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On the Application of Separable $R_0R_1R_2\cdots R_a$-Cyclic DNA Codes to DNA Computing

Yasemin Cengellenmis

(Department of Mathematics, Trakya University, Edirne, Turkey)

Abdullah Dertli*

(Department of Mathematics, Ondokuz Mayıs University, Samsun, Turkey)

E-mail: ycengellenmis@gmail.com, abdullah.dertli@gmail.com

Abstract: In this paper, the $R_0R_1R_2\cdots R_a$-cyclic codes of block length $(\alpha_0,\cdots,\alpha_a)$ are studied, where $R_i = F_4[u_1,\cdots,u_i]/\langle u_j^2-u_j, u_ju_s-u_su_j\rangle$, $j,s=1,2,\cdots,i$, $j\neq s$ for $i=1,2,\cdots,a$ and $R_0=F_4$. Their generator polynomials are given. The structures of the separable $R_0R_1R_2\cdots R_a$-cyclic codes are determined. A necessary and sufficient conditions of the separable $R_0R_1R_2\cdots R_a$ cyclic codes to be reversible and reversible complement are determined. By introducing a map, the separable $R_0R_1R_2\cdots R_a$-cyclic DNA codes are mapped to DNA codes with some examples.

Key Words: $R_0R_1R_2\cdots R_a$-cyclic code, separable $R_0R_1R_2\cdots R_a$ cyclic code, generator polynomial, Gray map, DNA code.

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§1. Introduction

It is well known that DNA contains a genetic program for the biological development of life and has two strands which are linked by Watson-Crick pairing so that every A is linked with a T and every C with a G, and vice versa, where A,T,C,G are the four bases of a DNA sequence.

DNA computing started in 1994 when Adleman showed how to solve a computationally difficult problem (traveling salesman problem, a well-known NP-complete problem) by manipulations of DNA molecules in [2]. A DNA code C of length n is a subset of $S_{D_4}^n$, where $S_{D_4}=\{A,T,C,G\}$ is the DNA alphabet. Moreover a DNA code satisfies some constraints such as the Hamming constraint for minimum distance, the reverse-complement constraint, the reverse constraint, and the fixed GC content.

Designing the DNA codes for DNA computing has been a major topic of research since the beginning of the century. The authors used a lot of methods to obtain them. One of the methods is to use skew cyclic codes over some finite rings. In [4], they introduced the $F_4(F_4+vF_4)$-skew cyclic codes, where $v^2=v$. A characterization of the $F_4(F_4+vF_4)$-skew

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[2] *Corresponding author

cyclic codes, which are reversible complements, has been obtained.

The other method is to use the cyclic DNA codes. In [7], they introduced the F_4RS- cyclic codes, where $R = F_4 + uF_4, u^2 = u$, $S = F_4 + uF_4 + vF_4$ with $u^2 = u, v^2 = v, uv = vu = 0$ in order to construct the DNA code. They gave a one-to-one correspondence between DNA codons of the alphabet

$$\{A, C, T, G\}^2, \{A, C, T, G\}^3,$$

by using the Gray maps from R to F_4^2 and S to F_4^3, respectively. By using the structures of separable codes, they studied cyclic DNA codes. They constructed DNA codes from them.

Motivated by the work in [7], we decided to study separable $R_0R_1R_2\cdots R_a$-cyclic DNA codes to construct DNA codes.

This paper is organized as follows. Section 2 gives some basic knowledge and in Section 3, the structures of linear codes over R_i, for $i = 1, 2, \cdots, a$ and the Gray maps are presented. Section 4 gives the structures of $R_0R_1R_2\cdots R_a$-cyclic codes, and a necessary and sufficient conditions of $R_0R_1R_2\cdots R_a$-cyclic to be separable are determined. Section 5 determines a necessary and sufficient condition of cyclic codes over R_i to be reversible and reversible complement for $i = 2, 3, \cdots, a$, obtain a necessary and sufficient condition of separable $R_0R_1R_2\cdots R_a$-cyclic to be reversible and reversible complement. It is obtained the DNA codes by using a map and these type codes with some examples.

§2. **Preliminaries**

Let $R_0 = F_4 = \{0, 1, w, w^2 = w + 1\}$. A family of the finite rings $R_i = F_4[u_1, \cdots, u_i]/ < u_j^2 - u_j, u_ju_s - u_su_j >$, where $j, s = 1, 2, \cdots, i$ and $j \neq s$ for $i = 1, 2, \cdots, a$ is studied in [5]. If $i = 1$, then $R_1 = F_4 + u_1F_4$, where $u_1^2 = u_1$. If $i = 2$, the $R_2 = F_4 + u_1F_4 + u_2F_4 + u_1u_2F_4$, where $u_1^2 = u_1, u_2^2 = u_2, u_1u_2 = u_2u_1$. The rings in this family contain the commutative ring with 4^{2^i} elements and the characteristic 2, where $i = 1, 2, \cdots, a$.

Let $B \subseteq \{1, 2, \cdots, i\}$ and $u_B = \Pi_{j\in B}u_j$, where $i = 1, 2, \cdots, a$. In particular $u_\emptyset = 1$. Each element of R_i is of the form $\sum_{B\in P_i} \alpha_B u_B$, where $\alpha_B \in F_4, P_i$ is the power set of the set $\{1, 2, \cdots, i\}$ for $i = 1, \cdots, a$. For $A, B \subseteq \{1, 2, \cdots, i\}$, we have that $u_Au_B = u_{A\cup B}$ which gives that

$$\sum_{B\in P_i} \alpha_B u_B . \sum_{C\in P_i} \beta_C u_C = \sum_{D\in P_i} \left(\sum_{B\cup C=D} \alpha_B\beta_C \right) u_D$$

for integers $i = 1, \cdots, a$.

The finite rings of the family are also written as recursively

$$R_i = R_{i-1} + u_iR_{i-1},$$

where $i = 1, 2, \cdots, a$.

The set $R_0R_1R_2\cdots R_a = \{(r_0, \cdots, r_a)|r_i \in R_i, i = 0, \cdots, a\}$ forms an R_a module under the componentwise addition and the following multiplication. For any elements $z \in R_a, r =$

$(r_0,\cdots,r_a)\in R_0R_1R_2\cdots R_a$, the multiplication is defined as

$$\begin{aligned}\bullet \quad &: \quad R_a\times R_0R_1R_2...R_a \longrightarrow R_0R_1R_2\cdots R_a\\ (z,r) \quad &\mapsto \quad z\bullet r=(\rho_0(z)r_0,\cdots,\rho_{a-1}(z)r_{a-1},zr_a),\end{aligned}$$

where

$$\begin{aligned}\rho_i \quad &: \quad R_a\longrightarrow R_i\\ \sum_{B\in P_a}\alpha_B u_B \quad &\mapsto \quad \sum_{B\in P_i}\alpha_B u_B\end{aligned}$$

ring homomorphisms for $i=0,\cdots,a-1$ and

$$\begin{aligned}\rho_0 \quad &: \quad R_a\longrightarrow R_0\\ \sum_{B\in P_a}\alpha_B u_B \quad &\mapsto \quad \alpha_\emptyset.\end{aligned}$$

This multiplication can be extended componentwise on $R_{\alpha_0\alpha_1..\alpha_a}=R_0^{\alpha_0}\times R_1^{\alpha_1}\times\cdots\times R_a^{\alpha_a}$ as for any $z\in R_a$ and $\mathbf{r}=(\mathbf{r}_0,\mathbf{r}_1,\cdots,\mathbf{r}_a)\in R_{\alpha_0\alpha_1\cdots\alpha_a}$, where $\mathbf{r}_i=(r_0^i,r_1^i,\cdots,r_{\alpha_i-1}^i)\in R_i^{\alpha_i}$ for $i=0,1,\cdots,a$,

$$\begin{aligned}\bullet \quad &: \quad R_a\times R_{\alpha_0\alpha_1..\alpha_a}\longrightarrow R_{\alpha_0\alpha_1\cdots\alpha_a}\\ (z,\mathbf{r}) \quad &\mapsto \quad z\bullet\mathbf{r}=(\rho_0(z)\mathbf{r_0},\rho_1(z)\mathbf{r_1},\cdots,\rho_{a-1}(z)\mathbf{r}_{a-1},z\mathbf{r}_a)\end{aligned}$$

So $R_{\alpha_0\alpha_1\cdots\alpha_a}$ is an R_a-module. A non empty subset C of $R_{\alpha_0\alpha_1..\alpha_a}$ is called an $R_0R_1R_2\cdots R_a$-linear code of block length $(\alpha_0,\cdots,\alpha_a)$ if C is an R_a- submodule of $R_{\alpha_0\alpha_1\cdots\alpha_a}$.

Definition 2.1 *An $R_0R_1R_2\cdots R_a$-linear code C of block length $(\alpha_0,\cdots,\alpha_a)$ is called an $R_0R_1R_2\cdots R_a$-cyclic code if its cyclic shift $\sigma(\mathbf{r})=(\sigma(\mathbf{r}_0),\sigma(\mathbf{r}_1),\cdots,\sigma(\mathbf{r}_a))\in C$, for any $\mathbf{r}=(\mathbf{r}_0,\mathbf{r}_1,\cdots,\mathbf{r}_a)\in C$.*

Definition 2.2 *Let $\mathbf{r}=(r_0^0,r_1^0,\cdots,r_{\alpha_0-1}^0,r_0^1,r_1^1,\cdots,r_{\alpha_1-1}^1,\cdots,r_0^a,r_1^a,\cdots,r_{\alpha_a-1}^a)$ and $\mathbf{r}'=((r_0^0)',(r_1^0)',\cdots,(r_{\alpha_0-1}^0)',(r_0^1)',(r_1^1)',\cdots,(r_{\alpha_1-1}^1)',\cdots,(r_0^a)',(r_1^a)',\cdots,(r_{\alpha_a-1}^a)')$ be two elements of $R_{\alpha_0\alpha_1\cdots\alpha_a}$. Then, the inner product is defined as*

$$\begin{aligned}\mathbf{r}\cdot\mathbf{r}' \quad = \quad & u_1u_2\cdots u_a\sum_{j_0=0}^{\alpha_0-1}r_{j_0}^0(r_{j_0}^0)'+u_2u_3\cdots u_a\sum_{j_1=0}^{\alpha_1-1}r_{j_1}^1(r_{j_1}^1)'\\ & +\cdots+u_a\sum_{j_{a-1}=0}^{\alpha_{a-1}-1}r_{j_{a-1}}^{a-1}(r_{j_{a-1}}^{a-1})'+\sum_{j_a=0}^{\alpha_a-1}r_{j_a}^a(r_{j_a}^a)'\end{aligned}$$

Theorem 2.3 *Let C be an $R_0R_1R_2\cdots R_a$-cyclic code of block length $(\alpha_0,\cdots,\alpha_a)$. Then, its dual $C^\perp$ is also an $R_0R_1R_2\cdots R_a$-cyclic code.*

Proof For any $\mathbf{r}' = (\mathbf{r}'_0, \mathbf{r}'_1, \cdots, \mathbf{r}'_a) \in C^\perp$, we have to show that $\sigma(\mathbf{r}') \in C^\perp$. For this, take $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_a) \in C$, where C is a $R_0R_1R_2\cdots R_a$-cyclic code of block length $(\alpha_0, \cdots, \alpha_a)$, then we have

$$\begin{aligned}\mathbf{r}\sigma(\mathbf{r}') &= u_1u_2\cdots u_a \sum \mathbf{r}_0\sigma(\mathbf{r}'_0) + u_2u_3\cdots u_a \sum \mathbf{r}_1\sigma(\mathbf{r}'_1) \\ &\quad + \cdots + u_a \sum \mathbf{r}_{a-1}\sigma(\mathbf{r}'_{a-1}) + \sum \mathbf{r}_a\sigma(\mathbf{r}'_a).\end{aligned}$$

Since C is an $R_0R_1R_2\cdots R_a$-cyclic, if we take $lcm(\alpha_0, \cdots, \alpha_a) = d$, then we have $\sigma^{d-1}(\mathbf{r}) = (\sigma^{d-1}(\mathbf{r}_0), \cdots, \sigma^{d-1}(\mathbf{r}_0)) \in C$, for any $\mathbf{r} \in C$. By taking the inner product of $\sigma^{d-1}(\mathbf{r}) \in C$ and $\mathbf{r}' \in C^\perp$, we get

$$\begin{aligned}\sigma^{d-1}(\mathbf{r})\mathbf{r}' &= u_1u_2\cdots u_a \sum \sigma^{d-1}(\mathbf{r}'_0)\mathbf{r}'_0 + u_2u_3\cdots u_a \sum \sigma^{d-1}(\mathbf{r}'_1)\mathbf{r}'_1 \\ &\quad + \cdots + u_a \sum \sigma^{d-1}(\mathbf{r}'_{a-1})\mathbf{r}'_{a-1} + \sum \sigma^{d-1}(\mathbf{r}'_a)\mathbf{r}'_a.\end{aligned}$$

By comparing the coefficients, we have

$$\sigma^{d-1}(\mathbf{r}'_i)\mathbf{r}'_i = \mathbf{r}_i\sigma(\mathbf{r}'_i),$$

where $i = 0, 1, \cdots, a$. Hence $\mathbf{r}\sigma(\mathbf{r}') = 0$, which shows that $\sigma(\mathbf{r}') \in C^\perp$. □

With the map $\Upsilon_{\alpha_0\alpha_1\cdots\alpha_a}$ from $R_{\alpha_0\alpha_1\cdots\alpha_a}$ to

$$\Omega_{\alpha_0\cdots\alpha_a} = R_0[x]/\langle x^{\alpha_0} - 1\rangle \times \cdots \times R_a[x]/\langle x^{\alpha_a} - 1\rangle,$$

and to any element $\mathbf{r} = (r^0_0, r^0_1, \cdots, r^0_{\alpha_0-1}, r^1_0, r^1_1, \cdots, r^1_{\alpha_1-1}, \cdots, r^a_0, r^a_1, \cdots, r^a_{\alpha_a-1}) \in R_{\alpha_0\alpha_1\cdots\alpha_a}$ corresponds to the element $\mathbf{r}(x) = (r^0_0 + r^0_1x + \cdots + r^0_{\alpha_0-1}x^{\alpha_0-1}, r^1_0 + r^1_1x + \cdots + r^1_{\alpha_1-1}x^{\alpha_1-1}, \cdots, r^a_0 + r^a_1x + \cdots + r^a_{\alpha_a-1}x^{\alpha_a-1}) = (r_0(x), r_1(x), \cdots, r_a(x)) \in \Omega_{\alpha_0\cdots\alpha_a}$.

The multiplication of any element $f(x) = f_0 + f_1x + \cdots + f_sx^s \in R_a[x]$ with the element $\mathbf{r}(x) = (r_0(x), \cdots, r_a(x)) \in \Omega_{\alpha_0\cdots\alpha_a}$ is defined as

$$f(x) \star (r_0(x), \cdots, r_a(x)) = (\rho_0(f(x))r_0(x), \cdots, \rho_{a-1}(f(x))r_{a-1}(x), f(x)r_a(x))$$

where $\rho_i(f(x)) = \rho_i(f_0) + \rho_i(f_1)x + \cdots + \rho_i(f_s)x^s$ for $i = 0, \cdots, a-1$. So the set $\Omega_{\alpha_0,\cdots,\alpha_a}$ forms an $R_a[x]$-module with respect to usual addition an multiplication $\star$. For any $\mathbf{r}(x) \in \Omega_{\alpha_0\cdots\alpha_a}$, the vector $x \star \mathbf{r}(x)$ is a cyclic shift of the corresponding vector of $\mathbf{r}(x)$.

Theorem 2.4 *A linear code C is called an $R_0R_1R_2\cdots R_a$-cyclic code of block length $(\alpha_0, \cdots, \alpha_a)$ if and only if $\Upsilon_{\alpha_0\alpha_1\cdots\alpha_a}(C)$ is an $R_a[x]$-submodule of $\Omega_{\alpha_0\cdots\alpha_a}$.*

§3. Linear Codes Over R_i

In [6], an idempotent decomposition of the finite ring $B_i = F_{p^r}[u_1, \cdots, u_i]/ < u_j^2 - u_j, u_ju_s - u_su_j >$ for $j, s = 1, \cdots, i$ was given. By taking $p = 2, r = 2$, a idempotent decomposition of R_i

is written as follows, where $i = 1, \cdots, a$.

$$e_{u_\emptyset} = e_1 = 1 + \sum_{B\in P_i} u_B$$

and the number of $e_{u_\emptyset}$ is $\binom{i}{0}$,

$$e_{u_{k_1}} = u_{k_1} + \sum_{k_1\in B\in P_i, |B|\geq 2} u_B$$

for $k_1 = 1, 2, \cdots, i$ and the number of $e_{u_{k_1}}$ is $\binom{i}{1}$.

$$e_{u_{k_1}u_{k_2}} = \begin{matrix} u_{k_1}u_{k_2} \\ k_1 < k_2 \end{matrix} + \sum_{k_1,k_2\in B\in P_i, |B|\geq 3} u_B$$

for $k_1, k_2 = 1, 2, \cdots, i$ and the number of $e_{u_{k_1}u_{k_2}}$ is $\binom{i}{2}$,

$$e_{u_{k_1}u_{k_2}u_{k_3}} = \begin{matrix} u_{k_1}u_{k_2}u_{k_3} \\ k_1 < k_2 < k_3 \end{matrix} + \sum_{k_1,k_2,k_3\in B\in P_i, |B|\geq 4} u_B$$

for $k_1, k_2, k_3 = 1, 2, \cdots, i$ and the number of $e_{u_{k_1}u_{k_2}u_{k_3}}$ is $\binom{i}{3}$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$$e_{u_1u_2\cdots u_i} = u_1u_2\cdots u_i$$

and the number of $e_{u_1u_2\cdots u_i}$ is $\binom{i}{i}$, where $B \subseteq \{1, 2, \cdots, i\}$ and P_i is the power set of $\{1, 2, \cdots, i\}, i = 1, 2, \cdots, a$.

Then, we have

$$\sum_{B\in P_i} e_{u_B} = 1, \quad (e_{u_B})^2 = e_{u_B} \quad \text{and} \quad e_{u_B}e_{u_A} = 0$$

if $A \neq B$ for any $A, B \subseteq \{1, 2, \cdots, i\}$. Hence,

$$R_i = \bigoplus_{B\in P_i} R_i e_{u_B} \cong \bigoplus_{B\in P_i} F_4 e_{u_B}$$

for $i = 1, \cdots, a$. So every element r_i of R_i can be uniquely expressed as

$$r_i = \sum_{B\in P_i} r_{i,u_B} e_{u_B},$$

where $r_{i,u_B} \in F_4$.

Example 3.1 Let $a = 3$. Then, $R_3 = F_4 + u_1F_4 + u_2F_4 + u_3F_4 + u_1u_2F_4 + u_1u_3F_4 + u_2u_3F_4 + u_1u_2u_3F_4$. Consider the idempotent elements of R_3 below

$$\begin{aligned}
e_{u_\emptyset} &= e_1 = 1 + u_1 + u_2 + u_3 + u_1u_2 + u_1u_3 + u_2u_3 + u_1u_2u_3, \\
e_{u_1} &= u_1 + u_1u_2 + u_1u_3 + u_1u_2u_3, \\
e_{u_2} &= u_2 + u_1u_2 + u_2u_3 + u_1u_2u_3, \\
e_{u_3} &= u_3 + u_1u_3 + u_2u_3 + u_1u_2u_3
\end{aligned}$$

and

$$\begin{aligned}
e_{u_1u_2} &= u_1u_2 + u_1u_2u_3, \quad e_{u_1u_3} = u_1u_3 + u_1u_2u_3, \\
e_{u_2u_3} &= u_2u_3 + u_1u_2u_3, \quad e_{u_1u_2u_3} = u_1u_2u_3.
\end{aligned}$$

Moreover, any element r_3 of R_3 is written as

$$r_3 = r_{3,1}e_1 + r_{3,u_1}e_{u_1} + r_{3,u_2}e_{u_2} + \cdots + r_{3,u_1u_2u_3}e_{u_1u_2u_3}.$$

If C_{α_i} is a linear code of length α_i over R_i , the C_{α_i} is written as

$$C_{\alpha_i} = \bigoplus_{B \in P_i} C_{\alpha_i,u_B} e_{u_B},$$

where C_{α_i,u_B} are codes over F_4 for integers $i = 1, 2, \cdots, a$ with

$$\begin{aligned}
C_{\alpha_i,u_\emptyset} &= C_{\alpha_i,1} = \big\{\mathbf{r}_{i,u_\emptyset} \in F_4^{\alpha_i} | \exists \mathbf{r}_{i,u_B}, B \neq \emptyset \in F_4^{\alpha_i}, \sum_{B \in P_i} \mathbf{r}_{i,u_B} e_{u_B} \in C_{\alpha_i}\big\}, \\
C_{\alpha_i,u_1} &= \{\mathbf{r}_{i,u_1} \in F_4^{\alpha_i} | \exists \mathbf{r}_{i,u_B,B\neq\{1\}} \in F_4^{\alpha_i}, \sum_{B \in P_i} \mathbf{r}_{i,u_B} e_{u_B} \in C_{\alpha_i}\}, \\
C_{\alpha_i,u_2} &= \{\mathbf{r}_{i,u_2} \in F_4^{\alpha_i} | \exists \mathbf{r}_{i,u_B,B\neq\{2\}} \in F_4^{\alpha_i}, \sum_{B \in P_i} \mathbf{r}_{i,u_B} e_{u_B} \in C_{\alpha_i}\}, \\
\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
C_{\alpha_i,u_i} &= \{\mathbf{r}_{i,u_i} \in F_4^{\alpha_i} | \exists \mathbf{r}_{i,u_B,B\neq\{i\}} \in F_4^{\alpha_i}, \sum_{B \in P_i} \mathbf{r}_{i,u_B} e_{u_B} \in C_{\alpha_i}\}
\end{aligned}$$

and

$$\begin{aligned}
C_{\alpha_i,u_1u_2} &= \{\mathbf{r}_{i,u_1u_2} \in F_4^{\alpha_i} | \exists \mathbf{r}_{i,u_B,B\neq\{1,2\}} \in F_4^{\alpha_i}, \sum_{B \in P_i} \mathbf{r}_{i,u_B} e_{u_B} \in C_{\alpha_i}\}, \\
\cdots & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
C_{\alpha_i,u_1u_2\cdots u_i} &= \{\mathbf{r}_{i,u_1u_2\cdots u_i} \in F_4^{\alpha_i} | \exists \mathbf{r}_{i,u_B,B\neq\{1,\cdots,i\}} \in F_4^{\alpha_i}, \sum_{B \in P_i} \mathbf{r}_{i,u_B} e_{u_B} \in C_{\alpha_i}\}
\end{aligned}$$

where $i = 1, \cdots, a$. And moreover,

$$|C_{\alpha_i}| = \prod_{B\in P_i} |C_{\alpha_i,u_B}| \text{ and } d_G(C_{\alpha_i}) = min\{d_H(C_{\alpha_i,u_B})\}$$

for $i = 1,\cdots,a$. In [5], the map on R_i was defined as

$$\begin{aligned} \phi_i \quad &: \quad R_i \longrightarrow R_{i-1}^2 \\ x_{i-1}+u_iy_{i-1} \quad &\longmapsto \quad (x_{i-1}, x_{i-1}+y_{i-1}) \end{aligned}$$

where $i = 2,\cdots,a$ and

$$\begin{aligned} \phi_1 \quad &: \quad R_1 \longrightarrow R_0^2 \\ x_0+u_1y_0 \quad &\longmapsto \quad (x_0, x_0+y_0) \end{aligned}$$

So, the Gray map is defined as follows

$$\begin{aligned} \phi = \phi_1\phi_2\cdots\phi_i \quad &: \quad R_i \longrightarrow R_0^{2^i} \\ x_{i-1}+u_iy_{i-1} \quad &\longmapsto \quad \phi\left(x_{i-1}+u_iy_{i-1}\right) \end{aligned}$$

where $i = 1,\cdots,a$.

If we take the element of R_i as $\sum_{B\in P_i} r_{i,u_B}e_{u_B}$, the Gray map defined also as

$$\begin{aligned} \psi_i \quad &: \quad R_i \longrightarrow R_0^{2^i} \\ \sum_{B\in P_i} r_{i,u_B}e_{u_B} \quad &\longmapsto \quad \left(r_{i,u_B}\right)_{B\in P_i} \end{aligned}$$

where $i = 1,\cdots,a$.

Example 3.2 Let $a = 2$. The Gray map on R_2 is defined as

$$\begin{aligned} \psi_2 \quad &: \quad R_2 \longrightarrow R_0^4 \\ \sum_{B\in P_2} r_{2,u_B}e_{u_B} \quad &\longmapsto \quad (r_{2,1}, r_{2,u_1}, r_{2,u_2}, r_{2,u_1u_2}) \end{aligned}$$

This can be extended from $R_i^{\alpha_i}$ to $R_0^{2^i\alpha_i}$ as

$$\begin{aligned} \psi_i \quad &: \quad R_i^{\alpha_i} \longrightarrow R_0^{2^i\alpha_i} \\ \mathbf{r}_i = (r_0^i, r_1^i, \cdots, r_{\alpha_i-1}^i) \quad &\longmapsto \quad (r_{j,i,u_B})_{B\in P_i} \end{aligned}$$

where $(r_0^i, r_1^i, \cdots, r_{\alpha_i-1}^i) \in R_i^{\alpha_i}$ and $r_j^i = \sum_{B\in P_i} r_{j,i,u_B}e_{u_B}$, for $j = 0,\cdots,\alpha_i-1, i = 1,\cdots,a$.

We define the Gray weight $wt_G\left(r_j^i\right)$ of an element $r_j^i \in R_i$ is defined as

$$wt_G\left(r_j^i\right) \;=\; wt_H\left(\psi_i\left(r_j^i\right)\right),$$

where wt_H denotes the Hamming weight over F_4, $j = 0, \cdots, \alpha_i - 1, i = 1, \cdots, a$. For any $\mathbf{r}_i, \mathbf{r}'_i \in R_i^{\alpha_i}$, the Gray distance between $\mathbf{r}_i, \mathbf{r}'_i$ is defined as

$$d_G(\mathbf{r}_i, \mathbf{r}'_i) = w_H(\psi_i(\mathbf{r}_i - \mathbf{r}'_i)).$$

It can be easily seen that the Gray map is a F_4-linear and distance preserving map from $R_i^{\alpha_i}$(Gray distance) to $F_4^{2^i\alpha_i}$ (Hamming distance) where $i = 1, \cdots, a$.

Notice that a linear code over F_4 of length α_0 is a subspace of $F_4^{\alpha_0}$ and a linear code over R_i of length α_i is a R_i-submodule of $R_i^{\alpha_i}$ for $i = 1, 2, \cdots, a$.

For any element $(r_0, \cdots, r_a) = (r_0, r_{1,1}(1 + u_1) + r_{1,u_1}(u_1), r_{2,1}(1 + u_1 + u_2 + u_1u_2) + r_{2,u_1}(u_1 + u_1u_2) + r_{2,u_2}(u_2 + u_1u_2) + r_{2,u_1u_2}(u_1u_2), \cdots, \sum_{B\in P_a} r_{a,u_B}e_{u_B}) \in R_0R_1R_2\cdots R_a$, we define a Gray map from $R_0R_1R_2\ldots R_a$ to $F_4^{2^{a+1}-1}$ as

$$\begin{aligned} \varphi \quad &: \quad R_0R_1R_2\cdots R_a \longrightarrow F_4^{2^{a+1}-1} \\ (r_0, \cdots, r_a) \quad &\longmapsto \quad \varphi(r_0, \cdots, r_a) = (r_0, \psi_1(r_1), \cdots, \psi_a(r_a)) = \epsilon, \end{aligned}$$

where

$$\epsilon = (r_0, r_{1,1}, r_{1,u_1}, r_{2,1}, r_{2,u_1}, r_{2,u_2}, r_{2,u_1u_2}, \cdots, r_{a,1}, r_{a,u_1}, \cdots, r_{a,u_a}, r_{a,u_1u_2}, \cdots, r_{a,u_1u_2,\cdots,u_a}).$$

It is seen that the map φ is a F_4 -linear, which can be extended on $R_{\alpha_0\alpha_1\cdots\alpha_a}$ as follow

$$\varphi \ : \ R_{\alpha_0\alpha_1\cdots\alpha_a} \longrightarrow F_4^{\sum_{i=0}^a 2^i\alpha_i}$$

$$(\mathbf{r}_0, \cdots, \mathbf{r}_a) = (r_0^0, \cdots r_{\alpha_0-1}^0, r_0^1, \cdots, r_{\alpha_1-1}^1, \cdots, r_0^a, \cdots, r_{\alpha_a-1}^a) \mapsto \varepsilon$$

with $\varepsilon = (r_0^0, \cdots r_{\alpha_0-1}^0, r_{0,1,1}, r_{0,1,u_1}, r_{1,1,1}, r_{1,1,u_1}, \cdots, (r_{j,a,u_B})_{B\in P_a})$, $(r_0^i, \cdots, r_{\alpha_i-1}^i) \in R_i^{\alpha_i}$ and

$$r_j^i \ = \ \sum_{B\in P_i} r_{j,i,u_B}e_{u_B},$$

for $j = 0, \cdots, \alpha_i - 1, i = 1, \cdots, a$.

The Gray weight of any element $(\mathbf{r}_0, \cdots, \mathbf{r}_a) \in R_{\alpha_0\alpha_1..\alpha_a}$ is defined as $w_G((\mathbf{r}_0, \cdots, \mathbf{r}_a))$ $= w_H(\mathbf{r}_0) + w_G(\mathbf{r}_1) + \cdots + w_G(\mathbf{r}_a)$ where w_H represents the Hamming weight over F_4. Moreover, it can be defined Gray distance between any two elements $\mathbf{r}, \acute{\mathbf{r}} \in R_{\alpha_0\alpha_1\cdots\alpha_a}$ as

$$d_G(\mathbf{r}, \acute{\mathbf{r}}) = w_G(\mathbf{r} - \acute{\mathbf{r}})$$

Proposition 3.3 *A Gray map φ is F_4-linear and distance preserving map from $R_{\alpha_0\alpha_1\cdots\alpha_a}$(Gray distance) to $F_4^{\sum_{i=0}^a 2^i\alpha_i}$ (Hamming distance).*

Proof Let $\mathbf{r} = (\mathbf{r}_0, \cdots, \mathbf{r}_a), \mathbf{r}' = (\mathbf{r}'_0, \cdots, \mathbf{r}'_a) \in R_{\alpha_0\cdots\alpha_a}$, where $\mathbf{r}_0 = (r_0, \cdots, r_{\alpha_0-1}), \mathbf{r}_0 =$

$(r'_0,\cdots,r'_{\alpha_0-1})\in F_4^{\alpha_0}$, $\mathbf{r}_1=\mathbf{r}_{1,1}e_1+\mathbf{r}_{1,u_1}e_{u_1},\mathbf{r}'_1=\mathbf{r}'_{1,1}e_1+\mathbf{r}'_{1,u_1}e_{u_1}\in R_1^{\alpha_1}$, where

$$\begin{aligned}
\mathbf{r}_{1,1} &= (r_{0,1,1},\cdots,r_{\alpha_1-1,1,1}),\\
\mathbf{r}_{1,u_1} &= (r_{0,1,u_1},\cdots,r_{\alpha_1-1,1,u_1}),\\
\mathbf{r}'_{1,1} &= (r'_{0,1,1},\cdots,r'_{\alpha_1-1,1,1}),\\
\mathbf{r}'_{1,u_1} &= (r'_{0,1,u_1},\cdots,r'_{\alpha_1-1,1,u_1})\in F_4^{\alpha_1},\\
\cdots\ &\cdots\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\mathbf{r}_a &= \sum_{B\in P_a}\mathbf{r}_{a,u_B}e_{u_B}\quad\text{and}\quad \mathbf{r}'_a=\sum_{B\in P_a}\mathbf{r}'_{a,u_B}e_{u_B}\in R_a^{\alpha_a},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{r}_{a,1} &= (r_{0,a,1},\cdots,r_{\alpha_a-1,a,1}),\\
\mathbf{r}_{a,u_1} &= (r_{0,a,u_1},\cdots,r_{\alpha_a-1,a,u_1}),\\
\cdots\ &\cdots\ \cdots\cdots\cdots\cdots\cdots,\\
\mathbf{r}_{a,u_1\cdots u_a} &= (r_{0,a,u_1\cdots u_a},\cdots,r_{\alpha_a-1,a,u_1\cdots u_a}),\\
\mathbf{r}'_{a,1} &= (r'_{0,a,1},\cdots,r'_{\alpha_a-1,a,1}),\\
\mathbf{r}'_{a,u_1} &= (r'_{0,a,u_1},\cdots,r'_{\alpha_a-1,a,u_1}),\\
\cdots\ &\cdots\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
\mathbf{r}'_{a,u_1\cdots u_a} &= (r'_{0,a,u_1\cdots,\alpha_a},\cdots,r'_{\alpha_a-1,a,u_1,\cdots,u_a})\in F_4^{\alpha_a}
\end{aligned}$$

For any $\mathbf{r},\mathbf{r}'\in R_{\alpha_0\cdots\alpha_a},\lambda\in F_4$, it is hold that $\varphi(\mathbf{r}+\mathbf{r}')=\varphi(\mathbf{r})+\varphi(\mathbf{r}')$ and $\varphi(\lambda\mathbf{r})=\lambda\varphi(\mathbf{r})$. By using the map φ is F_4 linear, we get $d_G(\mathbf{r},\mathbf{r}')=\mathbf{w_G}(\mathbf{r}-\mathbf{r}')=\mathbf{w_H}(\varphi(\mathbf{r}-\mathbf{r}'))=\mathbf{d_H}(\varphi(\mathbf{r}),\varphi(\mathbf{r}'))$. □

Proposition 3.4 *If C is an $R_0R_1R_2\cdots R_a$-cyclic code of block length $(\alpha_0,\cdots,\alpha_a)$ with $|C|=M$ and minimum distance d_G, them $\varphi(C)$ is a $(\sum_{i=0}^a 2^i\alpha_i,M,d_H)$ linear code over F_4, where $d_G=d_H$.*

Proof By using the fact that the map φ is F_4 linear, distance preserving and bijective, we have $\varphi(C)$ is a

$$(\sum_{i=0}^{a}2^i\alpha_i,M,d_H)$$

linear code over F_4. □

§4. Structures of the Cyclic Codes Over R_i and the $R_0R_1R_2\cdots R_a$-Cyclic Codes

With the map ζ_i from $R_i^{\alpha_i}$ to $R_i[x]/\langle x^{\alpha_i}-1\rangle$, to any element $\mathbf{r_i}=(r_0^i,r_1^i,\cdots,r_{\alpha_i-1}^i)\in R_i^{\alpha_i}$ corresponds to the elements $\mathbf{r_i}(x)=r_0^i+r_1^ix+\cdots+r_{\alpha_i-1}^ix^{\alpha_i-1}\in R_i[x]/\langle x^{\alpha_i}-1\rangle$. If C_{α_i} is a cyclic code over R_i of length α_i, then $\zeta_i(C_{\alpha_i})$ is an ideal in $R_i[x]/\langle x^{\alpha_i}-1\rangle$, for $i=0,1,\cdots,a$.

Theorem 4.1([8]) *Let C_{α_0} be a cyclic code of length α_0 over F_4. Then there exists a unique monic polynomial $f(x) \in F_4[x]/\langle x^{\alpha_0}-1\rangle$ such that $\zeta_0(C_{\alpha_0}) = \langle f(x)\rangle$ and $f(x)$ divides $x^{\alpha_0}-1$. Moreover C_{α_0} has 4^{k_1} codewords, where $k_1 = \alpha_0 - degf(x)$ and the set $\{f(x), xf(x), \cdots, x^{k_1-1}f(x)\}$ forms a basis of C_{α_0}.*

In [3], the structures and properties of cyclic codes over R_1 are given. Similarly, we have the following results for cyclic codes over R_i for $i = 2, 3, \cdots, a$.

Theorem 4.2 *Let $C_{\alpha_i} = \oplus_{B\in P_i} C_{\alpha_i,u_B} e_{u_B}$ be a linear code of length α_i over R_i for $i = 2, \cdots, a$. Then,*

1) C_{α_i} is a cyclic code of length α_i over R_i if and only if C_{α_i,u_B} are cyclic codes of length α_i over R_0, for $i = 2, \cdots, a$;

2) If C_{α_i} is a cyclic code of length α_i over R_i, then its dual

$$C_{\alpha_i}^{\perp} = \bigoplus_{B\in P_i} C_{\alpha_i,u_B}^{\perp} e_{u_B}$$

is also cyclic code over R_i, $B \in P_i$, for $i = 2, \cdots, a$;

3) If C_{α_i} is a cyclic code of length α_i over R_i, then $\zeta_i(C_{\alpha_i}) = \langle c_i(x)\rangle$, where

$$c_i(x) = \sum (c_{i,u_B}(x)) e_{u_B}$$

with $c_i(x)|x^{\alpha_i}-1$ and $\zeta_0(C_{\alpha_i,u_B}) = \langle c_{i,u_B}(x)\rangle$, $B \in P_i$, for $i = 2, \cdots, a$. Moreover

$$|C_{\alpha_i}| = 4^{2^i \alpha_i - \sum deg(c_{i,u_B}(x))}.$$

From [4], the following theorem can be given as follows:

Theorem 4.3 *Let C be an R_0R_1-cyclic code of block length (α_0, α_1). Then, $\Upsilon_{\alpha_0\alpha_1}(C) = \langle (f_{0,0}(x), 0), (f_{0,1}(x), f_{1,0}(x))\rangle$ where $f_{0,0}(x)|x^{\alpha_0}-1$, $f_{1,0}(x) = (1+u_1)f_{1,0,1}(x) + u_1 f_{1,0,u_1}(x)$, $f_{1,0}(x)|x^{\alpha_1}-1$ and $f_{0,1}(x) \in R_0[x]$.*

Theorem 4.4 *Let C be an $R_0R_1R_2$-cyclic code of block length $(\alpha_0, \alpha_1, \alpha_2)$. Then $\Upsilon_{\alpha_0\alpha_1\alpha_2}(C) = \langle (f_{0,0}(x), 0, 0), (f_{0,1}(x), f_{1,0}(x), 0), (f_{0,2}(x), f_{1,1}(x), f_{2,0}(x))\rangle$, where $f_{0,0}(x)|x^{\alpha_0}-1$, $f_{1,0}(x) = (1+u_1)f_{1,0,1}(x) + u_1 f_{1,0,u_1}(x)$, $f_{1,0}(x)|x^{\alpha_1}-1$, $f_{2,0}(x) = (1+u_1+u_2+u_1u_2)f_{2,0,1}(x) + (u_1 + u_1u_2)f_{2,0,u_1}(x) + (u_2+u_1u_2)f_{2,0,u_2}(x) + (u_1u_2)f_{2,0,u_1u_2}(x)$ with $f_{2,0}(x)|x^{\alpha_2}-1$ and $f_{0,1}(x), f_{0,2}(x) \in R_0[x]$, $f_{1,1}(x) \in R_1[x]$.*

Proof By defining a homomorphism $\Xi_{\alpha_0\alpha_1\alpha_2}$ between $\Upsilon_{\alpha_0\alpha_1\alpha_2}(C)$ and $R_2[x]/\langle x^{\alpha_2}-1\rangle$ by $\Xi_{\alpha_0\alpha_1\alpha_2}(r_0(x), r_1(x), r_2(x)) = r_2(x)$, we can obtain $\Xi_{\alpha_0\alpha_1\alpha_2}(\Upsilon_{\alpha_0\alpha_1\alpha_2}(C)) = \langle f_{2,0}(x)\rangle$, where

$$\begin{aligned} f_{2,0}(x) &= (1+u_1+u_2+u_1u_2)f_{2,0,1}(x) + (u_1+u_1u_2)f_{2,0,u_1}(x) + (u_2+u_1u_2)f_{2,0,u_2}(x) \\ &\quad +(u_1u_2)f_{2,0,u_1u_2}(x) \end{aligned}$$

and

$$Ker\Xi_{\alpha_0\alpha_1\alpha_2}=\{(r_0(x),r_1(x),0)\in\Omega_{\alpha_0\alpha_1\alpha_2}|(r_0(x),r_1(x),r_2(x))\in\Upsilon_{\alpha_0\alpha_1\alpha_2}(C)\}.$$

By defining R_1-submodule $I_1=\{(r_0(x),r_1(x))\in\Omega_{\alpha_0\alpha_1}|(r_0(x),r_1(x),0)\in Ker\Xi_{\alpha_0\alpha_1\alpha_2}\}$ and by using Theorem 5 in [7], we can say that I_1 has the generator polynomials of the form $\langle(f_{0,0}(x),0),(f_{0,1}(x),f_{1,0}(x))\rangle$ where $f_{0,0}(x)|x^{\alpha_0}-1, f_{1,0}(x)=(1+u_1)f_{1,0,1}(x)+u_1f_{1,0,u_1}(x)$ with $f_{1,0}(x)|x^{\alpha_1}-1$ and $f_{0,1}(x)\in R_0[x]$. Since we have $(r_0(x),r_1(x))\in I_1$, for any $(r_0(x),r_1(x),0)\in Ker\Xi_{\alpha_0\alpha_1\alpha_2}$, then there exist some $m_t(x)\in R_t[x]$ for $t=0,1$ such that

$$(r_0(x),r_1(x))=m_0(x)*(f_{0,0}(x),0)+m_1(x)*(f_{0,1}(x),f_{1,0}(x)).$$

Therefore

$$(r_0(x),r_1(x),0)=m_0(x)*(f_{0,0}(x),0,0)+m_1(x)*(f_{0,1}(x),f_{1,0}(x),0)$$

and we have

$$Ker\Xi_{\alpha_0\alpha_1\alpha_2}=\langle(f_{0,0}(x),0,0),(f_{0,1}(x),f_{1,0}(x),0)\rangle\,.$$

From first isomorphism theorem, we have

$$\Upsilon_{\alpha_0\alpha_1\alpha_2}(C)/Ker\Xi_{\alpha_0\alpha_1\alpha_2}\cong\Xi_{\alpha_0\alpha_1\alpha_2}(\Upsilon_{\alpha_0\alpha_1\alpha_2}(C))=\langle f_{2,0}(x)\rangle\,.$$

and let $(f_{0,2}(x),f_{1,1}(x),f_{2,0}(x))\in\Upsilon_{\alpha_0\alpha_1\alpha_2}(C)$ with

$$\Xi_{\alpha_0\alpha_1\alpha_2}(f_{0,2}(x),f_{1,1}(x),f_{2,0}(x))=f_{2,0}(x).$$

So, an $R_0R_1R_2$-cyclic code is generated by elements of form $(f_{0,0}(x),\ 0,0),(f_{0,1}(x),f_{1,0}(x),0)$ and $(f_{0,2}(x),f_{1,1}(x),f_{2,0}(x))$. □

Theorem 4.5 *Let C be an $R_0R_1R_2R_3$-cyclic code of block length $(\alpha_0,\alpha_1,\alpha_2,\alpha_3)$. Then,*

$$\Upsilon_{\alpha_0\alpha_1\alpha_2\alpha_3}(C)=\langle A_1,A_2,A_3,A_4\rangle\,,$$

where

$$\begin{aligned}A_1&=(f_{0,0}(x),0,0,0),\\A_2&=(f_{0,1}(x),f_{1,0}(x),0,0),\\A_3&=(f_{0,2}(x),f_{1,1}(x),f_{2,0}(x),0),\\A_4&=(f_{0,3}(x),f_{1,2}(x),f_{2,1}(x),f_{3,0}(x))\end{aligned}$$

and $f_{0,0}(x)|x^{\alpha_0}-1, f_{1,0}(x)=(1+u_1)f_{1,0,1}(x)+u_1f_{1,0,u_1}(x),\ f_{1,0}(x)|x^{\alpha_1}-1, f_{2,0}(x)=(1+u_1+u_2+u_1u_2)f_{2,0,1}(x)+(u_1+u_1u_2)f_{2,0,u_1}(x)+(u_2+u_1u_2)f_{2,0,u_2}(x)+(u_1u_2)f_{2,0,u_1u_2}(x)$, $f_{2,0}(x)|x^{\alpha_2}-1,\ f_{3,0}(x)=(1+u_1+u_2+u_3+u_1u_2+u_1u_3+u_2u_3+u_1u_2u_3)f_{3,0,1}(x)+(u_1+u_1u_2+u_1u_3+u_1u_2u_3)f_{3,0,u_1}(x)+(u_2+u_1u_2+u_2u_3+u_1u_2u_3)f_{3,0,u_2}(x)+(u_3+u_1u_3+$

$u_2u_3+u_1u_2u_3)f_{3,0,u_3}(x)+(u_1u_2+u_1u_2u_3)f_{3,0,u_1u_2}(x)+(u_1u_3+u_1u_2u_3)f_{3,0,u_1u_3}(x)+(u_2u_3+u_1u_2u_3)f_{3,0,u_2u_3}(x)+u_1u_2u_3f_{3,0,u_1u_2u_3}(x)$ with

$$f_{0,1}(x), f_{0,2}(x), f_{0,3}(x) \in R_0[x], f_{1,1}(x), f_{1,2}(x) \in R_1[x], f_{2,1}(x) \in R_2[x].$$

Proof Similarly, to proof of the Theorem 4.4, by defining a homomorphism $\Xi_{\alpha_0\alpha_1\alpha_2\alpha_3}$ between $\Upsilon_{\alpha_0\alpha_1\alpha_2\alpha_3}(C)$ and $R_3[x]/\langle x^{\alpha_3}-1\rangle$ by $\Xi_{\alpha_0\alpha_1\alpha_2\alpha_3}(r_0(x),r_1(x),r_2(x),r_3(x)) = r_3(x)$, by defining R_2-submodule I_2, the desired result is obtained. □

From all the above discussion, by using the same process an induction on a, we get the following corollary.

Corollary 4.6 *Let C be an $R_0R_1R_2\cdots R_i$-cyclic code of block length $(\alpha_0,\alpha_1,\alpha_2,...\cdots,\alpha_i)$. Then,*

$$\begin{aligned}\Upsilon_{\alpha_0\alpha_1\cdots\alpha_i}(C) \quad = \quad & \big\langle (f_{0,0}(x),0,0,0,\cdots,0),(f_{0,1}(x),f_{1,0}(x),0,0,\cdots,0),(f_{0,2}(x),\\ & f_{1,1}(x),f_{2,0}(x),0,\cdots,0),(f_{0,3}(x),f_{1,2}(x),f_{2,1}(x),f_{3,0}(x),0,\cdots,0),\\ & \cdots,(f_{0,i}(x),f_{1,i-1}(x),f_{2,i-2}(x),f_{3,i-3}(x),\cdots,f_{i,0}(x))\big\rangle,\end{aligned}$$

where $f_{i,0}(x)|x^{\alpha_i}-1$ for $i=0,1,\cdots,a$,

$$f_{i,0}(x) = \sum_{B\in P_i} f_{i,0,u_B}(x)e_{u_B}$$

for $i=1,\cdots,a$ and $f_{i,1}(x),f_{i,2}(x),\cdots,f_{i,(a-i)}(x)\in R_i[x]$ for $i=0,1,2,\cdots,a-1$.

In [7], some lemmas and theorems about the R_0R_1S cyclic codes and the separable codes were given, where $S=F_4+uF_4+vF_4$ and $u^2=u, v^2=v, uv=vu=0$. Similarly, we get the following lemmas and theorems.

Lemma 4.7 *Let*

$$\begin{aligned}\Upsilon_{\alpha_0\alpha_1\cdots\alpha_a}(C) \quad = \quad & \big\langle (f_{0,0}(x),0,0,0,\cdots,0),(f_{0,1}(x),f_{1,0}(x),0,0,\cdots,0),(f_{0,2}(x),\\ & f_{1,1}(x),f_{2,0}(x),0,\cdots,0),(f_{0,3}(x),f_{1,2}(x),f_{2,1}(x),f_{3,0}(x),0,\cdots,0),\\ & \cdots,(f_{0,a}(x),f_{1,a-1}(x),f_{2,a-2}(x),f_{3,a-3}(x),\cdots,f_{a,0}(x))\big\rangle\end{aligned}$$

be an $R_0R_1R_2\cdots R_a$-cyclic code. Then, we assume that $\deg f_{i,j}(x) < \deg f_{i,0}(x)$ *for $i=0,\cdots,a-1$,$j=1,2,\cdots,a-i$.*

An $R_0R_1R_2\cdots R_a$-linear code C of length $(\alpha_0,\alpha_1,\alpha_2,\cdots,\alpha_i)$ is called a separable code if $C=C'_{\alpha_0}\times\cdots\times C'_{\alpha_a}$, while considering C'_{α_i} as punctured codes over C by deleting the coordinates outside the α_i components, for $i=0,\cdots,a$.

Lemma 4.8 *Let* $\Upsilon_{\alpha_0\alpha_1\cdots\alpha_a}(C) = \big\langle (f_{0,0}(x),0,0,0,\cdots,0),(f_{0,1}(x),f_{1,0}(x),0,0,\cdots,0),(f_{0,2}(x),$

$f_{1,1}(x), f_{2,0}(x), 0, \cdots, 0)$, $(f_{0,3}(x)$, $f_{1,2}(x)$, $f_{2,1}(x)$, $f_{3,0}(x), 0, \cdots, 0), \cdots$, $(f_{0,a}(x), f_{1,a-1}(x)$, $f_{2,a-2}(x), f_{3,a-3}(x), \cdots, f_{a,0}(x))\big\rangle$ *be an* $R_0R_1R_2\cdots R_a$*-cyclic code of length* $(\alpha_0, \cdots, \alpha_a)$*. Then,*

$$\begin{aligned}
\zeta_0(C'_{\alpha_0}) &= \langle gcd(f_{0,0}(x), f_{0,1}(x), \cdots, f_{0,a}(x))\rangle\,,\\
\zeta_1(C'_{\alpha_1}) &= \langle gcd(f_{1,0}(x), f_{1,1}(x), \cdots, f_{1,a-1}(x))\rangle\,,\\
\cdots\ &\cdots\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,,\\
\zeta_{a-1}(C'_{\alpha_{a-1}}) &= \langle gcd(f_{a-1,0}(x), f_{a-1,1}(x))\rangle\,,\\
\zeta_a(C'_{\alpha_a}) &= \langle f_{a,0}(x)\rangle\,.
\end{aligned}$$

Lemma 4.9 *Let*

$$\begin{aligned}
\Upsilon_{\alpha_0\alpha_1\cdots\alpha_a}(C) = \big\langle &(f_{0,0}(x), 0, 0, 0, \cdots, 0), (f_{0,1}(x), f_{1,0}(x), 0, 0, \cdots, 0),\\
&(f_{0,2}(x), f_{1,1}(x), f_{2,0}(x), 0, \cdots, 0), (f_{0,3}(x), f_{1,2}(x), f_{2,1}(x),\\
&f_{3,0}(x), 0, \cdots, 0), \cdots, (f_{0,a}(x), f_{1,a-1}(x), f_{2,a-2}(x), f_{3,a-3}(x), \cdots, f_{a,0}(x))\big\rangle
\end{aligned}$$

be an $R_0R_1R_2\cdots R_a$*-cyclic code. Then* $f_{i,0}(x)|f_{i,j}(x)$ *if and only if* $f_{i,j}(x) = 0$ *for* $i = 0, 1, \cdots, a-1, j = 1, 2, \cdots, a-i$.

Theorem 4.10 *Let*

$$\begin{aligned}
\Upsilon_{\alpha_0\alpha_1\cdots\alpha_a}(C) = \big\langle &(f_{0,0}(x), 0, 0, 0, \cdots, 0), (f_{0,1}(x), f_{1,0}(x), 0, 0, \cdots, 0), (f_{0,2}(x),\\
&f_{1,1}(x), f_{2,0}(x), 0, \cdots, 0), (f_{0,3}(x), f_{1,2}(x), f_{2,1}(x), f_{3,0}(x), 0, \cdots, 0),\\
&\cdots, (f_{0,a}(x), f_{1,a-1}(x), f_{2,a-2}(x), f_{3,a-3}(x), \cdots, f_{a,0}(x))\big\rangle
\end{aligned}$$

be an $R_0R_1R_2...R_a$*-cyclic code. Then, the following assertions are equivalent:*

1) *C is a separable code;*
2) $f_{0,0}(x)|f_{0,i}(x)$, $f_{1,0}(x)|f_{1,i-1}(x), \cdots, f_{i-1,0}(x)|f_{i-1,1}(x)$, *for* $i = 1, 2, \cdots, a$;
3) $\zeta_i(C'_{\alpha_i}) = \langle f_{i,0}(x)\rangle$ *for* $i = 0, \cdots, a$;
4) $\Upsilon_{\alpha_0\alpha_1\cdots\alpha_a}(C) = \langle(f_{0,0}(x), 0, \cdots, 0), (0, f_{1,0}(x), \cdots, 0), \cdots, (0, \cdots, 0, f_{a,0}(x))\rangle\,.$

We get $\zeta_i(C^{'}_{\alpha_i}) =< f_{i,0}(x) >= \zeta_i(C_{\alpha_i})$ for $i = 0, \cdots, a$.

Theorem 4.11 *Let* $C = C_{\alpha_0} \times C_{\alpha_1} \times \cdots \times C_{\alpha_a}$ *be an* $R_0R_1R_2\cdots R_a$*-linear code of block length* $(\alpha_0, \cdots, \alpha_a)$*. Then, C is a separable* $R_0R_1R_2\cdots R_a$*-cyclic code of block length* $(\alpha_0, \cdots, \alpha_a)$ *if and only if* C_{α_i} *are cyclic codes of length* α_i *over* R_i*, for* $i = 0, 1, \cdots, a$*, respectively.*

Proof Let C be an $R_0R_1R_2\cdots R_a$-cyclic code of block length $(\alpha_0, \cdots, \alpha_a)$. Take $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_a) \in C$, where $\mathbf{r}_i = (r^i_0, r^i_1, \cdots, r^i_{\alpha_i-1}) \in C_{\alpha_i}$ for $i = 0, 1, \cdots, a$. By using the fact that C is an $R_0R_1R_2\cdots R_a$-cyclic code of block length $(\alpha_0, \cdots, \alpha_a)$, we have

$$(\sigma(\mathbf{r}_0), \sigma(\mathbf{r}_1), \cdots, \sigma(\mathbf{r}_a)) \in C,$$

which shows that $\sigma(\mathbf{r}_i) \in C_{\alpha_i}$ where $i = 0, 1, 2, \cdots, a$. Therefore, C_{α_i} are cyclic codes of length

α_i over R_i, for $i = 0, 1, \cdots, a$, respectively. Conversely, let C_{α_i} be cyclic codes of length α_i over R_i, for $i = 0, 1, \cdots, a$, respectively and $\mathbf{r}_{\alpha_i} \in C_{\alpha_i}$ where $i = 0, 1, \cdots, a$. By using the fact that C_{α_i} are cyclic, then $\sigma(\mathbf{r}_i) \in C_{\alpha_i}$ for $i = 0, 1, \cdots, a$. Take $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_a) \in C$. So $\sigma(\mathbf{r}) = (\sigma(\mathbf{r}_0), \sigma(\mathbf{r}_1), \cdots, \sigma(\mathbf{r}_a)) \in C$. ☐

Theorem 4.12 *Let* $C = C_{\alpha_0} \times \cdots \times C_{\alpha_a}$ *be a separable* $R_0R_1R_2 \cdots R_a$*- cyclic of block length* $(\alpha_0, \cdots, \alpha_a)$*, where* $\zeta_0(C_{\alpha_0}) = \langle f_{0,0}(x) \rangle, \cdots, \zeta_a(C_{\alpha_a}) = \langle f_{a,0}(x) \rangle$*. Then,* $\Upsilon_{\alpha_0\alpha_1\cdots\alpha_a}(C) = \langle f_{0,0}(x) \rangle \times \cdots \times \langle f_{a,0}(x) \rangle$*.*

Example 4.13 Let a be 3. Let $f_{0,0}(x) = x^2 + (w+1)x + w$. Then $\zeta_0(C_{\alpha_0}) = \langle f_{0,0}(x) \rangle$ is a cyclic code over R_0 with length $\alpha_0 = 3$ and $|C_{\alpha_0}| = 4, d = 3$.

Let $f_{1,0,1}(x) = f_{1,0,u_1}(x) = x^2 + (w+1)x + 1$. Then $\zeta_0(C_{\alpha_1,1}) = \zeta_0(C_{\alpha_1,u_1}) = \langle f_{1,0,1}(x) \rangle$ are cyclic codes over R_0 with length 5 and $|C_{\alpha_1,1}| = 4^3, d = 3$. Therefore, $\zeta_1(C_{\alpha_1}) = \langle f_{1,0}(x) \rangle$ is a cyclic code over R_1 with length $\alpha_1 = 5$ with cardinality $4^{10-4} = 4^6, d = 3$.

Let $f_{2,0,1}(x) = f_{2,0,u_1}(x) = x^3 + x + 1, f_{2,0,u_2}(x) = f_{2,0,u_1u_2}(x) = x^3 + x^2 + 1$. Then $\zeta_0(C_{\alpha_2,1}) = \zeta_0(C_{\alpha_2,u_1}) = \langle f_{2,0,1}(x) \rangle, \zeta_0(C_{\alpha_2,u_2}) = \zeta_0(C_{\alpha_2,u_1u_2}) = \langle f_{2,0,u_2}(x) \rangle$ are cyclic codes over R_0 with length 7 and $|C_{\alpha_2,1}| = |C_{\alpha_2,u_2}| = 4^4, d = 3$. Therefore, $\zeta_2(C_{\alpha_2}) = \langle f_{2,0}(x) \rangle$ is a cyclic code over R_2 with length $\alpha_2 = 7$ with cardinality $4^{28-12} = 4^{16}, d = 3$.

Let $f_{3,0,1}(x) = f_{3,0,u_1}(x) = f_{3,0,u_2}(x) = f_{3,0,u_3}(x) = x^6 + wx^4 + wx^3 + x^2 + (w+1)x + 1, f_{3,0,u_1u_2}(x) = f_{3,0,u_1u_3}(x) = f_{3,0,u_2u_3}(x) = f_{3,0,u_{12}u_3}(x) = x^6 + (w+1)x^5 + x^4 + wx^3 + wx^2 + 1$. Then $\zeta_0(C_{\alpha_3,1}) = \zeta_0(C_{\alpha_3,u_1}) = \zeta_0(C_{\alpha_3,u_2}) = \zeta_0(C_{\alpha_3,u_3}) = \langle f_{3,0,1}(x) \rangle, \zeta_0(C_{\alpha_3,u_1u_2}) = \zeta_0(C_{\alpha_3,u_1u_3}) = \zeta_0(C_{\alpha_3,u_2u_3}) = \zeta_0(C_{\alpha_3,u_1u_2u_3}) = \langle f_{3,0,u_1u_2}(x) \rangle$ are cyclic codes over R_0 with length 35 and $|C_{\alpha_3,1}| = |C_{\alpha_3,u_1u_2}| = 4^{29}, d = 3$. Therefore, $\zeta_3(C_{\alpha_3}) = \langle f_{3,0}(x) \rangle$ is a cyclic code over R_3 with length $\alpha_3 = 35$ with cardinality $4^{280-48} = 4^{232}, d = 3$.

Hence, $\Upsilon_{\alpha_0\alpha_1..\alpha_a}(C) = \langle f_{0,0}(x) \rangle \times \langle f_{1,0}(x) \rangle \times \langle f_{2,0}(x) \rangle \times \langle f_{3,0}(x) \rangle$ is a separable $R_0R_1R_2R_3$-cyclic code of block length $(3, 5, 7, 35)$. Moreover, $|C| = 4^{255}$, $d = 3$.

§5. DNA Codes

In this section, some basic definitions and details about cyclic DNA codes over R_0 and R_1 in literature will be given. Later the necessary and sufficient conditions cyclic codes over R_i for $i = 1, \cdots, a$ and separable $R_0R_1R_2 \cdots R_a$ cyclic codes to be reversible and reversible complement will be discussed.

It is well known that DNA has two strands that are linked that Watson-Crick pairing, every A is linked with a T and every C is linked with a G, and vice versa, where A, T, C, and G are four bases of DNA sequences. i.e. one writes $\overline{A} = T, \overline{T} = A, \overline{C} = G$ and $\overline{G} = C$. The $\overline{A}$ denotes complement of A.

Let M be a finite commutative ring and $\mathbf{C}$ be a linear code of length n over M. Let $\mathbf{a} = (a_1, \cdots, a_n)$ be a codeword in $\mathbf{C}$. The reverse of $\mathbf{a}$ is $\mathbf{a}^R = (a_n, a_{n-1}, \cdots, a_1)$. The complement of $\mathbf{a}$ is $\mathbf{a}^C = (\overline{a}_1, \overline{a}_2, \cdots, \overline{a}_n)$. The reverse complement of $\mathbf{a}$ is $\mathbf{a}^{RC} = (\overline{a}_n, \overline{a}_{n-1}, \cdots, \overline{a}_1)$, where $\overline{a_i}$ denotes complement of a_i, for $i = 1, \cdots, n$.

Definition 5.1 *Let* $\mathbf{C}$ *be a linear code of length* n *over* M*. Then* $\mathbf{C}$ *is called reversible if*

$\mathbf{a}^R \in \mathbf{C}$, *for any* $\mathbf{a} \in \mathbf{C}$, $\mathbf{C}$ *is called complement if* $\mathbf{a}^C \in \mathbf{C}$, *for any* $\mathbf{a} \in \mathbf{C}$ *and* $\mathbf{C}$ *is called reversible complement if* $\mathbf{a}^{RC} \in \mathbf{C}$, *for any* $\mathbf{a} \in \mathbf{C}$.

Definition 5.2 *Let* $\mathbf{C}$ *be a linear code of length* n *over* M. *Then* $\mathbf{C}$ *is said to be cyclic DNA codes if* $\mathbf{C}$ *is a cyclic and reversible complement.*

Definition 5.3 *For any polynomial* $s(x) = s_0 + s_1x + \cdots + s_tx^t \in M[x]$ *with* $s_t \neq 0$, *the reciprocal polynomial of* $s(x)$ *is defined as*

$$s^*(x) = x^t s(1/x)$$

If $s^*(x) = s(x)$, *then* $s(x)$ *is called self reciprocal.*

Lemma 5.4([9]) *Let* $\zeta_0(C_{\alpha_0}) = \langle f(x)\rangle$ *be a cyclic code of length* α_0 *over* $R_0 = F_4$. *Then* C_{α_0} *is reversible if and only if* $f(x)$ *is self reciprocal.*

In [1], they studied cyclic DNA code over $R_0 = F_4$ and used the bijection map between the set of DNA alphabet $S_{D_4} = \{A, T, C, G\}$ and $R_0 = F_4 = \{0, 1, w, w^2\}$, with $0 \mapsto A, 1 \mapsto T, w \mapsto C, w + 1 \mapsto G$.

Lemma 5.5([1]) *Let* $\zeta_0(C_{\alpha_0}) = \langle f(x)\rangle$*be a cyclic code of length* α_0 *over* $R_0 = F_4$. *Then* C_{α_0} *is a complement if and only if* $f(x)$ *is not divisible by* $x - 1$.

Theorem 5.6([7]) *Let* $\zeta_0(C_{\alpha_0}) = \langle f(x)\rangle$*be a cyclic code of length* α_0 *over* $R_0 = F_4$. *Then,* C_{α_0} *is reversible complement if and only if* $f(x)$ *is self reciprocal and* $f(x)$ *is not divisible by* $x - 1$.

In [7], they extended the map from R_1 to $S_{D_4}^2$, by using the Gray map ψ_1 as follows:

$r_1 \in R_1$	Gray Images $\psi_1(r_1)$	Codon $\gamma_1(r_1)$
0	$(0,0)$	AA
u_1	$(0,1)$	AT
u_1w	$(0,w)$	AC
u_1w^2	$(0,w^2)$	AG
1	$(1,1)$	TT
$1+u_1$	$(1,0)$	TA
$1+u_1w^2$	$(1,w)$	TC
$1+u_1w$	$(1,w^2)$	TG
w	(w,w)	CC
u_1w+w	$(w,0)$	CA
u_1w^2+w	$(w,1)$	CT
u_1+w	(w,w^2)	CG

$w^2 + u_1 w^2$	$(w^2, 0)$	GA
$w^2 + u_1$	(w^2, w)	GC
$w^2 + u_1 w$	$(w^2, 1)$	GT
w^2	(w^2, w^2)	GG

Definition 5.7([7]) *Let C_{α_1} be a linear code of length α_1 over R_1 and $\mathbf{r_1} = (r_0, \cdots, r_{\alpha_1 - 1}) \in C_{\alpha_1}$. By using the table, the map Λ_1 is defined as follows*

$$\begin{aligned} \Lambda_1 \quad &: \quad C_{\alpha_1} \longrightarrow S_{D_4}^{2\alpha_1} \\ \mathbf{r_1} = (r_0, \cdots, r_{\alpha_1 - 1}) \quad &\mapsto \quad \Lambda_1(\mathbf{r_1}) = (\gamma_1(r_0), \cdots, \gamma_1(r_{\alpha_1 - 1})). \end{aligned}$$

Theorem 5.8([7]) *Let $C_{\alpha_1} = C_{\alpha_1,1} e_1 + C_{\alpha_1,u_1} e_{u_1}$ be a cyclic code of length α_1 over R_1. Then, C_{α_1} is reversible over R_1 if and only if $C_{\alpha_1,1}, C_{\alpha_1,u_1}$ are reversible over F_4.*

Lemma 5.9([7]) *For any $r_1 \in R_1$, $\overline{r_1} + \overline{0} = r_1$.*

Theorem 5.10([7]) *Let $C_{\alpha_1} = C_{\alpha_1,1} e_1 + C_{\alpha_1,u_1} e_{u_1}$ be a cyclic code of length α_1 over R_1. Then, C_{α_1} is reversible complement over R_1 if and only if C_{α_1} is reversible and $(\overline{0}, \cdots, \overline{0}) \in C_{\alpha_1}$.*

Theorem 5.11 *Let C_{α_1} be a cyclic DNA code of length α_1 over the ring R_1 and minimum Hamming distance d. Then, $\Lambda_1(C_{\alpha_1})$ is a DNA code of length $2\alpha_1$ over the alphabet $\{A, T, G, C\}$ with minimum Hamming distance at least d.*

Similarly, we define a bijection map between R_2 to $S_{D_4}^4$ as follows, by considering the Gray images of elements of R_2.

$r_2 \in R_2$	Gray Images $\psi_2(r_2)$	Codon $\gamma_2(r_2)$
0	$(0, 0, 0, 0)$	$AAAA$
u_1	$(0, 1, 0, 1)$	$TATA$
$1 + w$	(w^2, w^2, w^2, w^2)	$GGGG$
1	$(1, 1, 1, 1)$	$TTTT$
w	(w, w, w, w)	$CCCC$
$\cdots$	$\cdots$	$\cdots$

Definition 5.12 *Let C_{α_2} be a linear code of length α_2 over R_2 and $\mathbf{r_2} = (r_0, \cdots, r_{\alpha_2 - 1}) \in C_{\alpha_2}$. By using the table, the map Λ_2 is defined as follows*

$$\begin{aligned} \Lambda_2 \quad &: \quad C_{\alpha_2} \longrightarrow S_{D_4}^{4\alpha_2} \\ \mathbf{r_2} = (r_0, \cdots, r_{\alpha_2 - 1}) \quad &\mapsto \quad \Lambda_2(\mathbf{r_2}) = (\gamma_2(r_0), \cdots, \gamma_2(r_{\alpha_2 - 1})). \end{aligned}$$

Theorem 5.13 *Let $C_{\alpha_2} = C_{\alpha_2,1} e_1 + C_{\alpha_2,u_1} e_{u_1} + C_{\alpha_2,u_2} e_{u_2} + C_{\alpha_2,u_1 u_2} e_{u_1 u_2}$ be a cyclic code of* *length α_2 over R_2. Then C_{α_2} is reversible over R_2 if and only if $C_{\alpha_2,1}, C_{\alpha_2,u_1}, C_{\alpha_2,u_2}, C_{\alpha_2,u_1u_2}$ are reversible over F_4.*

Lemma 5.14 *For any $r_2 \in R_2$, $\overline{r_2} + \overline{0} = r_2$.*

Theorem 5.15 *Let $C_{\alpha_2} = C_{\alpha_2,1}e_1 + C_{\alpha_2,u_1}e_{u_1} + C_{\alpha_2,u_2}e_{u_2} + C_{\alpha_2,u_1u_2}e_{u_1u_2}$ be a cyclic code of length α_2 over R_2. Then C_{α_2} is reversible complement over R_2 if and only if C_{α_2} are reversible over R_2 and $(\overline{0}, \cdots, \overline{0}) \in C_{\alpha_2}$.*

Theorem 5.16 *Let C_{α_2} be a cyclic DNA code of length α_2 over the ring R_2 and minimum Hamming distance d. Then, $\Lambda_2(C_{\alpha_2})$ is a DNA code of length $4\alpha_2$ over the alphabet $\{A, T, G, C\}$ with minimum Hamming distance at least d.*

Similarly, we define a bijection map between R_i to $S_{D_4}^{2^i}$ as follows, by considering the Gray images of elements of R_i for $i = 3, \cdots, a$.

$r_i \in R_i$	Gray Images $\psi_i(r_i)$	Codon $\gamma_i(r_i)$
0	$(0, 0, 0, \cdots, 0)$	$AAA \cdots A$
$1 + w$	$(w^2, w^2, w^2, \cdots, w^2)$	$GGG \cdots G$
1	$(1, 1, 1, \cdots, 1)$	$TTT \cdots T$
w	$(w, w, w, \cdots, w)$	$CCC \cdots C$
$\cdots$	$\cdots$	$\cdots$

Definition 5.17 *Let C_{α_i} be a linear code of length α_i over R_i and $\mathbf{r_i} = (r_0, \cdots, r_{\alpha_i - 1}) \in C_{\alpha_i}$ for $i = 3, .., a$. By using the table, the map Λ_i is defined as follows*

$$\begin{array}{rcl} \Lambda_i & : & C_{\alpha_i} \longrightarrow S_{D_4}^{2^i \alpha_i} \\ \mathbf{r_i} = (r_0, \cdots, r_{\alpha_i - 1}) & \mapsto & \Lambda_i(\mathbf{r_i}) = (\gamma_i(r_0), \cdots, \gamma_i(r_{\alpha_i - 1})) \end{array}$$

Theorem 5.18 *Let $C_{\alpha_i} = \oplus_{B \in P_i} C_{\alpha_i, u_B} e_{u_B}$ be a cyclic code of length α_i over R_i for $i = 3, \cdots, a$. Then C_{α_i} is reversible over R_i if and only if C_{α_i, u_B} are reversible over $R_0 = F_4$, where all $B \in P_i$ for $i = 3, \cdots, a$.*

Proof It is proven as proof of Theorem 10 in [7]. □

From Lemma 5.9 and Lemma 5.14, the following lemma can be obtained.

Lemma 5.19 *The following conditions hold.*

i) For any $r_i \in R_i$, $\overline{r_i} = \overline{(x_{i-1} + u_i y_{i-1})} = \overline{(x_{i-1})} + u_i y_{i-1}$, where $x_{i-1}, y_{i-1} \in R_{i-1}, i = 3, 2, \cdots, a$;

ii) For any $r_i \in R_i$, $\overline{r_i} + \overline{0} = r_i$ for $i = 3, \cdots, a$.

Theorem 5.20 *Let $C_{\alpha_i} = \oplus_{B \in P_i} C_{\alpha_i, u_B} e_{u_B}$ be a cyclic code of length α_i over R_i, for $i = 3, \cdots, a$. Then, C_{α_i} is reversible complement over R_i if and only if C_{α_i} are reversible over R_i*

and $(\overline{0},\cdots,\overline{0})\in C_{\alpha_i}$, *where all* $B\in P_i$ *for* $i=3,\cdots,a$.

Proof It is proven as proof of Theorem 11 in [7]. □

Theorem 5.21 *Let* C_{α_i} *be a cyclic DNA code of length* α_i *over the ring* R_i *and minimum Hamming distance* d. *Then,* $\Lambda_i(C_{\alpha_i})$ *is a DNA code of length* $2^i\alpha_i$ *over the alphabet* $\{A,T,G,C\}$ *with minimum Hamming distance at least* d, *for* $i=3,\cdots,a$.

Example 5.22 Let $R_1=F_4+u_1F_4$,

$$\begin{aligned}x^{11}-1 &= (x+1)\left(x^5+wx^4+x^3+x^2+(w+1)\,x+1\right)\\ &\quad \left(x^5+(w+1)\,x^4+x^3+x^2+wx+1\right)\\ &= h_1(x)h_2(x)h_3(x)\in F_4\left[x\right]\end{aligned}$$

and let $f_{1,0,1}\left(x\right)=f_{1,0,u_1}\left(x\right)=h_2(x)h_3(x)$. Then $\zeta_0(C_{\alpha_1,1})=\zeta_0(C_{\alpha_1,u_1})=\langle f_{1,0,1}\left(x\right)\rangle$ are cyclic codes over R_0 with length 11 and $|C_{\alpha_1,1}|=4, d=11$. Hence, $\zeta_1(C_{\alpha_1})=\langle f_{1,0}\left(x\right)\rangle$ is a cyclic code over R_1 with length 11 with $d=11$. As $f_{1,0,1}\left(x\right)$ and $f_{1,0,u_1}\left(x\right)$ are self-reciprocal polynomials, by Lemma 5.24, $C_{\alpha_1,1}, C_{\alpha_1,u_1}$ are reversible over R_0. Therefore, C_{α_1} is reversible over R_1. Also, C_{α_1} has $4^{22-20}=16$ codewords. As C_{α_1} is reversible and $\left(\overline{0},\overline{0},\cdots,\overline{0}\right)\in C_{\alpha_1}$, then C_{α_1} is a reversible complement code over R_1. Moreover, C_{α_1} is a cyclic DNA code and the image of the C_{α_1} under the map Δ is a DNA code of length 22, size 16 and minimum distance $d=11$. The DNA codewords are given in the following

A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
A	C	A	C	A	C	A	C	A	C	A	C	A	C	A	C	A	C	A	C	A	C
A	G	A	G	A	G	A	G	A	G	A	G	A	G	A	G	A	G	A	G	A	G
A	T	A	T	A	T	A	T	A	T	A	T	A	T	A	T	A	T	A	T	A	T
C	A	C	A	C	A	C	A	C	A	C	A	C	A	C	A	C	A	C	A	C	A
C	G	C	G	C	G	C	G	C	G	C	G	C	G	C	G	C	G	C	G	C	G
C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C	C
C	T	C	T	C	T	C	T	C	T	C	T	C	T	C	T	C	T	C	T	C	T

T	C	T	C	T	C	T	C	T	C	T	C	T	C	T	C	T	C	T	C	T	C
T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T
T	A	T	A	T	A	T	A	T	A	T	A	T	A	T	A	T	A	T	A	T	A
T	G	T	G	T	G	T	G	T	G	T	G	T	G	T	G	T	G	T	G	T	G
G	T	G	T	G	T	G	T	G	T	G	T	G	T	G	T	G	T	G	T	G	T
G	A	G	A	G	A	G	A	G	A	G	A	G	A	G	A	G	A	G	A	G	A
G	C	G	C	G	C	G	C	G	C	G	C	G	C	G	C	G	C	G	C	G	C
G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G	G

Example 5.23 Let $R_2=F_4+u_1F_4+u_2F_4+u_1u_2F_4, \alpha_2=41$ and $f_{2,0,1}\left(x\right)=f_{2,0,u_1}\left(x\right)=f_{2,0,u_2}\left(x\right)=f_{2,0,u_1u_2}\left(x\right)=x^{20}+wx^{19}+wx^{17}+(w+1)\,x^{16}+x^{13}+(w+1)\,x^{12}+x^{11}+(w+1)\,x^{10}+x^9+(w+1)\,x^8+x^7+(w+1)\,x^4+wx^3+wx+1$. Then $\zeta_0(C_{\alpha_3,u_1u_2})=\zeta_0(C_{\alpha_3,u_1u_3})=\zeta_0(C_{\alpha_3,u_2u_3})=\zeta_0(C_{\alpha_3,u_1u_2u_3})$ are cyclic codes over R_0 with length 41 and $|C_{\alpha_1,1}|=4^{21}, d=$

11. Hence, $\zeta_2(C_{\alpha_2}) = \langle f_{2,0}(x)\rangle$ is a cyclic code over R_2 with length 41 with $d = 11$. As $f_{2,0,1}(x), f_{2,0,u_1}(x), f_{2,0,u_2}(x)$ and $f_{2,0,u_1u_2}(x)$ are self-reciprocal polynomials, by Lemma 5.24, $C_{\alpha_3,u_1u_2}, C_{\alpha_3,u_1u_3}, C_{\alpha_3,u_2u_3}, C_{\alpha_3,u_1u_2u_3}$ are reversible codes over R_0. Therefore, C_{α_2} is a reversible code over R_2. Also, C_{α_2} has $4^{164-80} = 4^{84}$ codewords. As C_{α_2} is a reversible code and $(\overline{0}, \overline{0}, \cdots, \overline{0}) \in C_{\alpha_2}$, C_{α_2} is a reversible complement code over R_2. Moreover, C_{α_2} is a cyclic DNA code and the image of the C_{α_2} under the map Δ is a DNA code of length 164, size 4^{84} and minimum distance $d = 11$.

Definition 5.24 *Let D be an $R_0R_1R_2\cdots R_a$-linear code of block length $(\alpha_0, \cdots, \alpha_a)$. Then D is said to be reversible, if $\mathbf{r}^R = (\mathbf{r}_0^R, \cdots, \mathbf{r}_a^R) \in D$, for any $\mathbf{r} = (\mathbf{r}_0, \cdots, \mathbf{r}_a) \in D$, where $\mathbf{r}_i = (r_0^i, r_1^i, \cdots, r_{\alpha_i-1}^i) \in R_i^{\alpha_i}$ for $i = 0, 1, \cdots, a$.*

Definition 5.25 *Let D be an $R_0R_1R_2\cdots R_a$ linear code of block length $(\alpha_0, \cdots, \alpha_a)$. Then D is said to be complement, if $\mathbf{r}^C = (\mathbf{r}_0^C, \cdots, \mathbf{r}_a^C) \in D$, for any $\mathbf{r} = (\mathbf{r}_0, \cdots, \mathbf{r}_a) \in D$, where $\mathbf{r}_i = (r_0^i, r_1^i, \cdots, r_{\alpha_i-1}^i) \in R_i^{\alpha_i}$ for $i = 0, 1, \cdots, a$.*

Definition 5.26 *Let D be an $R_0R_1R_2\cdots R_a$ linear code of block length $(\alpha_0, \cdots, \alpha_a)$. Then, D is said to be reversible complement, if $\mathbf{r}^{RC} = (\mathbf{r}_0^{RC}, \cdots, \mathbf{r}_a^{RC}) \in D$, for any $\mathbf{r} = (\mathbf{r}_0, \cdots, \mathbf{r}_a) \in D$, where $\mathbf{r}_i = (r_0^i, r_1^i, \cdots, r_{\alpha_i-1}^i) \in R_i^{\alpha_i}$ for $i = 0, 1, \cdots, a$.*

Theorem 5.27 *Let $C = C_{\alpha_0} \times \cdots \times C_{\alpha_i}$ be separable $R_0R_1R_2\cdots R_i$-cyclic code of block length $(\alpha_0, \cdots, \alpha_i)$, where C_{α_i} are cyclic codes of length α_i over R_i, for $i = 0, 1, 2, \cdots, a$. Then C is reversible if and only if $C_{\alpha_0}, C_{\alpha_1}, \cdots, C_{\alpha_i}$ are reversible codes over R_i, for $i = 0, 1, 2, \cdots, a$, respectively.*

Proof Let C be a reversible code and let $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_a) \in C$, $\mathbf{r}_i = (r_0^i, r_1^i, \cdots, r_{\alpha_i-1}^i) \in C_{\alpha_i}$ for $i = 0, 1, \cdots, a$. By using the fact that C is a reversible, we have $\mathbf{r}^R = (\mathbf{r}_0^R, \mathbf{r}_1^R, \cdots, \mathbf{r}_a^R) \in C$. So this shows that $\mathbf{r}_i^R \in C_{\alpha_i}$ where $i = 0, 1, 2, \cdots, a$. Therefore, C_{α_i} are reversible codes of length α_i over R_i, for $i = 0, 1, \cdots, a$, respectively. Conversely, let C_{α_i} be reversible codes of length α_i over R_i, for $i = 0, 1, \cdots, a$, respectively and take $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_a) \in C$ where $\mathbf{r}_i \in C_{\alpha_i}$ with $i = 0, 1, \cdots, a$. By using the fact that C_{α_i} are reversible, then $\mathbf{r}_i^R \in C_{\alpha_i}$ for $i = 0, 1, \cdots, a$. So $\mathbf{r}^R = (\mathbf{r}_0^R, \mathbf{r}_1^R, \cdots, \mathbf{r}_a^R) \in C$. Hence C is reversible. □

Theorem 5.28 *Let $C = C_{\alpha_0} \times \cdots \times C_{\alpha_i}$ be separable $R_0R_1R_2\cdots R_i$-cyclic code of block length $(\alpha_0, \cdots, \alpha_i)$, where C_{α_i} are cyclic codes of length α_i over R_i, for $i = 0, 1, 2, \cdots, a$. Then, C is a reversible complement code if and only if $C_{\alpha_0}, C_{\alpha_1}, \cdots, C_{\alpha_i}$ are reversible complement codes over R_i, for $i = 0, 1, 2, \cdots, a$, respectively.*

Proof Let C be a reversible complement code. Take $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_a) \in C$, where $\mathbf{r}_i = (r_0^i, r_1^i, \cdots, r_{\alpha_i-1}^i) \in C_{\alpha_i}$ for $i = 0, 1, \cdots, a$. By using the fact that C is a reversible complement, we have $\mathbf{r}^{RC} = (\mathbf{r}_0^{RC}, \mathbf{r}_1^{RC}, \cdots, \mathbf{r}_a^{RC}) \in C$. So this shows that $\mathbf{r}_i^{RC} \in C_{\alpha_i}$ where $i = 0, 1, 2, \cdots, a$. Therefore C_{α_i} are reversible complement codes of length α_i over R_i, for $i = 0, 1, \cdots, a$, respectively. Conversely, let C_{α_i} be reversible complement codes of length α_i over R_i, for $i = 0, 1, \cdots, a$, respectively and take $\mathbf{r} = (\mathbf{r}_0, \mathbf{r}_1, \cdots, \mathbf{r}_a) \in C$ where $\mathbf{r}_i \in C_{\alpha_i}$ where $i = 0, 1, \cdots, a$. By using the fact that C_{α_i} are reversible complement, then $\mathbf{r}_i^{RC} \in C_{\alpha_i}$ for

$i = 0, 1, \cdots, a$. So $\mathbf{r}^{RC} = (\mathbf{r}_0^{RC}, \mathbf{r}_1^{RC}, \cdots, \mathbf{r}_a^{RC}) \in C$. Hence, C is a reversible complement. □

Definition 5.29 *Let C be an $R_0R_1R_2\cdots R_a$-linear code of block length $(\alpha_0, \cdots, \alpha_a)$ and $\mathbf{r} = (\mathbf{r}_0, \cdots, \mathbf{r}_a) = (r_0^0, \cdots, r_{\alpha_0-1}^0, r_0^1, \cdots, r_{\alpha_1-1}^1, \cdots, r_0^a, \cdots, r_{\alpha_a-1}^a) \in R_{\alpha_0,\cdots,\alpha_a}$. By using the table, the map Δ is defined as follows*

$$\begin{aligned} \Delta \quad &: \quad R_{\alpha_0,\cdots,\alpha_a} \longrightarrow S_{D_4}^{\alpha_0+2\alpha_1+\cdots+2^a\alpha_a} \\ \mathbf{r} = (\mathbf{r}_0, \cdots, \mathbf{r}_a) \quad &\mapsto \quad \Delta(\mathbf{r}) = (\Lambda_0(\mathbf{r}_0), \cdots, \Lambda_a(\mathbf{r}_a)). \end{aligned}$$

Theorem 5.30 *Let C be a separable $R_0R_1R_2\cdots R_a$ cyclic DNA code of block length $(\alpha_0, \cdots, \alpha_a)$ with $|C| = M$ and minimum Hamming distance d. Then, $\Delta(C)$ is a DNA code of length*

$$\alpha_0 + 2\alpha_1 + \cdots + 2^a\alpha_a$$

over the alphabet $\{A, T, G, C\}$ with the minimum Hamming distance at least d.

Example 5.31 Let $\Upsilon_{\alpha_0\alpha_1\alpha_2\alpha_3}(C) = \langle f_{0,0}(x)\rangle \times \langle f_{1,0}(x)\rangle \times \langle f_{2,0}(x)\rangle \times \langle f_{3,0}(x)\rangle$ is a separable $R_0R_1R_2R_3$-cyclic code of block length $(9, 3, 5, 35)$. Let $f_{0,0}(x) = x^6 + x^3 + 1$. Then,

$$\zeta_0(C_{\alpha_0}) = \langle f_{0,0}(x)\rangle$$

is a cyclic codes over R_0 with length 9. As $f_{0,0}(x)$ is self-reciprocal polynomial and $\left(\overline{0}, \overline{0}, \cdots, \overline{0}\right) \in C_{\alpha_0}$, C_{α_0} is reversible complement over R_0. Similarly, if we take $\zeta_1(C_{\alpha_1}) = \langle f_{1,0}(x)\rangle = \langle x+1\rangle, \zeta_2(C_{\alpha_2}) = \langle f_{2,0}(x)\rangle = \left\langle x^2 + wx + 1\right\rangle, \zeta_3(C_{\alpha_3}) = \langle f_{3,0}(x)\rangle = \left\langle x^2 + (w+1)x + 1\right\rangle$, C_{α_i}'s will be reversible complement codes over R_i, for $i = 1, 2, 3$. Hence, by Theorem 5.28, we get C as a reversible complement code . Then $\Delta(C)$ is a DNA code of length 315, size 4^{35}.

§6. Conclusion

In this paper, the structures of $R_0R_1R_2\cdots R_a$-cyclic codes are obtained with their generator polynomials, and a new inner product is defined. It was shown that if C is an $R_0R_1R_2\cdots R_a$-cyclic code, then $C^\perp$ is an $R_0R_1R_2\cdots R_a$-cyclic code, and a separable $R_0R_1R_2\cdots R_a$-cyclic codes are introduced with necessary and sufficient conditions for them being reversible and reversible complement, and also the DNA codes can be constructed from them with examples.

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