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Odd Vertex Equitable Even Labeling of Duplication and Product Graphs

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Abstract: Let G be a graph with p vertices and q edges and $A = \{1, 3, \cdots, q\}$ if q is odd or $A = \{1, 3, \cdots, q+1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \cdots, 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph. In this paper, we find some new results on odd vertex equitable even labeling and establish that some standard graphs admit odd vertex equitable even labeling.

Key Words: Vertex equitable labeling, odd vertex equitable even labeling, odd vertex equitable even graph, Smarandachely k-vertex equitable labeling, Smarandachely odd k-vertex equitable even labeling.

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§1. Introduction

All graphs considered here are simple, finite and undirected. Let $G(V, E)$ be a graph with p vertices and q edges. We follow the basic notations and terminology of graph theory as in [2]. We denote the vertex set and edge set of a graph by $V(G)$ and $E(G)$ respectively. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. A pioneering paper on graph labeling problems was published in 1967 by Rosa [9]. Over the last five decades, many types of graph labeling techniques have been introduced and studied by several authors. All these graph labeling techniques are beautifully classified and updated in his survey by Gallian [1]. Vertex equitable labeling, introduced by Lourdusamy and Seenivasan [7] is one among the labelings and it is classified under miscellaneous labelings in Gallian survey.

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Motivated by the concept of vertex equitable labeling Jeyanthi, Maheswari and Vijayalakshmi [3] extended this concept and introduced a new labeling called odd vertex equitable even labeling and proved that the graphs path, $P_n \odot P_m (n, m \geq 1)$, $K_{1,n} \cup K_{1,n-2} (n \geq 3)$, $K_{2,n}$, Tp-tree, cycle $C_n (n \equiv 0 \text{ or } 1 (\bmod 4))$, quadrilateral snake Q_n, ladder L_n, $L_n \odot K_1$, arbitrary super subdivision of any path P_n, are odd vertex equitable even graphs. In addition, they established that if every edge of a graph G is an edge of a triangle, then G is not an odd vertex equitable even graph. Further more, the same authors gave odd vertex equitable even labelings for cycle snake related families of graphs in [4], ladder related families of graphs in [5] and cycle related families of graphs in [6]. Lourdusamy et al. gave odd vertex equitable even labelings for quadrilateral snake related families of graphs in [8].

It is yet another study on odd vertex equitable even labeling with the objective to find some new results on odd vertex equitable even labeling. The following definitions are useful for the present study.

Definition 1.1([7]) *Let G be a graph with p vertices and q edges and $A = \{0, 1, 2, \cdots, \lceil \frac{q}{2} \rceil\}$. A graph G is said to be vertex equitable if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $1, 2, 3, \cdots, q$, where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. The vertex labeling f is known as vertex equitable labeling. A graph G is said to be a vertex equitable if it admits vertex equitable labeling.*

Generally, a vertex equitable labeling f is said to be Smarandachely k-vertex equitable labeling for any integer $k \leq |G|$ if there is a vertex subset $V' \subset V(G)$ with $|V'| = k$ with $|v_f(a) - v_f(b)| \leq 1$ in G.

Definition 1.2([3]) *A graph G with p vertices and q a edges and $A = \{1, 3, \cdots, q\}$ if q is odd or $A = \{1, 3, \cdots, q + 1\}$ if q is even. A graph G is said to admit an odd vertex equitable even labeling if there exists a vertex labeling $f : V(G) \to A$ that induces an edge labeling f^* defined by $f^*(uv) = f(u) + f(v)$ for all edges uv such that for all a and b in A, $|v_f(a) - v_f(b)| \leq 1$ and the induced edge labels are $2, 4, \cdots, 2q$ where $v_f(a)$ be the number of vertices v with $f(v) = a$ for $a \in A$. A graph that admits an odd vertex equitable even labeling is called an odd vertex equitable even graph.*

Generally, an odd vertex equitable even labeling f is said to be Smarandachely odd k-vertex equitable even labeling for an integer $k \leq |G|$ if there is a vertex subset $V' \subset V(G)$ with $|V'| = k$ with $|v_f(a) - v_f(b)| \leq 1$ in G.

Clearly, if $k = |V(G)|$, a Smarandachely k-vertex equitable labeling or a Smarandachely odd k-vertex equitable even labeling f is noting else but the vertex equitable labeling of G, a Smarandachely 0-vertex equitable labeling and a Smarandachely odd 0-vertex equitable even labeling f are respectively the vertex equitable labeling or odd vertex equitable even labeling f with $|v_f(a) - v_f(b)| \geq 2$, and for integer $k' \leq k$, a Smarandache k-vertex equitable labeling f is also a Smarandache k'-vertex equitable labeling, a Smarandachely odd k-vertex equitable even labeling of G is odd vertex equitable even labeling by definition.

Definition 1.3 *The direct product of G and H is the graph denoted $G \times H$ whose vertex* *set is $V(G) \times V(H)$ and for which vertices (g,h) and $(g^{'},h^{'})$ are adjacent precisely if $gg^{'} \in E(G)$ and $hh^{'} \in E(H)$. Then $V(G \times H) = \{(g,h)/g \in V(G)$ and $h \in V(H)\}$, $E(G \times H) = \{(g,h),(g^{'},h^{'})/gg^{'} \in E(G)$ and $hh^{'} \in E(H)\}$.*

Notice that the graph $P_m \times P_n$ is a disconnected graph with two components.

Definition 1.4 *The graph $P_n \times P_2$ is called a ladder graph.*

Definition 1.5 *Let G be a graph and v be any vertex of G. A new vertex $v^{'}$ is said to be duplication of v if all the vertices which are adjacent to v are adjacent to $v^{'}$. The graph obtained by duplication v is denoted by $D(G,v^{'})$.*

Definition 1.6 *A quadrilateral snake Q_n is a graph obtained from a path P_n with vertices $u_1,u_2,\cdots,u_n$ by joining u_i,u_{i+1} to the new vertices v_i,w_i respectively and then joining v_i and w_i. That is, every edge of the path is replaced by a cycle C_4.*

Definition 1.7 *The double quadrilateral snake $D(Q_n)$ is a graph obtained from a path P_n with vertices $u_1,u_2,\cdots,u_n$ by joining u_i and u_{i+1} to the new vertices v_i,x_i and w_i,y_i respectively and then joining v_i,w_i and x_i,y_i for $i=1,2,\cdots,n-1$.*

Definition 1.8 *The subdivision graph S(G) is obtained from G by subdividing each edge of G with a vertex.*

§2. Main Results

In this section, we prove that the graphs $D(L_n,v_1^{'})$, $D(SL_n,v_1^{'})$, $D(Q_n,u_1^{'})$, $D(D(Q_n),v_1^{'})$, $D(S(Q_n),u_1^{''})$ and $P_3 \times P_n$ (n *is odd*, $n \geq 3$) are odd vertex equitable even graphs.

Theorem 2.1 *The duplicate graph $D(L_n,v_1^{'})$ of a ladder L_n is an odd vertex equitable even graph.*

Proof Let the vertex set of $D(L_n,v_1^{'})$ be $\{u_1,u_2,\cdots,u_n,v_1,v_2,\cdots,v_n,v_1^{'}\}$ and the edge set of $D(L_n,v_1^{'})$ be $\{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_iv_i/1 \leq i \leq n\} \cup \{v_1^{'}v_2, v_1^{'}u_1\}$. Clearly, $D(L_n,v_1^{'})$ has $2n+1$ vertices and $3n$ edges.

Let $A = \begin{cases} 1,3,5,\cdots,3n \text{ if n is odd} \\ 1,3,5,\cdots,3n+1 \text{ if n is even.} \end{cases}$

Define a vertex labeling $f : V(D(L_n,v_1^{'})) \to A$ as follows:

$f(u_1) = 1,\ f(v_1^{'}) = 1,$

$$f(u_i) = \begin{cases} 3i-1 & ;1 \leq i \leq n, \quad i \text{ is even} \\ 3i & ;3 \leq i \leq n, \quad i \text{ is odd} \end{cases} \quad \text{and}$$

$$f(v_i) = \begin{cases} 3i+1 & ;1 \leq i \leq n, \quad i \text{ is even} \\ 3i & ;1 \leq i \leq n, \quad i \text{ is odd.} \end{cases}$$

It can be verified that the induced edge labels of $D(L_n, v_1^{'})$ are $2, 4, \cdots, 6n$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(L_n, v_1^{'})$. Thus $D(L_n, v_1^{'})$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(L_4, v_1^{'})$ is shown in Figure 1.

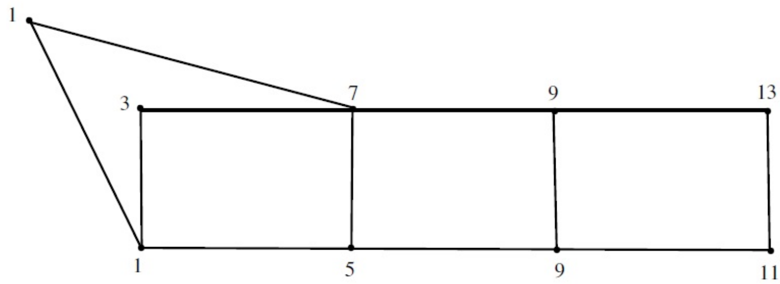

Figure 1. Odd vertex equitable even labeling of $D(L_4, v_1^{'})$

Theorem 2.2 *The duplicate graph $D(SL_n, v_1^{'})$ of a subdivision SL_n is an odd vertex equitable even graph.*

Proof Let the vertex set of $D(SL_n, v_1^{'})$ be $\{u_1, u_2, \cdots, u_n, v_1, v_2 \cdots, v_n\} \cup \{u_1^{'}, u_2^{'}, \cdots, u_{n-1}^{'}\} \cup \{w_1^{'}, w_2^{'}, \cdots, w_n^{'}\} \bigcup \{v_1^{''}, v_2^{''}, \cdots, v_{n-1}^{''}\} \cup \{v_1^{'}\}$ and the edge set of $D(SL_n, v_1^{'})$ be $\{u_i u_i^{'}, v_i v_i^{''}/1 \leq i \leq n-1\} \cup \{u_i^{'} u_{i+1}, v_i^{''} v_{i+1}/1 \leq i \leq n-1\} \cup \{u_i w_i^{'}, v_i w_i^{'}/1 \leq i \leq n\} \cup \{v_1^{'} w_1^{'}, v_1^{'} v_1^{''}\}$. Clearly, $D(SL_n, v_1^{'})$has $5n-1$ vertices and $6n-2$ edges.

Let $A = \{1, 3, 5, \cdots, 6n-1\}$. Define a vertex labeling $f : V(D(SL_n, v_1^{'})) \to A$ as follows:

$f(u_1) = 5, f(v_1^{'}) = 7, f(u_2) = 9, f(v_2) = f(v_2^{''}) = 11, f(w_1) = 1, f(v_1^{''}) = 3,$

$f(u_{2i+1}) = f(w_{2i+1}) = 12i + 3$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(u_{2i+2}) = 12i + 7$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$

$f(w_{2i}) = 12i - 3$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor,$

$f(u_{2i-1}^{'}) = 12i - 5$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor,$

$f(u_{2i}^{'}) = 12i + 5$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(v_{2i-1}) = 12i - 11$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil,$

$f(v_{2i+2}) = 12i + 9$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$

$f(v_{2i+1}^{''}) = 12i + 11$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$

$f(v_{2i+2}^{''}) = 12i + 13$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1.$

It can be verified that the induced edge labels of $D(SL_n, v_1^{'})$ are $2, 4, \cdots, 12n-4$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(SL_n, v_1^{'})$. Thus, $D(SL_n, v_1^{'})$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(SL_4, v_1^{'})$ is shown in Figure 2.

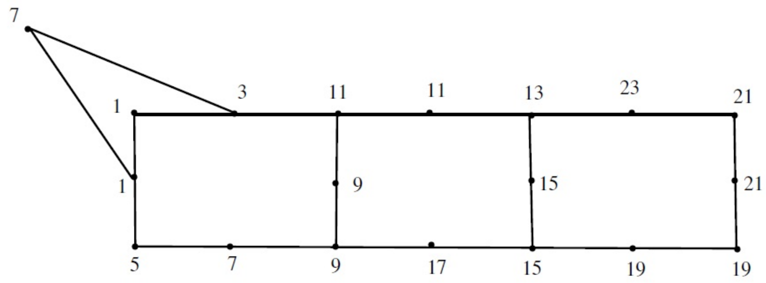

Figure 2. Odd vertex equitable even labeling of $D(SL_4, v_1^{'})$

Theorem 2.3 *The duplicate graph $D(Q_n, u_1^{'})$ of a quadrilateral Q_n is an odd vertex equitable even graph.*

Proof Let the vertex set of $D(Q_n, u_1^{'})$ be $\{u_i : 1 \leq i \leq n\} \cup \{v_i, w_i : 1 \leq i \leq n-1\} \cup \{u_1^{'}\}$ and the edge set of $D(Q_n, u_1^{'})$ be $\{u_i u_{i+1}/1 \leq i \leq n-1\} \cup \{u_i v_i, u_{i-1} w_i, v_i w_i/1 \leq i \leq n-1\} \cup \{u_1^{'} v_1, u_1^{'} u_2\}$. Clearly, $D(Q_n, u_1^{'})$has $3n-1$ vertices and $4n-2$ edges.

Let $A = \{1, 3, 5, \cdots, 4n-1\}$. Define a vertex labeling $f : V(D(Q_n, u_1^{'})) \to A$ as follows:

$f(u_1^{'}) = 3, f(v_1) = f(w_1) = 1, f(u_1) = 5,$

$f(u_{2i}) = 8i - 1$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor,$

$f(u_{2i+1}) = 8i + 3$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(v_{2i}) = 8i - 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(v_{2i+1}) = 8i + 3$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1,$

$f(w_{2i}) = 8i + 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1,$

$f(w_{2i-1}) = 8i + 5$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1.$

It can be verified that the induced edge labels of $D(Q_n, u_1^{'})$ are $2, 4, \cdots, 8n-4$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(Q_n, u_1^{'})$. Thus, $D(Q_n, u_1^{'})$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(Q_5, u_1^{'})$ is shown in Figure 3.

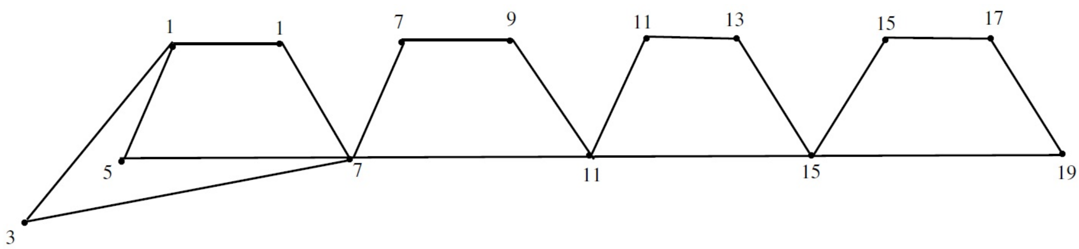

Figure 3. Odd vertex equitable even labeling of $D(Q_5, u_1^{'})$

Theorem 2.4 *The duplicate graph $D(D(Q_n), v_1^{'})$ of a quadrilateral $D(Q_n)$ is an odd vertex equitable even graph.*

Proof Let the vertex set of $D(D(Q_n), v_1^{'})$ be $\{u_i : 1 \leq i \leq n\} \cup \{v_i, x_i, w_i y_i : 1 \leq i \leq$

$n-1\} \cup \{v_1'\}$ and the edge set of $D(D(Q_n), v_1')$ be $\{u_iu_{i+1}/1 \leq i \leq n-1\} \cup \{u_{i+1}x_i, u_{i+1}y_i : 1 \leq i \leq n-1\} \cup \{w_ix_i, v_iy_i/1 \leq i \leq n-1\} \cup \{u_iv_i, u_iw_i/1 \leq i \leq n-1\} \cup \{v_1'y_1, v_1'u_1\}$. Clearly, $D(D(Q_n), v_1')$has $5n-3$ vertices and $7n-5$ edges.

Let $A = \begin{cases} 1, 3, 5, \cdots, 7n-4 \text{ if n is odd} \\ 1, 3, 5, \cdots, 7n-5 \text{ if n is even.} \end{cases}$

Define a vertex labeling $f : V(D(D(Q_n), v_1')) \to A$ as follows:

$f(u_1) = f(w_1) = 1, f(v_1') = 5, f(v_1) = 7, f(x_1) = 3,$
$f(u_{2i}) = 14i - 5$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,
$f(u_{2i+}) = 14i + 3$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,
$f(v_{2i}) = 14i - 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,
$f(v_{2i+1}) = 14i + 5$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,
$f(y_{2i-1}) = 14i - 5$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,
$f(y_{2i}) = 14i + 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,
$f(w_{2i+1}) = 14i + 3$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,
$f(w_{2i}) = 14i - 3$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,
$f(x_{2i}) = 14i - 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,
$f(x_{2i+1}) = 14i + 7$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$.

It can be verified that the induced edge labels of $D(D(Q_n), v_1')$ are $2, 4, \cdots, 14n-10$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(D(Q_n), v_1')$. Thus, $D(D(Q_n), v_1')$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(D(Q_4), v_1')$ is shown in Figure 4.

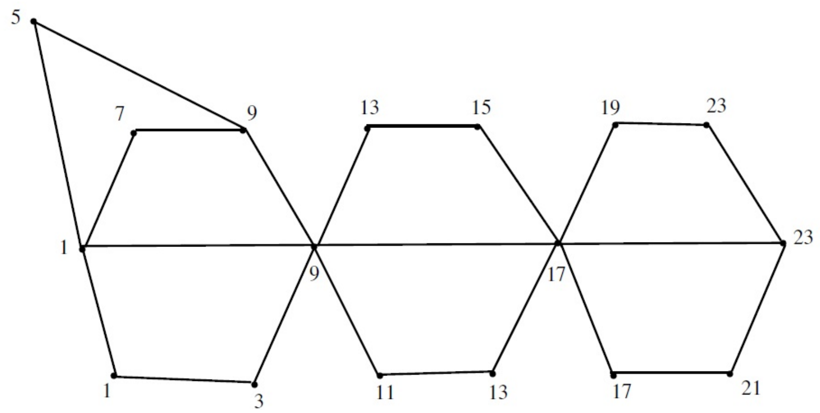

Figure 4 Odd vertex equitable even labeling of $D(D(Q_4), v_1')$

Theorem 2.5 *The duplicate graph $D(S(Q_n), u_1'')$ of a quadrilateral $S(Q_n)$ is an odd vertex equitable even graph.*

Proof Let P_n be the path $u_1, u_2, \cdots, u_n$ and let $V(Q_n) = \{v_i, w_i : 1 \leq i \leq n-1\} \bigcup \{u_i : 1 \leq i \leq n\}$. Let the vertex set of $D(S(Q_n), u_1'')$ be $\{x_i, y_i, z_i, u_i' : 1 \leq i \leq$

$n-1\} \cup \{u_1^{''}\} \bigcup V(Q_n)$ and the edge set of $D(S(Q_n), u_1^{''})$ be $\{u_iu_i^{'}, u_ix_i, u_i^{'}u_{i+1}, y_iu_{i+1} : 1 \leq i \leq n-1\} \bigcup \{x_iv_i, v_iz_i, z_iw_i, w_iy_i / 1 \leq i \leq n-1\} \bigcup \{x_1u_1^{''}, u_1^{'}u_1^{''}\}$. Clearly, $D(S(Q_n), u_1^{''})$has $7n-5$ vertices and $8n-6$ edges.

Let $A = \{1, 3, 5, \cdots, 8n-1\}$. Define a vertex labeling $f : V(D(S(Q_n), u_1^{''})) \to A$ as follows:

$f(u_1) = f(x_1) = 1, f(u_2) = 11, f(u_1^{'}) = 3, f(x_2) = 11, f(v_1) = 5, f(v_2) = 13,$

$f(w_1) = 9, f(z_1) = 7, f(z_2) = 13,$

when $n > 3$, $f(u_{2i-1}) = 16i - 15$ if $2 \leq i \leq \lceil \frac{n}{2} \rceil$,

$f(u_{2i}) = 16i - 17$ if $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$f(u_{2i+1}^{'}) = 16i + 11$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(u_{2i}^{'}) = 16i + 3$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(x_{2i+1}) = 16i + 5$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(x_{2i}) = 16i - 3$ if $n > 4, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(y_{2i-1}) = 16i - 7$ if $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$,

$f(y_{2i}) = 16i + 1$ if $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(v_{2i+1}) = 16i + 3$ if $n > 3, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(v_{2i}) = 16i - 5$ if $n > 4, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(w_{2i+1}) = 16i + 7$ if $n > 3, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(w_{2i}) = 16i - 1$ if $n \geq 3, 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$,

$f(z_{2i+1}) = 16i + 7$ if $n > 3, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$,

$f(z_{2i}) = 16i - 1$ if $n > 4, 2 \leq i \leq \lceil \frac{n}{2} \rceil - 1$.

It can be verified that the induced edge labels of $D(S(Q_n), u_1^{''})$ are $2, 4, \cdots, 16n - 12$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of $D(S(Q_n), u_1^{''})$. Thus, $D(S(Q_n), u_1^{''})$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $D(S(Q_4), u_1^{''})$ is shown in Figure 5.

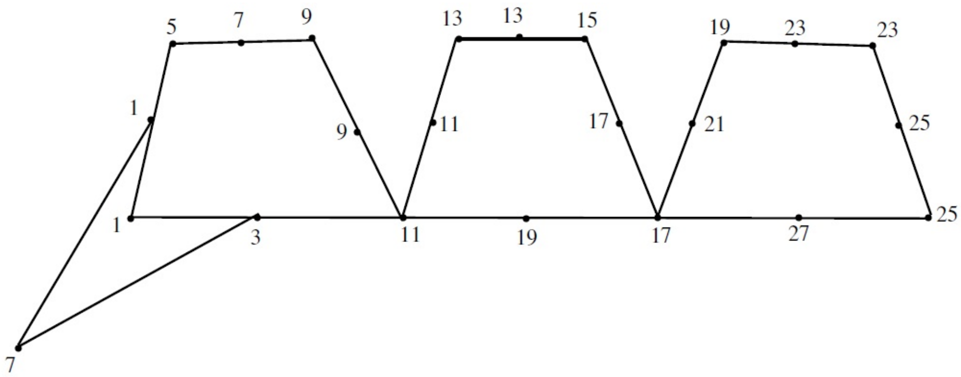

Figure 5. Odd vertex equitable even labeling of $D(S(Q_4), u_1^{''})$

Theorem 2.6 *The graph $P_3 \times P_n$ is an odd vertex equitable even graph for any $n \geq 3$ and n is odd.*

Proof Let the vertex set of $P_3 \times P_n$ be $\{u_{ij} : 1 \leq i \leq 3, 1 \leq j \leq n\}$ and the edge set of $P_3 \times P_n$ be $\{u_{1(2j)}u_{2(2j-1)}, u_{1(2j)}u_{2(2j+1)}, u_{3(2j)}u_{2(2j-1)}, u_{3(2j)}u_{2(2j+1)} : 1 \leq j \leq \lfloor \frac{n}{2} \rfloor\} \cup \{u_{11}u_{22}, u_{31}u_{22}\} \cup \{u_{1n}u_{2(n-1)}, u_{3n}u_{2(n-1)}\} \cup \{u_{1(2j+1)}u_{2(2j)}, u_{1(2j+1)}u_{2(2j+2)} : 1 \leq j \leq \lfloor \frac{n}{2} \rfloor - 1\} \cup \{u_{3(2j+1)}u_{2(2j)}, u_{3(2j+1)}u_{2(2j+2)} : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1\}$. Clearly $P_3 \times P_n$ has $3n$ vertices and $4(n-1)$ edges.

Let $A = \{1, 3, 5, \cdots, 4n-3\}$. Define a vertex labeling $f : V(P_3 \times P_n) \to A$ as follows:

$$f(u_{1j}) = \begin{cases} 2j-1 & if \;\; j \text{ is even} \\ 2n+2j-5 & if \;\; j \text{ is odd} \end{cases},$$

$$f(u_{2j}) = \begin{cases} 2j-1 & if \;\; j \text{ is odd} \\ 2n+2j-3 & if \;\; j \text{ is even} \end{cases},$$

$$f(u_{3j}) = \begin{cases} 2j-3 & if \;\; j \text{ is even} \\ 2n+2j-3 & if \;\; j \text{ is odd} \end{cases}.$$

It can be verified that the induced edge labels of $P_3 \times P_n$ are $2, 4, \cdots, 8n-8$ and $|v_f(i) - v_f(j)| \leq 1$ for all $i, j \in A$. Clearly f is odd vertex equitable even labeling of$P_3 \times P_n$. Thus, $P_3 \times P_n$ is an odd vertex equitable even graph. □

An example of odd vertex equitable even labeling of $P_3 \times P_7$ is shown in Figure 6.

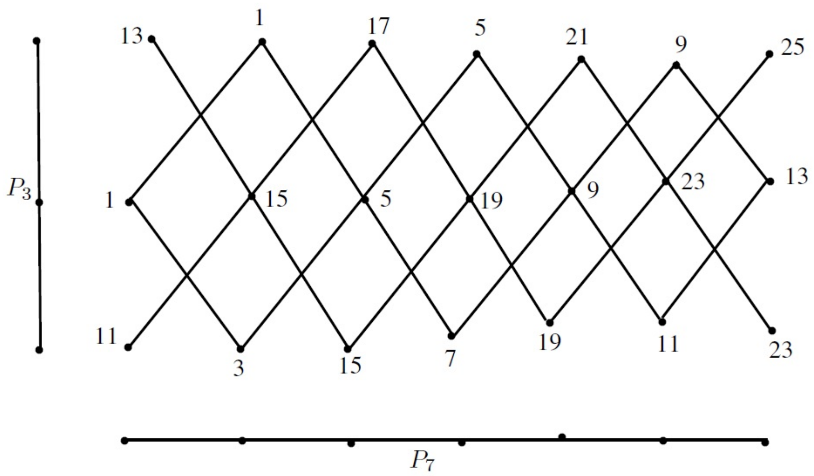

Figure 6. Odd vertex equitable even labeling of $P_3 \times P_7$

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