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Obstructions for
Connected Tree-width and Connected Path-width

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Abstract: Graph theory investigates networks of vertices and edges and provides powerful tools for analyzing their structural properties. A central approach to quantifying graph complexity is through *graph parameters*. A graph is *connected* if every pair of vertices is joined by a path, ensuring mutual reachability. Among these measures, *graph width parameters* capture structural complexity, typically defined via decompositions, separators, or connectivity restrictions. Understanding how such parameters behave under connectivity constraints is a well-established line of research. In this paper, we introduce and investigate connected obstructions, structural certificates that determine the values of connected tree-width and connected path-width.

Key Words: Connected tree-width, connected path-width, tree-width, bramble.

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§1. **Introduction**

1.1 Graphs and Applications

Graph theory studies networks of vertices and edges and analyzes their paths, structures, and properties [1]. Because graphs provide both visual and conceptual representations of relationships among real-world entities, they have become indispensable across many disciplines, including the natural and life sciences as well as the social sciences and engineering [2].

1.2 Graph Width Parameters

A *graph parameter* is a numerical invariant that assigns to each finite graph a value preserved under graph isomorphisms. A *graph width parameter* measures the structural complexity of graphs, typically through decompositions, separators, or connectivity restrictions.

A large body of research has investigated structural complexity via *graph parameters*. Among the most influential are the width-type parameters: tree-width [3], band-width [4, 5], hypertree-width [6], superhypertree-width [7], rank-width [8], branch-width [9], twin-width [10], mim-width [11], proper path-width [12], path-distance-width [13], and path-width [14, 15].

Two of the most widely studied width parameters are *treewidth* and *path-width*. Tree-width

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measures how close a graph is to a tree structure, controlling decomposition bag size, guiding efficient algorithms, and influencing parameterized complexity [16, 17]. Path-width measures how close a graph is to a path structure, restricting decomposition sequence width, enabling simplified dynamic programming, and informing structural graph analysis.

These parameters are central to algorithm design: many problems that are otherwise intractable become fixed-parameter tractable or solvable in polynomial time on classes of graphs of bounded width [18, 19]. Moreover, they play a key role in practical applications ranging from machine learning to network optimization [20, 21].

1.3 Connected Width Parameters

A graph is *connected* if every pair of vertices is joined by a path, ensuring mutual reachability [22]. Understanding how width parameters behave under explicit connectivity constraints is an established theme [23, 24]. In particular, tree-width and path-width have connected variants-*connected tree-width* [25] and *connected path-width* [26, 27] that have attracted considerable recent attention. Connected width parameters ensure decompositions remain cohesive across components, preserving structural integrity, improving interpretability, and enhancing algorithmic efficiency for graph analysis.

1.4 Obstructions for Width Parameters

For many width notions, *obstructions* serve as min-max certificates that characterize large width and guide algorithms [28, 3]. Classical examples include tangles [3, 29], blockages [30], brambles [29, 31], and ultrafilters [32]. Obstructions provide concise min-max characterizations, certify large width, guide algorithmic design, unify structural insights, and extend naturally to broader combinatorial and connectivity frameworks. These ideas extend beyond undirected graphs to directed graphs [33], matroids [9, 34], and general connectivity systems constrained by symmetric submodularity [32, 35], with implications for both algorithms [36] and game-theoretic viewpoints [37].

1.5 Our Contribution

From the above, research on graph-width parameters and their related topics is highly significant. However, studies on Connected Obstructions for width parameters have not yet been fully developed. This paper develops *connected obstructions* tailored to the connected variants of width. We introduce and investigate *connected brambles* for connected tree-width and *connected blockages* for connected path-width, providing tools to certify large values of these parameters and to clarify their structural behavior.

§2. Basic Notation

This section fixes the terminology used throughout the paper. Unless otherwise stated, all graphs considered in this paper are finite, undirected, and simple. Furthermore, the empty set is regarded as a subset of every set.

Definition 2.1(Undirected graph) *An undirected graph is a pair* $G = (V, E)$ *where* V *is a finite set whose elements are vertices and*

$$E \subseteq \{ \{u, v\} \subseteq V \mid u \neq v \}$$

is a finite set of unordered pairs of distinct vertices, called edges*. If* $\{u, v\} \in E$*, then* u *and* v *are* adjacent*, written* $u \sim v$*.*

Definition 2.2(Tree, leaf, node) *A tree is a connected, acyclic graph. If a tree has* $n := |V|$ *vertices, then it has exactly* $n - 1$ *edges. A leaf is a vertex of degree* 1*. In this paper, the terms* vertex *and* node *are synonymous, a node may be a leaf or an internal (nonleaf) vertex depending on its degree.*

Definition 2.3(Connectedness and components) *Let* $G = (V, E)$ *be a graph. A path from* u *to* v *is a finite sequence of vertices* $u = v_0, v_1, \cdots, v_k = v$ *such that* $\{v_i, v_{i+1}\} \in E$ *for all* i*. The graph* G *is connected if for every* $u, v \in V$ *there exists a path in* G *from* u *to* v*. If* G *is not connected, it is disconnected and decomposes into connected components, i.e., maximal connected subgraphs of* G*.*

We use the standard notion of a (vertex) *separation.*

Definition 2.4(Separation and order) *A separation of a graph* $G = (V, E)$ *is a pair* (A, B) *of vertex sets with*

$$A \cup B = V \qquad \text{and} \qquad E\big(G[A \setminus B], G[B \setminus A]\big) = \varnothing.$$

Its order is $|A \cap B|$*. We write* (B, A) *for the opposite orientation.*

Example 2.1(Separation and its order on a cycle) Let $G = C_4$ be the cycle $v_1 – v_2 – v_3 – v_4 – v_1$. Define

$$A = \{v_1, v_2, v_3\}, \qquad B = \{v_1, v_3, v_4\}.$$

Then $A \cup B = \{v_1, v_2, v_3, v_4\} = V(G)$. Moreover, there are no edges between $A \setminus B = \{v_2\}$ and $B \setminus A = \{v_4\}$. Thus (A, B) is a separation of G. Its order is

$$|A \cap B| = |\{v_1, v_3\}| = 2.$$

The opposite orientation is (B, A).

Definition 2.5(Induced subgraph, vertex deletion, and edge deletion) *Let* $G = (V(G), E(G))$ *be a finite undirected graph.*

(1) *For* $X \subseteq V(G)$*, the* induced subgraph *on* X *is*

$$G[X] := \big(X, \{\{u, v\} \in E(G) \mid u, v \in X\}\big).$$

(2) *For* $X \subseteq V(G)$*, the graph obtained by deleting* X *is*

$$G - X \;:=\; G[V(G) \setminus X].$$

(3) *For* $F \subseteq E(G)$*, the graph obtained by deleting the edges in* F *is*

$$G - F \;:=\; \bigl(V(G),\, E(G) \setminus F\bigr).$$

§3. **Obstructions for Connected Tree-Width**

In this section we recall tree-decompositions and connected tree-decompositions, and fix notation for subgraphs. Standard references include [3, 17, 38], for broader surveys see [16, 39].

Definition 3.1(Tree-decomposition and tree-width, [3]) *A tree-decomposition of* G *is a pair* $(T, (B_t)_{t \in V(T)})$ *where* T *is a tree and each* $B_t \subseteq V(G)$ *is a bag, such that*

(1) Vertex coverage: $V(G) = \bigcup_{t \in V(T)} B_t$*;*

(2) Edge coverage: *For every* $\{u, v\} \in E(G)$ *there exists* $t \in V(T)$ *with* $\{u, v\} \subseteq B_t$*;*

(3) Running intersection: *For every* $v \in V(G)$*, the index set* $T_v := \{\, t \in V(T) \mid v \in B_t \,\}$ *induces a connected subtree of* T*.*

The width of the decomposition is $\max_{t \in V(T)}(|B_t| - 1)$*. The tree-width of* G *is*

$$\mathrm{tw}(G) \;:=\; \min\{\, \max_{t \in V(T)} (|B_t| - 1) \;:\; (T, (B_t)) \textit{ is a tree-decomposition of } G \,\}.$$

Example 3.1(Tree-decomposition and tree-width of a cycle) Let $G = C_4$ with vertices v_1, v_2, v_3, v_4 and edges $v_1v_2, v_2v_3, v_3v_4, v_4v_1$. Take the tree T to be a single edge with nodes t_1–t_2, and define the bags

$$B_{t_1} = \{v_1, v_2, v_3\}, \qquad B_{t_2} = \{v_1, v_3, v_4\}.$$

The vertex coverage holds since $B_{t_1} \cup B_{t_2} = \{v_1, v_2, v_3, v_4\}$. Edge coverage holds: $v_1v_2, v_2v_3 \subseteq B_{t_1}$ and $v_3v_4, v_4v_1 \subseteq B_{t_2}$. For the running intersection, each vertex appears in a connected set of bags: v_1, v_3 appear in both B_{t_1}, B_{t_2}, while v_2 (resp. v_4) appears only in B_{t_1} (resp. B_{t_2}). Thus this is a tree-decomposition of width $\max(|B_{t_i}| - 1) = 3 - 1 = 2$. Since graphs of tree-width 1 are forests and C_4 contains a cycle, $tw(C_4) \geqslant 2$. Therefore $tw(C_4) = 2$.

Definition 3.2(Connected tree-decomposition and connected tree-width, [40]) *A connected tree-decomposition of* G *is a tree-decomposition* $(T, (B_t)_{t \in V(T)})$ *with the additional requirement*

$$\forall\, t \in V(T) \textit{ the induced subgraph } G[B_t] \textit{ is connected.}$$

Its width is again $\max_{t \in V(T)}(|B_t| - 1)$ *and the connected tree-width of* G *is*

$$\mathrm{ctw}(G) \;:=\; \min\{\, \max_{t \in V(T)} (|B_t| - 1) \;:\; (T, (B_t)) \textit{ is a connected tree-decomposition of } G \,\}.$$

Example 3.1(Connected tree-decomposition and connected tree-width of a path) Let $G = P_4$ be the path $v_1 - v_2 - v_3 - v_4$. Take a tree T that is a path on nodes $t_1 - t_2 - t_3$ and define bags

$$B_{t_1} = \{v_1, v_2\}, \qquad B_{t_2} = \{v_2, v_3\}, \qquad B_{t_3} = \{v_3, v_4\}.$$

Vertex and edge coverage hold, and for each vertex the index set of bags containing it is contiguous (e.g., v_2 appears exactly in B_{t_1}, B_{t_2}). Each bag induces a connected subgraph (an edge), so this is a *connected* tree-decomposition. The width is $\max_t(|B_t| - 1) = 2 - 1 = 1$. Hence, $ctw(P_4) = 1$.

Definition 3.3(Connected bramble) *A connected bramble in G is a family $\mathcal{B}$ of vertex sets $X \subseteq V(G)$ such that*

(1) Connectivity of elements: *$G[X]$ is connected for every $X \in \mathcal{B}$;*

(2) Touching: *For all $X, Y \in \mathcal{B}$, either $X \cap Y \neq \varnothing$ or there exists an edge $\{u, v\} \in E(G)$ with $u \in X$ and $v \in Y$.*

A set $S \subseteq V(G)$ hits *$\mathcal{B}$ if $S \cap X \neq \varnothing$ for all $X \in \mathcal{B}$, and the order of $\mathcal{B}$ is*

$$\operatorname{ord}(\mathcal{B}) \;:=\; \min\{\, |S| \mid S \subseteq V(G) \textit{ hits } \mathcal{B} \,\}.$$

Remark 3.1 *The condition (1) simply means each element $X \in \mathcal{B}$ induces a connected subgraph $G[X]$ and the condition (2) is the usual "touching" requirement; thus a connected bramble is the standard notion of a bramble with the connectivity of its elements emphasized explicitly.*

Example 3.2(A connected bramble of order 2 on a path) Again let $G = P_4$ with vertices $v_1 - v_2 - v_3 - v_4$. Consider

$$X_1 = \{v_1, v_2\}, \qquad X_2 = \{v_2, v_3\}, \qquad X_3 = \{v_3, v_4\}.$$

Each $G[X_i]$ is connected. Moreover, $X_1 \cap X_2 = \{v_2\} \neq \varnothing$ and $X_2 \cap X_3 = \{v_3\} \neq \varnothing$, while X_1 and X_3 touch via the edge $v_2 v_3$. Thus $\mathcal{B} := \{X_1, X_2, X_3\}$ is a connected bramble. Any single vertex fails to meet all three sets (e.g., v_2 misses X_3), but $\{v_2, v_3\}$ hits all. Hence $ord(\mathcal{B}) = 2$.

We establish a duality theorem for *connected* tree-width via the notions of connected partial $(< k)$-decompositions and k-flaps (cf. [14]). Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$

Definition 3.4(Connected partial $(< k)$-decomposition, cf. [41]) *A connected partial $(< k)$-decomposition of G is a pair (T, ℓ) consisting of a tree T and a labelling $\ell : V(T) \to 2^V$, satisfying with $U := \bigcup_{t \in V(T)} \ell(t)$,*

(1) Coverage inside U: *For every edge $\{u, v\} \in E(G[U])$ there exists $t \in V(T)$ with $\{u, v\} \subseteq \ell(t)$;*

(2) Running intersection: *For each $v \in U$, the set $\{t \in V(T) \mid v \in \ell(t)\}$ induces a connected subtree of T;*

(3) Bag-connectivity: *$G[\ell(t)]$ is connected for all $t \in V(T)$;*

(4) Size bound on internal bags: *If t is not a leaf of T, then $|\ell(t)| \leqslant k$;*

(5) Nontriviality: *At least one bag has size $\leqslant k$.*

We call U the covered vertex set of (T, ℓ). Note that U may be a proper subset of V.

Definition 3.5(k-flap, cf. [41]) *Let (T, ℓ) be a connected partial $(< k)$-decomposition of G. If x is a leaf of T with unique neighbor x', the set*

$$X \ := \ \ell(x) \setminus \ell(x')$$

is a k-flap provided $|\ell(x')| \leqslant k$. It is a connected k-flap if, in addition, $G[X]$ is connected.

Example 3.3(Connected partial $(< k)$-decomposition) Let $G = P_5$ be the path v_1–v_2–v_3–v_4–v_5 and fix $k = 2$. Take the tree T to be the path $t_1 - t_2 - t_3$ and label

$$\ell(t_1) = \{v_1, v_2\}, \qquad \ell(t_2) = \{v_2\}, \qquad \ell(t_3) = \{v_2, v_3\}.$$

The covered set is $U = \{v_1, v_2, v_3\} \subsetneq V(G)$. Inside U, each edge is covered ($v_1 v_2 \in \ell(t_1)$, $v_2 v_3 \in \ell(t_3)$); for every $w \in U$, the bags containing w form a connected subtree of T; each $G[\ell(t_i)]$ is connected; the unique internal node is t_2 with $|\ell(t_2)| = 1 \leqslant k$, and at least one bag has size $\leqslant k$. Hence (T, ℓ) is a connected partial $(< k)$-decomposition of G.

Example 3.4(k-flap) In the decomposition above, t_1 is a leaf with neighbor t_2. Then

$$X = \ell(t_1) \setminus \ell(t_2) = \{v_1, v_2\} \setminus \{v_2\} = \{v_1\}.$$

Since $|\ell(t_2)| = 1 \leqslant k$ and $G[X]$ is connected (a single vertex), X is a connected $k$flap. (Analogously, using the other leaf t_3 yields the flap $\{v_3\}$.)

We use the standard notion of *touching* for vertex sets: for $A, B \subseteq V$ we say that A and B *touch* if either $A \cap B \neq \varnothing$ or there exists an edge $\{u, v\} \in E$ with $u \in A$ and $v \in B$.

Lemma 3.1(Gluing along a nested neighborhood, cf. [41]) *Let (T_X, ℓ^X) and (T_Y, ℓ^Y) be connected partial $(< k)$-decompositions of a connected graph G. Let X (resp. Y) be a connected k-flap of (T_X, ℓ^X) (resp. (T_Y, ℓ^Y)). If $S := N_G(X) \subseteq N_G(Y)$, then identifying the leaves that carry X and Y, and relabelling the identified leaf by S, yields a connected partial $(< k)$-decomposition of G.*

A Sketch of Proof Relabelling the two leaf bags by S preserves bag-connectivity and the running-intersection property (attach S along the unique paths from the old leaves to bags containing its vertices). Internal bags remain of size $\leqslant k$, and coverage inside the new U is unchanged. The operation merely replaces the two flaps by the small separator S. ☐

Lemma 3.2(Separating non-touching flaps) *Let G be connected and let X and Y be connected k-flaps of connected partial $(< k)$-decompositions (T_X, ℓ^X) and (T_Y, ℓ^Y), respectively. If X and Y do not touch, then there exists a connected partial $(< k)$-decomposition (T, ℓ) of G whose connected k-flaps are contained in those of (T_X, ℓ^X) and (T_Y, ℓ^Y) but exclude X and Y.*

Proof Since X and Y do not touch, there is a vertex separator $S \subseteq V(G)$ such that no component of $G - S$ meets both X and Y. Choose S of minimum size. By Menger's theorem, $|S| \leqslant |N_G(X)| \leqslant k$. Let A be the union of S and all components of $G - S$ that meet X, and set $B := (V(G) \setminus A) \cup S$. Then $X \subseteq A \setminus S$ and $Y \subseteq B \setminus S$, with $A \cap B = S$.

Trim (T_X, ℓ^X) to $G[B]$ by setting, for each $t \in V(T_X)$,

$$\begin{aligned} \ell^{X'}(t) \ : \ &= \ (\ell^X(t) \bigcap B) \\ &\bigcup \{\, s \in S \mid t \text{ lies on the unique path from the } X\text{-leaf to some node containing } s \,\}. \end{aligned}$$

The bag-connectivity and the running-intersection property are preserved; internal bags stay of size $\leqslant k$ (new vertices are added only along paths while at least one vertex of the corresponding $X \to S$ path is removed from the bag), and the trimmed family covers all edges of $G[B]$. Thus $(T_X, \ell^{X'})$ is a connected partial $(< k)$-decomposition of $G[B]$ whose connected k-flaps are among those of (T_X, ℓ^X) but X is replaced by the small leaf S. Symmetrically trim (T_Y, ℓ^Y) to $G[A]$. Finally, glue the two trimmed decompositions along the common small leaf S to obtain (T, ℓ) as required. □

Using the above Lemma, we can prove the following theorem.

Theorem 3.1 *For $k \in \mathbb{N}$ and a graph G, the following are equivalent:*

$$ctw(G) \ \geqslant k \quad \Longleftrightarrow \quad G \textit{ contains a connected bramble of order} > k.$$

Proof $\Rightarrow$. Assume $ctw(G) \geqslant k$. Among all connected partial $(< k)$-decompositions of G, choose one whose set $\mathcal{F}$ of connected k-flaps is inclusion-minimal. By Lemma 3.2, any two members of $\mathcal{F}$ must touch (else we could strictly reduce the flap set). Hence, $\mathcal{F}$ is a family of pairwise touching connected sets.

We claim that $ord(\mathcal{F}) > k$. Suppose towards a contradiction that some $S \subseteq V(G)$ with $|S| \leqslant k$ hits $\mathcal{F}$. Gluing repeatedly along S (Lemma 3.1) collapses all large leaves into the small set S, producing a connected partial $(< k)$-decomposition that covers more vertices (eventually all of $V(G)$). This yields a *connected* tree-decomposition of width $\leqslant k$, contradicting $ctw(G) \geqslant k$. Therefore, $ord(\mathcal{F}) > k$, and $\mathcal{F}$ is the desired connected bramble.

$\Leftarrow$. Assume G contains a connected bramble $\mathcal{B}$ of order $> k$. Let $(T, (B_t)_{t \in V(T)})$ be any connected tree-decomposition of G. For each edge $e = t_1 t_2 \in E(T)$, deleting e splits T into components T_1, T_2 with vertex unions $U_i := \bigcup_{t \in V(T_i)} B_t$ $(i = 1, 2)$ and separator $X_e := B_{t_1} \cap B_{t_2}$. If X_e hits $\mathcal{B}$, then $|X_e| > k$ and the width is $> k$. Otherwise, some $B \in \mathcal{B}$ is disjoint from X_e; since $\mathcal{B}$ consists of connected sets and bags cover all edges, $B \subseteq U_1$ or $B \subseteq U_2$. In this case, orient e towards that side.

Orient every edge of T by this rule. A sink node t^* (which exists in any finite oriented tree) has the property that its bag B_{t^*} hits $\mathcal{B}$: if some $B \in \mathcal{B}$ were disjoint from B_{t^*}, then the unique edge of T on the path from t^* towards the side containing B would point away from t^*, contradicting sinkness. Hence $|B_{t^*}| \geqslant ord(\mathcal{B}) > k$, so the width of the decomposition is $> k$. As the decomposition was arbitrary, $ctw(G) \geqslant k$. □

§4. **Obstructions for Connected Path-Width**

We study the relationship between (connected) path-width and a connected analogue of blockage. We first recall path decompositions and connected path decompositions, see, e.g., [26].

Definition 4.1(Path decomposition and (connected) path-width, cf. [26]) *Let* $G = (V(G), E(G))$ *be a finite simple graph. For* $X \subseteq V(G)$ *write*

$$G[X] \ := \ \big(X,\ \{\{u,v\} \in E(G) \mid u,v \in X\}\big).$$

A path decomposition of G *is a finite sequence of bags*

$$P \ = \ (X_1, \cdots, X_m) \qquad (X_i \subseteq V(G)),$$

such that

(1) Vertex coverage: $\bigcup_{i=1}^{m} X_i = V(G)$*;*

(2) Edge coverage: *For every* $\{u,v\} \in E(G)$ *there exists* i *with* $\{u,v\} \subseteq X_i$*;*

(3) Running intersection: *For all* $1 \leqslant i \leqslant j \leqslant k \leqslant m$*,* $X_i \cap X_k \subseteq X_j$ *(equivalently: for each* $v \in V(G)$*, the set* $\{i \mid v \in X_i\}$ *is an interval of* $\{1, \cdots, m\}$*).*

The width of P *is* $\max_i(|X_i| - 1)$*. The path-width of* G *is*

$$pw(G) \ := \ \min\{\ \max_i(|X_i| - 1)\ :\ P \text{ a path decomposition of } G\ \}.$$

A path decomposition $P = (X_1, \cdots, X_m)$ *is connected if*

$$G[X_1 \cup \cdots \cup X_i] \ \text{ is connected for every } i \in \{1, \cdots, m\}.$$

The connected path-width of G *is*

$$cpw(G) \ := \ \min\{\ \max_i(|X_i| - 1)\ :\ P \text{ a connected path decomposition of } G\ \}.$$

Example 4.1(Connected partial ($< k$)-decomposition on a path) Let $G = P_5$ be the path $v_1 - v_2 - v_3 - v_4 - v_5$ and fix $k = 2$. Take the tree T with nodes $t_1 - t_2 - t_3$ (a path), and label

$$\ell(t_1) = \{v_1, v_2\}, \qquad \ell(t_2) = \{v_2\}, \qquad \ell(t_3) = \{v_2, v_3\}.$$

The covered set is $U = \{v_1, v_2, v_3\}$. Coverage inside U holds since the edges v_1v_2 and v_2v_3 lie in $\ell(t_1)$ and $\ell(t_3)$, respectively. For each vertex, the bags containing it form a connected subtree of T (v_2 appears in all three nodes; v_1 only in t_1; v_3 only in t_3). Thus, each $G[\ell(t_i)]$ is connected. The unique internal node is t_2 and $|\ell(t_2)| = 1 \leqslant k$; at least one bag has size $\leqslant k$. Thus (T, ℓ) is a connected partial ($< k$)-decomposition of G that covers $U \subsetneq V(G)$.

Example 4.2((Connected) path-width of a star) Let $G = K_{1,3}$ with center c and leaves a, b, d.

Define the path decomposition $P = (X_1, X_2, X_3)$ by

$$X_1 = \{c, a\}, \qquad X_2 = \{c, b\}, \qquad X_3 = \{c, d\}.$$

The vertex/edge coverage is immediate, and each vertex appears in a contiguous block of bags. Every prefix union is connected, $G[X_1] = G[\{c, a\}]$, $G[X_1 \cup X_2] = G[\{c, a, b\}]$, and $G[X_1 \cup X_2 \cup X_3] = G$. Hence, P is a *connected* path decomposition of width $\max_i(|X_i| - 1) = 2 - 1 = 1$, so $pw(G) = cpw(G) = 1$.

We next introduce a connected version of blockage tailored to connected path-width.

Definition 4.2(Attachment and connected blockage) *For $X \subseteq V(G)$ define the attachment of X in G by*

$$att_G(X) \ := \ \{\, x \in X \mid \exists\, y \in V(G) \setminus X \ with \ \{x, y\} \in E(G) \,\},$$

and set $\alpha_G(X) := |att_G(X)|$. The connected complement of X is

$$X^{\mathsf{C}} \ := \ \bigl(V(G) \setminus X\bigr) \ \cup \ att_G(X).$$

A connected blockage of order k in G is a family $\mathcal{B} \subseteq 2^{V(G)}$ satisfying

(1) Size bound: *For every $X \in \mathcal{B}$, $\alpha_G(X) \leqslant k$;*

(2) Heredity (downward, under small attachment): *If $X \in \mathcal{B}$ and $Y \subseteq X$ with $\alpha_G(Y) \leqslant k$, then $Y \in \mathcal{B}$;*

(3) Complementarity: *If $X_1, X_2 \subseteq V(G)$ are* complementary *in the sense that $X_1^{\mathsf{C}} \subseteq X_2$ or $X_2^{\mathsf{C}} \subseteq X_1$, and $|X_1 \cap X_2| \leqslant k$, then exactly one of X_1, X_2 lies in $\mathcal{B}$;*

(4) Connectivity: *Each $X \in \mathcal{B}$ induces a connected subgraph $G[X]$.*

Example 4.3(Connected blockage of order 1 on a path) Let $G = P_4$ be $v_1 - v_2 - v_3 - v_4$. For $X \subseteq V$ let $att_G(X) = \{x \in X : \exists\, y \notin X \text{ with } xy \in E(G)\}$. Set

$$\mathcal{B} = \{X_1, X_2, X_3\}, \quad X_1 = \{v_1\},\ X_2 = \{v_1, v_2\},\ X_3 = \{v_1, v_2, v_3\}.$$

Each $G[X_i]$ is connected. Moreover $att_G(X_1) = \{v_1\}$, $att_G(X_2) = \{v_2\}$, $att_G(X_3) = \{v_3\}$, so $|att_G(X_i)| = 1 \leqslant 1$ (size bound for order 1). Heredity holds within the chain $X_1 \subset X_2 \subset X_3$. For each i, the connected complement $X_i^{\mathsf{C}} = (V \setminus X_i) \cup att_G(X_i)$ satisfies $|X_i \cap X_i^{\mathsf{C}}| = |att_G(X_i)| = 1$, and exactly one of the complementary pair $\{X_i, X_i^{\mathsf{C}}\}$ is included in $\mathcal{B}$ (namely X_i). Thus $\mathcal{B}$ is a connected blockage of order 1 in P_4.

Lemma 4.1(Prefix attachments are small) *Let $P = (X_1, \cdots, X_m)$ be a connected path decomposition of width at most $k - 1$. For $i \in \{1, \cdots, m-1\}$ put $W_i := X_1 \cup \cdots \cup X_i$ and $S_i := X_i \cap X_{i+1}$. Then,*

$$att_G(W_i) \ \subseteq \ S_i \qquad and \qquad |att_G(W_i)| \ \leqslant \ |S_i| \ \leqslant \ k.$$

Proof Take $x \in att_G(W_i)$. Then $x \in W_i$ and there is $y \notin W_i$ with $\{x, y\} \in E(G)$. Let j be the minimal index with $y \in X_j$. Then $j \geqslant i + 1$. Since $\{x, y\}$ is covered by some bag and the running-intersection property holds, we must have $x \in X_j$ as well. As $x \in W_i$, the interval property for appearances of x forces $x \in X_i \cap X_{i+1} = S_i$. Hence $att_G(W_i) \subseteq S_i$, and $|S_i| \leqslant \min(|X_i|, |X_{i+1}|) \leqslant k$. ☐

Lemma 4.2(Prefix pairs are complementary) *With W_i as in Lemma* 4.1*, the pair (W_i, W_i^{C}) is complementary and $|W_i \cap W_i^C| = \alpha_G(W_i) \leqslant k$.*

Proof By definition, $W_i^C = (V(G) \setminus W_i) \cup att_G(W_i)$, so $W_i^C \subseteq W_i^C$ and thus $W_i^C \subseteq W_i^C$ witnesses complementarity of (W_i, W_i^C). Moreover, $W_i \cap W_i^C = att_G(W_i)$, which has size at most k by Lemma 4.1. ☐

We can now state the obstruction theorem.

Theorem 4.1 *If $cpw(G) \leqslant k - 1$, then G admits no connected blockage of order k.*

Proof Suppose to the contrary that $cpw(G) \leqslant k-1$ and $\mathcal{B}$ is a connected blockage of order k. Fix a connected path decomposition $P = (X_1, \cdots, X_m)$ of width at most $k - 1$ and write $W_i := X_1 \cup \cdots \cup X_i$ for $i = 1, \cdots, m$. By connectedness of P, each $G[W_i]$ is connected.

For each $i \in \{1, \cdots, m - 1\}$, Lemma 4.2 shows that (W_i, W_i^C) is a complementary pair with $|W_i \cap W_i^C| \leqslant k$. The complementarity axiom of $\mathcal{B}$ thus forces, for each such i, *exactly one* of W_i and W_i^C to lie in $\mathcal{B}$.

Define $I := \{\, i \in \{1, \ldots, m-1\} \mid W_i \in \mathcal{B} \,\}$. The heredity axiom implies that I is *downward closed*: if $i \in I$ and $1 \leqslant j \leqslant i$, then $W_j \subseteq W_i$, $G[W_j]$ is connected and $\alpha_G(W_j) \leqslant k$ by Lemma 4.1, hence $j \in I$.

Observe that $m \notin I$: if $W_m = V(G) \in \mathcal{B}$, then heredity (applied to $Y = \emptyset$ with $\alpha_G(\emptyset) = 0$) would force $\emptyset \in \mathcal{B}$, contradicting the connectivity requirement on members of $\mathcal{B}$. Thus, I is a proper (possibly empty) initial segment of $\{1, \ldots, m - 1\}$. Let $i^* := \max I$ if $I \neq \emptyset$; if $I = \emptyset$ set $i^* := 0$.

We now reach a contradiction in either case.

Case 1. $I = \emptyset$.

Then, for every $i \in \{1, \cdots, m - 1\}$ the complementarity axiom forces $W_i^C \in \mathcal{B}$. Pick $i = 1$. Since G is connected (as P is a connected path decomposition), $W_1 = X_1 \neq \emptyset$, and $W_1^C = (V \setminus X_1) \cup att_G(X_1)$ induces a subgraph that has at least two components (the part outside X_1 and the vertices of $att_G(X_1)$ that lie in X_1), contradicting the connectivity axiom of $\mathcal{B}$. Hence $I \neq \emptyset$.

Case 2. $I \neq \varnothing$.

Let $i^* = \max I$. Then, $W_{i^*} \in \mathcal{B}$ but $W_{i^*+1} \notin \mathcal{B}$. By Lemma 4.2, $(W_{i^*+1}, W_{i^*+1}^C)$ is a complementary pair with $|W_{i^*+1} \cap W_{i^*+1}^C| \leqslant k$, hence $W_{i^*+1}^C \in \mathcal{B}$. Now consider the pair

$$A := W_{i^*} \qquad \text{and} \qquad B := W_{i^*+1}^C.$$

We have $A \subsetneq W_{i^*+1}$ and $B \subseteq V \setminus W_{i^*} \cup (X_{i^*} \cap X_{i^*+1})$. A direct verification using Lemma 4.1 shows that

$$A^C \subseteq B \qquad \text{and} \qquad |A \cap B| = |att_G(A)| \leqslant k,$$

so A, B form a complementary pair of order at most k. By the complementarity axiom, exactly one of A, B may lie in $\mathcal{B}$, but we already have $A = W_{i^*} \in \mathcal{B}$ and $B = W_{i^*+1}^C \in \mathcal{B}$, a contradiction.

Thus no connected blockage of order k can exist when $cpw(G) \leqslant k-1$. □

§5. Connected Tangles and Their Relation to Connected Brambles

This section develops a *connected* version of tangles and proves a duality with connected brambles. We shall work with the (equivalent) *haven* formulation of tangles, adapted to enforce connectedness of the "large side" chosen by the tangle.

Definition 5.1(Connected haven and connected tangle (order $k+1$)) *Let $k \in \mathbb{N}$. A* connected haven of order $k+1$ *on G is a map*

$$\beta : \{ S \subseteq V(G) \mid |S| \leqslant k \} \longrightarrow \{\textit{vertex sets of components of } G - S\}$$

such that

(1) Connectivity: *For every S with $|S| \leqslant k$, the subgraph $G[\beta(S)]$ is a (nonempty) connected component of $G - S$;*

(2) Monotonicity: *If $S \subseteq T$ and $|T| \leqslant k$, then $\beta(T) \subseteq \beta(S)$.*

Such a haven induces an orientation T of all separations of order at most k by the rule

$$(A, B) \in T \iff \beta(A \cap B) \subseteq B.$$

We call any orientation T obtained this way a connected tangle of order $k+1$.

Example 5.1(A connected haven and its induced connected tangle on a path) Let $G = P_4$ with vertices $v_1 - v_2 - v_3 - v_4$ and fix $k = 1$ (haven of order $k+1 = 2$). Define β on all $S \subseteq V(G)$ with $|S| \leqslant 1$ by

$$\beta(\emptyset) = \{v_1, v_2, v_3, v_4\}, \quad \beta(\{v_1\}) = \{v_2, v_3, v_4\}, \quad \beta(\{v_2\}) = \{v_3, v_4\},$$

$$\beta(\{v_3\}) = \{v_4\}, \quad \beta(\{v_4\}) = \{v_1, v_2, v_3\}.$$

Each $\beta(S)$ is the vertex set of a (nonempty) component of $G - S$ and hence induces a connected subgraph. Monotonicity holds since the only proper inclusion with $|T| \leqslant 1$ is $\emptyset \subset \{v_i\}$, and $\beta(\{v_i\}) \subseteq \beta(\emptyset)$.

This haven induces an orientation T of all separations of order at most 1 by $(A, B) \in T \iff \beta(A \cap B) \subseteq B$. For example, for the separation (A, B) with $A = \{v_1, v_2, v_3\}$, $B = \{v_3, v_4\}$ (order 1, separator $\{v_3\}$), we have $\beta(A \cap B) = \beta(\{v_3\}) = \{v_4\} \subseteq B$; hence (A, B) is oriented toward B. Thus β is a connected haven of order 2 on P_4, and T is the corresponding connected tangle.

We now prove the connected tangle-connected bramble duality (parameter shift $k{+}1 \leftrightarrow {>}k$), which parallels the classical tangle-bramble correspondence.

Theorem 5.1(Connected tangle-connected bramble duality) *Let $k \in \mathbb{N}$ and G be a graph. The following are equivalent.*

(1) *G admits a connected tangle of order $k{+}1$ (equivalently, a connected haven β of order $k{+}1$);*

(2) *G contains a connected bramble of order $> k$.*

Proof (1) $\Rightarrow$ (2). Let β be a connected haven of order $k{+}1$. Define

$$\mathcal{B}_\beta \;:=\; \bigl\{\, \beta(S) \;\bigm|\; S \subseteq V(G),\ |S| \leqslant k \,\bigr\}.$$

Each $\beta(S)$ is connected by definition, so $\mathcal{B}_\beta$ consists of connected sets.

Pairwise touching. Let $S_1, S_2 \subseteq V(G)$ with $|S_i| \leqslant k$. By monotonicity,

$$\beta(S_1 \cup S_2) \;\subseteq\; \beta(S_1) \;\cap\; \beta(S_2).$$

Since $\beta(S_1 \cup S_2)$ is a (nonempty) component of $G-(S_1 \cup S_2)$, the intersection $\beta(S_1) \cap \beta(S_2)$ is nonempty. Hence any two members of $\mathcal{B}_\beta$ meet (and thus touch).

Order $> k$. Let $S \subseteq V(G)$ with $|S| \leqslant k$. Because $\beta(S) \subseteq V(G) \setminus S$ is a component of $G - S$, we have

$$S \cap \beta(S) \;=\; \varnothing.$$

Therefore S does *not* hit $\mathcal{B}_\beta$. Since this holds for every S of size at most k, any hitting set must have size at least $k{+}1$, i.e., $ord(\mathcal{B}_\beta) > k$. Thus $\mathcal{B}_\beta$ is a connected bramble of order $> k$.

(2) $\Rightarrow$ (1). Let $\mathcal{B}$ be a connected bramble with $ord(\mathcal{B}) > k$. For any $S \subseteq V(G)$ with $|S| \leqslant k$, every set $X \in \mathcal{B}$ lies in some component of $G - S$; since $|S| < ord(\mathcal{B})$, no single vertex set S hits $\mathcal{B}$, hence at least one component of $G - S$ meets *all* members of $\mathcal{B}$. Because sets in $\mathcal{B}$ are pairwise touching, such a component is unique: if two distinct components of $G - S$ both met all of $\mathcal{B}$, then taking $X \in \mathcal{B}$ meeting the first and $Y \in \mathcal{B}$ meeting the second would violate touching. Define $\beta(S)$ to be the vertex set of this unique component of $G - S$ that meets every member of $\mathcal{B}$.

By construction, $G[\beta(S)]$ is a (nonempty) connected component of $G - S$. If $S \subseteq T$ with $|T| \leqslant k$, then $G - T$ is obtained from $G - S$ by deleting additional vertices, so the component $\beta(T)$ of $G - T$ must lie inside the component $\beta(S)$ of $G - S$; hence $\beta(T) \subseteq \beta(S)$. Thus β is a connected haven of order $k{+}1$.

Finally, orient every separation (A, B) of order at most k toward the side containing $\beta(A \cap B)$, i.e., $(A, B) \in T \iff \beta(A \cap B) \subseteq B$. This T is a connected tangle of order $k{+}1$ witnessed by β. $\square$

Remark 5.1(Parameter shift and connection to connected tree-width) *The equivalence in Theorem* 5.1 *uses the standard shift: order $k{+}1$ tangles correspond to brambles of order $> k$.*

Combined with the connected bramble-connected tree-width duality proved earlier,

$$ctw(G) \geqslant k \iff G \textit{ has a connected bramble of order } > k-1$$

$$\iff G \textit{ has a connected tangle of order } k.$$

§6. Conclusion

In this work, we have examined the possibility of defining *connected obstructions* that characterize the values of connected tree-width and connected path-width.

Future research will further explore tree-width, path-width, and related width parameters in the context of biconnected graphs [42, 43] and triconnected graphs [44, 45]. In particular, following [46], we will study the tree-width of biconnected graphs under the name *biconnected tree-width*, along with corresponding notions such as *biconnected path-width* and *biconnected bramble*. Analogously, we plan to investigate *triconnected tree-width* and *triconnected path-width*. We also aim to extend tree-width, path-width, and other width parameters to bidirected acyclic graphs (BAGs) [47, 48], which generalize directed acyclic graphs (DAGs). Specifically, we will investigate *bag-tree-width* and *bag-path-width*, in analogy with the existing notions of *dag-tree-width* [49, 50] and *dag-path-width* [51].

Finally, we intend to generalize the graph-parameter concepts developed in this paper to two broader settings:

(*i*) uncertainty-aware graph models, including fuzzy graphs [52], intuitionistic fuzzy graphs [53], neutrosophic graphs [54, 55], quadripartitioned neutrosophic graphs [56], pentapartitioned neutrosophic graphs [57], and plithogenic graphs [58].

(*ii*) hierarchical graph models such as hypergraphs [59] and superhypergraphs [60, 61]. The study of width parameters within these extended frameworks promises new insights into structural graph theory under uncertainty and hierarchical modeling.

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