International J.Math. Combin. Vol.3 (2008), 105-110

On the AVSDT-Coloring of $S_m + W_n$

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Abstract: For any vertex $u \in V(G)$, let $T_N(u) = \{u\} \cup \{uv|uv \in E(G), v \in V(G)\} \cup \{v \in V(G)|uv \in E(G)\}$ and f a total $k-$coloring on G. The total-color neighbor of a vertex u of G is the color set $C_f(x) = \{f(x)|x \in T_N(u)\}$. For any adjacent vertices x and y of $V(G)$ such that $C_f(x) \neq C_f(y)$, we refer to f as a k *AVSDT-coloring of G* (the abbreviation of *adjacent-vertex-strongly-distinguishing total coloring of G*). The AVSDT-coloring number of G, denoted by $\chi_{ast}(G)$ is the minimal number of colors required for an AVSDT-coloring of G. A Smarandachely total coloring of a graph G is an AVSDT-coloring with $|C_f(x)\backslash C_f(y)| \geq 2$ and $|C_f(y)\backslash C_f(x)| \geq 2$. In this paper, we calculate the AVSDT-coloring number of S_m+W_n.

Keywords: Smarandachely total coloring, join graph, AVSDT-coloring number.

AMS(2000): 05C15, 94C15.

§1. Introduction

Graph coloring is a very important part of graph theory[1]. Recently, the central part is to get the chromatic number of the relate coloring. While it is a NP-problem to get the relate chromatic number for most graphs. Now these vertex distinguishing edge coloring[2], adjacent-strongly edge coloring[3], $D(\beta)$-vertex distinguishing edge coloring[4], adjacent-vertex-distinguishing total coloring[5], $D(\beta)$-vertex distinguishing total coloring[6], vertex-distinguishing total coloring[7], adjacent-vertex-strongly-distinguishing total coloring[8], and a Smarandachely total coloring etc. for a graph, are becoming an interesting research objects for researchers coming from information or computer sciences.

Definition 1.1[8] *Let $G(V,E)$ be a simple connected graph with $|V(G)| \geq 3$, k is a positive integer, f is a mapping from $V(G) \cup E(G)$ to $\{1, 2, \cdots, k\}$. If f satisfies*

(1) *for any $uv \in E(G)$, we have $f(u) \neq f(v), f(u) \neq f(uv) \neq f(v)$;*

(2) *for any adjacent edges $uv, uw \in E(G)(v \neq w)$, we have$f(uv) \neq f(uw)$;*

(3) *for any edge $uv \in V(G)$, we have $C(u) \neq C(v)$,*

where $C(u) = \{f(u)\} \cup \{f(v)|uv \in E(G)\} \cup \{f(uv)|uv \in E(G)\}$, then f is called adjacent-vertex-strongly-distinguishing total coloring of G, denoted by k-AVSDTC of G for short, and

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[1] Supported by NNSFC(10771091 and 10661007).

[2] Received July 8, 2008. Accepted September 26, 2008.

the number

$$\chi_{ast}(G) = min\{k|k - AVSDTC \quad of \quad G\}$$

is called the AVSDT-number of G.

Definition 1.2[1] *For a graph G and H with $V(G) \cap V(H) = \emptyset, E(G) \cap E(H) = \emptyset$, a new graph $G + H$ called the join of G and H is constructed by*

$$V(G+H) = V(G) \cup V(H), E(G+H) = E(G) \cup E(H) \cup \{uv|u \in V(G), v \in V(H)\}.$$

We once presented a conjecture on the AVSDT-number of a simple connected graph in [8] following.

Conjecture *Let $G(V,E)$ be a simple connected graph with $V(G) \geq 3$. Then $\chi_{ast}(G) \leq n + \lceil \log_2^n \rceil + 1$, where $\lceil \log_2^n \rceil$ denotes the minimal integer not less than log_2^n, and the equality holds if and only if G is a complete graph K_n with $n = 2^k - 2$.*

For terminologies and notations not defined here, we refer to the reference [1].

§2. Main results

Lemma 2.1 [8] *For a simple graph with no isolated edge, if $uv \in E(G)$ and $d(u) = d(v) = \Delta(G)$, then*

$$\chi_{ast}(G) \geq \Delta(G) + 2.$$

Remark When min{m,n}≤4, the AVSDTC-number of $S_m + W_n$ can be obtained easily.

Theorem 2.2 *Suppose $min\{m,n\} \geq 5$, then*

$$\chi_{ast}(S_m + W_n) = m + n + 3.$$

Proof We can know that if $\Delta(u_0) = \Delta(v_0) = m+n+1$, then $\chi_{ast}(S_m+W_n) \geq m+n+3$ by Lemma 2.1. In order to prove $\chi_{ast}(S_m+W_n) = m+n+3$, we only need to give a $(m+n+3)-AVSDTC-coloring\ of\ S_m+W_n$. Suppose $V(S_m) = \{u_i|i = 0, 1, \cdots, m\}$, $E(S_m) = \{u_0u_i|i = 1, 2, \cdots, m\}, V(W_n) = \{v_i|i = 0, 1, \cdots, n\}, E(W_n) = \{v_0v_i|i = 1, 2, \cdots, n\} \bigcup \{v_1v_2, v_2v_3, \cdots, v_nv_1\}$.

Our discussion is divided into two cases by constructing the mapping $f : V(G) \cup E(G) \rightarrow \{1, 2, \cdots, m+n+3\}$.

Case 1. $m \geq n \geq 5$

In this case, define

$f(u_0) = 1; f(u_i) = 2, i = 1, 2, \cdots, m; f(v_j) = j + 3, j = 0, 1, \cdots, n;$

$f(u_0v_j)=j+2, j=0,1,\cdots,n-2; f(u_0v_{n-1})=m+n+1; f(u_0v_n)=m+n+2;$

$f(u_0u_i)=n+2+i, i=1,2,\cdots,m-2; f(u_0u_{m-1})=n+1; f(u_0u_m)=n+2;$

$f(u_iv_j)=i+j+3, i=1,2,\cdots,m-1, j=0,1,2,\cdots,n-2;$

$f(u_iv_j)=i+j+4, i=1,2,\cdots,m-4, j=n-1,n;$

$f(u_{m-3}v_{n-1})=m+n; f(u_{m-3}v_n)=3; f(u_{m-2}v_{n-1})=3; f(u_{m-2}v_n)=4;$

$f(u_{m-1}v_{n-1})=4; f(u_{m-1}v_n)=5; f(u_mv_j)=m+j+3, j=0,1,\cdots,n-3;$

$f(u_mv_{n-2})=4; f(u_mv_{n-1})=5; f(u_mv_n)=6;$

$f(v_0v_j)=m+j+4, j=1,2,\cdots,n-1; f(v_0v_n)=1; f(v_1v_n)=m+n+3.$

For $j=1,2,\cdots,n-1$, if $n\equiv 0(mod2)$, then let $f(v_jv_{j+1})=\begin{cases}2, & j\equiv 1(mod2)\\ 1, & j\equiv 0(mod2)\end{cases}$ and if $n\equiv 1(mod2)$, then let $f(v_jv_{j+1})=\begin{cases}1, & j\equiv 1(mod2);\\ 2, & j\equiv 0(mod2).\end{cases}$

We can find classes C by the construction f following.

$C(u_0)=\{1,2,\cdots,m+n+2\};$

$C(u_i)=\{1,2,\cdots,n+i+4\}, i=1,2,\cdots,m-5;$

$C(u_i)=\{1,2,\cdots,m+n\}, i=m-4,m-3,\cdots,m;$

$C(v_0)=\{1,2,\cdots,m+3,m+5,m+6,\cdots,m+n+3\}; C(v_1)=\{1,2,\cdots,m+5,m+n+3\};$

$C(v_i)=\{1,2,3,i+2,i+3,\cdots,m+i+2,m+i+3,m+i+4\}, i=2,3,\cdots,n-3;$

$C(v_{n-2})=\{1,2,3,4,n,n+1,\cdots,m+n,m+n+2\};$

$C(v_{n-1})=\{1,2,3,4,5,n+1,n+2,\cdots,m+n-1,m+n,m+n+1,m+n+3\};$

$C(v_n)=\{1,2,3,4,5,6,n+2,n+3,n+5,n+6,\cdots,m+n,m+n+2,m+n+3\}.$

Obviously, $C(u)\neq C(v)$ for all the vertices in S_m+W_n for $\forall uv\in E(S_m+W_n)$. So $\chi_{ast}(S_m+W_n)=m+n+3$.

Case 2. $n>m\geq 5$

Define f as follows in this case.

$f(u_0)=1; f(u_i)=2, i=1,2,\cdots,m; f(v_j)=j+3, j=0,1,\cdots,n;$

$f(u_0v_j)=j+2, j=0,1,\cdots,n-2; f(u_0v_{n-1})=m+n+1; f(u_0v_n)=m+n+2;$

$f(u_0u_i)=n+2+i, i=1,2,\cdots,m-2; f(u_0u_{m-1})=n+1; f(u_0u_m)=n+2;$

$f(u_iv_j)=i+j+3, i=1,2,\cdots,m-2, j=1,2,\cdots,n-2;$

$f(u_iv_j)=i+j+4, i=1,2,\cdots,m-4, j=n-1,n;$

$f(u_{m-3}v_{n-1})=m+n; f(u_{m-3}v_n)=3; f(u_{m-2}v_{n-1})=3; f(u_{m-2}v_n)=4;$

$f(u_{m-1}v_j)=n+2+j, j=0,1,\cdots,m-2;\ f(u_{m-1}v_j)=5+j-m, j=m-1,m,\cdots,n;$

$f(u_mv_j)=n+3+j, j=0,1,\cdots,m-3;\ f(u_mv_j)=6+j-m, j=m-2,m,\cdots,n.$

For $j = 1, 2, \cdots, n-1$, if $n \equiv 0(mod2)$, let $f(v_jv_{j+1}) = \begin{cases} 2, & j \equiv 1(mod2) \\ 1, & j \equiv 0(mod2) \end{cases}$ and if $n \equiv 1(mod2)$, then $f(v_jv_{j+1}) = \begin{cases} 1, & j \equiv 1(mod2); \\ 2, & j \equiv 0(mod2). \end{cases}$

When $n - m = 1$, define

$f(v_0v_j) = m + j + 4, j = 1, 2, \cdots, n-2;$

$f(v_0v_{n-1}) = n + 1; f(v_0v_n) = 1; f(v_1v_n) = m + n + 3.$

When $n - m \geq 2$, define

$f(v_0v_j) = m + j + 2, j = 1, 2, \cdots, n - m - 1; f(v_0v_{n-m}) = m + n + 1;$

$f(v_0v_{n-m+1}) = m + n + 2; f(v_0v_j) = m + j + 2, j = n - m + 2, n - m + 3, \cdots, n - 2;$

$f(v_0v_{n-1}) = m + n + 3; f(v_0v_n) = 1; f(v_1v_n) = m + n + 3,$

Then we know these classes C following by the definition of f.

$C(u_0) = \{1, 2, \cdots, m + n + 2\};$

$C(u_i) = \{1, 2, \cdots, n + i + 4\}, i = 1, 2, \cdots, m - 5;$

$C(u_i) = \{1, 2, \cdots, m + n\}, i = m - 4, m - 3, \cdots, m;$

$C(v_0) = \{1, 2, \cdots, m + n + 3\}.$

In the case of $n - m = 1$, we find

$C(v_1) = \{1, 2, \cdots, m + 2, n + 3, n + 4, n + 5, m + n + 3\};$

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, n+j+2, n+j+3, n+4+j\}, i = 2, 3, \cdots, n-4;$

$C(v_{n-3}) = \{1, 2, 3, 4, n - 1, n - 2, \cdots, m + n - 2, 2n - 1, 2n + 1\};$

$C(v_{n-2}) = \{1, 2, 3, 4, 5, n, n + 1, \cdots, m + n - 1, m + n + 3\};$

$C(v_{n-1}) = \{1, 2, 3, , 5, 6, n + 1, n + 2, \cdots, m + n - 1, m + n, m + n + 1\};$

$C(v_n) = \{1, 2, 3, 4, 6, 7, n + 2, n + 3, n + 5, n + 6, \cdots, m + n, m + n + 2, m + n + 3\}.$

In the case of $n - m \geq 2$, we have

$C(v_1) = \{1, 2, \cdots, m + 3, n + 3, n + 4, m + n + 3\}.$

If n-m-1>m-3, then

$C(v_j) = \{1, 2, 3, j + 2, j + 3, \cdots, m + j + 2, n + j + 2, n + j + 3\}, j = 2, 3, \cdots, m - 3;$

$C(v_{m-2}) = \{1, 2, 3, 4, m, m + 1, \cdots, 2m - 1, m + n, 2m\};$

$C(v_j) = \{1, 2, 3, j + 2, j + 3, \cdots, m + j + 1, 5 + j - m, 6 + j - m, m + j + 2\}, j = m - 1, m, \cdots, n - m - 1;$

$C(v_{n-m}) = \{1, 2, 3, n - m + 2, n - m + 3, \cdots, n + 1, 5 + n - 2m, 6 + n - 2m, m + n + 1\};$

$C(v_{n-m+1}) = \{1, 2, 3, n - m + 3, n - m + 4, \cdots, n + 2, 6 + n - 2m, 7 + n - 2m, m + n + 2\};$

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, 5+j-m, 6+j-m, m+j+2\}, j = n-m+2, \cdots, n-2;$

$C(v_{n-1}) = \{1, 2, 3, n+1, n+2, \cdots, m+n-1, m+n, m+n+1, n+4-m, n+5-m, m+n+3\};$

$C(v_n) = \{1, 2, 3, 4, n+2, n+3, n+5, n+6, \cdots, m+n, n-m+5, n-m+6, m+n+2, m+n+3\}$.

If n-m-1<m-3, then we get

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+2, n+j+2, n+j+3\}, j = 2, 3, \cdots, n-m-1$;

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, j+m+1, n+2+j, n+3+j, m+n+1\}, j = n-m$;

and if n-m+1<m-2, we get

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, j+m+1, n+2+j, n+3+j, m+n+2\}, j = n-m+1$;

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, n+2+j, n+3+j, m+j+2\}, j = n-m+2, \cdots, m-3$;

$C(v_{m-2}) = \{1, 2, 3, 4, m, m+1, \cdots, 2m-1, m+n, 2m\}$.

Now if $n-m+1 = m-2$, we know

$C(v_{m-2}) = \{1, 2, 3, 4, m, m+1, \cdots, 2m-1, m+n, n+m+2\}$;

and for other cases, we get

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, 5+j-m, 6+j-m, m+j+2\}, j = m-1, m, \cdots, n-2$;

$C(v_{n-1}) = \{1, 2, 3, n+1, n+2, \cdots, m+n-1, m+n, m+n+1, n+4-m, n+5-m, m+n+3\}$;

$C(v_n) = \{1, 2, 3, 4, n+2, n+3, n+5, n+6, \cdots, m+n, n-m+5, n-m+6, m+n+2, m+n+3\}$.

Finally, if $n-m-1 = m-3$, we obtain

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+2, n+j+2, n+j+3\}, j = 2, 3, \cdots, m-3$;

$C(v_j) = \{1, 2, 3, 4, j+2, j+3, \cdots, j+m+1, n+2+j, n+3+j, m+n+1\}, j = n-m$;

$C(v_j) = \{1, 2, 3, 4, 5, j+2, j+3, \cdots, j+m+1, n+3+j, m+n+2\}, j = n-m+1$;

$C(v_j) = \{1, 2, 3, j+2, j+3, \cdots, m+j+1, 5+j-m, 6+j-m, m+j+2\}, j = n-m+2, \cdots, n-2$;

$C(v_{n-1}) = \{1, 2, 3, n+1, n+2, \cdots, m+n-1, m+n, m+n+1, n+4-m, n+5-m, m+n+3\}$;

$C(v_n) = \{1, 2, 3, 4, n+2, n+3, n+5, n+6, \cdots, m+n, n-m+5, n-m+6, m+n+2, m+n+3\}$.

Obviously, $C(u) \neq C(v)$ for all the vertices in $S_m + W_n$ for $\forall uv \in E(S_m + W_n)$. So $\chi_{ast}(S_m + W_n) = m + n + 3$. □

Acknowledgements

The authors wish to express their appreciation of comments and suggestions from the referees.

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