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On Sears's Basic Hypergeometric Series Transformation Formula

K. Shivashankara and G. Vinay

(Department of Mathematics, Yuvaraja's College, University of Mysore, Mysuru, 570005, India)

E-mail: vinaytalakad@gmail.com, drksshankara@gmail.com

Abstract: In this paper, we provide an alternative simple proof to Sears's ${}_3\varphi_2$ transformation formula using Gauss summation formula.

Key Words: Basic hypergeometric series, transformation formula, Gauss summation.

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§1. Introduction

For any complex number a and for any q with $|q| < 1$, $(a;q)_\infty$ or simply $(a)_\infty$ is defined as follows:

$$(a)_\infty = (a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$

For any integer n, we define $(a;q)_n$ or simply $(a)_n$ as

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty}, \tag{1.1}$$

provided the denominator is well defined. For any two integers m and n, the following holds good

$$(a)_n(aq^n)_m = (a)_m(aq^m)_n. \tag{1.2}$$

The basic hypergeometric series ${}_{s+1}\varphi_s$ is defined by

$${}_{s+1}\varphi_s\left[\begin{matrix} a_1, a_2, \cdots, a_{s+1} \\ b_1, b_2, \cdots, b_s \end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{s+1})_n}{(q)_n (b_1)_n (b_2)_n \cdots (b_s)_n} z^n,$$

where $a_1, a_2, \cdots, a_{s+1}, b_1, b_2, \cdots, b_s$ are any complex numbers, except that $(b_j)_n \neq 0$, $1 \leq j \leq s, 0 \leq n \leq \infty$. This series converges for all z with $|z| < 1$.

In his paper [3], D. B. Sears deduced the following ${}_3\varphi_2$ transformation formula

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{(d)_n (e)_n (q)_n} \left(\frac{de}{abc}\right)^n = \frac{(b)_\infty \left(\frac{de}{ab}\right)_\infty \left(\frac{de}{bc}\right)_\infty}{(d)_\infty (e)_\infty \left(\frac{de}{abc}\right)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{d}{b}\right)_n \left(\frac{e}{b}\right)_n \left(\frac{de}{abc}\right)_n}{\left(\frac{de}{ab}\right)_n \left(\frac{de}{bc}\right)_n (q)_n} b^n, \tag{1.3}$$

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where, $\left|\frac{dc}{abc}\right| < 1$, $|b| < 1$, and $|q| < 1$. Sears deduced the above transformation by letting one of the parameters in his ${}_4\varphi_3$ transformation to infinity. This identity has been widely in areas of special functions and number theory. See for instance [1] and [2]. In this paper, we give a simple proof to (1.3) using q-analogous of Gauss summation formula. This proof seems to be new in the literature. We close this section by recalling the q-analogous of Gauss summation formula and prove (1.3) in Section 2 following.

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m (q)_m} \left(\frac{c}{ab}\right)^m = \frac{\left(\frac{c}{a}\right)_\infty \left(\frac{c}{b}\right)_\infty}{(c)_\infty \left(\frac{c}{ab}\right)_\infty}, \quad \left|\frac{c}{ab}\right| < 1. \tag{1.4}$$

§2. Proof of (1.3)

From (1.4), it follows that

$$\frac{(A)_\infty (B)_\infty}{(C)_\infty \left(\frac{AB}{C}\right)_\infty} = \sum_{n=0}^{\infty} \frac{\left(\frac{C}{A}\right)_n \left(\frac{C}{B}\right)_n}{(C)_n (q)_n} \left(\frac{AB}{C}\right)^n, \qquad \left|\frac{AB}{C}\right| < 1.$$

Taking $A = dq^m, B = eq^m$ and $C = \frac{deq^m}{b}$ in the above, we obtain

$$\frac{(dq^m)_\infty (eq^m)_\infty}{\left(\frac{deq^m}{b}\right)_\infty (bq^m)_\infty} = \sum_{n=0}^{\infty} \frac{\left(\frac{d}{b}\right)_n \left(\frac{e}{b}\right)_n}{(q)_n \left(\frac{deq^m}{b}\right)_n} (bq^m)^n. \tag{2.1}$$

Using (1.1), we can write

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m}{(d)_m (e)_n (q)_m} \left(\frac{de}{abc}\right)^m = \frac{(b)_\infty}{(d)_\infty (e)_\infty} \sum_{m=0}^{\infty} \frac{(a)_m (c)_m \left(\frac{deq^m}{b}\right)_\infty}{(q)_m} \left(\frac{de}{abc}\right)^m \frac{(dq^m)_\infty (eq^m)_\infty}{\left(\frac{deq^m}{b}\right)_\infty (bq^m)_\infty}.$$

Using (2.1) in the above, we obtain

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m}{(d)_m (e)_n (q)_m} \left(\frac{de}{abc}\right)^m = \frac{(b)_\infty \left(\frac{de}{b}\right)_\infty}{(d)_\infty (e)_\infty} \sum_{m=0}^{\infty} \frac{(a)_m (c)_m}{(q)_m \left(\frac{de}{b}\right)_m} \left(\frac{de}{abc}\right)^m \sum_{n=0}^{\infty} \frac{\left(\frac{d}{b}\right)_n \left(\frac{e}{b}\right)_n}{(q)_n \left(\frac{deq^m}{b}\right)_n} (bq^m)^n.$$

Interchanging the order of summation in the above and using (1.2), we obtain

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m}{(d)_m (e)_m (q)_m} \left(\frac{de}{abc}\right)^m = \frac{(b)_\infty \left(\frac{de}{b}\right)_\infty}{(d)_\infty (e)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{d}{b}\right)_n \left(\frac{e}{b}\right)_n}{(q)_n \left(\frac{de}{b}\right)_n} (b)^n \sum_{m=0}^{\infty} \frac{(a)_m (c)_m}{(q)_m \left(\frac{deq^n}{b}\right)_m} \left(\frac{deq^n}{abc}\right)^m.$$

Now applying (1.4) to the inner series on the right hand side, we obtain

$$\sum_{m=0}^{\infty} \frac{(a)_m (b)_m (c)_m}{(d)_m (e)_n (q)_m} \left(\frac{de}{abc}\right)^m = \frac{(b)_\infty \left(\frac{de}{b}\right)_\infty}{(d)_\infty (e)_\infty} \sum_{n=0}^{\infty} \frac{\left(\frac{d}{b}\right)_n \left(\frac{e}{b}\right)_n}{(q)_n \left(\frac{de}{b}\right)_n} (b)^n \frac{\left(\frac{deq^n}{ab}\right)_\infty \left(\frac{deq^n}{bc}\right)_\infty}{\left(\frac{deq^n}{b}\right)_\infty \left(\frac{deq^n}{abc}\right)_\infty}.$$

Notice that the equation (1.3) follows directly from the above on using (1.1). This completes the proof. □

References

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