International J.Math. Combin. Vol.3(2024), 19-32

On Derivative of Eta Quotients of Levels 12 and 16

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Abstract: Z. S. Aygin and P. C. Toh have deduced a technique using the theory of modular forms to determine all eta quotients whose derivative is also an eta quotient up to level 36. This paper aims to find a technique without using the theory of modular forms to deduce all the identities of Aygin and Toh of levels 12 and 16.

Key Words: Eisenstein series, Dedekind eta function, eta quotients.

AMS(2010): 11M36, 11F20.

§1. Introduction

The Dedekind eta function is defined by

$$\eta(\tau) \ = \ q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = e^{2\pi i \tau}$, with $Im(\tau) > 0$. A Dedekind eta function identity is said to be of level n, if it involves Dedekind eta functions $\eta(d_1\tau), \eta(d_2\tau), \ldots, \eta(d_k\tau)$, where the least common multiple of $d_1, d_2, \cdots, d_k$ is n.

Recently Z. S. Aygin and P. C. Toh [2] determined all eta quotients whose derivative is also an eta quotient up to level 36 by employing the theory of modular forms. In fact, they have obtained one hundred of level 12 and four of level 16 of above said type. Further they have conjectured that these are the only identities of level 12 and 16 of this nature. Also they have shown application of these identities to the theory of partitions, integral representation of eta functions and many more. Some of the identities of Aygin and Toh [2] exist before their discovery, see for example [6], [5] and [10]. The purpose of this article is to give an elementary proof for level 12 and 16 eta quotient identities by using the theory developed in [5] and [10].

In Section 2, we provide alternative proof for level 12 identities. In Section 3, we give proof for level 16 identities. We close this section by recalling the definitions, notations and certain existing eta function identities which are required to prove the above said identities.

Let

$$k \ = \ \frac{\eta_2^5 \eta_3^2 \eta_{12}^2}{\eta_1^2 \eta_4^2 \eta_6^5}.$$

[1]Received March 15, 2024, Accepted August 15,2024.

We require following eta function identities:

$$k^2-1 = 4\frac{\eta_2^3\eta_3^3\eta_{12}^6}{\eta_1\eta_4^2\eta_6^9}, \tag{1.1}$$

$$k^2+1 = 2\frac{\eta_2^3\eta_3^6\eta_{12}^3}{\eta_1^2\eta_4\eta_6^9}, \tag{1.2}$$

$$3-k^2 = 2\frac{\eta_1^2\eta_2\eta_3^2\eta_{12}^3}{\eta_4\eta_6^7} \tag{1.3}$$

and

$$3+k^2 = 4\frac{\eta_2\eta_3^3\eta_4^2\eta_{12}^2}{\eta_1\eta_6^7}, \tag{1.4}$$

where $\eta_k = \eta(k\tau)$. The proofs of the above four identities are found in [4], [8], [3]. S. Ramanujan recorded two of the above four theta function identities in the form of modular equations in his notebook [7, p.230]. We denote

$$\eta_n[k_1, k_2, \cdots, k_l] = \eta_1^{k_1}\eta_{d_1}^{k_2}\eta_{d_2}^{k_3}\eta_{d_3}^{k_4}, \cdots, \eta_{d_s}^{k_{l-1}}\eta_n^{k_l},$$

where $d_1, d_2, \cdots, d_s$ are proper divisors of n and $k_1, k_2, \ldots, k_l \in \mathbb{Z}$.

§2. Level 12 Identities

Theorem 2.1 *Let* $k = \dfrac{\eta_2^5\eta_3^2\eta_{12}^2}{\eta_1^2\eta_4^2\eta_6^5}$. *Then we have*

$$q\frac{d}{dq}(\log k) = 2\frac{\eta_1^2\eta_4^2\eta_3^2\eta_{12}^2}{\eta_2^2\eta_6^2}, \tag{2.1}$$

$$\frac{\eta_2^6\eta_3^9\eta_{12}^9}{\eta_1^3\eta_4^3\eta_6^{18}} = \frac{k^4-1}{8}, \tag{2.2}$$

$$\frac{\eta_1^4\eta_4^4\eta_6^4}{\eta_2^4\eta_3^4\eta_{12}^4} = \frac{9-k^4}{k^4-1} \tag{2.3}$$

and

$$\frac{\eta_2^{24}\eta_3^{12}\eta_{12}^{12}}{\eta_1^{12}\eta_4^{12}\eta_6^{24}} = \frac{k^4(k^4-1)}{9-k^4}. \tag{2.4}$$

The parameter k is almost the same as the p defined in [1]. In fact $k = 2p+1$, where $p = \frac{1}{2}\left[\frac{\eta_2^{10}\eta_3^4\eta_{12}^4}{\eta_1^4\eta_4^4\eta_6^{10}} - 1\right]$. The (2.1) is due to Ramanujan and the proof of the same was given by B. C. Berndt [4]. The proof of (2.2) – (2.4) are found in [5].

From the above, one can easily deduce that

$$\frac{\eta_2^{48}}{\eta_1^{24}\eta_4^{24}} = \frac{8^4k^{12}}{(k^4-1)(9-k^4)^3}, \tag{2.5}$$

$$\frac{\eta_6^{48}}{\eta_3^{24}\eta_{12}^{24}} = \frac{8^4k^4}{(k^4-1)^3(9-k^4)}, \tag{2.6}$$

and

$$\frac{\eta_2^6}{\eta_6^6} = \frac{k^2(9-k^4)}{k^4-1}. \tag{2.7}$$

Also, from [9], we have

$$x := \frac{\eta_1^4\eta_6^2}{\eta_2^2\eta_3^4} = \frac{3-k^2}{1+k^2}. \tag{2.8}$$

From [5], we have

$$\frac{\eta_1^{12}\eta_6^{12}}{\eta_2^{12}\eta_3^{12}} = \frac{x^2(1-x^2)}{9-x^2} \tag{2.9}$$

and

$$\frac{\eta_1^9\eta_6^3}{\eta_2^9\eta_3^3} = \frac{8x^2}{9-x^2}. \tag{2.10}$$

From (2.8), (2.9) and (2.10), one can easily deduce that

$$\frac{\eta_1^{24}}{\eta_2^{24}} = \frac{(1+k^2)^2(3-k^2)^6}{k^6(3+k^2)^3(k^2-1)} \tag{2.11}$$

and

$$\frac{\eta_3^{24}}{\eta_6^{24}} = \frac{(3-k^2)^2(1+k^2)^6}{k^2(3+k^2)(k^2-1)^3}. \tag{2.12}$$

From (2.5), (2.6), (2.11) and (2.12), we have

$$\frac{\eta_2^{24}}{\eta_4^{24}} = \frac{8^4k^6(1+k^2)(3-k^2)^3}{(3+k^2)^6(k^2-1)^2} \tag{2.13}$$

and

$$\frac{\eta_6^{24}}{\eta_{12}^{24}} = \frac{8^4k^2(1+k^2)^3(3-k^2)}{(k^2-1)^6(3+k^2)^2}. \tag{2.14}$$

Now, we prove two out of one hundred level-12 identities.

Theorem 2.2 *If* $X = \eta_{12}[10, -36, 18, 8, 0, 0]$, *then*

$$q\frac{d}{dq}(\log(X)) = \eta_{12}[10, -7, -6, 1, 9, -3].$$

Proof By the definition of X, we have

$$X^{12} = \left(\frac{\eta_1^{24}}{\eta_2^{24}}\right)^5 \left(\frac{\eta_6^6}{\eta_2^6}\right)^{36} \left(\frac{\eta_4^{24}}{\eta_2^{24}}\right)^4 \left(\frac{\eta_3^{24}}{\eta_6^{24}}\right)^9. \tag{2.15}$$

Employing (2.11), (2.7), (2.13) and (2.12) in the above, we find that

$$X = \frac{(k^2-1)(1+k^2)^8}{16\,k^{12}(3+k^2)^3}. \tag{2.16}$$

Taking logarithm on both sides and differentiating with respect to q, we obtain

$$q\frac{d}{dq}(\log(X)) = \frac{4(3-k^2)^2}{(1+k^2)(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}. \tag{2.17}$$

Using (1.1), (1.2), (1.3), (1.4) and (2.1) in the right hand side of the above, we find that

$$q\frac{d}{dq}(\log(X)) = \eta_{12}[10,-7,-6,1,9,-3]. \tag{2.18}$$

This completes the proof. □

Theorem 2.3 *If* $Y = \eta_{12}[-18,0,-10,0,36,-8]$, *then*

$$q\frac{d}{dq}(\log(Y)) = 3\eta_{12}[-6,9,10,-3,-7,1].$$

Proof By the definition of Y, we have

$$Y^{12} = \left(\frac{\eta_2^{24}}{\eta_1^{24}}\right)^9 \left(\frac{\eta_6^{24}}{\eta_3^{24}}\right)^5 \left(\frac{\eta_6^{24}}{\eta_{12}^{24}}\right)^4 \left(\frac{\eta_6^{6}}{\eta_2^{6}}\right)^{36}. \tag{2.19}$$

Employing (2.7), (2.11), (2.12) and (2.14) in the above, we find that

$$Y = \frac{16(k^2-1)^3}{(3-k^2)^8(3+k^2)}. \tag{2.20}$$

Taking logarithm on both sides and differentiating with respect to q, we obtain

$$q\frac{d}{dq}(\log(Y)) = \frac{12k^2(1+k^2)^2}{(k^2-1)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}. \tag{2.21}$$

Using (1.1), (1.2), (1.3), (1.4) and (2.1) in the right hand side of the above, we find that

$$q\frac{d}{dq}(\log(Y)) = 3\eta_{12}[-6,9,10,-3,-7,1]. \tag{2.22}$$

This completes the proof. □

We proved the remaining 98 identities of level 12 [2], in the same way. Let

$$f(\tau) = \eta_n(k_1, k_2, \cdots, k_l).$$

We first express $f(\tau)$ in terms of product of powers of $k, k^2 \pm 1, k^2 \pm 3$ and then, we display the q times of logarithmic differentiation of $f(\tau)$ in terms of $k, k^2 \pm 1, k^2 \pm 3$ and $q\frac{dk}{dq}$, and finally we represent

$$q\frac{d}{dq}log(f)$$

in terms of $\eta_n(k_1, k_2, \cdots, k_l)$ in the following Table 1- Table 7.

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
1	$\eta_{12}[4,-18,0,5,0,9]$	$\frac{(k^2-1)^4\sqrt{(k^2+1)}}{2^7k^6(3-k^2)^{3/2}}$	$\frac{4(k^2+3)^2}{(k^4-1)(3-k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$4\eta_{12}[1,-7,-3,10,9,-6]$
2	$\eta_{12}[0,0,-4,-9,18,-5]$	$\frac{2^7(1+k^2)^{3/2}}{\sqrt{(3-k^2)(3+k^2)^4}}$	$\frac{6k^2(k^2-1)^2}{(k^2+1)(3-k^2)(k^2+3)}\,\frac{q}{k}\,\frac{dk}{dq}$	$12\eta_{12}[-3,9,1,-6,-7,10]$
3	$\eta_{12}[-2,4,6,0,-16,8]$	$\frac{(k^2-1)(3+k^2)}{2^4}$	$\frac{4k^2(k^2+1)}{(k^2-1)(3+k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$\eta_{12}[-2,7,6,-3,-5,1]$
4	$\eta_{12}[6,-16,-2,8,4,0]$	$\frac{(k^2-1)(3+k^2)}{2^4k^4}$	$\frac{4(k^2-3)}{(k^2-1)(k^2+3)}\,\frac{q}{k}\,\frac{dk}{dq}$	$\eta_{12}[6,-5,-2,1,7,-3]$
5	$\eta_{12}[-4,8,0,-3,-2,1]$	$\frac{2k^2}{\sqrt{(1+k^2)(3-k^2)}}$	$\frac{4(k^2+3)}{(k^2+1)(3-k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$4\eta_{12}[1,-5,-3,6,7,-2]$
6	$\eta_{12}[0,-2,-4,1,8,-3]$	$\frac{2}{\sqrt{(1+k^2)(3-k^2)}}$	$\frac{4k^2(k^2-1)}{(1+k^2)(3-k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$4\eta_{12}[-3,7,1,-2,-5,6]$
7	$\eta_{12}[2,0,-6,-8,12,0]$	$\frac{(k^2-1)}{(3+k^2)^3}$	$\frac{4k^2(3-k^2)}{(k^2-1)(3+k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$\eta_{12}[2,5,2,-3,-3,1]$
8	$\eta_{12}[6,-12,-2,0,0,8]$	$\frac{(k^2-1)^3}{2^4k^4(3+k^2)}$	$\frac{12(1+k^2)}{(k^2-1)(3+k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$3\eta_{12}[2,-3,2,1,5,-3]$
9	$\eta_{12}[-4,0,0,1,6,-3]$	$2\sqrt{\frac{(1+k^2)}{(3-k^2)^3}}$	$\frac{4k^2(3+k^2)}{(1+k^2)(3-k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$4\eta_{12}[-3,5,1,2,-3,2]$
10	$\eta_{12}[0,-6,4,3,0,-1]$	$\sqrt{\frac{(1+k^2)^3}{2^2k^4(3-k^2)}}$	$\frac{(k^2-1)}{(1+k^2)(3-k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$12\eta_{12}[1,-3,-3,2,5,2]$
11	$\eta_{12}[6,-12,-18,24,0,0]$	$\frac{(k^2-1)(3+k^2)^9}{8^4(1+k^2)^8}$	$\frac{4k^2(3-k^2)^2}{(k^2-1)(3+k^2)(1+k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$\eta_{12}[6,3,-2,-3,-1,1]$
12	$\eta_{12}[-6,0,2,0,-4,8]$	$\sqrt[3]{\frac{(k^2-1)^9(3+k^2)}{k^4(3-k^2)^8}}$	$\frac{12(1+k^2)^2}{(k^2-1)(3+k^2)(3-k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$3\eta_{12}[-2,-1,6,1,3,-3]$
13	$\eta_{12}[-12,6,0,-3,0,9]$	$\sqrt{\frac{(k^2-1)^8}{8^2(1+k^2)(3-k^2)^9}}$	$\frac{2k^2(3+k^2)^2}{(k^2-1)(1+k^2)(3-k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$4\eta_{12}[-3,3,1,6,-1,-2]$
14	$\eta_{12}[0,0,-4,3,2,-1]$	$\sqrt[6]{\frac{k^4(3-k^2)(3+k^2)^8}{8^4(1+k^2)^9}}$	$\frac{36(k^2-1)^2}{(3-k^2)(3+k^2)(1+k^2)}\,\frac{q}{k}\,\frac{dk}{dq}$	$12\eta_{12}[1,-1,-3,-2,3,6]$
15	$\eta_{12}[9,-30,9,12,0,0]$	$\frac{(1+k^2)^4(k^2-1)}{k^9}$	$\frac{(3-k^2)^2}{(1+k^2)(k^2-1)}\,\frac{q}{k}\,\frac{dk}{dq}$	$\eta_{12}[9,-6,-3,3,2,-1]$

Table 1

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
16	$\eta_{12}[-3,0,-3,0,10,-4]$	$\sqrt[3]{\frac{8^2k}{(3-k^2)^4(3+k^2)}}$	$\frac{3(1+k^2)^2}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-3,2,9,-1,-6,3]$
17	$\eta_{12}[12,-30,0,9,0,9]$	$\frac{(1+k^2)(k^2-1)^4}{k^9}$	$\frac{(3+k^2)^2}{(k^2-1)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[3,-6,-1,9,2,-3]$
18	$\eta_{12}[0,0,-4,-3,10,-3]$	$\sqrt[3]{\frac{8^3k}{(3-k^2)(3+k^2)^4}}$	$\frac{3(k^2-1)^2}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$12\eta_{12}[-1,2,3,-3,-6,9]$
19	$\eta_{12}[-8,-2,-8,0,26,-8]$	$\frac{8^4(k^2-1)^3}{(3-k^2)^{12}(3+k^2)^3}$	$\frac{8k^4(1+k^2)}{(k^2-1)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-8,16,8,-6,-8,2]$
20	$\eta_{12}[8,-26,8,8,2,0]$	$\frac{(1+k^2)^4(k^2-1)}{k^8(3+k^2)}$	$\frac{8(3-k^2)}{(1+k^2)(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[8,-8,-8,2,16,-6]$
21	$\eta_{12}[4,-13,0,4,1,4]$	$\sqrt{\frac{(k^2-1)^4(1+k^2)}{k^8(3-k^2)}}$	$\frac{4(3+k^2)}{(k^2-1)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[2,-8,-6,8,16,-8]$
22	$\eta_{12}[0,-1,-4,-4,13,-4]$	$\sqrt{\frac{8^4(1+k^2)^3}{(3-k^2)^3(3+k^2)^{12}}}$	$\frac{4k^4(k^2-1)}{(1+k^2)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-6,16,2,-8,-8,8]$
23	$\eta_{12}[-4,6,12,4,-30,12]$	$\frac{(k^2-1)(3+k^2)^3}{2^8}$	$\frac{8k^4}{(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-4,14,4,-6,-6,2]$
24	$\eta_{12}[-2,-3,-6,2,15,-6]$	$\frac{1}{\sqrt{(1+k^2)(3-k^2)^3}}$	$\frac{4k^4}{(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-6,14,2,-4,-6,4]$
25	$\eta_{12}[12,-30,-4,12,6,4]$	$\frac{(3+k^2)(k^2-1)^3}{2^8k^8}$	$\frac{24}{(3+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[4,-6,-4,2,14,-6]$
26	$\eta_{12}[-6,15,-2,-6,-3,2]$	$\sqrt{\frac{k^8}{(1+k^2)^3(3-k^2)}}$	$\frac{12}{(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[2,-6,-6,4,14,-4]$
27	$\eta_{12}[-1,-3,-9,-5,27,-9]$	$\sqrt{\frac{2^{14}(1+k^2)}{(3-k^2)^3(3+k^2)^6}}$	$\frac{8k^6}{(1+k^2)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-9,23,3,-10,-9,6]$
28	$\eta_{12}[-10,-6,-18,-2,54,-18]$	$\frac{2^{10}(k^2-1)}{(3-k^2)^6(3+k^2)^3}$	$\frac{16k^6}{(k^2-1)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-10,23,6,-9,-9,3]$
29	$\eta_{12}[18,-54,10,18,6,2]$	$\frac{(1+k^2)^6(k^2-1)^3}{k^{16}(3+k^2)}$	$\frac{48}{(1+k^2)(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[6,-9,-10,3,23,-9]$

Table 2

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
30	$\eta_{12}[9,-27,1,9,3,5]$	$\sqrt{\frac{(1+k^2)^3(k^2-1)^6}{2^{14}k^{16}(3-k^2)}}$	$\frac{24}{(1+k^2)(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[3,-9,-9,6,23,-10]$
31	$\eta_{12}[-6,6,18,18,-54,18]$	$\frac{(k^2-1)(3+k^2)^9}{(1+k^2)^2}$	$\frac{16k^6}{(k^2-1)(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-6,21,2,-9,-7,3]$
32	$\eta_{12}[-9,-3,-9,3,27,-9]$	$\sqrt[4]{\frac{8^4(k^2-1)^2}{(1+k^2)(3-k^2)^9}}$	$\frac{8k^6}{(k^2-1)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-9,21,3,-6,-7,2]$
33	$\eta_{12}[-3,9,-3,-3,-1,1]$	$\sqrt[3]{\frac{8^2k^{16}(3+k^2)^2}{(1+k^2)^9(3-k^2)}}$	$\frac{144}{(3+k^2)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[3,-7,-9,2,21,-6]$
34	$\eta_{12}[6,-18,-2,6,2,6]$	$\sqrt[3]{\frac{(3+k^2)(k^2-1)^9}{8^6k^{16}(3-k^2)^2}}$	$\frac{48}{(3+k^2)(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[2,-7,-6,3,21,-9]$
35	$\eta_{12}[-12,-12,-36,-12,108,-36]$	$\frac{8^8(k^2-1)(1+k^2)}{(3+k^2)^9(3-k^2)^9}$	$\frac{32k^8}{(k^2-1)(1+k^2)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-12,30,4,-12,-10,4]$
36	$\eta_{12}[12,-36,4,12,4,4]$	$\sqrt[3]{\frac{(1+k^2)^9(k^2-1)^9}{8^8k^{32}(3+k^2)(3-k^2)}}$	$\frac{(1+k^2)^9(k^2-1)^9}{8^8k^{32}(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[4,-10,-12,4,30,-12]$
37	$\eta_{12}[-1,6,-9,4,0,0]$	$\frac{k^3(3+k^2)^3}{2^2(1+k^2)^4}$	$\frac{(3-k^2)^2}{(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[9,-4,-3,-1,0,3]$
38	$\eta_{12}[-9,0,-1,0,6,4]$	$\frac{(k^2-1)^3}{k(3-k^2)^4}$	$\frac{3(1+k^2)^2}{(3-k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-3,0,9,3,-4,-1]$
39	$\eta_{12}[-4,-6,0,1,0,9]$	$\frac{(k^2-1)^4}{2^5k^3(3-k^2)^3}$	$\frac{(3+k^2)^2}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[-1,-4,3,9,0,-3]$
40	$\eta_{12}[0,0,-4,9,-6,1]$	$\frac{k(3+k^2)^4}{2^5(1+k^2)^3}$	$\frac{3(k^2-1)^2}{(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$12\eta_{12}[3,0,-1,-3,-4,9]$
41	$\eta_{12}[7,-21,3,8,3,0]$	$\frac{(1+k^2)^2(k^2-1)}{2^4k^6}$	$\frac{2(3-k^2)}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[7,-7,-5,4,9,-4]$
42	$\eta_{12}[8,-21,0,7,3,3]$	$\frac{(1+k^2)(k^2-1)^2}{2^5k^6}$	$\frac{2(3+k^2)}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[4,-7,-4,7,9,-5]$
43	$\eta_{12}[-3,-3,-7,0,21,-8]$	$\frac{2^4}{(3-k^2)^2(3+k^2)}$	$\frac{6k^2(1+k^2)}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-5,9,7,-4,-7,4]$

Table 3

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
44	$\eta_{12}[0,-3,-8,-3,21,-7]$	$\frac{2^5}{(3+k^2)^2(3-k^2)}$	$\frac{4k^4}{(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[-4,9,4,-5,-7,7]$
45	$\eta_{12}[-1,4,-1,-4,2,0]$	$\frac{2k}{(3+k^2)}$	$\frac{(3-k^2)}{(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[5,-2,1,-1,-2,3]$
46	$\eta_{12}[1,-2,1,0,-4,4]$	$\frac{(k^2-1)}{k}$	$\frac{(1+k^2)}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[1,-2,5,3,-2,-1]$
47	$\eta_{12}[0,-2,4,1,-4,1]$	$\frac{(1+k^2)}{2^2k}$	$\frac{(k^2-1)}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[3,-2,-1,1,-2,5]$
48	$\eta_{12}[-4,4,0,-1,2,-1]$	$\frac{2^2k}{(3-k^2)}$	$\frac{(3+k^2)}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[-1,-2,3,5,-2,1]$
49	$\eta_{12}[5,-12,-3,4,6,0]$	$\frac{(k^2-1)}{2^2k^3}$	$\frac{(3-k^2)}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[5,-4,1,3,0,-1]$
50	$\eta_{12}[-3,6,5,0,-12,4]$	$\frac{k(3+k^2)}{2^2}$	$\frac{3(1+k^2)}{(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[1,0,5,-1,-4,3]$
51	$\eta_{12}[-4,12,0,-5,-6,3]$	$\frac{2k^3}{(1+k^2)}$	$\frac{(3+k^2)}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$4\eta_{12}[3,-4,-1,5,0,1]$
52	$\eta_{12}[0,-6,-4,3,12,-5]$	$\frac{2^2}{k(3-k^2)}$	$\frac{3(k^2-1)}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$12\eta_{12}[-1,0,3,1,-4,5]$
53	$\eta_{12}[-7,0,-3,1,12,-3]$	$\frac{2(k^2-1)}{(3-k^2)^3}$	$\frac{4k^4}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-7,14,5,-3,-6,1]$
54	$\eta_{12}[-1,0,3,7,-12,3]$	$\frac{(3+k^2)^2}{2^5(1+k^2)}$	$\frac{4k^4}{(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-3,14,1,-7,-6,5]$
55	$\eta_{12}[3,-12,-1,3,0,7]$	$\frac{(k^2-1)^3}{2^5k^4(3-k^2)^2}$	$\frac{12}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[1,-6,-3,5,14,-7]$
56	$\eta_{12}[-3,12,-7,-3,0,1]$	$\frac{2^2k^4(3+k^2)}{(1+k^2)^3}$	$\frac{12}{(3+k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[5,-6,-7,1,14,-3]$
57	$\eta_{12}[-1,3,3,-8,3,0]$	$\frac{2^4(1+k^2)^2}{(3+k^2)^3}$	$\frac{2k^2(3-k^2)}{(1+k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[3,5,-1,-4,-3,4]$

Table 4

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
58	$\eta_{12}[-8,3,0,-1,3,3]$	$\frac{(k^2-1)^2}{2(3-k^2)^3}$	$\frac{2(3+k^2)k^2}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-4,5,4,3,-3,-1]$
59	$\eta_{12}[-3,-3,1,0,-3,8]$	$\frac{(k^2-1)^3}{2^4k^2(3-k^2)}$	$\frac{6(1+k^2)}{(3-k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-1,-3,3,4,5,-4]$
60	$\eta_{12}[0,-3,8,-3,-3,1]$	$\frac{(1+k^2)^3}{k^2(3+k^2)^2}$	$\frac{6(k^2-1)}{(1+k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[4,-3,-4,-1,5,3]$
61	$\eta_{12}[-1,-2,-5,-1,14,-5]$	$\frac{8}{(3-k^2)(3+k^2)}$	$\frac{4k^4}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-7,16,5,-7,-8,5]$
62	$\eta_{12}[5,-14,1,5,2,1]$	$\frac{(1+k^2)(k^2-1)}{8k^4}$	$\frac{4}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[5,-8,-7,5,16,-7]$
63	$\eta_{12}[-1,3,3,-2,-9,6]$	$\frac{(k^2-1)}{4}$	$\frac{2k^2}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-1,5,3,0,-3,0]$
64	$\eta_{12}[-2,3,6,-1,-9,3]$	$\frac{(1+k^2)}{16}$	$\frac{2k^2}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[0,5,0,-1,-3,3]$
65	$\eta_{12}[-3,9,1,-6,-3,2]$	$\frac{4k^2}{(3+k^2)}$	$\frac{6}{(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[3,-3,-1,0,5,0]$
66	$\eta_{12}[-6,9,2,-3,-3,1]$	$\frac{2k^2}{(3-k^2)}$	$\frac{6}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[0,-3,0,3,5,-1]$
67	$\eta_{12}[-3,6,9,-3,-18,9]$	$\frac{(1+k^2)(k^2-1)}{8}$	$\frac{4k^4}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-3,12,1,-3,-4,1]$
68	$\eta_{12}[-3,6,1,-3,-2,1]$	$\sqrt[3]{\frac{8k^4}{(3-k^2)(3+k^2)}}$	$\frac{12}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[1,-4,-3,1,12,-3]$
69	$\eta_{12}[-4,-1,0,0,1,4]$	$\frac{(k^2-1)^2}{2^6k(3-k^2)^2}$	$\frac{(3+k^2)(1+k^2)}{(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-2,-2,6,6,-2,-2]$
70	$\eta_{12}[0,-1,4,-4,1,0]$	$\frac{2^6(1+k^2)^2}{k(3+k^2)^2}$	$\frac{(k^2-1)(3-k^2)}{(1+k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[6,-2,-2,-2,-2,6]$

Table 5

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
71	$\eta_{12}[4,-9,0,2,-3,6]$	$\frac{(k^2-1)^2}{2^4k^3}$	$\frac{(3+k^2)}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[2,-4,2,6,0,-2]$
72	$\eta_{12}[-2,9,-6,-4,3,0]$	$\frac{2^2k^3}{(1+k^2)^2}$	$\frac{(3-k^2)}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[6,-4,-2,2,0,2]$
73	$\eta_{12}[-6,3,-2,0,9,-4]$	$\sqrt{\frac{2^4k}{(3-k^2)^2}}$	$\frac{3(1+k^2)}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[-2,0,6,2,-4,2]$
74	$\eta_{12}[0,-3,4,6,-9,2]$	$\frac{(3+k^2)^2}{2^4k}$	$\frac{3(k^2-1)}{(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[2,0,2,-2,-4,6]$
75	$\eta_{12}[4,-6,-12,8,6,0]$	$\frac{(3+k^2)^3(k^2-1)}{2^4(1+k^2)^4}$	$\frac{8k^2(3-k^2)}{(3+k^2)(k^2-1)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[4,2,-4,-2,6,-2]$
76	$\eta_{12}[-4,3,0,-2,-3,6]$	$\frac{(k^2-1)^4}{2^4(1+k^2)(3-k^2)^3}$	$\frac{4(3+k^2)k^2}{(k^2-1)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-2,2,-2,4,6,-4]$
77	$\eta_{12}[-12,6,4,0,-6,8]$	$\frac{(k^2-1)^3(3+k^2)}{2^8(3-k^2)^4}$	$\frac{24k^2(1+k^2)}{(k^2-1)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-4,6,4,-2,2,-2]$
78	$\eta_{12}[0,-3,-4,6,3,-2]$	$\frac{(3+k^2)^4}{2^4(3-k^2)(1+k^2)^3}$	$\frac{12k^2(k^2-1)}{(3+k^2)(3-k^2)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[-2,6,-2,-4,2,4]$
79	$\eta_{12}[12,-33,0,12,9,0]$	$\frac{(1+k^2)^2(k^2-1)^2}{8^2k^9}$	$\frac{(3-k^2)(3+k^2)}{(1+k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[6,-6,-2,6,2,-2]$
80	$\eta_{12}[0,-3,-4,0,11,-4]$	$\sqrt[3]{\frac{8^2}{k(3-k^2)^2(3+k^2)^2}}$	$\frac{3(k^2-1)(1+k^2)}{(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$6\eta_{12}[-2,2,6,-2,-6,6]$
81	$\eta_{12}[1,0-3,-1,0,3]$	$\frac{(k^2-1)}{2(1+k^2)}$	$\frac{4k^2}{(k^2-1)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[1,2,-3,1,6,-3]$
82	$\eta_{12}[-3,0,1,3,0,-1]$	$\frac{(3+k^2)}{(3-k^2)}$	$\frac{12k^2}{(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-3,6,1,-3,2,1]$
83	$\eta_{12}[-1,1,-1,-1,-1,3]$	$\sqrt{\frac{(k^2-1)^2}{4(1+k^2)(3-k^2)}}$	$\frac{8k^2}{(k^2-1)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-1,1,-5,2,13,-6]$
84	$\eta_{12}[2,-2,-6,2,2,2]$	$\frac{(3+k^2)(k^2-1)}{(1+k^2)^2}$	$\frac{16k^2}{(3+k^2)(k^2-1)(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[2,1,-6,-1,13,-5]$

Table 6

SI.No	eta quotient (f)	k-parameter representation [f(k)]	$q\frac{d}{dq}(\log f(k))$	logarithmic derivative of f
85	$\eta_{12}[-1,-1,-1,3,1,-1]$	$\sqrt{\frac{(3+k^2)^2}{2^2(1+k^2)(3-k^2)}}$	$\frac{4k^4}{(3+k^2)(1+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-5,13,-1,-6,1,2]$
86	$\eta_{12}[-6,2,2,2,-2,2]$	$\frac{(k^2-1)(3+k^2)}{(3-k^2)^2}$	$\frac{16k^4}{(k^2-1)(3+k^2)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-6,13,2,-5,1,-1]$
87	$\eta_{12}[-2,6,6,-10,-6,6]$	$\frac{2^2(1+k^2)^2(k^2-1)}{(3+k^2)^3}$	$\frac{16k^4}{(1+k^2)(k^2-1)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-2,11,-2,-5,3,-1]$
88	$\eta_{12}[-5,3,3,-1,-3,3]$	$\sqrt{\frac{2^4(1+k^2)(k^2-1)^2}{(3-k^2)^3}}$	$\frac{8k^4}{(1+k^2)(k^2-1)(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-5,11,-1,-2,3,-2]$
89	$\eta_{12}[-3,3,5,-3,-3,1]$	$\sqrt{\frac{2^2(1+k^2)^3}{(3-k^2)(3+k^2)^2}}$	$\frac{24k^2}{(1+k^2)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-1,3,-5,-2,11,-2]$
90	$\eta_{12}[-6,6,2,-6,-6,10]$	$\frac{(k^2-1)^3}{(3-k^2)^2(3+k^2)}$	$\frac{48k^2}{(k^2-1)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$3\eta_{12}[-2,3,-2,-1,11,-5]$
91	$\eta_{12}[-4,4,4,-4,-4,4]$	$\frac{(1+k^2)(k^2-1)}{(3-k^2)(3+k^2)}$	$\frac{32k^4}{(1+k^2)(k^2-1)(3-k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-4,10,-4,-4,10,-4]$
92	$\eta_{12}[3,-7,-1,2,1,2]$	$\frac{(k^2-1)}{4k^2}$	$\frac{2}{(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[3,-5,-1,4,7,-4]$
93	$\eta_{12}[-1,1,3,2,-7,2]$	$\frac{(3+k^2)}{4}$	$\frac{k^2}{2(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-1,7,3,-4,-5,4]$
94	$\eta_{12}[-2,-1,-2,1,7,-3]$	$\frac{2}{(3-k^2)}$	$\frac{2k^2}{(3-k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[-4,7,4,-1,-5,3]$
95	$\eta_{12}[-2,7,-2,-3,-1,1]$	$\frac{2k^2}{(1+k^2)}$	$\frac{2}{(1+k^2)}\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[4,-5,-4,3,7,-1]$
96	$\eta_{12}[-3,2,1,-1,-2,3]$	$\frac{(k^2-1)}{2(3-k^2)}$	$\frac{4k^2}{(3-k^2)(k^2-1)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[-3,4,1,1,4,-3]$
97	$\eta_{12}[-1,2,3,-3,-2,1]$	$\frac{2(1+k^2)}{(3+k^2)}$	$\frac{4k^2}{(1+k^2)(3+k^2)}\frac{q}{k}\frac{dk}{dq}$	$\eta_{12}[1,4,-3,-3,4,1]$
98	$\eta_{12}[-2,5,2,-2,-5,2]$	k	$\frac{q}{k}\frac{dk}{dq}$	$2\eta_{12}[2,-2,2,2,-2,2]$

Table 7

§3. Level 16 Identities

Let

$$h = \frac{\eta_2\eta_{16}^2}{\eta_1^2\eta_8}. \tag{3.1}$$

From [10], we have

$$z = q\frac{d}{dq}(\log(h)) = \frac{\eta_2\eta_4^6\eta_8}{\eta_1^2\eta_{16}^2}, \tag{3.2}$$

$$1+2h = \frac{\eta_2\eta_8^5}{\eta_1^2\eta_4^2\eta_{16}^2}, \tag{3.3}$$

$$1+4h = \frac{\eta_2^6}{\eta_1^4\eta_4^2}, \tag{3.4}$$

$$1+6h+8h^2 = \frac{\eta_2^7\eta_8^5}{\eta_1^6\eta_4^4\eta_{16}^2}, \tag{3.5}$$

$$1+4h+8h^2 = \frac{\eta_4^{10}}{\eta_1^4\eta_2^2\eta_8^4}, \tag{3.6}$$

$$\eta_1^{24} = z^6\frac{h}{(1+2h)^5(1+4h)^2(1+4h+8h^2)^5}, \tag{3.7}$$

$$\eta_2^{24} = z^6\frac{h^2(1+4h)^2}{(1+2h)^4(1+4h+8h^2)^4}, \tag{3.8}$$

$$\eta_4^{24} = z^6\frac{h^4}{(1+2h)^2(1+4h)^2(1+4h+8h^2)^2}, \tag{3.9}$$

$$8_8^{24} = z^6\frac{h^8(1+2h)^2}{(1+4h)^4(1+4h+8h^2)^4}, \tag{3.10}$$

and

$$\eta_{16}^{24} = z^6\frac{h^{16}}{(1+2h)^2(1+4h)^5(1+4h+8h^2)^5}. \tag{3.11}$$

Now, we prove one out of four level-16 identities.

Theorem 3.1 *Let* $Z = \eta_{16}[2,-5,2,-1,2]$ *then, prove that*

$$q\frac{d}{dq}(\log(Z)) = \eta_{16}[2,-5,8,1,-2].$$

Proof By the definition of Z, we have

$$Z^{24} = \frac{\eta_1^{48}\eta_4^{48}\eta_{16}^{48}}{\eta_2^{120}\eta_8^{24}}. \tag{3.12}$$

Employing $(3.7), (3.8), (3.9), (3.10)$ and (3.11) in the above, we find that

$$Z = \frac{h}{1+4h}. \tag{3.13}$$

Taking logarithm on both sides and differentiating with respect to q, we obtain

$$q\frac{d}{dq}(\log(Z)) = \frac{8h^2}{1+4h}\frac{q}{h}\frac{dh}{dq}. \tag{3.14}$$

Using (3.1), (3.2) and (3.4) in the right hand side of the above, we find that

$$q\frac{d}{dq}(\log(Z)) = \eta_{12}[10,-7,-6,1,9,-3]. \tag{3.15}$$

This completes the proof. □

We proved the remaining 3 identities of level 16 [10], in the same way. Let $g(\tau) = \eta_n(k_1, k_2, \cdots, k_l)$. We first express $f(\tau)$ in terms of product of powers of $h, 1+2h, 1+4h$, and then we display the q times of logarithmic differentiation of $g(\tau)$ in terms of $h, 1+2h, 1+4h$and $q\frac{dh}{dq}$, and finally we represent $q\frac{d}{dq}log(g)$ in terms of $\eta_n(k_1, k_2, \cdots, k_l)$ in the following Table 8.

SI.No	eta quotient (f)	h-parameter representation [f(h)]	$q\frac{d}{dq}(\log f(h))$	logarithmic derivative of f
1	$\eta_{16}[-2,1,-2,5,-2]$	$1+2h$	$\frac{2h}{1+2h}\frac{q}{h}\frac{dh}{dq}$	$2\eta_{16}[-2,1,8,-5,2]$
2	$\eta_{16}[-2,1,0,-1,2]$	h	$\frac{q}{h}\frac{dh}{dq}$	$\eta_{16}[-2,1,6,1,-2]$
3	$\eta_{16}[-2,5,0,-5,2]$	$\frac{1+4h}{1+2h}$	$\frac{2h}{(1+2h)(1+4h)}\frac{q}{h}\frac{dh}{dq}$	$2\eta_{16}[2,-5,10,-5,2]$

Table 8

Acknowledgement

The authors would like to thank the anonymous referee. The second author is supported by grant No.09/119(0224)/2021-EMR-I (ref. No: 16/06/2019(i)EU-V)by the funding agency CSIR, INDIA, under CSIR-JRF/SRF. The author is grateful to the funding agency.

References

[1] A. Alaca, S. Alaca, K. S. Williams, On the two-dimensional theta functions of the Borweins, *Acta Arithmetica*, 124.2 (2006).

[2] Z. S. Aygin, P. C. Toh, When is the derivative of an eta quotient another eta quotient? *J. Math. Anal. Appl.*, **480(1)** (2019).

[3] N. D. Baruah, R. Barman, Certain theta-function identities and Ramanujan's modular equations of degree 3, *Indian. J. Math.,* 48(1) (2006), 113-133.

[4] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[5] E. N. Bhuvan, On some Eisenstein series identities associated with Borwien's cubic theta functions, *Indian J. Pure Appl. Math.,* 49(4) (2018), 689-703.

[6] S. Cooper, D. Ye, The level 12 analogue of Ramanujan's function k, *J. Aust. Math. Soc.,* 101 (2016), 29-53.

[7] S. Ramanujan, *Notebooks (Volume 2)*, Tata Institute of Fundamental Research, Bombay,

1957.

[8] L. C. Shen, On the modular equations of degreee 3, *Proc. Amer. Math. Soc.,* 122 (1994), 1101-1114.

[9] K. R. Vasuki, T. G. Sreeramamurthy, Some evaluations of Ramanujan's cubic continued fraction, *Indian J. Pure Appl. Math.,* 35(8) (2004), 1003-1025.

[10] D. Ye, Level 16 analogue of Ramanujan's theories of elliptic functions to alternative bases, *J. Number Theory*, 164 (2016), 191-207.