

On Isomorphism Theorems of Torian Algebras

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Abstract: In this paper, the notions of ideals and congruences in torian algebras are used to construct quotient torian algebras. The Fundamental Theorem of homomorphisms of torian algebras is established. Moreover, the three isomorphism Theorems of torian algebras are also presented.

Key Words: Torian algebras, Smarandachely torian algebras, congruences, ideals.

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§1. Introduction

Obic algebras were introduced in 2019 by Ilojide in [6]. Homomorphisms and krib maps as well as monics of obic algebras were studied. Properties of implicative obic algebras were also investigated. In [7], torian algebras were introduced. The class of torian algebras is a wider class than the class of obic algebras. It was shown that with a suitably defined binary relation, torian algebras are partially ordered sets. The partial ordering was used to investigate some of their properties. In [8], ideals of torian algebras were studied. Their properties were investigated. Moreover, the dual and nuclei of ideals as well as congruences developed on ideals of torian algebras were also studied. Right distributive torian algebras were studied in [9]. It was shown that every right distributive torian algebra fixes its zero element. Moreover, necessary and sufficient conditions for a torian algebra to be right distributive were also established. In this paper, the study of torian algebras is continued. The notions of ideals and congruences in torian algebras are used to construct quotient torian algebras. The Fundamental Theorem of homomorphisms of torian algebras is established. Moreover, the three isomorphism Theorems of torian algebras are also presented.

§2. Preliminaries

Definition 2.1([6]) A triple $(X; *, 0)$; where X is a non-empty set, $*$ a binary operation on X , and 0 a constant element of X is called an obic algebra if the following axioms hold for all $x, y, z \in X$:

$$(1) x * 0 = x;$$

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- (2) $[x * (y * z)] * x = x * [y * (z * x)];$
- (3) $x * x = 0.$

Example 2.1([6]) Consider the multiplicative group $G = \{1, -1, i, -i\}$. Define a binary operation $*$ on G by $a * b = ab^{-1}$. Then $(G; *, 1)$ is an obic algebra.

Lemma 2.1([6]) Let X be an obic algebra. Then for all $x, y \in X$, the following holds

$$x * y = [x * (y * x)] * x.$$

Definition 2.2([6]) A non-empty subset S of an obic algebra X is called a subalgebra if S is an obic algebra with respect to the binary operation in X .

Definition 2.3([6]) An obic algebra X is said to have the weak property (WP) if $x * y = 0$ and $y * x = 0$ imply that $x = y$.

Definition 2.4([6]) An equivalence relation \sim^* on an obic algebra X is called a congruence if $(x \sim^* y)$ and $(u \sim^* v) \Rightarrow (x * u) \sim^* (y * v)$.

Definition 2.5([6]) Let $(X; *, 0)$ and $(Y; \circ, 0')$ be obic algebras. A function $f : X \rightarrow Y$ is called an obic homomorphism if $f(a * b) = f(a) \circ f(b)$ for all $a, b \in X$.

Let $(X; *, 0)$ and $(Y; \circ, 0')$ be obic algebras, and let $f : X \rightarrow Y$ be a homomorphism. The set $\{x \in X : f(x) = 0'\}$ is called the kernel of f . It is denoted by $Ker(f)$. The set $\{y \in Y : y = f(x); x \in X\}$ is called the image of f . It is denoted by $Im(f)$. If f is injective, then f is called a monomorphism. If f is surjective, then f is called an epimorphism. If f is both injective and surjective, then f is called an isomorphism. If f is an isomorphism, then X is said to be isomorphic to Y .

Definition 2.6([7]) An obic algebra X is called torian if $[(x * y) * (x * z)] * (z * y) = 0$ for all $x, y, z \in X$. Otherwise, if there are $x, y, z \in X$, such that $[(x * y) * (x * z)] * (z * y) \neq 0$, such an obic algebra X is called Smarandachely torian.

Definition 2.7([7]) A torian algebra which has the weak property is called a weak property torian algebra (WPTA).

Definition 2.8([8]) Let X be a torian algebra. A non-empty set S of X is called a left ideal of X if the following holds:

- (1) $0 \in S;$
- (2) If $x, y \in X$ such that $x, [[y * (x * y)] * y] \in S$, then $y \in S$.

Definition 2.9([8]) If a left ideal S of X is such that $[[x * (y * x)] * x] \in S$ for all $x, y \in X$, then S is said to be a complete left ideal of X or that S is complete in X .

Proposition 2.10([8]) Let $(X; *, 0)$ and $(Y; \odot, 0')$ be torian algebras. Let $f : X \rightarrow Y$ be a homomorphism. Then $Ker(f)$ is a complete left ideal of X .

§3. Main Results

In this section, we establish the first, second and third isomorphism Theorems of torian algebras. But before that, some preliminary results are presented. The following Theorem is taken from [8]. It is needed in proving Theorem 3.2, which is our first preliminary result.

Theorem 3.1 *Let S be a left ideal of a torian algebra X . Let \sim^1 be a relation on X defined by $x \sim^1 y \Leftrightarrow [[x * (y * x)] * x]$ and $[[y * (x * y)] * y] \in S$ for all $x, y \in X$. Then \sim^1 is a congruence on X .*

Remark 3.1 Let X^S denote the collection of equivalence classes in the equivalence relation in Theorem 3.1. An equivalence class of $x \in X$ is denoted by $[x]$.

Definition 3.1 *Let S be a left ideal of a torian algebra X equipped with a congruence \sim^1 . Let $[x], [y] \in X^S$. Define a binary operation \odot on X^S by $[x] \odot [y] = [x * y]$ for all $x \in X$.*

Theorem 3.2 *Let S be a left ideal of a torian algebra X equipped with a congruence \sim^1 . Let $[0]$ be the zero equivalence class. Then $(X^S; \odot, [0])$ is a torian algebra.*

Proof Let $[x], [y], [z] \in X^S$. Clearly, $[0] \in X^S$. Notice that $[x] \odot [0] = [x * 0] = [x]$. Also, $[x] \odot [x] = [x * x] = [0]$. Now, $(([x] \odot [y]) \odot ([x] \odot [z])) \odot ([z] \odot [y]) = ([x * y] \odot [x * z]) \odot [z * y] = (((x * y) * (x * z))) \odot [z * y] = [(((x * y) * (x * z)) * (z * y))] = [0]$.

Finally, notice that $([x] \odot ([y] \odot [z])) \odot [x] = ([x] \odot [y * z]) \odot [x] = [(x * (y * z))] \odot [x] = [(x * (y * (z * x)))] = [x] \odot [y * (z * x)] = [x] \odot [[y] \odot ([z] \odot [x])]$. So, $(X^S; \odot, [0])$ satisfies all the axioms of a torian algebra as required. \square

The following corollary follows from Theorem 3.2.

Corollary 3.1 *Let S be a left ideal of a WPTA X equipped with a congruence \sim^1 . Let $[0]$ be the zero equivalence class. Then $(X^S; \odot, [0])$ is a WPTA.*

Definition 3.2 *Let S be a left ideal of a torian algebra X equipped with a congruence \sim^1 , and let $[0]$ be the zero equivalence class. The torian algebra $(X^S; \odot, [0])$ is called the quotient torian algebra induced by the left ideal S .*

Remark 3.2 The torian algebra X^S is also denoted by X/S .

Theorem 3.3 *Let S be a left ideal of a torian algebra X equipped with a congruence \sim^1 . Then $[0]$ is a left ideal of X .*

Proof Clearly, $0 \in [0]$. So, $[0]$ is not empty. Let $x, [[y * (x * y)] * y] \in [0]$. Then $x \sim^1 0$ and $y * x = [[y * (x * y)] * y] \sim^1 0$. So, $y * x \sim^1 0$. Now, by the reflexivity of \sim^1 , we have $y \sim^1 y$. Since \sim^1 is a congruence, combining $y \sim^1 y$ and $x \sim^1 0$, we have $y * x \sim^1 y$. Since \sim^1 is symmetric, we have $y \sim^1 y * x$. By the transitivity of \sim^1 , combining $y \sim^1 y * x$ and $y * x \sim^1 0$, we have $y \sim^1 0$. Hence, $y \in [0]$ as required. \square

Theorem 3.4 *Let S be a left ideal of a torian algebra X equipped with a congruence \sim^1 . Then*

the map $\alpha : X \rightarrow X^S$ such that $\alpha(x) = [x]$ for all $x \in X$ is an epimorphism.

Proof By Theorem 3.2, $(X^S; \odot, [0])$ is a torian algebra. Let $x, y \in X$. Then $\alpha(x * y) = [x] \odot [y] = \alpha(x) \odot \alpha(y)$. Hence, α is a homomorphism. Clearly, α is surjective. Therefore, α is an epimorphism. \square

Definition 3.3 Let S be a left ideal of a torian algebra X equipped with a congruence \sim^1 . Then the epimorphism $\alpha : X \rightarrow X^S$ such that $\alpha(x) = [x]$ for all $x \in X$ is called the natural epimorphism induced by the left ideal S .

The following corollary is immediate from Proposition 2.1 and Theorem 3.4.

Corollary 3.2 Let X be a torian algebra equipped with a congruence \sim^1 . Let $f : X \rightarrow X$ be a homomorphism with $\text{Ker}(f) = K$. Then $\alpha : X \rightarrow X^K$ is a homomorphism.

Theorem 3.5 Let $(X; *, 0)$ and $(Y; \circ, 0')$ be torian algebras, and let $f : X \rightarrow Y$ be a homomorphism with $\text{Ker}(f) = K$. Then there exists a homomorphism $\phi : X^K \rightarrow Y$ such that the following holds:

- (1) $f = \phi\alpha$; where α is the natural homomorphism induced by K ;
- (2) ϕ is unique;
- (3) ϕ is a monomorphism.

Proof By Proposition 2.1 and Theorem 3.2, X^K is a torian algebra. Now, define $\phi : X^K \rightarrow Y$ by $\phi([x]) = f(x)$ for all $[x] \in X^K$. Let $[x], [y] \in X^K$. Then $\phi([x] \odot [y]) = \phi([x * y]) = f(x * y) = f(x) \circ f(y) = \phi([x]) \circ \phi([y])$. So, ϕ is a homomorphism. Notice that $\phi\alpha(x) = \phi(\alpha(x)) = \phi([x]) = f(x)$ for all $x \in X$. Hence, $f = \phi\alpha$.

We now show that ϕ is unique. Let $\phi' : X^K \rightarrow Y$ such that $f = \phi'\alpha$, and let $[x] \in X^K$. Then $\phi'([x]) = \phi'(\alpha(x)) = f(x) = \phi\alpha(x) = \phi(\alpha(x)) = \phi([x])$. So, $\phi' = \phi$. Hence, ϕ is unique.

We now show that ϕ is a monomorphism. Let $[x], [y] \in X^K$ such that $\phi([x]) = \phi([y])$. Then $f(x) = f(y)$. Therefore, $f(((x * (y * x)) * x)) = f(x * y) = f(x) \circ f(y) = f(x) \circ f(x) = 0'$. Thus, $((x * (y * x)) * x) \in K$. Also notice that $f(((y * (x * y)) * y)) = f(y * x) = f(y) \circ f(x) = 0'$. Thus, $((y * (x * y)) * y) \in K$. Since $((x * (y * x)) * x) \in K$ and $((y * (x * y)) * y) \in K$, then $x \sim^1 y$. Hence, $[x] = [y]$. Therefore, ϕ is a monomorphism. \square

Remark 3.3 Theorem 3.5 is the fundamental theorem of homomorphism of torian algebras.

Theorem 3.6 Let X and Y be torian algebras, and let $f : X \rightarrow Y$ be a homomorphism with $\text{Ker}(f) = K$. Then X^K is isomorphic to $\text{Im}(f)$.

Proof By Theorem 3.5, the map $\phi : X^K \rightarrow Y$ defined by $\phi([x]) = f(x)$ for all $[x] \in X^K$ is a monomorphism. Now, let $y = f(x) \in \text{Im}(f)$. Then there exists $x \in X$ such that $\phi([x]) = y$. So, for each $f(x) \in \text{Im}(f)$, there exists $[x] \in X^K$ such that $\phi([x]) = f(x)$. So, ϕ is an epimorphism, and hence an isomorphism. Therefore, X^K is isomorphic to $\text{Im}(f)$ as required. \square

Remark 3.4 Theorem 3.6 is the first isomorphism theorem of torian algebras.

The following corollary is immediate from Theorem 3.6.

Corollary 3.3 *Let X and Y be torian algebras, and let $f : X \rightarrow Y$ be an epimorphism with $\text{Ker}(f) = K$. Then X^K is isomorphic to Y .*

Theorem 3.7 *Let P be a subalgebra of a torian algebra X . Let S be a complete left ideal of X . Then $P/(P \cap S)$ is isomorphic to PS/S .*

Proof Let the map $f : P \rightarrow PS/S$ be defined by $f(p) = [p]$ for all $p \in P$. Then f is a homomorphism. We claim that f is an epimorphism such that $\text{Ker}(f) = P \cap S$. Now, let $[x] \in PS/S$. Then $x \in PS$. Now, there exists $p \in P$ such that $x \in [p]$, and so $[x] = [p]$. Therefore, $f(p) = [p] = [x]$. So, f is an epimorphism. Now, let $p \in P$ such that $p \in \text{Ker}(f)$. Then $f(p) = [0]$. Since by definition of f , $f(p) = [p]$, we then have $[p] = [0]$. Hence, $p = p*0 \in S$. Therefore, $p \in P \cap S$. Thus, $\text{Ker}(f)$ is contained in $P \cap S$. Now, let $p \in P \cap S$. Then $p \in P$ and $p \in S$. Since S is complete, then $[p] = [0]$. Thus, $f(p) = [p] = [0]$. So, $p \in \text{Ker}(f)$. Hence, $P \cap S$ is contained in $\text{Ker}(f)$. Therefore, $\text{Ker}(f) = P \cap S$. By Corollary 3.3, $P/(P \cap S)$ is isomorphic to PS/S as required. \square

Remark 3.5 Theorem 3.7 is the second isomorphism theorem of torian algebras.

Theorem 3.8 *Let X be a torian algebra. Let K be a complete left ideal of X and let A be a left ideal of K . Then X^K is isomorphic to X^A/K^A .*

Proof Since A is a left ideal of K and K is a left ideal of X , then A is a left ideal of X . By Theorem 3.2, X^K and X^A are torian algebras. Let \odot and \odot' be the binary operations of X^A and X^K respectively. Let the map $\phi : X^A \rightarrow X^K$ be defined by $\phi([x]_A) = [x]_K$ for all $[x]_A \in X^A$. We show that ϕ is an isomorphism with $\text{Ker}(\phi) = K^A$ and $\text{Im}(\phi) = X^K$. Now, let $[x]_A, [y]_A \in X^A$. Then $\phi([x]_A \odot [y]_A) = \phi([x*y]_A) = [x*y]_K = [x]_K \odot' [y]_K = \phi([x]_A) \odot \phi([y]_A)$. So ϕ is a homomorphism. Clearly, ϕ is a bijection. Now, notice that $\text{Ker}(\phi) = \{[x]_A \in X^A : \phi([x]_A) = [0]_K\} = \{[x]_A \in X^A : [x]_K = [0]_K\} = \{[x]_A : x \in K\} = K^A$. Notice also that $\text{Im}(\phi) = \{[x]_K \in X^K : x \in X\} = X^K$. By Theorem 3.6, therefore, X^K is isomorphic to X^A/K^A as required. \square

Remark 3.6 Theorem 3.8 is the third isomorphism theorem of torian algebras.

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