International J.Math. Combin. Vol.2(2020), 88-100

On Quotient of Randić and Sum-Connectivity Energy of Graphs

Puttaswamy and C. A. Bhavya

(Department of Mathematics, P.E.S. college of Engineering, Mandya-571401, India)

E-mail: prof.puttaswamy@gmail.com, cabhavya212@gmail.com

Abstract: In this paper we define the quotient of Randić and sum-connectivity energy of a graph. Then we obtain upper and lower bounds for $E_{qrs}(G)$, quotient of Randić and sum-connectivity energy of a graph. Further we compute the quotient of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the $(S_m \wedge P_2)$ graph.

Key Words: Quotient of Randić, eigenvalues of the sum-connectivity matrix, sum-connectivity energy.

AMS(2010): 05C50.

§1. Introduction

In 2010, Bo Zhou and Nenad Trinajstic [3] have introduced the sum-connectivity energy of a graph as follows. Let G be a simple graph and let $v_1, v_2, \cdots, v_n$ be its vertices. For $i = 1, 2, \ldots, n$, let d_i denote the degree of the vertex v_i. Then the sum-connectivity matrix of G is defined as $R = (R_{ij})$, where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i + d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The sum-connectivity energy of G is defined as the sum of absolute values of the eigenvalues of the sum-connectivity matrix of G arranged in a non-increasing order.

In the same year, Burcu Bozkurt, Dilek Güngör, Gutman and Sinan Çevik [2], have defined the Randić energy of a graph G as the sum of the absolute values of the eigenvalues of the Randić matrix (R_{ij}) where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

---

[1] Received March 13, 2020, Accepted June 11, 2020.

Motivated by these works, we introduce the Quotient of Randić and sum-connectivity energy of a simple graph G as follows. Let a and b be two nonnegative real number with $a \neq 0$. The quotient of Randić and sum-connectivity adjacency matrix of G is the $n \times n$ matrix $A_{qrs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} 0, & \text{ifı} = j, \\ \frac{1}{\sqrt{\frac{a(d_i+d_j)}{b(d_i d_j)}}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The eigenvalues of the graph G are the eigenvalues of A_{qrs}. Since A_{qrs} is real and symmetric, its eigenvalues are real numbers which are denoted by $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n$. Then the Quotient of Randić and sum-connectivity energy of G is defined as

$$E_{qrs}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

Since A_{qsr} is a real symmetric matrix, we have

$$\sum_{i=1}^{n} \lambda_i = tr(A_{qsr}) = 0 \tag{1}$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = tr(A_{qsr}^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = 2 \sum_{i \sim j} \frac{b(d_i d_j)}{a(d_i + d_j)} \tag{2}$$

In this paper we obtain the upper and lower bounds for $E_{qsr}(G)$ and compute the $E_{qrs}(G)$ of complete graph, star graph, complete bipartite graph, the $(S_m \wedge P_2)$ graph.

§2. Upper and Lower Bounds for $E_{qrs}(G)$

In this section we obtain Upper and lower bounds for $E_{qsr}(G)$.

Theorem 2.1 *Let G be a simple graph of order n with no isolated vertices and let a, b be two nonnegative real number with $a \neq 0$. Then*

$$E_{qrs}(G) \leq \sqrt{2n \sum_{i \sim j} \frac{b(d_i d_j)}{a(d_i + d_j)}}. \tag{3}$$

Proof Let $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, be the eigenvalues of A_{qrs}. Then using (2) and the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i \right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 \right) \cdot \left(\sum_{i=1}^{n} b_i^2 \right)$$

with $a_i = 1$, $b_i =\mid \lambda_i \mid$. We obtain

$$E_{qrs}(G) = \sum_{i=1}^{n} |\lambda_i| = \sqrt{(\sum_{i=1}^{n} |\lambda_i|)^2} \leq \sqrt{n \sum_{i=1}^{n} \lambda_i^2} = \sqrt{2n \sum_{i\sim j} \frac{b(d_i d_j)}{a(d_i + d_j)}}. \qquad \square$$

Theorem 2.2 *Let G be a simple graph of order n with no isolated vertices and let a, b be two nonnegative real number with $a \neq 0$. Then*

$$E_{qrs}(G) \geq 2\sqrt{\sum_{i\sim j} \frac{b(d_i d_j)}{a(d_i + d_j)}}. \tag{4}$$

Proof From (1), we have

$$\sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1\leq i<j\leq n} \lambda_i \lambda_j = 0$$

and therefore

$$-\sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{1\leq i<j\leq n} \lambda_i \lambda_j. \tag{5}$$

Thus

$$\begin{aligned} (E_{qrs}(G))^2 &= \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1\leq i<j\leq n} |\lambda_i \lambda_j| \\ &\geq \sum_{i=1}^{n} \lambda_i^2 + 2 \left| \sum_{1\leq i<j\leq n} \lambda_i \lambda_j \right| = 2\sum_{i=1}^{n} \lambda_i^2 \end{aligned}$$

on using (5). This together with (2) implies that

$$(E_{qrs}(G))^2 \geq 4 \sum_{i\sim j} \frac{b(d_i d_j)}{a(d_i + d_j)},$$

which gives (4). □

§3. Quotient of Randić and Sum-Connectivity Energies of Some Families of Graphs

We begin with some basic definitions and notations.

Definition 3.1([4]) *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 3.2([4]) *A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a* *bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

Definition 3.3([5]) *The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m)$ and $v_j v_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Now we compute Quotient of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the $(S_m \wedge P_2)$ graph.

Theorem 3.4 *Let a and b be two nonnegative real number with $a \neq 0$. Then the quotient of Randić and sum-connectivity energy of the complete bipartite graph $K_{m,n}$ is*

$$2\sqrt{\frac{b(mn)^2}{a(m+n)}}.$$

Proof Let the vertex set of the complete bipartite graph be $V(K_{m,n}) = \{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$. Then the Quotient of Randić and sum-connectivity matrix of complete bipartite graph is given by

$$A_{qrs} = \begin{pmatrix} 0 & \cdots & 0 & \sqrt{\frac{b(mn)}{a(m+n)}} & \cdots & \sqrt{\frac{b(mn)}{a(m+n)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \sqrt{\frac{b(nm)}{a(m+n)}} & \cdots & \sqrt{\frac{b(mn)}{a(m+n)}} \\ \sqrt{\frac{b(mn)}{a(m+n)}} & \cdots & \sqrt{\frac{b(mn)}{a(m+n)}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{b(mn)}{a(m+n)}} & \cdots & \sqrt{\frac{b(mn)}{a(m+n)}} & 0 & \cdots & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{qrs}| \quad = \quad \begin{vmatrix} \lambda I_m & -\sqrt{\frac{b(mn)}{a(m+n)}} J^T \\ -\sqrt{\frac{b(mn)}{a(m+n)}} J & \lambda I_n \end{vmatrix},$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\sqrt{\frac{b(mn)}{a(m+n)}} J^T \\ -\sqrt{\frac{b(mn)}{a(m+n)}} J & \lambda I_n \end{vmatrix} = 0,$$

which can be written as

$$|\lambda I_m|\left|\lambda I_n - \left(-\sqrt{\frac{b(mn)}{a(m+n)}}J\right)\frac{I_m}{\lambda}\left(-\sqrt{\frac{b(mn)}{a(m+n)}}J^T\right)\right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{\left(\frac{a(m+n)}{b(mn)}\right)^n}\left|\frac{a(m+n)}{b(mn)}\lambda^2 I_n - JJ^T\right| = 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{\left(\frac{a(m+n)}{b(mn)}\right)^n}P_{JJ^T}\left(\frac{a(m+n)}{b(mn)}\lambda^2\right) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_mJ_n$. Thus, we have

$$\frac{\lambda^{m-n}}{\left(\frac{a(m+n)}{b(mn)}\right)^n}\left(\left(\frac{a(m+n)}{b(mn)}\right)\lambda^2 - mn\right)\left(\frac{a(m+n)}{b(mn)}\lambda^2\right)^{n-1} = 0,$$

which is same as

$$\lambda^{m+n-2}\left(\lambda^2 - \frac{b(mn)^2}{a(m+n)}\right) = 0.$$

Therefore, the spectrum of $K_{m,n}$ is given by

$$Spec\,(K_{m,n}) = \begin{pmatrix} 0 & \sqrt{\frac{b(mn)^2}{a(m+n)}} & -\sqrt{\frac{b(mn)^2}{a(m+n)}} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Hence the quotient of Randić and sum-connectivity energy of the complete bipartite graph is

$$E_{qrs}(K_{m,n}) = 2\sqrt{\frac{b(mn)^2}{a(m+n)}},$$

as desired. □

Theorem 3.5 *Let a and b be two nonnegative real number with $a \neq 0$. Then the quotient of Randić and sum-connectivity energy of the S_n is*

$$2\sqrt{\frac{b(n-1)^2}{an}}.$$

Proof Let the vertex set of star graph be given by $V(S_n) = \{v_1, v_2, \ldots, v_n\}$. Then the quotient of Randić and sum-connectivity matrix of the star graph S_n is given by

$$A_{qrs}=\begin{pmatrix} 0 & \sqrt{\frac{b(n-1)}{an}} & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} & \sqrt{\frac{b(n-1)}{an}} \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{qrs}| = \begin{vmatrix} \lambda & -\sqrt{\frac{b(n-1)}{an}} & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} \\ -\sqrt{\frac{b(n-1)}{an}} & \lambda & 0 & \cdots & 0 \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \left(\sqrt{\frac{b(n-1)}{an}}\right)^{n} \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},$$

where, $\mu = \lambda\sqrt{\frac{an}{b(n-1)}}$. Then

$$|\lambda I - A_{qrs}| = \phi_n(\mu)\left(\sqrt{\frac{b(n-1)}{a(m+n)}}\right)^{n},$$

where

$$\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.$$

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\mu) = (\mu\phi_{n-1}(\mu) - \mu^{n-2}).$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1)).$$

Therefore

$$|\lambda I - A_{qrs}| = \left(\sqrt{\frac{b(n-1)}{an}}\right)^n \left[\left((\frac{an}{b(n-1)})\lambda^2 - (n-1)\right)\left(\lambda\sqrt{\frac{b(n-1)}{an}}\right)^{n-2}\right].$$

Thus the characteristic equation is given by

$$\lambda^{n-2}\left(\lambda^2 - \frac{b(n-1)^2}{an}\right) = 0.$$

Hence

$$Spec\,(S_n) = \begin{pmatrix} 0 & \sqrt{\frac{b(n-1)^2}{an}} & -\sqrt{\frac{b(n-1)^2}{an}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence the quotient of Randić and sum-connectivity energy of S_n is

$$E_{qrs}(S_n) = 2\sqrt{\frac{b(n-1)^2}{an}}. \qquad \square$$

Theorem 3.6 *Let a and b be two nonnegative real number with $a \neq 0$. Then the quotient of Randić and sum-connectivity energy of K_n is*

$$2(n-1)\sqrt{\frac{(n-1)b}{a2}}.$$

Proof Let the vertex set of Complete graph be given by $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Then the quotient of Randić and sum-connectivity energy of matrix of the complete graph K_n is given by

$$A_{qrs} = \begin{pmatrix} 0 & \sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \cdots & \sqrt{\frac{(n-1)^2b}{a2(n-1)}} \\ \sqrt{\frac{(n-1)^2b}{a2(n-1)}} & 0 & \cdots & \sqrt{\frac{(n-1)^2b}{a2(n-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \sqrt{\frac{(n-1)^2b}{a2(n-1)}} & \cdots & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{qrs}| = \begin{vmatrix} \lambda & -\sqrt{\frac{(n-1)^2 b}{a2(n-1)}} & \cdots & -\sqrt{\frac{(n-1)^2 b}{a2(n-1)}} \\ -\sqrt{\frac{(n-1)^2 b}{a2(n-1)}} & \lambda & \cdots & -\sqrt{\frac{(n-1)^2 b}{a2(n-1)}} \\ \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{(n-1)^2 b}{a2(n-1)}} & -\sqrt{\frac{(n-1)^2 b}{a2(n-1)}} & \cdots & \lambda \end{vmatrix}$$

$$= \left(\sqrt{\frac{(n-1)^2 b}{a2(n-1)}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{\frac{a2(n-1)}{(n-1)^2 b}}$. Then

$$|\lambda I - A_{qrs}| = \phi_n(\mu)\left(\sqrt{\frac{(n-1)^2 b}{a2(n-1)}}\right)^n,$$

where

$$\begin{aligned} \phi_n(\mu) &= \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix} = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ 0 & 0 & 0 & \cdots & -1-\mu & \mu+1 \end{vmatrix} \\ &= (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} + (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}. \end{aligned}$$

A calculation shows that

$$\begin{aligned} \phi_n(\mu) &= -(\mu+1)^{n-1} + (\mu+1)\left[(\mu+1)^{n-2}(\mu-(n-2))\right] \\ &= -(\mu+1)^{n-1} + (\mu+1)^{n-1}(\mu-(n-2)). \end{aligned}$$

Iterating this, we obtain $\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1))$. Thus, the characteristic equation is given by

$$\left(\sqrt{\frac{(n-1)^2 b}{a2(n-1)}}\right)^n (\mu+1)^{n-1}(\mu - (n-1)) = 0.$$

Hence the quotient of Randić and sum-connectivity energy of K_n is

$$E_{qrs}(K_n) = 2(n-1)\sqrt{\frac{(n-1)b}{a2}}. \qquad \square$$

Theorem 3.7 *Let a and b be two nonnegative real number with $a \neq 0$. Then the quotient of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ is*

$$4\sqrt{\frac{b(n-1)^2}{an}}.$$

Proof Let the vertex set of $(S_m \wedge P_2)$ graph be given by $V(S_m \wedge P_2) = \{v_1, v_2, \cdots, v_{2m+2}\}$. Then the quotient of Randić and sum-connectivity matrix of $(S_m \wedge P_2)$ graph is given by

$$A_{qrs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} \\ 0 & 0 & \cdots & 0 & \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & \sqrt{\frac{b(n-1)}{an}} & \cdots & \sqrt{\frac{b(n-1)}{an}} & 0 & 0 & \cdots & 0 \\ \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where, $m+1=n$.

Its characteristic polynomial is given by

$$|\lambda I - A_{qrs}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} \\ 0 & \lambda & \cdots & 0 & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & -\sqrt{\frac{b(n-1)}{an}} & \cdots & -\sqrt{\frac{b(n-1)}{an}} & \lambda & 0 & \cdots & 0 \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence, the characteristic equation is given by

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}\begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

where

$$\Lambda = \sqrt{\frac{na}{(n-1)b}}\lambda.$$

Let

$$\begin{aligned} \phi_{2n}(\Lambda) &= \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} \\ &= (-1)^{2n+2n}\Lambda\begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)} \end{aligned}$$

$$+(-1)^{2n+2}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+2}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain, after some simplifications

$$\Psi_{2n-1}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda),$$

where,

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}.$$

Then,

$$\phi_{2n}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned}\phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+2}\Psi_{2n-2}(\Lambda)+(-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda)\\ &= -\Lambda^{n-3}\Theta_n(\Lambda)+\Lambda\phi_{2n-2}(\Lambda).\end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda)=-(n-1)\Lambda^{n-2}\Theta_n(\Lambda)+\Lambda^{(n-1)}\xi_{n+1}(\Lambda),$$

where,

$$\xi_{n+1}(\Lambda)=\begin{vmatrix}\Lambda & 0 & 0 & \cdots & 0\\ 0 & \Lambda & 0 & \cdots & -1\\ 0 & 0 & \Lambda & \cdots & -1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & -1 & -1 & \cdots & \Lambda\end{vmatrix}_{(n+1)\times(n+1)}.$$

A calculation shows that

$$\begin{aligned}\phi_{2n}(\Lambda) &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda)+\Lambda^{n-1}\Lambda\Theta_n(\Lambda)\\ &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda)+\Lambda^{n}\Theta_n(\Lambda)\\ &= \left(\Lambda^n-(n-1)\Lambda^{n-2}\right)\Theta_n(\Lambda).\end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda)=\Lambda^n-(n-1)\Lambda^{n-2}.$$

Thus,

$$\phi_{2n}(\Lambda)=\left(\Lambda^n-(n-1)\Lambda^{n-2}\right)^2.$$

Hence characteristic equation becomes

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}\phi_{2n}(\Lambda)=0,$$

which is the same as

$$\left(\sqrt{\frac{b(n-1)}{an}}\right)^{2n}\left(\Lambda^n-(n-1)\Lambda^{n-2}\right)^2=0$$

and reduces to

$$\lambda^{2n-4}\left(\frac{na}{b(n-1)}\lambda^2-(n-1)\right)^2=0.$$

Therefore,

$$Spec\,((S_m \wedge P_2)) = \begin{pmatrix} 0 & \sqrt{\frac{b(n-1)^2}{na+(n-1)b}} & -\sqrt{\frac{b(n-1)^2}{na}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the quotient of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ graph is

$$E_{qrs}((S_m \wedge P_2)) = 4\sqrt{\frac{b(n-1)^2}{na}}. \qquad \square$$

References

- [1] C. Adiga, Abdelmejid Bayad, Ivan Gutman and Shrikanth Avant Srinivas , The minimum covering energy of a graphs, *Kragujevac j. Sci.*, 34 (2012), 39-56.
- [2] Ş. Burcu Bozkurt, A. Dilek Güngör, I. Gutman and A. Sinan Çevik, Randić matrix and Randić energy, *MATCH Commun. Math. Comput. Chem.*, 64 (2010), 239-250.
- [3] Bo Zhou and Nenad Trinajstic , On sum-connectivity matrix and sum-connectivity energy of (molecular) graphs *Acta Chim. Slov.*, 57 (2010), 518-523.
- [4] F. Harary, *Graph Theory*, 10^{th} Reprint, Narosa Publishing House, New Delhi, 2001.
- [5] Ş. M. A. Soeud, Huda Jaber, Some families of strongly*-graphs(Communicated).