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On Generalized Integral Type $\alpha - \tilde{\mathcal{F}}$ Contraction Mappings in Partial Metric Spaces

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Abstract: This study introduces generalized integral type $\alpha - \tilde{\mathcal{F}}$-contraction mappings in partial metric spaces that combine $\tilde{\mathcal{F}}$-contraction, integral transformations, and α-admissible mappings. It also investigates the existence and uniqueness of fixed points in the context of partial metric spaces. We provide some examples to corroborate our findings.

Key Words: Generalized integral type $\alpha - \tilde{\mathcal{F}}$ contraction mapping, $\tilde{\mathcal{F}}$-contraction mapping, α-admissible mapping, partial metric spaces.

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§1. Introduction

In 2002, Branciari [2] introduced the integral contraction as follows.

Theorem 1.1 *Let* (Ω_s, d) *be a complete metric space,* $k \in (0,1)$ *and let* $\Upsilon : \Omega_s \to \Omega_s$ *be a mapping such that for each* $\gamma_s, \zeta_s \in \Omega_s$

$$\int_0^{d(\Upsilon\gamma_s, \Upsilon\zeta_s)} \xi(t)\, dt \leq k \int_0^{d(\gamma_s, \zeta_s)} \xi(t)\, dt$$

where $\varphi : [0,\infty) \to [0,\infty)$ *is a Lebesgue-integrable map which is summable, (i.e., with finite integral) on each compact subset of* $[0,\infty)$*, nonnegative, and such that for each* $\epsilon > 0$, $\int_0^{\epsilon} \xi(t)\, dt > 0$, *then* Υ *has a unique fixed point.*

For some motivated results on integral type contractions, see [10, 12, 5].

In 2012, Samet et al. [13] introduced $\alpha - \psi$ contractive type mappings and shown some fixed point results for them. Wardowski [15, 16, 17] identified a new sort of contraction mapping called $\tilde{\mathcal{F}}$-contraction and shown that this mapping is a Banach contraction. Wardowski's result has been generalized by many authors (see [9, 1, 4, 14, 6]).

We begin by recalling a few definitions and lemmas. In 1992, Matthews [7] presented the concept of partial metric space (PMS) as follows:

Definition 1.1 *Let* Ω_s *be a non-empty set. A function* $\varrho_{pm} : \Omega_s \times \Omega_s \to [0,\infty)$ *is said to be a*

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partial metric on Ω_s *if the following conditions hold:*

$(PMS1)$ $\gamma_s = \zeta_s \Leftrightarrow \varrho_{pm}(\gamma_s, \gamma_s) = \varrho_{pm}(\zeta_s, \zeta_s) = \varrho_{pm}(\gamma_s, \zeta_s)$;
$(PMS2)$ $\varrho_{pm}(\gamma_s, \gamma_s) \leq \varrho_{pm}(\gamma_s, \zeta_s)$;
$(PMS3)$ $\varrho_{pm}(\gamma_s, \zeta_s) = \varrho_{pm}(\zeta_s, \gamma_s)$;
$(PMS4)$ $\varrho_{pm}(\gamma_s, \zeta_s) \leq \varrho_{pm}(\gamma_s, \iota_s) + \varrho_{pm}(\iota_s, \zeta_s) - \varrho_{pm}(\iota_s, \iota_s)$. *for all* $\gamma_s, \zeta_s, \iota_s \in \Omega_s$.

Lemma 1.2([7]) *Let* (Ω_s, ϱ_{pm}) *be a partial metric space.*

(*a*) *A sequence* $\{\gamma_{s_n}\}$ *in* (Ω_s, ϱ_{pm}) *converges to a point* $\gamma_s \in \Omega_s \Leftrightarrow$

$$\varrho_{pm}(\gamma_s, \gamma_s) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_s).$$

(*b*) *A sequence* $\{\gamma_{s_n}\}$ *in* (Ω_s, ϱ_{pm}) *is a Cauchy sequence if* $\lim_{m,n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m})$ *exists and finite.*

(*c*) (Ω_s, ϱ_{pm}) *is complete if every Cauchy* $\{\gamma_{s_n}\}$ *in* Ω_s *converges to a point* $\gamma_s \in \Omega_s$, *such that*

$$\varrho_{pm}(\gamma_s, \gamma_s) = \lim_{m,n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_s) = \varrho_{pm}(\gamma_s, \gamma_s).$$

Lemma 1.3([7],[8]) *Let* ϱ_{pm} *be a partial metric on* Ω_s, *then the function* $d^{\varrho}_{pm} : \Omega_s \times \Omega_s \to \mathbb{R}^+$ *such that*

$$d^{\varrho}_{pm}(\gamma_s, \zeta_s) = 2\varrho_{pm}(\gamma_s, \zeta_s) - \varrho_{pm}(\gamma_s, \gamma_s) - \varrho_{pm}(\zeta_s, \zeta_s)$$

is metric on Ω_s. *Let* (Ω_s, ϱ_{pm}) *be a partial metric space. Then,*

(1) *A sequence* $\{\gamma_{s_n}\}$ *in* (Ω_s, ϱ_{pm}) *is a Cauchy sequence* $\Leftrightarrow$ $\{\gamma_{s_n}\}$ *is a Cauchy sequence in the metric space* $(\Omega_s, d^{\varrho}_{pm})$.

(2) (Ω_s, ϱ_{pm}) *is complete* $\Leftrightarrow$ $(\Omega_s, d^{\varrho}_{pm})$ *is complete. Moreover,*

$$\lim_{n\to\infty} d^{\varrho}_{pm}(\gamma_{s_n}, \gamma_s) = 0 \Leftrightarrow \varrho_{pm}(\gamma_s, \gamma_s) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_s) = \lim_{n,m\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}).$$

Lemma 1.4([11]) *Assume that* $\gamma_{s_n} \to \iota_s$ *as* $n \to \infty$ *in a partial metric space* (Ω_s, ϱ_{pm}) *such that* $\varrho_{pm}(\iota_s, \iota_s) = 0$ *Then* $\lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \zeta_s) = \varrho_{pm}(\iota_s, \zeta_s)$ *for every* $\zeta_s \in \Omega_s$.

Lemma 1.5([3]) *Let* (Ω_s, ϱ_{pm}) *be a partial metric space.*

(1) *if* $\varrho_{pm}(\gamma_s, \zeta_s) = 0$ *then* $\gamma_s = \zeta_s$.
(2) *If* $\gamma_s \neq \zeta_s$ *then* $\varrho_{pm}(\gamma_s, \zeta_s) > 0$.

Samet et al. [13] introduced α-admissible mapping as follows:

Definition 1.6 *Let* $\Upsilon : \Omega_s \to \Omega_s$ *and* $\alpha : \Omega_s \times \Omega_s \to [0, \infty)$. Υ *is said to* α*-admissible if*

$$\alpha(\gamma_s, \zeta_s) \ \geq \ 1 \ \Rightarrow \ \alpha(\Upsilon\gamma_s, \Upsilon\zeta_s) \geq 1$$

for all $\gamma_s, \zeta_s \in \Omega_s$.

Wardowski [15] presented a new class of contraction mappings as follows:

Definition 1.7 *Let* $\Delta_{\mathfrak{F}}$ *be family of all functions* $\tilde{\mathcal{F}} : \mathbb{R}^+ \to \mathbb{R}$ *satisfying*

$(F1)$ $\tilde{\mathcal{F}}$ *is strictly increasing, i.e. for all* $\omega, \upsilon \in \mathbb{R}^+$ *if* $\omega < \upsilon$ *then* $\tilde{\mathcal{F}}(\omega) < \tilde{\mathcal{F}}(\upsilon)$*;*

$(F2)$ *for each sequence* $\{\omega_n\}$ *of positive numbers,*

$$\lim_{n\to\infty} \omega_n = 0 \;\Leftrightarrow\; \lim_{n\to\infty} \tilde{\mathcal{F}}(\omega_n) = -\infty;$$

$(F3)$ *there exists* $\lambda \in (0,1)$ *such that*

$$\lim_{\omega\to 0^+} \omega^{\lambda} \tilde{\mathcal{F}}(\omega) = 0.$$

Wardowski [15] defined $\tilde{\mathcal{F}}$-contraction as follows:

Definition 1.8 *Let* (Ω_s, d) *be a metric space, then the mapping* $\Upsilon : \Omega_s \to \Omega_s$ *is said to be an* $\tilde{\mathcal{F}}$*-contraction, if there exist* $F \in \Delta_{\mathfrak{F}}$ *and* $\tau > 0$ *such that for all* $\gamma_s, \zeta_s \in \Omega_s$ *with* $d(\Upsilon\gamma_s . \Upsilon\zeta_s) > 0$ *we have*

$$\tau + \tilde{\mathcal{F}}(d(\Upsilon\gamma_s, \Upsilon\zeta_s)) \le \tilde{\mathcal{F}}(d(\gamma_s, \zeta_s)).$$

§2. Main Results

Let Φ be family of all functions $\varphi : [0,\infty) \to [0,\infty)$ such that φ is a Lebesgue-integrable mapping which is summable on each compact subset of $[0,\infty)$, nonnegative and for each $\epsilon > 0$

$$\int_0^{\epsilon} \xi(t)\, dt > 0$$

Definition 2.1 *Let* (Ω_s, ϱ_{pm}) *be partial metric space and let* $\Upsilon : \Omega_s \to \Omega_s$ *be a self map.Then* Υ *is said to be generalized integral type* $\alpha - \tilde{\mathcal{F}}$*-contractive mapping if there exists two functions* $\alpha : \Omega_s \times \Omega_s \to [0,\infty)$ *and* $\tilde{\mathcal{F}} \in \Delta_{\mathfrak{F}}$ *such that for* $\tau > 0$ *with* $\varrho_{pm}(\Upsilon\gamma_s . \Upsilon\zeta_s) > 0$

$$\tau + \tilde{\mathcal{F}}\bigg(\alpha(\gamma_s, \zeta_s) \int_0^{\varrho_{pm}(\Upsilon\gamma_s, \Upsilon\zeta_s)} \xi(t)\, dt\bigg) \le \tilde{\mathcal{F}}\bigg(\int_0^{\Lambda(\gamma_s,\zeta_s)} \xi(t)\, dt\bigg), \tag{2.1}$$

where $\varphi \in \Phi$ *and*

$$\Lambda(\gamma_s, \zeta_s) = \max\{\varrho_{pm}(\gamma_s, \zeta_s), \varrho_{pm}(\gamma_s, \Upsilon\gamma_s), \varrho_{pm}(\zeta_s, \Upsilon\zeta_s)\}$$

Theorem 2.1 *Let* (Ω_s, ϱ_{pm}) *be a complete partial metric space and* $\Upsilon : \Omega_s \to \Omega_s$ *be self mapping. Suppose* $\alpha : \Omega_s \times \Omega_s \to [0,\infty)$ *be the mapping satisfying the conditions:*

(i) Υ *is* α*-admissible mapping;*

(ii) Υ *is generalized integral type* $\alpha - \tilde{\mathcal{F}}$*-contractive mapping;*

(iii) *There exists* $\gamma_{s_0} \in \Omega_s$ *such that* $\alpha(\gamma_{s_0}, \Upsilon\gamma_{s_0}) \ge 1$*;*

(iv) Υ *is continuous,*

then Υ *has a fixed point in* Ω_s.

Proof Let γ_{s_0} be an arbitrary point such that $\alpha(\gamma_{s_0}, \Upsilon\gamma_{s_0}) \geq 1$. Consider a sequence $\{\gamma_{s_n}\}$ in Ω_s such that $\gamma_{s_{n+1}} = \Upsilon\gamma_{s_n}$ for all $n \in \mathbb{N}$.

If $\gamma_{s_n} = \gamma_{s_{n+1}}$ for some $n \in \mathbb{N}$, γ_{s_n} is a fixed point of Υ, completing the existence proof. Assume $\gamma_{s_n} \neq \gamma_{s_{n+1}}$ for every $n \in \mathbb{N}$, Lemma 1.5 states that

$$\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) = \varrho_{pm}(\Upsilon\gamma_{s_{n-1}}, \Upsilon\gamma_{s_n}) > 0.$$

Now, since Υ is α-admissible, so

$$\alpha(\Upsilon\gamma_{s_0}, \Upsilon\gamma_{s_1}) = \alpha(\gamma_{s_1}, \gamma_{s_2}) \geq 1$$
$$\alpha(\Upsilon\gamma_{s_1}, \Upsilon\gamma_{s_2}) = \alpha(\gamma_{s_2}, \gamma_{s_3}) \geq 1$$

and using induction we have $\alpha(\gamma_{s_n}, \gamma_{s_{n+1}}) \geq 1$ for all $n \in \mathbb{N}$.

Now, Using the property (*F1*) we get

$$\begin{aligned}\tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg) &\leq \tau + \tilde{\mathcal{F}}\bigg(\alpha(\gamma_{s_n}, \gamma_{s_{n+1}}) \int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg) \\ &= \tau + \tilde{\mathcal{F}}\bigg(\alpha(\gamma_{s_n}, \gamma_{s_{n+1}}) \int_0^{\varrho_{pm}(\Upsilon\gamma_{s_{n-1}},\Upsilon\gamma_{s_n})} \xi(t)\,dt\bigg) \\ &\leq \tilde{\mathcal{F}}\bigg(\int_0^{\Lambda(\gamma_{s_{n-1}},\gamma_{s_n})} \xi(t)\,dt\bigg) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\Lambda(\gamma_{s_{n-1}}, \gamma_{s_n}) &= \max\{\varrho_{pm}(\gamma_{s_{n-1}},, \gamma_{s_n}), \varrho_{pm}(\gamma_{s_{n-1}}, \Upsilon\gamma_{s_{n-1}}), \varrho_{pm}(\gamma_{s_n}, \Upsilon\gamma_{s_n})\} \\ &= \max\{\varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n}), \varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n}), \varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})\} \\ &= \max\{\varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n}), \varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}})\}. \end{aligned} \tag{2.3}$$

Now, using (2.3) in (2.2) we get that

$$\tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg) \leq \tilde{\mathcal{F}}\bigg(\int_0^{\max\{\varrho_{pm}(\gamma_{s_{n-1}},\gamma_{s_n}),\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})\}} \xi(t)\,dt\bigg). \tag{2.4}$$

Now, if $\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) > \varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n})$, then we get

$$\tau + \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg) \leq \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})} \xi(t)\,dt\bigg),$$

which is a contradiction, Therefore

$$\Lambda(\gamma_{s_{n-1}}, \gamma_{s_n}) = \varrho_{pm}(\gamma_{s_{n-1}}, \gamma_{s_n}). \tag{2.5}$$

Again, Using (2.5) in (2.4) we get

$$\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})}\xi(t)\,dt\bigg) \leq \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_{n-1}},\gamma_{s_n})}\xi(t)\,dt\bigg) - \tau. \tag{2.6}$$

Continuing in the same way, we obtain

$$\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n-1}})}\xi(t)\,dt\bigg) \leq \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_{n-1}},\xi_{p_{n-2}})}\xi(t)\,dt\bigg) - \tau. \tag{2.7}$$

Using (2.7) in (2.6) we get that

$$\begin{aligned}\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})}\xi(t)\,dt\bigg) &\leq \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_{n-1}},\gamma_{s_n})}\xi(t)\,dt\bigg) - \tau\\ &\leq \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_{n-1}},\xi_{p_{n-2}})}\xi(t)\,dt\bigg) - 2\tau\end{aligned}$$

On generalizing

$$\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})}\xi(t)\,dt\bigg) < \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\xi_{p_0},\xi_{p_1})}\xi(t)\,dt\bigg) - n\tau. \tag{2.8}$$

Letting the limit $n \to \infty$ in (**??**) and using the definition of $\tilde{\mathcal{F}}$ we get

$$\lim_{n\to\infty}\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})}\xi(t)\,dt\bigg) = -\infty \Leftrightarrow \lim_{n\to\infty}\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}}) = 0. \tag{2.9}$$

Consequently, we get

$$\lim_{n\to\infty}\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}}) = 0. \tag{2.10}$$

Now, we show that $\{\gamma_{s_n}\}$ is a Cauchy sequence in Ω_s, i.e., we prove that

$$\lim_{n,m\to\infty}\varrho_{pm}(\gamma_{s_n},\gamma_{s_m}) = 0.$$

Put $e_n = \varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})$ for $n \in \mathbb{N}$. Then, from the property (F_3) of $\tilde{\mathcal{F}}$ contraction there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} e_n^k\tilde{\mathcal{F}}(e_n) = 0. \tag{2.11}$$

Following (2.8) for all $n \in \mathbb{N}$ we obtain

$$e_n^k\Bigg(\tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})}\xi(t)\,dt\bigg) - \tilde{\mathcal{F}}\bigg(\int_0^{\varrho_{pm}(\xi_{p_0},\xi_{p_1})}\xi(t)\,dt\bigg)\Bigg) \leq -e_n^k n\tau \leq 0. \tag{2.12}$$

Considering (2.10), (2.11) and letting $n \to \infty$ in (2.12) we get

$$\lim_{n\to\infty}(n(\varrho_{pm}(\gamma_{s_n},\gamma_{s_{n+1}})^k) = 0. \tag{2.13}$$

Since (2.13) holds, there exists $n_p \in \mathbb{N}$ such that $n(\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}))^k \leq 1$ for all $n \geq n_p$ or

$$\varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) \leq \frac{1}{n^{\frac{1}{k}}} \tag{2.14}$$

for all $n \geq n_p$.

Using *PMS4* (triangular inequality) and (2.14) we obtain that for $m > n \geq n_p$,

$$\begin{aligned}\varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) &\leq \varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) + \varrho_{pm}(\gamma_{s_{n+1}}, \gamma_{s_{n+2}}) + \cdots + \varrho_{pm}(\gamma_{s_{m-1}}, \gamma_{s_m}) \\ &\quad - [\varrho_{pm}(\gamma_{s_{n+1}}, \gamma_{s_{n+1}}) + \varrho_{pm}(\gamma_{s_{n+2}}, \gamma_{s_{n+2}}) + \cdots + \varrho_{pm}(\gamma_{s_{m-1}}, \gamma_{s_{m-1}})] \\ &\leq \varrho_{pm}(\gamma_{s_n}, \gamma_{s_{n+1}}) + \varrho_{pm}(\gamma_{s_{n+1}}, \gamma_{s_{n+2}}) + \cdots + \varrho_{pm}(\gamma_{s_{m-1}}, \gamma_{s_m}) \\ &= \sum_{i=n}^{m-1} \varrho_{pm}(\gamma_{s_i}, \gamma_{s_{i+1}}) \leq \sum_{i=n}^{\infty} \varrho_{pm}(\gamma_{s_i}, \gamma_{s_{i+1}}) \leq \sum_{i=n}^{\infty} \frac{1}{n^{\frac{1}{k}}}\end{aligned}$$

Since $k \in (0,1)$, the series $\sum_{i=n}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ is convergent, so

$$\lim_{n,m\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) = 0.$$

This implies that $\{\gamma_{s_n}\}$ is a Cauchy sequence in (Ω_s, ϱ_{pm}). Due to Lemma 1.3, $\{\gamma_{s_n}\}$ is a Cauchy sequence in $(\Omega_s, d^{\varrho}_{pm})$ which is complete. Therefore the sequence $\{\gamma_{s_n}\}$ is convergent in the space $(\Omega_s, d^{\varrho}_{pm})$ as a result there exist $\iota_s \in \Omega_s$ such that $\lim_{n\to\infty} d^{\varrho}_{pm}(\gamma_{s_n}, \iota_s) = 0$. Again from Lemma 1.2, we get

$$\varrho_{pm}(\iota_s, \iota_s) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \iota_s) = \lim_{m,n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) = 0. \tag{2.15}$$

Moreover, As Υ is continuous, we have

$$\iota_s = \lim_{n\to\infty} \gamma_{s_{n+1}} = \lim_{n\to\infty} \Upsilon\gamma_{s_n} = \Upsilon\iota_s$$

This completes the proof. □

Theorem 2.2 *Let (Ω_s, ϱ_{pm}) be a complete partial metric space and $\Upsilon : \Omega_s \to \Omega_s$ be self mapping. Suppose $\alpha : \Omega_s \times \Omega_s \to [0, \infty)$ be the mapping satisfying the conditions:*

(i) Υ is α-admissible mapping;

(ii) Υ is integral type generalized $\alpha - \tilde{\mathcal{F}}$-contractive mapping;

(iii) There exists $\gamma_{s_0} \in \Omega_s$ such that $\alpha(\gamma_{s_0}, \Upsilon\gamma_{s_0}) \geq 1$;

(iv) If $\{\gamma_{s_n}\}$ s a sequence in Ω_s such that $\alpha(\gamma_{s_n}, \gamma_{s_{n+1}}) \geq 1$ for all n and $\gamma_{s_n} \to \iota_s \in \Omega_s$ as $n \to \infty$, then there exists a subsequence $\gamma_{s_{n(i)}}$ of $\{\gamma_{s_n}\}$ such that $\alpha(\gamma_{s_{n(i)}}, \iota_s) \geq 1$ for all i;

(v) $\tilde{\mathcal{F}}$ is continuous,

then Υ has a fixed point in Ω_s.Further if ι_s, ι_t are fixed points of Υ with $\alpha(\iota_s, \iota_t) \geq 1$, then Υ has a unique fixed point in Ω_s.

Proof From the proof of the Theorem 2.1, the sequence $\{\gamma_{s_n}\}$ defined by $\gamma_{s_{n+1}} = \Upsilon\gamma_{s_n}$ is a Cauchy sequence in (Ω_s, ϱ_{pm}). Due to Lemma 1.3, $\{\gamma_{s_n}\}$ is a Cauchy sequence in $(\Omega_s, d^{\varrho}_{pm})$ which is complete. Therefore the sequence $\{\gamma_{s_n}\}$ is convergent in the space $(\Omega_s, d^{\varrho}_{pm})$ as a result there exist $\iota_s \in \Omega_s$ such that $\lim_{n\to\infty} d^{\varrho}_{pm}(\gamma_{s_n}, \iota_s) = 0$. Again from Lemma 1.2, we get

$$\varrho_{pm}(\iota_s, \iota_s) = \lim_{n\to\infty} \varrho_{pm}(\gamma_{s_n}, \iota_s) = \lim_{m,n\to\infty} \varrho_{pm}(\gamma_{s_n}, \gamma_{s_m}) = 0. \tag{2.16}$$

We now prove that Υ has a fixed point.

On contrary we suppose that $(\Upsilon\iota_s, \iota_s) > 0$. Then from condition (iii) there exists a subsequence $\gamma_{s_{n(i)}}$ of $\{\gamma_{s_n}\}$ such that $\alpha(\gamma_{s_{n(i)}}, \iota_s) \geq 1$ for all i. By Using given contractive condition (2.1) for $\gamma_s = \gamma_{s_{n(i)}}$ and $\zeta_s = \iota_s$ and property of $\tilde{\mathcal{F}}$ we have

$$\begin{aligned}
\tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\xi_{p_{n(i)+1}}, \Upsilon\iota_s)} \xi(t)\,dt\right) &= \tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\Upsilon\gamma_{s_{n(i)}}, \Upsilon\iota_s)} \xi(t)\,dt\right) \\
&\leq \tau + \tilde{\mathcal{F}}\left(\alpha(\gamma_{s_{n(i)}}, \iota_s)\int_0^{\varrho_{pm}(\Upsilon\gamma_{s_{n(i)}}, \Upsilon\iota_s)} \xi(t)\,dt\right) \\
&\leq \tilde{\mathcal{F}}\left(\int_0^{\Lambda(\gamma_{s_{n(i)}}, \iota_s)} \xi(t)\,dt\right) \qquad (2.17)
\end{aligned}$$

where

$$\begin{aligned}
\Lambda(\gamma_{s_{n(i)}}, \iota_s) &= \max\{\varrho_{pm}(\gamma_{s_{n(i)}}, \iota_s), \varrho_{pm}(\gamma_{s_{n(i)}}, \Upsilon\gamma_{s_{n(i)}}), \varrho_{pm}(\iota_s, \Upsilon\iota_s)\} \\
&= \max\{\varrho_{pm}(\gamma_{s_{n(i)}}, \iota_s), \varrho_{pm}(\gamma_{s_{n(i)}}, \xi_{p_{n(i)+1}}), \varrho_{pm}(\iota_s, \Upsilon\iota_s)\}. \qquad (2.18)
\end{aligned}$$

Taking $n \to \infty$ in (2.18) and using (2.16) we get that

$$\lim_{n\to\infty} \Lambda(\gamma_{s_{n(i)}}, \iota_s) = \varrho_{pm}(\iota_s, \Upsilon\iota_s). \tag{2.19}$$

Now, Letting $n \to \infty$ in (2.17) and using (2.19) and the continuity of $\tilde{\mathcal{F}}$ we get that

$$\tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\iota_s, \Upsilon\iota_s)} \xi(t)\,dt\right) \leq \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\iota_s, \Upsilon\iota_s)} \xi(t)\,dt\right)$$

which is a contradiction since $\tau > 0$, Thus we have $\Upsilon\iota_s = \iota_s$. This shows that ι_s is a fixed point of Υ. Further, suppose ι_s and ι_t be two fixed point of Υ such that $\varrho_{pm}(\iota_s, \iota_t) > 0$. From (2.1) we have

$$\begin{aligned}
\tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\iota_s, \iota_t)} \xi(t)\,dt\right) &= \tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\Upsilon\iota_s, \Upsilon\iota_t)} \xi(t)\,dt\right) \\
&\leq \tau + \tilde{\mathcal{F}}\left(\alpha(\iota_s, \iota_t)\int_0^{\varrho_{pm}(\Upsilon\iota_s, \Upsilon\iota_t)} \xi(t)\,dt\right) \\
&\leq \tilde{\mathcal{F}}\left(\int_0^{\Lambda(\iota_s, \iota_t)} \xi(t)\,dt\right), \qquad (2.20)
\end{aligned}$$

where

$$\begin{aligned}\Lambda(\iota_s,\iota_t) &= \max\{\varrho_{pm}(\iota_s,\iota_t),\varrho_{pm}(\iota_s,\Upsilon\iota_s),\varrho_{pm}(\iota_t,\Upsilon\iota_t)\}\\ &= \max\{\varrho_{pm}(\iota_s,\iota_t),\varrho_{pm}(\iota_s,\iota_s),\varrho_{pm}(\iota_t,\iota_t)\} = \varrho_{pm}(\iota_s,\iota_t).\end{aligned} \tag{2.21}$$

Putting (2.21) in (2.20) we get

$$\tau + \tilde{\mathcal{F}}\Big(\int_0^{\varrho_{pm}(\iota_s,\iota_t)} \xi(t)\,dt\Big) \le \tilde{\mathcal{F}}\Big(\int_0^{\varrho_{pm}(\iota_s,\iota_t)} \xi(t)\,dt\Big), \tag{2.22}$$

which is a contradiction. Hence Υ has a unique fixed point. $\square$

Following are consequences of the theorems.

Corollary 2.3 *Let* (Ω_s,ϱ_{pm}) *be a complete partial metric space and let* $\Upsilon:\Omega_s\to\Omega_s$ *be a self map. Suppose that there exist* $\tilde{\mathcal{F}}\in\Delta_{\mathfrak{F}}$ *and* $\tau>0$ *with* $\varrho_{pm}(\Upsilon\gamma_s.\Upsilon\zeta_s)>0$ *be such that*

$$\tau + \tilde{\mathcal{F}}\Big(\int_0^{\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)} \xi(t)\,dt\Big) \le \tilde{\mathcal{F}}\Big(\int_0^{\varrho_{pm}(\gamma_s,\zeta_s)} \xi(t)\,dt\Big) \tag{2.23}$$

for all $\gamma_s,\zeta_s\in\Omega_s$ *and* $\varphi:[0,\infty)\to[0,\infty)$ *is a Lebesgue-integrable mapping which is summable on each compact subset of* $[0,\infty)$*, nonnegative and for each* $\epsilon>0$

$$\int_0^{\epsilon}\xi(t)\,dt>0$$

and $\tilde{\mathcal{F}}$ *or* Υ *is continuous. Then* Υ *has a unique fixed point in* Ω_s*.*

Corollary 2.4 *Let* (Ω_s,ϱ_{pm}) *be a complete partial metric space and let* $\Upsilon:\Omega_s\to\Omega_s$ *be a continuous self map. Suppose that there exist* $k\in(0,1)$ *with* $\varrho_{pm}(\Upsilon\gamma_s.\Upsilon\zeta_s)>0$ *such that*

$$\int_0^{\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)} \xi(t)\,dt \le k\int_0^{\varrho_{pm}(\gamma_s,\zeta_s)} \xi(t)\,dt \tag{2.24}$$

and $\varphi:[0,\infty)\to[0,\infty)$ *is a Lebesgue-integrable mapping which is summable on each compact subset of* $[0,\infty)$*, nonnegative and for each* $\epsilon>0$

$$\int_0^{\epsilon}\xi(t)\,dt>0,$$

then Υ *has a unique fixed point in* Ω_s*.*

Example 2.5 Let $\Omega_s=[0,1]$ and define $\varrho_{pm}:\Omega_s\times\Omega_s\to\mathbb{R}^+$ by $\varrho_{pm}(\gamma_s,\zeta_s)=\max\{\gamma_s,\zeta_s\}$. Then (Ω_s,ϱ_{pm}) is a complete partial metric space. Consider the mapping $\Upsilon:\Omega_s\to\Omega_s$ defined by $\Upsilon(\iota_s)=\frac{\iota_s}{4}$. Suppose that $\xi(t)=2t$. Define the function $\tilde{\mathcal{F}}:\mathbb{R}^+\to\mathbb{R}$ by $\tilde{\mathcal{F}}(a)=\ln a$ for all $a\in\mathbb{R}^+>0$ and $\alpha:\Omega_s\times\Omega_s\to[0,\infty)$ by $\alpha(\gamma_s,\zeta_s)=4$ for all $\gamma_s,\zeta_s\in\Omega_s$.

We show that contractive condition of Theorem 2.1 is satisfied. Let $\gamma_s,\zeta_s\in\Omega_s$, without loss of generality we assume that $\gamma_s\ge\zeta_s$. Suppose that $\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)>0$ and let $\tau=\ln(2)$, then

$$\begin{aligned}\tau + \tilde{\mathcal{F}}\left(\alpha(\gamma_s,\zeta_s)\int_0^{\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)}\xi(t)\,dt\right) &= \tau + \tilde{\mathcal{F}}\left(4\int_0^{\varrho_{pm}(\frac{\gamma_s}{4},\frac{\zeta_s}{4})}2t\,dt\right) = \tau + \tilde{\mathcal{F}}(\frac{\gamma_s^2}{4}) \\ &= \ln(2) + \ln(\frac{\gamma_s^2}{4}) = \ln(\frac{\gamma_s^2}{2}) \\ &\leq \ln(\gamma_s^2) = \tilde{\mathcal{F}}(\gamma_s^2) = \tilde{\mathcal{F}}\left(\int_0^{\Lambda(\gamma_s,\zeta_s)}\xi(t)\,dt\right). \qquad (2.25)\end{aligned}$$

Hence, Υ has a fixed point, which in this case is 0.

Example 2.6 Let $\Omega_s = [0,1]$ and define $\varrho_{pm} : \Omega_s \times \Omega_s \to \mathbb{R}^+$ by $\varrho_{pm}(\gamma_s,\zeta_s) = \max\{\gamma_s,\zeta_s\}$. Then (Ω_s,ϱ_{pm}) is a complete partial metric space. Consider the mapping $\Upsilon : \Omega_s \to \Omega_s$ defined by $\Upsilon(\iota_s) = \frac{\iota_s^2+0.045}{12}$. Suppose that $\tau = \ln(1.5)$ and $\xi(t) = 1$ for $t > 0$. Define the function $\tilde{\mathcal{F}} : \mathbb{R}^+ \to \mathbb{R}$ by $\tilde{\mathcal{F}}(a) = \ln(a)$ for all $a \in \mathbb{R}^+ > 0$.

We show that contractive condition of Corollary 2.3 is satisfied. Let $\gamma_s, \zeta_s \in \Omega_s$, without loss of generality we assume that $\gamma_s \geq \zeta_s$. Suppose that $\Upsilon\gamma_s \neq \Upsilon\zeta_s$, then

$$\begin{aligned}\tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\Upsilon\gamma_s,\Upsilon\zeta_s)}\xi(t)\,dt\right) &= \tau + \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}\left(\frac{\gamma_s^2+0.045}{12},\frac{\zeta_s^2+0.045}{12}\right)}dt\right) \\ &= \tau + \tilde{\mathcal{F}}\left(\frac{\gamma_s^2+0.045}{12}\right) \\ &= \ln(1.5) + \ln\left(\frac{\gamma_s^2+0.045}{12}\right) \\ &\leq \ln(\gamma_s) = \tilde{\mathcal{F}}\left(\int_0^{\varrho_{pm}(\gamma_s,\zeta_s)}\xi(t)\,dt\right). \qquad (2.26)\end{aligned}$$

Therefore, it satisfies the condition of Corollary 2.3. Hence Υ has a fixed point, which in this case is 0.003751.

§3. Conclusion

In this article, we prove fixed point theorems for generalized integral type $\alpha - \tilde{\mathcal{F}}$ contraction in complete partial metric spaces and provide corollaries of the results. We also provided some examples to validate the results. This article extends and generalises previous research findings.

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