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On Grundy Coloring of Degree Splitting Graphs

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Abstract: A Grundy n-coloring of a graph G is a proper vertex coloring in which every vertex in $V(G)$ colored with C_n is adjacent with all C_{n-1} colors. The Grundy cloring or Grundy number $\Gamma(G)$ is the maximum number which can also be predicted by greedy coloring strategy by choosing some vertex order to obtain maximum colors. In this paper, we provide some exact values for Grundy coloring of degree splitting graph of wheel graph, helm graph, sunlet graph, crown graph and Friendship graph which are denoted by $[DS(W_n)]$, $[DS(H_n)]$, $[DS(S_n)]$, $[DS(H_{n,n})]$ and $\Gamma[DS(F_n)]$ respectively.

Key Words: Proper coloring, Grundy coloring, Smarandachely Grundy coloring, greedy algorithm, degree splitting graph.

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§1. Introduction

Throughout this, we consider only a simple, finite, undirected & connected graph. Graph coloring is the allocation of colors to the vertices of a graph G. A proper k-coloring is defined by the mapping $\sigma : V(G) \to C_s$ where $\sigma(f) \neq \sigma(g)$ for $\forall f \sim g$, $(f, g) \in V(G)$ [1, 10]. The Grundy k-coloring is a proper k-coloring in which $f \sim C_1$, $f \sim C_2$, $f \sim C_3$, $\cdots$, $f \sim C_{s-1}$ for $\forall$ $\sigma(f) = C_s$. This Grundy number $\Gamma(G)$ was initially studied by P.M.Grundy for directed version in 1939 but the undirected version was introduced by Christen and Selkow in 1979 [1, 2, 5]. This can also be predicted by using greedy algorithm which consider the vertices in some sequence and assign them its first available color. We know that, $\mu(G) \leq \chi(G) \leq \Gamma(G) \leq \triangle(G) + 1$ where $\mu(G)$ is the clique number [3].

§2. Preliminaries

A Grundy n-coloring of G is an n-coloring of G such that $\forall$ color C_t, every node colored with C_t is adjacent to at least one node colored with C_s, $\forall$ $C_s < C_t$ and the Grundy chromatic number $\Gamma(G)$ is the maximum number n such that G is Grundy n-coloring [3]. Generally, if $G \setminus H$ is Grundy n-colourable for a typical subgraph $H \prec G$ such as a path P_s, cycle C_s or $K_{1,s}$ for an integer $s \geq 1$, then G is said to be Smarandachely Grundy n-colourable on H. Clearly, such a

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Smarandachely Grundy n-colouring is nothing else but a Grundy n-colouring if $H = \emptyset$.

A graph with $V(G) = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of all vertices of same degree with at least two elements and $T = V(G) \backslash \{S_1 \cup S_2 \cup \cdots \cup S_t\}$. The degree splitting graph $DS(G)$, is obtained from G by adding vertices $w_1, w_2, \cdots, w_t$ and joining w_i to each vertex of S_i for $1 \leq i \leq t$ [8, 10].

For any integer $n \geq 4$, the wheel graph W_n is the n-vertex graph obtained by joining a vertex v_1 to each of the n-1 vertices $w_1, w_2, \cdots, w_{n-1}$ of the cycle graph C_{n-1} [11].

A helm graph H_n is a graph formed from a wheel W_n by attaching a pendant edge to each terminal vertex [7].

An n-sunlet graph on $2n$ vertices is obtained by attaching n-pendant edges to the cycle C_n and is denoted by S_n [11].

A crown graph(also known as a cocktail party graph) $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching [6] and the friendship graph F_n is the n-collection of cycle C_3 with a common vertex.

§3. Main Results

Here, we concentrate on exact values of Grundy Coloring of Degree Splitting graph of wheel graphs, helm graphs, sunlet graphs, crown graphs and friendship graphs which are symbolised by $[DS(W_n)]$, $[DS(H_n)]$, $[DS(S_n)]$, $[DS(H_{n,n})]$ and $\Gamma[DS(F_n)]$ respectively.

Theorem 3.1 *For $n \geq 4$, the grundy coloring for degree splitting graph of wheel graph W_n is given by*

$$\Gamma[DS(W_n)] = \begin{cases} n+1, & n = 4, \\ n-2, & n = 5, \\ 4, & n \geq 6. \end{cases}$$

Proof Consider a wheel graph W_n with vertex set

$$V(W_n) = \bigcup_{i=1}^{n} v_i$$

where v_1 is the hub vertex and edge set

$$E(W_n) = \{v_1 v_i : i \in (1,n]\} \cup \{v_i v_{i+1} : i \in (1,n)\} \bigcup \{v_2 v_n\}$$

such that $\mid V(W_n) \mid = n$ and $\mid E(W_n) \mid = 2n-2$. Moreover, $\triangle(W_n) = n-1$ and $\delta(W_n) = 3$. We have $T = \{v_i : i \in [1,n]\}$ for $n = 4$ otherwise $T = \{v_i : i \in (1,n]\}$.

Thus, by the construction of degree splitting graph, we introduce a new vertex w corresponding to the vertex set T and hve $V[DS(W_n)] = \{v_i : i \in [1,n]\} \cup \{w\}$ and $E[DS(W_n)] = \{v_2 v_n\} \cup \{v_1 v_i : i \in (1,n]\} \cup \{v_i v_{i+1} : i \in (1,n)\} \cup \{w v_i : i \in [1,n]\}$ for $n = 4$ otherwise $E[DS(W_n)] = \{v_2 v_n\} \cup \{v_1 v_i : i \in (1,n]\} \cup \{v_i v_{i+1} : i \in (1,n)\} \cup \{w v_i : i \in (1,n]\}$ where

$\mid V[DS(W_n)] \mid= n+1$ and

$$\mid E[DS(W_n)] \mid= \begin{cases} 3n-2, & n=4, \\ 3n-3, & n \neq 4 \end{cases}$$

provided $\delta[DS(W_n)] = 4$ and

$$\triangle[DS(W_n)] = \begin{cases} n, & n=4, \\ n-1, & n \neq 4. \end{cases}$$

Consider the colors $C_1, C_2, C_3, \cdots$ and assign the colors as follows.

Case 1. $n = 4$

In this case, assign the colors by using the mapping $\pi : V[DS(W_n)] \to \{C_k : 1 \leq k \leq 5\}$ such that

$$\bullet\ \pi(w) = C_5,$$

$$\bullet\ \pi(v_i) = \begin{cases} C_4, & i=1, \\ C_3, & i=2, \\ C_2, & i=3, \\ C_1, & i=4. \end{cases}$$

Thus, $\Gamma[DS(W_n)] = 5$ for $n = 4$ where $\Gamma[DS(W_n)] > 5$ is not possible since $\Gamma \leq \triangle + 1$ and suppose $\Gamma[DS(W_n)] < 5$, eventhough it satisfies the definition of Grundy coloring it is not maximum. Hence, $\Gamma[DS(W_n)] = n + 1$ for $n = 4$.

Case 2. $n = 5$

In this case, consider the mapping $\phi : V[DS(W_n)] \to \{C_k : 1 \leq k \leq 3\}$ and assign the colors as follows.

$$\bullet\ \phi(w) = C_3,$$

$$\bullet\ \phi(v_i) = \begin{cases} C_3, & i=1, \\ C_2, & i \equiv 0(\text{mod})2, \\ C_1, & i \equiv 1(\text{mod})2. \end{cases}$$

Thus, $\Gamma[DS(W_n)] = 3$ for $n = 5$.

Suppose $\Gamma[DS(W_n)] > 3$, then it makes the vertex v_3 colored with C_2 not adjacent with C_1 which contradicts grundy coloring for the mapping $\phi(w) = \phi(v_1) = C_4$,

$$\phi(v_i) = \begin{cases} C_3, & i \equiv 0(\text{mod})2, \\ C_2, & i=3, \\ C_1, & i=5 \end{cases}$$

and suppose $\Gamma[DS(W_n)] < 3$, it contradicts the definition of proper coloring. Therefore, $\Gamma[DS(W_n)] = n - 2$ for $n = 5$.

Case 3. $n \geq 6$

Let us consider the mapping $\psi : V[DS(W_n)] \rightarrow \{C_k : 1 \leq k \leq 4\}$ and assign the colors as follows.

Subcase 3.1 $n \equiv 0(\text{mod})2$

$$\begin{aligned} \psi(w) &= \psi(v_1) = C_4, \\ \psi(v_2) &= C_3, \\ \psi(v_i) &= \begin{cases} C_2, & i \equiv 1(\text{mod})2, \\ C_1, & i \equiv 0(\text{mod})2. \end{cases} \end{aligned}$$

Subcase 3.2 $n \equiv 1(\text{mod})2$,

$$\begin{aligned} \psi(w) &= \psi(v_1) = C_4, \\ \psi(v_2) &= \psi(v_{n-1}) = C_3, \\ \psi(v_i) &= \begin{cases} C_2, & i \equiv 1(\text{mod})2, \\ C_1, & i \equiv 0(\text{mod})2 \ \text{ and } \ i = n. \end{cases} \end{aligned}$$

Thus, from the above subcases, $\Gamma[DS(W_n)] = 4$ for $n \geq 6$.

Suppose $\Gamma[DS(W_n)] > 4$, then it makes some vertex colored with C_k not adjacent with all C_{k-1} colors. For instance, $\Gamma[DS(W_6)] = 5$ in which the vertex v_2 colored with C_4 and v_3 colored with C_3 are not adjacent with the color C_1 for the mapping $\psi(w) = \psi(v_1) = C_5$, $\psi(v_2) = C_4$, $\psi(v_3) = C_3$, $\psi(v_4) = \psi(v_6) = C_2$ and $\psi(v_5) = C_1$. This contradicts Grundy coloring. Similarly $4 < \Gamma[DS(W_n)] \leq n$ leads to contradiction of grundy coloring. And suppose $\Gamma[DS(W_n)] < 4$, eventhough it satisfies grundy coloring it is not maximum. Therefore, $\Gamma[DS(W_n)] = 4$ for $n \geq 6$.

Thus, from all above cases, we have

$$\Gamma[DS(W_n)] = \begin{cases} n+1, & n = 4, \\ n-2, & n = 5, \\ 4, & n \geq 6 \end{cases} \qquad \square$$

Theorem 3.2 *For $n \geq 3$, the Grundy coloring for degree splitting graph of sunlet graph S_n is given by*

$$\Gamma[DS(S_n)] = \begin{cases} 4, & n = 4, \\ 5, & n \neq 4 \end{cases}$$

Proof Consider a sunlet graph S_n with vertex set $V(S_n) = \{u_i : i \in [1, n]\} \cup \{v_j : j \in [1, n]\}$

and edge set $E(S_n) = \{u_iu_{i+1} : i \in [1,n)\} \cup \{u_1u_n\} \cup \{u_iv_j : i,j \in [1,n] \text{ and } i = j\}$ such that $| V(S_n) |=| E(S_n) |= 2n$. Moreover, $\triangle(S_n) = 3$ and $\delta(S_n) = 1$. Hence, we have $T_1 = \{u_i : i \in [1,n]\}$ and $T_2 = \{v_j : j \in [1,n]\}$.

Thus, by the construction of degree splitting graph, we introduce new vertices $\{w_1, w_2\}$ corresponding to vertex set T_1 and T_2 and therefore, $V[DS(S_n)] = \{u_i : i \in [1,n]\} \cup \{v_j : j \in [1,n]\} \cup \{w_k : k \in [1,2]\}$ and $E[DS(S_n)] = \{u_iu_{i+1} : i \in [1,n)\} \cup \{u_1u_n\} \cup \{u_iv_j : i,j \in [1,n] \text{ and } i = j\} \cup \{u_iw_1 : i \in [1,n]\} \cup \{v_jw_2 : j \in [1,n]\}$ where $| V[DS(S_n)] |= 2n+2$ and $| E[DS(S_n)] |= 4n$ provided $\delta[DS(S_n)] = 2$ and

$$\triangle[DS(S_n)] = \begin{cases} n+1, & n = 3, \\ n, & n \neq 3. \end{cases}$$

Consider the colors $C_1, C_2, C_3, \cdots$ and assign the colors as follows.

Case 1. $n = 4$

Assign the colors by using the mapping $\rho : V[DS(S_n)] \to \{C_t : 1 \leq t \leq 4\}$ such that

- $\rho(w_k) = \begin{cases} C_4, & k = 1, \\ C_3, & k = 2; \end{cases}$
- $\rho(u_i) = C_3$ for $i \equiv 1(\text{mod})2$;
- for $i \equiv 0(\text{mod})2$, $\rho(u_i) = \begin{cases} C_2, & i = 2, \\ C_1, & i = 4; \end{cases}$;
- for $1 \leq i \leq n$, $\rho(v_j) = \begin{cases} C_2, & j = n, \\ C_1, & 1 \leq j \leq n-1 \end{cases}$.

Thus, $\Gamma[DS(S_n)] = 4$ for $n = 4$.

Suppose $\Gamma[DS(S_n)] > 4$ then it makes the vertex v_4 colored with C_2 not adjacent with C_1 which contradicts grundy coloring for the mapping $\rho(u_i) = C_i$, $\rho(v_1) = \rho(v_n) = C_2$, $\rho(v_2) = \rho(v_3) = C_1$, $\rho(w_1) = C_5$ and $\rho(w_2) = C_3$ and suppose $\Gamma[DS(S_n)] < 4$, Even through it satisfies the definition of Grundy coloring it is not maximum. Thus, $\Gamma[DS(S_n)] = 4$ for $n = 4$.

Case 2. $n \neq 4$

Let us consider the mapping $\lambda : V[DS(S_n)] \to \{C_t : 1 \leq t \leq 5\}$ and assign the colors as follows.

Subcase 2.1 $n = 3$

- $\lambda(u_i) = C_{i+2}, \forall 1 \leq i \leq n$;
- $\lambda(v_j) = C_2, \forall 1 \leq j \leq n$;
- $\lambda(w_k) = C_1, \forall k \in \left[1, \left\lceil \frac{n}{2} \right\rceil\right]$.

Subcase 2.2 $n \geq 5$

- $\lambda(w_1) = C_5$ and $\lambda(w_2) = C_3$;
- $\lambda(u_i) = C_{i+2}, \forall i \in [1, 2]$;
- for odd n, $\lambda(u_i) = \begin{cases} C_2, & i \equiv 1(\text{mod})2, \\ C_1, & i \equiv 0(\text{mod})2; \end{cases}$

$$\lambda(v_j) = \begin{cases} C_2, & j \equiv 0(\text{mod})2, \\ C_1, & j = 2 \text{ and } j \equiv 1(\text{mod})2; \end{cases}$$

- for even n, $\lambda(u_i) = \begin{cases} C_3, & i = n-1, \\ C_2, & i \equiv 1(\text{mod})2 \text{ and } i = n, \\ C_1, & i \equiv 0(\text{mod})2; \end{cases}$

$$\lambda(v_j) = \begin{cases} C_2, & j \equiv 0(\text{mod})2 \text{ and } 4 \leq j < n, \\ C_1, & j = 2, n \text{ and } j \equiv 1(\text{mod})2. \end{cases}$$

Thus, from all above subcases, $\Gamma[DS(S_n)] = 5$ for $n \neq 4$.

Suppose $\Gamma[DS(S_n)] > 5$, it is not possible for $n = 3$ since $\Gamma \leq \triangle + 1$ whereas for $n \geq 5$, it makes some vertex colored with C_t not adjacent with all C_{t-1} colors. For instance, $\Gamma[DS(S_5)] = 6$ in which the vertex w_1 colored with C_6 is not adjacent with the color C_5 for the mapping $\lambda(w_1) = C_6$, $\lambda(w_2) = C_3$,

$$\lambda(v_j) = \begin{cases} C_2, & j = 1, \\ C_1, & 2 \leq j \leq 5 \end{cases} \quad \text{and} \quad \lambda(u_i) = \begin{cases} C_i, & 1 \leq i \leq 4, \\ C_2, & i = 5. \end{cases}$$

This contradicts Grundy coloring. Similarly $7 \leq \Gamma[DS(S_n)] \leq \triangle[DS(S_n)] + 1$ for $n \geq 6$ leads to contradiction and suppose $\Gamma[DS(S_n)] < 5$. Even though it satisfies Grundy coloring it is not maximum. We get $\Gamma[DS(S_n)] = 5$ for $n \neq 4$.

Thus, from all above cases, we have

$$\Gamma[DS(S_n)] = \begin{cases} 4, & n = 4, \\ 5, & n \neq 4. \end{cases}$$

This completes the proof. □

Theorem 3.3 *For $n \geq 3$, the grundy coloring for degree splitting graph of helm graph H_n is given by*

$$\Gamma[DS(H_n)] = 5.$$

Proof Consider a helm graph H_n with vertex set $V(H_n) = \{v_i : i \in [0, n]\} \cup \{u_j : j \in [1, n]\}$

and edge set $E(H_n) = \{v_0v_i : i \in [1,n]\} \cup \{v_1v_n\} \cup \{v_iv_{i+1} : i \in [1,n)\} \cup \{v_iu_j : i,j \in [1,n] and i = j\}$ such that $| V(H_n) |= 2n+1$ and $| E(H_n) |= 3n$. Moreover,

$$\triangle(H_n) = \begin{cases} 4, & n = 3,4, \\ n, & n \geq 5 \end{cases} \quad \text{and} \quad \delta(H_n) = d(u_j : j \in [1,n]) = 1.$$

Hence, we have

$$T_1 = \{v_i : i \in [0,n]\} \text{ and } T_2 = \{u_j : j \in [1,n]\}.$$

Thus, by the construction of degree splitting graph, we introduce a new set of vertices $\{w_1, w_2\}$ corresponding to vertex set T_1 and T_2. Consequently, $V[DS(H_n)] = \{v_i : i \in [0,n]\} \cup \{u_j : j \in [1,n]\} \cup \{w_k : k \in [1,2]\}$ and $E[DS(H_n)] = \{v_0v_i : i \in [1,n]\} \cup \{v_1v_n\} \cup \{v_iv_{i+1} : i \in [1,n)\} \cup \{v_iu_j : i,j \in [1,n]$ and $i = j\} \cup \{v_iw_1 : i \in [0,n]\} \cup \{u_jw_2 : j \in [1,n]\}$ for $n = 4$. Otherwise, $E[DS(H_n)] = \{v_0v_i : i \in [1,n]\} \cup \{v_1v_n\} \cup \{v_iv_{i+1} : i \in [1,n)\} \cup \{v_iu_j : i,j \in [1,n]$ and $i = j\} \cup \{v_iw_1 : i \in [1,n]\} \cup \{u_jw_2 : j \in [1,n]\}$ where $| V[DS(H_n)] |= 2n+3$ and

$$| E[DS(H_n)] |= \begin{cases} 5n+1, & n = 4, \\ 5n, & \text{Otherwise} \end{cases}$$

provided

$$\triangle[DS(H_n)] = \begin{cases} 5, & n = 3,4, \\ n, & n \geq 5 \end{cases}$$

and $\delta[DS(H_n)] = 2$.

Consider the colors $C_1, C_2, C_3, \cdots$ and assign the colors by using the mapping $\eta : V[DS(H_n)] \to \{C_t : 1 \leq t \leq 5\}$.

Case 1. $n = 3$

- $\eta(v_0) = \eta(w_k : k \in [1,2]) = C_1;$
- $\eta(u_j) = C_2, \forall j \in [1,n];$
- $\eta(v_i) = C_{i+2}, \forall i \in [1,n].$

Case 2. $n = 4$

- $\eta(w_1) = C_5$ and $\eta(w_2) = \eta(v_0) = C_1;$
- $\eta(u_j) = \begin{cases} C_3, & j = 2, \\ C_2, & \text{Otherwise, for } \forall j \in [1,n]; \end{cases}$
- $\eta(v_i) = \begin{cases} C_3, & i = 1, \\ C_i, & i \geq 2 \text{ for } \forall i \in [1,n]. \end{cases}$

Case 3. $n \geq 5$

- $\eta(v_0) = \eta(w_1) = C_5$ and $\eta(w_2) = C_3$;
- $\eta(u_j) = \begin{cases} C_2, & j = 1, \\ C_1, & j \geq 2 \ \forall j \in [1, n]; \end{cases}$
- $\eta(v_i) = \begin{cases} C_i, & 1 \leq i \leq 4, \\ C_2, & i \equiv 1(\text{mod})2, \\ C_3, & i \equiv 0(\text{mod})2 \text{ for } \forall i \in [1, n]. \end{cases}$

Thus, from all above cases, $\Gamma[DS(H_n)] = 5$.

Suppose $\Gamma[DS(H_n)] > 5$, then it makes some vertex v_i colored with C_t is not adjacent with all C_{t-1} colors. For instance, $\Gamma[DS(H_3)] = 6$ in which the vertex v_0 colored with C_3 is not adjacent with C_2 and C_1 for the mapping $\eta(w_k : k \in [1, 2]) = C_1$, $\eta(v_i : i \in [0, 3]) = C_{i+3}$ and $\eta(u_j : j \in [1, 3]) = C_2$. This leads to the contradiction of Grundy coloring. Similarly, $7 \leq \Gamma[DS(H_n)] \leq n + 1$ for $n \geq 5$ leads to contradiction and suppose $\Gamma[DS(H_n)] < 5$. Even though it satisfies the definition of Grundy coloring it is not maximum, i.e., $\Gamma[DS(H_n)] = 5$ for $n \geq 3$. □

Theorem 3.4 *For $n \geq 2$, the Grundy coloring for degree splitting graph of crown graph $H_{n,n}$ is given by*

$$\Gamma[DS(H_{n,n})] = n + 1.$$

Proof Consider a crown graph $H_{n,n}$ with vertex set $V(H_{n,n}) = \{u_i : i \in [1, n]\} \cup \{v_j : j \in [1, n]\}$ and edge set $E(H_{n,n}) = \{u_i v_j : i, j \in [1, n] \text{ and } i \neq j\}$ such that $| V(H_{n,n}) |= 2n$ and $| E(H_{n,n}) |= n(n - 1)$. Moreover, $\triangle(H_{n,n}) = \delta(H_{n,n}) = n - 1$, i.e., we have $T = u_i \cup v_j$ where $i, j \in [1, n]$.

Thus, by the construction of degree splitting graph, we introduce a new vertex w corresponding to the vertex set T, and so $V[DS(H_{n,n})] = \{u_i : i \in [1, n]\} \cup \{v_j : j \in [1, n]\} \cup \{w\}$ and $E[DS(H_{n,n})] = \{u_i v_j : i, j \in [1, n] \text{ and } i \neq j\} \cup \{u_i w : i \in [1, n]\} \cup \{v_j w : j \in [1, n]\}$ where $| V[DS(H_{n,n})] |= 2n + 1$ and $| E[DS(H_{n,n})] |= n(n + 1)$ provided $\triangle[DS(H_{n,n})] = 2n$ and $\delta[DS(H_{n,n})] = n$.

Consider the colors $C_1, C_2, \cdots$ and assign colors by using the mapping $\sigma : V[DS(H_{n,n})] \to \{C_k : k = 1, 2, 3, \cdots\}$ as follows:

- $\sigma(w) = C_{n+1}$;
- $\sigma(u_i) = \sigma(v_j) = C_i, \quad \forall i, j \in [1, n]$.

Thus, $\Gamma[DS(H_{n,n})] = n + 1$.

Suppose $\Gamma[DS(H_{n,n})] > n + 1$, then some vertex u_i or v_j colored with C_k is not adjacent with all C_{k-1} colors. For instance, $\Gamma[DS(H_{n,n})] = n+2$ for $n = 2$ in which the vertex u_2 colored with C_3 and the vertex v_1 colored with C_2 are not adjacent with C_1 for the mapping $\sigma(w) = C_4$,

$\sigma(u_1) = \sigma(v_1) = C_2$, $\sigma(u_n) = C_{n+1}$ and $\sigma(v_n) = C_{n-1}$. This leads to the contradiction of Grundy coloring. Similarly, $n+3 \leq \Gamma[DS(H_{n,n})] \leq 2n+1$ leads to contradiction. And suppose $\Gamma[DS(H_{n,n})] < n+1$, then it contradicts the definition of proper coloring, i.e., $\Gamma[DS(H_{n,n})] = n+1$ for $n \geq 2$. □

Theorem 3.5 *For $n \geq 1$, the Grundy coloring for degree splitting graph of friendship graph F_n is given by*

$$\Gamma[DS(F_n)] = \begin{cases} 4, & n = 1, \\ 3, & n \neq 1 \end{cases}$$

Proof Consider a friendship graph F_n with vertex set $V(F_n) = \bigcup_{i=0}^{2n}\{v_i\}$ and edge set $E(F_n) = \{v_0v_i : i \in [1, 2n]\} \cup \{v_iv_{i+1} : i \equiv 1(\text{mod})2\}$ such that $\mid V(F_n) \mid= 2n+1$ and $\mid E(F_n) \mid= 3n$. Moreover,

$$\triangle(F_n) = \begin{cases} 2, & n = 1, \\ 2n, & n \neq 1 \end{cases}$$

and $\delta(F_n) = 2$. We have $T = \bigcup_{i=0}^{2n}\{v_i\}$ for $n = 1$ otherwise $T = \bigcup_{i=1}^{2n}\{v_i\}$ for $n \geq 2$.

Thus, by the construction of degree splitting graph, we introduce a new vertex w corresponding to vertex set T, i.e., $V[DS(F_n)] = \{v_i : i \in [0, 2n]\} \cup \{w\}$ and $E[DS(F_n)] = \{v_0v_i : i \in [1, 2n]\} \cup \{v_iv_{i+1} : i \equiv 1(\text{mod})2\} \cup \{v_iw : i \in [0, 2n]\}$ for $n = 1$ otherwise $E[DS(F_n)] = \{v_0v_i : i \in [1, 2n]\} \cup \{v_iv_{i+1} : i \equiv 1(\text{mod})2\} \cup \{v_iw : i \in (0, 2n]\}$ for $n \geq 2$ where

$$\mid V[DS(F_n)] \mid= 2(n+1) \quad \text{and} \quad \mid E[DS(F_n)] \mid= \begin{cases} 3(n+1), & n = 1, \\ 5n, & n \neq 1 \end{cases}$$

provided

$$\triangle[DS(F_n)] = \begin{cases} 3, & n = 1, \\ 2n, & n \neq 1 \end{cases}$$

and $\delta[DS(F_n)] = 3$.

Consider the colors $C_1, C_2, C_3, \cdots$ and assign the colors as follows.

Case 1. $n = 1$

Let us consider the mapping $\zeta : V[DS(F_n)] \to \{C_s : 1 \leq s \leq 4\}$ such that

- $\zeta(w) = C_4$;
- $\zeta(v_i) = C_{i+1}$ for $0 \leq i \leq 2n$.

Obviously, $\Gamma[DS(F_n)] = 4$ for $n = 1$.

Case 2. $n \geq 1$

Assume the mapping $\tau : V[DS(F_n)] \to \{C_t : 1 \le t \le 3\}$ and assign the colors as follows.

- $\tau(w) = \tau(v_0) = C_3$;
- $\tau(v_i) = \begin{cases} C_2, & i \equiv 0(\text{mod})2, \\ C_1, & i \equiv 1(\text{mod})2 \text{ for } 1 \le i \le 2n. \end{cases}$

Thus, $\Gamma[DS(F_n)] = 3$. Suppose $\Gamma[DS(F_n)] > 3$ then the vertex v_i colored with C_t is not adjacent with all C_{t-1} colors. For instance, $\Gamma[DS(F_n)] = 4$, the vertex v_0 colored with C_4 is not adjacent with C_1 for the mapping $\tau(v_0) = C_4$,

$$\tau(v_i) = \begin{cases} C_3, & i \equiv 1(\text{mod})2, \\ C_2, & i \equiv 0(\text{mod})2 \end{cases}$$

for $\forall i \in [1, 2n]$ and $\tau(w) = C_1$. This leads to the contradiction of Grundy coloring. Similarly, $5 \le \Gamma[DS(F_n)] \le 2n + 1$ leads to contradiction. And suppose $\Gamma[DS(F_n)] < 3$, it contradicts the definition of proper coloring. Therefore, $\Gamma[DS(F_n)] = 3$ for $n \ne 1$.

From all above cases, we have

$$\Gamma[DS(F_n)] = \begin{cases} 4, & n = 1, \\ 3, & n \ne 1. \end{cases}$$

This completes the proof. □

References

[1] Brice Effantin, A note on Grundy coloring of central graphs, *Australasian Journal of Combinatorics*, 68(3), 346-356, 2017.

[2] Brice Effantin and Hamamache Kheddouci, Grundy number of graphs, *Discussiones Mathematicae Graph Theory*, 27, 5 – 18, 2007.

[3] Claude A. Christen and Stanley M. Selkow, Some perfect coloring properties of graphs, *Journal of Combinatorial Theory*, Series B, 27, 49-59, 1979.

[4] R. Kalaivani and D. Vijayalakshmi, On dominator coloring of degree splitting graph of some graphs, *Journal of Physics: Conference Series*, 1139 (1), 012081, 2018.

[5] Manouchehr Zaker, Results on the Grundy chromatic number of graphs, *Discrete Mathematics*, 306, 3166 – 3173, 2006.

[6] Marc Glen, Sergey Kitaev and Artem Pyatkin, On the representation number of a crown graph, *Discrete Applied Mathematics*, 244, 89-93, 2018.

[7] A.R. Maulidia and Purwanto, Elegant labelling of sun graphs and helm graphs,*Journal of Physics: Conference Series*, 2021.

[8] P. Ponraj and S. Somasundaram, On degree splitting graph of a graph, *National Academy Science Letters*, 27(7-8), 275-278, 2004.

[9] A. Rohini, M. Venkatachalam and R. Sangamithra, Irregular colorings of derived graphs

of flower graph, *Sema Journal*, 77(1), 47-57, 2019.

[10] S.K. Vaidya and Rakhimol V. Issac, The b-chromatic number of some degree splitting graphs, An International Journal of Mathematical Sciences with Computer Applications, *Malaya Journal of Matematik*, 2(3), 249-253, 2014.

[11] J. Vernold Vivin and M. Vekatachalam, On b-chromatic number of sunlet graph and wheel graph families, *Journal of the Egyptian Mathematical Society*, 23, 215-218, 2015.