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On Equitable Coloring of Weak Product of Odd Cycles

Tayo Charles Adefokun

(Department of Computer and Mathematical Sciences, Crawford University, Nigeria)

Deborah Olayide Ajayi

(Department of Mathematics, University of Ibadan, Ibadan, Nigeria)

E-mail: tayo.adefokun@gmail.com, olayide.ajayi@mail.ui.edu.ng

Abstract: In this article, we present algorithms for equitable weak product graph of cycles C_m and C_n, $C_m \times C_n$ such that it has an equitable chromatic value, $\chi_=(C_m \times C_n) = 3$, with mn odd and m or n is not a multiple of 3.

Key Words: Equitable coloring, equitable chromatic number, weak product, direct product, cross product.

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§1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. A k-coloring on G is a function $f : V(G) \to [1, k] = \{1, 2, \cdots, k\}$, such that if $uv \in E(G), u, v \in V(G)$ then $f(u) \neq f(v)$. A value $\chi(G) = k$, the chromatic number of G is the smallest positive integer for which G is k-colorable. G is said to be equitably k-colorable if for a proper k-coloring of G with vertex color class $V_1, V_2 \cdots V_k$, then $|(|V_i| - |V_j|)| \leq 1$ for all $i, j \in [i, k]$. Suppose n is the smallest integer such that G is equitably k-colorable, then n is the equitable chromatic number, $\chi_=(G)$, of G.

The notion of equitable coloring of a graph was introduced in [6] by Meyer. Notable work on the subject includes [7] where outer planar graphs were considered and [8] where general planar graphs were investigated. In [1] equitable coloring of the product of trees was considered. Chen et al. in [2] showed that for $m, n \geq 3$, $\chi_=(C_m \times C_n = 2)$ if mn is even and $\chi_=(C_m \times C_n = 3)$ if mn is odd. Recent work include [4], [5]. Furmanczyk in [3] discussed the equitable coloring of product graphs in general, following [2], where the authors separated the proofs of mn into various parts including the following:

1. m, n odd with $n = 0 \mod 3$

2. m, n odd, with

   (a) either m or n, say n satisfying $n - 1 \equiv 0 \mod 3$

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(b) either m or n, say n satisfying $n-2 \equiv 0 \bmod 3$.

In this paper we present equitable coloring schemes which

1. improve the proof in (b) above and
2. can be employed in developing the equitable 3-coloring for $C_m \times C_n$ with mn odd.

§2. Preliminaries

Let G_1 and G_2 be two graphs with $V(G_1)$ and $E(G_1)$ as the vertex and edge sets for G_1 respectively and $V(G_2)$ and $E(G_2)$ as the vertex and edge sets of G_2 respectively. The weak product of G_1 and G_2 is the graph $G_1 \times G_2$ such that $V(G_1 \times G_2) = \{(u,v) = u \in V(G) \text{and } u \in V(G_2)\}$ and $E(G_1 \times G_2) =$
$\{(u_1v_1)(u_2v_2) : u_1u_2 \in E(G_1) \text{and } v_1v_2 \in E(G_2)\}$. A graph $P_m = u_0u_1u_2 \cdots u_{m-1}$ is a path of length $m-1$ if for all $u_i, v_j \in V(P_m), i \neq j$. A graph $C_m = u_0u_1u_2 \cdots u_{m-1}$ is a cycle of length m if for all $u_i, v_j \in V(C_m), i \neq j$ and $u_0u_{m-1} \in E(C_m)$.

The following results due to Chen et al gives the equitable chromatic numbers of product of cycles.

Theorem 2.1([2]) *Let $m, n \geq 3$. Then*

$$\chi_{=}(C_m \times C_n) = \begin{cases} 2 & \text{if } mn \quad \text{is} \quad \text{even} \\ 3 & \text{if } mn \quad \text{is} \quad \text{odd.} \end{cases}$$

We require the following lemma in the main result.

Lemma 2.2 *Let n be any odd integer and let $n-1 \equiv 0 \bmod 3$. Then $n-1 \equiv 0 \bmod 6$.*

Proof Since n is odd, then there exists a positive integer m, such that $n = 2m+1$. Now since n is odd then, $n-1$ is even. Let $2m \equiv 0 \bmod 3$. Clearly, $n \geq 3$. Now $2m = 3k$ where k is an even positive integer. Thus $2m = 3(2k')$ for some positive integer k' and thus $2m = 6k'$. Hence $n-1 = 6k'$. □

§3. Main Results

In this section, we present the algorithms for the equitable 3-coloring of $C_m \times C_n$ with where m and n are odd with say $n-1 \equiv 0 \bmod 3$ and $n-2 \equiv 0 \bmod 3$.

Algorithm 1 Let $C_m \times C_n$ be product graph and let mn be odd, with $n-1 = 0 \bmod 3$.

Step 1 Define the following coloring for $u_iv_j \in V(C_m \times C_n)$.

$$f(u_iv_j) = \begin{cases} \alpha_2 & \text{for } \{u_iv_j : j \in [n-1]; j \geq 5; j+1 = 0 \bmod 3\} \\ \alpha_1 & \text{for } \{u_iv_j : j \in [n-1]; j+2 = 0 \bmod 3\} \cup \{u_iv_2 : i \in [m-1]\} \\ \alpha_3 & \text{for } \{u_iv_j : j \in [n-1]; j \geq 6; j = 0 \bmod 3\} \cup \{u_iv_1, i \in [m-1]\}. \end{cases}$$

Step 2 For all $u_iv_0; i \in [2]$, define the following coloring:

(a)

$$f(u_iv_0) = \begin{cases} \alpha_1 & \text{for } i = 1 \\ \alpha_2 & \text{for } i = 0, 2 \end{cases}$$

(b)

$$f(u_iv_3) = \begin{cases} \alpha_2 & \text{for } i = 1, 2 \\ \alpha_3 & \text{for } i = 0 \end{cases}$$

Step 3 Repeat Step 2(a) and Step 2(b) for all u_iv_0 and u_iv_3 for each $i \in [x, x+2]$ where $x = 0 \mod 3$.

Proof of Algorithm 1 Suppose n is odd and $n - 1 = 0 \mod 3$. From Lemma 2.2 above, $n - 1 = 0 \mod 6$ and consequently, $n - 4 = 0 \mod 3$. Suppose $\frac{n-4}{3} = n'$, where n is a positive integer. Let $P_m \times P_{n-4}$ be a subgraph of $P_m \times P_n$, where $P_{n-4} = v_4v_5 \cdots v_{n-1}$.. For all $u_iv_j \in V(P_m \times P_{n-4})$, let

$$f(u_iv_j) = \begin{cases} \alpha_1 & \text{for } \{u_iv_j : j \in [n-1], j+2 = 0 \mod 3\} \\ \alpha_2 & \text{for } \{u_iv_j : j \in [n-1], j \geq 5; j+1 = 0 \mod 3\} \\ \alpha_3 & \text{for } \{u_iv_j : j \in [n-1]; j \geq 6; j = 0 \mod 3\}. \end{cases}$$

From $f(u_iv_j)$ defined above, we see that $P_m \times P_{n-4}$ is equitably 3-colorable with color set $\{\alpha_1, \alpha_2, \alpha_3\} \equiv [1, 3]$, where $|V_{\alpha_1}| = |V_{\alpha_2}| = |V_{\alpha_3}| = mn'$. Next we show that there exists a 3-coloring of $P_m \times P_4$ that merges with $P_m \times P_{n-4}$ whose 3-coloring is defined by $f(u_iv_j)$ above. First, let $F(P_3 \times P_4)$ be the 3- coloring such that

$$F(P_3 \times P_4) = \begin{matrix} \alpha_2 & \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_1 & \alpha_3 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 & \alpha_1 & \alpha_3 \end{matrix}$$

From $F(P_3 \times P_4)$ we observe for all $j \in [3]$, that for $F(u_0v_j) \subset F(P_3 \times P_4), |V_{\alpha_1}| = 1, |V_{\alpha_2}| = 1, |V_{\alpha_3}| = 2$; for $F(u_1v_j) \subset F(P_3 \times P_4), |V_{\alpha_1}| = 2, |V_{\alpha_2}| = 1, |V_{\alpha_3}| = 1$; and for $F(u_0v_j) \subset F(P_3 \times P_4), |V_{\alpha_1}| = 1, |V_{\alpha_2}| = 2, |V_{\alpha_3}| = 1$.

We observe, over all, that for $F(P_3 \times P_4), |V_{\alpha_1}| = |V_{\alpha_2}| = |V_{\alpha_3}| = 4$. These confirm that $P_3 \times P_4$ is equitably 3-colorable at every stage of $i \in [2]$ and that $F(P_2 \times P_4) \subset F(P_3 \times P_4)$ is an equitable 3-coloring of $P_2 \times P_4$ for both $P_2 \times P_4 \subset P_3 \times P_4$. Now the equitable 3-coloring of $P_m \times P_4$ is now obtainable by repeating $F(P_3 \times P_4)$ at each interval $[x, x+2]$, where $x = 0 \mod 3$, until we reach m. Clearly, $F(u_iv_3) \cap F(u_iv_4) = \emptyset$ since $\alpha_1 \notin F(u_iv_3)$. Thus $P_m \times P_n$ is equitably 3-colorable based on the colorings defined earlier. Likewise, $F(u_iv_0) \cap F(u_iu_{n-1}) = \emptyset$ since $\alpha_3 \notin F(u_iv_0)$. Thus $P_m \times C_n$ is equitably 3-colorable based on the coloring defined above for $P_m \times P_n$.

Finally, for any $m \geq 3$, the equitable 3-coloring of $P_m \times P_{n-4}$ with respect to $F(P_m \times P_{n-4})$ above is equivalent to the equitable 3-coloring of $C_m \times C_{n-4}$ since $u_iv_ju_iv_{j+1} \notin E(P_m \times P_{m-4})$ for all $j \in [n-5]$. Also, for $m \geq 3$ the equitable 3-coloring of $P_m \times P_4$ with respect to

$F(P_m \times P_4)$ above is equivalent to the equitably 3- coloring of $C_m \times C_4$ by mere observation. Thus, $C_m \times C_n$ is equitably 3- colorable or all positive integer m and odd positive integer n such that $n-1=0 \bmod 3$.

Algorithm 2 Let m or n, say n be odd such that $n-2=0 \bmod 3$.

Step 1 Define the following coloring:

$$f(u_iv_j)=\begin{cases}\alpha_1 & \text{for } \{u_iv_j : j\in[n-1], j+1=0 \bmod 3\}\\ \alpha_2 & \text{for } \{u_iv_j : j\in[n-1], j=0 \bmod 3\}\\ \alpha_3 & \text{for } \{u_iv_j : j\in[n-1], j-1=0 \bmod 3\}\end{cases}$$

Step 2(a) For all $i\in[2]$, let $f(u_iv_0)=\alpha_1,\alpha_2,\alpha_1$ respectively $\alpha_1,\alpha_2\in[2]$.

Step 2(b) For all $i\in[2]$, let $f(u_iv_1)=\alpha_3,\alpha_2,\alpha_3$ respectively, $\alpha_3\in[2]$.

Step 3 Repeat step 2(a) and Step 2(b) above for all $i\in[x,x+2]$, where x is a positive integer and $x=0 \bmod 3$.

Proof of Algorithm 2 Let n be odd and let $n-2=0 \bmod 3$. By $f(u_iv_j)$ in step 1, $P_m\times P_{n-2}$, where $P_{n-2}=v_2v_3\cdots v_{n-1}$, is equitably 3-colorable with $|V_{\alpha_1}|=|V_{\alpha_2}|=|V_{\alpha_3}|=mn''$ where $n''=\frac{n-2}{3}$ and $F(u_iv_2)\cap F(u_iv_{n-1})=\emptyset$ for all $i\in[m-1]$. Now, let

$$F(P_3\times P_2)=\begin{matrix}\alpha_1 & \alpha_3\\ \alpha_2 & \alpha_2\\ \alpha_1 & \alpha_3\end{matrix}$$

It is clear that $F(P_3\times P_2)$ above follows from the coloring defined in step 2 of the algorithm and that $F(P_3\times P_2)$ is an equitable 3-coloring of $P_3\times P_2$ where $|V_{\alpha_1}|=|V_{\alpha_2}|=|V_{\alpha_3}|=2$. It is also clear that $F(P_3\times P_2)$ has an equitable coloring at $P_1\times P_2$ with $|V_{\alpha_1}|=1,|V_{\alpha_2}|=0,|V_{\alpha_3}|=1$ and at $P_2\times P_2$ with $|V_{\alpha_1}|=1,|V_{\alpha_2}|=2,|V_{\alpha_3}|=1$. Now, let with $x=0 \bmod 3$. For all $x\in[m-1]$, let $f(u_xv_j)=\alpha_1,\alpha_3$ for both $j=0,1$ respectively; for $x+1\in[m-1]$, let $f(u_{x+1}v_j)=\alpha_2$, for $j=0,1$ and for $x+2\in[m-1]$, let $f(u_{x+2}v_j)=\alpha_1,\alpha_3$ for $j=0,1$. With this last scheme, we have $P_m\times P_2$ that has an equitable 3- coloring for any value of m.

Finally, we can see that $P_m\times P_2$, for any m, so equitably, 3-colored merges with $P_m\times P_{n-2}$ that is equitably 3-colored earlier by $f(u_iv_j)$, such that $F(u_iv_1)\cap F(u_iv_2)=\emptyset$ for all $i\in[m-1]$ (by a similar argument as in the proof of Algorithm 1) and $F(u_iv_0)\cap F(u_iv_{n-1})=\emptyset$ for all $i\in[m-1]$ (by a similar argument as in the proof of Algorithm 1). □

Likewise $C_m\times C_n$ is equitable 3-colorable (by a similar argument as in the proof of Algorithm 1). Therefore, $C_m\times C_n$ is equitably 3-colorable for any $m\geq 3$ and odd n, such that $n-2=0 \bmod 3$.

§4. Examples

In Fig.1, we demonstrate how our algorithms equitably color graphs $C_5\times C_5$ and $C_5\times C_7$, which are two cases that illustrate $n-2=0 \bmod 3$ and $n-1=0 \bmod 3$ respectively. In the first case, we see that $\chi_=(C_5 \times C_5) = 3$, with $|V_1| = 8$ $|V_2| = 9$ and $|V_3| = 8$ and in the second case, $\chi_=(C_5 \times C_7) = 3$, with $|V_1| = 12$ $|V_2| = 11$ and $|V_3| = 12$. (Note that the first coloring takes care of the third instance in subcase 2.4 of [2] where it is a special case.)

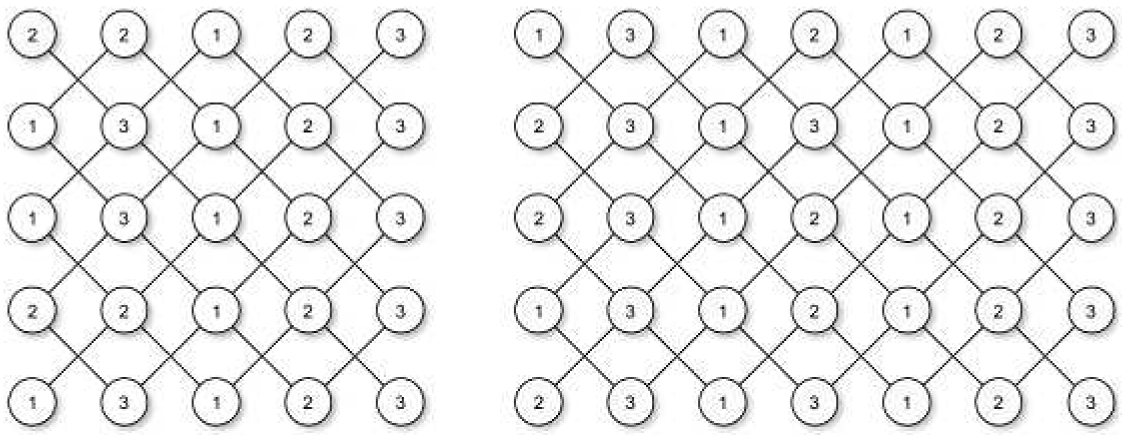

Fig.1 Equitable coloring of graphs $C_5 \times C_5$ and $C_5 \times C_7$

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