*International J.Math. Combin. Vol.3(2021), 35-48*

# On Laplacian of Product of Randić and Sum-Connectivity Energy of Graphs

R.Poojitha and Dr.Puttaswamy

(Department of Mathematics, P.E.S. college of Engineering, Mandya-571401, India)

E-mail: pooju24@gmail.com, prof.puttaswamy@gmail.com

**Abstract**: In this paper, we define the Laplacian of product of Randić and sum-connectivity energy of a graph. Then, we compute the Laplacian of product of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the crown graph, the $(S_m \wedge P_2)$ graph.

**Key Words**: Laplacian matrix, Laplacian of product of Randić and sum-connectivity energy, graph.

**AMS(2010)**: 05C50.

## §1. Introduction

In [6] we define product of Randić and sum-connectivity energy of a simple graph $G$ as follows:

Let $a$ and $b$ be two nonnegative real number with $a \neq 0$. The product of Randić and sum-connectivity adjacency matrix of $G$ is the $n \times n$ matrix $A_{prs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \dfrac{1}{\sqrt{\frac{1}{a(d_i+d_j)b(d_i d_j)}}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The eigenvalues of the graph $G$ are the eigenvalues of $A_{prs}$. Since $A_{prs}$ is real and symmetric, its eigenvalues are real numbers which are denoted by $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$, where

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n.$$

Then the product of Randić and sum-connectivity energy of $G$ is defined as

$$E_{prs}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

[1]Received August 5, 2021, Accepted August 30, 2021.

In 2004, D. Vukičević and Gutman [5] defined the Laplacian matrix of the graph $G$, denoted by $L = (L_{ij})$, as a square matrix of order $n$ whose elements are defined by

$$L_{ij} = \begin{cases} \delta_i, & \text{if } i = j, \\ -1, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are adjacent}, \\ 0, & \text{if } i \neq j \text{ and the vertices } v_i, v_j \text{ are not adjacent}, \end{cases}$$

where $\delta_i$ is the degree of vertex $v_i$. The eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$ of $L$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are called the Laplacian eigenvalues of $G$. In 2006, Gutman and B. Zhou [2] have defined the Laplacian energy of $LE(G)$ of $G$ as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where $m$ is number of edges and $n$ is number of vertices of $G$.

Motivated by these works, we introduce the Laplacian of product of Randić and sum-connectivity energy of a simple graph $G$ as follows. Let $a$ and $b$ be two nonnegative real number with $a \neq 0$. The Laplacian of product of Randić and sum-connectivity adjacency matrix of $G$ is the $n \times n$ matrix $A_{lprs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \delta_i, & \text{if } i = j, \\ \frac{1}{\sqrt{\frac{1}{a(d_i+d_j)b(d_i d_j)}}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent}, \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent}. \end{cases}$$

where $\delta_i$ is the degree of vertex $v_i$. Where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ are called the eigenvalues of $A_{lprs}$. Then Laplacian of product of Randić and sum-connectivity energy of $G$ is

$$E_{lprs}(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where $m$ is number of edges and $n$ is number of vertices of $G$.

## §2. Laplacian of Product of Randić and Sum-Connectivity Energies of Some Families of Graphs

We begin with some basic definitions and notations.

**Definition** 2.1([3]) *A graph $G$ is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on $n$ vertices is denoted by $K_n$.*

**Definition** 2.2([1]) *The Crown graph $S_n^0$ for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$ and edge set $\{u_i v_j; 1 \leq i, j \leq n, i \neq j\}$. $S_n^0$ is therefore equivalent* *to the complete bipartite graph $K_{n,n}$ from which the edges of perfect matching have been removed.*

**Definition** 2.3([3]) *A bigraph or bipartite graph $G$ is a graph whose vertex set $V(G)$ can be partitioned into two subsets $V_1$ and $V_2$ such that every line of $G$ joins $V_1$ with $V_2$. $(V_1, V_2)$ is a bipartition of $G$. If $G$ contains every line joining $V_1$ and $V_2$, then $G$ is a complete bigraph. If $V_1$ and $V_2$ have $m$ and $n$ points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.*

**Definition** 2.4([4]) *The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and $P_2$ is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m)$ and $v_j v_l \in E(P_2)$ and $1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.*

Now we compute Laplacian of product of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the Crown graph, the $(S_m \wedge P_2)$ graph.

**Theorem** 2.5 *Let $a$ and $b$ be as defined above. Then the Laplacian of product of Randić and sum-connectivity energy of the complete bipartite graph $K_{n,n}$ is $2\sqrt{\frac{1}{2abn}}$.*

*Proof* Let the vertex set of the complete bipartite graph be $V(K_{n,n}) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. Then, the Laplacian of product of Randić and sum-connectivity matrix of complete bipartite graph is given by

$$A_{lprs} = \begin{pmatrix} n & \cdots & 0 & \sqrt{\frac{1}{a(n+n)b(n^2)}} & \cdots & \sqrt{\frac{1}{a(n+n)b(n^2)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & n & \sqrt{\frac{1}{a(n+n)b(n^2)}} & \cdots & \sqrt{\frac{1}{a(n+n)b(n^2)}} \\ \sqrt{\frac{1}{a(n+n)b(n^2)}} & \cdots & \sqrt{\frac{1}{a(n+n)b(n^2)}} & n & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{1}{a(n+n)b(n^2)}} & \cdots & \sqrt{\frac{1}{a(n+n)b(n^2)}} & & \cdots & n \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{lprs}| = \begin{vmatrix} (\lambda - n) I_n & -\sqrt{\frac{1}{b(n^2)a(n+n)}} J^T \\ -\sqrt{\frac{1}{b(n^2)a(n+n)}} J & (\lambda - n) I_n \end{vmatrix},$$

where $J$ is an $n \times n$ matrix with all the entries are equal to 1. Hence, the characteristic equation is given by

$$\begin{vmatrix} \Lambda I_n & -\sqrt{\frac{1}{a(n+n)b(n^2)}} J^T \\ -\sqrt{\frac{1}{a(n+n)b(n^2)}} J & \Lambda I_n \end{vmatrix} = 0,$$

where $\Lambda = \lambda - n$ and which can be written as

$$|\Lambda I_n| \left| \Lambda I_n - \left( -\sqrt{\frac{1}{a(n+n)b(n^2)}} J \right) \frac{I_n}{\Lambda} \left( -\sqrt{\frac{1}{a(n+n)b(n^2)}} J^T \right) \right| = 0.$$

By simplification, we obtain

$$\frac{\Lambda^{n-n}}{((a(n+n)b(n^2))^n}\left|a(n+n)b(n^2)\Lambda^2 I_n - JJ^T\right| = 0,$$

which can be written as

$$\frac{\Lambda^{n-n}}{(a(n+n)b(n^2))^n}P_{JJ^T}(a(n+n)b(n^2))\Lambda^2) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_nJ_n$. Thus, we have

$$\frac{\Lambda^{n-n}}{(a(n+n)bn^2)^n}((a(n+n)b(n^2))\Lambda^2 - n^2)(a(n+n)b(n^2)\Lambda^2)^{n-1} = 0,$$

which is the same as

$$\Lambda^{n+n-2}(\Lambda^2 - \frac{n^2}{b(n)^2 a(n+n)}) = 0.$$

Therefore, the spectrum of $K_{n,n}$ is given by

$$Spec\,(K_{n,n}) = \begin{pmatrix} n & n+\sqrt{\frac{1}{2abn}} & n-\sqrt{\frac{1}{2abn}} \\ n+n-2 & 1 & 1 \end{pmatrix}.$$

Hence, the Laplacian of product of Randić and sum-connectivity energy of the complete bipartite graph is

$$E_{lprs}(K_{n,n}) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|,$$

and

$$E_{lprs}(K_{n,n}) = 2\sqrt{\frac{1}{2abn}}$$

as desired. □

**Theorem** 2.6 *Let a and b be as defined above. Then the Laplacian of product of Randić and sum-connectivity energy of the $S_n$ is*

$$\begin{aligned} E_{lprs}(S_n) &= \frac{(n-2)^2}{n} \\ &+ \left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{an^3b - 4(anb(n-1)-1)}{2nab}}\right| \\ &+ \left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{an^3b - 4(anb(n-1)-1)}{2nab}}\right|. \end{aligned}$$

*Proof* Let the vertex set of star graph be given by $V(S_n) = \{v_1, v_2, \cdots, v_n\}$. Then the Laplacian of product of Randić and sum-connectivity matrix of the star graph $S_n$ is given by

$$A_{lprs}=\begin{pmatrix} n-1 & \sqrt{\frac{1}{anb(n-1)}} & \sqrt{\frac{1}{anb(n-1)}} & \cdots & \sqrt{\frac{1}{anb(n-1)}} & \sqrt{\frac{1}{anb(n-1)}} \\ \sqrt{\frac{1}{anb(n-1)}} & 1 & 0 & \cdots & 0 & 0 \\ \sqrt{\frac{1}{anb(n-1)}} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{\frac{1}{anb(n-1)}} & 0 & 0 & \cdots & 1 & 0 \\ \sqrt{\frac{1}{anb(n-1)}} & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hence, its characteristic polynomial is given by

$$\begin{aligned} |\lambda I - A_{lprs}| &= \begin{vmatrix} \lambda-(n-1) & -\sqrt{\frac{1}{anb(n-1)}} & -\sqrt{\frac{1}{anb(n-1)}} & \cdots & -\sqrt{\frac{1}{anb(n-1)}} \\ -\sqrt{\frac{1}{anb(n-1)}} & \lambda-1 & 0 & \cdots & 0 \\ -\sqrt{\frac{1}{anb(n-1)}} & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{1}{anb(n-1)}} & 0 & 0 & \cdots & \lambda-1 \end{vmatrix} \\ &= \left(\sqrt{\frac{1}{anb(n-1)}}\right)^n \begin{vmatrix} \gamma & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}, \end{aligned}$$

where $\mu=(\lambda-1)\sqrt{anb(n-1)}$ and $\gamma=(\lambda-(n-1))\sqrt{anb(n-1)}$. Then,

$$|\lambda I - A_{lprs}| = \phi_n(\mu)\left(\sqrt{\frac{1}{a(n)b(n-1)}}\right)^n,$$

where

$$\phi_n(\mu)=\begin{vmatrix} \gamma & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}.$$

Applying the properties of determinant, we obtain after some simplifications $\phi_n(\mu) =$

$(\mu\phi_{n-1}(\mu) - \mu^{n-2})$ and iterating this formula, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu\gamma - (n-1)).$$

Therefore,

$$|\lambda I - A_{lprs}| = \left(\sqrt{\frac{1}{anb(n-1)}}\right)^n [((anb(n-1))(\lambda-1)(\lambda-(n-1)) \\ -(n-1))\left((\lambda-1)\sqrt{\frac{1}{anb(n-1)}}\right)^{n-2}\Bigg].$$

Thus, the characteristic equation is given by

$$(\lambda-1)^{n-2}\left((\lambda-1)(\lambda-(n-1)) - \frac{1}{anb}\right) = 0.$$

We know that

$$Spec\,(S_n) = \begin{pmatrix} 1 & n+\sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}} & n-\sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence, the Laplacian of product of Randić and sum-connectivity energy of $S_n$ is

$$E_{lprs}(S_n) = \frac{(n-2)^2}{n} + \left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}}\right| \\ + \left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}}\right|.$$

This completes the proof. □

**Theorem** 2.7 *Let $a$ and $b$ be as defined above. Then the Laplacian of product of Randić and sum-connectivity energy of $K_n$ is* $2\sqrt{\frac{1}{a2b(n-1)}}$ .

*Proof* Let the vertex set of Complete graph be given by $V(K_n) = \{v_1, v_2, \cdots, v_n\}$. Then the Laplacian of product of Randić and sum-connectivity energy of matrix of the complete graph $K_n$ is given by

$$A_{lprs} = \begin{pmatrix} n-1 & \sqrt{\frac{1}{a2b(n-1)^3}} & \cdots & \sqrt{\frac{1}{a2b(n-1)^3}} \\ \sqrt{\frac{1}{a2b(n-1)^3}} & n-1 & \cdots & \sqrt{\frac{1}{a2b(n-1)^3}} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\frac{1}{a2b(n-1)^3}} & \sqrt{\frac{1}{a2b(n-1)^3}} & \cdots & n-1 \end{pmatrix}.$$

Hence, its characteristic polynomial is given by

$$|\lambda I - A_{lprs}| = \begin{vmatrix} \lambda-(n-1) & -\sqrt{\frac{1}{a2b(n-1)^3}} & \cdots & -\sqrt{\frac{1}{a2b(n-1)^3}} \\ -\sqrt{\frac{1}{a2b(n-1)^3}} & \lambda-(n-1) & \cdots & -\sqrt{\frac{1}{a2b(n-1)^3}} \\ \vdots & \vdots & \ddots & \vdots \\ -\sqrt{\frac{1}{a2b(n-1)^3}} & -\sqrt{\frac{1}{a2b(n-1)^3}} & \cdots & \lambda-(n-1) \end{vmatrix}$$

$$= \left(\sqrt{\frac{1}{a2b(n-1)^3}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix},$$

where,

$$\mu = (\lambda-(n-1))\sqrt{a2(n-1)(n-1)^2b}.$$

Then

$$|\lambda I - A_{lprs}| = \phi_n(\mu)\left(\sqrt{\frac{1}{a2b(n-1)^3}}\right)^n$$

with

$$\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}$$

$$= \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ 0 & 0 & 0 & \cdots & -1-\mu & \mu+1 \end{vmatrix}$$

$$= \quad (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} + (\mu+1)\begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}$$

and

$$\begin{aligned} \phi_n(\mu) &= -(\mu+1)^{n-1} + (\mu+1)\left[(\mu+1)^{n-2}(\mu-(n-2))\right] \\ &= -(\mu+1)^{n-1} + (\mu+1)^{n-1}(\mu-(n-2)). \end{aligned}$$

Iterating this formula, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1)),$$

thus the characteristic equation is given by

$$\left(\sqrt{\frac{1}{a2b(n-1)^3}}\right)^n (\mu+1)^{n-1}(\mu-(n-1)) = 0.$$

Hence, the Laplacian of product of Randić and sum-connectivity energy of $K_n$ is

$$E_{lprs}(K_n) = \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right|, \text{ i.e., } E_{lprs}(K_n) = 2\sqrt{\frac{1}{a2b(n-1)}}. \qquad \square$$

**Theorem** 2.8 *Let the vertex set $V(S_n^0)$ of the crown graph be given by $V(S_n^0) = \{u_1, u_2, \cdots, u_n, v_1, v_2, \cdots, v_n\}$. Then, the Laplacian of product of Randić and sum-connectivity energy of the crown graph is $4\sqrt{\frac{1}{a2(n-1)b}}$.*

*Proof* The Laplacian of product of Randić and sum-connectivity energy matrix of crown graph is given by

$$A_{lprs} = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 & \alpha & \cdots & \alpha \\ 0 & n-1 & \cdots & 0 & \alpha & 0 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n-1 & \alpha & \alpha & \cdots & 0 \\ 0 & \alpha & \cdots & \alpha & n-1 & 0 & \cdots & 0 \\ \alpha & 0 & \cdots & \alpha & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \cdots & 0 & 0 & 0 & \cdots & n-1 \end{pmatrix},$$

where $\alpha = \sqrt{\frac{1}{ab(2(n-1)^3)}}$. Its characteristic polynomial is

$$|\lambda I - A_{lprs}| = \begin{vmatrix} (\lambda-(n-1))I_n & -\sqrt{\frac{1}{ab(2(n-1)^3)}}K^T \\ \sqrt{\frac{1}{ab(2(n-1)^3)}}K & (\lambda-(n-1)I_n \end{vmatrix},$$

where $K$ is an $n\times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \Lambda I_n & -\sqrt{\frac{1}{ab(2(n-1)^3)}}K^T \\ \sqrt{\frac{1}{ab(2(n-1)^3)}}K & \Lambda I_n. \end{vmatrix} = 0,$$

where $\Lambda = (\lambda-(n-1))$. This is the same as

$$|\Lambda I_n| \left| \Lambda I_n - \left(-\sqrt{\frac{1}{ab(2(n-1)^3)}}K\right)\frac{I_n}{\Lambda}\left(-\sqrt{\frac{1}{ab(2(n-1)^3)}}K^T\right)\right| = 0,$$

which can be written as

$$\left(\frac{1}{ab(2(n-1)^3)}\right)^n P_{KK^T}((ab(2(n-1)^3))\Lambda^2) = 0,$$

where $P_{KK^T(\Lambda)}$ is the characteristic polynomial of the matrix $KK^T$. Thus we have

$$\left(\frac{1)}{ab(2(n-1)^3)}\right)^n [ab(2(n-1)^3))\Lambda^2 - (n-1)^2][ab(2(n-1)^3))\Lambda^2 - 1]^{n-1} = 0,$$

which is the same as

$$\left(\Lambda^2 - \frac{1}{2ab(n-1)}\right)\left(\Lambda^2 - \frac{1}{a2b(n-1)^3}\right)^{n-1} = 0.$$

Therefore

$$Spec\,(S_n^0) = \begin{pmatrix} \beta & -\beta & \beta & -\beta \\ 1 & 1 & n-1 & n-1 \end{pmatrix},$$

where, $\beta = \sqrt{\frac{1}{2ab(n-1)}} + (n-1)$. Hence, the Laplacian of product of Randić and sum-connectivity energy of crown graph is

$$E_{lprs}(S_n^0) = 4\sqrt{\frac{1}{a2b(n-1)}}.$$ □

**Theorem** 2.9 *Let $a$ and $b$ be as defined above. Then the Laplacian of product of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ is*

$$\begin{aligned} \frac{(2n-4)(n-2)}{n} &+ 2\left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{an^3b - 4(anb(n-1)-1)}{2nab}}\right| \\ &+ 2\left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{an^3b - 4(anb(n-1)-1)}{2nab}}\right|. \end{aligned}$$

*Proof* Let the vertex set of $(S_m \wedge P_2)$ graph be given by $V(S_m \wedge P_2) = \{v_1, v_2, \cdots, v_{2m+2}\}$. Then the Laplacian of product of Randić and sum-connectivity matrix of $(S_m \wedge P_2)$ graph is given by

$$A_{lprs} = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 & \omega & \cdots & \omega \\ 0 & 1 & \cdots & 0 & \omega & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \omega & 0 & \cdots & 0 \\ 0 & \omega & \cdots & \omega & n-1 & 0 & \cdots & 0 \\ \omega & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \end{pmatrix}_{2n\times 2n},$$

where, $m+1=n$ and $\omega = \sqrt{\frac{1}{anb(n-1)}}$. Its characteristic polynomial is given by

$$|\lambda I - A_{lprs}| = \begin{vmatrix} \lambda-(n-1) & 0 & \cdots & 0 & 0 & -\omega & \cdots & -\omega \\ 0 & \lambda-1 & \cdots & 0 & -\omega & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda-1 & -\sqrt{\frac{b(n-1)}{an}} & 0 & \cdots & 0 \\ 0 & -\omega & \cdots & -\omega & \lambda-(n-1) & 0 & \cdots & 0 \\ -\omega & 0 & \cdots & 0 & 0 & \lambda-1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\omega & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda-1 \end{vmatrix}_{2n\times 2n}.$$

Hence, its characteristic equation is given by

$$(\omega)^{2n} \begin{vmatrix} \omega & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \omega & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

where $\Lambda = \sqrt{na(n-1)b}(\lambda-1)$ and $\gamma = \sqrt{na(n-1)b}(\lambda-(n-1))$.

Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \gamma & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}.$$

$$= (-1)^{2n+2n}\Lambda \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \gamma & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+2} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

and let

$$\begin{aligned}\Psi_{2n-1}(\Lambda) &= (-1)^{2n+2} \\ &\times \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \gamma & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.\end{aligned}$$

Applying the properties of determinant, we obtain

$$\Psi_{2n-1}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda),$$

after some simplifications, where

$$\Theta_n(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \gamma \end{vmatrix}_{n\times n}.$$

Then,

$$\phi_{2n}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as the above we obtain

$$\begin{aligned}\phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+2}\Psi_{2n-2}(\Lambda) \\ &\quad +(-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= -\Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda)\end{aligned}$$

and continuously like this, we obtain

$$\phi_{2n}(\Lambda) = -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda)$$

at the $(n-1)^{th}$ step, where

$$\xi_{n+1}(\Lambda) = \begin{vmatrix} \gamma & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}.$$

$$\begin{aligned} \phi_{2n}(\Lambda) &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\gamma\Theta_n(\Lambda) \\ &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\gamma\Theta_n(\Lambda) \\ &= (\Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2})\Theta_n(\Lambda). \end{aligned}$$

Applying the properties of determinants, we obtain

$$\Theta_n(\Lambda) = \Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2}.$$

Therefore

$$\phi_{2n}(\Lambda) = (\Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2})^2.$$

Hence, characteristic equation becomes

$$\left(\sqrt{\frac{1}{anb(n-1)}}\right)^{2n} \phi_{2n}(\Lambda) = 0,$$

which is the same as

$$\left(\sqrt{\frac{1}{anb(n-1)}}\right)^{2n} (\Lambda^{n-1}\gamma - (n-1)\Lambda^{n-2})^2 = 0,$$

which can be reduced to

$$(\lambda-1)^{2n-4}((nab(n-1)(\lambda-1)(\lambda-(n-1)) - (n-1))^2 = 0.$$

Therefore,

$$Spec\,((S_m \wedge P_2)) = \begin{pmatrix} 1 & n + \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}} & n - \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence, the Laplacian of product of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ graph is

$$E_{lprs}((S_m \wedge P_2)) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$$

Whence,

$$
\begin{aligned}
E_{lqrs}((S_m \wedge P_2)) &= \frac{(2n-4)(n-2)}{n} + 2\left|\frac{n^2-2(n-1)}{n} + \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}}\right| \\
&\quad +2\left|\frac{n^2-2(n-1)}{n} - \sqrt{\frac{an^3b-4(anb(n-1)-1)}{2nab}}\right|.
\end{aligned}
$$

This completes the proof. □

## References


[1] C. Adiga, Abdelmejid Bayad, Ivan Gutman and Shrikanth Avant Srinivas, The minimum covering energy of a graphs, *Kragujevac j. Sci.*, 34(2012), 39-56.

[2] I. Gutman and B. Zhou, Laplacian energy of a graph, *Linear Algebra Appl.*, 414(2006), 29-37.

[3] F. Harary, *Graph Theory*, $10^{th}$ Reprint, Narosa Publishing House, New Delhi, 2001.

[4] Ş. M. A. Soeud, Huda Jaber, Some Families of Strongly*-graphs (Communicated).

[5] D. Vukičević and I. Gutman, Laplacian matrix and distance in trees, *Kragujevac J. Maths.*, 26(2004), 19-24.

[6] Poojitha and Puttaswamy , On product of Randić and sum-connectivity energies of some graphs (Communicated).