International J.Math. Combin. Vol.1(2016), 8-17

On Dual Curves of Constant Breadth According to Dual Bishop Frame in Dual Lorentzian Space $\mathbb{D}_1^3$

Süha Yılmaz[1], Yasin Ünlütürk[2] and Ümit Ziya Savcı[3]

1. Dokuz Eylül University, Buca Educational Faculty, 35150, Buca-Izmir, Turkey
2. Kırklareli University, Department of Mathematics, 39100 Kırklareli, Turkey
3. Celal Bayar University, Department of Mathematics Education, 45900, Manisa-Turkey

E-mail: suha.yilmaz@deu.edu.tr, yasinunluturk@klu.edu.tr, ziyasavci@hotmail.com

Abstract: In this work, dual curves of constant breadth according to Bishop frame are defined, and applications of their differential equations are solved for special cases in dual Lorentzian space $\mathbb{D}_1^3$. Some characterizations of closed dual curves of constant breadth according to Bishop frame are presented in dual Lorentzian space $\mathbb{D}_1^3$. These characterizations are made by obtaining special solutions of differential equations which characterize closed dual curves of constant breadth according to Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$.

Key Words: Dual Lorentzian space, dual curve, dual curves of constant breadth, Bishop frame, differential equations.

AMS(2010): 53A35, 53A40, 53B25.

§1. Introduction

Bishop frame is used in engineering. This special frame has been particulary used in the study of DNA, and tubular surfaces and made in robot. Most of the literature on canal surfaces within the CAGD context has been motivated by the observation that canal surfaces with the rational spine curve and rational radius function are rational, and it is therefore natural to ask for methods which allow one to construct a rational parameterization of canal surface from its spin curve and radius function [8]. The construction of the Bishop frame is due to L. R. Bishop in [2]. That is why he defined this frame that curvature may vanish at some points on the curve. That is, second derivative of the curve may be zero. In this situation, an alternative frame is needed for non continously differentiable curves on which Bishop (parallel transport frame) frame is well defined and constructed in Euclidean and its ambient spaces [4, 18].

Curves of constant breadth have been studied in pure mathematics, optimization, mechanical engineering, physics and related directions. Basic properties of curves of constant breadth can be explained to someone without having any mathematical background knowledge. The existence of non-circular curves of constant breadth in the standard Euclidean plane has been known since the time of Euler; e.g., the Reuleaux triangle was presented by Reuleaux to horn-

[1]Received May 22, 2015, Accepted February 4, 2016.

blower, the founder of the compound steam-engine. In recent years, mathematical properties of the Reuleaux triangle have led to some very important applications. Since a curve of constant breadth can be freely rotated in a square always maintaining contact to all four sides of the square, a Reuleaux triangle can be used for drilling holes of maximum area into squares. Another application is given by the basic single-rotor Wankel engine. Its oval-shaped housing surrounds a three-sided rotor similar to a Reuleaux triangle. As the rotor rotates and orbitally revolves, each side of the rotor gets closer and farther from the wall of the housing, as also described above, in view of drilling holes into squares. A Reuleaux triangle is also used in the gear for driving a movie film [12].

In the classical theory of curves in differential geometry, curves of constant breadth have a long history as a research matter [3, 5, 9]. First it was introduced by Euler in [5]. Then Fujivara obtained a problem to determine whether there exist space curves of constant breadth or not, and he defined the concept "breadth" for space curves on a surface of constant breadth [6]. Furthermore, Blaschke defined the curve of constant breadth on the sphere [3]. Reuleaux gave a method to obtain these kinds of curves and applied the results he had by using his method, in kinematics and engineering [14]. Some geometric properties of plane curves of constant breadth were given by Köse in [11]. And, in another work of Köse [10], these properties were studied in the Euclidean 3-space $\mathbb{E}^3$. In Minkowski 3-space as an ambient space, some characterizations of timelike curves of constant breadth were given by Yılmaz and Turgut in [17]. Also, Yılmaz dealt with dual timelike curves of constant breadth in dual Lorentzian space in [16].

Dual numbers were introduced by W. K. Clifford as a tool for his geometrical investigations. Then dual numbers and vectors were used on line geometry and kinematics by Eduard Study. He devoted a special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping: The set of oriented lines in a three-dimensional Euclidean space $\mathbb{E}^3$ is one to one correspondence with the points of a dual space $\mathbb{D}^3$ of triples of dual numbers [7].

In this paper, we study dual curves of constant breadth according to Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$. We give some characterizations of dual curves of constant breadth according to Bishop frame in $\mathbb{D}_1^3$. Then we characterize these kinds of curves by obtaining special solutions of their differential equations in $\mathbb{D}_1^3$.

§2. Preliminaries

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three dimensional real vector space $\mathbb{E}^3$ with the metric

$$\langle dx,dx\rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{E}^3$. An arbitrary vector x of $\mathbb{E}_1^3$ is said to be spacelike if $\langle x, x\rangle > 0$ or $x = 0$, timelike if $\langle x, x\rangle < 0$ and lightlike or null if $\langle x, x\rangle = 0$ and $x \neq 0$. A timelike or light-like vector in $\mathbb{E}_1^3$ is said to be causal. For $x \in \mathbb{E}_1^3$ the norm is defined by $\|x\| = \sqrt{|\langle x, x\rangle|}$, then the vector x is called a spacelike unit vector if $\langle x, x\rangle = 1$ and a timelike unit vector if $\langle x, x\rangle = -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [13].

Dual numbers are given with the set

$$\mathbb{D} = \{\widehat{x} = x + \xi x^*; x, x^* \in \mathbb{E}\},$$

the symbol ξ designates the dual unit with the property $\xi^2 = 0$ for $\xi \neq 0$. Dual angle is defined as $\widehat{\theta} = \theta + \xi\theta^*$, where θ is the projected angle between two spears and θ^* is the shortest distance between them. The set $\mathbb{D}$ of dual numbers is commutative ring the the operations $+$ and $\cdot$. The set

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \left\{\widehat{\varphi} = \varphi + \xi\varphi^*; \varphi, \varphi^* \in \mathbb{E}^3\right\}$$

is a module over the ring $\mathbb{D}$ [15].

For any $\widehat{a} = a + \xi a^*$, $\widehat{b} = b + \xi b^* \in \mathbb{D}^3$, if the Lorentzian inner product of $\widehat{a}$ and $\widehat{b}$ is defined by

$$< \widehat{a}, \widehat{b} >=< a, b > +\xi(< a^*, b > + < a, b^* >),$$

then the dual space $\mathbb{D}^3$ together with this Lorentzian inner product is called the dual Lorentzian space and denoted by $\mathbb{D}^3_1$ [1]. For $\widehat{\varphi} \neq 0$, the norm $\|\widehat{\varphi}\|$ of $\widehat{\varphi}$ is defined by

$$\|\widehat{\varphi}\| = \sqrt{< \widehat{\varphi}, \widehat{\varphi} >}.$$

A dual vector $\widehat{\omega} = \omega + \xi\omega^*$ is called dual spacelike vector if $\langle\widehat{\omega}, \widehat{\omega}\rangle > 0$ or $\widehat{\omega} = 0$, dual timelike vector if $\langle\widehat{\omega}, \widehat{\omega}\rangle < 0$ and dual null (lightlike) vector if $\langle\widehat{\omega}, \widehat{\omega}\rangle = 0$ for $\widehat{\omega} \neq 0$. Therefore, an arbitrary dual curve which is a differential mapping onto $\mathbb{D}^3_1$, can locally be dual spacelike, dual timelike or dual null if its velocity vector is dual spacelike, dual timelike or dual null, respectively. Also, for the dual vectors $\widehat{a}, \widehat{b} \in \mathbb{D}^3_1$, Lorentzian vector product of these dual vectors is defined by

$$\widehat{a} \times \widehat{b} = a \times b + \xi(a^* \times b + a \times b^*)$$

where $a \times b$ is the classical cross product according to the signature $(+, +, -)$ [1].

The dual arc length of the curve $\widehat{\varphi}$ from t_1 to t is defined as

$$s = \int_{t_1}^{t} \|\widehat{\varphi}'(t)\| \, dt = \int_{t_1}^{t} \|\varphi'(t)\| \, dt + \varepsilon \int_{t_1}^{t} \langle t, \varphi'\rangle \, dt = s + \xi s^*,$$

where t is a unit tangent vector of $\varphi(t)$. From now on we will take the arc-length s of $\varphi(t)$ as the parameter instead of t [9].

Let $\widehat{\varphi} : I \subset \mathbb{E} \to \mathbb{D}^3_1$ be a dual spacelike curve with the arc-length parameter s. The Bishop derivative formula of dual spacelike curve $\widehat{\varphi}$ is expressed as

$$\begin{cases} \widehat{T}' = & \widehat{k}_1\widehat{N}_1 - \widehat{k}_2\widehat{N}_2, \\ \widehat{N}_1' = & -\varepsilon\widehat{k}_1\widehat{T}, \\ \widehat{N}_2' = & -\varepsilon\widehat{k}_2\widehat{T}, \end{cases} \tag{1}$$

where $\left\langle \widehat{T}, \widehat{T} \right\rangle = 1, \left\langle \widehat{N}_1, \widehat{N}_1 \right\rangle = \varepsilon = \pm 1, \left\langle \widehat{N}_2, \widehat{N}_2 \right\rangle = -\varepsilon$ and $\widehat{k}_1, \widehat{k}_2$ are Bishop curvatures. Here $\widehat{\tau} = \dfrac{d\widehat{\theta}}{ds}$ and $\widehat{\kappa} = \sqrt{\left|\widehat{k}_1^2 - \widehat{k}_2^2\right|}$. Thus, Bishop curvatures are defined by ([1], [2])

$$\widehat{k}_1 = \widehat{\kappa}(s) \cosh \widehat{\theta}(s), \quad \widehat{k}_2 = \widehat{\kappa}(s) \sinh \widehat{\theta}(s) \ .$$

Let $\widehat{\varphi} : I \subset \mathbb{E} \to \mathbb{D}_1^3$ be a dual timelike curve with the arc-length parameter s. The Bishop derivative formula of dual spacelike curve $\widehat{\varphi}$ is expressed as

$$\begin{cases} \widehat{T}' = & \widehat{k}_1 \widehat{N}_1 + \widehat{k}_2 \widehat{N}_2, \\ \widehat{N}_1' = & \widehat{k}_1 \widehat{T}, \\ \widehat{N}_2' = & \widehat{k}_2 \widehat{T}, \end{cases} \tag{2}$$

where $\left\langle \widehat{T}, \widehat{T} \right\rangle = -1, \left\langle \widehat{N}_1, \widehat{N}_1 \right\rangle = 1, \left\langle \widehat{N}_2, \widehat{N}_2 \right\rangle = 1$ and $\widehat{k}_1, \widehat{k}_2$ are Bishop curvatures. Here $\widehat{\tau} = \dfrac{d\widehat{\theta}}{ds}$ and $\widehat{\kappa} = \sqrt{\left|\widehat{k}_1^2 - \widehat{k}_2^2\right|}$. Thus, Bishop curvatures are defined by ([1], [2])

$$\widehat{k}_1 = \widehat{\kappa}(s) \cosh \widehat{\theta}(s), \quad \widehat{k}_2 = \widehat{\kappa}(s) \sinh \widehat{\theta}(s)$$

§3. Main Results

In this section, we give some characterizations of dual spacelike (timelike) curves of constant breadth according to Bishop frame in the dual Lorentzian space $\mathbb{D}_1^3$. First, we give the definition of dual spacelike (timelike) curves of constant breadth in $\mathbb{D}_1^3$. Then we characterize these kinds of curves by obtaining special solutions of their differential equations in $\mathbb{D}_1^3$.

Definition 3.1 *Let (C_1) be a dual spacelike (timelike) curve with position vector $\widehat{\varphi} = \widehat{\varphi}(s)$ in $\mathbb{D}_1^3$. If (C) has parallel tangents in opposite directions at corresponding points $\widehat{\varphi}(s)$ and $\widehat{\alpha}(s_\alpha)$ and the distance between these points is always constant, then (C_1) is called a dual spacelike (timelike) curve of constant breadth. Moreover, a pair of dual curves (C_1) and (C_2) for which the tangents at the corresponding points $\widehat{\varphi}(s)$ and $\widehat{\alpha}(s_\alpha)$, respectively, are parallel and in opposite directions, and the distance between these points is always constant are called a dual (timelike) curve pair of constant breadth.*

3.1 Dual Spacelike Curves of Constant Breadth According to Dual Bishop Frame

Let $\widehat{\varphi} = \widehat{\varphi}(s)$ be a simple closed dual spacelike curve in $\mathbb{D}_1^3$. We consider a dual spacelike curve in the class Γ as in [6] having parallel tangents $\widehat{T}_\varphi$ and $\widehat{T}_\alpha$ in opposite directions at the opposite points $\widehat{\varphi}$ and $\widehat{\alpha}$ of the curve according to Bishop frame. A simple closed dual spacelike curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to dual Bishop frame by the equation

$$\widehat{\alpha} = \widehat{\varphi} + \widehat{\gamma}\widehat{T} + \widehat{\delta}\widehat{N}_1 + \widehat{\lambda}\widehat{N}_2, \tag{3}$$

where $\widehat{\gamma}, \widehat{\delta}$ and $\widehat{\lambda}$ are arbitrary functions of s. Differentiating both sides of (4), we get

$$\frac{d\widehat{\alpha}}{ds_\alpha}\frac{ds_\alpha}{ds} = \left(\frac{d\widehat{\gamma}}{ds} - \varepsilon\widehat{\delta}\widehat{k}_1 - \varepsilon\widehat{\lambda}\widehat{k}_2 + 1\right)\widehat{T} + \left(\widehat{\gamma}\widehat{k}_1 + \frac{d\widehat{\delta}}{ds}\right)\widehat{N}_1 + \left(-\widehat{\gamma}\widehat{k}_2 + \frac{d\widehat{\lambda}}{ds}\right)\widehat{N}_2. \tag{4}$$

Considering $\widehat{T}_\alpha = -\widehat{T}_\varphi$ by the definition 3.1, we have the following system of equations

$$\begin{cases} \dfrac{d\widehat{\gamma}}{ds} = & \varepsilon\widehat{\delta}\widehat{k}_1 + \varepsilon\widehat{\lambda}\widehat{k}_2 - 1 - \dfrac{ds_\alpha}{ds}, \\ \dfrac{d\widehat{\delta}}{ds} = & -\widehat{\gamma}\widehat{k}_1, \\ \dfrac{d\widehat{\lambda}}{ds} = & \widehat{\gamma}\widehat{k}_2. \end{cases} \tag{5}$$

If we call $\widehat{\theta}$ as the angle between the tangent of the curve C at point $\widehat{\varphi}$ with a given direction and taking $\dfrac{d\widehat{\theta}}{ds} = \widehat{\tau}, \dfrac{d\widehat{\theta}}{ds_\alpha} = \widehat{\tau}^*$ into account, the equation (5) turns into

$$\begin{cases} \dfrac{d\widehat{\gamma}}{d\widehat{\theta}} = & \varepsilon\widehat{\delta}\dfrac{\widehat{k}_1}{\widehat{\tau}} + \varepsilon\widehat{\lambda}\dfrac{\widehat{k}_2}{\widehat{\tau}} - f(\widehat{\theta}), \\ \dfrac{d\widehat{\delta}}{d\widehat{\theta}} = & -\widehat{\gamma}\dfrac{\widehat{k}_1}{\widehat{\tau}}, \\ \dfrac{d\widehat{\lambda}}{d\widehat{\theta}} = & \widehat{\gamma}\dfrac{\widehat{k}_2}{\widehat{\tau}}, \end{cases} \tag{6}$$

where $f(\widehat{\theta}) = \dfrac{1}{\widehat{\tau}} + \dfrac{1}{\widehat{\tau}^*}$.

Let $\widehat{K}_1 = \dfrac{\widehat{k}_1}{\widehat{\tau}}, \widehat{K}_2 = \dfrac{\widehat{k}_2}{\widehat{\tau}}$ and using the system of ordinary differential equations (6), we have the following dual third order differential equation with respect to $\widehat{\gamma}$ as;

$$\begin{aligned} &\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3} + \varepsilon(\widehat{K}_1^2 - \widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}} + 3\varepsilon\left(\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}} - \widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}}\right)\widehat{\gamma} \\ &+\varepsilon\left(\textstyle\int \widehat{\gamma}\widehat{K}_1 d\widehat{\theta}\right)\frac{d^2\widehat{K}_1}{d\widehat{\theta}^2} - \varepsilon\left(\textstyle\int \widehat{\gamma}\widehat{K}_2 d\widehat{\theta}\right)\frac{d^2\widehat{K}_2}{d\widehat{\theta}^2} + \frac{d^2 f(\widehat{\theta})}{d\widehat{\theta}^2} = 0 \end{aligned} \tag{7}$$

We can give the following corollary.

Corollary 3.1.1 *The dual differential equation of third order given in (7) is a characterization of the simple closed dual spacelike curve* $\widehat{\alpha}$ *according to Bishop frame in* $\mathbb{D}_1^3$.

Since position vector of a simple closed dual spacelike curve can be determined by solution of the equation (7), let us investigate solution of the equation (7) in a special case. Let $\widehat{K}_1, \widehat{K}_2$

and $f(\widehat{\theta})$ be constants. Then the equation (7) turns to the following form

$$\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}+\varepsilon(\widehat{K}_1^2-\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}=0. \tag{8}$$

Solution of equation (8) yields the components

$$\begin{cases} \widehat{\gamma}=\widehat{A}+\widehat{B}\cos(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})+\widehat{C}\sin(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta}) \\ \widehat{\delta}=-\int\left\{\widehat{A}+\widehat{B}\cos(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})+\widehat{C}\sin(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})\right\}\widehat{K}_1 d\widehat{\theta} \\ \widehat{\lambda}=\int\left\{\widehat{A}+\widehat{B}\cos(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})+\widehat{C}\sin(\sqrt{\widehat{K}_1^2-\widehat{K}_2^2}\widehat{\theta})\right\}\widehat{K}_2 d\widehat{\theta}. \end{cases} \tag{9}$$

Corollary 3.1.2 *Position vector of a simple dual spacelike closed curve with constant dual curvature and constant dual torsion according to Bishop frame is obtained in terms of the values of $\widehat{\gamma}$, $\widehat{\delta}$ and $\widehat{\lambda}$ as in the equation (9).*

If the distance between opposite points of $\widehat{\varphi}$ and $\widehat{\alpha}$ is constant, then we can write that

$$\|\widehat{\alpha}-\widehat{\varphi}\|=-\widehat{\gamma}^2+\widehat{\delta}^2+\widehat{\lambda}^2=\text{constant}. \tag{10}$$

Differentiating (10) with respect to $\widehat{\theta}$ gives

$$-\widehat{\gamma}\frac{d\widehat{\gamma}}{d\widehat{\theta}}+\widehat{\delta}\frac{d\widehat{\delta}}{d\widehat{\theta}}+\widehat{\lambda}\frac{d\widehat{\lambda}}{d\widehat{\theta}}=0. \tag{11}$$

By virtue of (6), the differential equation (11) yields

$$-\widehat{\delta}\widehat{K}_1(1+\varepsilon)+\widehat{\lambda}\widehat{K}_2(1-\varepsilon)+f(\widehat{\theta})=0, \widehat{\gamma}=0. \tag{12}$$

There are two cases for the equation (12), we study these cases as follows:

Case 1. If $\widehat{K}_1=0$ and $\widehat{K}_2=0$ then we find that the components $\widehat{\delta}$ and $\widehat{\lambda}$ are constants and $f(\widehat{\theta})=0$.

Hence, Dual spacelike curves of constant breadth according to Bishop frame can be written as

$$\widehat{\alpha}=\widehat{\varphi}+\widehat{l}_1\widehat{T}+\widehat{l}_2\widehat{N}_1+\widehat{l}_3\widehat{N}_2, \tag{13}$$

where $\widehat{\gamma}=\widehat{l}_1, \widehat{\delta}=\widehat{l}_2, \widehat{\lambda}=\widehat{l}_3$; $\widehat{l}_1, \widehat{l}_2, \widehat{l}_3$ are constants.

Case 2. If $f(\widehat{\theta})=0$, then we have a relation among radii of curvatures as

$$\frac{1}{\widehat{\tau}}-\frac{1}{\widehat{\tau}^*}=0. \tag{14}$$

For this case, the equation (7) turns into

$$\begin{aligned}&\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}+\varepsilon(\widehat{K}_1^2-\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}+3\varepsilon(\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}}-\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}})\widehat{\gamma}\\&+\varepsilon(\textstyle\int\widehat{K}_1d\widehat{\theta})\widehat{\gamma}\frac{d^2\widehat{K}_1}{d\widehat{\theta}^2}-\varepsilon(\textstyle\int\widehat{K}_2d\widehat{\theta})\widehat{\gamma}\frac{d^2\widehat{K}_2}{d\widehat{\theta}^2}=0\end{aligned} \tag{15}$$

The equation (15) is a characterization for the components. However, its general solution of has not been found. Due to this, we investigate its solutions in special cases.

Let us suppose that $\widehat{K}_1=\widehat{K}_2=0$, then we rewrite the equation (15) as

$$\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}=0. \tag{16}$$

By this way, we have the components as follows:

$$\begin{cases}\widehat{\gamma}=\widehat{c}_1+\widehat{c}_2\widehat{\theta}+\widehat{c}_3\widehat{\theta}^2,\\ \widehat{\delta}=\text{constant},\\ \widehat{\lambda}=\text{constant}.\end{cases} \tag{17}$$

3.2 Dual Timelike Curves of Constant Breadth According to Dual Bishop Frame

Let $\widehat{\varphi}=\widehat{\varphi}(s)$ be a simple closed dual timelike curve in $\mathbb{D}_1^3$. We consider a dual timelike curve in the class Γ as in [6] having parallel tangents $\widehat{T}_\varphi$ and $\widehat{T}_\alpha$ in opposite directions at the opposite points $\widehat{\varphi}$ and $\widehat{\alpha}$ of the curve according to Bishop frame. A simple closed dual timelike curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to dual Bishop frame by the equation

$$\widehat{\alpha}=\widehat{\varphi}+\widehat{\gamma}\widehat{T}+\widehat{\delta}\widehat{N}_1+\widehat{\lambda}\widehat{N}_2, \tag{18}$$

where $\widehat{\gamma},\widehat{\delta}$ and $\widehat{\lambda}$ are arbitrary functions of s. Differentiating both sides of (18), we get

$$\frac{d\widehat{\alpha}}{ds_\alpha}\frac{ds_\alpha}{ds}=(\frac{d\widehat{\gamma}}{ds}+\widehat{\delta}\widehat{k}_1+\widehat{\lambda}\widehat{k}_2+1)\widehat{T}+(\widehat{\gamma}\widehat{k}_1+\frac{d\widehat{\delta}}{ds})\widehat{N}_1+(\widehat{\gamma}\widehat{k}_2+\frac{d\widehat{\lambda}}{ds})\widehat{N}_2. \tag{19}$$

Considering $\widehat{T}_\alpha=-\widehat{T}_\varphi$ by the Definition 3.1, we have the following system of equations

$$\begin{cases}\frac{d\widehat{\gamma}}{ds}=\frac{ds_\alpha}{ds}-\widehat{\delta}\widehat{k}_1-\widehat{\lambda}\widehat{k}_2-1,\\ \frac{d\widehat{\delta}}{ds}=-\widehat{\gamma}\widehat{k}_1,\\ \frac{d\widehat{\lambda}}{ds}=-\widehat{\gamma}\widehat{k}_2.\end{cases} \tag{20}$$

If we call $\widehat{\theta}$ as the angle between the tangent of the curve C at point $\widehat{\varphi}$ with a given direction and taking $\dfrac{d\widehat{\theta}}{ds}=\widehat{\tau}, \dfrac{d\widehat{\theta}}{ds_\alpha}=\widehat{\tau}^*$ into account, we have (20) as follow;

$$\begin{cases} \dfrac{d\widehat{\gamma}}{d\widehat{\theta}} = & -\widehat{\delta}\dfrac{\widehat{k}_1}{\widehat{\tau}}-\widehat{\lambda}\dfrac{\widehat{k}_2}{\widehat{\tau}}-f(\widehat{\theta}), \\ \dfrac{d\widehat{\delta}}{d\widehat{\theta}} = & -\widehat{\gamma}\dfrac{\widehat{k}_1}{\widehat{\tau}}, \\ \dfrac{d\widehat{\lambda}}{d\widehat{\theta}} = & -\widehat{\gamma}\dfrac{\widehat{k}_2}{\widehat{\tau}}, \end{cases} \tag{21}$$

where $f(\widehat{\theta})=\dfrac{1}{\widehat{\tau}}-\dfrac{1}{\widehat{\tau}^*}$.

Let $\widehat{K}_1=\dfrac{\widehat{k}_1}{\widehat{\tau}}, \widehat{K}_2=\dfrac{\widehat{k}_2}{\widehat{\tau}}$ and using the system of ordinary differential equations (21), we have the following dual third order differential equation with respect to $\widehat{\gamma}$ as;

$$\begin{aligned} &\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}-(\widehat{K}_1^2+\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}-3\varepsilon(\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}}+\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}})\widehat{\gamma} \\ &-(\textstyle\int \widehat{K}_1 d\widehat{\theta})\widehat{\gamma}\dfrac{d^2\widehat{K}_1}{d\widehat{\theta}^2}-(\textstyle\int \widehat{K}_2 d\widehat{\theta})\widehat{\gamma}\dfrac{d^2\widehat{K}_2}{d\widehat{\theta}^2}-\dfrac{d^2 f(\widehat{\theta})}{d\widehat{\theta}^2}=0. \end{aligned} \tag{22}$$

We can give the following corollary.

Corollary 3.2.1 *The dual differential equation of third order given in (22) is a characterization of the simple closed dual timelike curve* $\widehat{\alpha}$ *according to Bishop frame in* $\mathbb{D}_1^3$.

Since position vector of a simple closed dual timelike curve can be determined by solution of (22), let us investigate solution of the equation (22) in a special case. Let $\widehat{K}_1, \widehat{K}_2$ and $f(\widehat{\theta})$ be constants. Then the equation (22) turns into the following form

$$\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}-(\widehat{K}_1^2+\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}=0. \tag{23}$$

Solution of equation (23) yields the components

$$\begin{cases} \widehat{\gamma}=\widehat{A}+\widehat{B}e^{(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}+\widehat{C}e^{-(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}, \\ \\ \widehat{\delta}=-\int\left\{\widehat{A}+\widehat{B}e^{(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}+\widehat{C}e^{-(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}\right\}\widehat{K}_1 d\widehat{\theta}, \\ \\ \widehat{\lambda}=-\int\left\{\widehat{A}+\widehat{B}e^{(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}+\widehat{C}e^{-(\widehat{K}_1^2+\widehat{K}_2^2)\widehat{\theta}}\right\}\widehat{K}_2 d\widehat{\theta}. \end{cases} \tag{24}$$

Corollary 3.2.3 *Position vector of a simple dual timelike closed curve with constant dual curvature and constant dual torsion according to Bishop frame is obtained in terms of the values of* $\widehat{\gamma}$, $\widehat{\delta}$ *and* $\widehat{\lambda}$ *in the equation (24).*

If the distance between opposite points of $\widehat{\varphi}$ and $\widehat{\alpha}$ is constant, then we can write that

$$\|\widehat{\alpha}-\widehat{\varphi}\| = -\widehat{\gamma}^2+\widehat{\delta}^2+\widehat{\lambda}^2 = \text{constant}. \tag{25}$$

Differentiating (25) with respect to $\widehat{\theta}$ gives

$$-\widehat{\gamma}\frac{d\widehat{\gamma}}{d\widehat{\theta}}+\widehat{\delta}\frac{d\widehat{\delta}}{d\widehat{\theta}}+\widehat{\lambda}\frac{d\widehat{\lambda}}{d\widehat{\theta}} = 0. \tag{26}$$

By virtue of (21), the differential equation (26) yields

$$\widehat{\gamma}f(\widehat{\theta}) = 0. \tag{27}$$

There are two cases for the equation (27), we study these cases as follows:

Case 1. If $\widehat{\gamma}=0$ then we find that the components $\widehat{\delta}$ and $\widehat{\lambda}$ are constants.

Hence, Dual timelike curves of constant breadth according to Bishop frame can be written as

$$\widehat{\alpha} = \widehat{\varphi}+\widehat{l}_1\widehat{T}+\widehat{l}_2\widehat{N}_1+\widehat{l}_3\widehat{N}_2, \tag{28}$$

where $\widehat{\gamma}=\widehat{l}_1, \widehat{\delta}=\widehat{l}_2, \widehat{\lambda}=\widehat{l}_3$; $\widehat{l}_1, \widehat{l}_2, \widehat{l}_3$ are constants.

Case 2. If $f(\widehat{\theta})=0$, then we have a relation among radii of curvatures as

$$\frac{1}{\widehat{\tau}}-\frac{1}{\widehat{\tau}^*} = 0. \tag{29}$$

For this case, the equation (22) turns into

$$\begin{aligned} &\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3}-(\widehat{K}_1^2+\widehat{K}_2^2)\frac{d\widehat{\gamma}}{d\widehat{\theta}}-3\varepsilon(\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}}+\widehat{K}_1\frac{d\widehat{K}_1}{d\widehat{\theta}})\widehat{\gamma} \\ &-(\textstyle\int\widehat{K}_1 d\widehat{\theta})\widehat{\gamma}\frac{d^2\widehat{K}_1}{d\widehat{\theta}^2}-(\textstyle\int\widehat{K}_2 d\widehat{\theta})\widehat{\gamma}\frac{d^2\widehat{K}_2}{d\widehat{\theta}^2} = 0. \end{aligned} \tag{30}$$

The equation (30) is a characterization for the components. However, its general solution has not been found. Due to this, we investigate its solutions in special cases.

Let us suppose that $\widehat{K}_1=\widehat{K}_2=0$, then we rewrite the equation (30) as

$$\frac{d^3\widehat{\gamma}}{d\widehat{\theta}^3} = 0. \tag{31}$$

By this way, we have the components as follows:

$$\begin{cases} \widehat{\gamma} = \widehat{c}_1+\widehat{c}_2\widehat{\theta}+\widehat{c}_3\widehat{\theta}^2, \\ \widehat{\delta} = \text{constant}, \\ \widehat{\lambda} = \text{constant}. \end{cases} \tag{32}$$

References

[1] N.Ayyildiz, A.C. Çöken, and A.A. Yücesan, A characterization of dual Lorentzian spherical curves in the dual Lorentzian space, *Taiwanese J. Math.* 11 (4) (2007), 999–1018.

[2] L.R.Bishop, There is More Than One Way to Frame a Curve, *Amer. Math. Monthly* 82 (1975), 246-251.

[3] W.Blaschke, Konvexe Bereiche Gegebener Konstanter Breite und Kleinsten Inhalts. *Math. Ann.* 76 (1915), 504-513.

[4] B.Bükçü and M.K.Karacan, The slant helices according to bishop frame of the spacelike curve in Lorentzian space, *Jour. of Inter. Math.* 12(5) (2009), 691-700.

[5] L.Euler, De curvis trangularibus, *Acta Acad Petropol*, (1870), 1870.

[6] M.Fujivara, On Space Curves of Constant Breadth, *Tohoku Math. J.* (5) (1963), 179-184.

[7] H.Guggenheimer, *Differential Geometry*, McGraw Hill, New York, 1963.

[8] T.Körpınar, E.Turhan, On characterization of image-canal surfaces in terms of biharmonic image-slant helices according to Bishop frame in Heisenberg group $Heis^3$*, J. Math. Anal. Appl.* 382 (1) (2011) 57–65.

[9] Ö. Köse, Ş. Nizamoglu, and M. Sezer, An explicit characterization of dual spherical curves, *Doğa Turk. J. Math.*, 12 (3) (1988), 105–113.

[10] Ö. Köse, On Space Curves of Constant Breadth, *Doğa Turk. J. Math.* (10) 1 (1986) 11-14.

[11] Ö. Köse, Some Properties of Ovals and Curves of Constant Width in a Plane, *Doğa Turk. J. Math.* 8 (1984), 119-126.

[12] H.Martini, Z.Mustafaev, A new construction of curves of constant width, *Comp. Aid. Geom. Des.* 25 (9) (2008) 751-755.

[13] B.O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

[14] F.Reuleaux, *The Kinematics of Machinery*, Dover Publications, New York, 1963.

[15] G.R. Veldkamp, On the use of dual numbers, vectors and matrices in instantaneous spatial kinematics, *Mech. Math. Theory*, 11 (1976), 141-156.

[16] S.Yılmaz, Timelike dual curves of constant breadth in dual Lorentzian space, *IBSU Sci. J.* 2 (2008) 129- 136.

[17] S.Yılmaz, M.Turgut, On the time-like curves of constant breadth in Minkowski 3-Space, *Int. J. Math. Combin.*, 3 (2008) 34-39.

[18] S.Yılmaz, Bishop spherical images of a spacelike curve in Minkowski 3-space, *Int. Jour. of the Phys. Scien.* 5(6) (2010) 898-905.