

## On Equitable Coloring of Helm Graph and Gear Graph

Kaliraj.K

Department of Mathematics, R.V.S College of Engineering and Technology  
Coimbatore- 641 402 and Part time Research scholar, Anna University of Technology, Coimbatore-641 047

Vernold Vivin.J

Department of Mathematics,  
University College of Engineering Nagercoil Anna University of Technology Tirunelveli (Nagercoil Campus)  
Nagercoil-629 004, Tamil Nadu, India

Email: k\_kaliraj@yahoo.com, vernold\_vivin@yahoo.com

**Abstract:** The notion of equitable coloring was introduced by Meyer in 1973. In this paper we obtain interesting results regarding the equitable chromatic number  $\chi_=(G)$  for the Helm Graph  $H_n$ , line graph of Helm graph  $L(H_n)$ , middle graph of Helm graph  $M(H_n)$ , total graph of Helm graph  $T(H_n)$ , Gear graph  $G_n$ , line graph of Gear graph  $L(G_n)$ , middle graph of Gear graph  $M(G_n)$ , total graph of Gear graph  $T(G_n)$ .

**Key Words:** Smarandachely equitable  $k$ -colorable graph, Equitable coloring, Helm graph, Gear graph, line graph, middle graph and total graph.

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### §1. Introduction

Let  $G = (V, E)$  be a graph. If the vertices of  $G$  can be partitioned into  $m$  classes  $V_1, V_2, \dots, V_m$  such that each  $V_i$ , ( $1 \leq i \leq m$ ) are independent and  $||V_i| - |V_j|| \leq 1$  holds for every pair ( $1 \leq i, j \leq k$ ), where  $k \leq m$ , then the graph  $G$  is said to be *Smarandachely equitable  $k$ -colorable*. Particularly, if  $k = m$ , we abbreviated it to *equitably  $k$ -colorable*. The smallest integer  $k$  for which  $G$  is Smarandachely equitable  $k$ -colorable or equitable  $k$ -colorable is known as the *Smarandachely equitable chromatic number* or *equitable chromatic number* [1,3,7-10] of  $G$  and denoted by  $\chi_=(G)$ , respectively.

This model of graph coloring has many applications. Every time when we have to divide a system with binary conflicting relations into equal or almost equal conflict-free subsystems we can model such situation by means of equitable graph coloring. This subject is widely discussed in literature [3,8-10]. In general, the problem of optimal equitable coloring, in the sense of the number color used, is NP-hard. So we have to look for simplified structure of graphs allowing polynomial-time algorithms. This paper gives such solution for Helm graph and Gear graph families: Helm graph, its line, middle and total graphs; Gear graph, its line, middle and total graphs.

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## §2. Preliminaries

The Helm graph  $H_n$  is the graph obtained from an  $n$ -wheel graph by adjoining a pendent edge at each node of the cycle.

The Gear graph  $G_n$ , also known as a bipartite wheel graph, is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle.

The line graph [2,5] of  $G$ , denoted by  $L(G)$  is the graph with vertices are the edges of  $G$  with two vertices of  $L(G)$  adjacent whenever the corresponding edges of  $G$  are adjacent.

The middle graph [4] of  $G$ , is defined with the vertex set  $V(G) \cup E(G)$  where two vertices are adjacent iff they are either adjacent edges of  $G$  or one is the vertex and the other is an edge incident with it and it is denoted by  $M(G)$ .

Let  $G$  be a graph with vertex set  $V(G)$  and edge set  $E(G)$ . The total graph [2,4,5] of  $G$ , denoted by  $T(G)$  is defined in the following way. The vertex set of  $T(G)$  is  $V(G) \cup E(G)$ . Two vertices  $x, y$  in the vertex set of  $T(G)$  are adjacent in  $T(G)$  in case one of the following holds: (i)  $x, y$  are in  $V(G)$  and  $x$  is adjacent to  $y$  in  $G$ . (ii)  $x, y$  are in  $E(G)$  and  $x, y$  are adjacent in  $G$ . (iii)  $x$  is in  $V(G)$ ,  $y$  is in  $E(G)$ , and  $x, y$  are incident in  $G$ . Additional graph theory terminology used in this paper can be found in [2,5,9].

## §3. Equitable Coloring on Helm Graph

**Theorem 3.1** *If  $n \geq 4$  the equitable chromatic number of Helm graph  $H_n$ ,*

$$\chi_=(H_n) = \begin{cases} 3 & \text{if } n \text{ is even,} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

*Proof* Let  $H_n$  be the Helm graph obtained by attaching a pendant edge at each vertex of the cycle. Let  $V(H_n) = \{v\} \cup \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\}$  where  $v_i$ 's are the vertices of cycles taken in cyclic order and  $u_i$ 's are pendant vertices such that each  $v_i u_i$  is a pendant edge and  $v$  is a hub of the cycle.

**Case i:** If  $n$  is even.

**Case i-a:** If  $n = 6k - 2$  for some positive integer  $k$ , then set the partition of  $V$  as below.

$V_1 = \{v\} \cup \{u_i : 2k + 1 \leq i \leq 6k - 2\}$ ;  $V_2 = \{v_{2i-1} : 1 \leq i \leq 3k - 1\} \cup \{u_{2i} : 1 \leq i \leq k\}$ ;  $V_3 = \{v_{2i} : 1 \leq i \leq 3k - 1\} \cup \{u_{2i-1} : 1 \leq i \leq k\}$ . Clearly  $V_1, V_2, V_3$  are independent sets of  $V(H_n)$ . Also  $|V_1| = |V_2| = |V_3| = 4k - 1$ , it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .

**Case i-b:** If  $n = 6k$  for some positive integer  $k$ , then set the partition of  $V$  as below.

$V_1 = \{v\} \cup \{u_i : 2k + 2 \leq i \leq 6k\}$ ;  $V_2 = \{v_{2i-1} : 1 \leq i \leq 3k\} \cup \{u_{2i} : 1 \leq i \leq k\}$ ;  $V_3 = \{v_{2i} : 1 \leq i \leq 3k\} \cup \{u_{2i-1} : 1 \leq i \leq k + 1\}$ . Clearly  $V_1, V_2, V_3$  are independent sets of  $V(H_n)$ . Also  $|V_1| = |V_2| = 4k$  and  $|V_3| = 4k + 1$ , it holds the inequality  $||V_i| - |V_j|| \leq 1$  for

every pair  $(i, j)$ .

**Case i-c:** If  $n = 6k + 2$  for some positive integer  $k$ , then set the partition of  $V$  as below.

$V_1 = \{v\} \cup \{u_i : 2k + 3 \leq i \leq 6k + 2\}$ ;  $V_2 = \{v_{2i-1} : 1 \leq i \leq 3k + 1\} \cup \{u_{2i} : 1 \leq i \leq k + 1\}$ ;  
 $V_3 = \{v_{2i} : 1 \leq i \leq 3k + 1\} \cup \{u_{2i-1} : 1 \leq i \leq k + 1\}$ . Clearly  $V_1, V_2, V_3$  are independent sets of  $V(H_n)$ .  $|V_1| = 4k + 1$  and  $|V_2| = |V_3| = 4k + 2$ , it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .

In all the three subcases of Case i,  $\chi_=(H_n) \leq 3$ . Since  $\chi(H_n) \geq 3$ ,  $\chi_=(H_n) \geq \chi(H_n) \geq 3$ ,  $\chi_=(H_n) \geq 3$ . Therefore  $\chi_=(H_n) = 3$ .

**Case ii:** If  $n$  is odd.

**Case ii-a:** If  $n = 6k - 3$  for some positive integer  $k$ , then set the partition of  $V$  as below.

$V_1 = \{v\} \cup \{u_i : 3k + 1 \leq i \leq 5k - 1\}$ ;  $V_2 = \{v_{3i-2} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i} : 1 \leq i \leq k\}$ ;  
 $V_3 = \{v_{3i-1} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-2} : 1 \leq i \leq k\}$ ;  $V_4 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-1} : 1 \leq i \leq k\}$ . Clearly  $V_1, V_2, V_3$  are independent sets of  $V(H_n)$ . Also  $|V_1| = 3k - 2$  and  $|V_2| = |V_3| = |V_4| = 3k - 1$ , it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .

**Case ii-b:** If  $n = 6k - 1$  for some positive integer  $k$ , then set the partition of  $V$  as below.

$V_1 = \{v\} \cup \{u_i : 3k + 2 \leq i \leq 6k - 1\}$ ;  $V_2 = \{v_{3i-2} : 1 \leq i \leq 2k\} \cup \{u_{3i-1} : 1 \leq i \leq 2k - 1\}$ ;  
 $V_3 = \{v_{3i-1} : 1 \leq i \leq 2k\} \cup \{u_{3i} : 1 \leq i \leq 2k - 1\}$ ;  $V_4 = \{v_{3i} : 1 \leq i \leq 2k - 1\} \cup \{u_{3i-2} : 1 \leq i \leq 2k\}$ . Clearly  $V_1, V_2, V_3$  are independent sets of  $V(H_n)$ . Also  $|V_1| = 4k - 2$  and  $|V_2| = |V_3| = |V_4| = 4k - 1$ , it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .

**Case ii-c:** If  $n = 6k + 1$  for some positive integer  $k$ , then set the partition of  $V$  as below.

$V_1 = \{v\} \cup \{u_i : 3k + 2 \leq i \leq 6k + 1\}$ ;  $V_2 = \{v_{3i-2} : 1 \leq i \leq 2k\} \cup \{u_{3i-1} : 1 \leq i \leq 2k - 1\}$ ;  
 $V_3 = \{v_{3i-1} : 1 \leq i \leq 2k\} \cup \{v_n\} \cup \{u_{3i} : 1 \leq i \leq 2k - 1\}$ ;  $V_4 = \{v_{3i} : 1 \leq i \leq 2k\} \cup \{u_1\} \cup \{u_{3i+1} : 1 \leq i \leq 2k - 1\}$ . Clearly  $V_1, V_2, V_3$  are independent sets of  $V(H_n)$ .  $|V_1| = |V_3| = |V_4| = 4k$  and  $|V_2| = 4k - 1$ , it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .

In all the three subcases of cases (ii),  $\chi_=(H_n) \leq 4$ . Since  $\chi(H_n) \geq 4$ ,  $\chi_=(H_n) \geq \chi(H_n) \geq 4$ ,  $\chi_=(H_n) \geq 4$ . Therefore  $\chi_=(H_n) = 4$ .  $\square$

#### §4. Equitable Coloring on Line graph, Middle Graph and Total Graph of Helm Graph

**Theorem 4.1** *If  $n \geq 4$  the equitable chromatic number on line graph of Helm graph  $L(H_n)$ ,  $\chi_=(L(H_n)) = n$ .*

*Proof* Let  $V(H_n) = \{v\} \cup \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\}$  and  $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$  where  $e_i$  is the edge  $vv_i$  ( $1 \leq i \leq n$ ),  $e'_i$  is the edge  $v_i v_{i+1}$  ( $1 \leq i \leq n-1$ ),  $e'_n$  is the edge  $v_n v_1$  and  $s_i$  is the edge  $v_i u_i$  ( $1 \leq i \leq n$ ). By the definition of line graph  $V(L(H_n)) = E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$ .

Now, we partition the vertex set of  $V(L(H_n))$  as below.

$V_1 = \{e_1, e'_2, s_n\}$ ;  $V_i = \{e_i, e'_{i+1}, s_{i-1} : 2 \leq i \leq n-1\}$ ;  $V_n = \{e_n, e'_1, s_{n-1}\}$ . Clearly  $V_1, V_i, V_n$  ( $2 \leq i \leq n-1$ ) are independent sets of  $L(H_n)$ . Also  $|V_1| = |V_i| = |V_n| = 3$  ( $2 \leq i \leq n-1$ ), it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .  $\chi_=(L(H_n)) \leq n$ . Since  $e_i$  ( $1 \leq i \leq n$ ) forms a clique of order  $n$ ,  $\chi(L(H_n)) \geq n$ ,  $\chi_=(L(H_n)) \geq \chi(L(H_n)) \geq n$ ,  $\chi_=(L(H_n)) \geq n$ . Therefore  $\chi_=(L(H_n)) = n$ .  $\square$

**Theorem 4.2** *If  $n \geq 5$  the equitable chromatic number on middle graph of Helm graph  $M(H_n)$ ,  $\chi_=(M(H_n)) = n + 1$ .*

*Proof* Let  $V(H_n) = \{v\} \cup \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\}$  and  $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$  where  $e_i$  is the edge  $vv_i$  ( $1 \leq i \leq n$ ),  $e'_i$  is the edge  $v_i v_{i+1}$  ( $1 \leq i \leq n-1$ ),  $e'_n$  is the edge  $v_n v_1$  and  $s_i$  is the edge  $v_i u_i$  ( $1 \leq i \leq n$ ). By the definition of middle graph  $V(M(H_n)) = V(H_n) \cup E(H_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$ .

Now, we partition the vertex set of  $V(M(H_n))$  as below.

$V_1 = \{e_1, e'_2, u_1, s_n\}$ ;  $V_i = \{v_{i-1}, u_i, e_i, e'_{i+1} : 2 \leq i \leq n-1\} \cup \{s_{i-2} : 3 \leq i \leq n+1\}$ ;  $V_n = \{v_{n-1}, s_{n-2}, e_n, e'_1\}$ ;  $V_{n+1} = \{v, v_n, s_{n-1}, u_n\}$ . Clearly  $V_1, V_2, \dots, V_n, V_{n+1}$  are independent sets of  $M(H_n)$ . Also  $|V_1| = |V_2| = |V_n| = |V_{n+1}| = 4$  and  $|V_i| = 5$  ( $3 \leq i \leq n-1$ ), it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .  $\chi_=(M(H_n)) \leq n+1$ . Since  $ve_i$  ( $1 \leq i \leq n$ ) forms a clique of order  $n+1$ ,  $\chi(M(H_n)) \geq n+1$ ,  $\chi_=(M(H_n)) \geq \chi(M(H_n)) \geq n+1$ ,  $\chi_=(M(H_n)) \geq n+1$ . Therefore  $\chi_=(M(H_n)) = n + 1$ .  $\square$

**Theorem 4.3** *If  $n \geq 5$  the equitable chromatic number on total graph of Helm graph  $T(H_n)$ ,  $\chi_=(T(H_n)) = n + 1$ .*

*Proof* Let  $V(H_n) = \{v\} \cup \{v_1, v_2, \dots, v_n\} \cup \{u_1, u_2, \dots, u_n\}$  and  $E(H_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n-1\} \cup \{e'_n\} \cup \{s_i : 1 \leq i \leq n\}$  where  $e_i$  is the edge  $vv_i$  ( $1 \leq i \leq n$ ),  $e'_i$  is the edge  $v_i v_{i+1}$  ( $1 \leq i \leq n-1$ ),  $e'_n$  is the edge  $v_n v_1$  and  $s_i$  is the edge  $v_i u_i$  ( $1 \leq i \leq n$ ). By the definition of total graph  $V(T(H_n)) = V(H_n) \cup E(H_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$ .

Now, we partition the vertex set of  $V(T(H_n))$  as below.

$V_1 = \{e_1, e'_2, u_3, v_n\}$ ;  $V_2 = \{e_2, v_2, e'_3, s_n, u_4\}$ ;  $V_i = \{e_i, v_{i-1}, e'_{i+1}, s_{i-2}, u_{i+2} : 3 \leq i \leq n-2\}$ ;  $V_{n-1} = \{e_{n-1}, v_{n-2}, e'_n, s_{n-3}\}$ ;  $V_n = \{e_n, v_{n-1}, e'_1, s_{n-2}\}$ ;  $V_{n+1} = \{v, s_{n-1}, u_1, u_2\}$ . Clearly  $V_1, V_2, V_i, V_{n-1}, V_n, V_{n+1}$  ( $3 \leq i \leq n-2$ ) are independent sets of  $T(H_n)$ . Also  $|V_1| = |V_n| = |V_{n+1}| = 4$  and  $|V_2| = |V_i| = 5$  ( $3 \leq i \leq n-2$ ), it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ ,  $\chi_=(T(H_n)) \leq n + 1$ . Since  $ve_i$  ( $1 \leq i \leq n$ ) forms a clique of order  $n + 1$ ,  $\chi(T(H_n)) \geq n + 1$ ,  $\chi_=(T(H_n)) \geq \chi(T(H_n)) \geq n + 1$ ,  $\chi_=(T(H_n)) \geq n + 1$ . Therefore  $\chi_=(T(H_n)) = n + 1$ .

## §5. Equitable Coloring on Gear Graph, Line Graph, Middle Graph and Total Graph

**Theorem 5.1** *If  $n \geq 3$  the equitable chromatic number of gear graph  $G_n$ ,  $\chi_=(G_n) = 2$ .*

*Proof* Let  $V(G_n) = \{v\} \cup \{v_1, v_2, \dots, v_{2n}\}$  where  $v_i$ 's are the vertices of cycles taken in cyclic order and  $v$  is adjacent with  $v_{2i-1}$  ( $1 \leq i \leq n$ ).

Now, we partition the vertex set of  $V(G_n)$  as below.

$V_1 = \{v\} \cup \{v_{2i} : 1 \leq i \leq n-1\}$ ;  $V_2 = \{v_{2i-1} : 1 \leq i \leq n\}$ . Clearly  $V_1, V_2$  are independent sets of  $(G_n)$ . Also  $|V_1| = n+1$  and  $|V_2| = n$ , it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .  $\chi_=(G_n) \leq 2$ .  $\chi(G_n) \geq 2$ ,  $\chi_=(G_n) \geq \chi(G_n) \geq 2$ ,  $\chi_=(G_n) \geq 2$ . Therefore  $\chi_=(G_n) = 2$ .  $\square$

## §6. Equitable Coloring on Line Graph, Middle Graph and Total Graph of Gear Graph

**Theorem 6.1** *If  $n \geq 3$  the equitable chromatic number on line graph of Gear graph  $L(G_n)$ ,  $\chi_=(L(G_n)) = n$ .*

*Proof* Let  $V(G_n) = \{v\} \cup \{v_1, v_2, \dots, v_{2n}\}$  and  $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e'_n\}$  where  $e_i$  is the edge  $vv_{2i-1}$  ( $1 \leq i \leq n$ ),  $e'_i$  is the edge  $v_i v_{i+1}$  ( $1 \leq i \leq 2n-1$ ), and  $e'_n$  is the edge  $v_{2n-1} v_1$ . By the definition of line graph  $V(L(G_n)) = E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$ .

Now, we partition the vertex set of  $V(L(G_n))$  as below.

$V_1 = \{e_1, e'_2, e'_{2n-1}\}$ ;  $V_i = \{e_i, e'_{2i}, e'_{2i-3} : 2 \leq i \leq n\}$ . Clearly  $V_1, V_2, \dots, V_n$  are independent sets of  $L(G_n)$ . Also  $|V_1| = |V_i| = 3$  ( $2 \leq i \leq n$ ), it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .  $\chi_=(L(G_n)) \leq n$ . Since  $e_i$  ( $1 \leq i \leq n$ ) forms a clique of order  $n$ ,  $\chi(L(G_n)) \geq n$ ,  $\chi_=(L(G_n)) \geq \chi(L(G_n)) \geq n$ ,  $\chi_=(L(G_n)) \geq n$ . Therefore  $\chi_=(L(G_n)) = n$ .  $\square$

**Theorem 6.2** *If  $n \geq 5$  the equitable chromatic number on middle graph of Gear graph  $M(G_n)$ ,  $\chi_=(M(G_n)) = n+1$ .*

*Proof* Let  $V(G_n) = \{v\} \cup \{v_1, v_2, \dots, v_{2n}\}$  and  $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e'_n\}$  where  $e_i$  is the edge  $vv_{2i-1}$  ( $1 \leq i \leq n$ ),  $e'_i$  is the edge  $v_i v_{i+1}$  ( $1 \leq i \leq 2n-1$ ), and  $e'_n$  is the edge  $v_{2n-1} v_1$ . By the definition of middle graph  $V(M(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$ .

Now, we partition the vertex set of  $V(M(G_n))$  as below.

$V_1 = \{e_1, v_2, e'_{2n-4}, e'_{2n-1}\}$ ;  $V_2 = \{e_2, v_1, v_4, e'_{2n-2}\}$ ;  $V_3 = \{e_3, v_3, e'_1, v_6, e'_{2n}\}$ ;  $V_i = \{e_i, e'_{2i-5}, v_{2i-3}, v_{2i} : 4 \leq i \leq n\} \cup \{e'_{2i-8} : 5 \leq i \leq n\}$ ;  $V_{n+1} = \{v, e'_{2n-6}, v_{2n-1}, e'_{2n-3}\}$ . Clearly  $V_1, V_2, \dots, V_n, V_{n+1}$  are independent sets of  $M(G_n)$ . Also  $|V_1| = |V_2| = |V_{n+1}| = 4$  and  $|V_i| = |V_3| = 5$  ( $5 \leq i \leq n$ ), it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ .  $\chi_=(M(G_n)) \leq n+1$ . Since  $ve_i$  ( $1 \leq i \leq n$ ) forms a clique of order  $n+1$ ,  $\chi(M(G_n)) \geq n+1$ ,  $\chi_=(M(G_n)) \geq \chi(M(G_n)) \geq n+1$ ,  $\chi_=(M(G_n)) \geq n+1$ . Therefore  $\chi_=(M(G_n)) = n+1$ .  $\square$

**Theorem 6.3** *If  $n \geq 5$  the equitable chromatic number on total graph of Gear graph  $T(G_n)$ ,  $\chi_=(T(G_n)) = n+1$ .*

*Proof* Let  $V(G_n) = \{v\} \cup \{v_1, v_2, \dots, v_{2n}\}$  and  $E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n-1\} \cup \{e'_n\}$  where  $e_i$  is the edge  $vv_{2i-1}$  ( $1 \leq i \leq n$ ),  $e'_i$  is the edge  $v_i v_{i+1}$  ( $1 \leq i \leq 2n-1$ ),

and  $e'_{2n}$  is the edge  $v_{2n-1}v_1$ . By the definition of total graph  $V(T(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq 2n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq 2n\}$ .

Now, we partition the vertex set of  $V(T(G_n))$  as below.

$V_1 = \{e_1, e'_2, v_4, v_{2n-1}\}$ ;  $V_2 = \{e_2, v_1, e'_4, v_6\}$ ;  $V_i = \{e_i, e'_{2i-5}, v_{2i-3}, e'_{2i}, v_{2i+2} : 3 \leq i \leq n-1\}$ ;  $V_n = \{e_n, e'_{2n-5}, v_{2n-3}, e'_{2n}\}$ ;  $V_{n+1} = \{v, v_2, e'_{2n-1}, e'_{2n-3}\}$ .

Clearly  $V_1, V_2, V_i, V_n, V_{n+1}$  ( $3 \leq i \leq n-1$ ) are independent sets of  $T(G_n)$ . Also  $|V_1| = |V_2| = |V_n| = |V_{n+1}| = 4$  and  $|V_i| = 5$  ( $3 \leq i \leq n-1$ ), it holds the inequality  $||V_i| - |V_j|| \leq 1$  for every pair  $(i, j)$ ,  $\chi_=(T(G_n)) \leq n+1$ . Since  $v e_i$  ( $1 \leq i \leq n$ ) forms a clique of order  $n+1$ ,  $\chi(T(G_n)) \geq n+1$ ,  $\chi_=(T(G_n)) \geq \chi(T(G_n)) \geq n+1$ ,  $\chi_=(T(G_n)) \geq n+1$ . Therefore  $\chi_=(T(G_n)) = n+1$ .  $\square$

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