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On Parallel Ruled Surfaces in Minkowski 3-Space and Their Retractions

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Abstract: In this paper, new types of retractions on ruled surfaces and parallel ruled surfaces as directrix retraction and ruling retraction in Minkowski 3-space has been presented. The topological folding of surfaces is deduced. The limit of these retractions and foldings have been obtained. The relations between the fundamental forms, Gaussian and mean curvatures before and after folding are discussed.

Key Words: Minkowski 3-space, parallel surfaces ruled surfaces, retractions, directrix and ruling retractions, folding, Frenet equations.

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§1. Introduction

Parallel surface something like that a surface M^r, whose points are at a constant distance along the normal from another surface M is said to be parallel to M. In theory of surfaces. There are some special surfaces such as ruled surfaces, minimal surfaces, and surfaces of constant curvatures. So, there are infinite numbers of surfaces because we choose the constant distance along the normal arbitrarily. Parallel surface can be regarded as the locus of a point which are on the normals to M at a non-zero constant distance r from M. A surface M is ruled if through every point of M there is a straight line that lies on M. The most familiar examples are the plane and the curved surface of a cylinder or cone. A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. For example, a cone is formed by keeping one point of a line fixed whilst moving another point along a circle. A surface that can be (locally) unrolled onto a flat plane without tearing or stretching it is called a developable surface [8-16].

§2. Preliminary Notes

The Minkowski 3-space E_1^3 is the Euclidean space E^3 provided with the Lorentzian inner product

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$$\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3,$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in E_1^3$.

We say that a vector v in E_1^3 is space-like if $\langle v, v\rangle > 0$, time like if $\langle v, v\rangle < 0$ and light-like (null) if $\langle v, v\rangle = 0$ and $v \neq \mathbf{0}$. The norm of the vector $v \in E_1^3$ is defined [12,17] by

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

A subset$A$ of a topological space X is called retract of X if there exists a continuous map $r : X \to A$ called a retraction [5, 6] such that $r(a) = a$ for any $a \in A$.

Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ if f does not preserve the length it is called topological folding [1-4].

Definition 2.1([12,17]) *A surface M in the Minkowski 3-space E_1^3 is said to be space-like, time-like surface if, respectively the induced metric on the surface is a positive definite Riemannian metric, Lorentz metric. In other words, the normal vector on the space like (time like) surface is a time- like (space like) vector.*

Definition 2.2 *Let M be a surface. An immersion $x : M \to E_1^3$ is called space-like (resp. timelike, light-like) if all tangent planes $\left(T_pM, x^*\left(\langle\ ,\ \rangle_p\right)\right)$ are space-like (resp. time-like, light-like). A non-degenerate surface is a space like or time-like surface.*

An immersion $X : M \to L^3$ from a connected surface M into L^3 is said to be a time-like surface If the induced metric via ψ is a Lorentzian metric on M, which will be also denoted by $\langle\ ,\ \rangle$ or by I.

Definition 2.3 *A ruled surface is a surface that can be swept out by moving a line in space if therefor has a parameterization of the form*

$$c(u, v) = b(u) + v\delta(u),$$

where $b(u)$ is called the ruled surface directrix or the base curve and $\delta(u)$ is the director curve.

Alternatively, the surface can be represented a ruling joining corresponding points on two space curves. This is represented by

$$c(u, v) = (1 - v)c_A(u) + vc_B(u),\ \ 0 \leq u, v \leq 1,$$

where $c_A(u)$ and $c_B(u)$ are directrices. The two representations are identical if

$$b(u) = c_A(u) \ \text{ and } \ \delta(u) = c_B(u) - c_A(u).$$

The straight lines themselves are called rulings.

Definition 2.4 *A surface M in E_1^3 is constant mean curvature if and only if $g(H, H) = c$,* *where $c \in \mathbb{R}$, g is the standard metric in and H the mean curvature vector field. If $c = 0$ then $H = 0$, which means that M is a minimal surface.*

Let S be an orientable surface and let n be the unit normal vector field of S, the surface $\overline{S}$ is parallel to S at distance δ if the point $\overline{p}(u, v)$ are defined by

$$\overline{p}(u, v) = p(u, v) + \delta n(u, v),$$

where δ is constant positive real number and n is the unit normal vector field on M.

§3. Folding of Parallel Surfaces in Minkowski 3-Space E_1^3

Theorem 3.1 *Let M and M^δ are space-like surfaces in Minkowski 3-space and let h be a topological folding of M and M^δ, then, $h(M)$ and $h\left(M^\delta\right)$ are space-like surfaces in E_1^3 and, the folded surface $h\left(M^\delta\right)$ is parallel to the surface $h(M)$ if and only if M^δ is a parallel surface of M.*

Proof Let h be a topological folding of the space-like surfaces M and M^δ, where M^δ is parallel to M in E_1^3 and

$$h(M) = h(M(u, v))$$

for any space-like curve ψ on the space-like surface M. We have a curve $h(\psi)$ on the folded surface $h(M)$, where $h'(\psi)$ satisfies

$$\langle h'(\psi), h'(\psi)\rangle = h^{'2}(M(u, v))$$

since M is a space-like surface $\langle \psi', \psi'\rangle > 0$.

Also, this hold for any curve psi^δ on M^δ then $h(M)$ and $h\left(M^\delta\right)$ are space-like surfaces in E_1^3. Since

$$N = \frac{M_u \wedge M_v}{|M_u \wedge M_v|},$$

then

$$N_f = \frac{h(M)_u \wedge h(M)_v}{|h(M)_u \wedge h(M)_v|} = \frac{h^{'2}(M)(M_u \wedge h(M)_v}{|h^{'2}(M)M_u \wedge h(M)_v|} = \frac{h^{'2}(M_u \wedge h(M)_v}{|h^{'2}M_u \wedge h(M)_v|} = N$$

and

$$h\left(M^\delta\right) = h(M) + \delta_f N_f = h(M) + \delta_f N.$$

We can choose $\delta_f \leq \delta$. Then, the folded surface $h\left(M^\delta\right)$ is parallel to the surface $h(M)$. □

Corollary 3.1 *Let M be a space-like surface in E_1^3 and $h(M)$ be a topological folding of M. Then, the fundamental forms and the Gaussian and mean curvatures of $h(M)$ can be formed by the fundamental forms and the Gaussian and mean curvatures of M.*

Proof Let M be space-like surface in Minkowski 3-space and $h(M)$ be a topological folding of M. Then, by definition

$$\begin{aligned} h(M) &= h(M(u,v)), \\ I_h &= \langle dh(M), dh(M) \rangle, \\ I_h &= \left\langle \frac{\partial h(M)}{\partial u}, \frac{\partial h(M)}{\partial u} \right\rangle (du)^2 + 2 \left\langle \frac{\partial h(M)}{\partial u}, \frac{\partial h(M)}{\partial v} \right\rangle dudv + \left\langle \frac{\partial h(M)}{\partial v}, \frac{\partial h(M)}{\partial v} \right\rangle (dv)^2. \end{aligned}$$

Hence, we get the first fundamental form of the folded surface to be

$$I_h = E^h (du)^2 + 2F^h dudv + G^h (dv)^2 = h^{'2} I.$$

By definition,

$$\begin{aligned} II_h &= \left\langle -dh(M), N^h \right\rangle, \\ II_h &= -\left\langle \frac{\partial h(M)}{\partial u}, \frac{\partial n^h}{\partial u} \right\rangle - 2 \left\langle \frac{\partial h(M)}{\partial u}, \frac{\partial n^h}{\partial v} \right\rangle dudv - \left\langle \frac{\partial h(M)}{\partial v}, \frac{\partial n^h}{\partial v} \right\rangle (dv)^2. \end{aligned}$$

So, we get that the second fundamental form

$$II_h = e^h (du)^2 + 2f^h dudv + g^h (dv)^2 = h^{'2} II.$$

Similarly, by definition

$$\begin{aligned} III_h &= \left\langle dN^h, dN^h \right\rangle, \\ III_h &= \left\langle \frac{\partial N^h}{\partial u}, \frac{\partial N^h}{\partial u} \right\rangle + 2 \left\langle \frac{\partial N^h}{\partial u}, \frac{\partial N^h}{\partial v} \right\rangle dudv + \left\langle \frac{\partial N^h}{\partial v}, \frac{\partial N^h}{\partial v} \right\rangle (dv)^2. \end{aligned}$$

Notice that $N^h = N$, we get that

$$\begin{aligned} III_h &= \left\langle \frac{\partial N^h}{\partial u}, \frac{\partial N^h}{\partial u} \right\rangle + 2 \left\langle \frac{\partial N^h}{\partial u}, \frac{\partial N^h}{\partial v} \right\rangle dudv + \left\langle \frac{\partial N^h}{\partial v}, \frac{\partial N^h}{\partial v} \right\rangle (dv)^2 \\ &= \langle dN, dN \rangle = III \end{aligned}$$

and also, we have that

$$\begin{aligned} E_h &= h^{'2}(M) \langle M_u, M_u \rangle = h^{'2}(M) E, \\ G_h &= h^{'2}(M) \langle M_v, M_v \rangle = h^{'2}(M) G, \\ F_h &= h^{'2}(M) \langle M_u, M_v \rangle = h^{'2}(M) F \end{aligned}$$

and

$$\begin{aligned} e_h &= -\langle N_u, M_u \rangle = -h^{'2}(M) \langle N_u, M_u \rangle = h^{'2}(M) e; \\ f_h &= -\langle N_u, M_v \rangle = -h^{'2}(M) \langle N_u, M_v \rangle = h^{'2}(M) f; \\ g_h &= -\langle N_v, M_v \rangle = -h^{'2}(M) \langle N_v, M_v \rangle = h^{'2}(M) g, \end{aligned}$$

where the Gaussian curvature of the folded surface can be calculated by

$$K_h = \frac{e_h g_h - f_h^2}{E_h G_h - F_h^2} = \frac{h^{'4}\left(eg - f^2\right)}{h^{'4}\left(EG - F^2\right)} = \frac{eg - f^2}{EG - F^2} = K$$

and the mean curvature of the folded surface

$$H_h(p) = \frac{G_h e_h + E_h g_h - 2F_h f_h}{2\left(EG - F^2\right)} = \frac{h^{'4}\left(Ge + Eg - 2Ff\right)}{2h'^4\left(EG - F^2\right)} = H(p)$$

at a point $p \in h(M)$. □

Example 3.1 Consider $M = M(u, v)$ be a surface in Minkowski 3-space and $h(M(u, v))$ be a topological folding of m defined as $h(M) : M(u, v) \to \frac{M(u,v)}{m}, m \in \mathbb{N}$. Then, we get

$$\begin{aligned} h(M)_u &= h'(M)M_u, \quad h(M)_v = h'(M)M_v, \\ \langle h_u, h_u \rangle &= h'(M)\langle M_u, M_u \rangle, \quad \langle h_v, h_v \rangle = h'(M)\langle M_v, M_v \rangle \end{aligned}$$

and the normal vector of the folded surface is

$$N_h(p) = \frac{h_u(M) \wedge h_v(M)}{\|h_u(M) \wedge h_v(M)\|} = N(p).$$

We get that

$$\begin{aligned} I_h &= E_h(du)^2 + 2F_h du dv + G_h(dv)^2 \\ &= \frac{1}{m^2}\left(E^h(du)^2 + 2F^h du dv + G^h(dv)^2\right) = \frac{1}{m^2} I, \\ II_h &= e^h(du)^2 + 2f^h du dv + g^h(dv)^2 \\ &= e(du)^2 + 2f du dv + g(dv)^2 = \frac{1}{m^2} II, \\ III_h &= III. \end{aligned}$$

The Gaussian and mean curvatures of the folded surface can be calculated by

$$K_h = \frac{e_h g_h - f_h^2}{E_h G_h - F_h^2} = \frac{eg - f^2}{EG - F^2} = K \text{ and } H_h = H.$$

Theorem 3.2 *Let M and M^δ be two parallel surfaces in E_1^3 and $h(u, v)$ be a topological folding on M and M^δ. Then, the fundamental forms of $h(M^\delta)$ can be formed by the fundamental forms of $h(M)$ and we have $K_h = K_h^\delta$ and $H_h = H_h^\delta$, where K_h, K_h^δ, H_h and H_h^δ are the Gaussian and mean curvatures of the folded surfaces $h(M)$ and $h(M^\delta)$.*

Definition 3.1([15]) *A surface is called Weingarten surface or w-surface in E_1^3 if there is a nontrivial relation $\Phi(K, H) = 0$ or equivalently if the gradients of K and H are linearly dependent. In terms of the partial derivatives concerning u and v this is the equation $K_u H_v - K_v H_u = 0$, where K and H are Gaussian and mean curvatures of surface, respectively.*

Corollary 3.2 *Let M be a ruled surface in E_1^3 and $h(u,v)$ be an isometric folding of M. Then, $h(M)$ be a ruled surface and $h(M)$ can not be a developable surface.*

§4. Retraction of Ruled Surfaces in Minkowski 3-Space

Definition 4.1 *Let $X(u,v) = b(u) + v\delta(u)$ be a ruled surface in Minkowski 3-space with a retraction r defined by $r : X(u,v) \to X(u,v) - \{l(u)\}$, where $l(u)$ be the ruled surface directrix or the base curve which is called directrix retraction and the limit of this retraction be the base curve.*

Definition 4.2 *Let $X(u,v) = b(u) + v\delta(u)$, be a ruled surface in Minkowski 3-space with a retraction r defined by $r : X(u,v) \to X(u,v) - \{l(u)\}$, where $l(u)$ be a one of their rulings, which is called ruling retraction, and the limit of this retraction of X be a one of their rulings.*

Theorem 4.1 *The retraction of a ruled surface in E_1^3 is a ruled surface that can be swept out by moving a line in space if therefor has a parameterization of the form*

$$X(u,v) = b(u) + v\delta(u),$$

where $b(u)$ is called the ruled surface directrix or the base curve and $\delta(u)$ is the director curve.

Theorem 4.2 *The limit of retractions of a ruled surface in E_1^3 is a curve on the ruled surface.*

Proof Let $X(u,v) = b(u) + v\delta(u)$ be a ruled surface in E_1^3 and let $r = r(X(u,v))$ be a retraction of $X(u,v)$, where $r(u,v) : X(u,v) \to X(u,v) - \{l(u)\}$ with $l(u)$ being a curve on $X(u,v)$. Then, the limit of retractions of $X(u,v)$ is a curve on the surface $X(u,v)$ in E_1^3 and $r = r(X(u,v))$ satisfies

$$\begin{aligned}
r_1(u,v) : X(u,v) &\to X(u,v) - \{l_1(u)\},\\
r_2(u,v) : r_1(X) &\to r_1(X) - \{l_2(u)\},\\
r_3(u,v) : r_2(r_1(X)) &\to r_2(r_1(X)) - \{l_3(u)\},\\
\cdots\cdots\cdots\cdots\cdots\cdots &\cdots \cdots\cdots\cdots\cdots\cdots\cdots ,\\
r_n : r_{n-1}(r_{n-2}(\cdots r_1(X))\cdots) &\to r_{n-1}(r_{n-2}(\cdots r_1(X))\cdots) - \{l_n(u)\}.
\end{aligned}$$

Then $\lim_{n\to\infty} r_n(X) = m(u)$, which is a curve on the surface $X(u,v)$. □

Corollary 4.1 *The limit of retractions of a non-developable ruled surface in E_1^3 is a curve on the ruled surface.*

Corollary 4.2 *The limit of retractions of the helicoid $X(u,v) = (u\cos v, u\sin v, v)$ in Minkowski 3-space be either a helix or a one of its rulings.*

Proof Let $X(u,v) = (u\cos v, u\sin v, v)$ be a helicoid which is a non-developable surface in Minkowski 3-space and $r(u,v) : X(u,v) \to X(u,v) - \{l(u)\}$ be a retraction of $X(u,v)$. We discuss two cases following.

Case 1. The directrex retraction defined as

$$r(u,v) : X(u,v) \to X(u,v) - \{l(u)\},$$

where the curve $\{l_1(u)\}$ is a helix on the surface. In this case, we get

$$\begin{array}{rcl} r_1(u,v) : X(u,v) & \to & X(u,v) - \{l_1(u)\} = X_1(u,v), \\ r_2(u,v) : X_1(u,v) & \to & X_1(u,v) - \{l_2(u)\} = X_2(u,v), \\ r_3(u,v) : X_2(u,v) & \to & X_2(u,v) - \{l_3(u)\} = X_3(u,v), \\ \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots , \\ r_{n-1}(u,v) : X_{n-2}(u,v) & \to & X_{n-2}(u,v) - \{l_{n-1}(u)\} = X_{n-1}(u,v), \\ \lim_{n\to\infty} r_n\left(X_n(u,v)\right) & = & l(u) \end{array}$$

where $l(u)$ is a helix in E_1^3.

Case 2. The ruling retraction defined as

$$r(u,v) : X(u,v) \to X(u,v) - \{l^*(u)\},$$

where the curve $\{l_1^*(u)\}$ is a one of its rulings of the surface by using a sequence of retractions then we have that the limit of ruling retractions of the helicoid is one of its rulings. □

Theorem 4.3 *The limit of retractions of the developable $X(u,v)$ in Minkowski 3-space be either a helix or a one of its rulings.*

Proof The proof is similar to the proof of Corollary 4.2. □

Corollary 4.3 *The limit of directrix retractions of the double cone $X(u,v)$ in Minkowski 3-space be a cone in E_1^3 and the limit of directrix retractions of this cone is a point which all rulings meeting.*

Proof Let $X(u,v)$ be a double cone in E_1^3 which is developable ruled surface with the property that all rulings meet in a point $p \in X(u,v)$, the directrix retraction of the upper nappe of $X(u,v)$ can be defined as

$$\begin{array}{rcl} r_1(X((u,v)) : X(u,v) & \to & X(u,v) - \{s_1^1\}, \\ r_2(X(u,v)) : X(u,v) & \to & X(u,v) - \{s_2^1\}, \\ r_3(X(u,v)) : X(u,v) & \to & X(u,v) - \{s_3^1\}, \\ \cdots\cdots\cdots & \cdots & \cdots\cdots\cdots , \\ \lim_{n\to\infty} r_n\left(X_n(u,v)\right) & = & X^*(u,v) \end{array}$$

and we have the limit of directrix retractions of $X(u,v)$ to be a cone $X^*(u,v)$. We therefore get that the directrix retraction of a cone $X^*(u,v)$ can be defined as

$$\begin{array}{rcl} r_1\left(X^*(u,v)\right): X(u,v) & \to & X^*(u,v)-\{s_1^{*1}\}, \\ r_2\left(X^*(u,v)\right): X(u,v) & \to & X^*(u,v)-\{s_2^{*1}\}, \\ r_3\left(X^*(u,v)\right): X(u,v) & \to & X^*(u,v)-\{s_3^{*1}\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots, \\ \lim\limits_{n\to\infty} r_n\left(X_n^*(u,v)\right) & = & p. \end{array}$$

Then the limit of directrix retractions of a cone $X^*(u,v)$ is meeting the ruling point. □

§5. Parallel Ruled Surfaces in E_1^3 and Their Retractions

Theorem 5.1([15]) *Let M be a time like surface and let M^δ be a parallel surface of M in E_1^3. Then, we get*

$$\overline{K}=\frac{K}{1-2\delta H+\delta^2 K}$$

and

$$\overline{H}=\frac{H-\delta K}{1-2\delta H+\delta^2 K},$$

where the Gaussian and mean curvatures of M and M^δ are denoted by K,H and $\overline{K},\overline{H}$, respectively.

Theorem 5.2 *Let $X(u,v)$ and $Y(u,v)=X(u,v)+\delta N(u,v)$ be parallel time like ruled surfaces with time-like rulings in Minkowski 3-space E_1^3 with Gaussian and mean curvatures K,H and $\overline{K},\overline{H}$, respectively. Then, $X(u,v)$ and $Y(u,v)$ are time-like developable surface with time-like rulings if*

$$\overline{K}=K=0 \quad \textit{and} \quad \overline{H}=\frac{H}{1-2\delta H}.$$

Proof Let $X(u,v)$ and $Y(u,v)=X(u,v)+\delta N(u,v)$ be parallel ruled surfaces in Minkowski 3-space with Gaussian and mean curvatures K,H and $\overline{K},\overline{H}$, respectively. We find from the relation between the Gaussian and mean curvatures of the surfaces $X(u,v)$ and $Y(u,v)$ that

$$\overline{K}=\frac{K}{1-2\delta H+\delta^2 K}$$

and

$$\overline{H}=\frac{H+\delta K}{1-2\delta H+\delta^2 K}.$$

If $K=0$ we get that $\overline{K}=K=0$, and then $Y(u,v)$ are developable surfaces in E_1^3 and $1-2H\delta=c^*$, where c^* be constant. And then, we have

$$H=\frac{1-c^*}{2\delta}.$$

The relation between the mean curvatures of the surfaces $X(u,v)$ and $Y(u,v)$ is

$$\overline{H} = \frac{H}{1-2\delta H}.$$

Consequently, we know that

$$H = \frac{\overline{H}}{1+2\delta\overline{H}}.$$ □

Theorem 5.3([15]) *Let M be a space like surface and M^δ be a parallel surface of M in E_1^3. Then, the parallel surface of a space-like developable ruled surface is a space-like ruled surface.*

Theorem 5.4 *Let M be a developable space like ruled surface in E_1^3. Then, the ruling retraction $r(M) = M_r$ is a developable space like ruled surface, and the parallel surface M^δ of M is a space-like ruled Weingarten surface.*

Proof Since M_r is a ruled retraction of a developable space-like ruled surface M, M_r is a developable space-like ruled surface in E_1^3. From Theorem 3.1, the parallel surface M_r^δ of the developable space-like ruled surface M_r is a developable space-like ruled surface. Notice that M_r^δ and M_r both are developable surfaces. The Gaussian curvature of M_r^δ and M_r satisfy $K_r^\delta = K_r = 0$. We therefore get that $\Psi\left(K_r^\delta, H_r^\delta\right) = 0$. Thus, the parallel surface M_r^δ of M_r is a Weingarten surface. □

Corollary 5.1 *Let $r(M) = M_r$ be the directrix retraction of the space-like ruled surface M and let M_r^δ be a parallel ruled surface of a directrix retraction ruled surface M_r in E_1^3. Then the parallel surface M_r^δ is a Weingarten surface if M_r is a Weingarten surface.*

Theorem 5.5 *Let M and M^δ be two parallel surfaces in E_1^3 and $h(u,v)$ be a topological folding on M and M^δ if $h(M)$ is a minimal surface, then M^δ is a minimal surface also.*

Proof Let M and M^δ be two parallel surfaces in E_1^3. If M is a minimal surface, $H = 0$ and from

$$\overline{H} = \frac{H}{1-2\delta H},$$

we get $\overline{H} = 0$ and so, M^δ is a minimal surface. □

Corollary 5.2 *The directrix retraction $r(X(u,v))$ of the ruled surface $X(u,v)$ in Minkowski 3-space E_1^3 is a minimal surface if $k_1 = -k_2$, where k_1 and k_2 are the principal curvatures of the retracted ruled curve.*

Corollary 5.3 *Under the directrix retraction of the helicoid $X(u,v)$ in a Minkowski 3-space, the first fundamental forms, Gaussian and mean curvatures are invariant.*

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