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On Finsler Space with Randers Conformal Change
— Main Scalar, Geodesic and Scalar Curvature

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Abstract: Let M^n be an n-dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x, y), (y^i = \dot{x}^i)$ of M^n. In the present paper we define Randers conformal change as

$$L(x, y) \to L^*(x, y) = e^{\sigma(x)} L(x, y) + \beta(x, y)$$

where $\sigma(x)$ is a function of x and $\beta(x, y) = b_i(x)y^i$ is a 1- form on M^n.
This transformation is more general as it includes conformal, Randers and homothetic transformation as particular cases. In the present paper we have found out the expressions for scalar curvature and main scalar of two-dimensional Finsler space obtained by Randers conformal change of F^n. We have also obtained equation of geodesic for this transformed space.

Key Words: two-dimensional Finsler space, β-change, homothetic change, conformal change, one form metric, main scalar, scalar curvature, geodesic.

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§1. Introduction

Let M^n be an n-dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $L(x, y), (y^i = \dot{x}^i)$ of M^n. If a differential 1-form $\beta(x, y) = b_i(x)y^i$ is given on M^n, then M. Matsumoto [1] introduced another Finsler space whose fundamental function is given by

$$\bar{L}(x, y) = L(x, y) + \beta(x, y)$$

This change of Finsler metric has been called β-change [2,3].

The conformal theory of Finsler spaces has been initiated by M.S. Knebelman [4] in 1929 and has been investigated in detail by many authors [5-8] etc. The conformal change is defined as

$$L(x, y) \to e^{\sigma(x)} L(x, y),$$

where $\sigma(x)$ is a function of position only and known as conformal factor.

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In the present paper, we construct a theory which generalizes all the above mentioned changes. In fact, we consider a change of the form

$$L(x,y) \to L^*(x,y) = e^{\sigma(x)}L(x,y) + \beta(x,y), \tag{1}$$

where $\sigma(x)$ is a function of x and $\beta(x,y) = b_i(x)y^i$ is a 1- form on M^n, which we call a Randers conformal change. This change generalizes various types of changes. When $\beta = 0$, it reduces to a conformal change. When $\sigma = 0$, it reduces to a Randers change. When $\beta = 0$ and σ is a non-zero constant then it reduces to homothetic change.

In the present paper we have obtained the relations between

(1) the main scalars of F^2 and F^{*2};

(2) the scalar curvatures of F^2 and F^{*2}.

Further, we have derived the equation of geodesic for F^{*n}.

§2. Randers Conformal Change

Definition 2.1 *Let (M^n, L) be a Finsler space F^n, where M^n is an n-dimensional differentiable manifold equipped with a fundamental function L. A change in fundamental metric L, defined by equation (1), is called Randers conformal change, where $\sigma(x)$ is conformal factor and function of position only and $\beta(x,y) = b_i(x)y^i$ is a 1- form on M^n. A space equipped with fundamental metric $L^*(x,y)$ is called Randers conformally changed space F^{*n}.*

This change generalizes various changes studied by Randers [11], Matsumuto [12], Shibata [13], Pandey [10] etc. Differentiating equation (1) with respect to y^i, the normalized supporting element $l_i^* = \dot{\partial}_i L^*$ is given by

$$l_i^*(x,y) = e^{\sigma(x)}l_i(x,y) + b_i(x), \tag{2}$$

where $l_i = \dot{\partial}_i L$ is the normalized supporting element in the Finsler space F^n. Differentiating (2) with respect to y^j, the angular metric tensor $h_{ij}^* = L^*\dot{\partial}_i\dot{\partial}_j L^*$ is given by

$$h_{ij}^* = e^{\sigma(x)}\frac{L^*}{L}h_{ij} \tag{3}$$

where $h_{ij} = L\dot{\partial}_i\dot{\partial}_j L$ is the angular metric tensor in the Finsler space F^n .

Again the fundamental tensor $g_{ij}^* = \dot{\partial}_i\dot{\partial}_j\frac{L^{*2}}{2} = h_{ij}^* + l_i^*l_j^*$ is given by

$$g_{ij}^* = \tau g_{ij} + b_ib_j + e^{\sigma(x)}L^{-1}(b_iy_j + b_jy_i) - \beta e^{\sigma(x)}L^{-3}y_iy_j \tag{4}$$

where we put $y_i = g_{ij}(x,y)y^j$, $\tau = e^{\sigma(x)}\frac{L^*}{L}$ and g_{ij} is the fundamental tensor of the Finsler space F^n. It is easy to see that the $\det(g_{ij}^*)$ does not vanish, and the reciprocal tensor with components g^{*ij} is given by

$$g^{*ij} = \tau^{-1}g^{ij} + \phi y^iy^j - L^{-1}\tau^{-2}(y^ib^j + y^jb^i) \tag{5}$$

where $\phi = e^{-2\sigma(x)}(Le^{\sigma(x)}b^2 + \beta)L^{*-3}$, $b^2 = b_i b^i$, $b^i = g^{ij}b_j$ and g^{ij} is the reciprocal tensor of g_{ij}. Here it will be more convenient to use the tensors

$$h_{ij} = g_{ij} - L^{-2}y_i y_j, \qquad a_i = \beta L^{-2}y_i - b_i \tag{6}$$

both of which have the following interesting property:

$$h_{ij}y^j = 0, \qquad a_i y^i = 0 \tag{7}$$

Now differentiating equation (4) with respect to y^k and using relation (6), the Cartan covariant tensor C^* with the components $C^*_{ijk} = \dot{\partial}_k(\frac{g^*_{ij}}{2})$ is given as:

$$C^*_{ijk} = \tau[C_{ijk} - \frac{1}{2L^*}(h_{ij}a_k + h_{jk}a_i + h_{ki}a_j)] \tag{8}$$

where C_{ijk} is (h)hv-torsion tensor of Cartan's connection $C\Gamma$ of Finsler space F^n.

In order to obtain the tensor with the components C^*_{ijk}, paying attention to (7), we obtain from (5) and (8),

$$\begin{aligned} C^{*j}_{ik} &= C^j_{ik} - \frac{1}{2L^*}(h^j_i a_k + h^j_k a_i + h_{ik}a^j) \\ &\quad -(\tau L)^{-1}C_{ikr}y^j b^r - \frac{\tau^{-1}}{2LL^*}(2a_i a_j + a^2 h_{ij})y^j \end{aligned} \tag{9}$$

where $a_i a^i = a^2$.

Proposition 2.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the normalized supporting element l^*_i, angular metric tensor h^*_{ij}, fundamental metric tensor g^*_{ij} and (h)hv-torsion tensor C^*_{ijk} of F^{*n} are given by (2), (3), (4) and (8) respectively.*

§3. Main Scalar of Randers Conformally Changed Two-Dimensional Finsler Space

The (h)hv-torsion tensor for a two-dimensional Finsler space F^2 is given by [9]:

$$C_{ijk} = I m_i m_j m_k \tag{10}$$

where $I = C_{222}$ is the main scalar of F^2.

Similarly, the (h)hv-torsion tensor for a two-dimensional Finsler space F^{*2} is given by

$$C^*_{ijk} = I^* m^*_i m^*_j m^*_k \tag{11}$$

where I^* is the main scalar of F^{*2}, and m^*_i is unit vector orthogonal to l^*_i in two-dimensional Finsler space.

Putting $j = k$ in equation (9), we get

$$C^*_i = C_i - \frac{(n+1)}{2L^*}a_i \tag{12}$$

The normalized torsion vectors are $m^i = \frac{C^i}{C}$ in F^2 and $m^{*i} = \frac{C^{*i}}{C^*}$ in F^{*2}, where C and C^* are the lengths of C^i and C^{*i} in F^2 and F^{*2} respectively. The equation (12) can also be written as

$$m^*_i = \lambda m_i + \mu a_i \tag{13}$$

where $\lambda = \frac{C}{C^*}$ and $\mu = -\frac{(n+1)}{2C^*}L^{*-1}$.

Now

$$C^{*2} = g^{*ij}C^*_iC^*_j = \tau^{-1}[C^2 + \frac{(n+1)}{L^*}A_\gamma], \tag{14}$$

where $A_\gamma = C_\gamma + \frac{(n+1)}{4L^*}a^2$ and $C_\gamma = C_ib^i$ are scalars.

The contravariant components of l^*_i and m^*_i are given below:

$$l^{*i} = g^{*ij}l^*_j = Al^i + Bb^i \tag{15}$$

where $A = e^{\sigma(x)}\tau^{-1} - \tau - 2\beta e^{\sigma(x)} + \beta\phi L - b^2\tau^{-2} + e^{\sigma(x)}L^2$ and $B = (-e^{\sigma(x)}\tau^{-2} - \tau^{-1} - \beta L^{-1}\tau^{-2})$ are scalars, $l_il^i = 1$ and $b_il^i = b^il_i = L\beta$. Also

$$m^{*i} = Dm^i + El^i + Fa^i \tag{16}$$

where $D = \tau^{-1}\lambda$, $E = (-\tau^{-2}\lambda H - \tau^{-2}\mu(\beta^2L^{-1} - b^2))$, $F = \mu\tau^{-1}$ and $H = m_ib^i$ are scalars. Hence, we have

Proposition 3.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then contravariant components of the Berwald frame (l, m) in two-dimensional Finsler space are given by (15) and (16), whereas covariant components are given by (2) and (13) respectively.*

Proposition 3.2 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the relationship between the lengths of the components C_i and C^*_i is given by (14).*

Since the (h)hv-torsion tensor given by (8) can be rewritten in two-dimensional form as follows:

$$I^*m^*_im^*_jm^*_k = \tau[Im_im_jm_k - \frac{3}{2L^*}a_2m_im_jm_k] \tag{17}$$

where $h_{ij} = m_im_j$ and $a_i = a_1l_i + a_2m_i$, then $a_iy^i = 0 \Longrightarrow a_1 = 0$. So, $a_i = a_2m_i$, a_1 and a_2 are certain scalars.

From equations (13) and (17), we have

$$I^*(\lambda + \mu a_2)^3m_im_jm_k = \tau[Im_im_jm_k - \frac{3}{2L^*}a_2m_im_jm_k] \tag{18}$$

Contracting (18) by $m_im_jm_k$, we have

$$I^* = \frac{\tau}{(\lambda + \mu a_2)^3}[I - \frac{3}{2L^*}a_2] \tag{19}$$

Theorem 3.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the relationship between the Main scalars I^* and I of the Finsler space F^{*2} and F^2 is given by (19).*

Corollary 3.1 *For $\sigma(x) = 0$, i.e. for Randers change, the relationship between the Main scalars I^* and I of the Finsler space F^{*2} and F^2 is given by [10]:*

$$I^* = \frac{(L+\beta)L^{-1}}{(\lambda+\mu a_2)^3} I - \frac{3L^{-1}}{2(\lambda+\mu a_2)^3} a_2.$$

Corollary 3.2 *For $\beta = 0$, i.e. for conformal change, the relationship between the Main scalars I^* and I of the Finsler space F^{*2} and F^2 is given by*

$$I^* = \frac{e^{\sigma(x)}}{\lambda^3} I.$$

Corollary 3.3 *For $\beta = 0$ and σ = a non-zero constant i.e. for homothetic change, the relationship between the Main scalars I^* and I of the Finsler space F^{*2} and F^2 is given by*

$$I^* = \frac{e^{\sigma}}{\lambda^3} I.$$

§4. Geodesic of Randers Conformally Changed Space

Let s be the arc-length, then the equation of a geodesic [14] of $F^n = (M^n, L)$ is written in the well-known form:

$$\frac{d^2x^i}{ds^2} + 2G^i(x, \frac{dx}{ds}) = 0, \tag{20}$$

where functions $G^i(x, y)$ are given by

$$2G^i = g^{ir}(y^j \dot{\partial}_r \partial_j F - \partial_r F), \qquad F = \frac{L^2}{2}.$$

Now suppose s^* is the arc-length in the Finsler space $F^{*n} = (M^n, L^*)$, then the equation of geodesic in F^{*n} can be written as

$$\frac{d^2x^i}{ds^{*2}} + 2G^{*i}(x, \frac{dx}{ds^*}) = 0, \tag{21}$$

where functions $G^{*i}(x, y)$ are given by

$$2G^{*i} = g^{*ir}(y^j \dot{\partial}_r \partial_j F^* - \partial_r F^*), \qquad F^* = \frac{L^{*2}}{2}.$$

Since $ds^* = L^*(x, dx)$, this is also written as

$$ds^* = e^{\sigma(x)} L(x, dx) + b_i(x) dx^i = e^{\sigma(x)} ds + b_i(x) dx^i$$

Since $ds = L(x, dx)$, we have

$$\frac{dx^i}{ds} = \frac{dx^i}{ds^*}[e^{\sigma(x)} + b_i \frac{dx^i}{ds}] \tag{22}$$

Differentiating (22) with respect to s, we have

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{ds^{*2}}[e^{\sigma(x)} + b_i\frac{dx^i}{ds}]^2 + \frac{dx^i}{ds^*}(\frac{de^{\sigma(x)}}{ds} + \frac{db_i}{ds}\frac{dx^i}{ds} + b_i\frac{d^2x^i}{ds^2}).$$

Substituting the value of $\frac{dx^i}{ds^*}$ from (22), the above equation becomes

$$\begin{aligned}\frac{d^2x^i}{ds^2} &= \frac{d^2x^i}{ds^{*2}}[e^{\sigma(x)} + b_i\frac{dx^i}{ds}]^2 \\ &\quad + \frac{\frac{dx^i}{ds}}{[e^{\sigma(x)} + b_i\frac{dx^i}{ds}]}(\frac{de^{\sigma(x)}}{ds} + \frac{db_i}{ds}\frac{dx^i}{ds} + b_i\frac{d^2x^i}{ds^2})\end{aligned} \tag{23}$$

Since $2G^{*i} = g^{*ir}(y^j\dot{\partial}_r\partial_j\frac{L^{*2}}{2} - \partial_r\frac{L^{*2}}{2})$, we have

$$\begin{aligned}2G^*_i \quad &= e^{2\sigma(x)}G_i + y^j[e^{\sigma(x)}L\dot{\partial}_i(\partial_je^{\sigma(x)})L + e^{\sigma(x)}L\dot{\partial}_i\partial_j\beta + \\ &\beta\dot{\partial}_i(\partial_je^{\sigma(x)})L + \beta e^{\sigma(x)}\dot{\partial}_i\partial_jL + \beta\dot{\partial}_i\partial_j\beta + (e^{\sigma(x)}l_i + b_i)((\partial_je^{\sigma(x)})L \\ &+\partial_j\beta) + e^{\sigma(x)}b_r\partial_jL] - [e^{\sigma(x)}L(\partial_ie^{\sigma(x)})L + e^{\sigma(x)}L\partial_i\beta + \beta\partial_i(e^{\sigma(x)})L \\ &+\beta e^{\sigma(x)}\partial_iL + \beta\partial_i\beta]\end{aligned} \tag{24}$$

Now we have

$$2G^{*i} = g^{*ir}G^*_r = JG^i + M^i \tag{25}$$

where $J = e^{2\sigma(x)}\tau^{-1}$ and

$$\begin{aligned}M^i \quad = \quad & e^{2\sigma(x)}G_r[\phi y^iy^r - L^{-1}\tau^{-2}(y^ib^r + y^rb^i)] + [\tau^{-1}g^{ir} + \phi y^iy^r \\ & -L^{-1}\tau^{-2}(y^ib^r + y^rb^i)][y^j[e^{\sigma(x)}L\dot{\partial}_r(\partial_je^{\sigma(x)})L + e^{\sigma(x)}L\dot{\partial}_r\partial_j\beta \\ & +\beta\dot{\partial}_r(\partial_je^{\sigma(x)})L + \beta e^{\sigma(x)}\dot{\partial}_r\partial_jL + \beta\dot{\partial}_r\partial_j\beta + (e^{\sigma(x)}l_r + b_r)((\partial_je^{\sigma(x)})L \\ & +\partial_j\beta) + e^{\sigma(x)}b_r\partial_jL] - [e^{\sigma(x)}L(\partial_re^{\sigma(x)})L + e^{\sigma(x)}L\partial_r\beta + \beta\partial_r(e^{\sigma(x)})L \\ & +\beta e^{\sigma(x)}\partial_rL + \beta\partial_r\beta]\end{aligned} \tag{26}$$

Proposition 4.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the relationship between the Berwald connection function G^{*i} and G^i is given by (25).*

Theorem 4.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the equation of geodesic of F^{*n} is given by (21), where $\frac{d^2x^i}{ds^{*2}}$ and G^{*i} are given by (23) and (25) respectively.*

Corollary 4.1 *For $\sigma(x) = 0$, i.e. for Randers change, the equation of geodesic of F^{*n} is given by (21), where $\frac{d^2x^i}{ds^{*2}}$ and G^{*i} are given below [10]:*

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{ds^{*2}}[1 + b_i\frac{dx^i}{ds}]^2 + \frac{dx^i}{ds^*}(\frac{db_i}{ds}\frac{dx^i}{ds} + b_i\frac{d^2x^i}{ds^2})$$

and

$$\begin{aligned} 2G^{*i} &= L(L+\beta)^{-1}G^i + G_r[-L(L+\beta)^{-2}((y^ib^r+y^rb^i)+(Lb^2+\beta)(L+\beta)^{-3}y^iy^r] \\ &\quad +[L(L+\beta)^{-1}g^{ir} - L(L+\beta)^{-2}((y^ib^r+y^rb^i)+(Lb^2+\beta)(L+\beta)^{-3}y^iy^r][y^j(2L\partial_jb_r \\ &\quad +\beta\dot{\partial}_j\partial_rL+2\beta\partial_jb_r+b_r\partial_jL+)-(\beta l_r+(L+\beta)\partial_jb_ry^j)]. \end{aligned}$$

Corollary 4.2 *For $\beta = 0$, i.e. for conformal change, the equation of geodesic of F^{*n} is given by (21), where $\frac{d^2x^i}{ds^{*2}}$ and G^{*i} are given below:*

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{ds^{*2}}e^{2\sigma(x)} + \frac{dx^i}{ds^*}\frac{de^{\sigma(x)}}{ds}$$

and

$$2G^{*i} = G^i + e^{-2\sigma(x)}g^{ir}[y^j[e^{\sigma(x)}L\dot{\partial}_r(\partial_je^{\sigma(x)})L + e^{\sigma(x)}l_r(\partial_je^{\sigma(x)})L] - \quad e^{\sigma(x)}L(\partial_re^{\sigma(x)})L].$$

Corollary 4.3 *For $\beta = 0$ and $\sigma =$ a non-zero constant i.e. for homothetic change, the equation of geodesic of F^{*n} is given by (21), where $\dfrac{d^2x^i}{ds^{*2}}$ and G^{*i} are given below*

$$\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{ds^{*2}}e^{2\sigma}$$

and $2G^{*i} = G^i$.

§5. Scalar Curvature of Randers Conformally Changed Two-Dimensional Finsler Space

The (v)h-torsion tensor R^i_{jk} in two-dimensional Finsler space may be written as [9]

$$R^i_{jk} = LRm^i(l_jm_k - l_km_j), \tag{27}$$

where R is the h-scalar curvature in F^2.

Similarly the (v)h-torsion tensor R^{*i}_{jk} in Finsler space F^{*2} is given by

$$R^{*i}_{jk} = L^*R^*m^{*i}(l^*_jm^*_k - l^*_km^*_j), \tag{28}$$

where R^* is the h-scalar curvature in F^{*2}. If we are concerned with Berwald connection $B\Gamma$, the non-vanishing (v)h-torsion tensor R^i_{jk} [9] is given as

$$R^i_{jk} = \delta_kG^i_j - \delta_jG^i_k = \partial_kG^i_j - \partial_jG^i_k + G^r_jG^i_{rk} - G^r_kG^i_{rj}, \tag{29}$$

where $\delta_i = \partial_i - G^r_i\dot{\partial}_r$, $G^i_j = \dot{\partial}_jG^i$ and $G^i_{jk} = \dot{\partial}_kG^i_j$.

Similarly the (v)h-torsion tensor R^{*i}_{jk} for Berwald connection $B\Gamma$ in F^{*n} is

$$R^{*i}_{jk} = \delta_kG^{*i}_j - \delta_jG^{*i}_k = \partial_kG^{*i}_j - \partial_jG^{*i}_k + G^{*r}_jG^{*i}_{rk} - G^{*r}_kG^{*i}_{rj}, \tag{30}$$

where $\delta_i = \partial_i - G^{*r}_i \dot{\partial}_r$, $G^{*i}_j = \dot{\partial}_j G^{*i}$ and $G^{*i}_k = \dot{\partial}_k G^{*i}_j$.

Using relation (25) we have

$$G^{*i}_j = \dot{\partial}_j G^{*i} = \frac{1}{2}[JG^i_j + M^i_j], \tag{31}$$

where $\dot{\partial}_j M^i = M^i_j$, and

$$G^{*i}_{jk} = \dot{\partial}_k G^{*i}_j = \frac{1}{2}[JG^i_{jk} + M^i_{jk}], \tag{32}$$

where $\dot{\partial}_k M^i_j = M^i_{jk}$.

Using equation (30) and (31) in (29), we have

$$\begin{aligned} R^{*i}_{jk} &= \frac{J}{2}[\partial_k G^i_j - \partial_j G^i_k] + \frac{1}{2}[\partial_k M^i_j - \partial_j M^i_k] + \frac{J^2}{4}[G^r_j G^i_{kr} - G^r_k G^i_{jr}] \\ &\quad + \frac{J}{2}[G^r_j M^i_{kr} + M^r_j G^i_{kr} - G^r_k M^i_{jr} - M^r_k G^i_{jr}] + [M^r_j M^i_{kr} - M^r_k M^i_{jr}] \end{aligned} \tag{33}$$

From equation (27) we have

$$\frac{R^{*i}_{jk}}{R^*} = L^* m^{*i}(l^*_j m^*_k - l^*_k m^*_j).$$

In view of (1), (2), (13) and (16), we have

$$\begin{aligned} \frac{R^{*i}_{jk}}{L^* R^*} &= D\lambda e^{\sigma(x)} m^i (l_j m_k - l_k m_j) + D\lambda m^i (b_j m_k - b_k m_j) \\ &\quad + (El^i + \mu e^{\sigma(x)} m^i (l_j b_k - l_k b_j) \\ &\quad + Fa^i)[\lambda e^{\sigma(x)}(l_j m_k - l_k m_j)\mu e^{\sigma(x)}(l_j b_k - l_k b_j) + \lambda(b_j m_k - b_k m_j)] \end{aligned} \tag{34}$$

Using (26), (28), (32) and (33), we have

$$\begin{aligned} &\frac{1}{R^*}(\frac{J}{2}[\partial_k G^i_j - \partial_j G^i_k] + \frac{1}{2}[\partial_k M^i_j - \partial_j M^i_k] + \frac{J^2}{4}[G^r_j G^i_{kr} - G^r_k G^i_{jr}] \\ &+ \frac{J}{2}[G^r_j M^i_{kr} + M^r_j G^i_{kr} - G^r_k M^i_{jr} - M^r_k G^i_{jr}] + [M^r_j M^i_{kr} - M^r_k M^i_{jr}]) \\ &= \frac{D\lambda\tau}{R}(\partial_k G^i_j - \partial_j G^i_k + G^r_j G^i_{rk} - G^r_k G^i_{rj}) + (e^{\sigma(x)} L + \beta)(\mu e^{\sigma(x)} m^i (l_j b_k \\ &\quad - l_k b_j) + D\lambda m^i (b_j m_k - b_k m_j) + (El^i + Fa^i)[\lambda e^{\sigma(x)}(l_j m_k - l_k m_j) \\ &\quad + \mu e^{\sigma(x)}(l_j b_k - l_k b_j) + \lambda(b_j m_k - b_k m_j)]) \end{aligned} \tag{35}$$

Theorem 5.1 *Let $F^{*n} = (M^n, L^*)$ be an n-dimensional Finsler space obtained from the Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the relationship between scalar curvatures of the Finsler space F^{*2} and F^2 is given by (34).*

Corollary 5.1 *For $\sigma(x) = 0$, i.e. for Randers change, the relationship between scalar curvatures* *of the Finsler space* F^{*2} *and* F^2 *is given as [10]:*

$$\begin{aligned}
&\frac{1}{R^*}(\frac{J_1}{2}[\partial_k G^i_j - \partial_j G^i_k] + \frac{1}{2}[\partial_k M^i_{1j} - \partial_j M^i_{1k}] + \frac{J_1^2}{4}[G^r_j G^i_{kr} - G^r_k G^i_{jr}] \\
&+\frac{J_1}{2}[G^r_j M^i_{1kr} + M^r_{1j} G^i_{kr} - G^r_k M^i_{1jr} - M^r_{1k} G^i_{jr}] + [M^r_{1j} M^i_{1kr} - M^r_{1k} M^i_{1jr}]) \\
&= \frac{D_1 \lambda \tau_1}{R}(\partial_k G^i_j - \partial_j G^i_k + G^r_j G^i_{rk} - G^r_k G^i_{rj}) + (L+\beta)(\mu_1 m^i (l_j b_k - l_k b_j) \\
&+D_1 \lambda m^i (b_j m_k - b_k m_j) + (E_1 l^i + F_1 a^i)[\lambda(l_j m_k - l_k m_j) + \mu_1 (l_j b_k - l_k b_j) \\
&+\lambda(b_j m_k - b_k m_j)]]),
\end{aligned}$$

where

$$J_1 = \frac{L(L+\beta)^{-1}}{2}, \quad \tau_1 = \frac{L+\beta}{L}, \quad \mu_1 = -\frac{(n+1)}{2C^*}(L+\beta)^{-1},$$
$$D_1 = \frac{L}{L+\beta}\frac{C}{C^*}, \quad E_1 = -(\frac{L+\beta}{L})^{-2}(\lambda H + \mu_1(\beta^2 L^{-1} - b^2)), \quad F_1 = \mu_1 \frac{L}{L+\beta}$$

and

$$\begin{aligned}
M^i_1 \;\; = \;\; & \frac{1}{2}[G_r[-L(L+\beta)^{-2}((y^i b^r + y^r b^i) + (Lb^2+\beta)(L+\beta)^{-3} y^i y^r] \\
& +[L(L+\beta)^{-1} g^{ir} - L(L+\beta)^{-2}((y^i b^r + y^r b^i) + (Lb^2+\beta)(L+\beta)^{-3} y^i y^r] \\
& \times[y^j(2L\partial_j b_r + \beta\dot{\partial}_j \partial_r L + 2\beta \partial_j b_r + b_r \partial_j L) - (\beta l_r + (L+\beta)\partial_j b_r y^j)]],
\end{aligned}$$

$$M^i_{1j} = \dot{\partial}_j M^i_1, \quad M^i_{1jk} = \dot{\partial}_k M^i_{1j}.$$

Corollary 5.2 *For* $\beta = 0$*, i.e. for conformal change, the relationship between scalar curvatures of the Finsler space* F^{*2} *and* F^2 *is given as:*

$$\begin{aligned}
&\frac{1}{R^*}(\frac{1}{2}[\partial_k G^i_j - \partial_j G^i_k] + \frac{1}{2}[\partial_k M^i_{2j} - \partial_j M^i_{2k}] + \frac{1}{4}[G^r_j G^i_{kr} - G^r_k G^i_{jr}] \\
&+\frac{1}{2}[G^r_j M^i_{2kr} + M^r_{2j} G^i_{kr} - G^r_k M^i_{2jr} - M^r_{2k} G^i_{jr}] + [M^r_{2j} M^i_{2kr} - M^r_{2k} M^i_{2jr}]) \\
&= \frac{D_2 \lambda \tau_2}{R}(\partial_k G^i_j - \partial_j G^i_k + G^r_j G^i_{rk} - G^r_k G^i_{rj}),
\end{aligned}$$

where

$$\tau_2 = e^{\sigma(x)}, \quad D_2 = e^{-\sigma(x)}\frac{C}{C^*}$$

and

$$\begin{aligned}
&M^i_2 = e^{-2\sigma(x)} g^{ir}[y^j[e^{\sigma(x)} L \dot{\partial}_r (\partial_j e^{\sigma(x)})L + e^{\sigma(x)} l_r (\partial_j e^{\sigma(x)})L] - e^{\sigma(x)} L(\partial_r e^{\sigma(x)})L], \\
&M^i_{2j} = \dot{\partial}_j M^i_2, \quad M^i_{2jk} = \dot{\partial}_k M^i_{2j}.
\end{aligned}$$

Corollary 5.3 *For* $\beta = 0$ *and* $\sigma =$ *a non-zero constant i.e. for homothetic change, the relationship between scalar curvatures of the Finsler space* F^{*2} *and* F^2 *is given as:*

$$\frac{1}{R^*}(\frac{1}{2}[\partial_k G^i_j - \partial_j G^i_k] + \frac{1}{4}[G^r_j G^i_{kr} - G^r_k G^i_{jr}]) = \frac{D_3 \lambda \tau_3}{R}(\partial_k G^i_j - \partial_j G^i_k + G^r_j G^i_{rk} - G^r_k G^i_{rj}),$$

where

$$\tau_3 = e^{\sigma}, D_3 = e^{-\sigma}\frac{C}{C^*}.$$

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