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# On Involute and Evolute Curves of Spacelike Curve with a Spacelike Principal Normal in Minkowski 3-Space

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**Abstract** In this study, we have generalized the involute and evolute curves of the spacelike curve $\alpha$ with a spacelike principal normal in Minkowski 3-Space. Firstly, we have shown that, the length between the spacelike curves $\alpha$ and $\beta$ is constant. Furthermore, the Frenet frame of the involute curve $\beta$ has been found as depend on curvatures of the curve $\alpha$. We have determined the curve $\alpha$ is planar in which conditions. Secondly, we have found transformation matrix between the evolute curve $\beta$ and the curve $\alpha$. Finally, we have computed the curvatures of the evolute curve $\beta$.

**Key Words**: Spacelike curve, involutes, evolutes, Minkowski 3-space.

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## §1. Preliminaries

Let $R^3 = \{(x_1, x_2, x_3) | x_1, x_2, x_3 \in R\}$ be a 3-dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $IR^3$. The Lorentz scalar product of $x$ and $y$ is defined by

$$\langle x, y\rangle_L = -x_1y_1 + x_2y_2 + x_3y_3,$$

$E_1^3 = \left(R^3, \langle x, y\rangle_L\right)$ is called 3-dimensional Lorentzian space, Minkowski 3-Space or 3- dimensional semi-euclidean space. The vector $x$ in $IE_1^3$ is called a spacelike vector, null vector or a timelike vector if $\langle x, x\rangle_L > 0$ or $x = 0$, $\langle x, x\rangle_L = 0$ or $\langle x, x\rangle_L < 0$, respectively. For $x \in E_1^3$, the norm of the vector $x$ defined by $\|x\|_L = \sqrt{|\langle x, x\rangle_L|}$, and $x$ is called a unit vector if $\|x\|_L = 1$. For any $x$, $y \in E_1^3$, Lorentzian vectoral product of $x$ and $y$ is defined by

$$x \wedge_L y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve $\alpha(s)$. Then $T(s), N(s)$ and $B(s)$ are tangent, the principal normal and the binormal vector of the curve

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$\alpha(s)$, respectively. Depending on the causal character of the curve $\alpha$, we have the following Frenet-Serret formulae:

If $\alpha$ is a spacelike curve with a spacelike principal normal $N$,

$$T' = \kappa N,\ N = -\kappa T + \tau B,\ B' = \tau N \tag{1.1}$$

$$\langle T, T\rangle_L = \langle N, N\rangle_L = 1, \langle B, B\rangle_L = -1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0.$$

If $\alpha$ is a spacelike curve with a timelike principal normal $N$,

$$T' = \kappa N,\ N = \kappa T + \tau B,\ B' = \tau N \tag{1.2}$$

$$\langle T, T\rangle_L = \langle B, B\rangle_L = 1, \langle N, N\rangle_L = -1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0.$$

If $\alpha$ is a timelike curve and finally,

$$T' = \kappa N,\ N = \kappa T + \tau B,\ B' = -\tau N \tag{1.3}$$

$$\langle T, T\rangle_L = -1, \langle B, B\rangle_L = \langle N, N\rangle_L = 1, \langle T, N\rangle_L = \langle N, B\rangle_L = \langle T, B\rangle_L = 0.$$

known in [2]. If the curve $\alpha$ is non-unit speed, then

$$\kappa(t) = \frac{\left\| \alpha^{'}(t) \wedge_L \alpha^{''}(t) \right\|_L}{\left\| \alpha^{'}(t) \right\|_L^3},\ \ \tau(t) = \frac{\det\left(\alpha^{'}(t), \alpha^{''}(t), \alpha^{'''}(t)\right)}{\left\| \alpha^{'}(t) \wedge_L \alpha^{''}(t) \right\|_L^2}. \tag{1.4}$$

If the curve $\alpha$ is unit speed, then

$$\kappa(s) = \left\|\alpha''(s)\right\|_L,\ \tau(s) = \left\| B'(s)\right\|_L. \tag{1.5}$$

## §2. The involute of spacelike curve with a spacelike principal normal

**Definition** 2.1 *Let unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$ with a principal normal and spacelike curve $\beta : I \longrightarrow E_1^3$ with a spacelike principal normal be given. For $\forall s \in I$, then the curve $\beta$ is called the involute of the curve $\alpha$, if the tangent at the point $\alpha(s)$ to the curve $\alpha$ passes through the tangent at the point $\beta(s)$ to the curve $\beta$ and*

$$\langle T^*(s), T(s)\rangle_L = 0. \tag{2.1}$$

*Let the Frenet-Serret frames of the curves $\alpha$ and $\beta$ be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. In this case, the causal characteristics of the Frenet-Serret frames of the curves $\alpha$ and $\beta$ must be of the form.*

$$\{T \text{ spacelike},\ N \text{ spacelike},\ B \text{ timelike}\}$$

*and*

$$\{T^* \text{ spacelike},\ N^* \text{ spacelike},\ B^* \text{ timelike}\}.$$

**Theorem** 2.1 *Let the curve $\beta$ be involute of the the curve $\alpha$ and let $k$ be a constant real number. Then*

$$\beta(s) = \alpha(s) + (k - s)T(s). \tag{2.2}$$

*Proof* The curve $\beta(s)$ may be given as

$$\beta(s) = \alpha(s) + u(s)T(s) \tag{2.3}$$

If we take the derivative Eq. (2.3), then we have

$$\beta^{'}(s) = \left(1 + u^{'}(s)\right) T(s) + u(s)\kappa(s)N(s).$$

Since the curve $\beta$ is involute of the curve $\alpha$, $\langle T^{*}(s), T(s)\rangle_{L} = 0$ . Then, we get

$$1 + u^{'}(s) = 0 \text{ or } u(s) = k - s. \tag{2.4}$$

Thus we get

$$\beta(s) - \alpha(s) = (k - s)\, T(s) \tag{2.5}$$

□

**Corollary** 2.2 *The distance between the curves $\beta$ and $\alpha$ is $|k - s|$.*

*Proof* If we take the norm in Eq. (2.5), then we get

$$\left\| \beta(s) - \alpha(s) \right\|_{L} = |k - s| \,. \tag{2.6}$$

□

**Theorem** 2.3 *Let the curve $\beta$ be involute of the the curve $\alpha$, then*

$$\begin{bmatrix} T^{*} \\ N^{*} \\ B^{*} \end{bmatrix} = \left( \left| \kappa^{2} - \tau^{2} \right| \right)^{-1} \begin{bmatrix} 0 & 1 & 0 \\ \kappa & 0 & -\tau \\ -\tau & 0 & \kappa \end{bmatrix} \cdot \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

*Proof* If we take the derivative Eq. (2.5), we can write

$$\beta^{'}(s) = (k - s)\kappa(s)N(s)$$

and

$$\left\| \beta^{'}(s) \right\|_{L} = |(k - s)\kappa(s)| \,.$$

Furthermore, we get

$$T^{*}(s) = \frac{\beta^{'}(s)}{\|\beta'(s)\|_{L}} = \frac{(k - s)\kappa(s)}{|(k - s)\kappa(s)|} N(s).$$

From the last equation, we must have

$$T^{*}(s) = N(s) \text{ or } T^{*}(s) = -N(s) \,.$$

We assume that $T^*(s) = N(s)$. Let's denote the coordinate function on $IR$ by $x$. Then, for$\forall s \in IR$, $x(s) = s$, we get

$$\begin{aligned} \beta^{'}(s) &= (k-s)\kappa(s)N(s), \\ \beta^{'} &= (k-x)\kappa N. \end{aligned}$$

Thus, we have

$$\begin{aligned} \beta^{''} &= -\kappa N + (k-x)\kappa^{'} N + (k-x)\kappa(-\kappa T + \tau B) \\ \beta^{''} &= -(k-x)\kappa^2 T + \left[(k-x)\kappa^{'} - \kappa\right] N + (k-x)\kappa\tau B \end{aligned}$$

Hence, we have

$$\beta^{'} \wedge_L \beta^{''} = (k-x)^2\kappa^2 \left(-\tau T + \kappa B\right)$$

and

$$\left\| \beta^{'} \wedge_L \beta^{''} \right\|_L = |k-x|^2 \kappa^2 \sqrt{|\tau^2 - \kappa^2|}.$$

Furthermore, we get

$$B^* = \frac{\beta^{'} \wedge_L \beta^{''}}{\left\| \beta^{'} \wedge_L \beta^{''} \right\|} = \frac{(k-x)^2\kappa^2 \left(-\tau T + \kappa B\right)}{(k-x)^2\kappa^2 \sqrt{|\tau^2 - \kappa^2|}} = \frac{-\tau T + \kappa B}{\sqrt{|\kappa^2 - \tau^2|}}.$$

Since $N^* = B^* \wedge_L T^*$, then we obtain

$$N^* = \frac{\tau T - \kappa B}{\sqrt{|\tau^2 - \kappa^2|}}.$$

□

**Theorem** 2.4 *Let the curve $\beta$ be involute of the the curve $\alpha$. Let the curvature and torsion of the curve $\beta$ be $\kappa^*$ and $\tau^*$, respectively. Then*

$$\kappa^*(s) = \frac{\sqrt{\left|\left(\tau^2 - \kappa^2\right)(s)\right|}}{|k-s|\,\kappa(s)}\ ,\ \tau^*(s) = \frac{\kappa(s)\tau^{'}(s) - \kappa^{'}(s)\tau(s)}{|k-s|\,\kappa(s)\sqrt{\left|\left(\tau^2 - \kappa^2\right)(s)\right|}}.$$

*Proof* From Eq. (1.3) and Eq. (1.4), we have

$$\kappa^*(s) = \frac{|k-s|^2\,\kappa^2(s)}{|k-s|^3\,\kappa^3(s)} = \frac{\sqrt{\left|\left(\tau^2 - \kappa^2\right)(s)\right|}}{\kappa(s)\,|k-s|}$$

and

$$\begin{aligned}\beta^{'''} &= \left[\kappa^2 T-(k-x)2\kappa\kappa^{'}T-(k-x)\kappa^2(\kappa N)\right]\\ &+\left[-\kappa^{'}-\kappa^{'}+(k-x)\kappa^{''}\right]N\\ &+\left[-\kappa+(k-x)\kappa^{'})\right](-\kappa T+\tau B)\\ &+\left[-\kappa\tau+(k-x)\kappa^{'}\tau+(k-x)\kappa\tau^{'}\right]B\\ &+[(k-x)\kappa\tau](\tau N)\\ &= \left[2\kappa^2-3\left(k-x\right)\kappa\kappa^{'}\right]T\\ &+\left[-\left(k-x\right)\kappa^3-2\kappa^{'}+\left(k-x\right)\kappa^{''}+\left(k-x\right)\kappa\tau^2\right]N\\ &+\left(-2\kappa\tau+2\left(k-x\right)\kappa^{'}\tau+\left(k-x\right)\kappa\tau^{'}\right)B.\end{aligned}$$

Furthermore, since

$$\tau^*(s)=\frac{\det\left(\beta^{'}(s),\beta^{''}(s),\beta^{'''}(s)\right)}{\left\|\beta^{'}(s)\wedge_L\beta^{''}(s)\right\|_L^2},$$

we have

$$\begin{aligned}\Delta &= -(k-x)^2\kappa^2\begin{bmatrix}-\kappa & \tau\\ 2\kappa^2-3\left(k-x\right)\kappa\kappa^{'} & -2\kappa\tau+2\left(k-x\right)\kappa^{'}\tau+\left(k-x\right)\kappa\tau^{'}\end{bmatrix}\\ &= -(k-x)^2\kappa^2\left[2\kappa^2\tau-2\left(k-x\right)\kappa\kappa^{'}\tau-\left(k-x\right)\kappa^2\tau^{'}-2\kappa^2\tau+3\left(k-x\right)\kappa\kappa^{'}\tau\right]\\ &= (k-x)^3.\kappa^3\left(\kappa\tau^{'}-\kappa^{'}\tau\right)\\ \Delta &= \det\left(\beta^{'},\beta^{''},\beta^{'''}\right).\end{aligned}$$

Hence, we get

$$\begin{aligned}\tau^*(s) &= \frac{\kappa^3(k-s)^3\left(\kappa(s)\tau^{'}(s)-\kappa^{'}(s)\tau(s)\right)}{\kappa^4\left|k-s\right|^4\left(\tau^2(s)-\kappa^2(s)\right)},\\ \tau^*(s) &= \frac{\kappa(s)\tau^{'}(s)-\kappa^{'}(s)\tau(s)}{\kappa(s)\left|k-x\right|\left(\tau^2(s)-\kappa^2(s)\right)}.\end{aligned}$$

□

From the last equation, we have the following corollaries:

**Corollary** 2.5 *If the curve $\alpha$ is planar, then its involute curve $\beta$ is also planar.*

**Corollary** 2.6 *If the curvature $\kappa \neq 0$ and the torsion $\tau \neq 0$ of the curve $\alpha$ are constant, then the involute curve $\beta$ is planar, i.e., if the curve $\alpha$ is a ordinary helix, then its the involute curve $\beta$ is planar.*

**Corollary** 2.7 *If the curvature $\kappa \neq 0$ and the torsion $\tau \neq 0$ of the curve $\alpha$ are not constant but $\frac{\tau}{\kappa}$ is constant, then the involute curve $\beta$ is planar, i.e. if the curve $\alpha$ is a general helix, then their the involute curve $\beta$ is planar.*

**Theorem** 2.8 *Suppose that the planar curve $\alpha : I \longrightarrow E_1^3$ with arc-length parameter are given. Then, the locus of the center of the curvature of the curve $\alpha$ is the unique involute of the curve $\alpha$ which lies on the plane of the curve $\alpha$.*

*Proof* The locus of the center of the curvature of the curve $\alpha$ is

$$C(s) = \alpha(s) - \frac{1}{\kappa(s)} N(s), \ \ \kappa(s) \neq 0$$

If we take the derivative in the above equation, then we have

$$\begin{aligned} \frac{dC}{ds} &= \ T - \left(\frac{1}{\kappa}\right)^{'} N + \frac{1}{\kappa}\left(-\kappa T\right), \\ C^{'} &= -\left(\frac{1}{\kappa}\right)^{'} N, \\ \left\langle C^{'}, T\right\rangle_L &= -\left(\frac{1}{\kappa}\right)^{'} \langle N, T\rangle_L \,, \\ \left\langle C^{'}(s), T(s)\right\rangle_L &= 0. \end{aligned}$$

Therefore, the evolute $C$ of the spacelike curve $\alpha$ is the locus of the center of the curvature. Is the curve $C$ planar ? If the torsion of the curve $C$ is denoted by $\tau^*$, then

$$\tau^*(s) = \frac{\left(\kappa^{'}\tau - \kappa\tau^{'}\right)(s)}{\kappa(t)\left|k - s\right| . \left(\tau^2(s) - \kappa^2(s)\right)}.$$

If we take $\tau = 0$, then we have

$$\tau^*(s) = 0$$

Thus, the curve $C$ is planar. □

## §3. The evolute of spacelike curve with a spacelike principal normal

**Definition** 3.1 *Let the unit speed spacelike curve $\alpha$ with a spacelike principal normal and the spacelike curve $\beta$ with the same interval be given. For $\forall s \in I$, the tangent at the point $\beta(s)$ to the curve $\beta$ passes through the point $\alpha(s)$ and*

$$\langle T^*(s), T(s)\rangle_L = 0.$$

*Then, $\beta$ is called the evolute of the curve $\alpha$. Let the Frenet-Serret frames of the curves $\alpha$ and $\beta$ be $(T, N, B)$ and $(T^*, N^*, B^*)$, respectively.*

**Theorem** 3.1 *Let the curve $\beta$ be the evolute of the unit speed spacelike curve $\alpha$ , Then*

$$\beta(s) = \alpha(s) + \frac{1}{\kappa(s)} N(s) - \frac{1}{\kappa(s)} \left[\tanh\left(\varphi(s) + c\right)\right] B(s), \tag{3.1}$$

*where $c \in IR$ and $\varphi(s) + c = \int \tau(s)ds$ . Furthermore, in the normal plane of the point $\alpha(s)$ the measure of directed angle between $\beta(s) - \alpha(s)$ and $N(s)$ is*

$$\varphi(s) + c.$$

*Proof* The tangent of the curve $\beta$ at the point $\beta(s)$ is the line constructed by the vector $T^*(s)$ . Since this line passes through the point $\alpha(s)$, the vector $\beta(s) - \alpha(s)$ is perpendicular to the vector $T(s)$ . Then

$$\beta(s) - \alpha(s) = \lambda N(s) + \mu B(s). \tag{3.2}$$

If we take the derivative of Eq. (3.2), then we have

$$\beta^{'}(s) = \alpha^{'}(s) + \lambda^{'} N + \lambda(-\kappa T + \tau B) + \mu^{'} B(s) + \mu(\tau N)$$

$$\beta^{'}(s) = (1 - \lambda\kappa) T + \left(\lambda^{'} + \mu\tau\right) N + \left(\lambda\tau + \mu^{'}\right) B. \tag{3.3}$$

According to the definition of the evolute, since $\langle T^*(s), T(s)\rangle = 0$, from Eq. (3.3), we get

$$\lambda = \frac{1}{\kappa}\ , \tag{3.4}$$

and

$$\beta^{'} = \left(\lambda^{'} + \mu\tau\right) N + \left(\lambda\tau + \mu^{'}\right) B. \tag{3.5}$$

From the Eq. (3.2) and Eq. (3.5), the vector field $\beta^{'}$ is parallel to the vector field $\beta - \alpha$ . Then we have

$$\frac{\lambda^{'} + \mu\tau}{\lambda} = \frac{\lambda\tau + \mu^{'}}{\mu}.$$

After that, we have

$$\begin{aligned} \tau &= \frac{\lambda^{'}\mu - \lambda\mu^{'}}{\lambda^2 - \mu^2} \\ \tau &= -\frac{\left(\frac{\mu}{\lambda}\right)^{'}}{1 - \left(\frac{\mu}{\lambda}\right)^2}. \end{aligned}$$

If we take the integral the last equation, we get

$$\varphi(s) + c = -\arg\tanh\left(\frac{\mu(s)}{\lambda(s)}\right).$$

Hence, we find

$$\mu(s) = -\lambda(s)\tanh\left(\varphi(s) + c\right). \tag{3.6}$$

If we substitute Eq. (3.4) and Eq. (3.6) into Eq. (3.2), we have

$$\begin{aligned} \beta(s) &= \alpha(s) + \frac{1}{\kappa(s)} N(s) - \frac{1}{\kappa(s)}\left[\tanh\left(\varphi(s) + c\right)\right] B(s) \\ \beta(s) &= M(s) - \frac{1}{\kappa(s)}\tanh\left[\varphi(s) + c\right] B(s). \end{aligned}$$

Then, we obtain an evolute curve for each $c \in IR$ . Since

$$\left\langle \overrightarrow{M(s)\beta(s)}, \overrightarrow{M(s)\ \alpha(s)} \right\rangle_L = 0,$$

in the Lorentzian triangle which have corners $\beta(s)$, $M(s)$ and $\alpha(s)$ , the angle $M$ is right angle in the Lorentzian mean. In the same triangle, the tangent of the angle $\alpha(s)$ is

$$\frac{\frac{1}{\kappa(s)}\tanh\left[\varphi(s)+c\right]}{\frac{1}{\kappa(s)}}=\tanh\left[\varphi(s)+c\right]. \tag{3.7}$$

Then, the measure of the angle between the vectors $\beta(s)-\alpha(s)$ and $N(s)$ is $\varphi(s)+c$. □

**Theorem** 3.2 *Let the spacelike curve $\beta : I \longrightarrow E_1^3$ be evolute of the unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$. If the Frenet-Serret vector fields of the curve $\beta$ are $T^*$ (spacelike), $N^*$ (space), $B^*$ (timelike), then*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix}=\begin{bmatrix} 0 & \cosh(\varphi+c) & -\sinh(\varphi+c) \\ -1 & 0 & 0 \\ 0 & -\sinh(\varphi+c) & \cosh(\varphi+c) \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{3.8}$$

*Proof* Since the Frenet-Serret vector fields of the curve $\beta$ are $T^*$, $N^*$, $B^*$ and

$$\beta=\alpha+\rho N-\rho\tanh\left(\varphi+c\right)B,$$

we have

$$\begin{aligned}
\beta^{'}(s) &= \alpha^{'}+\rho^{'}N+\rho\left(-\kappa T+\tau B\right) \\
&\quad -\left[\rho^{'}\tanh\left(\varphi+c\right)B+\rho\varphi^{'}\sec\mathrm{h}^2\left(\varphi+c\right)B+\rho\tanh\left(\varphi+c\right)\tau N\right] \\
&= \left(1-\rho\kappa\right)T+\left(\rho^{'}-\rho\tau\tan\left(\varphi+c\right)\right)N \\
&\quad +\left[\left(\rho\tau-\rho\varphi^{'}\right)-\rho^{'}\tanh\left(\varphi+c\right)+\rho\varphi^{'}\tanh^2\left(\varphi+c\right)\right]B \\
&= \left[\rho^{'}-\rho\tau\tanh\left(\varphi+c\right)\right]N+\left[-\rho^{'}+\rho\tau\tanh\left(\varphi+c\right)\right]B\tanh\left(\varphi+c\right) \\
&= \left[\rho^{'}-\rho\tau\tanh(\varphi+c)\right]\left[N-\tanh\left(\varphi+c\right)B\right]
\end{aligned}$$

$$\beta^{'}(s)=\left[\frac{\rho^{'}-\rho\tau\tanh(\varphi+c)}{\cosh\left(\varphi+c\right)}\right]\left[\cosh\left(\varphi+c\right)N-\sinh\left(\varphi+c\right)B\right]. \tag{3.9}$$

If we take the norm in the Eq. (3.9), then we obtain

$$\begin{aligned}
\left\|\beta^{'}(s)\right\|_L &= \frac{\left|\rho^{'}-\rho\tau\tanh(\varphi+c)\right|}{\cosh\left(\varphi+c\right)} \\
&= \frac{\left|-\frac{\kappa^{'}}{\kappa^2}-\frac{1}{\kappa}\tau\frac{\sinh(\varphi+c)}{\cosh(\varphi+c)}\right|}{\cosh\left(\varphi+c\right)} \\
&= \frac{\left|\kappa^{'}\cosh\left(\varphi+c\right)+\kappa\tau\sinh(\varphi+c)\right|}{\kappa^2\cosh\left(\varphi+c\right)}.
\end{aligned}$$

Since $T^*=\frac{\beta^{'}}{\left\|\beta^{'}\right\|_L}$, then we get

$$T^*=\cosh\left(\varphi+c\right)N-\sinh\left(\varphi+c\right)B. \tag{3.10}$$

Therefore, we have obtained Eq. (3.9). The curve $\beta$ is not a unit speed curve. If we take the derivative of Eq. (3.10) with respect to $s$, we find

$$\begin{aligned}(T^*)' &= \left(\tau-\varphi'\right)\left[B\cosh(\varphi+c)+N\sinh(\varphi+c)\right]-\kappa T\cosh(\varphi+c)\\ &= -\kappa T\cosh(\varphi+c)\end{aligned}$$

Since $T' = \left\|\alpha'\right\|_L \kappa N$ , we have

$$(T^*)' = \left\|\beta'\right\|_L \kappa^* N^*.$$

Thus

$$\left\|\beta'\right\|_L \kappa^* N^* = -\kappa\cosh(\varphi+c)\,T.$$

Since the vectors $N^*$ and $T$ have the unit length, we get $N^* = -T$ or $N^* = T$. Since $B^* = N^* \wedge_L (-T^*)$, we have

$$B^* = -\sinh(\varphi+c)N+\cosh(\varphi+c)B. \tag{3.11}$$

Thus, the proof is completed. □

**Theorem** 3.3 *Let $\beta : I \longrightarrow E_1^3$ be the evolute of the unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$. Let the Frenet vector fields, curvature and torsion of the curve $\beta$ be $T^*, N^*, B^*$, $\kappa^*$ and $\tau^*$, respectively. Then*

$$\begin{aligned}\kappa^* &= \frac{\kappa^3\cosh^3(\varphi+c)}{\left|\kappa\tau\sinh(\varphi+c)+\kappa'\cos(\varphi+c)\right|},\ \kappa>0\\ |\tau^*| &= \frac{\kappa^3\cosh^2(\varphi+c)\left|\sinh(\varphi+c)\right|}{\left|\kappa\tau\sinh(\varphi+c)+\kappa'\cosh(\varphi+c)\right|}.\end{aligned}$$

*Proof* Since $N^*$ and $T$ have unit length, then taking norm from equility $\left\|\beta'\right\|_L \kappa^* N^* = -\kappa\cosh(\varphi+c)\,T$ .We can write have

$$|\kappa^*| = \frac{\kappa\cosh(\varphi+c)}{\|\beta'\|_L} \tag{3.12}$$

$$= \kappa\cosh(\varphi+c) : \frac{\left|\kappa'\cosh(\varphi+c)+\kappa\tau\sinh(\varphi+c)\right|}{\kappa^2\cosh(\varphi+c)},$$

$$|\kappa^*| = \frac{\kappa^3\cosh^3(\varphi+c)}{\kappa'\cosh(\varphi+c)+\kappa\tau\sinh(\varphi+c)}$$

If we take the derivative Eq. (3.11) with respect to $s$, then we have

$$\begin{aligned}(B^*)' &= \left(\varphi'-\tau\right)\left[N\cosh(\varphi+c)-B\sinh(\varphi+c)\right]+\kappa T\sinh(\varphi+c)\\ &= \kappa T\sin(\varphi+c).\end{aligned}$$

Since $(B^*)' = \left\|\beta'\right\|_L \tau^* N^*$, we get

$$\left\|\beta'\right\|_L \tau^* N^* = \kappa T\sin(\varphi+c).$$

From the last equation, we must have

$$T^*(s) = N(s) \text{ or } T^*(s) = -N(s)\,.$$

We assume that $T^*(s) = -N(s)$ then we find that

$$\begin{aligned} |\tau^*| &= \frac{\kappa\,|\sinh(\varphi + c)|}{\|\beta'\|} \qquad (3.13)\\ &= \kappa\,|\sinh(\varphi + c)| : \frac{\left|\kappa' \cosh(\varphi + c) + \kappa\tau \sinh(\varphi + c)\right|}{\kappa^2 \cosh(\varphi + c)}, \\ |\ \tau^*| &= \frac{\kappa^3 \cosh^2(\varphi + c)\,|\sinh(\varphi + c)|}{|\kappa' \cosh(\varphi + c) + \kappa\tau \sinh(\varphi + c)|}. \end{aligned}$$

□

**Theorem** 3.4 *Let $\beta : I \longrightarrow E_1^3$ be the evolute of the unit speed spacelike curve $\alpha : I \longrightarrow E_1^3$. Let the curvature and torsion of the curve $\beta$ be $\kappa^*$ and $\tau^*$, respectively. Then*

$$\left|\frac{\tau^*}{\kappa^*}\right| = |\tanh(\varphi + c)|\,. \qquad (3.14)$$

*Furthermore, we denote by $\beta^{(1)}$ and $\beta^{(2)}$, the evolute curves obtained by using $c_1$ and $c_2$ instead of $c$, respectively. The tangents of the curves $\beta^{(1)}$ and $\beta^{(2)}$ at the points $\beta^{(1)}(s)$ and $\beta^{(2)}(s)$ intersect at the point $\alpha(s)$ . The measure of the angle between the tangents is $c_1 - c_2$.*

*Proof* The Eq. (3.14) is obtained easily by using Eq. (3.12) and Eq. (3.13), i.e.,

$$\begin{aligned} \left|\frac{\tau^*}{\kappa^*}\right| &= \frac{\kappa\,|\sinh(\varphi + c)|}{\|\beta'\|_L} : \frac{\kappa \cosh(\varphi + c)}{\|\beta'\|_L} \\ &= |\tanh(\varphi + c)|\,. \end{aligned}$$

The measure of the angle between the vectors $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ and $V_2(s)$, and between the vectors $\overrightarrow{\alpha(s)\beta^{(2)}(s)}$ and $N(s)$ are $\varphi(s){+}c_1$ and $\varphi(s){+}c_2$, respectively. The vector $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ is parallel to the tangent of the curve $\beta^{(1)}$ at the point $\beta^{(1)}(s)$. The vector $\overrightarrow{\alpha(s)\beta^{(2)}(s)}$ is parallel to the tangent of the curve $\beta^{(2)}$ at the point $\beta^{(2)}(s)$. Furthermore, since $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ , $\overrightarrow{\alpha(s)\beta^{(1)}(s)}$ and $\overrightarrow{N}$ are perpendicular to the vector $T(s)$, these three vectors are planar. Then, the measure of the angle between the tangents of the curves $\beta^{(1)}$ and $\beta^{(2)}$ at the points $\beta^{(1)}(s)$ and $\beta^{(2)}(s)$ is

$$\varphi(s) + c_1 - [\varphi(s) + c_2] = c_1 - c_2.$$

So, the proof is completed. □

**Theorem** 3.5 *Suppose that, two different evolutes of the spacelike curve a spacelike principal normal curve $\alpha$ are given. Let the points on the evolutes of the curve $\alpha$ corresponding to the point $P$ be $P_1$ and $P_2$. Then the angle $\widehat{P_1 P P_2}$ is constant.*

*Proof* Let the evolutes of the curve $\alpha$ be $\beta$ and $\gamma$. Let the arc-length parameters of the $\alpha, \beta$ and $\gamma$ be $s$, $s^*$ and $\widehat{s}$ , respectively. Let the curvatures of the curves $\alpha$, $\beta$ and $\gamma$ be $k$, $k^*$ and

$\widehat{k}$ respectively. And let the Frenet vectors of the curves $\alpha$, $\beta$ and $\gamma$ be $\{T,N,B\}, \{T^*,N^*,B^*\}$ and $\left\{\widehat{T},\widehat{N},\widehat{B}\right\}$. Then

$$T = N^*, T = \widehat{N}. \tag{3.15}$$

Since the curves $\beta$ and $\gamma$ are evolute, then

$$\langle T,T^*\rangle_L = \left\langle T,\widehat{T}\right\rangle_L = 0 \tag{3.16}$$

Therefore, if $f(s) = \left\langle T^*,\widehat{T}\right\rangle_L$, then we have

$$\begin{aligned}
(f)^{'}(s) &= \left\langle (T^*)^{'},\widehat{T}\right\rangle_L + \left\langle T^*,\left(\widehat{T}\right)^{'}\right\rangle_L \\
&= \left\langle \kappa^* N^* \frac{ds^*}{ds},\widehat{T}\right\rangle_L + \left\langle T^*,\widehat{\kappa}\widehat{N}\frac{d\widehat{s}}{ds}\right\rangle_L \\
&= \kappa^*\frac{ds^*}{ds}\left\langle N^*,\widehat{T}\right\rangle_L + \widehat{\kappa}\frac{d\widehat{s}}{ds}\left\langle T^*,\widehat{N}\right\rangle_L \\
&= \kappa^*\frac{ds^*}{ds}\left\langle T,\widehat{T}\right\rangle_L + \widehat{\kappa}\frac{d\widehat{s}}{ds}\langle T^*,N^*\rangle_L \\
&= \kappa^*\frac{ds^*}{ds}.0 + \widehat{\kappa}\frac{d\widehat{s}}{ds}.0 \\
(f)^{'}(s) &= 0.
\end{aligned}$$

Therefore, we have $f(s) = \theta$ =constant. Hence, $m\left(\widehat{P_1PP_2}\right) = m\left(T^*,\widehat{T}\right) = \theta$ =constant. □

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