7}\right)^3\tau(G'_6), & \tau(G'_8) &= \tau(G'_7), \\
\tau(G'_9) &= \frac{12x_2+7}{3(12x_2+6)}\tau(G'_8), & \tau(G_1) &= \tau(G'_9).
\end{aligned}$$

Combining these ten transformations, we have

$$\tau(G_2) = 4(12x_2+7)^2\tau(G_1) \quad (3)$$

and further,

$$\tau(G_n) = \prod_{i=2}^n 4(12x_i+7)^2\tau(G_1) = 3 \times 4^{n-1}x_i^2 \left[\prod_{i=2}^n (12x_i+7) \right]^2, \quad (4)$$

where $x_{i-1} = \frac{19x_i+11}{12x_i+7}$, $2 \leq i \leq n$. Its characteristic equation is $12u^2 - 12u - 11 = 0$ which have two roots $u_1 = \frac{3+\sqrt{42}}{6}$ and $u_2 = \frac{3-\sqrt{42}}{6}$.

Subtracting these two roots into both sides of $x_{i-1} = \frac{19x_i+11}{12x_i+7}$, we get

$$x_{i-1} - \frac{(3-\sqrt{42})}{6} = \frac{19x_i+11}{12x_i+7} - \frac{3-\sqrt{42}}{6} = (13+2\sqrt{42}) \left[\frac{x_i - \left(\frac{1}{2} - \sqrt{\frac{7}{6}}\right)}{12x_i+7} \right] \quad (5)$$

$$x_{i-1} - \frac{(3+\sqrt{42})}{6} = \frac{19x_i+11}{12x_i+7} - \frac{3+\sqrt{42}}{6} = (13-2\sqrt{42}) \left[\frac{x_i - \left(\frac{1}{2} + \sqrt{\frac{7}{6}}\right)}{12x_i+7} \right] \quad (6)$$

Let $y_i = \frac{x_i - \left(\frac{1}{2} - \sqrt{\frac{7}{6}}\right)}{x_i - \left(\frac{1}{2} + \sqrt{\frac{7}{6}}\right)}$. Then by equations (5) and (6), we get

$$y_{i-1} = (337+52\sqrt{42})y_i, \quad y_i = (337+52\sqrt{42})^{n-i}y_n.$$

Therefore,

$$x_i = \frac{(337+52\sqrt{42})^{n-i} \left(\frac{1}{2} + \sqrt{\frac{7}{6}}\right) y_n - \left(\frac{1}{2} - \sqrt{\frac{7}{6}}\right)}{(337+52\sqrt{42})^{n-i}y_n - 1}.$$

Thus

$$x_1 = \frac{(337+52\sqrt{42})^{n-1} \left(\frac{1}{2} + \sqrt{\frac{7}{6}}\right) y_n - \left(\frac{1}{2} - \sqrt{\frac{7}{6}}\right)}{(337+52\sqrt{42})^{n-1}y_n - 1}. \quad (7)$$

Using the expression $x_{n-1} = \frac{19x_n+11}{12x_n+7}$ and denoting the coefficients of $19x_n+11$ and $12x_n+7$ as h_n and k_n , we have

$$\begin{aligned} 12x_n + 7 &= h_0(19x_n + 11) + k_0(12x_n + 7), \\ 12x_{n-1} + 7 &= \frac{h_1(19x_n + 11) + k_1(12x_n + 7)}{h_0(19x_n + 11) + k_0(12x_n + 7)}, \\ 12x_{n-2} + 7 &= \frac{h_2(19x_n + 11) + k_2(12x_n + 7)}{h_1(19x_n + 11) + k_1(12x_n + 7)}, \\ &\vdots \\ 12x_{n-i} + 7 &= \frac{h_i(19x_n + 11) + k_i(12x_n + 7)}{h_{i-1}(19x_n + 11) + k_{i-1}(12x_n + 7)}, \end{aligned} \quad (8)$$

$$12x_{n-(i+1)} + 11 = \frac{h_{i+1}(19x_n + 11) + k_{i+1}(12x_n + 7)}{h_i(19x_n + 11) + k_i(12x_n + 7)}, \quad (9)$$

$$\begin{aligned} &\vdots \\ 12x_2 + 7 &= \frac{h_{n-2}(19x_n + 11) + k_{n-2}(12x_n + 7)}{h_{n-3}(19x_n + 11) + k_{n-3}(12x_n + 7)}. \end{aligned} \quad (10)$$

Thus, we obtain

$$\tau(G_n) = 3 \times 4^{n-1} x_i^2 [h_{n-2}(19x_n + 11) + k_{n-2}(12x_n + 7)]^2, \quad (11)$$

where $h_0 = 0$, $k_0 = 1$ and $h_1 = 12$, $k_1 = 7$. By the expression $x_{n-1} = \frac{19x_n+11}{12x_n+7}$ and using equations (8) and (9), we have

$$h_{i+1} = 26h_i - h_{i-1}; k_{i+1} = 26k_i - k_{i-1}. \quad (12)$$

The characteristic equation of (12) is $\nu^2 - 26\nu + 1 = 0$ which have two roots $\nu_1 = 13 + 2\sqrt{42}$ and $\nu_2 = 13 - 2\sqrt{42}$. The general solutions of equation (12) are $h_i = a_1\nu_1^i + a_2\nu_2^i$; $k_i = b_1\nu_1^i + b_2\nu_2^i$.

Using the initial conditions $h_0 = 0$, $k_0 = 1$ and $h_1 = 12$, $k_1 = 7$, yields

$$\begin{aligned} h_i &= \frac{\sqrt{42}}{14}(13 + 2\sqrt{42})^i - \frac{\sqrt{42}}{14}(13 - 2\sqrt{42})^i, \\ k_i &= \left(\frac{42 - 3\sqrt{42}}{84}\right)(13 + 2\sqrt{42})^i + \left(\frac{42 + 3\sqrt{42}}{84}\right)(13 - 2\sqrt{42})^i. \end{aligned} \quad (13)$$

If $x_n = 1$, it means that G_n without any electrically equivalent transformation. Plugging equation (13) into equation (11), we have

$$\begin{aligned} \tau(G_n) &= 3 \times 4^{n-1} x_i^2 \\ &\times \left[\left(\frac{798 + 123\sqrt{42}}{84}\right)(13 + 2\sqrt{42})^{n-2} + \left(\frac{798 - 123\sqrt{42}}{84}\right)(13 - 2\sqrt{42})^{n-2} \right]^2 \end{aligned} \quad (14)$$

if $n \geq 2$. When $n = 1$, $\tau(G_1) = 3$ which satisfies Eq.(14) Therefore, the number of spanning

trees in the sequence of the graph G_n is given by

$$\tau(G_n) = 3 \times 4^{n-1} x_i^2 \times \left[\left(\frac{798 + 123\sqrt{42}}{84} \right) (13 + 2\sqrt{42})^{n-2} + \left(\frac{798 - 123\sqrt{42}}{84} \right) (13 - 2\sqrt{42})^{n-2} \right]^2, \quad (15)$$

if $n \geq 1$, where

$$x_i = \frac{(337 + 52\sqrt{42})^{n-1} (135 + 32\sqrt{42}) + 66 \left(\frac{1}{2} - \sqrt{\frac{7}{6}} \right)}{6(337 + 52\sqrt{42})^{n-1} (17 + 2\sqrt{42}) + 66}, \quad n \geq 2. \quad (16)$$

Inserting Eq.(16) into Eq.(15) we obtain the result.

Consider the sequence of graphs H_1, H_2, \dots, H_n constructed as shown in see Fig.3.

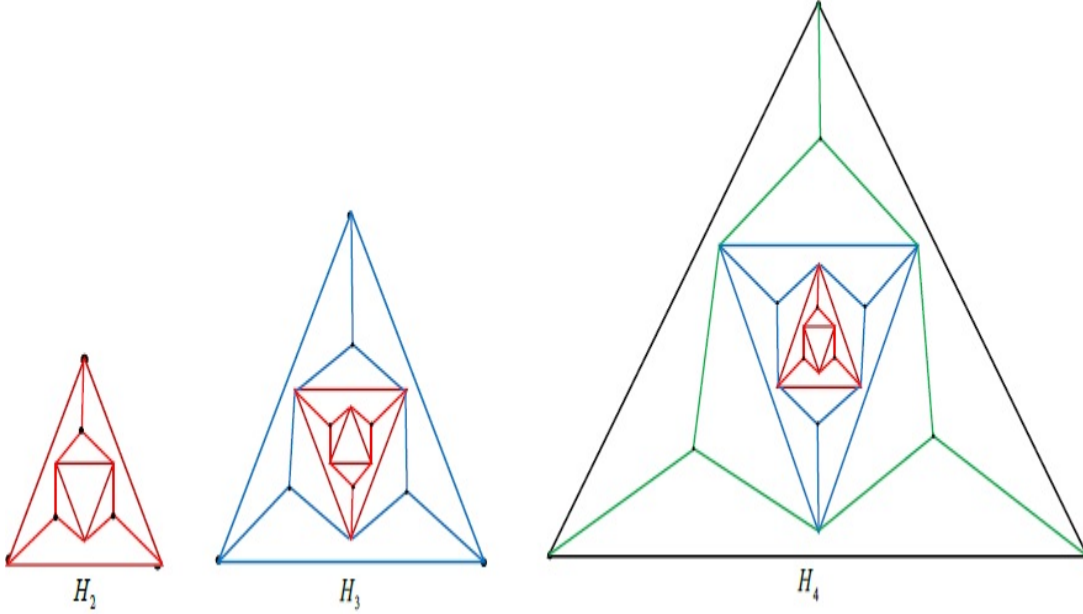


Figure 3

According to the construction, the number of total vertices $|V(H_n)|$ and edges $|E(H_n)|$ are $|V(H_n)| = 6n - 3$ and $|E(H_n)| = 12n - 9$, $n \geq 1$. It is clear that the average degree is approximately 4 for a large n . \square

Theorem 3.2 For $n \geq 1$, the number of spanning trees in the sequence of the graph H_n is given by

$$\frac{3 \times 2^{n-3} (2(8 + 3\sqrt{7})^n (98 + 37\sqrt{7}) - 3(8 - 3\sqrt{7})^n (1897 + 717\sqrt{7}) + (7 + 5\sqrt{7})(2024 + 765\sqrt{7})^n)^2}{49 (381 + 144\sqrt{7} + (11 + 4\sqrt{7})(127 + 48\sqrt{7})^n)^2}.$$

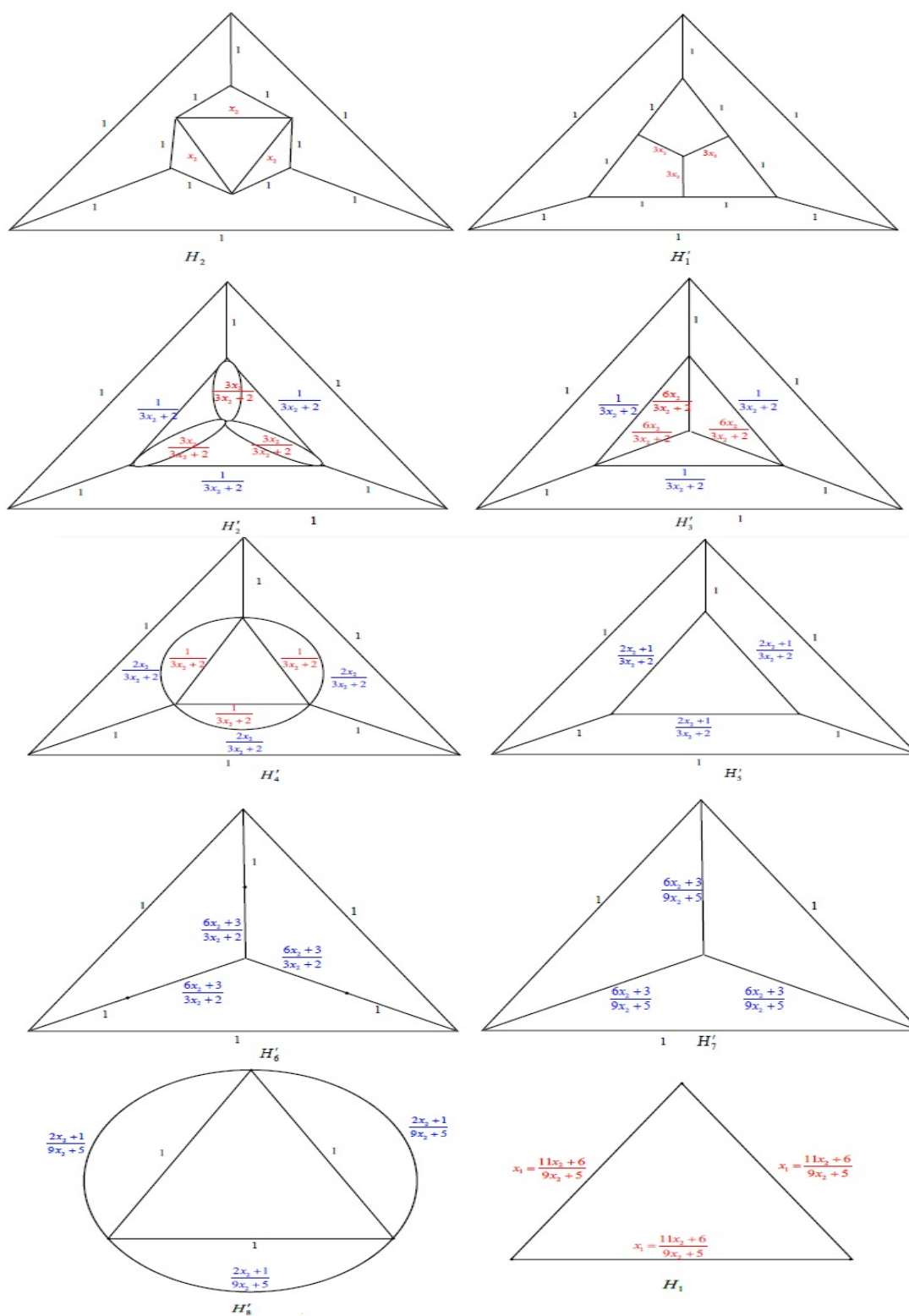


Figure 4 The transformations from H_2 to H_1

Proof We use the electrically equivalent transformation to transform H_i to H_{i-1} . Fig.4 illustrates the transformation process from H_2 to H_1 .

Using the properties given in section 2, we have the following the transformations:

$$\begin{aligned}\tau(H'_1) &= 9\tau(H_2), & \tau(H'_2) &= \left[\frac{1}{3x_2+2} \right]^3 \tau(H'_1), \\ \tau(H'_3) &= \tau(H'_2), & \tau(H'_4) &= \frac{3x_2+2}{18x_2} \tau(H'_3), \\ \tau(H'_5) &= \tau(H'_4), & \tau(H'_6) &= 9 \left(\frac{2x_2+1}{3x_2+2} \right) \tau(H'_5), \\ \tau(H'_7) &= \left[\frac{3x_2+2}{9x_2+5} \right]^3 \tau(H'_6), & \tau(H'_8) &= \frac{9x_2+5}{3(6x_2+3)} \tau(H'_6), \\ \tau(H_1) &= \tau(H'_8) & .\end{aligned}$$

Combining these nine transformations, we have

$$\tau(H_2) = 2(9x_2+5)^2 \tau(H_1). \quad (17)$$

Further,

$$\tau(H_n) = \prod_{i=2}^n 2(9x_i+5)^2 \tau(H_1) = 3 \times 2^{n-1} x_1^2 \left[\prod_{i=2}^n (9x_i+5) \right]^2 \quad (18)$$

where $x_{i-1} = \frac{11x_i+6}{9x_i+5}$, $i = 2, 3, \dots, n$. Its characteristic equation is $9u^2 - 6u - 6 = 0$, with roots $u_1 = \frac{1-\sqrt{7}}{3}$ and $u_2 = \frac{1+\sqrt{7}}{3}$.

Subtracting these two roots into both sides of $x_{i-1} = \frac{11x_i+6}{9x_i+5}$, we get

$$x_{i-1} - \frac{1-\sqrt{7}}{3} = \frac{11x_i+6}{9x_i+5} - \frac{1-\sqrt{7}}{3} = (8+3\sqrt{7}) \frac{x_i - \frac{1-\sqrt{7}}{3}}{9x_i+5}, \quad (19)$$

$$x_{i-1} - \frac{1+\sqrt{7}}{3} = \frac{11x_i+6}{9x_i+5} - \frac{1+\sqrt{7}}{3} = (8-3\sqrt{7}) \frac{x_i - \frac{1+\sqrt{7}}{3}}{9x_i+5}. \quad (20)$$

Let $y_i = \frac{x_i - \frac{1-\sqrt{7}}{3}}{x_i - \frac{1+\sqrt{7}}{3}}$. Then by Eqs.(19) and(20), we get $y_{i-1} = (127+48\sqrt{7})y_i$ and $y_{i-1} = (127+48\sqrt{7})^{n-i} y_i$.

Therefore,

$$x_i = \frac{(127+48\sqrt{7})^{n-i} \left(\frac{1+\sqrt{7}}{3} \right) y_n - \frac{1-\sqrt{7}}{3}}{(127+48\sqrt{7})^{n-i} y_n - 1}.$$

Thus,

$$x_i = \frac{(127+48\sqrt{7})^{n-1} \left(\frac{1+\sqrt{7}}{3} \right) y_n - \frac{1-\sqrt{7}}{3}}{(127+48\sqrt{7})^{n-1} y_n - 1}. \quad (21)$$

Using the expression $x_{n-1} = \frac{11x_n+6}{9x_n+5}$ and denoting the coefficients of $11x_n+6$ and $9x_n+5$

as h_n and k_n we have

$$\begin{aligned}
 9x_n + 5 &= h_0(11x_n + 6) + k_0(9x_n + 5), \\
 9x_{n-1} + 5 &= \frac{h_1(11x_n + 6) + k_1(9x_n + 5)}{h_0(11x_n + 6) + k_0(9x_n + 5)}, \\
 9x_{n-2} + 5 &= \frac{h_2(11x_n + 6) + k_2(9x_n + 5)}{h_1(11x_n + 6) + k_1(9x_n + 5)}, \\
 &\vdots \\
 9x_{n-i} + 5 &= \frac{h_i(11x_n + 6) + k_i(9x_n + 5)}{h_{i-1}(11x_n + 6) + k_{i-1}(9x_n + 5)}, \tag{22}
 \end{aligned}$$

$$9x_{n-(i+1)} + 5 = \frac{h_{i+1}(11x_n + 6) + k_{i+1}(9x_n + 5)}{h_i(11x_n + 6) + k_i(9x_n + 5)}, \tag{23}$$

$$\begin{aligned}
 &\vdots \\
 9x_2 + 5 &= \frac{h_{n-2}(11x_n + 6) + k_{n-2}(9x_n + 5)}{h_{n-3}(11x_n + 6) + k_{n-3}(9x_n + 5)} \tag{24}
 \end{aligned}$$

Thus, we obtain

$$\tau(H_n) = 3 \times 2^{n-1} x_1^2 [h_{n-2}(11x_n + 6) + h_{n-2}(9x_n + 5)]^2, \tag{25}$$

where $h_0 = 0$, $k_0 = 1$ and $h_1 = 9$, $k_1 = 5$. By the expression $x_{n-1} = \frac{11x_n + 6}{9x_n + 5}$ and Eqs.(23) and (24), we have

$$h_{i+1} = 16h_i - h_{i-1}; \quad k_{i+1} = 16k_i - k_{i-1}. \tag{26}$$

The characteristic equation of Eq.(26) is $v^2 - 16v + 1 = 0$ with roots $v_1 = 8 + 3\sqrt{7}$ and $v_2 = 8 - 3\sqrt{7}$. The general solutions of Eq. (26) are

$$h_i = a_1 v_1^i + a_2 v_2^i; \quad k_i = b_1 v_1^i + b_2 v_2^i.$$

Using the initial conditions $h_0 = 0$, $k_0 = 1$ and $h_1 = 9$, $k_1 = 5$, yields

$$\begin{aligned}
 h_i &= \frac{3\sqrt{7}}{14} (8 + 3\sqrt{7})^i - \frac{3\sqrt{7}}{14} (8 - 3\sqrt{7})^i \\
 k_i &= \left(\frac{7 - \sqrt{7}}{14} \right) (8 + 3\sqrt{7})^i + \left(\frac{7 + \sqrt{7}}{14} \right) (8 - 3\sqrt{7})^i. \tag{27}
 \end{aligned}$$

If $x_n = 1$, it means that H_n is without any electrically equivalent transformation. Plugging Eq.(27) into Eq.(25), we have

$$\tau(H_n) = 3 \times 2^{n-1} x_1^2 \left[\left(\frac{98 + 37\sqrt{7}}{14} \right) (8 + 3\sqrt{7})^{n-2} + \left(\frac{98 - 37\sqrt{7}}{14} \right) (8 - 3\sqrt{7})^{n-2} \right]^2 \tag{28}$$

if $n \geq 2$. When $n = 1$, $\tau(H_1) = 3$ which satisfies Eq.(28). Therefore, the number of spanning

trees in the sequence of the graph H_n is given by

$$\tau(H_n) = 3 \times 2^{n-1} x_1^2 \left[\left(\frac{98 + 37\sqrt{7}}{14} \right) (8 + 3\sqrt{7})^{n-2} + \left(\frac{98 - 37\sqrt{7}}{14} \right) (8 - 3\sqrt{7})^{n-2} \right]^2 \quad (29)$$

if $n \geq 1$, where

$$x_1 = \frac{(127 + 48\sqrt{7})^{n-1} (13 + 5\sqrt{7}) + (1 - \sqrt{7})}{(127 + 48\sqrt{7})^{n-1} (11 + 4\sqrt{7}) + 3}, \quad n \geq 1. \quad (30)$$

Inserting Eq.(30) into Eq.(29) we obtain the desired result. \square

Consider the sequence of graphs L_1, L_2, \dots, L_n constructed as shown in Fig.5

According to the construction, the number of total vertices $|V(L_n)|$ and edges $|E(L_n)|$ are $|V(L_n)| = 9n - 6, |E(L_n)| = 18n - 15, n = 1, 2, \dots$. It is clear that the average degree is approximately 4 for a large n .

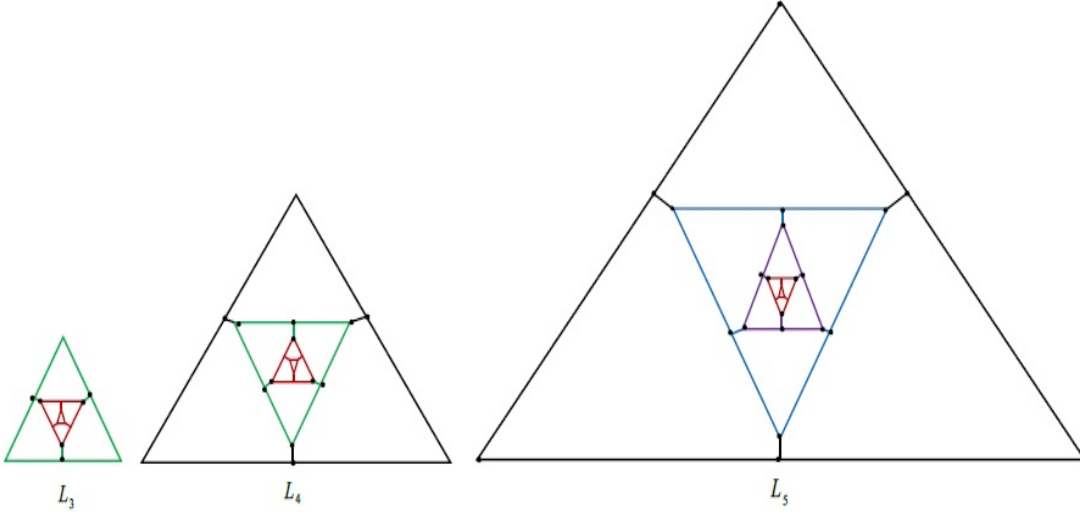


Figure 5 Some sequences of the graph L_n

Theorem 3.3 For $n \geq 1$, the number of spanning trees in the sequence of the graph L_n is given by

$$\frac{2^{-(n+1)} \left((11 + 5\sqrt{5}) (14 + 6\sqrt{5})^n + (14 - 6\sqrt{5})^n (123 + 55\sqrt{5}) - 2 (644 + 288\sqrt{5})^n \right)^2}{3 \left(2^n (47 + 21\sqrt{5}) - (3 + \sqrt{5}) (47 + 21\sqrt{5})^n \right)^2}.$$

Proof We use the electrically equivalent transformation to transform L_i to L_{i-1} . Fig.6 illustrates the transformation process from L_2 to L_1 .

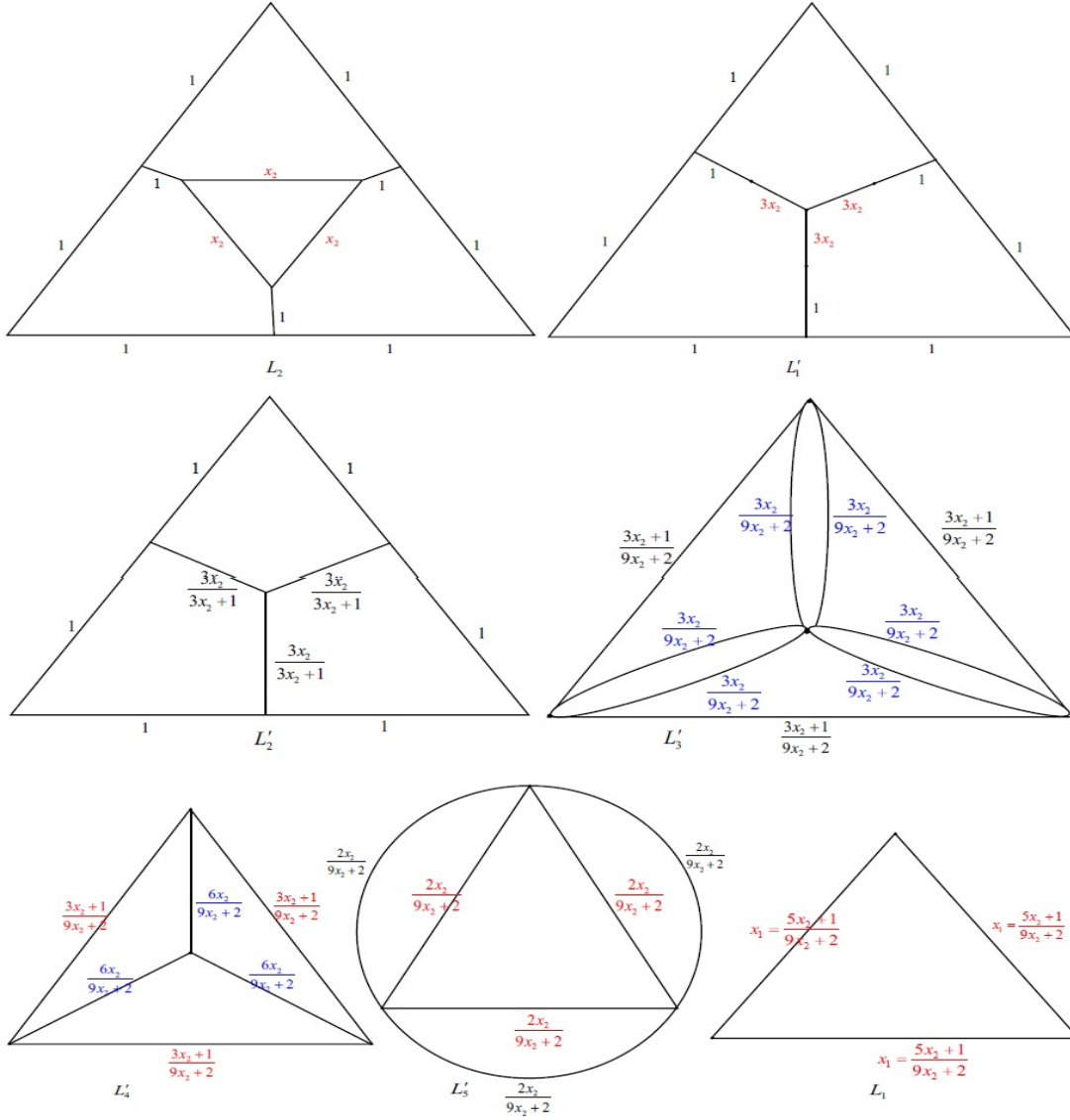


Figure 6 The transformations from L_2 to L_1

Using the properties given in section 2, we have the following transformations

$$\begin{aligned} \tau(L'_1) &= 9x_2 \tau(L_2), & \tau(L'_2) &= \left[\frac{1}{3x_2 + 1}\right]^3 \tau(L'_1), \\ \tau(L'_3) &= \left(\frac{3x_2 + 1}{9x_2 + 2}\right)^3 \tau(L'_2), & \tau(L'_4) &= \tau(L'_3), \\ \tau(L'_5) &= \left(\frac{9x_2 + 2}{18x_2}\right) \tau(L'_4), & \tau(L_1) &= \tau(L'_5). \end{aligned}$$

Combining these six transformations, we have

$$\tau(L_2) = 2 (9x_2 + 2)^2 \tau(L_1). \tag{31}$$

Further

$$\tau(L_n) = \prod_{i=2}^n 2(9x_i + 2)^2 \tau(L_1) = 3 \times 2^{n-1} x_1^2 \left[\prod_{i=2}^n (9x_i + 2) \right]^2, \quad (32)$$

where $x_{i-1} = \frac{5x_i+1}{9x_i+2}$, $i = 2, 3, \dots, n$. Its characteristic equation is $9u^2 - 3u - 1 = 0$ with roots $u_1 = \frac{1-\sqrt{5}}{6}$ and $u_2 = \frac{1+\sqrt{5}}{6}$.

Subtracting these two roots into both sides of $x_{i-1} = \frac{5x_i+1}{9x_i+2}$, we get

$$x_{i-1} - \frac{1-\sqrt{5}}{6} = \frac{5x_i+1}{9x_i+2} - \frac{1-\sqrt{5}}{6} = (7+3\sqrt{5}) \cdot \frac{x_i - (\frac{1-\sqrt{5}}{6})}{2(9x_i+2)} \quad (33)$$

$$x_{i-1} - \frac{1+\sqrt{5}}{6} = \frac{5x_i+1}{9x_i+2} - \frac{1+\sqrt{5}}{6} = (7-3\sqrt{5}) \cdot \frac{x_i - (\frac{1+\sqrt{5}}{6})}{2(9x_i+2)} \quad (34)$$

Let $y_i = \frac{x_i - \frac{1-\sqrt{5}}{6}}{x_i - \frac{1+\sqrt{5}}{6}}$. Then, by Eqs.(33) and (34), we get $y_{i-1} = (\frac{47+21\sqrt{5}}{2}) y_i$ and $y_i = y_{i-1} = (\frac{47+21\sqrt{5}}{2})^{n-i} y_n$. Therefore,

$$x_i = \frac{(\frac{47+21\sqrt{5}}{2})^{n-i} (d\frac{1+\sqrt{5}}{6}) y_n - \frac{1-\sqrt{5}}{6}}{(\frac{47+21\sqrt{5}}{2})^{n-i} y_n - 1}.$$

Thus,

$$x_1 = \frac{(\frac{47+21\sqrt{5}}{2})^{n-1} (\frac{1+\sqrt{5}}{6}) y_n - \frac{1-\sqrt{5}}{6}}{(\frac{47+21\sqrt{5}}{2})^{n-1} y_n - 1}. \quad (35)$$

Using the expression $x_{n-1} = \frac{5x_n+1}{9x_n+2}$ and denoting the coefficients of $5x_n+1$ and $9x_n+2$ as h_n and k_n , we have

$$\begin{aligned} 9x_n + 2 &= h_0(5x_n + 1) + k_0(9x_n + 2), \\ 9x_{n-1} + 2 &= \frac{h_1(5x_n + 1) + k_1(9x_n + 2)}{h_0(5x_n + 1) + k_0(9x_n + 2)}, \\ 9x_{n-2} + 2 &= \frac{h_2(5x_n + 1) + k_2(9x_n + 2)}{h_1(5x_n + 1) + k_1(9x_n + 2)}, \\ &\vdots \\ 9x_{n-i} + 2 &= \frac{h_i(5x_n + 1) + k_i(9x_n + 2)}{h_{i-1}(5x_n + 1) + k_{i-1}(9x_n + 2)}, \end{aligned} \quad (36)$$

$$9x_{n-(i+1)} + 2 = \frac{h_{i+1}(5x_n + 1) + k_{i+1}(9x_n + 2)}{h_i(5x_n + 1) + k_i(9x_n + 2)}, \quad (37)$$

$$\begin{aligned} &\vdots \\ 9x_2 + 2 &= \frac{h_{n-2}(5x_n + 1) + k_{n-2}(9x_n + 2)}{h_{n-3}(5x_n + 1) + k_{n-3}(9x_n + 2)}. \end{aligned} \quad (38)$$

Thus, we obtain

$$\tau(L_n) = 3 \times 2^{n-1} x_1^2 [h_{n-2}(5x_n + 1) + k_{n-2}(9x_n + 2)]^2, \quad (39)$$

where $h_0 = 0$, $k_0 = 1$ and $h_1 = 9$, $k_1 = 2$. By the expression $x_{n-1} = \frac{5x_n+1}{9x_n+2}$, Eqs.(36) and (37), we have

$$h_{i+1} = 7h_i - h_{i-1}; \quad k_{i+1} = 7k_i - k_{i-1} \quad (40)$$

The characteristic equation of Eq.(40) is $v^2 - 7v + 1 = 0$ with roots $v_1 = \frac{7+3\sqrt{5}}{2}$ and $v_2 = \frac{7-3\sqrt{5}}{2}$. The general solutions of Eq.(40) are

$$h_i = a_1 v_1^i + a_2 v_2^i; \quad k_i = b_1 v_1^i + b_2 v_2^i.$$

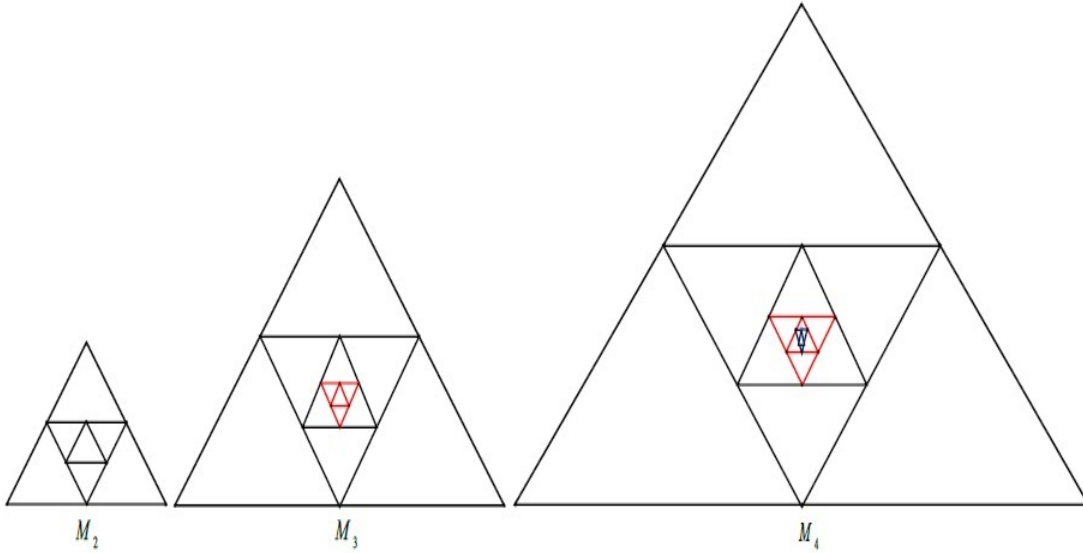


Figure 7 Some sequences of the graph M_n

Using the initial conditions $h_0 = 0$, $k_0 = 1$ and $h_1 = 9$, $k_1 = 2$, yields

$$\begin{aligned} h_i &= \frac{3\sqrt{5}}{5} \left(\frac{7+3\sqrt{5}}{2}\right)^i - \frac{3\sqrt{5}}{5} \left(\frac{7-3\sqrt{5}}{2}\right)^i; \\ k_i &= \left(\frac{5-\sqrt{5}}{10}\right) \left(\frac{7+3\sqrt{5}}{2}\right)^i + \left(\frac{5+\sqrt{5}}{10}\right) \left(\frac{7-3\sqrt{5}}{2}\right)^i. \end{aligned} \quad (41)$$

If $x_n = 1$, it means that L_n is without any electrically equivalent transformation. Plugging Eq.(41) into Eq.(39), we have

$$\begin{aligned} \tau(L_n) &= 3 \times 2^{n-1} x_1^2 \left[\left(\frac{11+5\sqrt{5}}{2}\right) \left(\frac{7+3\sqrt{5}}{2}\right)^{n-2} \right. \\ &\quad \left. + \left(\frac{11-5\sqrt{5}}{2}\right) \left(\frac{7-3\sqrt{5}}{2}\right)^{n-2} \right]^2 \end{aligned} \quad (42)$$

if $n \geq 2$. When $n = 1$, $\tau(L_1) = 3$ which satisfies Eq.(42). Therefore, the number of spanning

trees in the sequence of the graph is given by

$$\begin{aligned} \tau(L_n) = & 3 \times 2^{n-1} x_1^2 \left[\left(\frac{11+5\sqrt{5}}{2} \right) \left(\frac{7+3\sqrt{5}}{2} \right)^{n-2} \right. \\ & \left. + \left(\frac{11-5\sqrt{5}}{2} \right) \left(\frac{7-3\sqrt{5}}{2} \right)^{n-2} \right]^2 \end{aligned} \quad (43)$$

if $n \geq 1$, where

$$x_1 = \frac{2 \left(\frac{47+21\sqrt{5}}{2} \right)^{n-1} (2 + \sqrt{5}) - (1 - \sqrt{5})}{3 \left(\frac{47+21\sqrt{5}}{2} \right)^{n-1} (3 + \sqrt{5}) - 6}, \quad n \geq 1. \quad (44)$$

Inserting Eq.(44) into Eq.(43) we obtain the desired result. \square

Consider the sequence of graphs M_1, M_2, \dots, M_n constructed as shown in Fig.7.

Theorem 3.4 For $n \geq 1$, the number of spanning trees in the sequence of the graph M_n is given by

$$\frac{4^{n-2} \left((-3 + \sqrt{3}) (1351 + 780\sqrt{3})^n + (1713 + 989\sqrt{3}) (7 - 4\sqrt{3})^n \right)^2}{3 \left(97 + 56\sqrt{3} - (2 + \sqrt{3}) (97 + 56\sqrt{3})^n \right)^2}.$$

Proof We use the electrically equivalent transformation to transform M_i to M_{i-1} and the transformation process from M_2 to M_1 is illustrated in Fig.8.

Using the properties given in Section 2, we have the following transformations

$$\begin{aligned} \tau(M'_1) &= 9x_2 \tau(M_2), & \tau(M'_2) &= \left(\frac{1}{9x_2 + 2} \right)^3 \tau(M'_1), \\ \tau(M'_3) &= \tau(M'_2), & \tau(M'_4) &= \left(\frac{3x_2 + 2}{18x_2} \right) \tau(M'_3), \\ \tau(M'_5) &= \tau(M'_4), & \tau(M'_6) &= \left(\frac{18x_2 + 9}{3x_2 + 2} \right) \tau(M'_5), \\ \tau(M'_7) &= \left(\frac{3x_2 + 2}{12x_2 + 7} \right) \tau(M'_6), & \tau(M'_8) &= \tau(M'_7), \\ \tau(M'_9) &= d_9^1 \left(\frac{12x_2 + 7}{3x_2 + 2} \right) \tau(M'_8), & \tau(M_1) &= \tau(M'_9). \end{aligned}$$

Combining these ten transformations, we have

$$\tau(M_2) = 4(12x_2 + 7)^2 \tau(M_1). \quad (45)$$

Further,

$$\tau(M_n) = \prod_{i=2}^n 4(12x_i + 7)^2 \tau(M_1) = 3 \times 4^{n-1} x_1^2 \left[\prod_{i=2}^n (12x_i + 7) \right]^2, \quad (46)$$

where $x_{i-1} = \frac{7x_i + 4}{12x_i + 7}$, $i = 2, 3, \dots, n$. Its characteristic equation is $3u^2 - 1 = 0$ with roots $u_1 = \frac{-\sqrt{3}}{3}$ and $u_2 = \frac{\sqrt{3}}{3}$.

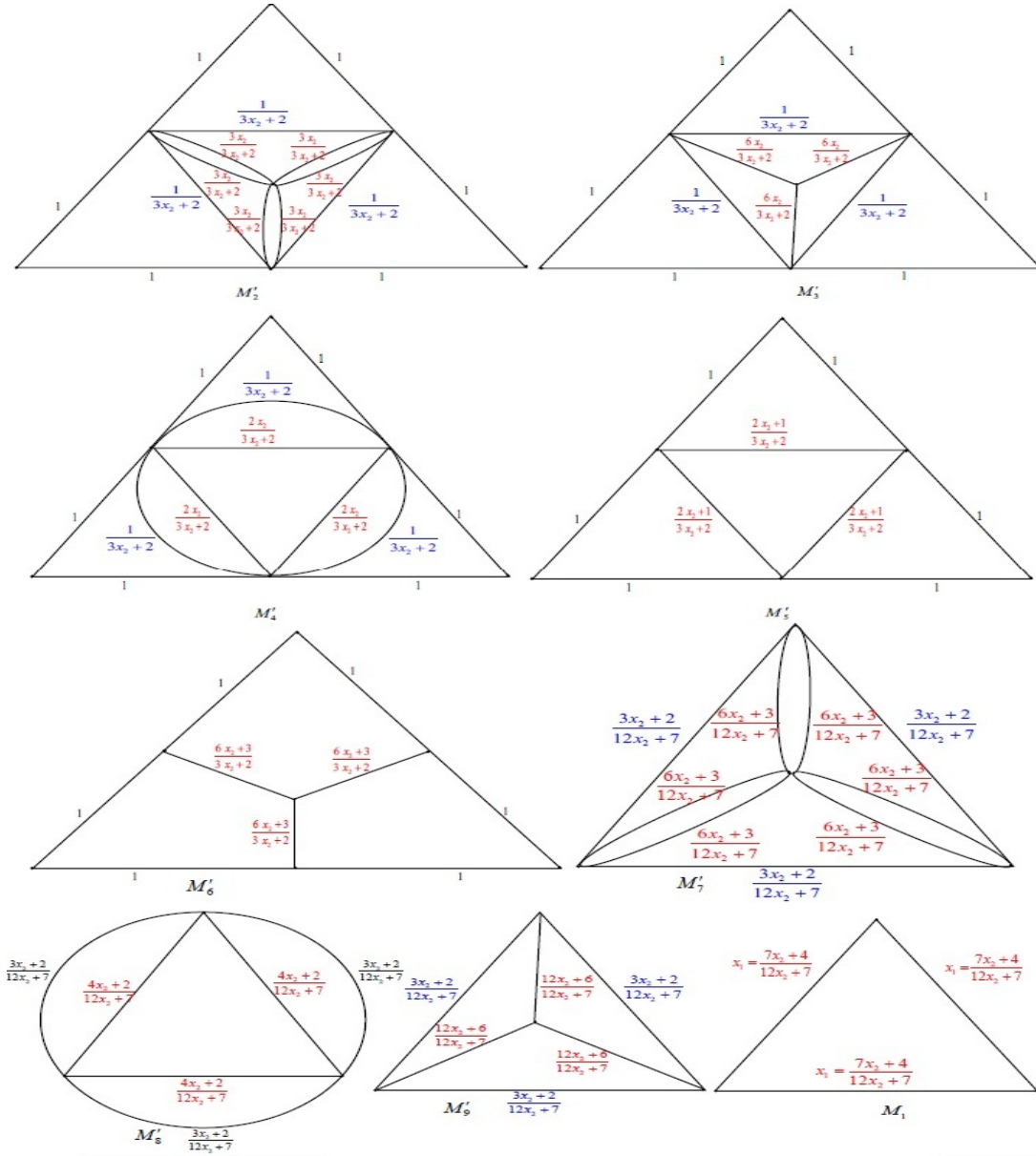


Figure 8 The transformations from M_2 to M_1

Subtracting these two roots into both sides of $x_{i-1} = \frac{7x_i+4}{12x_i+7}$, we get

$$x_{i-1} - \left(\frac{-\sqrt{3}}{3} \right) = \frac{7x_i+4}{12x_i+7} + \frac{\sqrt{3}}{3} = (7+4\sqrt{3}) \times \frac{x_i + \frac{\sqrt{3}}{3}}{12x_i+7}, \quad (47)$$

$$x_{i-1} - \frac{\sqrt{3}}{3} = \frac{7x_i+4}{12x_i+7} - \frac{\sqrt{3}}{3} = (7-4\sqrt{3}) \times \frac{x_i - \frac{\sqrt{3}}{3}}{12x_i+7}. \quad (48)$$

Let $y_i = \frac{x_i + \frac{\sqrt{3}}{3}}{x_i - \frac{\sqrt{3}}{3}}$. Then by Eqs.(47) and (48), we get $y_{i-1} = (97 + 56\sqrt{3})y_i$ and $y_i =$

$(97 + 56\sqrt{3})^{n-i} y_n$. Therefore,

$$x_i = \frac{(97 + 56\sqrt{3})^{n-i} \left(\frac{\sqrt{3}}{3}\right) y_n + \frac{\sqrt{3}}{3}}{(97 + 56\sqrt{3})^{n-i} y_n - 1}.$$

Thus,

$$x_1 = \frac{(97 + 56\sqrt{3})^{n-1} \left(\frac{\sqrt{3}}{3}\right) y_n + \frac{\sqrt{3}}{3}}{(97 + 56\sqrt{3})^{n-1} y_n - 1}. \quad (49)$$

Using the expression $x_{n-1} = \frac{7x_n+4}{12x_n+7}$ and denoting the coefficients of $7x_n + 4$ and $12x_n + 7$ as h_n and k_n , we have

$$\begin{aligned} 12x_n + 7 &= h_0(7x_n + 4) + k_0(12x_n + 7), \\ 12x_{n-1} + 7 &= \frac{h_1(7x_n + 4) + k_1(12x_n + 7)}{h_0(7x_n + 4) + k_0(12x_n + 7)}, \\ 12x_{n-2} + 7 &= \frac{h_2(7x_n + 4) + k_2(12x_n + 7)}{h_1(7x_n + 4) + k_1(12x_n + 7)}, \\ &\vdots \\ 12x_{n-i} + 7 &= \frac{h_i(7x_n + 4) + k_i(12x_n + 7)}{h_{i-1}(7x_n + 4) + k_{i-1}(12x_n + 7)}, \end{aligned} \quad (50)$$

$$12x_{n-(i+1)} + 7 = \frac{h_{i+1}(7x_n + 4) + k_{i+1}(12x_n + 7)}{h_i(7x_n + 4) + k_i(12x_n + 7)}, \quad (51)$$

$$\begin{aligned} &\vdots \\ 12x_2 + 7 &= \frac{h_{n-2}(7x_n + 4) + k_{n-2}(12x_n + 7)}{h_{n-3}(7x_n + 4) + k_{n-3}(12x_n + 7)}. \end{aligned} \quad (52)$$

Thus, we obtain

$$\tau(M_n) = 3 \times 4^{n-1} x_1^2 [h_{n-2}(7x_n + 4) + k_{n-2}(12x_n + 7)]^2, \quad (53)$$

where $h_0 = 0$, $k_0 = 1$ and $h_1 = 12$, $k_1 = 7$. By the expression $x_{n-1} = \frac{7x_n+4}{12x_n+7}$ and Eqs.(53) and (51), we have

$$h_{i+1} = 14h_i - h_{i-1}; \quad k_{i+1} = 14k_i - k_{i-1}. \quad (54)$$

The characteristic equation of Eq.(54) is $v^2 - 14v + 1 = 0$ with roots $v_1 = 7 + 4\sqrt{3}$ and $v_2 = 7 - 4\sqrt{3}$. The general solutions of Eq. (54) are

$$h_i = a_1 v_1^i + a_2 v_2^i; \quad k_i = b_1 v_1^i + b_2 v_2^i.$$

Using the initial conditions $h_0 = 0$, $k_0 = 1$ and $h_1 = 12$, $k_1 = 7$, yields

$$h_i = \frac{\sqrt{3}}{2} \left(7 + 4\sqrt{3}\right)^i - \frac{\sqrt{3}}{2} \left(7 - 4\sqrt{3}\right)^i; \quad k_i = \frac{1}{2} \left(7 + 4\sqrt{3}\right)^i + \frac{1}{2} \left(7 - 4\sqrt{3}\right)^i. \quad (55)$$

If $x_n = 1$, it means that M_n is without any electrically equivalent transformation. Plugging

Eq.(55) into Eq.(53), we have

$$\begin{aligned} \tau(M_n) = & 3 \times 4^{n-1} x_1^2 \left[\left(\frac{19 + 11\sqrt{3}}{2} \right) (7 + 4\sqrt{3})^{n-2} \right. \\ & \left. + \left(\frac{19 - 11\sqrt{3}}{2} \right) (7 - 4\sqrt{3})^{n-2} \right]^2 \end{aligned} \tag{56}$$

if $n \geq 2$. When $n = 1$, $\tau(M_1) = 3$ which satisfies Eq.(56). Therefore, the number of spanning trees in the sequence of the graph is given by

$$\begin{aligned} \tau(M_n) = & 3 \times 4^{n-1} x_1^2 \left[\left(\frac{19 + 11\sqrt{3}}{2} \right) (7 + 4\sqrt{3})^{n-2} \right. \\ & \left. + \left(\frac{19 - 11\sqrt{3}}{2} \right) (7 - 4\sqrt{3})^{n-2} \right]^2 \end{aligned} \tag{57}$$

if $n \geq 2$, where

$$x_1 = \frac{(97 + 56\sqrt{3})^{n-1} \left(\frac{3+2\sqrt{3}}{3} \right) + \frac{\sqrt{3}}{3}}{(97 + 56\sqrt{3})^{n-1} (2 + \sqrt{3}) - 1}. \tag{58}$$

Inserting Eq.(58) into Eq.(57) we obtain the desired result. \square .

Consider the sequence of graphs Q_1, Q_2, \dots, Q_n constructed as shown in Fig.9.

According to the construction, the number of total vertices $|V(Q_n)|$ and edges $|E(Q_n)|$ are $|V(Q_n)| = 6n - 3, |E(Q_n)| = 15n - 12, n = 1, 2, \dots$. It is clear that the average degree is approximately 5 for a large n .

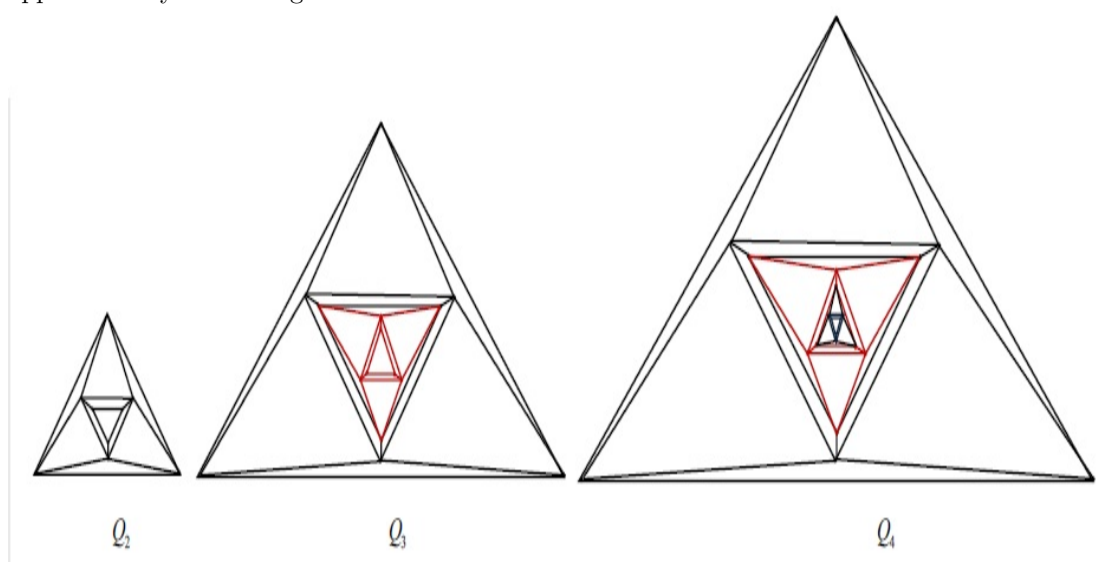


Figure 9 Some sequences of the graph Q_n

Theorem 3.5 For $n \geq 1$, the number of spanning trees in the sequence of the graph Q_n is

given by $\frac{C}{D}$, where

$$\begin{aligned}
 C &= 2^{n-5} \left((44 - 31\sqrt{2}) (17 + 12\sqrt{2})^n + (44 + 31\sqrt{2}) (17 - 12\sqrt{2})^n \right)^2 \\
 &\quad \times \left(-7 (239 + 169\sqrt{2}) (17 + 13\sqrt{2}) (577 + 408\sqrt{2})^n \right)^2 \\
 D &= 3 \left(7 (577 + 408\sqrt{2}) + (9 + 4\sqrt{2}) (577 + 408\sqrt{2})^n \right)^2
 \end{aligned}$$

Proof We use the electrically equivalent transformation to transform Q_i to Q_{i-1} . In the Fig.10 – 1 and 10 – 2, we illustrate the transformation process from Q_2 to Q_1 .

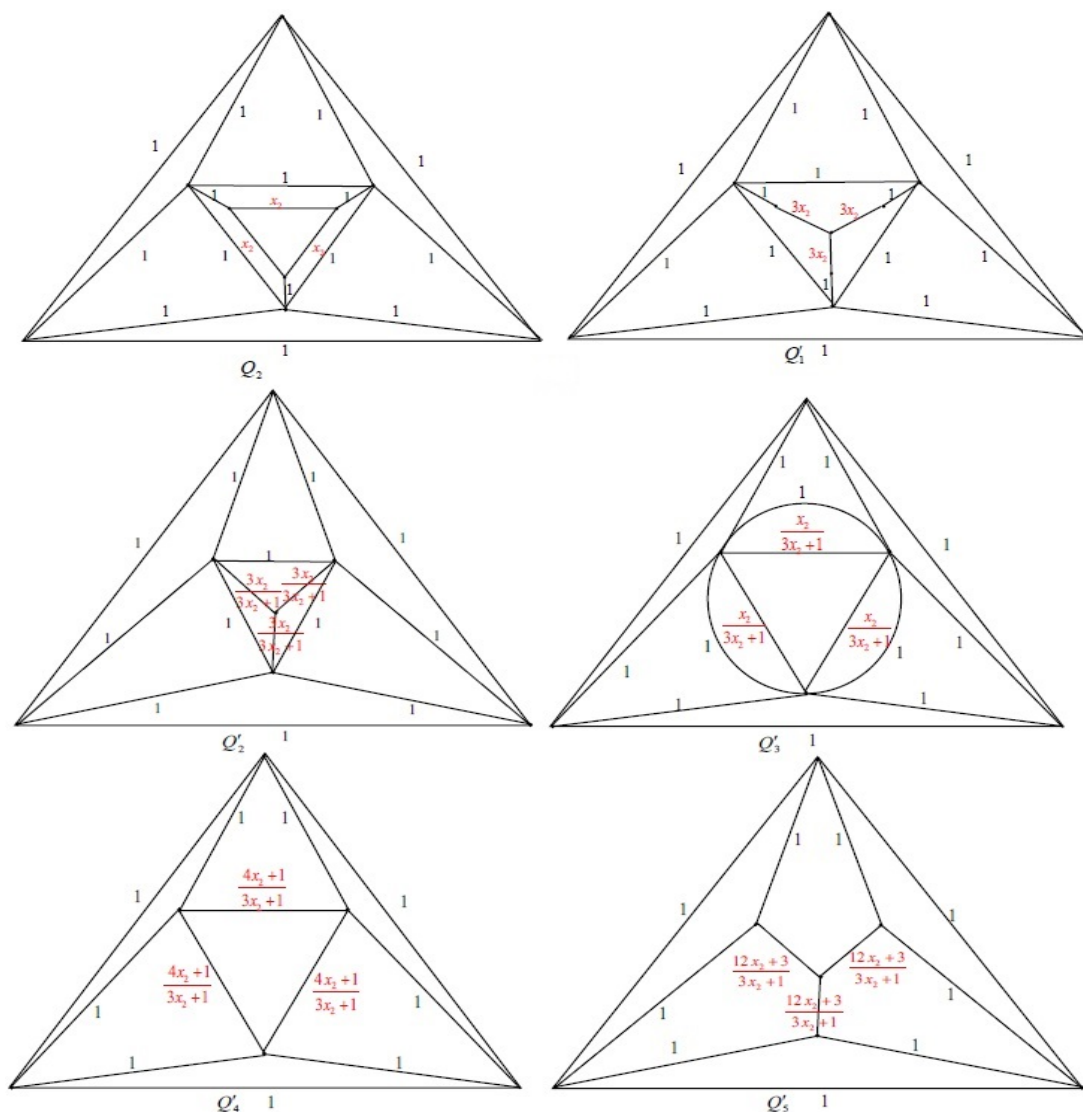


Figure 10-1 The transformations from Q_2 to Q_1

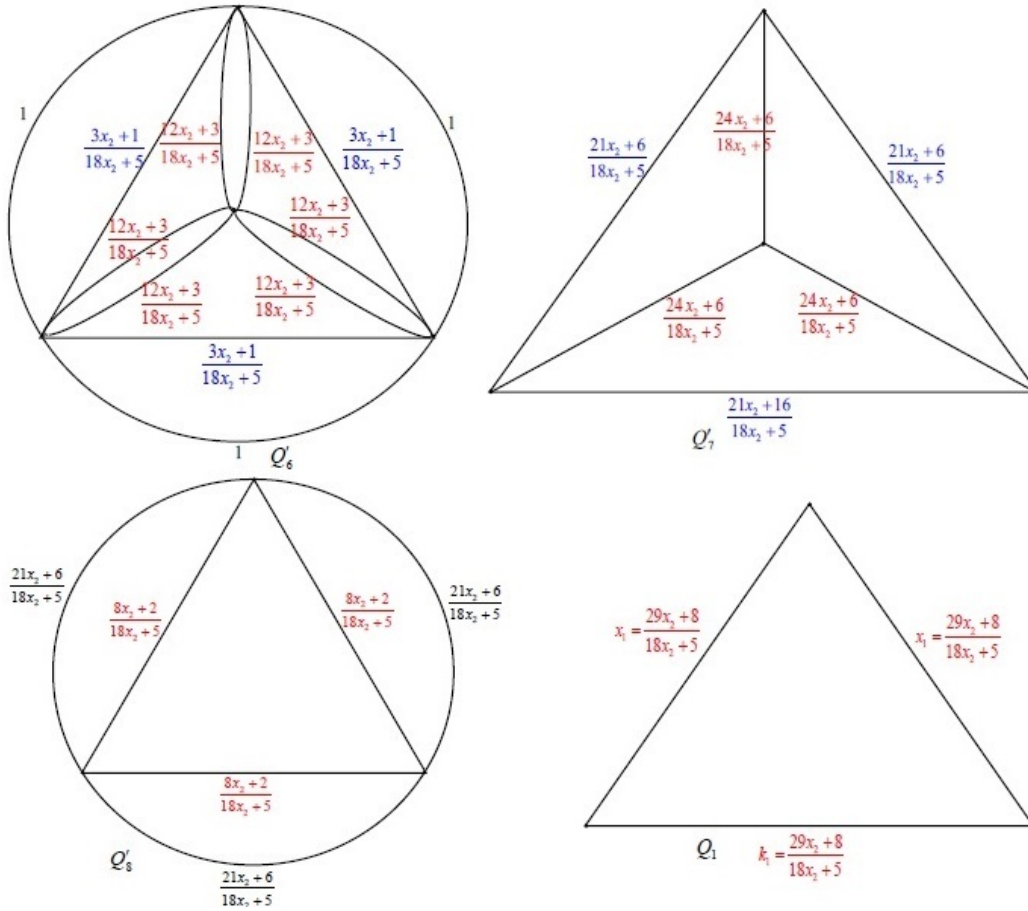


Figure 10-2 The transformations from Q_2 to Q_1

Using the properties given in Section 2, we have the following the transformations:

$$\begin{aligned} \tau(Q'_1) &= 9_2 \tau(Q_2), & \tau(Q'_2) &= \left(\frac{1}{3x_2 + 1}\right)^3 \tau(Q'_1), \\ \tau(Q'_3) &= \left(\frac{3x_2 + 1}{9x_2}\right) \tau(Q'_2), & \tau(Q'_4) &= \tau(Q'_3), \\ \tau(Q'_5) &= 9 \left(\frac{4x_2 + 1}{3x_2 + 1}\right) \tau(Q'_4), & \tau(Q'_6) &= \left(\frac{3x_2 + 1}{18x_2 + 5}\right)^3 \tau(Q'_5), \\ \tau(Q'_7) &= \tau(Q'_6), & \tau(Q'_8) &= \frac{1}{9} \left(\frac{18x_2 + 5}{8x_2 + 2}\right)^3 \tau(Q'_7) \end{aligned}$$

and $\tau(Q_1) = \tau(Q'_8)$.

Combining these nine transformations, we have

$$\tau(Q_2) = 2(18x_2 + 5)^2 \tau(Q_1). \tag{59}$$

Further,

$$\tau(Q_n) = \prod_{i=2}^n 2(18x_i + 5)^2 \tau(Q_1) = 3 \times 2^{n-1} x_1^2 \left[\prod_{i=2}^n (18x_i + 5) \right]^2, \quad (60)$$

where $x_{i-1} = \frac{29x_i+8}{18x_i+5}$, $i = 2, 3, \dots, n$. Its characteristic equation is $9u^2 - 12u - 4 = 0$ with roots $u_1 = \frac{2-2\sqrt{2}}{3}$ and $u_2 = \frac{2+2\sqrt{2}}{3}$.

Subtracting these two roots into both sides of $x_{i-1} = \frac{29x_i+8}{18x_i+5}$, we get

$$x_{i-1} - \frac{2-2\sqrt{2}}{3} = \frac{29x_i+8}{18x_i+5} - \frac{2-2\sqrt{2}}{3} = (17+12\sqrt{2}) \cdot \frac{x_i - (\frac{2-2\sqrt{2}}{3})}{18x_i+5}, \quad (61)$$

$$x_{i-1} - \frac{2+2\sqrt{2}}{3} = \frac{29x_i+8}{18x_i+5} - \frac{2+2\sqrt{2}}{3} = (17-12\sqrt{2}) \cdot \frac{x_i - (\frac{2+2\sqrt{2}}{3})}{18x_i+5}. \quad (62)$$

Let $y_i = \frac{x_i - \frac{2-2\sqrt{2}}{3}}{x_i - \frac{2+2\sqrt{2}}{3}}$. Then by Eqs.(61) and (62), we get $y_{i-1} = (577 + 408\sqrt{2}) y_i$ and $y_i = (577 + 408\sqrt{2})^{n-i} y_n$.

Therefore,

$$x_i = \frac{(577 + 408\sqrt{2})^{n-i} (\frac{2+2\sqrt{2}}{3}) y_n - \frac{2-2\sqrt{2}}{3}}{(577 + 408\sqrt{2})^{n-i} y_n - 1}.$$

Thus,

$$x_i = \frac{(577 + 408\sqrt{2})^{n-1} (\frac{2+2\sqrt{2}}{3}) y_n - \frac{2-2\sqrt{2}}{3}}{(577 + 408\sqrt{2})^{n-1} y_n - 1}. \quad (63)$$

Using the expression $x_{n-1} = \frac{29x_n+8}{18x_n+5}$ and denoting the coefficients of $29x_n+8$ and $18x_n+5$ as h_n and k_n , we have

$$\begin{aligned} 18x_n + 5 &= h_0(29x_n + 8) + k_0(18x_n + 5), \\ 18x_{n-1} + 5 &= \frac{h_1(29x_n + 8) + k_1(18x_n + 5)}{h_0(29x_n + 8) + k_0(18x_n + 5)}, \\ 18x_{n-2} + 5 &= \frac{h_2(29x_n + 8) + k_2(18x_n + 5)}{h_1(29x_n + 8) + k_1(18x_n + 5)}, \\ &\vdots \\ 18x_{n-i} + 5 &= \frac{h_i(29x_n + 8) + k_i(18x_n + 5)}{h_{i-1}(29x_n + 8) + k_{i-1}(18x_n + 5)}, \end{aligned} \quad (64)$$

$$18x_{n-(i+1)} + 5 = \frac{h_{i+1}(29x_n + 8) + k_{i+1}(18x_n + 5)}{h_i(29x_n + 8) + k_i(18x_n + 5)}, \quad (65)$$

$$\begin{aligned} &\vdots \\ 18x_2 + 5 &= \frac{h_{n-2}(29x_n + 8) + k_{n-2}(18x_n + 5)}{h_{n-3}(29x_n + 8) + k_{n-3}(18x_n + 5)}. \end{aligned} \quad (66)$$

Thus, we obtain

$$\tau(Q_n) = 3 \times 2^{n-1} x_1^2 [h_{n-2} (29x_n + 8) + k_{n-2} (18x_n + 5)]^2, \quad (67)$$

where $h_0 = 0$, $k_0 = 1$ and $h_1 = 18$, $k_1 = 5$. By the expression $x_{n-1} = \frac{29x_n + 8}{18x_n + 5}$ and Eqs.(65) and (66), we have

$$h_{i+1} = 34h_i - h_{i-1}; \quad k_{i+1} = 34k_i - k_{i-1}, \quad (68)$$

The characteristic equation of Eq.(68) is $v^2 - 34v + 1 = 0$ with roots $v_1 = 17 + 12\sqrt{2}$ and $v_2 = 17 - 12\sqrt{2}$. The general solutions of Eq. (68) are

$$h_i = \lambda a_1 v_1^i + a_2 v_2^i; \quad k_i = b_1 v_1^i + b_2 v_2^i.$$

Using the initial conditions $h_0 = 0$, $k_0 = 1$ and $h_1 = 18$, $k_1 = 5$, yields

$$\begin{aligned} h_i &= \frac{3\sqrt{2}}{8} (17 + 12\sqrt{2})^i - \frac{3\sqrt{2}}{8} (17 - 12\sqrt{2})^i; \\ k_i &= \left(\frac{2 - \sqrt{2}}{4}\right) (17 + 12\sqrt{2})^i + \left(\frac{2 + \sqrt{2}}{4}\right) (17 - 12\sqrt{2})^i. \end{aligned} \quad (69)$$

If $x_n = 1$, it means that Q_n is without any electrically equivalent transformation. Plugging Eq.(68) into Eq.(67), we have

$$\begin{aligned} \tau(Q_n) &= 3 \times 2^{n-1} x_1^2 \left[\left(\frac{92 + 65\sqrt{2}}{8}\right) (17 + 12\sqrt{2})^{n-2} \right. \\ &\quad \left. + \left(\frac{92 - 65\sqrt{2}}{8}\right) (17 - 12\sqrt{2})^{n-2} \right]^2 \end{aligned} \quad (70)$$

if $n \geq 2$. When $n = 1$, $\tau(Q_1) = 3$ which satisfies Eq.(69). Therefore, the number of spanning trees in the sequence of the graph is given by

$$\begin{aligned} \tau(Q_n) &= 3 \times 2^{n-1} x_1^2 \left[\left(\frac{92 + 65\sqrt{2}}{8}\right) (17 + 12\sqrt{2})^{n-2} \right. \\ &\quad \left. + \left(\frac{92 - 65\sqrt{2}}{8}\right) (17 - 12\sqrt{2})^{n-2} \right]^2, \end{aligned} \quad (71)$$

if $n \geq 1$, where

$$x_1 = \frac{(577 + 408\sqrt{2})^{n-1} \left(\frac{34 + 26\sqrt{2}}{21}\right) + \left(\frac{2 - 2\sqrt{2}}{3}\right)}{(577 + 408\sqrt{2})^{n-1} \left(\frac{9 + 4\sqrt{2}}{7}\right) + 1}. \quad (72)$$

Inserting Eq.(72) into Eq.(71) we obtain the desired result. \square

Consider the sequence of graphs R_1, R_2, \dots, R_n constructed as shown in Fig.11.

According to the construction, the number of total vertices $|V(R_n)|$ and edges $|E(R_n)|$

are $|V(R_n)| = 9n - 6, |E(R_n)| = 18n - 15, n = 1, 2, \dots$. It is clear that the average degree is approximately 4 for a large n .

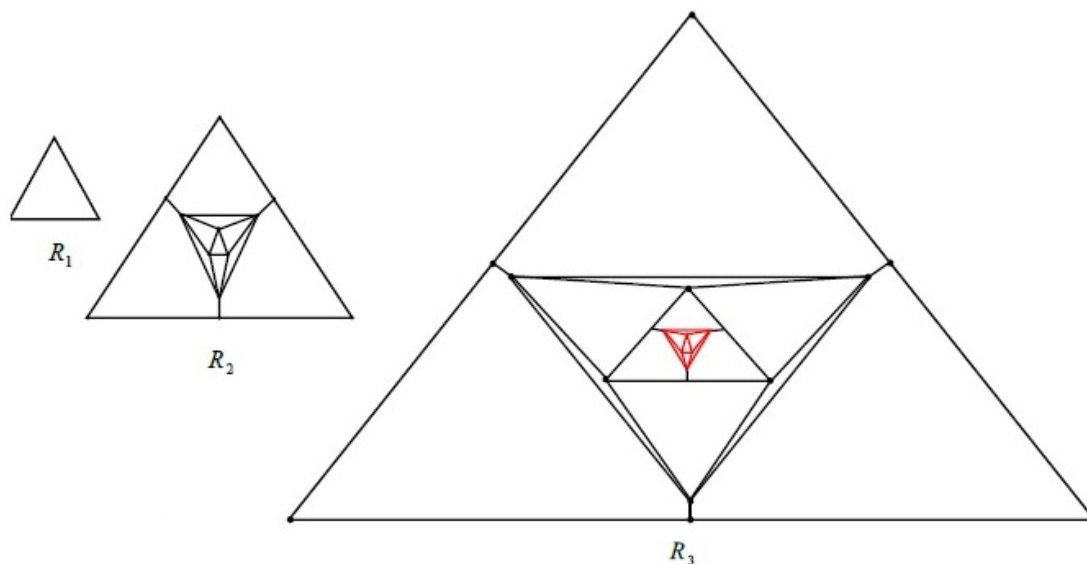


Figure 11 Some sequences of the graph R_n

Theorem 3.6 For $n \geq 1$, the number of spanning trees in the sequence of the graph R_n is given by $\frac{A}{B}$, where

$$A = (59 + \sqrt{3477}^{2n} (-111 \times 2^{n+1} (47519 + 806\sqrt{3477}) + 17(3479 + 59\sqrt{3477})^n \times (-511119 + 8467\sqrt{3477}) + 629(3479 - 59\sqrt{3477})^n (169194297 + 2869349\sqrt{3477})^2)$$

$$B = (16119372 (-629 \times 2^n (3479 + 59\sqrt{477}) + (2417 + 35\sqrt{3477}) (3479 + 59\sqrt{3477})^n)^2).$$

Proof We use the electrically equivalent transformation to transform R_i to R_{i-1} . In Figs.12 - 1, 12 - 2 and 12 - 3, we illustrate the transformation process from R_2 to R_1 .

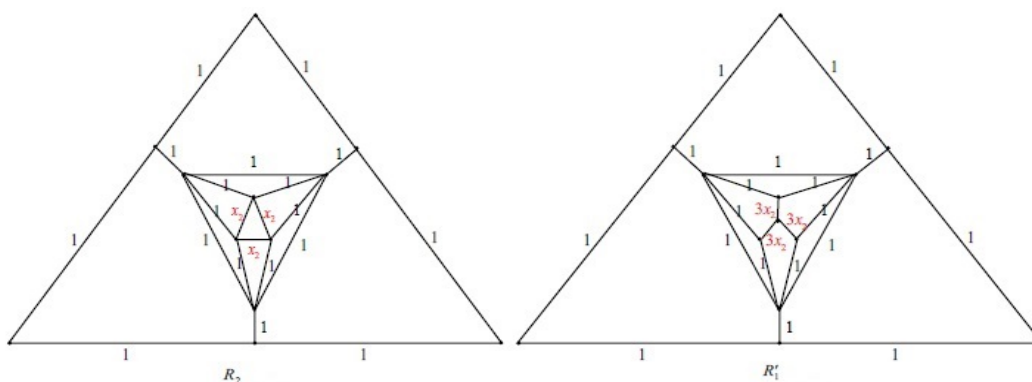


Figure 12-1 The transformations from R_2 to R_1

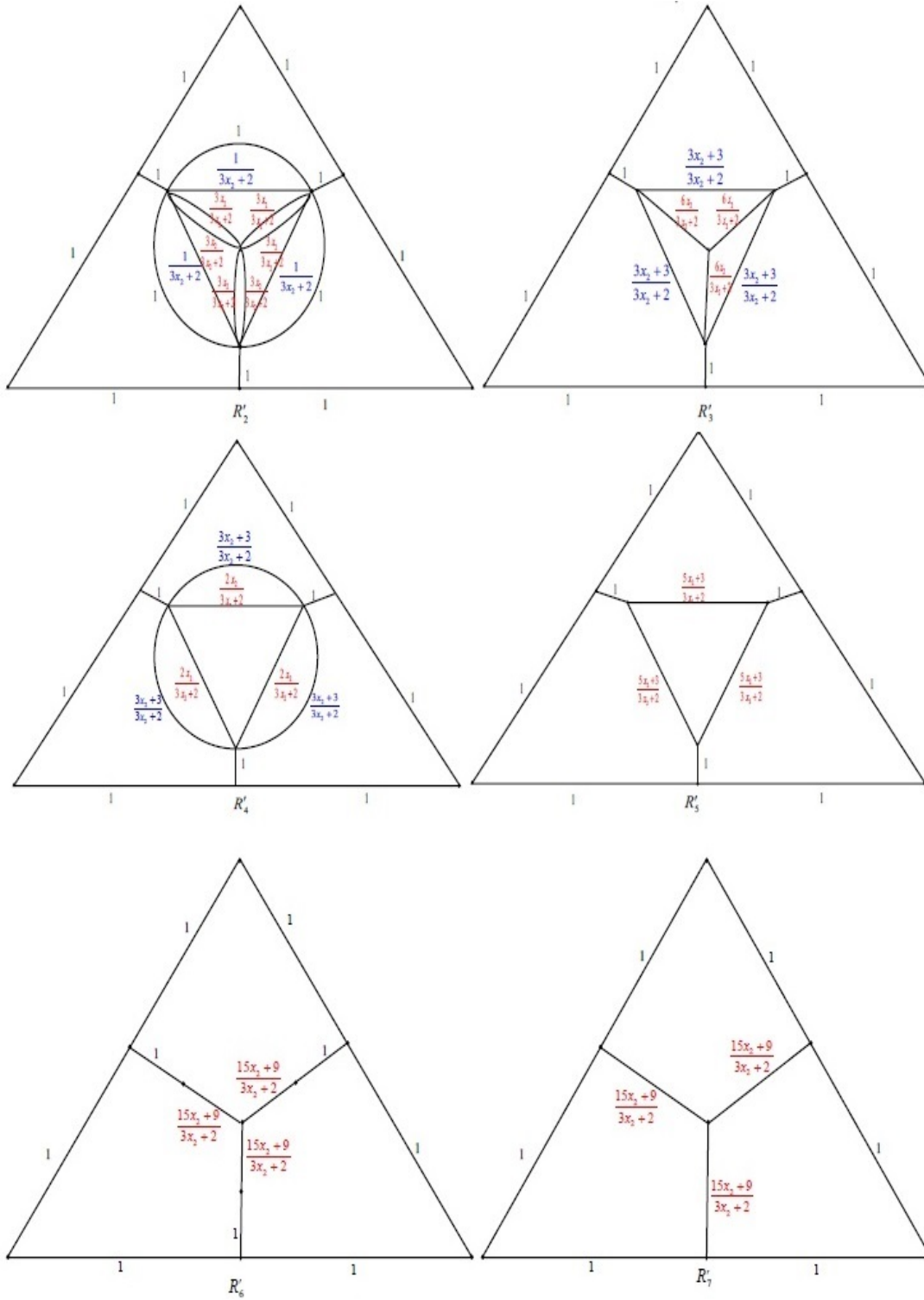


Figure 12-2 The transformations from R_2 to R_1

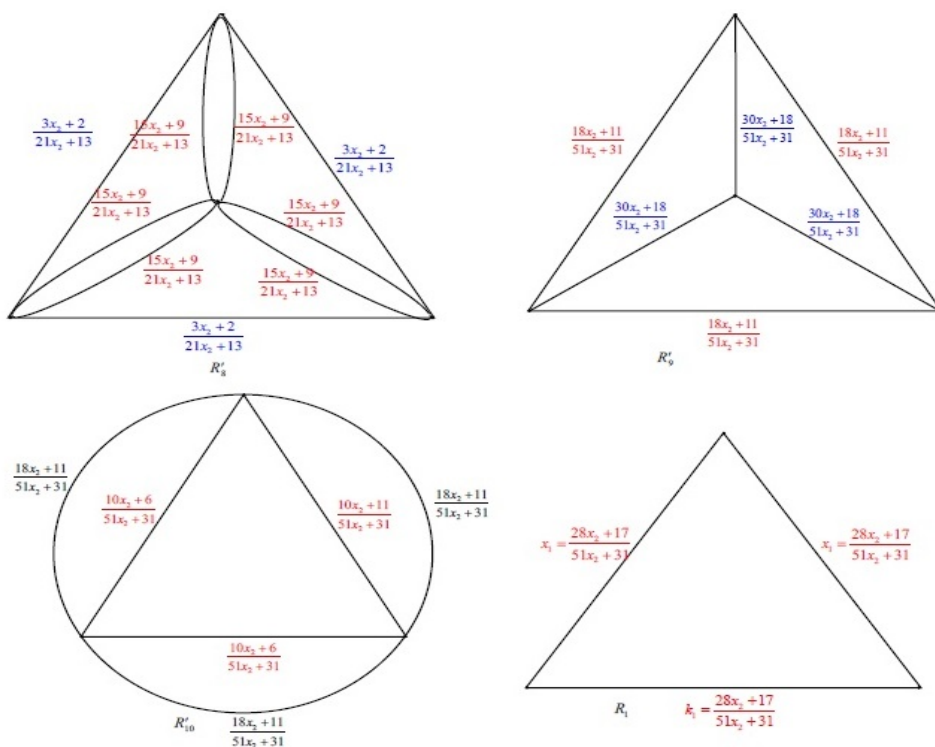


Figure 12-2 The transformations from R_2 to R_1

Using the properties given in section 2, we have the following the transformations:

$$\begin{aligned}
 \tau(R'_1) &= 9x_2 \tau(R_2), & \tau(R'_2) &= \left(\frac{1}{3x_2+2}\right)^3 \tau(R'_1), \\
 \tau(R'_3) &= \tau(R'_2), & \tau(R'_4) &= \frac{3x_2+2}{18x_2} \tau(R'_3), \\
 \tau(R'_5) &= \tau(R'_4), & \tau(R'_6) &= 9\left(\frac{5x_2+3}{3x_2+2}\right) \tau(R'_5), \\
 \tau(R'_7) &= \left(\frac{3x_2+2}{18x_2+11}\right)^3 \tau(R'_6), & \tau(R'_8) &= \left(\frac{18x_2+11}{51x_2+31}\right)^3 \tau(R'_7), \\
 \tau(R'_9) &= \tau(R'_8), & \tau(R'_{10}) &= \frac{51x_2+31}{90x_2+54} \tau(R'_9)
 \end{aligned}$$

and $\tau(R_1) = \tau(R'_{10})$. Combining these eleven transformations, we have

$$\tau(R_2) = 4(51x_2+31)^2 \tau(R_1). \quad (73)$$

Further,

$$\tau(R_n) = \prod_{i=2}^n 4(51x_i+31)^2 \tau(R_1) = 3 \times 4^{n-1} x_1^2 \left[\prod_{i=2}^n (51k_i+31) \right]^2, \quad (74)$$

where $x_{i-1} = \frac{28x_i+17}{51x_i+31}$, $i = 2, 3, \dots, n$. Its characteristic equation is $51u^2 + 3u - 17 = 0$ with

roots $u_1 = \frac{-3-\sqrt{3477}}{102}$ and $u_2 = \frac{-3+\sqrt{3477}}{102}$.

Subtracting these two roots into both sides of $x_{i-1} = \frac{28x_i+17}{51x_i+31}$, we get

$$x_{i-1} + \frac{3 + \sqrt{3477}}{102} = \frac{28x_i + 17}{51x_i + 31} + \frac{3 + \sqrt{3477}}{102} = (59 + \sqrt{3477}) \cdot \frac{x_i + \left(\frac{3 + \sqrt{3477}}{2}\right)}{2(51x_i + 31)}, \quad (75)$$

$$x_{i-1} - \frac{3 - \sqrt{3477}}{102} = \frac{28x_i + 17}{51x_i + 31} + \frac{3 - \sqrt{3477}}{102} = (59 - \sqrt{3477}) \cdot \frac{x_i + \left(\frac{3 - \sqrt{3477}}{2}\right)}{2(51x_i + 31)}. \quad (76)$$

Let $y_i = \frac{x_i + \frac{3+\sqrt{3477}}{102}}{x_i + \frac{3-\sqrt{3477}}{102}}$. Then by Eqs.(75) and (76), we get $y_{i-1} = \left(\frac{3479+59\sqrt{3477}}{2}\right)y_i$ and $y_i = \left(\frac{3479+59\sqrt{3477}}{2}\right)^{n-i} y_n$.

Therefore

$$x_i = \frac{\left(\frac{-3+\sqrt{3477}}{102}\right)\left(\frac{3479 + 59\sqrt{3477}}{2}\right)^{n-i} y_n + \frac{3+\sqrt{3477}}{102}}{\left(\frac{3479 + 59\sqrt{3477}}{2}\right)^{n-i} y_n - 1}.$$

Thus

$$x_i = \frac{\left(\frac{-3 + \sqrt{3477}}{102}\right)\left(\frac{3479 + 59\sqrt{3477}}{2}\right)^{n-1} y_n + \frac{3 + \sqrt{3477}}{102}}{\left(\frac{3479 + 59\sqrt{3477}}{2}\right)^{n-1} y_n - 1}. \quad (77)$$

Using the expression $x_{n-1} = \frac{28x_n+17}{51x_n+31}$ and denoting the coefficients of $28x_n + 17$ and $51x_n + 31$ as h_n and k_n , we have

$$\begin{aligned} 51x_n + 31 &= h_0(28x_n + 17) + k_0(51x_n + 31), \\ 51x_{n-1} + 31 &= \frac{h_1(28x_n + 17) + k_1(51x_n + 31)}{h_0(28x_n + 17) + k_0(18x_n + 31)}, \\ 51x_{n-2} + 31 &= \frac{h_2(28x_n + 17) + k_2(51x_n + 31)}{h_1(28x_n + 17) + k_1(51x_n + 31)}, \\ &\vdots \\ 51x_{n-i} + 31 &= \frac{h_i(28x_n + 17) + k_i(51x_n + 31)}{h_{i-1}(28x_n + 17) + k_{i-1}(51x_n + 31)}, \end{aligned} \quad (78)$$

$$51x_{n-(i+1)} + 31 = \frac{h_{i+1}(28x_n + 17) + k_{i+1}(51x_n + 31)}{h_i(28x_n + 17) + k_i(51x_n + 31)}, \quad (79)$$

$$\begin{aligned} &\vdots \\ 51x_2 + 31 &= \frac{h_{n-2}(28x_n + 17) + k_{n-2}(51x_n + 31)}{h_{n-3}(28x_n + 17) + k_{n-3}(51x_n + 31)}. \end{aligned} \quad (80)$$

Thus, we obtain

$$\tau(R_n) = 3 \times 4^{n-1} x_1^2 [h_{n-2}(28x_n + 17) + k_{n-2}(51x_n + 31)]^2, \quad (81)$$

where $h_0 = 0$, $k_0 = 1$ and $h_1 = 51$, $k_1 = 31$. By the expression $x_{n-1} = \frac{28x_n+17}{51x_n+31}$ and Eqs.(78)

and (79), we have

$$h_{i+1} = 59h_i - h_{i-1} ; k_{i+1} = 59k_i - k_{i-1}. \quad (82)$$

The characteristic equation of Eq.(82) is $v^2 - 59v + 1 = 0$ with roots $v_1 = \frac{59+\sqrt{3477}}{2}$ and $v_2 = \frac{59-\sqrt{3477}}{2}$. The general solutions of Eq.(82) are

$$h_i = a_1 v_1^i + a_2 v_2^i ; k_i = b_1 v_1^i + b_2 v_2^i.$$

Using the initial conditions $h_0 = 0, k_0 = 1$ and $h_1 = 51, k_1 = 31$, yields

$$\begin{aligned} h_i &= \frac{17\sqrt{3477}}{1159} \left(\frac{59 + \sqrt{3477}}{2}\right)^i - \frac{17\sqrt{3477}}{1159} \left(\frac{59 - \sqrt{3477}}{2}\right)^i; \\ k_i &= \left(\frac{1159 + \sqrt{3477}}{2218}\right) \left(\frac{59 + \sqrt{3477}}{2}\right)^i \\ &\quad + \left(\frac{1159 - \sqrt{3477}}{2218}\right) \left(\frac{59 - \sqrt{3477}}{2}\right)^i. \end{aligned} \quad (83)$$

If $x_n = 1$, it means that R_n is without any electrically equivalent transformation. Plugging Eq.(83) into Eq.(81), we have

$$\begin{aligned} \tau(R_n) &= 3 \times 4^{n-1} x_1^2 \left[\left(\frac{47519 + 806\sqrt{3477}}{1159}\right) \left(\frac{59 + \sqrt{3477}}{2}\right)^{n-2} \right. \\ &\quad \left. + \left(\frac{47519 - 806\sqrt{3477}}{1159}\right) \left(\frac{59 - \sqrt{3477}}{2}\right)^{n-2} \right]^2 \end{aligned} \quad (84)$$

if $n \geq 2$. When $n = 1$, $\tau(R_1) = 3$ which satisfies Eq.(84). Therefore, the number of spanning trees in the sequence of the graph is given by

$$\begin{aligned} \tau(R_n) &= 3 \times 4^{n-1} x_1^2 \left[\left(\frac{47519 + 806\sqrt{3477}}{1159}\right) \left(\frac{59 + \sqrt{3477}}{2}\right)^{n-2} \right. \\ &\quad \left. + \left(\frac{47519 - 806\sqrt{3477}}{1159}\right) \left(\frac{59 - \sqrt{3477}}{2}\right)^{n-2} \right]^2 \end{aligned} \quad (85)$$

if $n \geq 2$, where

$$x_1 = \frac{\left(\frac{3479 + 59\sqrt{3477}}{2}\right)^{n-1} \left(\frac{99 + 2\sqrt{3477}}{111}\right) + \left(\frac{3 + \sqrt{3477}}{102}\right)}{\left(\frac{3479 + 59\sqrt{3477}}{2}\right)^{n-1} \left(\frac{2417 + 35\sqrt{3477}}{1258}\right) - 1}. \quad (86)$$

Inserting Eq.(86) into Eq.(85) we obtain the desired result. \square

§4. Numerical Results

The following two tables illustrate some the values of the number of spanning trees in the graphs G_n, H_n, L_n, M_n, Q_n and R_n .

n	$\tau(G_n)$	$\tau(H_n)$	$\tau(L_n)$
1	3	3	3
2	10800	1734	216
3	29128368	881292	20172
4	78529953792	447690264	1895064
5	211716289555200	227423130672	178054848
6	570785860162301952	115529159623776	16729574496

Table 1. Some values of $\tau(G)_n$, $\tau(H_n)$ and $\tau(L_n)$.

n	$\tau(M_n)$	$\tau(Q_n)$	$\tau(R_n)$
1	3	3	3
2	1452	8214	24300
3	1123632	18960588	338098368
4	871902912	43761009624	4704976258752
5	676578632448	101000334380592	65474444206252800
6	525011068136448	233108596706389344	91114229029 4589960192

Table 2. Some values of $\tau(M_n)$, $\tau(O_n)$ and $\tau(R_n)$

§5. Spanning Tree Entropy

After having explicit Formulas for the number of spanning trees of the sequence of the six graphs G_n , H_n , L_n , M_n , Q_n and R_n , we can calculate its spanning tree entropy Z which is a finite number and a very interesting quantity characterizing the network structure, defined in [16,17] to be

$$Z(G) = \lim_{n \rightarrow \infty} \frac{\ln \tau(G)}{|V(G)|}. \tag{87}$$

for a graph G . We have known that

$$\begin{aligned} Z(G_n) &= \frac{1}{6} \left(\ln [4] + 2 \ln [13 + 2\sqrt{42}] \right) = 1.316587045, \\ Z(T_n) &= \frac{1}{6} \left(\ln [2] - 2 \ln [127 + 48\sqrt{7}] + 2 \ln [2024 + 765\sqrt{7}] \right) \\ &= 1.038410991, \\ Z(L_n) &= \frac{1}{6} \left(\ln [8] - 2 \ln [47 + 21\sqrt{5}] + 2 \ln [161 + 72\sqrt{5}] \right) \\ &= 0.757140296, \end{aligned}$$

$$\begin{aligned}
Z(M_n) &= \frac{1}{6} \left(\ln [4] - 2 \ln [97 + 56\sqrt{3}] + 2 \ln [1351 + 780\sqrt{3}] \right) \\
&= 1.0109020991, \\
Z(Q_n) &= \frac{1}{6} \left(\ln [2] + 2 \ln [17 + 12\sqrt{2}] \right) = 1.0290689313, \\
Z(R_n) &= \frac{2}{9} \left(\ln [59 + \sqrt{3477}] \right) = 1.060088273.
\end{aligned}$$

Now we compare the value of entropy in our graphs with other graphs. It is clear that the entropy of the graph G_n is larger than the entropy of the graph Q_n of the same average degree 5. Also the entropy of the graph M_n of average degree 4 is larger than the entropies of the graphs H_n, R_n and L_n of the same average degree 4. In addition the entropies of the graphs M_n and H_n are larger than the entropy of the fractal scale free lattice [18] which has the entropy 1.040 and has the same average degree 4, while the entropies of the graphs L_n and R_n are smaller than fractal scale free lattice. Also the entropies of the graphs M_n, H_n, L_n and R_n are smaller than the entropy of the two dimensional Sierpinski gasket [19] which has the entropy 1.166 of the same average degree 4. Moreover, the entropies of the graphs H_n, L_n and R_n are smaller than entropy of the 3-prism graph [20] which has the entropy 1.0445 but the graph M_n has entropy larger than 3-prism graph.

§6. Conclusions

In this paper, we calculate the number of spanning trees in the sequences of some graphs generated by triangle graph using electrically equivalent transformations. The feature of this technique lies in the parry of strenuous computation of Laplacian spectra that is prerequisite for a generic method for determining spanning trees. In addition, our results have shown that the entropy is related to the average degree of the graph.

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