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On Ruled Surfaces in Minkowski 3-Space

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Abstract: In this paper, we studied the timelike and the spacelike ruled surfaces in Minkowski 3-space by using the angle between unit normal vector of the ruled surface and the principal normal vector of the base curve. We obtained some characterizations on the ruled surfaces by using its rulings, the curvatures of the base curve, the shape operator and Gauss curvature.

Key Words: Minkowski space, ruled surface, striction curve, Gauss curvature

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§1. Introduction

It is safe to report that the many important studies in the theory of ruled surfaces in Euclidean and also in Minkowski and Galilean spaces. A surface M is ruled if through every point of M there is a straight line that lies on M. The most familiar examples are the plane and the curved surface of a cylinder or cone. Other examples are a conical surface with elliptical directrix, the right conoid, the helicoid, and the tangent developable of a smooth curve in space. A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. For example, a cone is formed by keeping one point of a line fixed whilst moving another point along a circle. A developable surface is a surface that can be (locally) unrolled onto a flat plane without tearing or stretching it. If a developable surface lies in three-dimensional Euclidean space, and is complete, then it is necessarily ruled, but the converse is not always true. For instance, the cylinder and cone are developable, but the general hyperboloid of one sheet is not. More generally, any developable surface in three-dimensions is part of a complete ruled surface, and so itself must be locally ruled. There are surfaces embedded in four dimensions which are however not ruled. (for more details see [1])(Hilbert & Cohn-Vossen 1952, pp. 341–342).

In the light of the existing literature, in [8,9,10] authors introduced timelike and spacelike ruled surfaces and they investigated invariants of timelike and spacelike ruled surfaces by Frenet-

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Serret frame vector fields in Minkowski space.

In this study, we investigated timelike ruled surfaces with spacelike rulings, timelike ruled surfaces with timelike rulings and spacelike ruled surfaces with spacelike rulings. Since unit normals of a ruled surface lies in normal planes of the curves on that surface then we investigated all of invariants of base curve of a ruled surface with respect to the angle between unit normal of surface and principal normal.

Now we review some basic concepts on classical differential geometry of space curves and ruled surfaces in Minkowski space. Let $\alpha : I \longrightarrow IR^3$ be a curve with $\alpha'(s) \neq 0$, where $\alpha'(s) = d\,\alpha(s)/ds$. The arc-length s of a curve $\alpha(s)$ is determined such that $\|\alpha'(s)\| = 1$. Let us denote $T(s) = \alpha'(s)$ and we call $T(s)$ a tangent vector of α at $\alpha(s)$. Its well known that there are three types curves in Minkowski space such that if $\langle \alpha', \alpha' \rangle > 0$, α is called spacelike curve, if $\langle \alpha', \alpha' \rangle < 0$, α is called timelike curve and if $\langle \alpha', \alpha' \rangle = 0$, α is called null curve. The curvature of α is defined by $\kappa(s) = \sqrt{\|\alpha''(s)\|}$. If $\kappa(s) \neq 0$, unit principal normal vector $N(s)$ of the curve at $\alpha(s)$ is given by $\alpha''(s) = \kappa(s) N(s)$. The unit vector $B(s) = T(s) \Lambda N(s)$ is called unit binormal vector of α at $\alpha(s)$. If α is a timelike curve, Frenet-Serret formulae is

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = -\tau N, \tag{1}$$

where $\tau(s)$ is the torsion of α at $\alpha(s)$ ([2]). If α is a spacelike curve with a spacelike or timelike principal normal N, the Frenet formulae is

$$T' = \kappa N, \quad N' = -\epsilon\kappa T + \tau B, \quad B' = \tau N, \tag{2}$$

where $\langle T, T \rangle = 1$, $\langle N, N \rangle = \epsilon = \pm 1$, $\langle B, B \rangle = -\epsilon$, $\langle T, N \rangle = \langle T, B \rangle = \langle N, B \rangle = 0$ ([4]).

A straight line X in IR^3, such that it is strictly connected to Frenet frame of the curve $\alpha(s)$, is represented uniquely with respect to this frame, in the form

$$X(s) = f(s)N(s) + g(s)B(s), \tag{3}$$

where $f(s)$ and $g(s)$ are the smooth functions. As X moves along $\alpha(s)$, it generates a ruled surface given by the regular parametrization

$$\varphi(s, v) = \alpha(s) + vX(s), \tag{4}$$

where the components f and g are differentiable functions with respect to the arc-lenght parameter of the curve $\alpha(s)$. This surface will be denoted by M. The curve $\alpha(s)$ is called a base curve and the various positions of the generating line X are called the rulings of the surface M.

If consecutive rulings of a ruled surface in IR^3 intersect, the surface is to be developable. All the other ruled surfaces are called skew surfaces. If there is a common perpendicular to two constructive rulings in the skew surface, the foot of the common perpendicular on the main ruling is called a striction point. The set of the striction points on the ruled surface defines the striction curve [3].

The striction curve of M can be written in terms of the base curve $\alpha(s)$ as

$$\overline{\alpha}(s) = \alpha(s) - \frac{\langle T, \nabla_T X\rangle}{\|\nabla_T X\|^2} X(s). \tag{5}$$

If $\|\nabla_T X\| = 0$, the ruled surface doesn't have any striction curves. This case characterizes the ruled surface as cylindrical. Thus, the base curve can be taken as a striction curve.

Let P_x be distribution parameter of M, then

$$P_X = \frac{det(T, X, \nabla_T X)}{\|\nabla_T X\|^2}, \tag{6}$$

where ∇ is Levi-Civita connection on E_v^3 [1]. If the base curve is periodic, M is a closed ruled surface. Let M be a closed ruled surface and W be Darboux vector, then the Steiner rotation and Steiner translation vectors are

$$D = \oint_{(\alpha)} W, \quad V = \oint_{(\alpha)} d\alpha \ , \tag{7}$$

respectively. Furthermore, the pitch of M and the angle of the pitch are

$$L_X = \langle V, X\rangle, \quad \lambda_X = \langle D, X\rangle \ , \tag{8}$$

respectively [3, 5, 6].

§2. Timelike Ruled Surfaces with Spacelike Rulings

Let $\alpha : I \to E_1^3$ be a regular timelike curve with the arc-lenght parameter s and $\{T, N, B\}$ be Frenet vectors. In generally, during one parametric spatial motion, each line X in moving space generates a timelike ruled surface. Since ξ is normal to T, $\xi \in Sp\{N, B\}$ and ξ can be choosen as $\xi = T\Lambda X$ along the spacelike line X depending on the orientation of M. Thus, ξ and X can be written as

$$\xi = -\sin\psi N + \cos\psi B, \quad X = \cos\psi N + \sin\psi B, \tag{9}$$

where $\psi = \psi(s)$ is the angle between ξ and N along α [6]. From (2) and (9), we write

$$\nabla_T X = \kappa\cos\psi T + (\psi' + \tau)\ \xi. \tag{10}$$

We obtain the distribution parameter of the timelike ruled surface M as

$$P_X = \frac{\psi' + \tau}{(\psi' + \tau)^2 - \kappa^2\cos^2\psi} \tag{11}$$

by using (6) and (10). It is well known that the timelike ruled surface is developable if and only if P_X is zero from [1], so we can state the following theorem.

Theorem 2.1 *A timelike ruled surface with the spacelike rulings is developable if and only if*

$$\psi = -\int \tau ds + c$$

is satisfied, where c is a constant.

In the case $\psi = (2k-1)\pi/2$ and $\psi = k\pi$, $k \in Z$, we get $P_X = P_B$ and $P_X = P_N$, respectively. Thus, the distribution parameters are

$$P_B = \frac{1}{\tau}, \quad P_N = \frac{\tau}{\tau^2 - \kappa^2}$$

and we obtain

$$\frac{P_B}{P_N} = 1 - \left(\frac{\kappa}{\tau}\right)^2.$$

Thus, we get a corollary following.

Corollary 2.2 *The base curve of the timelike ruled surface with the spacelike rulings is a timelike helice if and only if* $\frac{P_B}{P_N}$ *is a constant.*

On the other hand, from (5) the striction curve of M is

$$\overline{\alpha}(s) = \alpha(s) + \frac{\kappa \cos\psi}{(\psi' + \tau)^2 - \kappa^2 \cos^2\psi} X(s).$$

In the case that M is a cylindrical timelike ruled surface with the spacelike rulings, we get the theorem following.

Theorem 2.3 *i) If M is a cylindrical timelike ruled surface with the spacelike rulings, $\kappa \cos\psi = 0$. In the case $\kappa = 0$, the timelike ruled surface is a plane. In the case $\psi = k\pi$, $k \in Z$, unit normal vector of M and binormal vector of the base curve are on the same direction and both the striction curve and the base curve are geodesics of M.*

ii) A cylindrical timelike ruled surface with the spacelike rulings is developable if and only if

$$\kappa \cos\left(\int \tau ds + c\right) = 0$$

is satisfied. In this case, the base curve is a timelike planar curve.

On the other hand, the equation (4) indicates that $\varphi_v : I \times \{v\} \to M$ is a curve of M for each $v \in IR$. Let A be the tangent vector field of the curve φ_v then A is

$$A = (1 + v\kappa \cos\psi)\, T + v\,\{\tau + \psi'\}\,\xi. \tag{12}$$

Since the vector field A is normal to ξ, $\tau + \psi' = 0$ is satisfied. Thus, we get the theorem following.

Theorem 2.4 *The tangent planes of a timelike ruled surface with the spacelike rulings are the* *same along the spacelike generating lines if and only if*

$$\tau + \psi' = 0$$

is satisfied.

Theorem 2.5 *i) Let ψ be $\psi' \neq -\tau$ and M be a closed timelike ruled surface with the spacelike rulings as given in the form (4). The distance between spacelike generating lines of M is minimum along the striction curve.*

ii) Let $\overline{\alpha}(s)$ be a striction curve of a timelike ruled surface with the spacelike rulings, then

$$\frac{\kappa \cos\psi}{(\psi' + \tau)^2 - \kappa^2 \cos^2\psi}$$

is a constant.

*iii) Let M be a timelike ruled surface as given in the form (**??**), then $\varphi(s, v_o)$ is a striction point if and only if $\nabla_T X$ is normal to the tangent plane at that point on M, where*

$$v_o = \frac{\kappa \cos\psi}{(\psi' + \tau)^2 - \kappa^2 \cos^2\psi}.$$

Proof i) Let $X_{\alpha(s_1)}$ and $X_{\alpha(s_2)}$ be spacelike generating lines which pass from the points $\alpha(s_1)$ and $\alpha(s_2)$ of the base curve, respectively ($s_1, s_2 \in IR$ and $s_1 < s_2$). Distance between these spacelike generating lines along the orthogonal orbits is

$$J(v) = \int_{s_1}^{s_2} \|A\| \, ds.$$

So we obtain

$$J(v) = \int_{s_1}^{s_2} \left(2v\kappa \cos\psi - 1 + \left((\psi' + \tau)^2 - \kappa^2 \cos^2\psi\right) v^2\right)^{\frac{1}{2}} ds.$$

If $J(v)$ is minimum for v_0, $J'(v_o) = 0$ and we get

$$v_o = \frac{\kappa \cos\psi}{(\psi' + \tau)^2 - \kappa^2 \cos^2\psi}.$$

Thus, the orthogonal orbit is the striction curve of M for $v = v_o$.

ii) Since the tangent vector field of the striction curve is normal to X, $\left\langle X, \frac{d\overline{\alpha}}{ds}\right\rangle = 0$. Thus, we get

$$\frac{d}{ds}\left(\frac{\kappa \cos\psi}{(\psi' + \tau)^2 - \kappa^2 \cos^2\psi}\right) = 0$$

and so

$$\frac{\kappa \cos\psi}{(\psi' + \tau)^2 - \kappa^2 \cos^2\psi} = \text{constant}.$$

iii) We suppose that $\varphi(s, v_o)$ is a striction point on the timelike base curve $\alpha(s)$ of M, then we must show that $\langle \nabla_T X, X\rangle = 0$ and $\langle \nabla_T X, A\rangle = 0$. Since the vector field X is an unit vector, $T[\langle X, X\rangle] = 0$ and we get $\langle \nabla_T X, X\rangle = 0$.

On the other hand, from (10) and (12), we obtain

$$\langle \nabla_T X, A\rangle = \left\{(\psi' + \tau)^2 - \kappa^2\cos^2\psi\right\} v_o - \kappa\cos\psi.$$

Now, by applying

$$v_o = \frac{\kappa\cos\psi}{(\psi' + \tau)^2 - \kappa^2\cos^2\psi},$$

we get $\langle \nabla_T X, A\rangle = 0$. This means that $\nabla_T X$ is normal to the tangent plane at the striction point $\varphi(t, v_o)$ on M.

Conversely, since $\nabla_T X$ is normal to the tangent plane at the point $\varphi(s, v_o)$ on M, $\langle \nabla_T X, A\rangle = 0$ and from (10) and (12), we obtain

$$\left\{(\psi' + \tau)^2 - \kappa^2\cos^2\psi\right\} v_o - \kappa\cos\psi = 0.$$

Thus, we get $v_o = \dfrac{\kappa\cos\psi}{(\psi' + \tau)^2 - \kappa^2\cos^2\psi}$ and so $\varphi(s, v_o)$ is a striction point on M. □

Theorem 2.6 *Absolute value of the Gauss curvature of M is maximum at the striction points on the spacelike generating line X and it is*

$$|K|_{\max} = \frac{\left\{(\psi' + \tau)^2 - \kappa^2\cos^2\psi\right\}^2}{(\psi' + \tau)^2}.$$

Proof Let M be a timelike ruled surface as given in the form (4) and Φ be base of the tangent space which is spanned by the unit vectors A_o and X where A_o is the tangent vector of the curve $\varphi(s, v = \text{constant.})$ with the arc-length parameter s^*. Hence, we write $A_o = \dfrac{d\varphi}{ds^*} = \dfrac{d\varphi}{ds}\dfrac{ds}{ds^*}$ where $\dfrac{d\varphi}{ds} = A$, $A_o = \dfrac{1}{\|A\|}A$ and $\dfrac{ds}{ds^*} = \dfrac{1}{\|A\|}$. Thus, we obtain the following equations after the routine calculations.

$$\begin{aligned}
\nabla_{A_o} T &= \frac{\kappa}{\|A\|}\left\{\cos\psi X - \sin\psi\xi\right\}, \\
\nabla_{A_o}\xi &= \frac{1}{\|A\|}\left\{-\kappa\sin\psi T - (\psi' + \tau) X\right\}, \\
\nabla_{A_o} A &= \frac{1}{\|A\|}\left\{\begin{array}{c}
\left\{(1 + v\kappa\cos\psi)' - v(\psi' + \tau)\kappa\sin\psi\right\} T \\
+\left\{(1 + v\kappa\cos\psi)\kappa\cos\psi + v(\psi' + \tau)^2\right\} X \\
+\left\{(-v\kappa\sin\psi - 1)\kappa\sin\psi + (v(\psi' + \tau))'\right\}\xi
\end{array}\right\}.
\end{aligned}$$

On the other hand, we denote $\xi_{\varphi(s,v)}$ as the unit normal vector at the points $\varphi(s, v = \text{constant})$, then from (1), (9) and (12) we get

$$\xi_{\varphi(s,v)} = \frac{1}{\|A\|}\left\{v\left(\psi' + \tau\right)T + \left(1 + v\kappa\cos\psi\right)\xi\right\}. \tag{13}$$

By differentiating both side of (13) with respect to the parameter s, we get

$$\frac{d\xi_{\varphi(s,v)}}{ds} = \left\{\begin{array}{c} \left\{\begin{array}{c} \left(v\left(\psi' + \tau\right)' - \kappa\sin\psi\left(1 + v\kappa\cos\psi\right)\right)\frac{1}{\|A\|} \\ +v\left(\psi' + \tau\right)\left(\frac{1}{\|A\|}\right)' \end{array}\right\}T \\ +\left\{\left(1 + v\kappa\cos\psi\right)\left(\frac{1}{\|A\|}\right)' + \left(\begin{array}{c} \left(1 + v\kappa\cos\psi\right)' \\ -v\kappa\sin\psi\left(\psi' + \tau\right) \end{array}\right)\frac{1}{\|A\|}\right\}\xi \\ -\left(\psi' + \tau\right)\frac{1}{\|A\|}X \end{array}\right\}. \tag{14}$$

Let S be the shape operator of M at the points $\varphi(s,v)$, then we can obtain that the matrix S_Φ is following with respect to base Φ,

$$S_\Phi = \begin{bmatrix} -\left\langle S\left(A_o\right), A_o\right\rangle & \left\langle S\left(A_o\right), X\right\rangle \\ -\left\langle S\left(X\right), A_o\right\rangle & \left\langle S\left(X\right), X\right\rangle \end{bmatrix}.$$

Since $\left\langle S\left(X\right), X\right\rangle = 0$ and $\left\langle S\left(A_o\right), X\right\rangle = \left\langle S\left(X\right), A_o\right\rangle$, the Gauss curvature is

$$K(s,v) = \det S_\Phi = \left\langle S\left(A_o\right), X\right\rangle^2.$$

Suppose that s^* is arc-length parameter of A_o, then we get

$$S\left(A_o\right) = \nabla_{A_o}\xi_{\varphi(s,v)} = \frac{d\xi_{\varphi(s,v)}}{ds^*} = \frac{d\xi_{\varphi(s,v)}}{ds}\frac{ds}{ds^*} = \frac{1}{\|A\|}\frac{d\xi_{\varphi(s,v)}}{ds}.$$

From (12) and (14), we obtain

$$\begin{aligned} S\left(A_o\right) = {} & \frac{1}{\|A\|}\left\{\begin{array}{c} \left(v\left(\psi' + \tau\right)' - \kappa\sin\psi\left(1 + v\kappa\cos\psi\right)\right)\frac{1}{\|A\|} \\ +v\left(\psi' + \tau\right)\left(\frac{1}{\|A\|}\right)' \end{array}\right\}T \\ & +\frac{1}{\|A\|}\left\{\left(1 + v\kappa\cos\psi\right)\left(\frac{1}{\|A\|}\right)' + \left(\begin{array}{c} \left(1 + v\kappa\cos\psi\right)' \\ -v\kappa\sin\psi\left(\psi' + \tau\right) \end{array}\right)\frac{1}{\|A\|}\right\}\xi \\ & -\left(\psi' + \tau\right)\frac{1}{\|A\|^2}X. \end{aligned}$$

Hence, the Gauss curvature is

$$K(s,v) = \frac{\left(\psi' + \tau\right)^2}{\|A\|^4}. \tag{15}$$

We differentiate both side of (17) with respect to v for finding the maximum value of the

Gauss curvature along X on M. Thus, we obtain

$$\frac{\partial K(s,v)}{\partial v}=\frac{-4\left(\psi'+\tau\right)^2\left\{\left(v\left(\psi'+\tau\right)^2-\kappa^2\cos^2\psi\right)-\kappa\cos\psi\right\}}{\left\{v^2\left\{\left(\psi'+\tau\right)^2-\kappa^2\cos^2\psi\right\}-2v\kappa\cos\psi-1\right\}^3}=0$$

and $v=\dfrac{\kappa\cos\psi}{\left(\psi'+\tau\right)^2-\kappa^2\cos^2\psi}$. It is easy to see that $\varphi(s,v)$ is the striction point and we can say that the absolute value of the Gauss curvature of M is maximum at the striction points on X. Finally, by using (17), we get

$$|K|_{\max}=\frac{\left\{\left(\psi'+\tau\right)^2-\kappa^2\cos^2\psi\right\}^2}{\left(\psi'+\tau\right)^2}. \tag{16}$$

This completes the proof. □

By using (11) and (16), we can write the relation between the Gauss curvature and the distribution parameter as

$$|K|_{\max}=\frac{1}{\left(P_X\right)^2}. \tag{17}$$

Thus, we prove the following corollary too.

Corollary 2.7 *The distribution parameter of the timelike ruled surface with the spacelike rulings depends on the spacelike generating lines.*

Moreover, the Darboux frame of the surface along the timelike base curve is

$$\begin{bmatrix}\nabla_T T\\ \nabla_T X\\ \nabla_T \xi\end{bmatrix}=\begin{bmatrix}0 & \kappa\cos\psi & -\kappa\sin\psi\\ \kappa\cos\psi & 0 & \left(\psi'+\tau\right)\\ -\kappa\sin\psi & -\left(\psi'+\tau\right) & 0\end{bmatrix}\begin{bmatrix}T\\ X\\ \xi\end{bmatrix}$$

and the Darboux vector is

$$W=-\varepsilon_2\left(\psi'+\tau\right)T-\varepsilon_1\kappa\sin\psi X-\varepsilon_1\kappa\cos\psi\xi,$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_3$ are the signs of standart vectors e_1,e_2,e_3, respectively. Thus, we obtain the geodesic curvature, the geodesic torsion and the normal curvature of the timelike ruled surface with the spacelike rulings along its spacelike generating lines as

$$\kappa_g=\varepsilon_1\kappa\sin\psi,\quad \tau_g=-\varepsilon_2\left(\psi'+\tau\right),\quad \kappa_\xi=-\varepsilon_1\kappa\cos\psi\ ,$$

respectively. Note also that if the timelike ruled surface with the spacelike rulings is a constant curvature surface with a nonzero geodesic curvature, P_X is a constant and from (8) and (15), we obtain $\dfrac{\tau_g^2}{\tau_g^2-\kappa_\xi^2}$ =constant. Hence, we get the theorem following.

Theorem 2.8 *A timelike ruled surface with the spacelike rulings is a constant curvature surface* *with a nonzero geodesic curvature if and only if* $\dfrac{\tau_g^2}{\tau_g^2-\kappa_\xi^2}$ *is a constant. In the case that the base curve is one of the timelike geodesics of the timelike ruled surface, the timelike ruled surface is developable.*

On the other hand, the Steiner rotation vector is

$$D=-\varepsilon_2\left(\oint\limits_{(\alpha)}(\psi'+\tau)\,ds\right)T-\varepsilon_1\left(\oint\limits_{(\alpha)}\kappa\sin\psi ds\right)X-\varepsilon_1\left(\oint\limits_{(\alpha)}\kappa\cos\psi ds\right)\xi.$$

Furthermore, the angle of pitch of M is

$$\lambda_X=-\varepsilon_1\oint\limits_{(\alpha)}\kappa\sin\psi ds.$$

From (4), (5) and (17), we obtain that $L_N=\lambda_N=0$, $L_B=0$ and $\lambda_B=-\varepsilon_1\oint\limits_{(\alpha)}\kappa ds$ for the special cases, $X=N$ and $X=B$.

§3. Timelike Ruled Surfaces with Spacelike Rulings

Let $\alpha:I\to E^3$ be a regular spacelike curve with the arc-lenght parameter s. Since T and X are spacelike vectors, ξ is a timelike vector and the functions f and g satisfy

$$f^{\,2}-g^2=-\epsilon$$

along α ([6]). From (2) and (3), we write

$$\nabla_T X=-\epsilon f\kappa T+\{\tau+\epsilon(f'g-fg')\}\,\xi. \tag{18}$$

The distribution parameter of the timelike ruled surface with timelike rulings is obtained by a direct computation as

$$P_X=\frac{\tau+\epsilon(f'g-fg')}{f^2\kappa^2+\{\tau+\epsilon(f'g-fg')\}^{\,2}}. \tag{19}$$

Thus, we have the following result.

Theorem 3.1 *A timelike ruled surface with timelike rulings is developable if and only if*

$$\tau+\epsilon(f'g-fg')=0$$

is satisfied.

In the cases $f=0$, $g=1$ and $f=1$, $g=0$, we get $P_X=P_B$ and $P_X=P_N$, respectively.

Thus, the distribution parameters are

$$P_B = \frac{1}{\tau}, \quad P_N = \frac{\tau}{\kappa^2 + \tau}$$

and we obtain

$$\frac{P_B}{P_N} = 1 + \left(\frac{\kappa}{\tau}\right)^2 .$$

Thus, we get the following corollary.

Corollary 3.2 *The base curve of the timelike ruled surface with timelike rulings is a spacelike helice if and only if* $\frac{P_B}{P_N}$ *is a constant.*

On the other hand, from (5), the striction curve of M is

$$\overline{\alpha}(s) = \alpha(s) + \frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2} X(s).$$

In the case that M is a cylindrical timelike ruled surface with timelike rulings, we find the following result.

Theorem 3.3 *i) If M is a cylindrical timelike ruled surface with timelike rulings, $\kappa f = 0$. In the case $\kappa = 0$, the timelike ruled surface is a plane. In the case $\kappa \neq 0$, unit normal vector of M and binormal vector of the base curve are on the same direction and both the striction curve and the base curve are geodesics of M.*

ii) A cylindrical timelike ruled surface with timelike rulings is developable if and only if the base curve is a planar spacelike curve.

The tangent vector field of the curve $\varphi_v : I \times \{v\} \to M$ is

$$A = (1 - v\epsilon\kappa f)\, T + v\,\{\tau + \epsilon(f'g - fg')\}\,\xi \tag{20}$$

on M for each $v \in IR$. Since the vector field A is normal to ξ, $\tau + \epsilon(f'g - fg') = 0$ is satisfied along the curve φ_v. Thus, we have the following theorem.

Theorem 3.4 *The tangent planes of a timelike ruled surface with timelike rulings are the same along a timelike generating lines if and only if*

$$\tau + \epsilon(f'g - fg') = 0$$

is satisfied.

Theorem 3.5 *i) Let $\tau + \epsilon(f'g - fg') \neq 0$ and M be a closed timelike ruled surface with timelike rulings as given in the form (4). The distance between timelike generating lines of M is minimum along the striction curve.*

ii) Let $\overline{\alpha}(s)$ be a striction curve of M, then $\frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2}$ *is a constant.*

iii) Let M be a timelike ruled surface with timelike rulings as given in the form (4), then $\varphi(t, v_o)$ is a striction point if and only if $\nabla_T X$ is normal to the tangent plane at that point on M, where

$$v_o = \frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2}.$$

Proof *i)* Let $X_{\alpha(s_1)}$ and $X_{\alpha(s_2)}$ be timelike generating lines which pass from the points $\alpha(s_1)$ and $\alpha(s_2)$ of the base curve, respectively($s_1, s_2 \in IR$, *and* $s_1 < s_2$). Distance between these timelike generating lines along the orthogonal orbits is

$$J(v) = \int_{s_1}^{s_2} \|A\| \, ds.$$

Then, we obtain

$$J(v) = \int_{s_1}^{s_2} \left(\left\{ f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2 \right\} v^2 - 2v\epsilon f\kappa + 1 \right)^{\frac{1}{2}} ds.$$

If $J(v)$ is minimum for v_0, $J'(v_o) = 0$ and we get

$$v_o = \frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2}.$$

Thus, the orthogonal orbit is the striction curve of M for $v = v_o$.

ii) Since the tangent vector field of the striction curve is normal to X, $\left\langle X, \dfrac{d\overline{\alpha}}{ds} \right\rangle = 0$. Thus, we get

$$\frac{d}{ds}\left(\frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2} \right) = 0$$

and so

$$\frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2} = \text{constant}.$$

iii) We suppose that $\varphi(t, v_o)$ is a striction point on the spacelike base curve $\alpha(t)$ of M, then we must show that $\langle \nabla_T X, X \rangle = 0$ and $\langle \nabla_T X, A \rangle = 0$. Since the vector field X is an unit vector, $T[\langle X, X \rangle] = 0$ and we get $\langle \nabla_T X, X \rangle = 0$.

On the other hand, from (18) and (20), we obtain

$$\langle \nabla_T X, A \rangle = -\epsilon f\kappa (1 - \epsilon v_o \kappa f) + v_o \{\tau + \epsilon(f'g - fg')\}.$$

By using

$$v_o = \frac{-\kappa \cosh\psi}{(\psi' + \tau)^2 + \kappa^2 \cosh^2\psi}$$

we get $\langle \nabla_T X, A \rangle = 0$. This means that $\nabla_T X$ is normal to the tangent plane at the striction point $\varphi(s, v_o)$ on M.

Conversely, since $\nabla_T X$ is normal to the tangent plane at the point $\varphi(s, v_o)$ on M, $\langle \nabla_T X, A \rangle =$

0 and from (18) and (20), we obtain

$$\left\{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2\right\} v_o - \epsilon\kappa f = 0.$$

Thus, we get

$$v_o = \frac{\epsilon\kappa f}{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2}$$

and so $\varphi(s, v_o)$ is a striction point on M. □

Theorem 3.6 *Absolute value of the Gauss curvature of M is maximum at the striction points on the timelike generating line X and it is*

$$|K|_{\max} = \frac{\left\{f^2\kappa^2 + \{\tau + \epsilon(f'g - fg')\}^2\right\}^2}{\{\tau + \epsilon(f'g - fg')\}^2}.$$

Proof Let M be a timelike ruled surface with timelike rulings as given in the form (4) and Φ be base of the tangent space which is spanned by the unit vectors A_o and X where A_o is the tangent vector of the curve $\varphi(s, v = \text{constant.})$ with the arc-length parameter s^*. Hence, we write $A_o = \frac{d\varphi}{ds^*} = \frac{d\varphi}{ds}\frac{ds}{ds^*}$ where $\frac{d\varphi}{ds} = A$, $A_o = \frac{1}{\|A\|}A$ and $\frac{ds}{ds^*} = \frac{1}{\|A\|}$. Thus, we obtain the following equations after the routine calculations.

$$\begin{aligned}
\nabla_{A_o}T &= \frac{\epsilon\kappa}{\|A\|}\{-fX + g\xi\} \\
\nabla_{A_o}\xi &= \frac{1}{\|A\|}\{-\epsilon\kappa gT + \{\tau + \epsilon(f'g - fg')\}X\} \\
\nabla_{A_o}A &= \frac{1}{\|A\|}\left\{\begin{array}{c}
\left\{(1 - \epsilon v\kappa f)' - \epsilon v\kappa g\left(\tau + \epsilon(f'g - fg')\right)\right\}T \\
+\left\{-\epsilon\kappa f(1 - \epsilon v\kappa f) + v\{\tau + \epsilon(f'g - fg')\}^2\right\}X \\
+\left\{\epsilon\kappa g(1 - \epsilon v\kappa f) + \{v\{\tau + \epsilon(f'g - fg')\}\}'\right\}\xi
\end{array}\right\}
\end{aligned}$$

On the other hand, we denote $\xi_{\varphi(s,v)}$ as the unit normal vector at the points φ $(s, v =$ constant), then using (2), (3) and (20) we get

$$\xi_{\varphi(s,v)} = \frac{1}{\|A\|}\{-v\{\tau + \epsilon(f'g - fg')\}T + (1 - \epsilon v\kappa f)\xi\}. \tag{21}$$

By differentiating both side of (21) with respect to the parameter s, we get

$$\frac{d\xi_{\varphi(s,v)}}{ds} = \left\{\begin{array}{c}
\left\{\begin{array}{c}
-\left\{v\{\tau + \epsilon(f'g - fg')\}' + \epsilon\kappa g(1 - \epsilon v\kappa f)\right\}\frac{1}{\|A\|} \\
-v\{\tau + \epsilon(f'g - fg')\}\left(\frac{1}{\|A\|}\right)'
\end{array}\right\}T \\
+\left\{(1 - \epsilon v\kappa f)\left(\frac{1}{\|A\|}\right)' + \left(\begin{array}{c}
(1 - \epsilon v\kappa f)' \\
-\epsilon v\kappa g\{\tau + \epsilon(f'g - fg')\}
\end{array}\right)\frac{1}{\|A\|}\right\}\xi \\
+\{\tau + \epsilon(f'g - fg')\}\frac{1}{\|A\|}X
\end{array}\right\}. \tag{22}$$

From (20) and (22), we obtain

$$
\begin{aligned}
S\left(A_o\right) \;=\; & \frac{1}{\|A\|}\left\{\begin{array}{c} -\left\{v\left\{\tau+\epsilon(f'g-fg')\right\}'+\epsilon\kappa g\left(1-\epsilon v\kappa f\right)\right\}\frac{1}{\|A\|} \\ -v\left\{\tau+\epsilon(f'g-fg')\right\}\left(\frac{1}{\|A\|}\right)' \end{array}\right\}T \\
& +\frac{1}{\|A\|}\left\{\left(1-\epsilon v\kappa f\right)\left(\frac{1}{\|A\|}\right)'+\left(\begin{array}{c} \left(1-\epsilon v\kappa f\right)' \\ -\epsilon v\kappa g\left\{\tau+\epsilon(f'g-fg')\right\} \end{array}\right)\frac{1}{\|A\|}\right\}\xi \\
& +\left\{\tau+\epsilon(f'g-fg')\right\}\frac{1}{\|A\|^2}X.
\end{aligned}
$$

So the Gauss curvature is

$$K(s,v)=\frac{\left\{\tau+\epsilon(f'g-fg')\right\}^2}{\|A\|^4}. \tag{23}$$

We differentiate both side of (23) with respect to v for finding the maximum value of the Gauss curvature along X on M. Thus, we obtain

$$\frac{\partial K(s,v)}{\partial v}=\frac{4\left\{\tau+\epsilon(f'g-fg')\right\}^2\left\{v\left\{f^2\kappa^2+\left\{\tau+\epsilon(f'g-fg')\right\}^2\right\}-\epsilon\kappa f\right\}}{\left\{v^2\left\{\tau+\epsilon(f'g-fg')\right\}^2+\left(1-\epsilon v\kappa f\right)^2\right\}^3}=0$$

and $v=\dfrac{\epsilon\kappa f}{f^2\kappa^2+\left\{\tau+\epsilon(f'g-fg')\right\}^2}$. It is easy to see that $\varphi\left(s,v\right)$ is the striction point and we can say that the absolute value of the Gauss curvature of M is maximum at the striction points on X. Finally, by using (23), we get

$$|K|_{\max}=\frac{\left\{f^2\kappa^2+\left\{\tau+\epsilon(f'g-fg')\right\}^2\right\}^2}{\left\{\tau+\epsilon(f'g-fg')\right\}^2}. \tag{24}$$

This completes the proof. □

By using (19) and (24), we can write the relation between the Gauss curvature and the distribution parameter as similar to the equation (17). Thus, we prove the following corollary, too.

Corollary 3.7 *The distribution parameter of the timelike ruled surface with timelike rulings depends on the timelike generating lines.*

Moreover, the Darboux frame of the surface along the spacelike base curve is

$$\left[\begin{array}{c}\nabla_T T\\ \nabla_T X\\ \nabla_T \xi\end{array}\right]=\left[\begin{array}{ccc}0 & -\epsilon f\kappa & \epsilon g\kappa\\ -\epsilon f\kappa & 0 & \left\{\tau+\epsilon(f'g-fg')\right\}\\ -\epsilon g\kappa & \left\{\tau+\epsilon(f'g-fg')\right\} & 0\end{array}\right]\left[\begin{array}{c}T\\ X\\ \xi\end{array}\right]$$

and the Darboux vector is

$$W=-\varepsilon_2\left\{\tau+\epsilon(f'g-fg')\right\}T+\varepsilon_1\epsilon g\kappa X+\varepsilon_1\epsilon f\kappa\xi.$$

Thus, we obtain the geodesic curvature, the geodesic torsion and the normal curvature of the timelike ruled surface with timelike rulings along its timelike generating lines as

$$\kappa_g = \varepsilon_1 \epsilon g \kappa \quad \tau_g = -\varepsilon_2 \left\{ \tau + \epsilon(f'g - fg') \right\} \quad \kappa_\xi = \varepsilon_1 \epsilon f \kappa,$$

respectively. If the timelike ruled surface with timelike rulings is a constant curvature surface with a nonzero geodesic curvature, P_X is a constant and from (8) and (15), we obtain $\dfrac{\tau_g^2}{\kappa_g^2 + \tau_g^2} =$ constant. Hence, we get the following theorem.

Theorem 3.8 *A timelike ruled surface with timelike rulings is a constant curvature surface with a nonzero geodesic curvature if and only if* $\dfrac{\tau_g^2}{\kappa_g^2 + \tau_g^2}$ *is a constant. In the case that the spacelike base curve is one of the geodesics of the timelike ruled surface, the timelike ruled surface is developable.*

The Steiner rotation vector is

$$D = -\varepsilon_2 \left(\oint_{(\alpha)} \left\{ \tau + \epsilon(f'g - fg') \right\} ds \right) T + \varepsilon_1 \epsilon \left(\oint_{(\alpha)} g\kappa ds \right) X + \varepsilon_1 \epsilon \left(\oint_{(\alpha)} f\kappa ds \right) \xi.$$

The angle of pitch of M is

$$\lambda_X = -\varepsilon_1 \epsilon \oint_{(\alpha)} g\kappa ds.$$

From (4), (5) and (17), we obtain that $L_N = \lambda_N = 0$, $L_B = 0$ and $\lambda_B = -\varepsilon_1 \epsilon \oint_{(\alpha)} \kappa ds$ for the special cases, $X = N$ and $X = B$.

§4. Spacelike Ruled Surfaces with Spacelike Rulings

Let $\alpha : I \to E^3$ be a regular spacelike curve with the arc-length parameter s. Since T and X are spacelike vectors, ξ is a timelike vector and the functions f and g satisfy

$$f^2 - g^2 = \epsilon$$

along α ([6]). From (2) and (3), we write

$$\nabla_T X = -\epsilon f \kappa T + \left\{ \tau - \epsilon(f'g - fg') \right\} \xi. \tag{25}$$

The distribution parameter of the spacelike ruled surface with spacelike rulings is

$$P_X = \frac{-\left\{ \tau - \epsilon(f'g - fg') \right\}}{f^2\kappa^2 - \left\{ \tau - \epsilon(f'g - fg') \right\}^2}. \tag{26}$$

Thus we have the following result.

Theorem 4.1 *A spacelike ruled surface with spacelike rulings is developable if and only if*

$$\tau - \epsilon(f'g - fg') = 0$$

is satisfied.

In the cases $f = 0$, $g = 1$ and $f = 1$, $g = 0$, we get $P_X = P_B$ and $P_X = P_N$, respectively. Thus, the distribution parameters are

$$P_N = \frac{-\tau}{\kappa^2 - \tau}, \quad P_B = \frac{1}{\tau}$$

and we obtain

$$\frac{P_B}{P_N} = \left(\frac{\kappa}{\tau}\right)^2 - 1.$$

Thus, we get the following conclusion.

Corollary 4.2 *The base curve of the spacelike ruled surface is a spacelike helice if and only if* $\frac{P_B}{P_N}$ *is a constant.*

On the other hand, from (5), the striction curve of M is

$$\overline{\alpha}(s) = \alpha(s) + \frac{\epsilon\kappa f}{f^2\kappa^2 - \{\tau - \epsilon(f'g - fg')\}^{\,2}} X(s).$$

In the case that M is a cylindirical spacelike ruled surface, we know the result following.

Theorem 4.3 *i) If M is a cylindrical spacelike ruled surface, $\kappa f = 0$. In the case $\kappa = 0$, the spacelike ruled surface is a plane. In the case $f = 0$, unit normal vector of M and binormal vector of the base curve are on the same direction and both the striction curve and the spacelike base curve are geodesics of M.*

ii) A cylindrical spacelike ruled surface is developable if and only if the base curve is a planar spacelike curve.

The tangent vector field of the curve $\varphi_v : I \times \{v\} \to M$ is

$$A = (1 - v\epsilon\kappa f)\, T + v\,\{\tau - \epsilon(f'g - fg')\}\,\xi \tag{27}$$

on M for each $v \in IR$. Since the vector field A is normal to ξ, $\tau - \epsilon(f'g - fg') = 0$ is satisfied. Thus, the following theorem is true.

Theorem 4.4 *The tangent planes of a spacelike ruled surface are the same along a spacelike generating lines if and only if*

$$\tau - \epsilon(f'g - fg') = 0$$

is satisfied.

Theorem 4.5 *i) Let $\kappa f \neq 0$ and M be a closed spacelike ruled surface as given in the form (4).*

The distance between spacelike generating lines of M *is minimum along the spacelike striction curve.*

ii) Let $\overline{\alpha}(s)$ *be a striction curve of* M*, then* $\dfrac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}$ *is a constant.*

iii) Let M *be a spacelike ruled surface as given in the form (4), then* $\varphi(s,v_o)$ *is a striction point if and only if* $\nabla_T X$ *is normal to the tangent plane at that point on* M*, where*

$$v_o=\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}.$$

Proof *i)* Let $X_{\alpha(s_1)}$ and $X_{\alpha(s_2)}$ be spacelike generating lines which pass from the points $\alpha(s_1)$ and $\alpha(s_2)$ of the base curve, respectively$(s_1,s_2\in IR,\ \textit{and}\ s_1<s_2)$. The distance between these spacelike generating lines along the orthogonal orbits is

$$J(v)=\int_{s_1}^{s_2}\|A\|\,ds.$$

Then, we obtain

$$J(v)=\int_{s_1}^{s_2}\left(\left\{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2\right\}v^2-2v\epsilon f\kappa+1\right)^{\frac{1}{2}}ds.$$

If $J(v)$ is minimum for v_0, $J'(v_o)=0$ and we get

$$v_o=\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}.$$

Thus, the orthogonal orbit is the spacelike striction curve of M for $v=v_o$.

ii) Since the tangent vector field of the spacelike striction curve is normal to X, $\left\langle X,\frac{d\overline{\alpha}}{ds}\right\rangle=0$. Thus, we get

$$\frac{d}{ds}\left(\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}\right)=0$$

and so

$$\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2}=\text{constant}.$$

iii) We suppose that $\varphi(s,v_o)$ is a spacelike striction point on the spacelike base curve $\alpha(s)$ of M, then we must show that $\langle\nabla_T X,X\rangle=0$ and $\langle\nabla_T X,A\rangle=0$. Since the vector field X is an unit vector, $T[\langle X,X\rangle]=0$ and we get $\langle\nabla_T X,X\rangle=0$. On the other hand, from (25) and (27), we obtain

$$\langle\nabla_T X,A\rangle=-\epsilon f\kappa(1-\epsilon v_o\kappa f)-v_o\{\tau-\epsilon(f'g-fg')\}.$$

By using

$$v_o=\frac{\epsilon\kappa f}{f^2\kappa^2-\{\tau-\epsilon(f'g-fg')\}^2},$$

we get $\langle\nabla_T X,A\rangle=0$. This means that $\nabla_T X$ is normal to the tangent plane at the spacelike striction point $\varphi(s, v_o)$ on M.

Conversely, since $\nabla_T X$ is normal to the tangent plane at the point $\varphi(s, v_o)$ on M, $\langle \nabla_T X, A \rangle = 0$ and from (25) and (27), we obtain

$$-\epsilon f \kappa (1 - \epsilon v_o \kappa f) - v_o \{\tau - \epsilon(f'g - fg')\} = 0.$$

So we get $v_o = \dfrac{\epsilon \kappa f}{f^2 \kappa^2 - \{\tau - \epsilon(f'g - fg')\}^2}$ and so $\varphi(s, v_o)$ is a striction point on M. □

Theorem 4.6 *Absolute value of the Gauss curvature of M is maximum at the striction points on the spacelike generating line X and it is*

$$|K|_{\max} = \frac{\left\{ f^2 \kappa^2 - \{\tau - \epsilon(f'g - fg')\}^2 \right\}^2}{\{\tau - \epsilon(f'g - fg')\}^2}$$

Proof Let M be a spacelike ruled surface as given in the form (1) and Φ be base of the tangent space which is spanned by the unit vectors A_o and X where A_o is the tangent vector of the curve $\varphi(s, v = const.)$ with the arc-length parameter s^*. Hence, we write $A_o = \dfrac{d\varphi}{ds^*} = \dfrac{d\varphi}{ds}\dfrac{ds}{ds^*}$ where $\dfrac{d\varphi}{ds} = A$, $A_o = \dfrac{1}{\|A\|} A$ and $\dfrac{ds}{ds^*} = \dfrac{1}{\|A\|}$. Thus, we obtain the following equations after the routine calculations,

$$\begin{aligned}
\nabla_{A_o} T &= \frac{\epsilon \kappa}{\|A\|} \{ fX - g\xi \}, \\
\nabla_{A_o} \xi &= \frac{1}{\|A\|} \{ -\epsilon \kappa g T + \{\tau - \epsilon(f'g - fg')\} X \}, \\
\nabla_{A_o} A &= \frac{1}{\|A\|} \left\{ \begin{array}{c} \left\{ (1 - \epsilon v \kappa f)' - \epsilon v \kappa g \left(\tau - \epsilon(f'g - fg') \right) \right\} T \\ + \left\{ -\epsilon \kappa f (1 - \epsilon v \kappa f) + v \{\tau - \epsilon(f'g - fg')\}^2 \right\} X \\ + \left\{ -\epsilon \kappa g (1 - \epsilon v \kappa f) + \{ v \{\tau - \epsilon(f'g - fg')\} \}' \right\} \xi \end{array} \right\}.
\end{aligned}$$

We denote $\xi_{\varphi(s,v)}$ as the unit normal vector at the points φ $(s, v =$constant), then from (2), (3) and (27) we get

$$\xi_{\varphi(s,v)} = \frac{1}{\|A\|} \{ v \{\tau - \epsilon(f'g - fg')\} T + (1 - \epsilon v \kappa f) \xi \}. \tag{28}$$

By differentiating both side of (28) with respect to the parameter s, we get

$$\frac{d\xi_{\varphi(s,v)}}{ds} = \left\{ \begin{array}{c} \left\{ \begin{array}{c} \left\{ v \{\tau - \epsilon(f'g - fg')\}' - \epsilon \kappa g (1 - \epsilon v \kappa f) \right\} \frac{1}{\|A\|} \\ + v \{\tau - \epsilon(f'g - fg')\} \left(\frac{1}{\|A\|} \right)' \end{array} \right\} T \\ + \left\{ (1 - \epsilon v \kappa f) \left(\frac{1}{\|A\|} \right)' + \left(\begin{array}{c} (1 - \epsilon v \kappa f)' \\ -\epsilon v \kappa g \{\tau - \epsilon(f'g - fg')\} \end{array} \right) \frac{1}{\|A\|} \right\} \xi \\ + \{\tau - \epsilon(f'g - fg')\} \frac{1}{\|A\|} X \end{array} \right\}. \tag{29}$$

From (27) and (29), we obtain

$$\begin{aligned} S\left(A_o\right) &= \frac{1}{\|A\|}\left\{\begin{array}{c} \left\{v\left\{\tau-\epsilon(f'g-fg')\right\}'-\epsilon\kappa g\left(1-\epsilon v\kappa f\right)\right\}\frac{1}{\|A\|} \\ +v\left\{\tau-\epsilon(f'g-fg')\right\}\left(\frac{1}{\|A\|}\right)' \end{array}\right\}T \\ &+\frac{1}{\|A\|}\left\{(1-\epsilon v\kappa f)\left(\frac{1}{\|A\|}\right)'+\left(\begin{array}{c} (1-\epsilon v\kappa f)' \\ -\epsilon v\kappa g\left\{\tau-\epsilon(f'g-fg')\right\} \end{array}\right)\frac{1}{\|A\|}\right\}\xi \\ &+\left\{\tau-\epsilon(f'g-fg')\right\}\frac{1}{\|A\|^2}X. \end{aligned}$$

Hence, the Gauss curvature is

$$K(s,v)=\frac{\left\{\tau-\epsilon(f'g-fg')\right\}^2}{\|A\|^4}. \tag{30}$$

We differentiate both side of (30) with respect to v for finding the maximum value of the Gauss curvature along X on M. Thus, we obtain

$$\frac{\partial K(s,v)}{\partial v}=\frac{4\left\{\tau-\epsilon(f'g-fg')\right\}^2\left\{v\left\{f^2\kappa^2-\left\{\tau-\epsilon(f'g-fg')\right\}^2\right\}-\epsilon\kappa f\right\}}{\left\{v^2\left\{f^2\kappa^2-\left\{\tau-\epsilon(f'g-fg')\right\}^2\right\}-2\epsilon v\kappa f+1\right\}^3}=0$$

and $v=\dfrac{\epsilon\kappa f}{f^2\kappa^2-\left\{\tau-\epsilon(f'g-fg')\right\}^2}$. It is easy to see that $\varphi\left(s,v\right)$ is the spacelike striction point and we can say that the absolute value of the Gauss curvature of M is maximum at the striction points on X. Finally, by using (30), we get

$$|K|_{\max}=\frac{\left\{f^2\kappa^2-\left\{\tau-\epsilon(f'g-fg')\right\}^2\right\}^2}{\left\{\tau-\epsilon(f'g-fg')\right\}^2}. \tag{31}$$

This completes the proof. □

We can write the relation between the Gauss curvature and the distribution parameter as similar to (17) by using (26) and (31).Thus, we prove the following corollary, too.

Corollary 4.7 *The distribution parameter of the spacelike ruled surface depends on the spacelike generating lines.*

Moreover, the Darboux frame of the surface along the spacelike base curve is

$$\begin{bmatrix} \nabla_T T \\ \nabla_T X \\ \nabla_T \xi \end{bmatrix}=\begin{bmatrix} 0 & \epsilon f\kappa & -\epsilon g\kappa \\ -\epsilon f\kappa & 0 & \left\{\tau-\epsilon(f'g-fg')\right\} \\ -\epsilon g\kappa & \left\{\tau-\epsilon(f'g-fg')\right\} & 0 \end{bmatrix}\begin{bmatrix} T \\ X \\ \xi \end{bmatrix}$$

and the Darboux vector is

$$W=-\varepsilon_2\left\{\tau-\epsilon(f'g-fg')\right\}T-\varepsilon_1\epsilon g\kappa X-\varepsilon_1\epsilon f\kappa\xi.$$

Thus, we obtain the geodesic curvature, the geodesic torsion and the normal curvature of the ruled surface along its spacelike generating lines as

$$\kappa_g = -\varepsilon_1 \epsilon g \kappa \quad \tau_g = -\varepsilon_2 \left\{ \tau - \epsilon (f'g - fg') \right\} \quad \kappa_\xi = -\varepsilon_1 \epsilon f \kappa \ ,$$

respectively. Note also that if the ruled surface is a constant curvature surface with a nonzero geodesic curvature, P_X is a constant and from (8) and (15), we obtain $\dfrac{\tau_g^2}{\kappa_g^2 + \tau_g^2} =$ constant. Hence, we have the following theorem.

Theorem 4.8 *A spacelike ruled surface is a constant curvature surface with a nonzero geodesic curvature if and only if* $\dfrac{\tau_g^2}{\kappa_g^2 + \tau_g^2}$ *is a constant. In the case that the spacelike base curve is one of the geodesics of the spacelike ruled surface, the spacelike ruled surface is developable.*

On the other hand, the Steiner rotation vector is

$$D = -\varepsilon_2 \left(\oint\limits_{(\alpha)} \left\{ \tau - \epsilon (f'g - fg') \right\} ds \right) T - \varepsilon_1 \epsilon \left(\oint\limits_{(\alpha)} g \kappa ds \right) X - \varepsilon_1 \epsilon \left(\oint\limits_{(\alpha)} f \kappa ds \right) \xi.$$

The angle of pitch of M is

$$\lambda_X = -\varepsilon_1 \epsilon \oint\limits_{(\alpha)} g \kappa ds.$$

From (4), (5) and (17), we obtain that $L_N = \lambda_N = 0$, $L_B = 0$ and $\lambda_B = -\varepsilon_1 \epsilon \oint\limits_{(\alpha)} \kappa ds$ for the special cases, $X = N$ and $X = B$.

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