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On the Number of Graceful Trees

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Abstract: Applying a relation of graceful trees with permutations, we enumerate non-equivalent graceful trees and get a closed formula for such number in this paper.

Key Words: Graceful tree, labeling, permutation.

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§1. Introduction

For a simple graph $G = (V(G), E(G))$, a vertex *labeling* of G is a mapping $\theta : V(G) \to \mathbf{Z}$ of non-negative integers that induces for each edge xy a label depending on $\theta(x)$ and $\theta(y)$. A labeling is called a *graceful labeling* of a graph G if it satisfying three conditions following:

(i) $\forall u, v \in V(G)$, if $u \neq v$, then $\theta(u) \neq \theta(v)$;

(ii) $max\{\theta(v)|v \in V(G)\} = |E(G)|$;

(iii) For $\forall e = xy \in E(G)$, let $\theta(e) = |\theta(x) - \theta(y)|$. Then $\forall e_1, e_2 \in E(G)$, if $e_1 \neq e_2$, then $\theta(e_1) \neq \theta(e_2)$.

Many research works on graph labeling can be found in the reference [2], particularly, graceful graphs. Gracefulness of some graph families can be also seen in references $[4] - [10]$. In this paper, we concentrate on the enumeration problem of graceful trees with given order.

Let $K_n = (V, E)$ be a complete graph with n vertices $v_1, v_2, \cdots, v_n$. All edges of K_n can be denoted by $e_{ij} = v_iv_j$, where $i, j \in N = \{1, 2, \cdots, n\}$), $(i \neq j$. We denote the vertex labeling of v_i by $\theta(v_i)$, and label it with $\theta(v_i) = i$. Then all edges labeling are respective $\theta(v_nv_1) = n-1$, $\theta(v_nv_2) = n-2$, $\theta(v_{n-1}v_1) = n-2, \cdots, \theta(v_nv_{n-1}) = 1$, $\theta(v_{n-1}v_{n-2}) = 1, \cdots, \theta(v_2v_1) = 1$. Obviously, all edge labels $\theta(v_iv_j)$ make up $(n-1)!$ graceful graphs. Certainly, these graceful graphs include disconnected and isomorphic graphs.

If all edges e_{ij} correspond to coordinates (x_i, y_j) on a Euclidean plane by $x_i = i, y_j = j$ for $1 < i \leq n$, $1 \leq j < n$, then there is a bijection between e_{ij} and (x_i, y_j). Its diagram is a lower triangle with $y = x - a$ for $a = 1, 2, \cdots, n-1$, and the graceful label $\theta(e)$ of an edge e is on the oblique line $y = x - a$.

For example, let $G = K_6$. Its diagram can be found in Fig.1.1.

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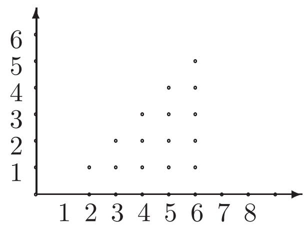


Fig.1.1

In this diagram, if $\theta(e) = 1$, then $\theta(e) \in \{|x-y| : 2-1, 3-2, 4-3, 5-4, 6-5\}$. If $\theta(e) = 2$, then $\theta(e) \in \{|x-y| : 3-1, 4-2, 5-3, 6-4\}$. $\cdots$, If $\theta(e) = 6-1$, then $\theta(e) \in \{|x-y| : 6-1\}$. In other words, there are 5 oblique lines on Fig.1 when $n = 6$. Suppose these lines are L_1, L_2, L_3, L_4, L_5. Let (x_{li}, y_{lj}) be a point on the plane with the coordinate (x_i, y_j) and l denotes $l-th$ oblique line. Then $\{(x_{16}, y_{11}) = (6,1)\} \in L_1$, $\{(x_{25}, y_{21}) = (5,1), (x_{26}, y_{22}) = (6,2)\} \in L_2$, $\{(x_{34}, y_{31}) = (4,1), (x_{35}, y_{32}) = (5,2), (x_{36}, y_{33}) = (6,3)\} \in L_3$, $\cdots$,$\{(x_{52}, y_{51}) = (2,1), (x_{53}, y_{52}) = (3,2), (x_{54}, y_{53}) = (4,3), (x_{55}, y_{54}) = (5,4), (x_{56}, y_{55}) = (6,5)\} \in L_5$. Moreover, we define

$$y_{11}(y_{21}+y_{22})\cdots(y_{n-1,1}+y_{n-1,2}+\cdots+y_{n-1,n-1}) = \sum y_{1j_1}y_{2j_2}\cdots y_{n-1,j_{n-1}}, \tag{1}$$

$$x_{1,n}(x_{2,n-1}+x_{2,n})\cdots(x_{n-1,2}+x_{n-1,3}+\cdots+x_{n-1,n}) = \sum x_{1j_1}x_{2j_2}\cdots x_{n-1,j_{n-1}}. \tag{2}$$

The expansion of these polynomials (1) and (2) both have $(n-1)!$ terms. Terms $\prod_{r=1}^{n-1} x_{s_r,i_r}$ and $\prod_{r=1}^{n-1} y_{s_r,j_r})$ in their expansion are called the *correspondent term pair*, denoted by $(x,y) = (\prod_{r=1}^{n-1} x_{s_r,i_r}, \prod_{r=1}^{n-1} y_{s_r,j_r})$. Then each pair (x,y) corresponds to a graceful graph as just explained.

In a labeling graph G, if a vertex labeling $v_i = n-i+1$ is replaced by $v_i = i$, then all edge labels are invariant. This kind of labeling are called *equivalent*, seeing in Fig 1.2 for details, in where, ($a \to a'$ and $b \to b'$).

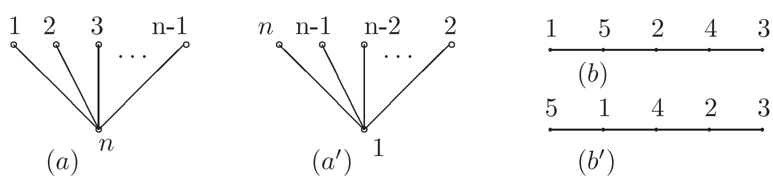


Fig.1.2

For instance, choose $n = 4$ in (1) and (2), i.e.,

$$\begin{aligned} &y_{11}(y_{21}+y_{22})(y_{31}+y_{32}+y_{33}) \\ &\quad = y_{11}y_{21}y_{31}+y_{11}y_{21}y_{32}+y_{11}y_{21}y_{33}+y_{11}y_{22}y_{31}+y_{11}y_{22}y_{32}+y_{11}y_{22}y_{33} \end{aligned}$$

$$x_{14}(x_{23}+x_{24})(x_{32}+x_{33}+x_{34})$$
$$= x_{14}x_{23}x_{32} + x_{14}x_{23}x_{33} + x_{14}x_{23}x_{34} + x_{14}x_{24}x_{32} + x_{14}x_{24}x_{33} + x_{14}x_{24}x_{34}$$

If $(x,y) = (x_{14}x_{23}x_{32}, y_{11}y_{21}y_{31})$, we get $x_{14} - y_{11} = 3, x_{23} - y_{21} = 2, x_{32} - y_{31} = 1$. Hence (x,y) is correspondent to a graceful star graph.

If $(x,y) = (x_{14}x_{23}x_{33}, y_{11}y_{21}y_{32})$, we find $x_{14} - y_{11} = 3, x_{23} - y_{21} = 2, x_{33} - y_{32} = 1$, which is correspondent to a graceful path graph.

If $(x,y) = (x_{14}x_{23}x_{34}, y_{11}y_{21}y_{33})$, we have $x_{14} - y_{11} = 3, x_{23} - y_{21} = 2, x_{34} - y_{33} = 1$. It is correspondent to a graceful triangular graph.

Notice that by definition, these two labeling in pairs $(x,y) = (x_{14}x_{24}x_{32}, y_{11}y_{22}y_{31})$ and $(x,y) = (x_{14}x_{23}x_{34}, y_{11}y_{21}y_{33})$, $(x,y) = (x_{14}x_{24}x_{33}, y_{11}y_{22}y_{32})$ and $(x,y) = (x_{14}x_{23}x_{33}, y_{11}y_{21}y_{32})$, $(x,y) = (x_{14}x_{24}x_{34}, y_{11}y_{22}y_{33})$ and $(x,y) = (x_{14}x_{23}x_{32}, y_{11}y_{21}y_{31})$ are equivalent.

§2. The Enumeration of Graceful Trees

For enumerating graceful trees, a well-known result is useful.

Lemma 2.1([3]) *Let $T = \{t_1, t_2, \cdots, t_{n-1}\}$ be a set of $n-1$ involutions on $N = \{1, 2, \cdots, n\}$. Then the product $t_1 t_2 \cdots t_{n-1}$ is an n-cyclic permutation if and only if (N, T) is a tree.*

From Lemma 2.1 we obtain a result in the following.

Theorem 2.1 *Let (x,y) be a correspondent term pair. If it is an n-cyclic permutation, then (x,y) corresponds to a graceful tree.*

Proof From the formulae (1) and (2), we have y_{11} and $x_{1n} \to (x_{1n}, y_{11})$, y_{21} and $x_{2,n-1} \to (x_{2,n-1}, y_{21})$, y_{22} and $x_{2,n} \to (x_{2,n}, y_{22})$, $\cdots$,etc.. They satisfy $y = x - a, a = 1, 2, \cdots, n-1$. So $(x,y) = (\prod_{r=1}^{n-1} x_{s_r,i_r}, \prod_{r=1}^{n-1} y_{s_r,j_r})$, namely $\{\theta(x,y)\} = \{1, 2, \cdots, n-1\}\}$. Now if it is n- cyclic permutation (not exist less than n), then it is correspondent to a connected graph of n vertices with $n-1$ edges by the Lemma 2.1. Therefore it is a graceful tree. □

Corollary 2.1 *A correspondent term pair (x,y) is a graceful tree only if*

$$\bigcup_{i=1}^{n-1} x_i \bigcup \bigcup_{j=1}^{n-1} y_j = \{1, 2, \cdots, n\}.$$

Define a matrix A by

$$A = [a_{xy}],$$

where $a_{xy} = (x,y)$. This matrix shows that there are $(n-1)!/2$ labeling ways on graceful graphs, but in which $(n-2)!/2$ labeling ways are equivalent. We need to delete the pair $(2,n)$ in the matrix A. This is tantamount to cancel equivalent labeling. In addition, the three pairs $(1,n), (1,n-1)$ and $(n-1,n)$ consist of a 3-cyclic with an edge set $\{e_{1n}, e_{1,n-1}, e_{n-1,n}\}$. In other words, there are $(n-2)!/2$ graceful graphs contain 3-cyclic with edge $e_{n-1,n}$, correspondent to the pair $(n-1,n)$. Hence cancel the pair $(n-1,n)$ in the matrix A. So we get a new matrix A' from A.

According to the previous discussions, define a permutation

$$T(n)=\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-1} \\ x_1 & x_2 & \cdots & x_{n-1} \end{pmatrix},$$

where $y_1=y_2=1, x_1=n, x_2=n-1$. Then we have the next result.

Theorem 2.2 *For an integer $n\geq 3$,*

(i) if $y_{i+1}=y_i$ or $y_{i+1}=y_i+1$ for all indexes i, then $T(n)$ corresponds to a graceful tree;

(ii) if there is an integer k such that $y_i=y_{i+1}$ and $y_{i+2}=y_i+1, y_{i+3}=y_i+2,\cdots,y_{i+k}=y_i+k-1$, rearrange y_j such that the j-th entry is $y'_j\leq j$ for $i+2\leq j\leq i+k$ and define $x'_j=y'_j+n-j$. Then the new pair (x',y') , namely

$$T'(n)=\begin{pmatrix} 1 & 1 & y'_3 & y'_4 & \cdots & y'_{n-1} \\ n & n-1 & x'_3 & x'_4 & \cdots & x'_{n-1} \end{pmatrix}$$

.

is still correspondent to a graceful tree.

Proof The case of $y_3=y_4=\cdots=y_{n-1}=1$ and $x_i=n-i+1, i=3,4,\cdots,n-1$ is trivial,which corresponds to a star tree.

We verify Theorem 2.2(i) in the first. When $y_1=y_2=1, x_1=n, x_2=n-1$, so v_1, v_{n-1} and v_n three vertices consist of a path. When $y_i=y_{i+1}, x_i=y_i+n-i$, then $x_{i+1}=y_i+n-i-1=x_i-1$. When $y_{i+1}=y_i+1, x_i=y_i+n-i$, then $x_{i+1}=x_i$. So for any integer $i, 1\leq i\leq n$, we know that $y_{i+1}=y_i+1\rightarrow x_{i+1}=x_i$; $y_{i+1}=y_i\rightarrow x_{i+1}=x_i-1$, i.e., $0\leq|y_{i+1}-y_i|\leq 1, 0\leq|x_{i+1}-x_i|\leq 1$ and $x_{n-1}-y_{n-1}=1$. Thereafter,

$$\bigcup_{i=1}^{n-1} y_i\bigcup\bigcup_{j=1}^{n-1} x_j=\{1,2,\cdots,n\}.$$

Because three vertices v_1, v_{n-1} and v_n consist of a path. When $y_3=y_2=y_1=1$, we obtain $x_3=n-2$. So v_{n-2} and v_1 are connected. Similarly, if $y_3=2, x_3=n-1$, v_2 and v_{n-1} are connected. In fact, for any integer $i, 1\leq i\leq n$, we have $y_{i+1}=y_i\rightarrow x_{i+1}=x_i+1$ or $y_{i+1}=y_i+1\rightarrow x_{i+1}=x_i$. If $y_{i+1}=y_i$, then y_{i+1} and y_i corresponds to same vertex v_s, x_{i+1} corresponds to vertex v_t, v_s and v_t are connected, by $x_i=y_i+n-i$. Similarly, if $x_{i+1}=x_i$, then x_{i+1} and x_i corresponds to same vertex v_t,y_{i+1} corresponds to vertex v_s, v_s and v_t are connected. we know that $T(n)$ corresponds to a graceful tree by Lemma 2.1.

For Theorem 2.2(ii), let $N=\{1,2,\cdots,n\}$. If $y_i=y_{i+1}, x_{i+1}=x_i-1$ and $y_{i+2}, y_{i+3},\cdots$, y_{i+k} are consecutive plus 1 of y_i, then $x_{i+2}=x_{i+3}=\cdots=x_{i+k}=x_{i+1}$. Since y_{i+1} does not participate in the rearrangement, we know that $x_{i+1}=y_{i+1}+n-i+1$. Notice that $y_{i+2}, y_{i+3},\cdots,y_{i+k}$ participating in the rearrangement do not change these labels of n vertices. Namely, the labeling set $\{1,2,\cdots,n\}$ is not dependent on $x_{i+2}, x_{i+3},\cdots,x_{i+k}$ by $x_{i+2}=x_{i+3}=\cdots=x_{i+k}=x_{i+1}$. In fact, $y_{i+2}, y_{i+3},\cdots,y_{i+k}$ correspond to $k-1$ leaves of a tree, and

$min\{x_i\} = x_{n-1} > max\{y_i\} = y_{n-1}, i = 1, 2, \cdots, n$. If y_{i+r} is replaced by $y_{i+r-j}(1 \leq j \leq r-2)$ for $2 \leq i \leq k$, then $y_{i+r} > y_{i+r-j}$. We obtain $x'_{i+r-j} > x_{i+r-j} = x_{i+1}$, since there exists an $x_s = x'_{i+r-j}(x_1 \geq x_s \geq x_{i+1})$ correspondent to a vertex of a tree, which does not change these y_{i+r} correspondent to leaves. If y_{i+r} is replaced by $y_{i+r+j}(1 \leq j \leq k-r)$, we obtain $x'_{i+r+j} = y_{i+r} + n - i - r - j < x_{i+r+j} = x_{i+1}$. If $x'_{i+r+j} \geq x_{n-1}$, there exists an $x_s = x'_{i+r+j}(x_{i+1} \geq x_s \geq x_{n-1})$. Now if $x'_{i+r+j} \leq y_{n-1}$, then there still exists a $y_t = x'_{i+r+j}$. Both of them do not change these y_{i+r} correspondent to leaves. Therefore,

$$T'(n) = \begin{pmatrix} 1 & 1 & y'_3 & y'_4 & \cdots & y'_{n-1} \\ n & n-1 & x'_3 & x'_4 & \cdots & x'_{n-1} \end{pmatrix}.$$

still corresponds to a graceful tree. □

According to Theorem 2.2, the rearrangement on y_i enable us to get new graceful tree, is not equivalent to the original tree. We enumerate all rearrangement labeling on graceful trees in the following.

Let $T(1^2, 2^2, 3^2, \cdots, k^{r_0})$ denote a permutation

$$\begin{pmatrix} 1 & 1 & y_3 & y_4 & \cdots & y_{n-1} \\ n & n-1 & x_3 & x_4 & \cdots & x_{n-1} \end{pmatrix},$$

in which, $y_1 = y_2, y_3 = y_4, \cdots, y_{2i-1} = y_{2i} = i$ for $i \leq k$. Let $E(T_n)$ denote the number of all non-equivalent graceful trees of n vertices, and $E(T_n, k^{r_0})$ denote the number of permutations on $k+1, k+2, \cdots, n-k-r_0+1$ satisfying $y_i \leq i$ and $x_i = y_i + n - i$ for $k+1 \leq i \leq n-k-r_0+1$. Applying Theorem 2.2 we find the following result.

Theorem 2.3 *For any integer $n > 2$, let $E(T_n, K) = \sum_{1 \leq k \leq \frac{n}{2}} E(T_n, k^{r_0})$. If $n \equiv 0(mod 2)$, then*

$$\begin{aligned} E(T_n, K) &= \sum_{i=2}^{\alpha} i^{n-3i+2}(i^{i-1} - 1) \cdot (i-2)! \\ &+ \sum_{i=1}^{\beta-1} (\alpha + i)((\alpha + i)^{\alpha - 2i + 1} - 1) \cdot (\alpha + i - 2)! \\ &+ \sum_{i=1}^{\gamma} (2i-1) \cdot (\frac{n}{2} - i)! + (\alpha - 1) \sum_{i=0}^{\lambda} i! \\ &+ \sum_{i=1}^{\beta} i(\alpha - 2i + \rho + 2) \cdot (\alpha - 2i + \rho)!, \end{aligned} \quad (3)$$

where,

$$\begin{cases} \alpha = \frac{n}{3}, \beta = \frac{n}{6}, \gamma = \frac{n}{6}, \lambda = \frac{n}{3} - 1, \rho = 0, & if\ n \equiv 0(mod 6); \\ \alpha = \frac{n-1}{3}, \beta = \frac{n+2}{6}, \gamma = \frac{n-4}{6}, \lambda = \frac{n-1}{3}, \rho = 1, & if\ n \equiv -2(mod 6); \\ \alpha = \frac{n+1}{3}, \beta = \frac{n-2}{6}, \gamma = \frac{n-2}{6}, \lambda = \frac{n-2}{3}, \rho = -1, & if\ n \equiv 2(mod 6). \end{cases}$$

If $n \equiv 1(mod2)$, *then*

$$\begin{aligned}
E(T_n, K) &= \sum_{i=2}^{\alpha'} i^{n-3i+2}(i^{i-1}-1)\cdot(i-2)! \\
&+ \sum_{i=1}^{\beta'}((\alpha'+i)^{\alpha'-2i}-1)\cdot(\alpha'+i-2)! \\
&+ \sum_{i=1}^{\gamma'}(2i-1)\cdot(\frac{n-1}{2}-i)! + (\alpha'-1)\sum_{i=0}^{\lambda'} i! \\
&+ \sum_{i=1}^{\beta'} i(\alpha'-2i+\rho'+2)\cdot(\alpha'-2i+\rho')! + \beta'+1, \qquad (4)
\end{aligned}$$

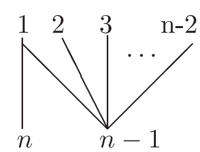
where,

$$\begin{cases}
\alpha' = \frac{n+1}{3}, \beta' = \frac{n-5}{6}, \gamma' = \frac{n+1}{6}, \lambda' = \frac{n+1}{3}-2, \rho' = -1, & if\ n \equiv -1(mod6);\\
\alpha' = \frac{n}{3}, \beta' = \frac{n-3}{6}, \gamma' = \frac{n-3}{6}, \lambda' = \frac{n}{3}-1, \rho' = 0, & if\ n \equiv 3(mod6);\\
\alpha' = \frac{n+2}{3}, \beta' = \frac{n-7}{6}, \gamma' = \frac{n-1}{6}, \lambda' = \frac{n-4}{3}, \rho' = -2, & if\ n \equiv 2(mod6).
\end{cases}$$

Proof Let $k=1, r_0=2$. Then

$$T(1^2) = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & \cdots & n-2 \\ n & n-1 & n-1 & n-1 & n-1 & \cdots & n-1 \end{pmatrix}.$$

In fact, it is correspondent to a graceful tree(see Fig.2.1 below).



$Fig.2.1$

If $y_3 \neq 2$, then $y_3 = 3$ because $x_i = y_i + n - i$ and $max\{x_i\} = n$. Similarly, if $y_4 \neq 2$ too, then $y_4 = 4$. If there is an integer r,$3 \leq r \leq n-1$ such that $y_r = 2$, then $y_i = i, x_i = n$ for $3 \leq i < r$. In other word, only $y_3 = 2$ or $y_3 = 3$, and y_4 is one element of the set $\{2,3,4\} - \{y_3\}$, y_5 is one element of the set $\{2,3,4,5\} - \{y_3, y_4\}$, $\cdots$. Continuing this process, y_{n-1} is uniquely determined at the final. Hence the number of permutations is $2 \times 2 \times 2 \times \cdots \times 2 \times 1 = 2^{n-4}$. When

$$T(1^3) = \begin{pmatrix} 1 & 1 & 1 & 2 & 3 & \cdots & n-3 \\ n & n-1 & n-2 & n-2 & n-2 & \cdots & n-2 \end{pmatrix},$$

then choose an element y_4 in the set $\{2,3,4\}$, an element y_5 in the set $\{2,3,4,5\} - \{y_4\}$, $\cdots$. Continuing in this manner, y_{n-2} and y_{n-1} are 2 selectable. So the number of such permutations is $3 \times 3 \times 3 \times \cdots \times 3 \times 2! = 3^{n-6} \cdot 2!$.

Similarly, When

$$T(1^4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 3 & \cdots & n-4 \\ n & n-1 & n-2 & n-3 & n-3 & n-3 & \cdots & n-3 \end{pmatrix},$$

we have $E(T_n, 1^4) = 4 \times 4 \times 4 \times \cdots \times 4 \times 3! = 4^{n-8} \cdot 3!$ and generally,

$$E(T_n, 1^r) = \begin{cases} r^{n-2r}(r-1)!, & \begin{cases} 2 \le r \le \frac{n}{2} - 1, & \text{n is even;} \\ 2 \le r \le \frac{n-1}{2}, & \text{n is odd.} \end{cases} \\ (n-r-1)!, & \begin{cases} \frac{n}{2} - 1 < r \le n-1, & \text{n is even;} \\ \frac{n-1}{2} < r \le n-1, & \text{n is odd.} \end{cases} \end{cases} \tag{5}$$

In general, if $k + \lceil \frac{k}{2} \rceil \le \frac{n}{2} - 1$

$$E(T_n, k^r) = \sum_{r=k+1}^{\frac{n}{2} - \lceil \frac{k}{2} \rceil} r^{n-2r-k+1} \cdot (r-1)! + \sum_{r=\frac{n}{2} - \lceil \frac{k}{2} \rceil + 1}^{n-k} (n-k-r)!, \text{n is even;}$$

$$E(T_n, k^r) = \sum_{r=k+1}^{\lceil \frac{n-k}{2} \rceil} r^{n-2r-k+1} \cdot (r-1)! + \sum_{r=\lceil \frac{n-k}{2} \rceil + 1}^{n-k} (n-k-r)!, \text{n is odd.} \tag{6}$$

If $k + \lceil \frac{k}{2} \rceil > \frac{n}{2} - 1$

$$E(T_n, k^r) = \begin{cases} \sum_{r=1}^{n-2k} (n-2k-r)!, & \text{n is even;} \\ \sum_{r=1}^{n-2k-1} (n-2k-r)!, & \text{n is odd.} \end{cases} \tag{7}$$

By (6) and (7), when n is even, define

$$f(k) = \sum_{r=k+1}^{\frac{n}{2} - \lceil \frac{k}{2} \rceil} r^{n-2r-k+1} \cdot (r-1)!$$

with $k \in \{\frac{n}{3} - 1, \frac{n-1}{3} - 1, \frac{n+1}{3} - 1\}$. Then we know that

(a) if $n \equiv 0(mod 6)$, $k = \frac{n}{3} - 1$, then

$$f(\frac{n}{3} - 1) = (\frac{n}{3})^2(\frac{n}{3} - 1)!;$$

(b) if $n \equiv -2(mod 6)$, $k = \frac{n-1}{3} - 1$, then

$$f(\frac{n-1}{3} - 1) = (\frac{n-1}{3})^3(\frac{n-1}{3} - 1)! + (\frac{n-1}{3} + 1)(\frac{n-1}{3})!;$$

(c) if $n \equiv 2(mod 6)$, $k = \frac{n}{3} - 1$, then

$$f(\frac{n+1}{3} - 1) = (\frac{n+1}{3})(\frac{n+1}{3} - 1)!.$$

Whence we obtain that

$$\sum_{i=2}^{r<\frac{n+1}{3}}(i^{n-2i}+i^{n-2i-1}+i^{n-2i-2}+\cdots+i^{n-3i+2})(i-1)!=\sum_{i=2}^{r<\frac{n+1}{3}}i^{n-3i+2}(i^{i-1}-1)(i-2)!. \quad (8)$$

When $n\equiv 0(mod6)$,

$$\begin{aligned}&\sum_{i=1}^{\frac{n}{6}-1}((\frac{n}{3}+i)^1+(\frac{n}{3}+i)^2+(\frac{n}{3}+i)^3+\cdots+(\frac{n}{3}+i)^{\frac{n}{3}-2i+1})(\frac{n}{3}+i-1)!\\&=\sum_{i=1}^{\frac{n}{6}-1}(\frac{n}{3}+i)((\frac{n}{3}+i)^{(\frac{n}{3}-2i+1)}-1)(\frac{n}{3}+i-2)!.\end{aligned}$$

We obtain that

$$\begin{aligned}&\sum_{k+\lceil\frac{k}{2}\rceil\le\frac{n}{2}-1}\sum_{r=k+1}^{\frac{n}{2}-\lceil\frac{k}{2}\rceil}r^{n-2r-k+1}(r-1)!\\&=\sum_{i=2}^{\frac{n}{3}}i^{n-3i+2}(i^{i-1}-1)(i-2)!+\sum_{i=1}^{\frac{n}{6}-1}(\frac{n}{3}+i)((\frac{n}{3}+i)^{(\frac{n}{3}-2i+1)}-1)(\frac{n}{3}+i-2)!. \quad (9)\end{aligned}$$

Similarly, when $n\equiv -2(mod6)$,

$$\begin{aligned}&\sum_{k+\lceil\frac{k}{2}\rceil\le\frac{n}{2}-1}\sum_{r=k+1}^{\frac{n}{2}-\lceil\frac{k}{2}\rceil}r^{n-2r-k+1}(r-1)!\\&=\sum_{i=2}^{\frac{n-1}{3}}i^{n-3i+2}(i^{i-1}-1)(i-2)!\\&+\sum_{i=1}^{\frac{n-4}{6}}(\frac{n-1}{3}+i)((\frac{n-1}{3}+i)^{(\frac{n-1}{3}-2i+1)}-1)(\frac{n-1}{3}+i-2)!, \quad (10)\end{aligned}$$

and when $n\equiv 2(mod6)$,

$$\begin{aligned}&\sum_{k+\lceil\frac{k}{2}\rceil\le\frac{n}{2}-1}\sum_{r=k+1}^{\frac{n}{2}-\lceil\frac{k}{2}\rceil}r^{n-2r-k+1}(r-1)!\\&=\sum_{i=2}^{\frac{n+1}{3}}i^{n-3i+2}(i^{i-1}-1)(i-2)!\\&+\sum_{i=1}^{\frac{n-8}{6}}(\frac{n+1}{3}+i)((\frac{n+1}{3}+i)^{(\frac{n+1}{3}-2i+1)}-1)(\frac{n+1}{3}+i-2)!. \quad (11)\end{aligned}$$

Now let

$$f_1(k) = \sum_{r=\frac{n}{2}-\lceil\frac{k}{2}\rceil+1}^{n-k} (n-k-r)!.$$

Similarly, we get that

(a) When $n \equiv 0(mod6)$, $k = \frac{n}{3} - 1$,

$$\sum_{r=\frac{n}{2}-\lceil\frac{k}{2}\rceil+1}^{n-k} (n-k-r)! = \sum_{i=1}^{\frac{n}{3}-1} f_1(i)$$
$$= \sum_{i=1}^{\frac{n}{6}} (2i-1)(\frac{n}{2}-i)! + (\frac{n}{3}-1)\sum_{i=0}^{\frac{n}{3}-1} i!. \quad (12)$$

(b) When $n \equiv -2(mod6)$, $k = \frac{n-1}{3} - 1$,

$$\sum_{r=\frac{n}{2}-\lceil\frac{k}{2}\rceil+1}^{n-k} (n-k-r)! = \sum_{i=1}^{\frac{n-1}{3}-1} f_1(i)$$
$$= \sum_{i=1}^{\frac{n-4}{6}} (2i-1)(\frac{n}{2}-i)! + (\frac{n-1}{3}-1)\sum_{i=0}^{\frac{n-1}{3}-1} i!. \quad (13)$$

(c) When $n \equiv 2(mod6)$, $k = \frac{n+1}{3} - 1$,

$$\sum_{r=\frac{n}{2}-\lceil\frac{k}{2}\rceil+1}^{n-k} (n-k-r)! = \sum_{i=1}^{\frac{n+1}{3}-1} f_1(i)$$
$$= \sum_{i=1}^{\frac{n-2}{6}} (2i-1)(\frac{n}{2}-i)! + (\frac{n+1}{3}-1)\sum_{i=0}^{\frac{n+1}{3}-1} i!. \quad (14)$$

When $k + \lceil\frac{k}{2}\rceil > \frac{n}{2} - 1$. Let

$$f_2(k) = \sum_{r=1}^{n-2k} (n-2k-r)!.$$

We know that

(a)When $n \equiv 0(mod6)$, $k > \frac{n}{3} - 1$,

$$\sum f_2(k \geq \frac{n}{3}) = \sum_{i=1}^{\frac{n}{6}} i(\frac{n}{3}-2i+2)(\frac{n}{3}-2i)!. \quad (15)$$

(b) When $n \equiv -2(mod6)$, $k > \frac{n-1}{3} - 1$,

$$\sum f_2(k \geq \frac{n-1}{3}) = \sum_{i=1}^{\frac{n+2}{6}} i(\frac{n-1}{3} - 2i + 3)(\frac{n-1}{3} - 2i + 1)!. \quad (16)$$

(c) When $n \equiv 2(mod6)$, $k > \frac{n+1}{3} - 1$,

$$\sum f_2(k \geq \frac{n+1}{3}) = \sum_{i=1}^{\frac{n-2}{6}} i(\frac{n+1}{3} - 2i + 1)(\frac{n+1}{3} - 2i - 1)!. \quad (17)$$

To sum up, we obtain (3) by formulae $(9), (10), (11), (12), (13), (14), (15), (16)$ and (17).

Similarly, the discussion for the case $n \equiv 1(mod2)$ can be divided into three subcases, i.e., $n \equiv -1(mod6), k = \frac{n-2}{3}$, $n \equiv 3(mod6), k = \frac{n}{3} - 1$ and $n \equiv 1(mod6), k = \frac{n-4}{3}$, and the formula (4) can be found as the formula (3). □

For example, $E(T_6, K) = 10$ when $n = 6$. We obtain 10 non-equivalent graceful trees by permutations following.

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 4 \\ 6 & 5 & 5 & 5 & 5 \end{pmatrix} \to \{e_{16}, e_{15}, e_{25}, e_{35}, e_{45}\}; \begin{pmatrix} 1 & 1 & 2 & 4 & 3 \\ 6 & 5 & 5 & 6 & 4 \end{pmatrix} \to \{e_{16}, e_{15}, e_{25}, e_{46}, e_{34}\};$$

$$\begin{pmatrix} 1 & 1 & 3 & 2 & 4 \\ 6 & 5 & 6 & 4 & 5 \end{pmatrix} \to \{e_{16}, e_{15}, e_{36}, e_{24}, e_{45}\}; \begin{pmatrix} 1 & 1 & 3 & 4 & 2 \\ 6 & 5 & 6 & 6 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{36}, e_{46}, e_{23}\};$$

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 3 \\ 6 & 5 & 4 & 4 & 4 \end{pmatrix} \to \{e_{16}, e_{15}, e_{14}, e_{24}, e_{34}\}; \begin{pmatrix} 1 & 1 & 1 & 3 & 2 \\ 6 & 5 & 4 & 5 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{14}, e_{35}, e_{23}\};$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ 6 & 5 & 4 & 3 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{14}, e_{13}, e_{23}\}; \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & 5 & 4 & 3 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{14}, e_{13}, e_{12}\};$$

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 6 & 5 & 5 & 4 & 4 \end{pmatrix} \to \{e_{16}, e_{15}, e_{25}, e_{24}, e_{34}\}; \begin{pmatrix} 1 & 1 & 2 & 2 & 2 \\ 6 & 5 & 5 & 4 & 3 \end{pmatrix} \to \{e_{16}, e_{15}, e_{25}, e_{24}, e_{23}\}.$$

When n is a large number, $E(T_n) >> E(T_n, K)$. Of course, there exist a lot of isomorphic trees in the previous enumeration. We have verified the number of non-isomorphic graceful paths P_n for $n \leq 13$ vertices in the following table.

n	2	3	4	5	6	7	8	9	10	11	12	13
E(P_n)	1	1	1	2	6	8	10	30	74	162	330	760

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