

## On the Order of a Meromorphic Matrix Valued Function on Annuli

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**Abstract:** In this paper, we derive some results for meromorphic matrix valued functions on annuli and also extended some basic results of Nevanlinna theory to matrix valued meromorphic functions on annuli.

**Key Words:** Value distribution theory, Nevanlinna theory, annuli.

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### §1. Introduction

In recent years much work has been done in generalizing theorems from complex function theory to matrix valued functions. A prime example is the work of Potapov [12], who provided the general formula for the factorization of a matrix valued inner function and factorization of matrix valued functions play an important role in many branches of analysis and engineering. In the year 2014, Bhoosnurmath proved some results concerning meromorphic matrix valued functions (see [14]). In 2005, A. Ya. Khrystyanyan and A. A. Kondratyuk have proposed on the Nevanlinna Theory for meromorphic functions on annuli (see [6],[7]) and after this work others have done lot of work in this area (see [1-4], [8-22],[23-35]). Thus it is interesting to consider some results for meromorphic matrix valued functions in multiply connected domains. By Doubly connected mapping theorem [5] each doubly connected domain is conformally equivalent to the annulus  $\{z : r < |z| < R\}$ ,  $0 \leq r < R \leq +\infty$ . We consider only two cases :  $r = 0$ ,  $R = +\infty$  simultaneously and  $0 \leq r < R \leq +\infty$ . In the latter case the homothety  $z \mapsto \frac{z}{rR}$  reduces the given domain to the annulus

$$\mathbb{A} = \mathbb{A}(R_0) = \mathbb{A} \left( \frac{1}{R_0}, R_0 \right) = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\},$$

where  $R_0 = \sqrt{\frac{R}{r}}$ . Thus, in both cases every annulus is invariant with respect to the inversion  $z \mapsto \frac{1}{z}$ . In this paper we derive some results for meromorphic matrix valued functions on annulus  $\mathbb{A}$ . However, the methods used here are different.

First, we define the order of a matrix function which is meromorphic function on the

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annulus

$$\mathbb{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}.$$

A complex number  $z$  is called a pole of  $A(z)$  on  $\mathbb{A}$  if it is a pole of one of the entries of  $A(z)$  on  $\mathbb{A}$ , and  $z$  is called a zero of  $A(z)$  on  $\mathbb{A}$  if it is a pole of  $A(z)^{-1}$  on  $\mathbb{A}$ . Let  $A(z)$  be a meromorphic  $m \times m$ -matrix valued function, then

$$m(R, A) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \|A(Re^{i\theta})\| d\theta \quad (1.1)$$

and

$$m\left(\frac{1}{R}, A\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{\|A(Re^{i\theta})\|} d\theta. \quad (1.2)$$

Here  $\|A(z)\| = \max_{\|x\|=1, x \in \mathbb{C}^m} \|A(z)x\|$ .

Set

$$N(R, A) = \int_0^R \frac{n(t, A) - n(0, A)}{t} dt + n(0, A) \log R, \quad (1.3)$$

$$N(R, f) = \int_0^R \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log R,$$

Therefore

$$T(R, A) = N(R, A) + m(R, A),$$

where  $\log^+ x = \max\{\log x, 0\}$ , and  $n(t, A)$  is the counting function of poles of the function  $f$  in  $\{z : |z| \leq t\}$ . Here we show the notations of the Nevanlinna theory for meromorphic  $m \times m$ -matrix valued function on annuli. Let

$$N_1(R, A) = \int_{\frac{1}{R}}^1 \frac{n_1(t, A)}{t} dt, \quad N_2(R, A) = \int_1^R \frac{n_2(t, A)}{t} dt$$

and

$$\begin{aligned} m_0(R, A) &= m(R, A) + m\left(\frac{1}{R}, A\right) - 2m(1, A), \\ N_0(R, A) &= N_1(R, A) + N_2(R, A), \end{aligned}$$

where  $n_1(t, A)$  and  $n_2(t, A)$  are the counting functions of the poles of  $m \times m$ -matrix valued function  $A$  in  $\{z : t < |z| \leq 1\}$  and  $\{z : 1 < |z| \leq t\}$ , respectively. The Nevanlinna characteristic of  $m \times m$ -matrix valued meromorphic function  $A(z)$  on the annulus  $\mathbb{A}$  is defined by

$$T_0(R, A) = m_0(R, A) + N_0(R, A). \quad (1.4)$$

The order  $\rho$  of  $A$  is defined by

$$\rho = \limsup_{R \rightarrow \infty} \frac{\log T_0(R, A)}{\log R}. \quad (1.5)$$

Suppose  $A(z)$   $m \times m$ -matrix valued meromorphic function we can decompose  $A(z)$  as

follows:

$$A(z) = E(z) \text{diag}((z - z_0)^{K_1} \dots (z - z_0)^{K_m}) F(z) \quad (1.6)$$

for each  $z \in \mathbb{C}$ , where  $E(z)$  and  $F(z)$  are analytic and invertible at  $z_0$  on  $\mathbb{A}$  and  $K_m \geq \dots \geq K_1$  are integers. The numbers  $|K_j|$  for which  $K_j < 0$  are called partial pole multiplicities of  $A$  at  $Z_0$  on  $\mathbb{A}$ , the numbers  $K_j$  for which  $K_j > 0$  are called the partial zero multiplicities of  $A$  at  $Z_0$  on  $\mathbb{A}$ . The function  $\text{diag}((z - z_0)^{K_j})_{j=1}^m$  is called local smith form of  $A(z)$  on  $\mathbb{A}$ .

Throughout this paper we assume that  $A(z)$  is  $m \times m$ -matrix valued meromorphic and regular function on the annulus  $\mathbb{A}$ , that is, there exist at least one point where  $A(z)$  is analytic and invertible on  $\mathbb{A}$ . Then  $A(z)^{-1}$  is also a  $m \times m$ -matrix valued meromorphic function  $A(z)$  on the annulus  $\mathbb{A}$ , as can be seen by applying Cramer's rule.

**Proposition 1.1** *Suppose  $A(z)$  is a  $m \times m$ -matrix valued meromorphic function on the annulus  $\mathbb{A}$  of finite order  $\rho$ . Let  $\rho_{i,j}$  denote the order of the  $ij$  entry  $a_{ij}$  of  $A(z)$ . Then*

$$\rho = \max_{1 \leq i, j \leq m} \rho_{i,j}. \quad (1.7)$$

*Proof* Note that

$$\begin{aligned} |a_{ij}(z)| &= |\langle A(z)e_j, e_i \rangle| \\ &\leq \|A(z)e_j\| \|e_i\| \leq \|A(z)\|. \end{aligned}$$

From this one sees that  $m_0(R, a_{ij}) \leq m_0(R, A)$ . Clearly  $N_0(R, a_{ij}) \leq N_0(R, A)$ , so that  $T_0(R, a_{ij}) \leq T_0(R, A)$ . This implies that

$$\max_{1 \leq i, j \leq m} \rho_{i,j} \leq \rho.$$

Conversely, the local smith form shows that the highest order of a pole that  $a_{ij}(z)$  can have at  $z_0$  is  $|K_1(z_0)|$  on  $\mathbb{A}$  and since  $E(z_0)$  and  $F(z_0)$  are invertible, at least one of the  $a_{ij}(z)$  will have a pole of order  $|K_1(z_0)|$  at  $z_0$  on  $\mathbb{A}$ . Then

$$\begin{aligned} n(t, A) &= \sum_{\{z: |z| \leq t\}} \sum_{K_j < 0} |K_j(z)| \leq \sum_{\{z: |z| \leq t\}} \{K_j < 0\} |K_1(z)| \\ &\leq m \sum_{\{z: |z| \leq t\}} |K_1(z)| \leq m \sum_{i=1}^m \sum_{j=1}^m n(t, a_{ij}), \end{aligned}$$

so that

$$N(R, A) \leq m \sum_{i,j=1}^m N(R, a_{ij}).$$

Similarly

$$N_1(R, A) \leq m \sum_{i,j=1}^m N_1(R, a_{ij}) \quad \text{and} \quad N_2(R, A) \leq m \sum_{i,j=1}^m N_2(R, a_{ij}).$$

Therefore

$$N_0(R, A) \leq m \sum_{i,j=1}^m N_0(R, a_{ij}).$$

Furthermore,

$$\begin{aligned} \|A(z)\| &= \max_{\|x\|=1} \|A(z)x\| \\ &\leq m^{\frac{1}{2}} \max_{\|x\|=1} \max_{1 \leq i \leq m} \sum_{j=1}^m |a_{ij}(z)x_j| \\ &\leq m^{\frac{1}{2}} \max_{x:|x_j| \leq 1} \max_{1 \leq i \leq m} \sum_{j=1}^m |a_{ij}(z)x_j| \\ &\leq m^{\frac{3}{2}} \max_{1 \leq i,j \leq m} |a_{ij}(z)|. \end{aligned}$$

Therefore

$$\begin{aligned} m_0(R, A) &\leq \log m^{\frac{3}{2}} + \max_{1 \leq i,j \leq m} m_0(R, a_i) \\ m_0(R, A) &\leq \log m^{\frac{3}{2}} + m \sum_{i,j=1}^m m_0(R, a_i). \end{aligned}$$

It follows that

$$T_0(R, A) \leq \log m^{\frac{3}{2}} + m \sum_{i,j=1}^m T_0(R, a_i).$$

Now for each  $\epsilon > 0$ , there are constants  $C_{ij}$  such that for all  $R$  sufficiently large

$$T_0(R, a_{ij}) \leq C_{ij} R^{\rho_{i,j} + \epsilon}.$$

Then for all sufficiently large  $R$ , we have

$$T_0(R, A) \leq C_{ij} R^{\max \rho_{i,j} + \epsilon}.$$

Hence the order  $\rho$  of  $m \times m$ -matrix valued meromorphic function  $A(z)$  is less than or equal to  $\max \rho_{i,j}$ .  $\square$

**Remark 1.1** Next, if  $A(z)$  is  $m \times m$ -matrix valued entire function on  $\mathbb{A}$  of order  $\widehat{\rho}$  is defined as follows : it is the infimum of the numbers  $\lambda$  for which there exists positive constants  $B$  and  $C$  for which

$$\|A(z)\| \leq A \exp(B|z|^\lambda) \tag{1.8}$$

for all  $|z|$  sufficiently large.

**Proposition 1.2** *If  $A(z)$  is an  $m \times m$ -matrix valued entire function on  $\mathbb{A}$ , then  $\rho = \widehat{\rho}$ .*

*Proof* Let  $\widehat{\rho}_{i,j}$  be the order of  $a_{ij}(z)$  as entire matrix valued function on  $\mathbb{A}$ , that is defined

similarly to (1.8). We claim that  $\widehat{\rho} = \max \widehat{\rho}_{i,j}$  for  $1 \leq i, j \leq m$ . Indeed, since  $|a_{ij}| \leq \|A(z)\|$  it follows that  $\max \widehat{\rho}_{i,j} \leq \widehat{\rho}$  for  $1 \leq i, j \leq m$ .

Conversely, suppose that  $\|A(z)\|^2 \leq \sum_{i,j=1}^m |a_{ij}(z)|^2$  to see that

$$\widehat{\rho} \leq \max \widehat{\rho}_{i,j} \text{ for } 1 \leq i, j \leq m.$$

Since it is well known that for scalar functions  $\widehat{\rho} = \rho_{i,j}$ , it follows that we can apply Proposition 1.1 to get the desired result.  $\square$

**Proposition 1.3** *Let  $A(z)$  be a regular meromorphic matrix valued function on  $\mathbb{A}$  of finite order  $\rho$ . Then  $A(z)^{-1}$  has order at most  $\rho$  on annuli  $\mathbb{A}$ .*

*Proof* We use the fact that if  $f$  and  $g$  are scalar meromorphic functions of order  $\rho_1$  and  $\rho_2$ , respectively, then  $f + g$ ,  $f \cdot g$  and  $\frac{f}{g}$  are functions having order at most  $\max(\rho_1, \rho_2)$ .

Compute  $A(z)^{-1}$  by Cramers rule,

$$A(z)^{-1} = \frac{\text{Adj } A(z)}{\det A(z)}$$

By Remark 1.1 and Proposition 1.1 each entry of  $A(z)^{-1}$  has order at most  $\rho$  on annuli  $\mathbb{A}$ . Proposition 1.1 yields that  $A(z)^{-1}$  has order at most  $\rho$  on  $\mathbb{A}$ .  $\square$

By the definition of order, one obtains the following result.

**Proposition 1.4** *Let  $A(z)$  and  $B(z)$  be regular meromorphic matrix valued functions on  $\mathbb{A}$  of finite order. Then the order of  $A(z)B(z)$  is at most the maximum of the order of  $A(z)$  and the order of  $B(z)$  on annuli  $\mathbb{A}$ .*

## §.2. Main Results

We use the following lemmas to prove our main result, which can be derived from the proof of Nevanlinna-Polya theorem in [13].

**Lemma 2.1** *Let  $n$  be an arbitrary fixed positive integer and for each  $k$  ( $k = 1, 2, \dots, n$ ). Let  $f_k$  and  $g_k$  be analytic functions of a complex variable  $z$  on a non-empty domain  $D$ .*

*If  $f_k$  and  $g_k$  ( $k = 1, 2, \dots, n$ ) satisfy*

$$\sum_{k=1}^n |f_k(z)|^2 = \sum_{k=1}^n |g_k|^2$$

*on  $D$  and if  $f_1, f_2, \dots, f_n$  are linearly independent on  $D$ , then there exists an  $n \times n$  unitary*

matrix  $C$ , where each of the entries of  $C$  is a complex constants such that

$$C \begin{bmatrix} f_1(z) \\ f_2(z) \\ \dots \\ f_n(z) \end{bmatrix} = \begin{bmatrix} g_1(z) \\ g_2(z) \\ \dots \\ g_n(z) \end{bmatrix}$$

holds on  $D$ .

**Lemma 2.2** Let  $A = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix}$  and  $B = \begin{bmatrix} g_1(z) \\ g_2(z) \end{bmatrix}$  be two meromorphic matrix valued functions on  $\mathbb{A}$ . If  $f_k$  and  $g_k$  ( $k = 1, 2$ ) satisfy

$$|f_1(z)|^2 + |f_2(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2, \quad (2.1)$$

on  $\mathbb{A}$ , then there exists a  $2 \times 2$  unitary matrix  $C$  where each of the entries of  $C$  is a complex constant such that

$$B = CA. \quad (2.2)$$

*Proof* We consider the following two cases.

**Case 1.** If  $f_1$  and  $f_2$  are linearly independent on  $\mathbb{A}$ , then the proof follows by Lemma 2.1.

**Case 2.** If  $f_1$  and  $f_2$  are linearly dependent on  $\mathbb{A}$ , then there exists two complex constants  $c_1$  and  $c_2$  not both zero such that

$$c_1 f_1(z) + c_2 f_2(z) = 0. \quad (2.3)$$

We discuss two subcases following.

**Subcase 2.1** If  $c_2 \neq 0$ , then by (2.3) we get

$$f_2(z) = -\frac{c_1}{c_2} f_1(z) \quad (2.4)$$

holds on  $\mathbb{A}$ .

If we set  $b = -\frac{c_1}{c_2}$ , then by (2.4) we have

$$f_2(z) = b f_1(z) \quad (2.5)$$

on  $\mathbb{A}$ . Hence from (2.1), we have

$$(1 + |b|^2) |f_1(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2. \quad (2.6)$$

We may assume that  $f_1 \not\equiv 0$  on  $\mathbb{A}$ . Otherwise the proof is trivial.

Hence by (2.6), we get

$$\left| \frac{g_1(z)}{f_1(z)} \right|^2 + \left| \frac{g_2(z)}{f_2(z)} \right|^2 = 1 + |b|^2. \quad (2.7)$$

Taking the Laplacians  $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$  of both sides of (2.7) with respect to  $z = x + iy$  ( $x, y$  real), we get

$$\left| \left( \frac{g_1(z)}{f_1(z)} \right)' \right|^2 + \left| \left( \frac{g_2(z)}{f_2(z)} \right)' \right|^2 = 0. \quad (2.8)$$

Since  $\Delta|P(z)|^2 = 4|P'(z)|^2$ , where  $P(z)$  is an analytic function of  $z$  on  $\mathbb{A}$ . By (2.8), we get

$$\left( \frac{g_1(z)}{f_1(z)} \right)' = 0$$

and

$$\left( \frac{g_2(z)}{f_2(z)} \right)' = 0.$$

Hence

$$g_1(z) = cf_1(z) \quad \text{and} \quad g_2(z) = df_2(z), \quad (2.9)$$

where  $c, d$  are complex constants.

Substituting (2.9) in (2.7), we get

$$|c|^2 + |d|^2 = 1 + |b|^2. \quad (2.10)$$

Let us define

$$U = \begin{bmatrix} \frac{1}{\sqrt{1+|b|^2}} & \frac{-\bar{b}}{1+\sqrt{|b|^2}} \\ \frac{b}{1+\sqrt{|b|^2}} & \frac{1}{1+\sqrt{|b|^2}} \end{bmatrix} \quad (2.11)$$

and

$$V = \begin{bmatrix} \frac{c}{1+\sqrt{|b|^2}} & \frac{-\bar{d}}{1+\sqrt{|b|^2}} \\ \frac{d}{1+\sqrt{|b|^2}} & \frac{\bar{c}}{1+\sqrt{|b|^2}} \end{bmatrix} \quad (2.12)$$

Then it is easy to prove, by using the definitions of a unitary matrix and multiplication of two  $2 \times 2$  matrices, that

$$U \begin{bmatrix} \sqrt{1+|b|^2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ b \end{bmatrix} \quad (2.13)$$

and

$$U \begin{bmatrix} \sqrt{1+|b|^2} \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \quad (2.14)$$

Set

$$C = VU^{-1}. \quad (2.15)$$

Since all  $2 \times 2$  unitary matrices form a group under the standard multiplication of matrices, by (2.15),  $C$  is a  $2 \times 2$  unitary matrix.

Now, by (2.13), we have

$$U^{-1} \begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} \sqrt{1+|b|^2} \\ 0 \end{bmatrix}. \quad (2.16)$$

Then from (2.5), (2.9), (2.14), (2.15) and (2.16), we have

$$\begin{aligned} C \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix} &= f_1(z) \begin{bmatrix} 1 \\ b \end{bmatrix} \\ &= f_1(z)V \left( U^{-1} \begin{bmatrix} 1 \\ b \end{bmatrix} \right) = f_1(z)V \begin{bmatrix} \sqrt{1+|b|^2} \\ 0 \end{bmatrix} \\ &= f_1(z) \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} g_1(z) \\ g_2(z) \end{bmatrix} \end{aligned}$$

Therefore (2.2) holds. Thus in this case the proof of the theorem is now completed.

**Subcase 2.2** Let  $c_2 = 0$  and  $c_1 \neq 0$ . In this case, by (2.3) we obtain  $f_1 \equiv 0$ .

Hence by (2.1),

$$|f_2(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2, \quad (2.17)$$

By (2.17) and a similar discussion to that of Subcase 1 ( $b$  becomes 0) we obtain the desired result.  $\square$

**Theorem 2.1** Let  $A = \begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix}$  and  $B = \begin{bmatrix} g_1(z) \\ g_2(z) \end{bmatrix}$  be two meromorphic matrix valued functions on  $\mathbb{A}$ . If  $f_k$  and  $g_k$  ( $k = 1, 2$ ) satisfy

$$|f_1(z)|^2 + |f_2(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2, \quad (2.18)$$

on  $\mathbb{A}$ , then

$$\rho_A = \rho_B, \quad (2.19)$$

where  $\rho_A$  and  $\rho_B$  are the orders of  $A$  and  $B$  respectively.

*Proof* By Lemma 2.2, we have  $B = CA$  where  $A$  and  $B$  are as defined in the Theorem 2.1. Therefore

$$T_0(R, B) = T_0(R, CA).$$

Using the basics of Nevanlinna theory on annuli, we can show that

$$T_0(R, B) \leq T_0(R, A)$$

as  $T_0(R, C) = o(T_0(R, f))$ . On further simplification, we get

$$\rho_B \leq \rho_A. \quad (2.20)$$

By integer changing  $f_k$  and  $g_k$  ( $k = 1, 2$ ) in Lemma 2.2, we get

$$A = CB,$$

which implies

$$T_0(R, A) = T_0(R, B),$$

and hence

$$\rho_A \leq \rho_B. \quad (2.21)$$

From (2.20) and (2.21), we get

$$\rho_A = \rho_B. \quad (2.22)$$

Hence the result.  $\square$

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